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Construction of Hermitian Self-Orthogonal Codes and Application

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Abstract: We introduce some methods for constructing quaternary Hermitian self-orthogonal (HSO) codes, and construct quaternary $[n,5]$ HSO for $342 \leq n \leq 492$. Furthermore, we present methods of constructing Hermitian linear complementary dual (HLCD) codes from known HSO codes, and obtain many HLCD codes with good parameters. As an application, 31 classes of entanglement-assisted quantum error correction codes (EAQECCs) with maximal entanglement can be obtained from these HLCD codes. These new EAQECCs have better parameters than those in the literature.

Keywords: Hermitian self-orthogonal code; Hermitian linear complementary dual code; quantum error-correcting code

MSC: 94B05; 11T71

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1. Introduction

Let $F_4 = \{0, 1, \omega, \omega^2\}$ be the field with four elements, where $\omega^2 = 1 + \omega$, and F_4^n be the n-dimensional row space over F_4. For $x \in F_4$, its conjugate is $\overline{x} = x^2$. A quaternary $[n,k]$ code $\mathcal{C}$ is a k-dimensional subspace of F_4^n, vectors in $\mathcal{C}$ are called as codewords of $\mathcal{C}$. If the minimal Hamming weights of non-zero codewords in $\mathcal{C}$ is d, then $\mathcal{C}$ is denoted as $\mathcal{C} = [n,k,d]$. A linear code $[n,k,d]$ is optimal if there is no $[n,k,d+1]$ code; such a code is denoted as $\mathcal{C} = [n,k,d_{op}(n,k)]$. For $u = (u_1, u_2 \cdots u_n)$, $v = (v_1, v_2 \cdots v_n) \in F_4^n$, their Hermitian inner product is $(u,v)_h = \sum_{i=1}^n u_i \cdot \overline{v_i} = \sum_{i=1}^n u_i \cdot v_i^2$. The Hermitian dual code $\mathcal{C}^{\perp_h}$ is defined as $\mathcal{C}^{\perp_h} = \{u \in F_4^n \mid (u,v)_h = 0, \forall v \in \mathcal{C}\}$. If $\mathcal{C} \subseteq \mathcal{C}^{\perp_h}$, then $\mathcal{C}$ is called as an HSO code. In particular, if $\mathcal{C} = \mathcal{C}^{\perp_h}$, then $\mathcal{C}$ is called a Hermitian self-dual code. If $\mathcal{C} \cap \mathcal{C}^{\perp_h} = \{0\}$, then $\mathcal{C}$ is called an HLCD code.

In the past 30 years, much work has been conducted concerning quaternary optimal linear codes for both theoretical and practical reasons (see, e.g., [1–5] and the references given therein). By 1996, the parameters of all optimal linear codes with $k \leq 4$ were determined [2–4]. Since 1996, people have paid much attention to optimal linear codes with $k = 5$; to date, there are yet 104 open cases on the parameters of optimal linear codes (see [3–6]). If $d_{op}(n,k)$ is not determined for a given n,k, a code $\mathcal{C}$ with the largest known distance $d_{bk}(n,k)$ is denoted as $\mathcal{C} = [n,k,d_{bk}(n,k)]$. Quaternary HSO codes and HLCD codes are special kinds of linear codes, these two kinds of codes have connections with many branches of mathematics and quantum information [7–19]. In recent years, there has been an increasing interest in optimal HSO and HLCD codes. A HSO (or HLCD) $[n,k,d]$ code is called optimal if there is no HSO (or HLCD) $[n,k,d']$ code for $d' > d$. Entanglement-assisted (EA) stabilizer formalism was devised by Brun et al. in [20]. It has been proven that each $[n,k,d]$ quaternary HLCD code gave a maximal entanglement-assisted quantum code with a parameter $[[n,k,d;n-k]]_2$ by [14,19,21]. Under this EA stabilizer formalism, any quaternary HLCD code can be transformed into a maximal EAQECC if the shared entanglement is available between sender and receiver. Hence, it is important to study optimal quaternary HLCD codes for constructing $[[n,k,d;n-k]]_2$ EAQECCs.

Recall that, from 1978 to 1998, people paid much attention to special HSO codes—self-dual codes with short lengths (see [7–10]). In 1998, Calderbank et al. [11] set up connections among quantum codes, binary symplectic codes, and HSO codes; this inspired people to study HSO codes over F_4 for general length n. Bouyukliev et al. [12] discussed the classification of optimal HSO codes over F_4 for length $n \leq 29$ and low dimensions. Ma et al. [13] determined the parameters of optimal HSO codes over F_4 for all n and $k \leq 3$. Recently, Refs. [17,18] determined the parameters of optimal $[n,4]$ HSO codes for all $n \geq 8$ and most $[n,5]$ HSO codes for $n \leq 341$. In [14], Lu et al. showed that an $[n,k,d]$ HLCD (also called zero radical) code can derive an $[[n,k,d;n-k]]_2$ EAQECC, they construct many good EAQECCs. According to [14–16], the parameters of optimal HLCD codes were determined for all n and $k \leq 3$. Refs. [14,19] discussed the construction of $[n,4]$ HLCD codes with $4 \leq n \leq 85$ and $[n,5]$ HLCD codes with $5 \leq n \leq 341$, respectively. They have given some optimal HLCD codes and good low bounds on the distance of optimal HLCD codes. According to [3], the parameters of optimal $[n,5]$ linear codes for $n \geq 492$ are known and can be constructed using a unified method of puncturing. Thus, the construction of $[n,5]$ optimal HSO codes for $n \geq 492$ can be conducted as we conducted in [18] for constructing $[n,4]$ HSO codes with $n \geq 124$. According to [6,14,18,19], when discussing $[n,5]$ HSO codes and HLCD codes, we should consider codes with length n such that $342 \leq n \leq 492$.

This paper is organized as follows. In Section 2, we prepare the definitions, notations, and basic results used in this paper. In Section 3, the construction of $[n,5]$ HSO codes is presented. In Section 4, we derive $[n,5]$ HLCD codes from known HSO codes and related EAQECCs. Finally, in Section 5, we conclude this paper.

2. Preliminaries

In this section, some notations, definitions, and basic results are given (for details, see Ref. [22]).

Throughout this paper, we use the following notations. We assume all codes are linear codes over F_4, and use 2 and 3 to represent ω and ω^2 in F_4, respectively. Let $\mathbf{1_n}=(1,1,\ldots,1)_{1\times n}$ and $\mathbf{0_n}=(0,0,\ldots,0)_{1\times n}$ denote the all-one vector and the all-zero vector of length n, respectively. Let 0 denote a zero matrix of appropriate size and I_k denote the identity matrix of order k. Let A^T denote the transpose of a matrix A, and let $A^\dagger$ denote the conjugate transpose of A.

Let $\mathcal{C} = [n,k]$. A $k \times n$ matrix G whose rows form a basis of $\mathcal{C}$ is called a generator matrix. The weight enumerator of $\mathcal{C}$ is $w(z) = \sum_{i=0}^{n} A_i z^i = A_0 + A_1 z + \cdots + A_n z^n$, where A_i is the number of codewords in $\mathcal{C}$ with weight equal to i for $0 \leq i \leq n$. We say that two $[n,k]$ codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent, provided there is a monomial matrix M such that $\mathcal{C}_2 = \mathcal{C}_1 M$. A code $\mathcal{C}$ is called an even code if all its codewords have even weights [22]. According to Ref. [11], $\mathcal{C}$ is a HSO code if and only if it is an even code. Using the generator matrix, one can give the following criterion for a code to be HSO or HLCD.

Proposition 1 ([15]). *Let G be a generator matrix of $\mathcal{C}$; then,*

(1) $\mathcal{C}$ is an HSO code if and only if $GG^\dagger = 0$.
(2) $\mathcal{C}$ is a HLCD code if and only if $GG^\dagger$ is nonsingular.

Definition 1. *If $A = A_{k,m}$ is a $k \times m$ matrix and the vectors formed by row linear combination of A have the largest weight δ, then A is called as an (m,δ) block. If $AA^\dagger = 0$, A is called an (m,δ) HSO block. If $AA^\dagger$ is nonsingular, A is called an (m,δ) HLCD block.*

We introduce some methods for constructing new codes from known ones. Lemmas 1 and 2 are directly obtained from [17] for linear codes. Lemma 3 and 4 can be derived from [18]. These four lemmas give the constructions of HSO codes by juxtaposition, pasting, puncturing, and shorting, respectively.

Lemma 1 ([17]). *Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are $[n_1, k, d_1]$ and $[n_2, k, d_2]$ HSO codes, respectively. If $\mathcal{C}_1$ and $\mathcal{C}_2$ have generator matrices G_1 and G_2, respectively, then $(G_1 \mid G_2)$ generates an $[n_1 + n_2, k, d_1 + d_2]$ HSO code.*

Lemma 2 ([17]). *Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are $[n_1, k, d_1]$ and $[n_2, k-1, d_2]$ HSO codes, respectively. If $\mathcal{C}_1$ contains a codeword of weight at least $d_1 + d_2$, then there exists an $[n_1 + n_2, k, d_1 + d_2]$ HSO code.*

Lemma 3. *Suppose $\mathcal{C} = [n, k, d]$ is an HSO code with generator matrix $G_{k,n}$ and $G_{k,n}$ has a $k \times m$ sub-matrix A. If A is an (m, δ) HSO (HLCD) block, then there is an $[n - m, k, d - \delta]$ HSO (HLCD) code.*

Proof. Let generator matrix $G_{k,n} = (A_{k,m} \mid B_{k,n-m})$, then

$$G_{k,n}G^{\dagger}_{k,n} = A_{k,m}A^{\dagger}_{k,m} + B_{k,n-m}B^{\dagger}_{k,n-m}$$

according to $\mathcal{C}$ is a HSO code.

Let $A_{k,m}$ generate an HSO code $\mathcal{C}_A$. α is a codeword in $\mathcal{C}_A$ with minimum Hamming weight δ. Since HSO code $\mathcal{C} = [n, k, d]$, there is a codeword β in $\mathcal{C}$ with minimum Hamming weight d. $B_{k,n-m} = (G_{k,n} \setminus A_{k,m})$. According to [22], $B_{k,n-m}$ generates an $[n - m, k, d_B]$ code with $d_B = d - \delta$.

(1) If $A_{k,m}$ is an (m, δ) HSO block, $rank(G_{k,n}G^{\dagger}_{k,n}) = rank(A_{k,m}A^{\dagger}_{k,m}) = 0$. Then, $rank(B_{k,n-m}B^{\dagger}_{k,n-m}) = 0$ and $B_{k,n-m}$ generates an $[n - m, k, d - \delta]$ HSO code.

(2) If $A_{k,m}$ is an (m, δ) HLCD block, $rank(G_{k,n}G^{\dagger}_{k,n}) = 0$ and $rank(A_{k,m}A^{\dagger}_{k,m}) = k$. Then, $rank(B_{k,n-m}B^{\dagger}_{k,n-m}) = k$, $B_{k,n-m}$ generates an $[n - m, k, d - \delta]$ HLCD code.

□

Lemma 4. *Suppose $n \geq 341 + 4$ and $\mathcal{C} = [n, 5, d]$ is an HSO code with $d \geq 6$. Then, there are $[n - i, 4, d - 2\lceil i/2 \rceil]$ HSO codes for $i = 1, 2, 3, 4$.*

3. Constructing HSO Codes

In this section, we discuss the construction of $[n, 5]$ HSO codes for $342 \leq n \leq 492$; our results are given in two subsections.

3.1. $[n, 5]$ HSO Codes for $342 \leq n \leq 407$

In [3], the authors introduced a dual transform method for constructing new codes from known codes; they derived the existence of three codes with parameters $[364, 5, 272]$, $[386, 5, 288]$, and $[407, 5, 304]$ from three known codes with parameters $[27, 5, 16]$, $[38, 5, 24]$, and $[28, 5, 16]$, respectively. Using Magma [23], we can check that these three codes $[364, 5, 272]$, $[386, 5, 288]$, $[407, 5, 304]$ have generator matrices $G_{5,364}$, $G_{5,386}$, and $G_{5,407}$ (see Appendix A.1) and weight enumerators $1 + 942x^{272} + 81x^{288}$, $1 + 213x^{288} + 42x^{304}$, and $1 + 924x^{304} + 99x^{320}$, respectively.

We try to find HSO blocks in $G_{5,364}$. It is not difficult to see that A_{01a} and A_{01b} have submatrices SA_{01a} and SA'_{01a} and SA_{01b} and SA'_{01b} as follows, where

$$SA_{01a} = \begin{pmatrix} 0000 \\ 1111 \\ 0000 \\ 0123 \\ 1111 \end{pmatrix}, SA'_{01a} = \begin{pmatrix} 0000 \\ 1111 \\ 1111 \\ 3333 \\ 0123 \end{pmatrix}, SA_{01b} = \begin{pmatrix} 0000 \\ 1111 \\ 2222 \\ 0123 \\ 3102 \end{pmatrix}, SA'_{01b} = \begin{pmatrix} 0000 \\ 1111 \\ 3333 \\ 0123 \\ 1032 \end{pmatrix},$$

Columns $(00010)^T$, $(00001)^T$, $(00012)^T$, and $(00011)^T$ from A_{00} are added to matrices SA_{01a}, SA'_{01a}, SA_{01b}, and SA'_{01b}, respectively. One can obtain four 5×5 matrices, X_1, X_2, X_3, and X_4, of $G_{5,364}$, respectively, and they all satisfy $X_i \cdot X^{\dagger}_i = 0$ for $i = 1, 2, 3, 4$. It is easy to see that these X_i are formed by different columns of $G_{5,364}$, and each is a $(5,4)$ HSO block; thus, $G_{5,364}$ has $(5,4)$, $(10,8)$, $(15,12)$, and $(20,16)$ HSO blocks. According to Lemma 3, by removing these blocks, in turn, from $G_{5,364}$, HSO codes $[364-5i,5,272-4i]$ for $i=0,1,2,3,4$ can be constructed from $G_{5,364}$. From previous discussions, using Lemmas 3 and 4, we can achieve the following theorem.

Theorem 1. *Based on the* $[364,5,272]$ *HSO code, HSO codes with the following parameters can be constructed:*

(1) $[364-5i,5,272-4i]$ *for* $i=0,1,2,3,4$;
(2) $[364-5i-j,5,272-4i-2\lceil\frac{j}{2}\rceil]$ *for* $i=0,1,2,3$ *and* $j=1,2,3,4$.

Similar to the above discussion, we can show that $G_{5,386}$ has four 5×5 submatrices, Y_1, Y_2, Y_3, and Y_4, and $G_{5,407}$ has four 5×5 submatrices, Z_1, Z_2, Z_3, and Z_4, where

$$Y_1=\begin{pmatrix}0000\ 0\\0000\ 0\\0111\ 1\\1012\ 3\\0333\ 3\end{pmatrix},Y_2=\begin{pmatrix}0111\ 1\\0000\ 0\\0000\ 0\\1012\ 3\\1012\ 3\end{pmatrix},Y_3=\begin{pmatrix}0111\ 1\\0000\ 0\\0222\ 2\\1012\ 3\\3031\ 2\end{pmatrix},Y_4=\begin{pmatrix}0111\ 1\\0111\ 1\\1012\ 3\\0111\ 1\\0000\ 0\end{pmatrix};$$
$$Z_1=\begin{pmatrix}0000\ 0\\0000\ 0\\0111\ 1\\0333\ 3\\1012\ 3\end{pmatrix},Z_2=\begin{pmatrix}0000\ 0\\0111\ 1\\0000\ 0\\1012\ 3\\1230\ 1\end{pmatrix},Z_3=\begin{pmatrix}0000\ 0\\0111\ 1\\0111\ 1\\1012\ 3\\2132\ 0\end{pmatrix},Z_4=\begin{pmatrix}0000\ 0\\0111\ 1\\0222\ 2\\1012\ 3\\3120\ 3\end{pmatrix}.$$

It is easy to see that these Y_i are formed by different columns of $G_{5,386}$, with each being a $(5,4)$ HSO block; these Z_i are formed by different columns of $G_{5,407}$, with each being a $(5,4)$ HSO block for $i=1,2,3,4$. Hence, both of $G_{5,386}$ and $G_{5,407}$ have $(5,4)$, $(10,8)$, $(15,12)$, and $(20,16)$ HSO blocks. According to Lemma 3, by removing these blocks, in turn, from $G_{5,386}$ and $G_{5,407}$, HSO codes $[386-5i,5,288-4i]$ and $[407-5i,5,304-4i]$ for $i=0,1,2,3,4$ can be constructed from $G_{5,386}$ and $G_{5,407}$, respectively. From previous discussions, by using Lemmas 3 and 4, we can achieve the following theorem.

Theorem 2. *Based on the* $[386,5,288]$ *and* $[407,5,304]$ *HSO codes, HSO codes with the following parameters can be constructed:*

(1) $[386-5i,5,288-4i]$ *for* $i=0,1,2,3,4$;
(2) $[386-5i-j,5,288-4i-2\lceil\frac{j}{2}\rceil]$ *for* $i=0,1,2,3$ *and* $j=1,2,3,4$;
(3) $[407-5i,5,304-4i]$ *for* $i=0,1,2,3,4$;
(4) $[407-5i-j,5,304-4i-2\lceil\frac{j}{2}\rceil]$ *for* $i=0,1,2,3$ *and* $j=1,2,3,4$.

3.2. $[n,5]$ HSO Codes for $408\le n\le 492$

In this subsection, we use the McDonald code $[256,5,192]$ and four known codes given in [4] to construct $[n,5]$ HSO codes for $408\le n\le 492$.

In [4], four optimal codes $[172,5,128]$, $[194,5,144]$, $[215,5,160]$, and $[236,5,176]$ and their generator matrices are given. It is easy to see that these four codes are HSO codes. Using column permutation (special equivalent transform M), we obtain four equivalent HSO codes with generator matrices $G_{5,172}=(I_5\mid A_{5,167})$, $G_{5,194}=(I_5\mid A_{5,189})$, $G_{5,215}=(I_5\mid A_{5,210})$, and $G_{5,236}=(I_5\mid A_{5,231})$, respectively, all these matrices are given in Appendix A.2.

Lemma 5. *If* $n=256+m$, $m\ge170$ *and there is an* $[m,5,d_{5,m}]$ *HSO code, then there are HSO codes with the following parameters:* $[n,5,d]=[256+m,5,192+d_{5,m}]$, $[n-5i,5,192+d_{5,m}-4i]$ *for* $i=1,2,3,4$ *and* $[n-5i-j,5,192+d_{5,m}-4i-2\lceil\frac{j}{2}\rceil]$ *for* $i=0,1,2,3,4$ *and* $j=1,2,3,4$.

Proof. Suppose $G_{5,n} = (M_5 \mid G_{5,m})$, where $G_{5,m} = (I \mid A_{5,m-5})$ is a generator matrix of an $[m, 5, d_{5,m}]$ HSO code. Then, $G_{5,n}$ generates an $[n, 5, d] = [256 + m, 5, 192 + d_{5,m}]$ HSO code.

There are four submatrices in $G_{5,256}$:

$$G_a = \begin{pmatrix} 0111 \\ 0000 \\ 1231 \\ 0000 \\ 1231 \end{pmatrix}, G_b = \begin{pmatrix} 1111 \\ 0123 \\ 0000 \\ 0000 \\ 1111 \end{pmatrix}, G_c = \begin{pmatrix} 1111 \\ 0000 \\ 0123 \\ 1111 \\ 1111 \end{pmatrix}, G_d = \begin{pmatrix} 0000 \\ 0000 \\ 0000 \\ 0111 \\ 1231 \end{pmatrix}.$$

Adding column vectors $(1, 0, 0, 0, 0)^T$, $(0, 1, 0, 0, 0)^T$, $(0, 0, 1, 0, 0)^T$, and $(0, 0, 0, 1, 0)^T$ from $G_{5,m}$ to submatrices G_a, G_b, G_c, and G_d, respectively, we obtain four 5×5 submatrices, U_1, U_2, U_3, and U_4, of $G_{5,n}$. It is obvious that these are formed by different columns of $G_{5,n}$, all U_i satisfy $U_i U_i^\dagger = 0$ and are $(5, 4)$ HSO blocks for $1 \le i \le 4$. Hence, $G_{5,n}$ have $(5, 4)$, $(10, 8)$, $(15, 12)$, and $(20, 16)$ HSO blocks.

By removing $U_i (1 \le i \le 4)$ from $G_{5,n}$, in turn, one can derive that there are $[n - 5i, 5, 192 + d_{5,m} - 4i]$ HSO codes for $0 \le i \le 4$. From $n - 5i \ge 345$, we can obtain $[428 - 5i - j, 5, 320 - 4i - 2\lceil \frac{j}{2} \rceil]$ HSO codes for $(0 \le i \le 4, 1 \le j \le 4)$. $\square$

Since there are four HSO codes $[172, 5, 128]$, $[194, 5, 144]$, $[215, 5, 160]$, and $[236, 5, 176]$, we have the following corollary.

Corollary 1. *There are four groups of HSO codes:*

(1) $[428 - 5i, 5, 320 - 4i]$ *for* $0 \le i \le 4$, *and* $[428 - 5i - j, 5, 320 - 4i - 2\lceil \frac{j}{2} \rceil]$ *for* $0 \le i \le 3$ *and* $1 \le j \le 4$;

(2) $[450 - 5i, 5, 336 - 4i]$ *for* $0 \le i \le 4$, *and* $[450 - 5i - j, 5, 336 - 4i - 2\lceil \frac{j}{2} \rceil]$ *for* $0 \le i \le 3$ *and* $1 \le j \le 4$;

(3) $[472 - 5i, 5, 352 - 4i]$ *for* $0 \le i \le 4$, *and* $[472 - 5i - j, 5, 352 - 4i - 2\lceil \frac{j}{2} \rceil]$ *for* $0 \le i \le 3$ *and* $1 \le j \le 4$;

(4) $[492 - 5i, 5, 368 - 4i]$ *for* $0 \le i \le 4$, *and* $[492 - 5i - j, 5, 368 - 4i - 2\lceil \frac{j}{2} \rceil]$ *for* $0 \le i \le 3$ *and* $1 \le j \le 4$.

Summarizing the above two subsections, we construct $[n, 5]$ HSO codes for each n with $342 \le n \le 492$.

4. Construction of HLCD Codes

In this section, we focus on constructing HLCD codes from known HSO codes in the last section by puncturing some HLCD blocks.

Lemma 6. *Let* A_i *be* $(5, 4)$ *HSO blocks for* $1 \le i \le 4$ *and* $A = (A_1, A_2, A_3, A_4)$. *If* $n \ge 341$, $G_{5,n} = (A \mid G_{5,n-20})$ *is a generator matrix of an* $[n, 5, d]$ *HSO code and* $G_{5,n-20}$ *has* (j, j) *HLCD blocks for* $5 \le j \le 9$. *Then, there are* $[n - 5i - j, 5, d - 4i - j]$ *HLCD codes for* $0 \le i \le 3$ *and* $5 \le j \le 9$.

Proof. Let B_j be (j, j) HLCD blocks of $G_{5,n-20}$ for $5 \le j \le 9$. Let $D_{5i+j} = (A_1, \cdots, A_i \mid B_j)$ for $1 \le i \le 4$ and $5 \le j \le 9$. Then, these D_{5i+j} are $(5i + j, 4i + j)$ HLCD blocks of $G_{5,n}$ for $1 \le i \le 4$ and $5 \le j \le 9$. Puncturing these blocks form $G_{5,n}$; then, one can obtain the generator matrix of $[n - 5i - j, 5, d - 4i - j]$ HLCD codes for $0 \le i \le 3$ and $5 \le j \le 9$. $\square$

According to Section 3.1, for $n = 364, 386, 407$, let $d = 272, 288, 304$, respectively; there are $[n, 5, d]$ HSO codes with generator matrices $G_{5,n} = (A \mid G_{5,n-20})$, where $A = (A_1, A_2, A_3, A_4)$, as shown in Section 3 , and $G_{5,n-20} = (G_{5,n} \setminus A)$. Thus, if we can find that each $G_{5,n-20}$ has (j, j) HLCD blocks $B_{m,j}$ for $5 \le j \le 9$ and $m = n - 20$, then we can obtain $[n - 5i - j, 5, d - 4i - j]$ HLCD codes. We check these facts in three cases.

Case 1. *Let $m = 344$ and $G_{5,344} = (G_{5,364} \setminus (X_1, X_2, X_3, X_4)$. It is easy to check that $G_{5,344}$ has five (j, j) HLCD blocks $B_{m,j}$ for $5 \leq j \leq 9$, as follows:*

$$B_{m,5} = \begin{pmatrix} 1000\ 0 \\ 2100\ 0 \\ 2011\ 1 \\ 1010\ 3 \\ 1333\ 0 \end{pmatrix}, B_{m,6} = \begin{pmatrix} 10000\ 0 \\ 21000\ 0 \\ 20111\ 1 \\ 10110\ 3 \\ 13233\ 0 \end{pmatrix}, B_{m,7} = \begin{pmatrix} 100000\ 0 \\ 210000\ 0 \\ 201111\ 1 \\ 101310\ 3 \\ 132133\ 0 \end{pmatrix}, B_{m,8} = \begin{pmatrix} 1000000\ 0 \\ 2100000\ 0 \\ 2011111\ 1 \\ 1011310\ 3 \\ 1302133\ 0 \end{pmatrix},$$
$$B_{m,9} = \begin{pmatrix} 10000000\ 0 \\ 21000000\ 0 \\ 20111111\ 1 \\ 10101310\ 3 \\ 13022133\ 0 \end{pmatrix}.$$

Case 2. *Let $m = 366$, $G_{5,366} = (G_{5,386} \setminus (Y_1, Y_2, Y_3, Y_4)$. It is easy to check that $G_{5,344}$ has five (j, j) HLCD blocks $B_{m,j}$ for $5 \leq j \leq 9$, as follows:*

$$B_{m,5} = \begin{pmatrix} 00011 \\ 00111 \\ 11313 \\ 23223 \\ 13231 \end{pmatrix}, B_{m,6} = \begin{pmatrix} 000011 \\ 100111 \\ 211313 \\ 323223 \\ 313231 \end{pmatrix}, B_{m,7} = \begin{pmatrix} 0011011 \\ 0011111 \\ 1133313 \\ 2333223 \\ 1311231 \end{pmatrix}, B_{m,8} = \begin{pmatrix} 00011011 \\ 10011111 \\ 21133313 \\ 32333223 \\ 31311231 \end{pmatrix},$$
$$B_{m,9} = \begin{pmatrix} 001111011 \\ 001111111 \\ 111313313 \\ 232323223 \\ 133131231 \end{pmatrix}.$$

Case 3. *Let $m = 387$ and $G_{5,387} = (G_{5,407} \setminus (Z_1, Z_2, Z_3, Z_4)$. It is easy to check that $G_{5,387}$ has five (j, j) HLCD blocks $B_{m,j}$ for $5 \leq j \leq 9$ as follows:*

$$B_{m,5} = \begin{pmatrix} 1000\ 0 \\ 1110\ 0 \\ 3201\ 1 \\ 2121\ 2 \\ 2002\ 2 \end{pmatrix}, B_{m,6} = \begin{pmatrix} 10000\ 0 \\ 11110\ 0 \\ 32201\ 1 \\ 23121\ 2 \\ 22002\ 2 \end{pmatrix}, B_{m,7} = \begin{pmatrix} 100000\ 0 \\ 111110\ 0 \\ 313201\ 1 \\ 213121\ 2 \\ 223002\ 2 \end{pmatrix}, B_{m,8} = \begin{pmatrix} 1000000\ 0 \\ 1111110\ 0 \\ 3313201\ 1 \\ 2113121\ 2 \\ 2223002\ 2 \end{pmatrix},$$
$$B_{m,9} = \begin{pmatrix} 10000000\ 0 \\ 11111110\ 0 \\ 33113201\ 1 \\ 22213121\ 2 \\ 23223002\ 2 \end{pmatrix}.$$

According to Section 3, for $n = 428, 450, 471, 492$, let $d = 320, 336, 352, 368$, respectively; there are $[n, 5, d]$ HSO codes with generator matrices $G_{5,n} = (U \mid G_{5,241} \mid A_{5,n-261})$, where $U = (U_1, U_2, U_3, U_4)$ and $G_{5,241} = ((G_{5,256} \mid I_5) \setminus U)$. Thus, if we can find that each $A_{5,n-261}$ has (j, j) HLCD blocks $D_{m,j}$ for $5 \leq j \leq 9$ and $m = n - 261$, then we can obtain $[n - 5i - j, 5, d - 4i - j]$ HLCD codes. We check these facts in four cases.

Case 4. *Let $m = 167$, and $A_{5,167}$ is given in Appendix A.2. It is easy to check that $A_{5,167}$ has five (j, j) HLCD blocks $D_{m,j}$ for $5 \leq j \leq 9$, as follows:*

$$D_{m,5} = \begin{pmatrix} 0221\ 3 \\ 2132\ 2 \\ 2000\ 0 \\ 1302\ 0 \\ 1100\ 0 \end{pmatrix}, D_{m,6} = \begin{pmatrix} 01221\ 3 \\ 22132\ 2 \\ 20000\ 0 \\ 13302\ 0 \\ 13100\ 0 \end{pmatrix}, D_{m,7} = \begin{pmatrix} 022021\ 3 \\ 211332\ 2 \\ 200000\ 0 \\ 123202\ 0 \\ 101000\ 0 \end{pmatrix}, D_{m,8} = \begin{pmatrix} 0122021\ 3 \\ 2211332\ 2 \\ 2000000\ 0 \\ 1323202\ 0 \\ 1301000\ 0 \end{pmatrix},$$
$$D_{m,9} = \begin{pmatrix} 00322021\ 3 \\ 21011332\ 2 \\ 20000000\ 0 \\ 11223202\ 0 \\ 13001000\ 0 \end{pmatrix}.$$

Case 5. *Let $m = 189$, and $A_{5,189}$ is given in Appendix A.2. It is easy to check that $A_{5,189}$ has five (j, j) HLCD blocks $D_{m,j}$ for $5 \le j \le 9$, as follows:*

$$D_{m,5} = \begin{pmatrix} 00313 \\ 32202 \\ 22200 \\ 03311 \\ 11111 \end{pmatrix}, D_{m,6} = \begin{pmatrix} 000313 \\ 321202 \\ 221200 \\ 030311 \\ 111111 \end{pmatrix}, D_{m,7} = \begin{pmatrix} 0011313 \\ 3200202 \\ 2212200 \\ 0303311 \\ 1111111 \end{pmatrix}, D_{m,8} = \begin{pmatrix} 02011313 \\ 30200202 \\ 23212200 \\ 02303311 \\ 11111111 \end{pmatrix},$$
$$D_{m,9} = \begin{pmatrix} 010011313 \\ 332100202 \\ 232112200 \\ 023003311 \\ 111111111 \end{pmatrix}.$$

Case 6. *Let $m = 210$, and $A_{5,210}$ is given in Appendix A.2. It is easy to check that $A_{5,210}$ has five (j, j) HLCD blocks $D_{m,j}$ for $5 \le j \le 9$, as follows:*

$$D_{m,5} = \begin{pmatrix} 33131 \\ 01211 \\ 10210 \\ 00200 \\ 22031 \end{pmatrix}, D_{m,6} = \begin{pmatrix} 333131 \\ 101211 \\ 310210 \\ 000200 \\ 122031 \end{pmatrix}, D_{m,7} = \begin{pmatrix} 3133131 \\ 1201211 \\ 3110210 \\ 0200200 \\ 1322031 \end{pmatrix}, D_{m,8} = \begin{pmatrix} 32133131 \\ 11201211 \\ 30110210 \\ 01200200 \\ 12322031 \end{pmatrix},$$
$$D_{m,9} = \begin{pmatrix} 132133131 \\ 211201211 \\ 030110210 \\ 201200200 \\ 212322031 \end{pmatrix}.$$

Case 7. *Let $m = 231$, and $A_{5,231}$ is given in Appendix A.2. It is easy to check that $A_{5,231}$ has five (j, j) HLCD blocks $D_{m,j}$ for $5 \le j \le 9$, as follows:*

$$D_{m,5} = \begin{pmatrix} 12210 \\ 30012 \\ 01021 \\ 20131 \\ 32332 \end{pmatrix}, D_{m,6} = \begin{pmatrix} 112210 \\ 230012 \\ 201021 \\ 120131 \\ 032332 \end{pmatrix}, D_{m,7} = \begin{pmatrix} 1112210 \\ 2230012 \\ 2201021 \\ 1120131 \\ 0032332 \end{pmatrix}, D_{m,8} = \begin{pmatrix} 11112210 \\ 22230012 \\ 02201021 \\ 31120131 \\ 20032332 \end{pmatrix},$$
$$D_{m,9} = \begin{pmatrix} 111112210 \\ 232230012 \\ 032201021 \\ 311120131 \\ 200032332 \end{pmatrix}.$$

Summarizing previous discussions, from seven HSO codes (which are also optimal codes), $[364, 5, 272]$, $[386, 5, 288]$, $[407, 5, 304]$, $[428, 5, 320]$, $[450, 5, 336]$, $[471, 5, 352]$, and $[492, 4, 368]$, we can derive seven groups of HLCD codes, as follows.

Theorem 3. *Let $0 \le i \le 4, 5 \le j \le 9$. There are seven groups of HLCD codes with lengths $342 \le n \le 487$:*

$[364 - 5i - j, 5, 272 - 4i - j]$, $[386 - 5i - j, 5, 288 - 4i - j]$, $[407 - 5i - j, 5, 304 - 4i - j]$, $[428 - 5i - j, 5, 320 - 4i - j]$, $[450 - 5i - j, 5, 336 - 4i - j]$, $[471 - 5i - j, 5, 352 - 4i - j]$, and $[492 - 5i - j, 4, 368 - 4i - j]$.

Comparing the parameters of the above new HLCD codes with those in [19], one can see that 31 of our HLCD codes have larger distances than those $[n, 5]$ of the same lengths in [19], and most of the others have the same distances as those in [19]. Table 1 shows our 31 HLCD codes and theirs.

Table 1. Comparison of HLCD codes.

No.	HLCD in [19]	Our HLCD Codes	No.	HLCD in [19]	Our HLCD Codes
1	$[359,5,266]$	$[359,5,267]$	17	$[466,5,346]$	$[466,5,347]$
2	$[397,5,294]$	$[397,5,295]$	18	$[472,5,350]$	$[472,5,351]$
3	$[400,5,296]$	$[400,5,297]$	19	$[475,5,352]$	$[475,5,353]$
4	$[401,5,297]$	$[401,5,298]$	20	$[476,5,353]$	$[476,5,354]$
5	$[402,5,298]$	$[402,5,299]$	21	$[477,5,354]$	$[477,5,355]$
6	$[417,5,309]$	$[417,5,310]$	22	$[478,5,354]$	$[478,5,355]$
7	$[418,5,310]$	$[418,5,311]$	23	$[479,5,355]$	$[479,5,356]$
8	$[420,5,311]$	$[420,5,312]$	24	$[480,5,356]$	$[480,5,357]$
9	$[421,5,312]$	$[421,5,313]$	25	$[481,5,357]$	$[481,5,358]$
10	$[422,5,313]$	$[422,5,314]$	26	$[482,5,357]$	$[482,5,359]$
11	$[423,5,314]$	$[423,5,315]$	27	$[483,5,358]$	$[483,5,359]$
12	$[440,5,326]$	$[440,5,327]$	28	$[484,5,359]$	$[484,5,360]$
13	$[460,5,341]$	$[460,5,342]$	29	$[485,5,360]$	$[485,5,361]$
14	$[461,5,342]$	$[461,5,343]$	30	$[486,5,361]$	$[486,5,362]$
15	$[464,5,344]$	$[464,5,345]$	31	$[487,5,361]$	$[487,5,363]$
16	$[465,5,345]$	$[465,5,346]$			

For each of our $[m,5,d]$ HLCD codes given in Table 1, we can derive $[341s+m,5,256s+d]$ HLCD codes for $s \geq 0$.

Theorem 4. *If $[m,5,d]$ is one of our 31 HLCD codes given in Table 1, then there are $[[341s+m,5,256s+d;341s+m-5]]_2$ EAQECCs for $s \geq 0$. Thus, we obtain 31 classes of EAQECCs better than those in [19] of the same lengths.*

5. Conclusions

In this paper, we have studied the construction of HSO codes and HLCD codes with good minimum distances from known codes and further constructed EAQECCs with good parameters.

The largest minimum distance d_{so} of HSO codes for $342 \leq n \leq 492$ has been given above. If $d_{op}(n,5)$ is determined for a given n, for any optimal linear code $[n,5,d_{op}(n,5)]$, an HSO code with $d_{so}(n,5) = 2\lfloor \frac{d_{op}(n,5)}{2} \rfloor$ could be constructed. If $d_{op}(n,5)$ is not determined for given n, for any linear code $[n,5,d_{bk}(n,5)]$, an HSO code with $d_{so}(n,5) = 2\lfloor \frac{d_{bk}(n,5)}{2} \rfloor$ could be constructed. The minimum distance has been optimized for all the above HSO codes.

Based on these HSO codes, we can further construct HLCD codes with lengths $342 \leq n \leq 492$. The parameters of these HLCD codes are as follows: $[n-5i-j,5,d_{so}(n,5)-4i-j]$ for $n = 364,386,407,428,450,471,492$, $0 \leq i \leq 4$ and $5 \leq j \leq 9$. By comparing with ones in the literature, it is easy to know that our 31 HLCD codes in Table 1 have better parameters. From these HLCD codes, we have obtained 31 classes of entanglement-assisted quantum codes with maximal entanglement.

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Abbreviations

The following abbreviations are used in this manuscript:

HSO	Hermitian self-orthogonal
HLCD	Hermitian linear complementary dual
EAQECC	Entanglement-assisted quantum error-correcting code

Appendix A. Generator Matrices of Some Special Optimal HSO Codes

Appendix A.1. Generator matrices $G_{5,386}$ and $G_{5,407}$ in Section 3.1

Let $G_{5,364} = (A_{00}, A_{01a}, A_{01b}, A_{10a}, A_{10b}, A_{11a}, A_{11b}, A_{12a}, A_{12b}, A_{13a}, A_{13b})$, where

$$A_{00} = \begin{pmatrix} 0000000000000000 \\ 0000000000000000 \\ 0000111111111111 \\ 0111001111223333 \\ 1012230123130122 \end{pmatrix}, A_{01a} = \begin{pmatrix} 0000000000000000000000000000 \\ 1111111111111111111111111111 \\ 0000000000001111111111111111 \\ 0001112223330001112222233333 \\ 0131231321230120231122301123 \end{pmatrix},$$

$$A_{01b} = \begin{pmatrix} 00000000000000000000000000000000 \\ 11111111111111111111111111111111 \\ 22222222222222223333333333333333 \\ 00011122222333330000111122223333 \\ 12312301223012330123012301231233 \end{pmatrix}, A_{10a} = \begin{pmatrix} 111111111111111111111111111\ 1 \\ 000000000000000000000000000\ 0 \\ 000000000000111111111111111\ 1 \\ 001111222333000111112222333\ 3 \\ 010123123013012012230123012\ 3 \end{pmatrix},$$

$$A_{10b} = \begin{pmatrix} 1111111111111111111111111111111\ 1 \\ 0000000000000000000000000000000\ 0 \\ 2222222222222222333333333333333\ 3 \\ 0000111122222333000011112222233\ 3 \\ 0123012301122123012301230122123\ 3 \end{pmatrix}, A_{11a} = \begin{pmatrix} 11111111111111111111111111111111111\ 1 \\ 11111111111111111111111111111111111\ 1 \\ 00000000000000001111111111111111111\ 1 \\ 00011111222233330000011111222233333\ 3 \\ 12301123012301230123300123012201223\ 3 \end{pmatrix},$$

$$A_{11b} = \begin{pmatrix} 111111111111111111111111111111111111111\ 1 \\ 111111111111111111111111111111111111111\ 1 \\ 222222222222222222223333333333333333333\ 3 \\ 000011111122222333330000011111222233333\ 3 \\ 012300112301223011230112211223012300123\ 3 \end{pmatrix}, A_{12a} = \begin{pmatrix} 11111111111111111111111111111111111\ 1 \\ 22222222222222222222222222222222222\ 2 \\ 00000000000000001111111111111111111\ 1 \\ 00001122222333330000011111222222333\ 3 \\ 12330200123011230122301233011233012\ 2 \end{pmatrix},$$

$$A_{12b} = \begin{pmatrix} 1111111111111111111111111111111111111111\ 1 \\ 2222222222222222222222222222222222222222\ 2 \\ 2222222222222222222233333333333333333333\ 3 \\ 0000011111222233333300011111222222333333\ 3 \\ 0112301223012300112221230123301123300112\ 2 \end{pmatrix}, A_{13a} = \begin{pmatrix} 11111111111111111111111111111111111\ 1 \\ 33333333333333333333333333333333333\ 3 \\ 00000000000000001111111111111111111\ 1 \\ 00011111222233330001111122222233333\ 3 \\ 01201123012301230230112301223300112\ 3 \end{pmatrix},$$

$$A_{13b} = \begin{pmatrix} 111111111111111111111111111111111111111\ 1 \\ 333333333333333333333333333333333333333\ 3 \\ 222222222222222222223333333333333333333\ 3 \\ 000111112222223333330000111111222223333\ 3 \\ 012012230011230122330123011223011231223\ 3 \end{pmatrix}.$$

Let $G_{5,386} = (B_{00}, B_{01a}, B_{01b}, B_{10a}, B_{10b}, B_{11a}, B_{11b}, B_{12a}, B_{12b}, B_{13a}, B_{13b})$, where

$$B_{00} = \begin{pmatrix} 00000000000000000\ 0 \\ 00000000000000000\ 0 \\ 00001111111111111\ 1 \\ 11110000111122233\ 3 \\ 01130123012311323\ 3 \end{pmatrix}, B_{01a} = \begin{pmatrix} 000000000000000000000000000\ 0 \\ 111111111111111111111111111\ 1 \\ 000000000000001111111111111\ 1 \\ 000111222333330001112222333\ 3 \\ 123013123012331231230123012\ 3 \end{pmatrix},$$

$$B_{01b} = \begin{pmatrix} 00000000000000000000000000000000000\ 0 \\ 11111111111111111111111111111111111\ 1 \\ 22222222222222222233333333333333333\ 3 \\ 00001111122223333300011122222222333\ 3 \\ 02230122301131223312330130012223022\ 3 \end{pmatrix}, B_{10a} = \begin{pmatrix} 1111111111111111111111111111111\ 1 \\ 0000000000000000000000000000000\ 0 \\ 0000000000000011111111111111111\ 1 \\ 0000111222233300011112222333333\ 3 \\ 0123012012301311201230122001122\ 3 \end{pmatrix},$$

$$B_{10b} = \begin{pmatrix} 1111111111111111111111111111111\ 1 \\ 0000000000000000000000000000000\ 0 \\ 2222222222222222223333333333333\ 3 \\ 0000111122222233330001112222333\ 3 \\ 0123012301122301230121230123012\ 2 \end{pmatrix}, B_{11a} = \begin{pmatrix} 111111111111111111111111111111111111111\ 1 \\ 111111111111111111111111111111111111111\ 1 \\ 000000000000000000111111111111111111111\ 1 \\ 000011111222233333000111122222222233333\ 3 \\ 012301123011301223123012200122333001123\ 3 \end{pmatrix},$$

$$B_{11b} = \begin{pmatrix} 111111111111111111111111111111111111111\ 1 \\ 111111111111111111111111111111111111111\ 1 \\ 222222222222222222333333333333333333333\ 3 \\ 000111112222222333000011111111222333333\ 3 \\ 013012330011223123001200112233123001112\ 3 \end{pmatrix}, B_{12a} = \begin{pmatrix} 11111111111111111111111111111111111\ 1 \\ 22222222222222222222222222222222222\ 2 \\ 00000000000000000011111111111111111\ 1 \\ 00011112222222333300000111122223333\ 3 \\ 01302330011223012301123122322330112\ 3 \end{pmatrix},$$

$$B_{12b} = \begin{pmatrix} 1111111111111111111111111111111111111111111\ 1 \\ 2222222222222222222222222222222222222222222\ 2 \\ 2222222222222222222222333333333333333333333\ 3 \\ 0001111111222222233333000000011111112222333\ 3 \\ 0230011223001223301223001122301122330123001\ 2 \end{pmatrix}, B_{13a} = \begin{pmatrix} 11111111111111111111111111111111111\ 1 \\ 33333333333333333333333333333333333\ 3 \\ 00000000000111111111111111111111111\ 1 \\ 00001112222333000011111112222222333\ 3 \\ 01121230123123012301122330112233011\ 3 \end{pmatrix},$$

$$B_{13b} = \begin{pmatrix} 1111111111111111111111111111111111111111111\ 1 \\ 3333333333333333333333333333333333333333333\ 3 \\ 2222222222222222222222333333333333333333333\ 3 \\ 0000011111112223333333000001111122223333333\ 3 \\ 0122300122331230011223011230122312230011223\ 3 \end{pmatrix}.$$

Let $G_{5,407} = (D_{00}, D_{01a}, D_{01b}, D_{10a}, D_{10b}, D_{11a}, D_{11b}, D_{12a}, D_{12b}, D_{13a}, D_{13b})$, where

$$D_{00} = \begin{pmatrix} 000000000000000000\ 0 \\ 000000000000000000\ 0 \\ 000011111111111111\ 1 \\ 011100111112222333\ 3 \\ 112323012231223012\ 3 \end{pmatrix}, D_{01a} = \begin{pmatrix} 00000000000000000000000000000\ 0 \\ 11111111111111111111111111111\ 1 \\ 00000000000000011111111111111\ 1 \\ 00011111222223300111122222333\ 3 \\ 12301223001231213122301223011\ 2 \end{pmatrix},$$

$$D_{01b} = \begin{pmatrix} 0000000000000000000000000000000000000\ 0 \\ 1111111111111111111111111111111111111\ 1 \\ 2222222222222222222333333333333333333\ 3 \\ 0001111112222233333000011111222223333\ 3 \\ 1230012330122301223012301223012330123\ 3 \end{pmatrix}, D_{10a} = \begin{pmatrix} 11111111111111111111111111111\ 1 \\ 00000000000000000000000000000\ 0 \\ 00000000001111111111111111111\ 1 \\ 00111223333000011111222223333\ 3 \\ 23023230123123300123012330012\ 3 \end{pmatrix},$$

$$D_{10b} = \begin{pmatrix} 1111111111111111111111111111111111111\ 1 \\ 0000000000000000000000000000000000000\ 0 \\ 2222222222222222222333333333333333333\ 3 \\ 0000011122222333333000001111122223333\ 3 \\ 0223312301233112233012230122301230122\ 3 \end{pmatrix}, D_{11a} = \begin{pmatrix} 1111111111111111111111111111111111111\ 1 \\ 1111111111111111111111111111111111111\ 1 \\ 0000000000000000111111111111111111111\ 1 \\ 0000011122222233333000011112222223333\ 3 \\ 0123302300123301123012301230112230122\ 3 \end{pmatrix},$$

$$D_{11b} = \begin{pmatrix} 111111111111111111111111111111111111111111111\ 1 \\ 111111111111111111111111111111111111111111111\ 1 \\ 222222222222222222222223333333333333333333333\ 3 \\ 000000111112222233333330000011112222222333333\ 3 \\ 011223012231122301122330122302330112233001122\ 3 \end{pmatrix},$$

$$D_{12a} = \begin{pmatrix} 11111111111111111111111111111111111111111\ 1 \\ 22222222222222222222222222222222222222222\ 2 \\ 00000000000000000001111111111111111111111\ 1 \\ 00000111222222233330000001111122222233333\ 3 \\ 01223012300223301220012330112301123300122\ 3 \end{pmatrix},$$

$$D_{12b} = \begin{pmatrix} 11111111111111111111111111111111111111111\ 1 \\ 22222222222222222222222222222222222222222\ 2 \\ 22222222222222222223333333333333333333333\ 3 \\ 00011111222222333330000001111112222223333\ 3 \\ 02302233001223012230112230122330112330012\ 3 \end{pmatrix},$$

$$D_{13a} = \begin{pmatrix} 11111111111111111111111111111111111111111\ 1 \\ 33333333333333333333333333333333333333333\ 3 \\ 00000000000000000001111111111111111111111\ 1 \\ 00011111222223333330000011111222222333333\ 3 \\ 12301233012330012230123300133112233001223\ 3 \end{pmatrix},$$

$$D_{13b} = \begin{pmatrix} 11111111111111111111111111111111111111111\ 1 \\ 33333333333333333333333333333333333333333\ 3 \\ 22222222222222222222222333333333333333333\ 3 \\ 00000111111222222233333000111222222233333\ 3 \\ 01233001133001123301123023012300112300112\ 3 \end{pmatrix}.$$

Appendix A.2. Generator Matrices $G_{5,172}$, $G_{5,194}$, $G_{5,215}$, and $G_{5,236}$ in Section 3.2

Let $G_{5,172} = (E_1, E_2)$, where

$$E_1 = \begin{pmatrix} 10000 & 02220313122330200013330311323322120001332022303103320202333210021033322320123101222331320\ 0 \\ 01000 & 02222232311012210101320131312031213223110033312013022113310120133330110002223310031302120\ 3 \\ 00100 & 0001131033333330330103000000000000220220222222222222222222122121111011111000100303303333333\ 3 \\ 00010 & 11102221101100202303131331120202200311311322322223031022010200113213232233211002001211032\ 1 \\ 00001 & 00000101111111111110111113313133300300300003030003233222233132221232222222322320323233202\ 3 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 20000212133221321200311130022333333220100110123232230301131022233102103220 21\ 3 \\ 322102131332000111320233122000103212312230110022332003001223210013202101133 2\ 2 \\ 333333333333333330133133131131011111111111111121211122212122222222222000000000\ 0 \\ 301322233222131103103102023023220120113231111013010132020200201011123122320 2\ 0 \\ 322330302033300023003003032032323313331113211311111213111111011111133300100 0\ 0 \end{pmatrix}.$$

Let $G_{5,194} = (H_1, H_2)$, where

$$H_1 = \begin{pmatrix} 10000 \quad 2221300123002332303203110331222130101212020320320012112113023311201123333101032032231322 1\ 0 \\ 01000 \quad 2221300110331001031021332113332130110303102102102231221220310311201123333102301301102100 3\ 2 \\ 00100 \quad 1033332211112233001100003322221112322210300112233332211000033333322111100003330111222333 2\ 2 \\ 00010 \quad 0122223300003322110011112233333111232221030011223333221100003300001122223333000322211100 01\ 1 \\ 00001 \quad 0000000000000000000000000000011111111111111111111111111111111111111111111111111111111111\ 1 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 2021320211330211302221130020022131203330322200333123123102110002330320320031312320013002111300113 1\ 3 \\ 0203131300221300213221130020022130312221233311211301301321223331003320320302021013331220333121002 0\ 2 \\ 2100033000011112222222333330111110000032221111122223300110000333322003322111233300011103332221122 0\ 0 \\ 1233300333322221111000111112333332222210003333300001122332222111100112233000322211100012223330033 1\ 1 \\ 1111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111\ 1 \end{pmatrix}.$$

Let $G_{5,215} = (K_1, K_2, K_3)$, where

$$K_1 = \begin{pmatrix} 10000 \quad 000300333033330300300000300011311313122222212100100010103030303333332303222202202112111 11\ 2 \\ 01000 \quad 032102013213132322323220322000310100011033303110011123221303020232222332200231030310313 11\ 0 \\ 00100 \quad 210111323230012210102320111111332032022011233102132211031300122201101333033311021222300 03\ 2 \\ 00010 \quad 122112111213111110310103000322022222232300030322322210103030303011011011133113313033020 32\ 0 \\ 00001 \quad 233212110101322111002332300333100013223122100330201100131211023022333030233110021212320 31\ 1 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} 12222221211122121101010220200333333330333011301100002022112333333220011131232232233033110011012 01\ 2 \\ 11002321201122133333233323201010100011000100010001111101321323333223210111011010023022223222323 32\ 2 \\ 30113101132222300302212321332022011033333000022201003310312201003213231022002332113323032022113 33\ 1 \\ 30022221211122123333333333311110022020022011231133330300110222222003300030131131133233002200231 23\ 1 \\ 02202110033333203021120223322300311222133111003333220133211211003120030032211120303032031222221 00\ 2 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 2222222221221332133 13\ 1 \\ 3332313302312111201 21\ 1 \\ 0320111033330330110 21\ 0 \\ 3322213212212001200 20\ 0 \\ 0330031101212112322 03\ 1 \end{pmatrix}.$$

Let $G_{5,236} = (L_1, L_2, L_3)$, where

$$L_1 = \begin{pmatrix} 10000 \quad 001203231211132102033003232222003303203230322333112113111131010333000001122011002232020 02\ 1 \\ 01000 \quad 112311020122312333223223031111332110312132232111332221322233203220011010100211230121020 13\ 0 \\ 00100 \quad 323102323333023301202011211000101133201233320010323013032201012010233023132121231303132 13\ 2 \\ 00010 \quad 322112322211300033021322132322322211120233230001322102132210102013000210310313102031211 30\ 2 \\ 00001 \quad 330101023111312031003331103322231231201020321222002221310012310012333123000231012001023 13\ 2 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 13133113333323011222222203233001033302102022102130103102000311003013310312011123321103333033220 22\ 1 \\ 12330230111031111012323232123321233223032022131012210212032311300123012101301112301111011203113 22\ 3 \\ 02100033101223132020210213020213022021302011101120322002331100131103203122312311031120133202332 01\ 1 \\ 13000132001333032033022131303120210202130333011130222003330000112222112031113013320220330022233 311\ 1 \\ 12312311332030123301200333230022323111323100023113120213212011323123010220113330100023300123110 00\ 2 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0222202022222320313132321013231001111111221\ 0 \\ 0223313032212021312302113021202232232230 01\ 2 \\ 1231023131003223101301123022323321032201 02\ 1 \\ 1230032131113231223023301311000102311120 13\ 1 \\ 1330022031123110233231220121321013200032 33\ 2 \end{pmatrix}.$$

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