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Review

The anholonomic frame and connection deformation method for constructing off-diagonal solutions in (modified) Einstein gravity and nonassociative geometric flows and Finsler–Lagrange–Hamilton theories

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Abstract This article is a status report on the anholonomic frame and connection deformation method, AFCDM, for constructing generic off-diagonal exact and parametric solutions in general relativity, GR, relativistic geometric flows and modified gravity theories, MGTs. Such models can be generalized to nonassociative and noncommutative star products on phase spaces and modelled equivalently as nonassociative Finsler–Lagrange–Hamilton geometries. Our approach involves a nonholonomic geometric reformulation of classical models of gravitational and matter fields described by Lagrange and Hamilton densities on relativistic phase spaces. Using nonholonomic dyadic variables, the Einstein equations in GR and MGTs can formulated as systems of nonlinear partial differential equations, PDEs, which can be decoupled and integrated in some general off-diagonal forms. In this approach, the Lagrange and Hamilton dynamics and related models of classical and quantum evolution, are equivalently described in terms of generalized Finsler-like or canonical metrics and (nonlinear) connection structures on deformed phase spaces defined by solutions of modified Einstein equations. New classes of exact and parametric solutions in (nonassociative) MGTs are formulated in terms of generating and integration functions and generating effective/matter sources. The physical interpretation of respective classes of solutions depends on the type of (non) linear symmetries, prescribed boundary/asymptotic conditions or posed Cauchy problems. We consider possible applications of the AFCDM with explicit examples of off-diagonal deformations of black holes, cylindrical metrics and wormholes, black ellipsoids and torus configurations. In general, such solutions encode nonassociative and/or with geometric flow variables. For another types of generic off-diagonal (nonassociative) solutions, we study models with nonholonomic cosmological solitonic and spheroid deformations involving vertices and solitonic vacua for voids. We emphasize that such new classes of generic off-diagonal solutions can not be considered, in general, in the framework of the Bekenstein–Hawking entropy paradigm. This motivates relativistic/nonassociative phase space extensions of the G. Perelman thermodynamic approach to geometric flows and MGTs defined by nonholonomic Ricci solitons. In Appendix, Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, summarize the AFCDM for various classes of quasi-stationary and cosmological solutions in MGTs with 4-d and 10-d spacetimes and (nonassociative) phase space variables on (co) tangent bundles.

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Contents

1 Introduction, historical remarks and preliminaries

It is well known that exact solutions play a very important role in gravity theories. For Einstein's gravity, there are important textbooks [1–4] summarizing most important physical solutions and the methods for constructing such solutions which typically are defined by certain diagonalizable ansatz for metrics with certain prescribed global and local symmetries. In this work, we review and provide a series of new results on geometric and analytic methods for constructing exact and parametric generic off-diagonal solutions in general relativity, GR, and modified gravity theories, MGTs. We note that a metric is generic off-diagonal if it can't be diagonalized by coordinate transforms in finite spacetime/phase space region. Such extra dimension (super) gravity, string gravity or nonassociative and noncommutative theories can be elaborated as effective ones defined on pseudo-Riemannian or metric-affine (with independent metric and linear connection structures) spaces of dimensions 4–11 and on relativistic eight-dimensional, 8-d, phase spaces. A spacetime in GR is modelled as a four-dimensional, 4-d, Lorentz spacetime manifold enabled with a symmetric metric field signature $(+, +, +, -)$ which must be a solution of the Einstein equations. MGTs can be formulated in abstract and adapted frame forms, in general, on higher dimension Lorentz manifolds and/or on (co) tangent Lorentz bundles. They may involve nonholonomic metric-affine structures with general nonsymmetric metrics and nonlinear connections, N-connections, characterized by nontrivial torsion and nonmetricity fields. For reviews of such results and methods, we cite [5–8]. Respective geometric constructions can be performed in coordinate free (or in some special coordinates which allow us to find solutions in explicit forms) for various low and high dimensions when physical theories are defined on phase spaces enabled with conventional velocity/momentum coordinates.

In this work, a geometric formalism with nonholonomic distributions defining N-connections structures and associated nonholonomic (co) frames defining conventional dyadic decompositions, i.e. (2+2)-splitting is outlined. Here we note that in mathematical and physical literature, there are used equivalent terms like anholonomic, i.e. non-integrable, variables/coordinates. In the first part of this paper, the most important results are formulated in abstract geometric form when necessary details and dyadic frame formulas are provided for 4-d spacetimes and (modified) gravity theories. In the second part of the paper and in appendix, we also explain how using abstract geometric/symbolic constructions, the approach can be extended for extra dimensions and/or on (co) tangent Lorentz bundles. In all cases, respective classes of off-diagonal solutions are generated for oriented shells of dyadic decompositions of type $(2 + 2 + 2 + \cdots)$. For our approach to nonassociative and noncommutative phase space theories, the geometric constructions are defined by star product R-flux deformations in string theory. To characterize the physical properties of new classes of solutions of physically important systems of nonlinear partial differential equations, PDEs, on such nonholonomic spacetime manifolds and phase spaces the geometric constructions have to be extended for theories of nonholonomic geometric flows on a real (temperature-like) parameter and corresponding statistical and geometric thermodynamic models. There are defined nonholonomic frame transforms and canonical deformations of linear connections for geometric constructions adapted to an N-connection splitting. In such nonholonomic variables, various physically important systems of nonlinear PDEs (for instance, modified Einstein and geometric flow equations) can be decoupled and integrated, i.e. solved, in certain general forms defining exact or parametric solutions determined by generic off-diagonal metrics and generalized connections. The Levi–Civita, LC, configurations with zero torsion can be extracted by imposing additional nonholonomic constraints. The coefficients of nonholonomic geometric objects constructed for such classes of generic off-diagonal solutions depend, in general, on all spacetime coordinates. During the last 25 years, in a series of tenths of our and co-authors' works, such a geometric technique was concluded as the anholonomic frame and connection method, AFCDM, for constructing solutions in geometric flow and gravity theories.

We note that the AFCDM [6–10] is very different from the other well-known geometric, analytic and numeric methods on constructing exact solutions in gravity outlined, for instance, in [1–4]. Usually, the books on GR and higher dimension gravity theories summarize the methods and physically important results developed for some diagonal ansatz of metrics when the Einstein equations are transformed into some systems of nonlinear ordinary differential equations, ODEs. In our approach, we elaborated on more general geometric and analytic methods for generating directly (not reducing to ODEs) off-diagonal solutions of nonlinear PDEs encoding general classes of nonholonomic deformations of gravitational and matter field equations in GR and MGTs. The main goal of this status report is to outline the AFCDM and related constructions for 4-d (modified) Einstein gravity and analyze a series of new and physically important examples of generic off-diagonal exact and parametric solutions. We also show how the approach can be extended in abstract geometric form to higher dimensions and on (co) tangent Lorentz bundles, for more general MGTs when generating functions and effective sources may encode nonassociative and noncommutative data for nonholonomic geometric flows and generalized Finsler variables. Such constructions are outlined in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 in the Appendix. The data from such tables can be generalized for other classes of effective sources (they may encode quantum deformations; supersymmetric or spinor variables; functional dependencies; thermodynamic variables; other type nonlinear and algebraic or group structures etc.) we can generate off-diagonal solutions for corresponding physically important systems of nonlinear PDEs. Additionally to the bibliography presented in reviews [5–8], we cite a series of works published during the last 30 years by authors from Eastern Europe. Such results and methods are less known in Western Countries and we include and summarize them in this status report (that why almost a half of citations are related to our contributions and collaborations). For nonassociative phase space theories and related canonical Finsler–Lagrange–Hamilton variables, we shall cite only the works (by other authors), which are closely related to applications and developments of the AFCDM.

Following only standard Lagrange or Hamilton formulations (for instance, using the Wheeler–DeWitt, or Ashtekar formulations) to construct in explicit form certain off-diagonal solutions of nonlinear PDEs and consider quantum deformations of such systems are not possible. In canonical dyadic and Finsler like variables, we can formulate new geometric and analytic methods of finding solutions of nonlinear classical and quantum functional physically important systems of nonlinear PDEs. The AFCDM provides a new geometric technique for constructing general forms of various classes of solutions of nonlinear systems PDEs for (nonassociative) geometric flows and MGTs. This approach uses not only special diagonal ansatz for metrics transforming PDEs into ordinary differential equations, ODEs, but also uses various off-diagonal ansatz for metrics and auxiliary connections, allowing direct integration of physically important systems. The geometric constructions are performed in abstract and adapted frame forms for 4-d and higher dimension Lorentz manifolds and their (co) tangent bundles. Such spacetimes and generalized (star product deformed) phase spaces can be endowed with conventional nonholonomic distributions defining dyadic splitting of type $2+2+2+\cdots$ of the total phase space and spacetime dimensions. The main idea is to define and use for such a splitting an auxiliary canonical distinguished connection, d-connection, and respective nonholonomic frames which allow us to decouple and integrate (modified) Einstein equations in general forms. Off-diagonal Levi–Civita, LC, configurations with zero torsion can be also extracted by imposing additional nonholonomic constraints on some more general classes of solutions.

1.1 Diagonal ansatz reducing (modified) Einstein equations to nonlinear ODEs

The most important geometric and analytic methods for constructing exact solutions (and/or with some decompositions on constant parameters) in GR are summarized and discussed in standard monographs, for instance, [1–4]. In [2], the Einstein equations are formulated in abstract geometric form,

$$En = Ric - \frac{1}{2} c \mathbf{g} Rsc + \Lambda \mathbf{g} = \frac{8\pi G}{c^4} Tm, \quad (1)$$

with a cosmological constant Λ and an energy–momentum tensor Tm. In 4-d, this consists a system of nonlinear PDEs for six independent components of the metric tensor $\mathbf{g} = g_{\alpha\beta}(u^\gamma) e^\alpha \otimes e^\beta$. In Eq. (1), the metric compatible and zero torsion Levi–Civita, LC, connection, $\nabla[\mathbf{g}]$, is used. We follow such conventions (see details in next sections): The systems of coordinates and indices are labeled as $u^\gamma = (u^1, u^2, u^3, u^4 = t)$. For higher dimensions, we can write $u^5, u^6, \ldots$ considering metrics of different signatures etc. We shall use also $u^\gamma = (x^i, u^5, u^6)$, for $i = 1, 2, 3, 4$; or $u = (x, u^5, u^6)$. Usually we state that the light velocity constant $c = 1$ excepting some formulas when it will be physically important to write c. Various types (not) primed, underlined etc. indices may run values of type $\alpha, \beta, \ldots, \alpha', \beta', \ldots = 1, 2, 3, 4$. The Einstein convention on up-low indices is used. Frame transforms are defined as $e^\alpha = e^\alpha_{\ \alpha'}(u^{\gamma'}) du^{\alpha'}$ and $e_\beta = e^{\beta'}_{\ \beta}(u^{\gamma'}) \partial_{\beta'}$, for $\partial_{\beta'} = \partial/\partial u^{\beta'}$ and $e^\alpha_{\ \alpha'} e^{\alpha'}_{\ \beta} = \delta^\alpha_\beta$, where δ^α_β is the Kronecker symbol. From 10 components of a symmetric tensor $g_{\alpha\beta}(u^\gamma)$, there are 6 independent ones because 4 of them can transformed in zero using coordinate transforms as consequence of the Bianchi identities for (pseudo) Riemannian spaces. The coefficients of the Ricci tensor for ∇ are $Ric = R_{\alpha\beta} e^\alpha \otimes e^\beta$, the curvature scalar $Rsc := g^{\alpha\beta} R_{\alpha\beta}$, and $Tm = \{T_{\alpha\beta}\}$ is the symmetric energy-momentum tensor for matter (with G being the gravitational/Newton constant). Usually, we shall use abstract index, or abstract not index, formulas as in [2]. Nevertheless certain necessary coordinate and abstract index formulas for general and N-adapted frames will be used when certain special types of indices/coordinates are important for constructing explicit classes of solutions. The bulk of known and physically important exact solutions of (1) were constructed for diagonal ansatz of metrics, motivated by certain assumptions on symmetries of gravitational and matter field interactions, when corresponding systems of nonlinear PDEs are transformed into systems of nonlinear ordinary differential equations, ODEs. Such equations can be integrated (i.e. solved) in certain general forms depending on respective integration constants and physical parameters. The integration constants can be related to certain physical constants using respective boundary/asymptotic conditions, some prescribed data for Cauchy systems etc. Physically important solutions are selected to define some well-defined and verifiable physical models. For instance, such constructions must be with relativistic causality, to satisfy some positive entropy conditions and allow to define thermodynamic variables. Typical models are elaborated, for instance, for some positive energy conditions, with the goal to avoid singularities at least in some observable regions, etc.

As the most important example of a solution generated by a diagonal ansatz and ODEs in GR, we can consider the

Schwarzschild black hole, BH, metric. It is for the vacuum Einstein spaces, when (1) transforms into $Ric = 0$. Using spherical coordinates $u^\alpha = (r, \theta, \varphi, t)$, such a diagonal static solution can be written as

$$g_{\alpha\beta} = diag\Big[g_1(r) = \left(1 - \frac{r_s}{r}\right)^{-1}, g_2(r,\theta) = r^2, g_3(u^\gamma) = r^2 \sin^2\theta, g_4(u^\gamma) = -f(r) = -\left(1 - \frac{r_s}{r}\right)\Big], \quad (2)$$

for the quadratic line element

$ds^2_{Sch} = g_\alpha(u^\gamma)[du^\alpha]^2 = g_1 dr^2 + g_2 d\theta^2 + g_3 d\varphi^2 + g_4 dt^2.$

In (2), the Schwarzschild (horizon) radius $r_s = 2Gm/c^2$ is determined by the condition that for $r \gg r_s$ such a metric defined the Newton gravitational potentials for a point mass m. We emphasize that in corresponding chosen coordinate bases the coefficients of the Schwarzschild metric do not depend on the time coordinate t, i.e. it posses a Killing symmetry on time like vector $\partial_4 = \partial_t$.

Another important example of a diagonal ansatz used for constructing homogeneous and isotropic cosmological models in GR and MGTs is that for the Friedman–Lemaître–Robertson–Walker, FLRW, spaces,

$$g_{\alpha\beta} = diag[g_1(r,t) = a^2(t)/(1 - \epsilon r^2), g_2(r,t) = a^2(t) r^2, g_3(r,\theta,t) = a^2(t) r^2 \sin^2\theta, g_4 = -1. \quad (3)$$

In this quadratic line element, the constant ϵ represents the curvature of the space (it can be taken $0, \pm 1$) and the "scale factor" $a(t)$ should be, for instance, a solution of the Einstein equations (1), when the energy–momentum tensor is taken in a form

$$T_{\alpha\beta} = diag[P, P, P, \rho] \quad (4)$$

for a fluid type matter with pressure P and energy density ρ.

In higher dimension gravity theories, the diagonal ansatz (2) and/or (3) were correspondingly generalized for extra dimension coordinates (spherical, cylindrical and other higher symmetry ones) but keeping the property to be diagonalizable by certain coordinate transforms. That allowed to construct a number of BH, wormhole (WH) and cosmological solutions in (super) string and MGTs and exploited, for instance, in modern cosmology and astrophysics.

It is very difficult to construct in explicit forms exact or parametric off-diagonal solutions of systems of coupled nonlinear PDEs of type (1) and their higher dimension generalizations, or in MGTs. The parametric solutions may be also exact for a fixed value and order of a physical parameter (like the Planck and string constants, or other ones with possible polarizations; in this work, we consider only classical models even certain nonassociative contributions may be determined by star products involving the imaginary unity). The main property of ansatz of types (2) and/or (3) is that they reduce the gravitational field equations to some systems of nonlinear ODEs, which can be integrated in certain general or approximate forms determined by integration constants. Their physical interpretation depends on the types of assumptions on symmetries, boundary/asymptotic conditions and/or how a corresponding Cauchy problem is solved (all such conditions are stated following certain geometric/physical considerations). The cosmological constant and the data for an energy momentum tensor can be considered respectively as effective/matter generating sources. Usually, this type of solutions in GR involve certain singularities and horizons.

The constructions can be generalized to spaces of higher dimension and for various modifications of gravity and matter field theories, with possible quantum corrections, additional terms with supersymmetric and superstring contributions, nonassociative/noncommutative generalizations etc. Such solutions were found, generalized, and studied intensively in GR and MGTs during the last 100 years. There were formulated a series of geometric and physically important theorems on BH singularities, cosmic censorships, conditions of stability, scenarios of evolution/inflation/acceleration etc. In this approach, the basic ideas and principles for constructing off-diagonal solutions of (modified) Einstein equations can be stated in this form:

Principles 1 (**reducing PDEs to ODEs and constructing diagonal solutions**) :

$$\begin{bmatrix} \text{system of nonlinear PDEs,} \\ \text{(modified) Einstein eqs. (1)} \end{bmatrix} \Rightarrow \begin{bmatrix} \text{frame/coordinate transforms and symmetries:} \\ \text{spherical/cylindrical, Killing, Lie algebras,} \\ \text{diagonal ansatz } g_\alpha(u^\gamma) \end{bmatrix} \Rightarrow \begin{bmatrix} \textbf{integrable systemsof} \\ \textbf{nonlinear ODEs} \end{bmatrix}$$
$$\Downarrow$$
$$\begin{bmatrix} \text{special generating functions } f(r), \text{ or } a(t), \\ \text{non-structure vacuum, or special generating sources : } \Lambda, diag[P, P, P, \rho]; \\ \text{integration constants determined by boundary/asymptotic conditions, Cauchy problems,} \\ \text{horizons, singularities, BH theorems, hyper-surface thermodynamics, cosmic censorship etc.} \end{bmatrix}.$$

Such methods allow us to elaborate on nonlinear physical models for gravitational and matter fields interactions determined by solutions of some classes of ODEs, when from six independent components of metrics there are chosen only a few diagonal components (maximum 4, for a 4-d spacetime) of "diagonalizable metrics". For instance, such a metric is defined by a function $f(r)$, or $a(t)$, when the presence of other coordinates is motivated by choosing certain spherical/cylindrical/ellipsoidal/toroid systems of coordinates and/or considering some frame/coordinate transforms. We emphasize that prescribing such an ansatz, we "cut" other possibilities to find more general classes of solutions depending, for instance, on all spacetime coordinates and when the metrics contain generic off-diagonal terms (for instance, with 6 independent coefficients). In GR, any metric can be diagonalized in a point, or along a geodesics, and represented as a standard diagonal Minkowski metric, $\eta_{\alpha\beta} = diag[1, 1, 1, -1]$. A general pseudo-Riemannian metric can be represented in a diagonal form with coefficients depending on spacetime coordinated with respect to certain nonholonomic frames. Some such generalized ansatz, for instance, with necessary $2 + 2$ decompositions can be convenient for constructing new classes of diagonal and off-diagonal solutions. This is the price we should pay in order to solve systems of nonlinear PDEs by reducing them to more simple systems of nonlinear ODEs. Nevertheless, even in such cases, there were elaborated a number of on physically important gravitational models with applications in modern astrophysics and cosmology. The bulk of experimental and observational verifications, theoretical constructions and applications in modern gravity/particle physics/cosmology were performed using systems of (non) linear wave/oscillator equations, respective BH and cosmological ODEs, and their (superpositions of) solutions. Modern approaches to accelerating cosmology, dark matter and energy physics and related plethora of MGTs request more advanced geometric, analytic and numeric methods for generating exact and parametric solutions in physically important nonlinear systems of PDEs not constraining the "geometric optics and methodology" only via ODEs.

1.2 Physical and geometric motivations for constructing off-diagonal solutions

In GR and MGTs, there were found more general classes of solutions with geometric and physical properties which are different than those stated above. For instance, the Kerr solution for rotating BHs contains odd-diagonal terms and ellipsoidal ergo-spheres, which are induced in rotation frames. There were found examples of solutions for black rotoids/toroids, wormholes etc. with various types of broken/nonlinear symmetries describing locally anisotropic matter interactions and respective inhomogeneous/anisotropic cosmological models (a number of examples are reviewed in monograph [1]). We cite [2–4] for the main concepts, methods, interpretations and discussions of most physical important solutions and models. In such monographs, there are provided certain examples of solutions for gravitational nonlinear waves and solitons when the coefficients of metrics depend on 2 or 3 spacetime coordinates, with parametric dependencies, and may involve certain off-diagonal terms. They were constructed using some special methods for generating solutions of nonlinear PDEs, for instance, with so-called LA symmetries and solitonic hierarchies, see details in [11–14] and references therein.

Nevertheless, during many years of research on mathematics and physics of gravitational field equations, it was not formulated a general geometric and analytic method for constructing generic off-diagonal solutions with dependence on all spacetime coordinates in GR and MGTs. The solutions with maximal, or with many, degrees of freedom (for 4-d gravity theories, being considered 6 independent components of metrics) are of crucial importance if we try to explore and solve a series of fundamental problems in nonlinear physics and elaborate quasi-classical models of gravity an quantum gravity, QG. For such models, generic off-diagonal interactions and nonholonomic constraints are important in the non-perturbative and nonlinear regimes which should test QG and higher dimension theories. Off-diagonal symmetric and non-symmetric metrics and generalized (non) linear connections are used for elaborating realistic and modified gravity models for inhomogeneous/ anisotropic/acceleration cosmology; to construct dark energy and dark matter theories with quasi-periodic structure and pattern forming, filaments, vortices, solitons etc. In such cases, we can't work only with "simplified" diagonal ansatz reducing gravitational and matter field equations to certain systems of nonlinear ODEs. We have to elaborate new methods which allow to construct generic off-diagonal solutions, with constraints and generating functions and sources, solving in direct form respective systems of nonlinear PDEs.

A series of our works were devoted to constructing new classes of exact solutions in GR and MGTs of 4-d, 5-d spacetimes and 8-d phase spaces with warped dimensions, and further nonassociative/supersymmetric generalizations for string and generalized Finsler geometry [5–10,15–30]. Considering nonholonomic $2(3)+2+2+\cdots$ splitting of dimensions by a so-called nonlinear connection, N-connection, structure, $\mathbf{N}$, we defined an auxiliary connection, called as the canonical distinguished, d, connection,

$$\widehat{\mathbf{D}}[\mathbf{g}] = \nabla[\mathbf{g}] + \widehat{\mathbf{Z}}[\mathbf{g}], \qquad (5)$$

with the canonical distortion d-tensor, $\widehat{\mathbf{Z}}$, when all three geometric objects are determined by the same metric structure $\mathbf{g}$. For geometric objects adapted to a N-connection, we use bold face symbols. In next section, we provide all necessary definitions and abstract/index formulas. With respect to so-called nonholonomic N-adapted frames, we can decouple in some general forms the modified Einstein equations,

$$\widehat{\mathbf{E}}n[\mathbf{g},\widehat{\mathbf{D}}] = \widehat{\mathbf{R}}ic - \frac{1}{2}\mathbf{g}\widehat{R}sc = \widehat{\mathbf{Y}}[\mathbf{g},\widehat{\mathbf{D}}]. \quad (6)$$

The geometric objects in such a nonlinear system of PDEs with nonholonomic variables and respective constraints, and canonical distortion (6) of the LC-connection ∇, are determined by geometric data $[\mathbf{g}, \widehat{\mathbf{D}}]$. The canonical effective source $\widehat{\mathbf{Y}}$ encodes possible deformations of the standard energy momentum tensor in (1), terms like $-\Lambda \mathbf{g}$ and distortions of the Einstein tensor determined by (5). For MGTs, there are included contributions from generalized gravitational and matter fields Lagrangians, higher dimension corrections, (non) associative/(non) commutative/supersymmetric contributions from string/M-theory, in Finsler–Lagrange–Hamilton gravity etc.

Having decoupled in a general form the system of nonlinear and nonholonomic PDEs (6), it is possible to solve it in certain exact or parametric forms. This way, we can construct generic off-diagonal solutions $\mathbf{g}$ determined by corresponding classes of generating and integration functions, effective generating sources $\widehat{\mathbf{Y}}$, and corresponding N-connection, $\mathbf{N}$, splitting. It should be noted that such solutions involve a canonical d-torsion structure, $\widehat{\mathbf{T}}s$, which is determined by nonzero anholonomy coefficients if $\mathbf{N}$ are nontrivial, and related off-diagonal terms. Such nonholonomic torsions are different from the torsion fields, for instance, in the Einstein-Cartan and/or string gravity. We can extract, in general, off-diagonal solutions $\mathbf{g}$ for the LC-connection $\nabla[\mathbf{g}]$ if we impose additional constraints on generating and integration functions which result in zero distortion d-tensors $\widehat{\mathbf{Z}}[\mathbf{g}]$ in (5),

$$\widehat{\mathbf{Z}} = 0, \text{ which is equivalent to } \widehat{\mathbf{D}}_{|\widehat{\mathbf{T}}s=0} = \nabla. \quad (7)$$

The basic ideas and principles are stated as

The first general goal of this article is to show how Principles 2–AFCDM can be performed in explicit form for 4-d Lorentz manifolds with nonholonomic 2+2 splitting and canonical distortion of the LC-connection. A corresponding new methodology of constructing generic off-diagonal solutions analyzing their possible physical implications in (modified) gravity will be outlined. We shall provide and discuss a series of important physical solutions related to BH physics and modern cosmology. Then, the second general goal is to outline in brief (using abstract geometric methods) that the AFCDM can be extended to higher dimensions using nonholonomic 2+2+2+ splitting. In the case of phase space theories, the geometric constructions involve additional velocity/momentum variables which is similar to Finsler–Lagrange–Hamilton geometry. Here, we emphasize that if we restrict our research only with "pure" Lagrange or Hamilton phase space theories (which are very important for quantization) we are not able to unify the spacetime and phase space geometry and physics and describe the constructions in terms of off-diagonal solutions of (modified) Einstein equations. In canonical nonholonomic variables, our approach can be generalized for nonassociative and noncommutative gravity and geometric flow theories.

We review an unified geometric abstract formalism for 4-d gravity theories in the Part 1 and extend the methods in abstract geometric forms for 8-d phase spaces with nonassociative geometric flows and Finsler–Hamilton–Lagrange variables in Part II. Finally, we outline and summarize in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 from the Appendix how the AFCDM can be applied for generating off-diagonal solutions in higher dimension theories and for (nonassociative) phase space models, with corresponding formulas for 8-d and 10-d quasi-stationary and locally anisotropic cosmological configurations.

Principles 2−AFCDM : off−diagonal solutions, generalized connections &LC − connections

$$\begin{bmatrix} \text{system of nonlinear PDEs,} \\ \text{distorted Einstein eqs. (6)} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \text{frame/coordinate transforms, N-adapted } \widehat{\mathbf{D}}[\mathbf{g}] \text{ (5)} \\ \text{nonlinear and Killing symmetries, effective sources} \\ \text{off-diagonal ansatz } g_{\alpha\beta}(u^{\gamma}) \end{bmatrix} \Rightarrow \begin{bmatrix} \textbf{decoupling and integrable} \\ \textbf{systems of nonlinear PDEs} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \text{generating and integration functions depending on spacetime coordinates} \\ \text{nontrivial vacuum, generating sources, effective cosmologial constants;} \\ \text{integration functions determined by boundary/asymptotic conditions, Cauchy problems,} \\ \text{horizons, singularities, G. Perelman thermodynamics, stability and flow evolution, etc.} \end{bmatrix}$$

$$\Downarrow \text{ nonholonomic sLC-conditions (7)}$$

$$\begin{bmatrix} \text{system of nonlinear PDEs,} \\ \text{Einstein eqs. (1)} \end{bmatrix}.$$

1.3 Nonassociative (co) tangent Lorentz bundles and Finsler–Lagrange–Hamilton geometry

The 4-d nonholonomic geometric constructions and the AFCDM can be extended to nonassociative and noncommutative theories defined on generalized spacetime and/or phase space models. We cite here a series of (related to our purposes) works on nonassociative gauge/membrane theory and double field theory constructed in the framework of string theory [31–39]. Such nonassociative structures also arise in the world volume of a D-brane, for open strings, and for the models with flux compactification for closed strings.

Our research program [5,6,9,10,40,41] on nonassociative geometry, physics and quantum information theory is based on the approach to nonassociative gravity formulated for $\star$-product (i.e. star-product) deformations determined by R-flux backgrounds in string gravity [38,39]. In a self-consistent form, such nonassociative and noncommutative theories are modelled on a conventional phase space ${}^{\shortparallel}\mathcal{M}$. To include the general relativity (GR) as a particular case we use model star deformed phase spaces on cotangent bundle, ${}^{\shortparallel}\mathcal{M} = T^{*}\mathbf{V},$ on spacetime Lorentz manifold, $\mathbf{V}$. In our works, the phase space dimension is $\dim \mathcal{M} = 8$, with total phase space local coordinates labelled in the form ${}^{\shortparallel}u^{\alpha_s} = (x^{i_s}, {}^{\shortparallel}p_{a_s})$, where ${}^{\shortparallel}$ on the left indicates a phase space with spacetime coordinates, $x^{j_s} = (x^j, t)$, and *complex* momentum coordinates, ${}^{\shortparallel}p_{a_s} = ip_{a_s} = (ip_a, iE)$, with $i^2 = -1$. Alternatively, we can consider also real momentum coordinates with ${}^{\shortmid}p_{a_s} = p_{a_s} = (p_a, E)$ as introduced in [39] which we modified for nonholonomic configurations with corresponding labels and boldface symbols. For classical models, we can study only real deformations of the geometric and physical objects which are nontrivial even, in general form, the star product structure involves the complex unity. The phase space will be denoted ${}^{\shortmid}\mathcal{M}$ if the momentum-like coordinates are real ones and labelled in the form ${}^{\shortmid}u^{\alpha_s} = (x^{i_s}, {}^{\shortmid}p_{a_s})$ (in brief, ${}^{\shortmid}u = (x, p)$). In original form, the nonassociative, vacuum, gravitational equations, ${}^{\shortparallel}Ric^{\star}[{}^{\shortparallel}\nabla^{\star}] = 0$, were postulated as phase space $\star$-deformations of the standard Ricci tensor Ric in GR. Here we note that well-defined nonassociative Ricci tensors allow us to formulate corresponding models of geometric flows as in [9,10,40,41]. For a prescribed star product (see definitions in next sections), the nonassociative tensor ${}^{\shortparallel}Ric^{\star}$ can be constructed for a unique nonassociative Levi-Civita (LC) connection, ${}^{\shortparallel}\nabla^{\star}$, which is torsionless and metric compatible with the respective $\star$-deformed symmetric, ${}^{\shortparallel}_{\star}\mathbf{g},$ and nonsymmetric, ${}^{\shortparallel}_{\star}\mathbf{q}$, metric structures. Using real phase space variables, we can write in the symbolic form ${}^{\shortmid}Ric^{\star}({}^{\shortmid}u) = {}^{\shortmid}Ric^{\star}(x,p)$ for $\left({}^{\shortmid}_{\star}\mathbf{g}({}^{\shortmid}u), {}^{\shortmid}_{\star}\mathbf{q}({}^{\shortmid}u)\right)$ and ${}^{\shortmid}\nabla^{\star}({}^{\shortmid}u)$, when the geometric objects depend additionally on momentum-like coordinates ${}^{\shortmid}u^{\alpha_s} = (x^{i_s}, p_{a_s})$. Such dependencies on velocity or momentum like coordinated are considered in Finsler–Lagrange–Hamilton geometry and gravity [30,42,43] (when the generating functions and respective nonlinear and linear connections are subjected to certain homogeneity and nonholonomic conditions on $\mathcal{M}$ or ${}^{\shortmid}\mathcal{M}$ and even on $\mathbf{V}$ with a corresponding nonholonomic). In this work, we use and abstract geometric formalism when those constructions can be extended by respective star products on $\mathcal{M}^{\star}$ or ${}^{\shortmid}\mathcal{M}^{\star}$.

Nonassociative and noncommutative modified MGTs were formulated as a type of bimetric gravity theory [44,45]. We cite [46,47], for commutative bimetric theories, and [48–51], for constructions when the second metric structure can be nonsymmetric. Further developments for commutative and nonassociative gravity were performed in [5,52]; when the geometric constructions on phase spaces $\mathcal{M}^{\star}$ or ${}^{\shortmid}\mathcal{M}^{\star}$ enabled with $\star$-product structure. Such theories may involve also nonassociative generalizations of relativistic and supersymmetric/(non) commutative Finsler–Lagrange–Hamilton spaces [30,42,43]. In this paper, we consider Finsler-like geometric objects and variables and generalize in nonassociative and noncommutative form such geometric models with but re-define the formulas for nonlinear quadratic elements in a form so that nonassociative phase space BH solutions can be generated as real configurations coming from star R-flux deformations. We also compute nontrivial phase space components of the nonsymmetric parts of the metric which can occur in nonassociative gravity.

The works [5,6,10,38,39] raised three important questions in nonassociative gravity:

1. How to formulate and understand the physical properties of generic off-diagonal solutions for 4-d (1) and nonholonomic (6) and their nonassociative phase space 8-d generalizations involving ${}^{\shortmid}Ric^{\star}(x,p)$ and respective ${}^{\shortmid}\widehat{\mathbf{R}}ic^{\star}(x,p)$ (in general, phase space systems of nonlinear PDEs may encode nontrivial nonassociative sources ${}^{\shortmid}\mathbf{J}^{\star}(x,p)$)?
2. How to construct in explicit form classes of exact, physically important solutions in nonassociative gravity and determine the physical meaning of such solutions? Here we note that such exact or parametric solutions, in general, are generic off-diagonal and when certain nontrivial effective sources ${}^{\shortmid}\mathbf{J}^{\star}(x,p)$ contain terms defined by nonassociative star product and R-flux data and, via nonlinear symmetries, relate various classes of generating functions and generating sources to certain effective cosmological constants ${}^{\shortmid}\Lambda$.
3. In [6,10,40,41], we proved that the AFCDM can generalized for nonassociative gravity but new classes of solutions are generic off-diagonal with nonassociative geometric objects of type $\left({}^{\shortmid}_{\star}\mathbf{g}(x,p), {}^{\shortmid}_{\star}\mathbf{q}(x,p)\right)$ and

${}^{\shortmid}\widehat{\mathbf{D}}^{\star}(x, p)$ with generic dependence on extra-dimension and/or momentum like variables. Such solutions can't be interpreted in the framework of the Bekenstein–Hawking paradigm [53–56] because, in general, they do not involve certain hypersurface, duality, or holographic properties. To characterize the thermodynamic and informational properties of such nonassociative solutions we must generalize the approach for nonassociative and quantum information flows and study generalized models of G. Perelman thermodynamics for Ricci flows [57].

1.4 Motivations and the main hypothesis on Finsler–Lagrange–Hamilton phase space geometries

Various types of phase space gravity theories have been elaborated with the aim of quantizing gravity and formulating quantum field theories, QGTs, on curved spaces using certain classical and quantum Lagrange and/or Hamilton formulations. Here we note the Arnowit-Deser-Misner, ADM, approach with 3+1 spacetime splitting (for an abstract geometric formulation see [2,58]) used in canonical quantum gravity and further developments [59–62]. There are quite different geometric and quantum theoretic formalisms involving corresponding Lagrange density, $L(x, v)$, and/or Hamilton density, $H(x, p)$, on respective phase spaces. Such constructions can be related via Legendre transforms, redefined for Poisson/almost symplectic structures etc. which was applied with certain success in formulating canonical, string, loop, gauge-like and other types of approaches to QG, geometric quantization and deformation quantization (DQ) etc. Nevertheless, even in the semi-classical limits, such theories involve certain variants of modified Lagrange, Hamilton, Hamilton–Jacoby, Wheeler–De Witt, WDW, gauge gravity and other type of nonlinear equations. It is not possible to decouple and solve in certain general forms such as nonlinear and/or quantum systems of (functional) PDEs. For instance, the WDW equation is a very sophisticated functional equation in the space of metrics, which has deep consequences of the arrow of time problem in quantum cosmology. A. Ashtekar introduced new variables by analogy to quantum electrodynamics for certain connections like gravitational potentials which was exploited in loop gravity and related theories. The existing methods do not convert, or connect, the above mentioned approaches with conventional phase space Lagrangians and Hamiltonians to the problem of constructing generic off-diagonal solutions of gravitational field equations in GR, or MGTs, formulated on phase spaces in certain analogous forms to the Einstein equations. For constructing models of QG, it is important to consider also quantum deformations and respective nonlinear functional equations, which in Finsler–Lagrange–Hamilton variables can be performed in explicit form as exact and parametric solutions of certain systems of nonlinear classical or quantum PDEs.

In a series of works [7,8,24,28,29,29,30,63], we proved that GR can be extended in off-diagonal integrable forms on phase spaces using nonholonomic dyadic variables with $2(3) + 2 + \cdots$ decompositions and auxiliary connections. The auxiliary connections are adapted to a N-connection structure as in modified Finsler geometry on (co) tangent Lorentz bundles, and defined as distortions of the so-called Cartan/Berwald/Chern or other connections in Finsler geometry, and can be nonholonomically constrained to the Levi–Civita, LC, connection. The main point is that we can consider toy models with nonholonomic $2 + 2$ splitting with N-connection and certain canonical connections defined by pseudo-Riemannian metrics in GR, but with distorted LC connections to nonholonomic data which allow construction of off-diagonal solutions in very general forms. Having defined a general class of solutions in "distorted GR", we can impose additional nonholonomic constraints and extract LC configurations for the standard GR. The approach works also for theories of higher dimension and for various models on phase spaces, including nonassociative star product deformations to $\mathcal{M}^{\star}$ or ${}^{\shortmid}\mathcal{M}^{\star}$. For all such MGTs and generalizations to (non) associative geometric flow evolution models, we can introduce canonical type dyadic variables and, equivalently, Finsler–Lagrange–Hamilton variables. If we work directly only with the Lagrange or Hamilton phase space configurations, we are not able to derive standard geometric metric and connection structures as in the (pseudo) Riemannian or metric-affine geometry described by corresponding Ricci, torsion, nonmetricity etc. tensors. Nevertheless, if we introduce the Hessians of $L(x, v)$ and or $H(x, p)$ as corresponding (co) vertical metrics, and corresponding Sasaki lifts to total metrics on $\mathcal{M}$ or ${}^{\shortmid}\mathcal{M}$, we preserve the priorities of phase space constructions in the Lagrange and Hamilton mechanics (with extensions to classical and quantum field theories) and the possibility to work with metrics, adapted frames, connections, curvatures etc. as in higher dimension gravity, which may have certain extra dimension coordinates as generalized velocity/momentum ones.

We explain the details of our alternative geometrization of mechanics and field theories and nonassociative star product deformation in Sect. 5. Here we emphasize that our geometric formalism involves modified Finsler connections (using canonical "hat" d-connections) which allow to apply the AFCDM and integrate in general form physically important systems of nonlinear PDEs. Working only with L- and/or H-models on phase spaces, without a definition of N-connections and related canonically deformed Finsler-like connections, we are not able to apply the AFCDM for finding off-diagonal solutions in GR and MGTs. Theories with conventional L- and/or H, and correspondingly related Poisson/almost symplectic formalisms are efficient for elaborating various methods of quantization when the quantum field theory (QFT) was developed using corresponding methods of linear functional analysis. From a rigorous mathematical viewpoint, such approaches do not work for QG because a general theory of nonlinear functional analysis has not been formulated yet and the existing methods have not been applied directly to physically important systems of PDEs like (modified) Einstein equations and geometric flow models. We note that QG has the property of asymptotic safety (also referred to as nonperturbative renormalizability) related to the modern Wilsonian viewpoint on QFT involving functional renormalization group equations [64]. Nevertheless, even in this approach, there are not used in direct form certain exact and parametric solutions for metric and connection variables; the flow equations of such a QG are the results of a higher-derivative Einstein–Hilbert truncation.

In this work, we follow the idea that Lagrangians and/or Hamiltonians can be used on phase spaces for defining new type geometric objects such as the N-connection structures (via corresponding semi-spray, i.e. nonlinear geodesic equations, which are equivalent to the Euler–Lagrange, or Hamilton equations) and certain total phase space s-metrics and canonical s-connections. Such a Finsler–Lagrange–Hamilton phase space gravity, with generalized Ricci s-tensors and scalar curvatures, can be formulated as a nonholonomic extra dimension generalization of the Einstein gravity (we can add also nonassociative star product deformations), which can be integrated in certain off-diagonal forms. Using corresponding classes of off-diagonal parametric solutions of (nonassociative) geometric flow and phase-modified gravitational equations, we can apply various methods of quantization which can be selected to be parametric renormalizable for a respective family of metric-affine configurations. This is not possible if we work only with some effective Lagrangians and Hamiltonians for quantizing undefined classes of metrics and connections, or for certain WDW functional equations, or the Ashtekar variables.

The main **Hypothesis** in this work on 4-d nonholonomic Einstein gravity and generalizations on 8-d co-tangent Lorentz bundle (phase space) for nonassociative and noncommutative gravity and geometric flow theories is that such *models can be formulated in nonholonomic canonical variables and, equivalently, in Finsler–Hamilton variables.* The corresponding N-connection structures can be defined from respective nonlinear geodesic equations which are equivalent to the Hamilton equations on a phase space, but there are derived also Ricci s-tensors for nonholonomic metric-affine phase spaces. In the so-called canonical form (with “hat” variables), physically important systems of nonlinear PDEs (for instance, modified geometric flow and Ricci soliton equations) possess general decoupling and integrability properties. This allows us to construct and study the physical properties of various classes of exact and parametric solutions defined by off-diagonal symmetric and nonsymmetric metrics. The corresponding AFCDM works for general connections (nonlinear and adapted ones, encoding Finsler-like and nonassociative and noncommutative distortions), when LC-configurations can be extracted for additional nonholonomic constraints which can be solved in explicit form. We argue (and we shall provide explicit examples in respective sections) that explicit criteria can be formulated when such solutions define 4-d , 8-d and 10-d nonholonomic Lorentz spacetime models and (nonassociative and/or noncommutative) black hole/ellipsoid, wormhole, toroid, and cosmological configurations described by modified dispersion relations, MDRs, and encoding nonholonomic Finsler–Hamilton structures, in general, in nonassociative forms.

1.5 The objectives and structure of the paper

This paper provides a status report on the AFCDM and applications with geometric computation examples on 4-d and extra dimension gravity theories (including 8-d nonassociative phase space models with geometric evolution). Details of the proofs are provided in Part I for nonholonomic (2+2) decompositions when the constructions for higher dimensions, in Part II, are derived in abstract geometric form to higher dimensions including and phase space models. The most important formulas and physically important systems of nonlinear PDEs can be modelled equivalently in canonical and/or Finsler–Hamilton variables; and for nonassociative theories of geometric and information flows and gravity, see related results and methods in [5,6,9,10,29,30,40,41]. In Appendix, we summarize the approach and provide abstract and N-adapted formulas for higher dimensions and (co) tangent Lorentz bundles which may encode Finsler–Lagrange–Hamilton structure and/or nonassociative data.

The main objectives are stated for respective parts of the article. For the Part I, there are three objectives:

The **first objective**, Obj1, is to review in Sect. 2 the geometry of N-connections defining (dyadic) nonholonomic (2+2)-splitting, and related adapted frames and distinguished connection, d-connection, structures. There are defined canonical d-connection and LC-connection determined by the same metric structure with a N-connection splitting and corresponding curvature and torsion d-tensors, Ricci and Einstein d-tensors. The (modified) gravitational equations are formulated in canonical nonholonomic variables. We show how to compute in explicit forms the N-adapted formulas for the canonical d-connections and derive the coefficient formulas for the canonical torsion and Ricci d-tensors, canonical distortion of the scalar curvature. There are considered necessary parametrizations in N-adapted forms of the effective and energy-momentum tensors.

We prove the general decoupling and integration properties of the Einstein equations for the canonical d-connection in Sect. 3 (this consists of the **second objective**, Obj2, of our work. For simplicity, the general formulas for generic off-diagonal solutions are derived for nonholonomic spacetimes with Killing d-vector symmetry. Corresponding N-adapted coefficients are expressed in terms of generating and integration functions and generating sources. We show that such classes of solutions possess nonlinear symmetries which allow to re-define the generating functions and introduce effective cosmological constants. There are provided the quadratic linear elements for off-diagonal solutions in using geometric data with (1) generating functions and generating sources; (2) re-defined generating functions and effective cosmological constants; (3) nonlinearly related to "original" generating functions and effective sources/cosmological constants. We also study deformations of prime d-metrics by so-called (4) gravitational polarization functions (which can be also considered as generating functions) into target d-metrics defining off-diagonal solutions of the Einstein equations. For (5) small parametric deformations of the polarization functions, we consider parametric deformations of the d-metrics and respective solutions. We consider a toy 2+2 model with effective momentum variables which allow a straightforward geometric generalizations for (co) tangent phase space models with conventional (2+2)+(2+2) splitting considered in Tables 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 from the Appendix. There are analyzed the conditions, and the possibility to solve such nonholonomic constraints, which are necessary for extracting LC-configurations.

The **third objective**, Obj3, stated for Sect. 4, is to provide and study explicit examples of new classes of exact/ parametric generic off-diagonal solutions constructed by using the AFCDM. For the first class of such quasi-stationary solutions, we consider examples of new Kerr de Sitter solution and their nonholonomic deformations to spheroidal configurations. Then the procedure of nonholonomic off-diagonal deformations of cylindrical systems in GR is formulated. Such constructions are applied for generating solutions describing locally anisotropic wormholes. We also provide new classes of generic off-diagonal solutions describing locally anisotropic black torus, BT, and black ellipsoid, BE, configurations. There are analyzed solutions involving both BT and BE nonholonomically deformed geometric objects. Then, we construct and analyze new classes of nonholonomic cosmological solitonic and spheroid deformations involving 2-d vertices and elaborate on models with small parametric off-diagonal cosmological deformations with solitonic vacua for voids.

For the Part II devoted to (nonassociative) canonical and Finsler–Hamilton phase space generalizations the main objectives are:

The **forth objective**, Obj4, is to outline in Sect. 5 the nonassociative geometric flow theory in canonical nonholonomic variables. Such models are determined by star product R-flux deformations in string theory which, in our approach, are formulated in canonical nonholonomic variables with dyadic (shell) splitting. This allows us to prove general decoupling and integrability properties of such theories using abstract geometric methods. We explain why "pure" Lagrange and/or Hamilton phase space models can't be used for generating integrable MGTs when equivalent formulations in terms of generalized Finsler–Lagrange–Hamilton phase space models, with N-connections and canonical s-connections, allow to construct off-diagonal solutions of (modified) Einstein equations.

The nonassociative Finsler–Lagrange–Hamilton geometric flow theories are formulated and studied in Sect. 6. This consists the **fifth objective**, Obj5, which aims to characterize such geometries using G. Perelman's F- and W-functionals (with respective generalizations and nonholonomic modifications) and derived statistical and geometric thermodynamic models encoding nonassociative and/or nonholonomic Finsler–Hamilton like data. The **sixth objective**, Obj6, in the same section, is to formulate the theory of nonassociative Finsler–Lagrange–Hamilton geometric flows. We define star product R-flux versions of such nonassociatie generalized Finsler geometries and postulate nonassociative versions of geometric evolution equations. Another very important **seventh objective**, Obj7, also in Sect. 6, is to formulate nonassociative Finsler like generalizations of G. Perelman thermodynamics for geometric flows which is important for future developments of thermofield and quantum gravity theories encoding nonassociative and modified dispersion data.

In Sect. 8, we show how the AFCDM for nonassociative geometric flows and gravity can be applied (using respective distortions of adapted Finsler–Lagrange–Hamilton structures and nonlinear transforms) to decouple and solve physically important systems of nonlinear PDEs for such models and physical theories. This consists the **eights objective**, Obj8, of this work. We distinguish two general classes of nonassociative Finsler like quasi-stationary solutions and locally anisotropic cosmological solutions. Respective nonlinear symmetries encoding nonassociative and noncommutative modified Finsler–Hamilton data are analyzed. In the same section, there is the **ninths objective**, Obj9, to construct and analyze explicit examples of physically important nonassociative and Finsler–Hamilton configurations. We provide exact parametric solutions describing nonassociative Finsler–Hamilton black ellipsoids and compute respective Bekenstein–Hawking entropy and G. Perelman thermodynamic variables.

We discuss and conclude the results and methods in Sect. 8. Here we also note that the **tenth objective**, Obj10, is stated for the Appendix. It aims a summary of the AFCDM (from 4-d GR till 10-d and nonassociative phase spaces of 8-d) stated as Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16. Such tables can be used for generating exact and parametric solutions in various types of gravity theories by prescribing physically important generating and integration functions and effective and matter field sources. They outline respective geometric ideas, typical off-diagonal ansatz, and the main results on decoupling of respective modified Einstein equations and classifications into quasi-stationary and locally anisotropic cosmological solutions for gravity theories on 4-d Lorentz manifolds (see Tables 1, 2, 3). Then the results are extended for 10-d Lorentz manifolds in Tables 4, 5, 6. There are also considered geometric summaries of the AFCDM for phase space models elaborated on (co) tangent Lorentz bundles of total dimension 8-d and for a 4-d Lorentz base spacetime manifold. The formulas are provided for (effective) sources which, in general, encode nonassociative Finsler–Lagrange–Hamilton data which allow us to construct exact/parametric solutions using canonical variables. The Tables 7, 8, 9, 10, 11 classify and outline possible versions for solutions determined by generic off-diagonal metrics and generalized connections depending both on base spacetime and velocity type coordinates.

In the final subsection of the Appendix, we summarize the results on solutions for gravity like models elaborated on nonholonomic cotangent Lorentz bundles encoding momentum like variables which may be with a fixed energy like parameter or depend like a "rainbow" on an energy type coordinate. Such solutions can be formulated both for commutative and nonassociative models, in general, with nonassociative Finsler–Lagrange–Hamilton geometric flow dependence. Projections of such nonholonomic phase space solutions on the base Lorentz spacetime manifolds may model quasi-stationary and/or locally anisotropic cosmological solutions. Such generic off-diagonal phase space solutions may encode various types of nonassociative/noncommutative/supersymmetric/nonmetric/nonsymmetric data/quasi-classical contributions computed (or introduced phenomenologically) for different models of modified geometric and information flow theories, superstring/M-theory and other MGTs.

1.6 Remarks on abbreviations and notations

In this review work, we use many abbreviations and notations which are standard for certain researchers in some directions on geometry and physics but unknown for others working in particle physics, cosmology and other directions. The principles and main formulas of abstract geometric and N-adapted coefficient calculus are outlined in respective subsections with explanations of Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 from the Appendix. We have to dub some terms and symbols in certain sections containing interdisciplinary methods. For instance, the abbreviations introduced new geometric methods of constructing parametric solutions are dubbed and explained again for applications in modern cosmology or nonassociative geometric flow thermodynamics. For convenience, we present here a list of abbreviations mentioning them in the order as they were introduced in a respective paragraph of a (sub) section.

GR – *general relativity* – Sect. 1, paragraph 1
MGTs – *modified gravity theories* – Sect. 1, paragraph 1
2-d, 4-d, or 8-d etc. – *two, four, or eight etc. dimenisonal/ dimensions* – Sect. 1, paragraph 1
N-connection(s) – *nonlinear connection(s)* – Sect. 1, paragraph 1
$2+2+2+\cdots$ – *dyadic decompostions/splitting of phase space or spacetime dimensions* – Sect. 1, paragraph 2
PDEs – *partial differential equations* – Sect. 1, paragraph 2
LC configuration/connection – *Levi–Civita configuration/connection* – Sect. 1, paragraph 2
AFCDM – the *anholonomic frame and connection deformation method* (of constructing off-diagonal solutions in geometric flow and gravity theories) – Sect. 1, paragraph 2
ODEs – *ordinary differential equations* – Sect. 1, paragraph 3
BH – *black hole* – Sect. 1.1, paragraph 2
FLRW – *Friedman–Lemaître–Robertson–Walker* (cosmology) – section 1.1, paragraph 3, formulas (3)
WH – *wormhole* - Sect. 1.1, paragraph 4
LA – *LA symmetries* (a standard term in the theory of solitons) – Sect. 1.2, paragraph 2
QG – *quantum gravity* – Sect. 1.2, paragraph 3
$\star$-product/-deformation – *star product/deformation* – Sect. 1.3, paragraph 1
D-brane – *Dirichlet brane* (a standard term in string/M-theory) – Sect. 1.3, paragraph 1
R-flux – a *standard term* in string/M-theory (explained in paragraphs related to formulas (155) – (157)) – Sect. 1.3, paragraph 2
ADM – *Arnowit–Deser–Misner* (a method with 3+1 splitting in GR and MGTs) – Sect. 1.4, paragraph 1
WDW – *Wheeler de Wit* (an important equation in GR and QG) – Sect. 1.4, paragraph 1
QFT – *quantum field theory* – Sect. 1.4, paragraph 3
Obj1 – *objective 1* (of this work) – Sect. 1.5, paragraph 1
d-connection, d-tensor, d-object – *geometric objects adapted to a N-connection structure* – Sect. 1.5, paragraph 3
BT – *black torus* – Sect. 1.5, paragraph 6
BE – *black ellipsoid* – Sect. 1.5, paragraph 6

F- and W-functionals – introduced by G. Perelman together with the concept of W-entropy (in his geometric flow thermodynamics) – Sect. 1.5, paragraph 8, related to Obj5, see details in Sect. 6.3

MDR – *modified dispersion relations* – Part II, paragraph 1

To avoid ambiguities certain abbreviations and abstract symbols are repeated, if necessary, in different parts/sections of the paper.

Part I Nonholonomic variables in Einstein gravity & MGTs

2 The geometry of GR & MGTs with nonholonomic (2+2)-splitting

We review the main ideas and proofs for a nonholonomic geometric formulation of the Einstein gravity (general relativity, GR) and modified gravity theories, MGTs, on 4-d metric-affine manifolds, which in canonical nonholonomic variables will allow us to prove in next section certain general decoupling and integrations properties of gravitational field equations.

2.1 Geometric and physical objects in nonholonomic 2+2 variables

2.1.1 Nonlinear connections and distinguished metrics

We shall work on a Lorentz spacetime enabled with standard geometric data $(V, \mathbf{g})$, where V is a 4-d pseudo-Riemannian manifold of necessary smooth (differentiability) class, defined by a symmetric metric tensor of signature $(+++-)$,

$$\mathbf{g} = g_{\alpha'\beta'}(u)e^{\alpha'} \otimes e^{\beta'}. \tag{8}$$

In this formula, we consider general co-frames $e^{\alpha'}$ which are dual to frame bases $e_{\alpha'}$. On a coordinate neighborhood $U \subset V$, we can always define local coordinates $u = \{u^{\alpha} = (x^i, y^a)\}$ involving a conventional $2+2$ splitting into h-coordinates, $x = (x^i)$, and v-coordinates, $y = (y^a)$, for indices $j, k, \ldots = 1, 2$ and $a, b, c, \ldots = 3, 4$, when $\alpha, \beta, \ldots = 1, 2, 3, 4$. A local coordinate basis and a co-base are written respectively as $e_{\alpha} = \partial_{\alpha} = \partial/\partial u^{\beta}$ and $e^{\beta} = du^{\beta}$. Transforms to arbitrary frames (tetrads/ vierbeinds) are defined as $e_{\alpha'} = e^{\alpha}_{\ \alpha'}(u)e_{\alpha}$ and $e^{\alpha'} = e_{\alpha}^{\ \alpha'}(u)e^{\alpha}$. Such (co) bases are orthonormal if $e_{\alpha}^{\ \alpha'}e^{\beta}_{\ \alpha'} = \delta^{\beta}_{\alpha}$, where δ^{β}_{α} is the Kronecker symbol.

In coordinate free form, a 2+2 decomposition can be introduced as a conventional nonlinear connection structure (N-connection), when for the tangent bundle $TV := \bigcup_u T_u V$ it is prescribed a non–integrable (equivalently, nonholonomic/anholonomic) conventional horizontal and vertical splitting, in brief, h- and v–decomposition into respective 2-d and 2-d subspaces, hV and vV. This is equivalent to the condition that a Whitney sum

$$\mathbf{N} : TV = hV \oplus vV \tag{9}$$

is globally defined for V and TV. For instance, in Finsler geometry [7,29,30], the N-connections are defined by splitting of type $TTV = hTV \oplus vTV$, involving the second tangent bundle TTV, or (in some equivalent forms) using nonholonomic distributions and splitting in exact sequences. On Lorentz manifolds, a N-connection (9) consist an example of nonholonomic distribution defining a fibered 2+2 structure. We can use the term nonholonomic Lorentz/pseudo-Riemannian manifold when a conventional h-v-splitting of some classes of local bases is defined at least on a neighbourhood $U \subset V$, $e_{\alpha'} = (e_{i'}, e_{a'})$ and $e^{\beta'} = (e^{i'}, e^{a'})$. Hereafter, we shall omit priming/underlying/overlying etc. of indices if that will not result in ambiguities. We also note that in our works we use "boldface" symbols in order to emphasize that certain spaces/geometric objects are enabled/adapted with/to a N-connection structures. In local coordinate form, a N-connection is defined by a nonholonomic distribution stated by a set of coefficients $N^a_i(u)$ when $\mathbf{N} = N^a_i(x, y)dx^i \otimes \partial/\partial y^a$.

Using N-connection coefficients $\mathbf{N} = \{N^a_i\}$ (9), we can define N-elongated (equivalently, N-adapted) local bases (partial derivatives), $\mathbf{e}_{\nu}$, and co-bases (differentials), $\mathbf{e}^{\mu}$, when

$$\mathbf{e}_{\nu} = (\mathbf{e}_i, e_a) = (\mathbf{e}_i = \partial/\partial x^i - N^a_i(u)\partial/\partial y^a,$$
$$e_a = \partial_a = \partial/\partial y^a), \text{ and} \tag{10}$$
$$\mathbf{e}^{\mu} = (e^i, \mathbf{e}^a) = (e^i = dx^i, \ \mathbf{e}^a = dy^a + N^a_i(u)dx^i), \tag{11}$$

are linear on N^a_i. For instance, a N-elongated basis (10) satisfies the nonholonomy relations

$$[\mathbf{e}_{\alpha}, \mathbf{e}_{\beta}] = \mathbf{e}_{\alpha}\mathbf{e}_{\beta} - \mathbf{e}_{\beta}\mathbf{e}_{\alpha} = W^{\gamma}_{\alpha\beta}\mathbf{e}_{\gamma}, \tag{12}$$

with (antisymmetric) nontrivial anholonomy coefficients

$$W^b_{ia} = \partial_a N^b_i, W^a_{ji} = \Omega^a_{ij} = \mathbf{e}_j \left(N^a_i\right) - \mathbf{e}_i(N^a_j), \tag{13}$$

where Ω^a_{ij} define the coefficients of N-connection curvature. If all anholonomic coefficients (13) are zero for a $\mathbf{e}_{\alpha}$, such a N-adapted base is holonomic and we can write it as a partial derivative ∂_{α}, with zero N-connection coefficients for corresponding coordinate transforms. In curved coordinates, for holonomic bases, the coefficients N^a_j may be non-zero even all $W^{\gamma}_{\alpha\beta} = 0$.

The geometric objects on a nonholonomic manifold $\mathbf{V}$ enabled with a N-connection structure $\mathbf{N}$ (and on extensions to tangent, $T\mathbf{V}$, and cotangent, $T^*\mathbf{V}$, bundles; and their tensor products, for instance, $T\mathbf{V} \otimes T^*\mathbf{V}$) are called distinguished (in brief, d-objects, d-vectors, d-tensors etc) if they are adapted to the N-connection structure via corresponding decompositions with respect to frames of type (10) and (11). For instance, we write a d-vector as $\mathbf{X} = (hX, vX)$.

Any spacetime metric $\mathbf{g}$ (8) can be represented equivalently as a d-metric $\mathbf{g} = (hg, vg)$, when

$$\begin{aligned}\mathbf{g} =\ & g_{ij}(x,y)\, e^i \otimes e^j \\ & + g_{ab}(x,y)\, \mathbf{e}^a \otimes \mathbf{e}^b, \text{ in N-adapted form;} \quad (14)\\ =\ & \underline{g}_{\alpha\beta}(u) du^\alpha \otimes du^\beta, \\ & \text{using tensor products of dual coordinate bases.} \quad (15)\end{aligned}$$

In above formulas $hg = \{ g_{ij}\}$ and $vg = \{g_{ab}\}$. Introducing coefficients of (11) into (14) and regrouping with respect to the coordinate dual basis, we obtain the formulas for the coefficients in (15),

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e g_{be} & g_{ab} \end{bmatrix} \quad (16)$$

for any prescribed set of coefficients N_i^a. A metric $\mathbf{g} = \{\underline{g}_{\alpha\beta}\}$ (16) is generic off-diagonal if the anholonomy coefficients $W_{\alpha\beta}^\gamma$ are not identical to zero (in 4-d, such a matrix can not be diagonalized via coordinate transforms, but we can consider such diagonalizations for 2-d and 3-d subspaces). Parameterizations of type (16) are used (1) in Kaluza–Klein theories when $N_j^e = A_j^e$ are identified as certain gauge fields after compactification on y-coordinates (usually, there are considered higher dimension spacetimes); (2) in Finsler like theories, when N_j^e are defined in certain forms which are used in respective Finsler–Lagrange–Hamilton theories, see details in [7,8,24]; and in GR when N-coefficients are treated as off-diagonal terms and used for N-adapted geometric constructions.

In this work, we prefer to work with d-metrics (14), d-tensors, d-connections etc., because in certain N-adapted forms for d-objects it is possible to prove the decoupling and integration properties of (modified) Einstein equations. In coordinate bases, the constructions are very cumbersome and a general decoupling is not possible if we use the LC-connection ∇. Finally, we note that the components of the inverse metric $\underline{g}^{\alpha\beta}$ (in general, with off-diagonal terms) are computed for nondegenerated metric structures following formulas $\underline{g}^{\alpha\beta}\underline{g}_{\gamma\beta} = \delta^\alpha_\beta$. In similar forms, there are defined and computed the inverse d-metrics and their h- and v-coefficients, $\mathbf{g}^{\alpha\beta} = (g^{ij}, g^{ab})$.

2.1.2 N-adapted connections, the canonical d-connection and fundamental geometric objects

Linear connection structures on a nonholonomic $\mathbf{V}$ can be defined in N-adapted or in a general form, which may be not N-adapted, for instance, in the case of the LC-connection ∇).

A **d-connection** $\mathbf{D} = (hD, vD)$ is a linear connection preserving under parallelism the N-connection splitting (9). It defines a covariant N-adapted derivative $\mathbf{D}_{\mathbf{X}}\mathbf{Y}$ of a d-vector field $\mathbf{Y}$ in the direction of a d-vector $\mathbf{X}$. With respect to N-adapted frames (10) and (11), any $\mathbf{D}_{\mathbf{X}}\mathbf{Y}$ can be computed as in GR and/or metric affine gravity but with the coefficients decomposed defined by h- and v-indices,

$$\begin{aligned}&\mathbf{D} = \{\mathbf{\Gamma}^\gamma_{\ \alpha\beta} = (L^i_{jk}, \acute{L}^a_{bk}; \acute{C}^i_{jc}, C^a_{bc})\}, \text{ where}\\ &hD = (L^i_{jk}, \acute{L}^a_{bk}) \text{ and } vD = (\acute{C}^i_{jc}, C^a_{bc}). \quad (17)\end{aligned}$$

By definition, any d-connection is characterized by three fundamental geometric d-objects, which (by definition in abstract forms) are:

$$\begin{aligned}&\mathcal{T}(\mathbf{X},\mathbf{Y}) := \mathbf{D}_{\mathbf{X}}\mathbf{Y} - \mathbf{D}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}],\\ &\qquad \text{torsion d-tensor, d-torsion;}\\ &\mathcal{R}(\mathbf{X},\mathbf{Y}) := \mathbf{D}_{\mathbf{X}}\mathbf{D}_{\mathbf{Y}} - \mathbf{D}_{\mathbf{Y}}\mathbf{D}_{\mathbf{X}} - \mathbf{D}_{[\mathbf{X},\mathbf{Y}]},\\ &\qquad \text{curvature d-tensor, d-curvature;}\\ &\mathcal{Q}(\mathbf{X}) := \mathbf{D}_{\mathbf{X}}\mathbf{g}, \text{nonmetricity d-fiels, d-nonmetricity} \quad (18)\end{aligned}$$

The N-adapted coefficients of such geometric d-objects are computed by introducing $\mathbf{X} = \mathbf{e}_\alpha$ and $\mathbf{Y} = \mathbf{e}_\beta$, defined by (10), and considering h-v-splitting (17) for $\mathbf{D} = \{\mathbf{\Gamma}^\gamma_{\ \alpha\beta}\}$ into above formulas, see details in [7,8,24],

$$\begin{aligned}&\mathcal{T} = \{\mathbf{T}^\gamma_{\ \alpha\beta} = \left(T^i_{\ jk}, T^i_{\ ja}, T^a_{\ ji}, T^a_{\ bi}, T^a_{\ bc}\right)\};\\ &\mathcal{R} = \{\mathbf{R}^\alpha_{\ \beta\gamma\delta} = \left(R^i_{\ hjk}, R^a_{\ bjk}, R^i_{\ hja}, R^c_{\ bja}, R^i_{\ hba}, R^c_{\ bea}\right)\};\\ &\mathcal{Q} = \{\mathbf{Q}^\gamma_{\ \alpha\beta} = \mathbf{D}^\gamma \mathbf{g}_{\alpha\beta} = (Q^k_{\ ij}, Q^c_{\ ij}, Q^k_{\ ab}, Q^c_{\ ab})\}.\end{aligned}$$

We say that any geometric data $(\mathbf{V}, \mathbf{N}, \mathbf{g}, \mathbf{D})$ define a N-adapted metric-affine structure (equivalently, metric-affine d-structure) determined by a d-metric and a d-connection stated independently, but both in N-adapted form $\mathbf{V}$.

Using a d-metric $\mathbf{g}$ (14), we can define two important linear connection structures (the Levi–Civita, LC, connection and the canonical d-connection):

$$(\mathbf{g}, \mathbf{N}) \rightarrow \begin{cases} \mathbf{\nabla} : \mathbf{\nabla}\mathbf{g} = 0;\ {}_{\nabla}\mathcal{T} = 0, \text{ LC–connection ;} \\ \widehat{\mathbf{D}} : \widehat{\mathbf{Q}} = 0;\ h\widehat{\mathcal{T}} = 0, v\widehat{\mathcal{T}} = 0, \\ h \quad \widehat{\mathcal{T}} \neq 0, \text{ the canonical d-connection} \end{cases} \quad (19)$$

In our work, "hat" labels are used for geometric d-objects written in such a canonical form. For any $\widehat{\mathbf{D}}$, we can define and compute the canonical fundamental geometric objects (18), $\widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^\alpha_{\ \beta\gamma\delta}\}$ etc. In a similar form, we can compute the fundamental geometric objects defined by $\mathbf{\nabla}$, for instance, ${}_{\nabla}\mathcal{R} = \{ {}_{\nabla}R^\alpha_{\ \beta\gamma\delta}\}$ (in such cases, boldface indices are not used). Considering the canonical distortion relation for linear connections (5), we can compute respective canonical distortions of fundamental geometric d-objects (18). Such formulas relate, for instance, two different curvature tensors, ${}_{\nabla}\mathcal{R} = \{{}_{\nabla}R^{\alpha}{}_{\beta\gamma\delta}\}$ and $\widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta}\}$ etc.

2.1.3 The canonical Ricci and Einstein d-tensors; the canonical d-torsion and LC-conditions

We can define the canonical Ricci d-tensor as the contraction on the 1st and 4th indices of the canonical curvature d-tensor,

$$\widehat{\mathbf{R}}ic = \{\widehat{\mathbf{R}}_{\ \beta\gamma} := \widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\alpha}\}. \tag{20}$$

It should be noted that this d-tensor, in general, is not symmetric, i.e. $\widehat{\mathbf{R}}_{\ \beta\gamma} \neq \widehat{\mathbf{R}}_{\ \gamma\beta}$. This is typical for nonholonomic geometric objects. The canonical scalar curvature is introduced as

$$\widehat{R}sc := \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\ \alpha\beta}.$$

These formulas allow to define the canonical (nonholonomic) Einstein d-tensor,

$$\widehat{\mathbf{E}}n := \widehat{\mathbf{R}}ic - \frac{1}{2}\mathbf{g}\widehat{R}sc = \{\widehat{\mathbf{R}}_{\ \beta\gamma} - \frac{1}{2}\mathbf{g}_{\ \beta\gamma}\widehat{R}sc\}. \tag{21}$$

Adapting for $\widehat{\mathbf{D}}$ the geometric abstract principles for deriving gravitational filed equations provided in [2] for ∇, and considering a nontrivial cosmological constant Λ, we can derive the canonical Einstein d-equations (6) using the canonical d-tensor (21). Such equations can be proved also in N-adapted variational form if we introduce conventional gravitational and matter fields Lagrange densities, ${}^{g}L(\widehat{\mathbf{R}}ic)$ (as in GR with ${}^{g}L(R)$) and postulating a ${}^{m}L(\varphi^{A}, \mathbf{g}_{\beta\gamma})$. In N-adapted form, the stress-energy d-tensor of matter fields φ^{A} (labeled by a general index A) is defined and computed

$$\mathbf{T}_{\alpha\beta} = -\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta({}^{m}L\sqrt{|\mathbf{g}_{\mu\nu}|})}{\delta\mathbf{g}^{\alpha\beta}}. \tag{22}$$

For such sources, we can define the trace $T := \mathbf{g}^{\alpha\beta}\mathbf{T}_{\alpha\beta}$ and the effective source $\widehat{\mathbf{Y}}[\mathbf{g},\widehat{\mathbf{D}}] \simeq \{\mathbf{T}_{\alpha\beta}\}$. In various physical theories, one consider more general ${}^{m}L$, for instance, depending on some covariant/spinor derivatives, and/or ${}^{g}L(f(T,R))$ determined by a functional $f(R,T)$ in MGTs. There are considered various nonsymmetric effective source, for instance, in massive gravity. For simplicity, in this work we omit such considerations (see details and review of results generalized with N-connection structures, in [7,8,24,27] and references therein).

In this work, we consider (effective) sources $\widehat{\mathbf{Y}}[\mathbf{g},\widehat{\mathbf{D}}] = \{\Upsilon^{\beta}_{\delta}\}$ parameterized with respect to N-adapted frames (10) and (11) in such forms (this can be done for very general classes of energy-momentum tensors using by respective frame/coordinate transforms):

$$\widehat{\Upsilon}^{\beta}_{\delta} = diag[\Upsilon_{\alpha} : \Upsilon^{1}_{1} = \Upsilon^{2}_{2} = {}^{h}\Upsilon(x^{k}); \\ \Upsilon^{3}_{3} = \Upsilon^{4}_{4} = {}^{v}\Upsilon(x^{k}, y^{a})]. \tag{23}$$

This assumption means that we shall work with such classes of nonholonomic transforms and constraints when the effective sources are determined by **two generating sources** ${}^{h}\Upsilon(x^{k})$ and ${}^{v}\Upsilon(x^{k}, y^{a})$. It imposes certain nonholonomic constraints on $\mathbf{T}_{\alpha\beta}$, cosmological constant Λ and possible splitting of such constants into h- and v-components; as well on distortion d-tensors $\widehat{\mathbf{Z}}[\mathbf{g}]$ (5) and other values included in $\widehat{\mathbf{Y}}$. To decouple and integrate in general explicit forms some physically important systems of PDEs (for geometric flows and gravitational and matter field equations) is possible if we consider that $\widehat{\mathbf{Y}}[\mathbf{g},\widehat{\mathbf{D}},\kappa]$ contains a small parameter κ, or if the gravitational and matter field dynamics is subjected to certain convenient classes of constraints. In such cases, the solutions can be constructed exactly and/or recurrently using power decompositions $\kappa^{0}, \kappa^{1}, \kappa^{2}, \ldots$ (for instance, κ can be a string constant, or other parameter for constructing ellipsoid deformations etc. We say that the corresponding classes of solutions are exact/parametric, for instance, for linear dependencies on κ^{0} and κ^{1}.

With respect to N-adapted frames (10) and (11), we can re-write equivalently the Einstein equations (1) for ∇ using the canonical d-connection $\widehat{\mathbf{D}}$,

$$\widehat{\mathbf{R}}^{\alpha}{}_{\beta} = \widehat{\mathbf{\Upsilon}}^{\alpha}{}_{\beta}, \tag{24}$$

$$\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta} = 0, \tag{25}$$

where generating sources $\widehat{\mathbf{\Upsilon}}^{\alpha}{}_{\beta} = [\,{}^{h}\Upsilon\delta^{i}{}_{j},\ {}^{v}\Upsilon\delta^{a}{}_{b}]$ (23) and the equations (25) are equivalent to (7), when $\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}[\mathbf{g},\mathbf{N},\widehat{\mathbf{D}}]\}$ is defined in abstract form as in (18). Here we note that, in general, $\widehat{\mathbf{D}}^{\beta}\widehat{\mathbf{E}}^{\alpha}{}_{\beta} \neq 0$ and $\widehat{\mathbf{D}}^{\beta}\widehat{\mathbf{\Upsilon}}^{\alpha}{}_{\beta} \neq 0$, which is typical for nonholonomic systems. For instance, in nonholonomic mechanics, the conservation laws are not standard ones. We have to introduce the so-called Lagrange multiples associated to certain classes of nonholonomic constraints. Solving the constraint equations, it is possible to re-define the variables, then to introduce new effective Lagrangians and, finally, to define standard conservation laws. This can be performed in explicit general forms only for some "toy" models. Using distortions of connections (5), we can rewrite (24) in terms of ∇, when $\nabla^{\beta}E^{\alpha}{}_{\beta} = \nabla^{\beta}T^{\alpha}{}_{\beta} = 0$. So, there are not conceptual problems with the definition of conservation laws for matter fields using two different linear connections (19) defined by the same metric structure $\mathbf{g}$. This is different, for instance, from the Einstein–Cartan and string theory with torsion fields when the second connection may be not defined by the same metric structures but for certain additional or effective gauge fields etc.

We conclude that all geometric constructions and physical theories derived for the geometric data $(\mathbf{g}, \nabla)$ can be equivalently modeled by the canonical geometric data $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ if we use canonical distortion relations (5), or we consider nonholonomic constraints of type (25), equivalently (7). The main result of the AFCDM (to be proven in next section) is that we can decouple in general forms the canonical nonholonomic Einstein equations (24) with $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ and for certain generic off-diagonal ansatz (16), re-written in adapted form as a d-metric (14). For such ansatz, we cannot decouple the standard Einstein equations (1) written in terms of geometric data $(\mathbf{g}, \nabla)$. The main geometric and analytic idea of the AFCDM is that we should search for certain classes of general solutions of gravitational field equations written for $\widehat{\mathbf{D}}$. After some classes of off-diagonal metrics are constructed in terms of generated and integration functions, we can extract LC-configurations for ∇ imposing additional nonholonomic constraints (25), equivalently (7). To solve such nonholonomic equations we have to restrict the classes of generating and integration functions as we shall prove in next section.

2.2 Coefficient formulas for the canonical d-connection and Ricci d-tensors

In this subsection, we provide some coefficient formulas which are important for proofs of decoupling (modified) gravitational field equations and generating solutions following the AFCDM, see details and proofs in [7,8,24,27].

With respect to N-adapted frames (10) and (11) for a h-v-splitting (17) of a d-connection $\mathbf{D}$, the fundamental d-objects (18) are defined by such coefficient formulas:

$$\begin{aligned}
&\text{d-curvature, } \mathcal{R} = \{\mathbf{R}^{\alpha}{}_{\beta\gamma\delta} = (R^{i}{}_{hjk}, R^{a}{}_{bjk}, P^{i}{}_{hja}, P^{c}{}_{bja}, S^{i}{}_{hba}, S^{c}{}_{bea})\},\\
&\text{for } R^{i}{}_{hjk} = \mathbf{e}_k L^{i}{}_{hj} - \mathbf{e}_j L^{i}{}_{hk} + L^{m}{}_{hj} L^{i}{}_{mk} - L^{m}{}_{hk} L^{i}{}_{mj} - C^{i}{}_{ha}\Omega^{a}{}_{kj},\\
&R^{a}{}_{bjk} = \mathbf{e}_k \acute{L}^{a}{}_{bj} - \mathbf{e}_j \acute{L}^{a}{}_{bk} + \acute{L}^{c}{}_{bj}\acute{L}^{a}{}_{ck} - \acute{L}^{c}{}_{bk}\acute{L}^{a}{}_{cj} - C^{a}{}_{bc}\Omega^{c}{}_{kj}, \qquad (26)\\
&P^{i}{}_{jka} = e_a L^{i}{}_{jk} - D_k \acute{C}^{i}{}_{ja} + \acute{C}^{i}{}_{jb} T^{b}{}_{ka},\\
&P^{c}{}_{bka} = e_a \acute{L}^{c}{}_{bk} - D_k C^{c}{}_{ba} + C^{c}{}_{bd} T^{c}{}_{ka},\\
&S^{i}{}_{jbc} = e_c \acute{C}^{i}{}_{jb} - e_b \acute{C}^{i}{}_{jc} + \acute{C}^{h}{}_{jb}\acute{C}^{i}{}_{hc} - \acute{C}^{h}{}_{jc}\acute{C}^{i}{}_{hb},\ S^{a}{}_{bcd}\\
&\qquad = e_d C^{a}{}_{bc} - e_c C^{a}{}_{bd} + C^{e}{}_{bc} C^{a}{}_{ed} - C^{e}{}_{bd} C^{a}{}_{ec};\\
&\text{d-torsion, } \mathcal{T} = \{\mathbf{T}^{\gamma}{}_{\alpha\beta} = (T^{i}{}_{jk}, T^{i}{}_{ja}, T^{a}{}_{ji}, T^{a}{}_{bi}, T^{a}{}_{bc})\},\\
&\text{for } T^{i}{}_{jk} = L^{i}_{jk} - L^{i}_{kj}, T^{i}{}_{jb} = C^{i}_{jb}, T^{a}{}_{ji} = -\Omega^{a}{}_{ji},\\
&T^{c}_{aj} = L^{c}_{aj} - e_a(N^{c}_{j}), T^{a}_{bc} = C^{a}_{bc} - C^{a}_{cb}; \qquad (27)\\
&\text{d-nonmetricity, } \mathcal{Q} = \{\mathbf{Q}_{\gamma\alpha\beta} = \big(Q_{kij}, Q_{kab}, Q_{cij}, Q_{cab}\big)\},\\
&\text{for } Q_{kij} = D_k g_{ij}, Q_{kab} = D_k g_{ab}, Q_{cij} = D_c g_{ij}, Q_{cab} = D_c g_{ab}. \qquad (28)
\end{aligned}$$

The h-v-coefficients of the Ricci d-tensor, $\mathbf{R}ic = \{\mathbf{R}\ _{\beta\gamma} := \mathbf{R}^{\alpha}{}_{\beta\gamma\alpha}\}$, split into four groups of coefficient, respectively defined by contacting respective indices in (26),

$$\mathbf{R}_{\alpha\beta} = \{R_{ij} := R^{k}{}_{ijk},\ R_{ia} := -R^{k}{}_{ika},\\ R_{ai} := R^{b}{}_{aib},\ R_{ab} := R^{c}{}_{abc}\}. \qquad (29)$$

Using the inverse d-tensor of a d-metric (14), we can compute the scalar curvature ${}^{s}\widehat{R}$ of $\widehat{\mathbf{D}}$ is by definition

$$Rsc := \mathbf{g}^{\alpha\beta}\mathbf{R}\ _{\alpha\beta} = g^{ij}R_{ij} + g^{ab}R_{ab}. \qquad (30)$$

A canonical d-connection (19) is defined by N-adapted coefficients $\widehat{\mathbf{D}} = \{\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = (\widehat{L}^{i}{}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}{}_{jc}, \widehat{C}^{a}_{bc})\}$, for

$$\begin{aligned}
\widehat{L}^{i}{}_{jk} &= \frac{1}{2} g^{ir}(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}),\\
\widehat{L}^{a}_{bk} &= e_b(N^{a}_{k}) + \frac{1}{2} g^{ac}(\mathbf{e}_k g_{bc} - g_{dc}\, e_b N^{d}_{k} - g_{db}\, e_c N^{d}_{k}),\\
\widehat{C}^{i}{}_{jc} &= \frac{1}{2} g^{ik} e_c g_{jk},\ \widehat{C}^{a}_{bc} = \frac{1}{2} g^{ad}(e_c g_{bd} + e_b g_{cd} - e_d g_{bc}),
\end{aligned} \qquad (31)$$

are computed for a d-metric $\mathbf{g} = [g_{ij}, g_{ab}]$ (14) using N-elongated partial derivatives (10). In a similar form, we can compute the coefficients of a LC-connection $\nabla = \{\Gamma^{\gamma}{}_{\alpha\beta}\}$, see general coefficient and/or N-adapted formulas in [2,24]. We note that symbols $\Gamma^{\gamma}{}_{\alpha\beta}$ are not boldface because ∇ is not a d-connection, i.e. it do not preserve a h- and v-splitting under parallelism. The N-adapted coefficients of the canonical distortion d-tensor in (5) can be found following formulas $\widehat{\mathbf{Z}} = \{\widehat{\mathbf{Z}}^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} - \Gamma^{\gamma}{}_{\alpha\beta}\}$.

Finally, we note that introducing the formulas $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ (31) in (26)–(30) (instead of the coefficients of a general d-connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta})$), we can compute the N-adapted coefficients of canonical fundamental d-objects, for instance, $\widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta} = (\widehat{R}^{i}{}_{hjk}, \widehat{R}^{a}{}_{bjk}, \ldots)\}$, $\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta} = (\widehat{T}^{i}{}_{jk}, \widehat{T}^{i}{}_{ja}, \ldots)\}$, for $\widehat{\mathcal{Q}} = \{\widehat{\mathbf{Q}}_{\gamma\alpha\beta} = (\widehat{Q}_{kij} = 0, \widehat{Q}_{kab} = 0) = 0$, and similarly for $\widehat{\mathbf{R}}_{\alpha\beta} = \{\widehat{R}_{ij} := \widehat{R}^{k}{}_{ijk},\ \ldots\}$ and $\widehat{R}sc := \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}\ _{\alpha\beta} = g^{ij}\widehat{R}_{ij} + g^{ab}\widehat{R}_{ab}$. Such formulas will be used in next section to prove decoupling properties of (modified) Einstein equations.

3 Decoupling and integration of (modified) Einstein equations

In this section we show how the canonical distorted Einstein equations (24) can be decoupled and integrated in general forms. We provide necessary N-adapted coefficient formulas, study respective nonlinear and dual symmetries and discuss most important and general variants for parameterising such solutions. The conditions of extracting LC-configurations and off-diagonal solutions in GR, i.e. for standard Einstein equations (1), are stated in explicit form.

3.1 Decoupling property

We prove that for very general classes of off-diagonal ansatz for d-metrics, the system of nonlinear PDEs (24) with generating sources (23) can be decoupled in general N-adapted form.

3.1.1 Off-diagonal ansatz with a Killing vector

Let us consider a quasi-stationary d-metric of type (14), when

$$\begin{aligned}\hat{\mathbf{g}} &= g_i(x^k)dx^i \otimes dx^i + h_3(x^k, y^3)\mathbf{e}^3 \otimes \mathbf{e}^3 \\ &\quad + h_4(x^k, y^3)\mathbf{e}^4 \otimes \mathbf{e}^4, \\ \mathbf{e}^3 &= dy^3 + w_i(x^k, y^3)dx^i, \ \mathbf{e}^4 = dy^4 + n_i(x^k, y^3)dx^i,\end{aligned} \tag{32}$$

with Killing symmetry on the time like coordinate $\partial_4 = \partial_t$. For such ansatz, the N-connection coefficients $\widehat{N}_i^3 = w_i(x^k, y^3)$ and $\widehat{N}_i^4 = n_i(x^k, y^3)$ and N-adapted coefficients of d-metric $\widehat{\mathbf{g}}_{\alpha\beta} = [\widehat{g}_{ij}(x^\kappa), \widehat{g}_{ab}(x^\kappa, y^3)]$ are functions of necessary smooth class. With respect to coordinate frames, the d-metric (32) transforms into an off-diagonal ansatz (16), when the coefficients of metrics do not depend on $y^4 = t$, but depend in certain general forms on space coordinates (x^i, y^3). We can prove decoupling properties for more general ansatz for d-metrics, when

$$\begin{aligned}\mathbf{g} &= g_i(x^k)dx^i \otimes dx^i \\ &\quad + \omega^2(x^k, y^a)[\underline{h}_3(x^k, y^4)h_3(x^k, y^3)\mathbf{e}^3 \otimes \mathbf{e}^3 \\ &\quad + h_4(x^k, y^3)\underline{h}_4(x^k, y^4)]\ \mathbf{e}^4 \otimes \mathbf{e}^4, \\ \mathbf{e}^3 &= dy^3 + [w_i(x^k, y^3) + \underline{n}_i(x^k, y^4)]dx^i, \\ \mathbf{e}^4 &= dy^4 + [n_i(x^k, y^3) + \underline{w}_i(x^k, y^4)]dx^i,\end{aligned}$$

does not have explicit Killing symmetries but may involve vertical co-space conformal transforms with factor $\omega(x^k, y^a)$, see examples in [24,27]. Such an ansatz results in more cumbersome formulas and additional technical difficulties for generating exact/parametric solutions. For simplicity, we do not provide such constructions in this work. We label a d-metric $\hat{\mathbf{g}}$ with a "hat" in order to emphasize that it is of type (32) with Killing symmetry on ∂_t. It is supposed that such a parametrization can be obtained under some frame/coordinate transforms even, in general, such $\hat{\mathbf{g}}(u)$ may depend on all spacetime coordinates.

We can consider a different ansatz for locally anisotropic cosmological d-metrics,

$$\begin{aligned}\underline{\mathbf{g}} &= g_i(x^k)dx^i \otimes dx^i + \underline{h}_3(x^k, t)\underline{\mathbf{e}}^3 \otimes \underline{\mathbf{e}}^3 + \underline{h}_4(x^k, t)\underline{\mathbf{e}}^4 \otimes \underline{\mathbf{e}}^4, \\ \underline{\mathbf{e}}^3 &= dy^3 + \underline{n}_i(x^k, t)dx^i, \quad \underline{\mathbf{e}}^4 = dy^4 + \underline{w}_i(x^k, t)dx^i,\end{aligned} \tag{33}$$

with Killing symmetry on the space like coordinate ∂_3. Correspondingly, we parameterize the N-connection coefficients $\underline{N}_i^3 = \underline{n}_i(x^k, t)$ and $\underline{N}_i^4 = \underline{w}_i(x^k, t)$ and the coefficients of a d-metric $\underline{\mathbf{g}}_{\alpha\beta} = [g_{ij}(x^\kappa), \underline{g}_{ab}(x^\kappa, t)]$, all such values being functions of necessary smooth class. With respect to coordinate frames, the d-metric (33) transforms into a different type off-diagonal ansatz (16), when the coefficients of metrics do not depend on y^3, but depend in certain general forms on spacetime coordinates $(x^i, y^4 = t)$.

For simplicity, we shall sketch proofs of general decoupling and integration properties for quasi-stationary d-metrics (32). To generate solutions for locally anisotropic d-metrics we can change in formal symbolic forms, respectively, $h_3(x^k, y^3) \rightarrow \underline{h}_4(x^k, t), h_4(x^k, y^3) \rightarrow \underline{h}_3(x^k, t)$ and $w_i(x^k, y^3) \rightarrow \underline{n}_i(x^k, t), n_i(x^k, y^3) \rightarrow \underline{w}_i(x^k, t)$. Such "dual" symmetries can be prescribed only for generic off-diagonal solutions with respective Killing symmetries on ∂_4, or ∂_3. Nevertheless, this allow to study main geometric and physical properties of generic off-diagonal metrics with possible nonholonomic constraints and deformations constructed as solutions of systems of nonlinear PDEs, depending, in principle, on 3 from 4 spacetime/space coordinates, not reducing the problem to finding solutions of systems of nonlinear ODEs.

3.1.2 N-adapted coefficients of quasi-stationary canonical d-connections

To simplify computations we use brief notations of partial derivatives, for instance, $\partial_1 q(u^\alpha) = q^\bullet, \partial_2 q(u^\alpha) = q'$, $\partial_3 q(u^\alpha) = q^*$ and $\partial_4 q(u^\alpha) = q^\diamond$.

There are such nontrivial coefficients of $\widehat{\mathbf{\Gamma}}^\gamma_{\ \alpha\beta}$ (31) computed for quasi-stationary d-metrics (32),

$$\begin{aligned}\widehat{L}^1_{11} &= \frac{g_1^\bullet}{2g_1}, \ \widehat{L}^1_{12} = \frac{g_1'}{2g_1}, \widehat{L}^1_{22} = -\frac{g_2^\bullet}{2g_1}, \\ \widehat{L}^2_{11} &= \frac{-g_1'}{2g_2}, \ \widehat{L}^2_{12} = \frac{g_2^\bullet}{2g_2}, \ \widehat{L}^2_{22} = \frac{g_2'}{2g_2}, \\ \widehat{L}^4_{4k} &= \frac{\boldsymbol{\partial}_k(h_4)}{2h_4} - \frac{w_k h_4^*}{2h_4}, \widehat{L}^3_{3k} = \frac{\boldsymbol{\partial}_k h_3}{2h_3} - \frac{w_k h_3^*}{2h_3}, \\ \widehat{L}^3_{4k} &= -\frac{h_4}{2h_3}n_k^*, \\ \widehat{L}^4_{3k} &= \frac{1}{2}n_k^*, \widehat{C}^3_{33} = \frac{h_3^*}{2h_3}, \widehat{C}^3_{44} = -\frac{h_4^*}{h_3}, \ \widehat{C}^4_{33} = 0, \\ \widehat{C}^4_{34} &= \frac{h_4^*}{2h_4}, \widehat{C}^4_{44} = 0.\end{aligned} \tag{34}$$

We shall need also the values

$$\widehat{C}_3 = \widehat{C}^3_{33} + \widehat{C}^4_{34} = \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}, \widehat{C}_4 = \widehat{C}^3_{43} + \widehat{C}^4_{44} = 0. \tag{35}$$

The formulas (34) and (35) are important for computing in explicit form the N-adapted coefficients of the canonical d-torsion and canonical Ricci and Einstein d-tensors.

3.1.3 Coefficients of the N-connection curvature, canonical d-torsion, and LC-conditions

For the N-connection coefficients in (32), the coefficients of the N-connection curvature $\widehat{\Omega}^a_{ij} = \widehat{\mathbf{e}}_j\left(\widehat{N}^a_i\right) - \widehat{\mathbf{e}}_i(\widehat{N}^a_j)$ used in (13) are computed

$$\widehat{\Omega}^a_{ij} = \mathbf{\partial}_j\left(\widehat{N}^a_i\right) - \partial_i(\widehat{N}^a_j) - w_i(\widehat{N}^a_j)^* + w_j(\widehat{N}^a_i)^*.$$

We find such nontrivial values

$$\widehat{\Omega}^3_{12} = -\widehat{\Omega}^3_{21} = \mathbf{\partial}_2 w_1 - \partial_1 w_2 - w_1 w_2^* + w_2 w_1^* = w_1' - w_2^\bullet - w_1 w_2{}^* + w_2 w_1^*;$$
$$\widehat{\Omega}^4_{12} = -\widehat{\Omega}^4_{21} = \mathbf{\partial}_2 n_1 - \partial_1 n_2 - w_1 n_2^* + w_2 n_1^* = n_1' - n_2^\bullet - w_1 n_2^* + w_2 n_1^*. \quad (36)$$

As a result, we can compute the nontrivial coefficients of the canonical d-torsion using formulas (27). We have $\widehat{T}^a_{\ ji} = -\Omega^a_{\ ji}$, with nontrivial coefficients (36), and $\widehat{T}^c_{aj} = \widehat{L}^c_{aj} - e_a(\widehat{N}^c_j)$. For other types of coefficients, we express

$$\widehat{T}^i_{\ jk} = \widehat{L}^i_{jk} - \widehat{L}^i_{kj} = 0,\ \widehat{T}^i_{\ ja} = \widehat{C}^i_{jb} = 0,\ \widehat{T}^a_{bc} = \widehat{C}^a_{bc} - \widehat{C}^a_{cb} = 0,$$
$$\widehat{T}^3_{3k} = \widehat{L}^3_{3k} - e_3(\widehat{N}^3_k) = \frac{\mathbf{\partial}_k h_3}{2h_3} - w_k\frac{h_3^*}{2h_3} - w_k^*,$$
$$\widehat{T}^3_{4k} = \widehat{L}^3_{4k} - e_4(\widehat{N}^3_k) = -\frac{h_4}{2h_3}n_k^*,$$
$$\widehat{T}^4_{3k} = \widehat{L}^4_{3k} - e_3(\widehat{N}^4_k) = \frac{1}{2}n_k^* - n_k^* = -\frac{1}{2}n_k^*,\ \widehat{T}^4_{4k} = \widehat{L}^4_{4k} - e_4(N^4_k) = \frac{\mathbf{\partial}_k h_4}{2h_4} - w_k\frac{h_4^*}{2h_4},$$
$$-\widehat{T}^3_{12} = w_1' - w_2^\bullet - w_1 w_2^* + w_2 w_1^*,$$
$$-\widehat{T}^4_{12} = n_1' - n_2^\bullet - w_1 n_2^* + w_2 n_1^*. \quad (37)$$

The LC-conditions (25) for zero canonical d-torsions, are satisfied if

$$\widehat{L}^c_{aj} = e_a(\widehat{N}^c_j),\ \widehat{C}^i_{jb} = 0,\ \widehat{\Omega}^a_{\ ji} = 0, \quad (38)$$

when in N-adapted frames $\widehat{\mathbf{\Gamma}}^\gamma_{\ \alpha\beta} = \Gamma^\gamma_{\ \alpha\beta}$ even, in general, $\widehat{\mathbf{D}} \neq \nabla$. This is possible because two different linear connections have different transformation laws under general frame/coordinate transforms (such values are not (d) tensor objects). In the LC-cases, all values (37) must vanish. We obtain nontrivial off-diagonal solutions with $h_4^* \neq 0$ and $w_k^* \neq 0$ but $n_k^* = 0$, for $w_k = \mathbf{\partial}_k h_4/h_4^*$. We can search for other types of LC-configurations with $n_k^* \neq 0$ and/or $h_3^* \neq 0$ but it is difficult to obtain explicit formulas for such classes of solutions (they may be also off-diagonal). Finally, it should be noted that conditions of type (38) can be imposed after a general class of quasi-stationary off-diagonal metrics is constructed in a general off-diagonal form involving a nonholonomic torsion. It should be noted that it is not possible to decouple in a general form the Einstein equations working from the very beginning with ∇ defined by a generic off-diagonal ansatz with coefficients depending on 2-4 coordinates.

3.1.4 N-adapted coefficients of the canonical Ricci d–tensor

The h-coefficients of a canonical variant of Ricci d-tensor (29) are computed for contractions of indices in (26), when $\widehat{R}_{ij} = \widehat{R}^k_{\ ijk}$, for

$$\widehat{R}^i_{\ hjk} = \mathbf{e}_k\widehat{L}^i_{.hj} - \mathbf{e}_j\widehat{L}^i_{hk} + \widehat{L}^m_{hj}\widehat{L}^i_{mk} - \widehat{L}^m_{hk}\widehat{L}^i_{mj} - \widehat{C}^i_{ha}\widehat{\Omega}^a_{jk}$$
$$= \mathbf{\partial}_k\widehat{L}^i_{.hj} - \partial_j\widehat{L}^i_{hk} + \widehat{L}^m_{hj}\widehat{L}^i_{mk} - \widehat{L}^m_{hk}\widehat{L}^i_{mj}. \quad (39)$$

We note that these formulas are for and quasi-stationary ansatz (32) and respectively computed values (34) when $\widehat{C}^i_{\ ha} = 0$ and

$$e_k\widehat{L}^i_{hj} = \partial_k\widehat{L}^i_{hj} + N^a_k\partial_a\widehat{L}^i_{hj} = \partial_k\widehat{L}^i_{hj} + w_k(\widehat{L}^i_{hj})^* + n_k(\widehat{L}^i_{hj})^\diamond = \partial_k\widehat{L}^i_{hj}$$

because $\widehat{L}^i_{hj}$ depend only in h-coordinates. Taking derivatives of (34), we obtain

$$\partial_1\widehat{L}^1_{\ 11} = (\frac{g_1^\bullet}{2g_1})^\bullet = \frac{g_1^{\bullet\bullet}}{2g_1} - \frac{(g_1^\bullet)^2}{2\,(g_1)^2},$$
$$\partial_1\widehat{L}^1_{\ 12} = (\frac{g_1'}{2g_1})^\bullet = \frac{g_1'^\bullet}{2g_1} - \frac{g_1^\bullet g_1'}{2\,(g_1)^2},$$
$$\partial_1\widehat{L}^1_{\ 22} = (-\frac{g_2^\bullet}{2g_1})^\bullet = -\frac{g_2^{\bullet\bullet}}{2g_1} + \frac{g_1^\bullet g_2^\bullet}{2\,(g_1)^2},$$
$$\partial_1\widehat{L}^2_{\ 11} = (-\frac{g_1'}{2g_2})^\bullet = -\frac{g_1'^\bullet}{2g_2} + \frac{g_1^\bullet g_2'}{2\,(g_2)^2},$$
$$\partial_1\widehat{L}^2_{\ 12} = (\frac{g_2^\bullet}{2g_2})^\bullet = \frac{g_2^{\bullet\bullet}}{2g_2} - \frac{(g_2^\bullet)^2}{2\,(g_2)^2},$$
$$\partial_1\widehat{L}^2_{\ 22} = (\frac{g_2'}{2g_2})^\bullet = \frac{g_2'^\bullet}{2g_2} - \frac{g_2^\bullet g_2'}{2\,(g_2)^2},$$
$$\partial_2\widehat{L}^1_{\ 11} = (\frac{g_1^\bullet}{2g_1})' = \frac{g_1^{\bullet\prime}}{2g_1} - \frac{g_1^\bullet g_1'}{2\,(g_1)^2},$$
$$\partial_2\widehat{L}^1_{\ 12} = (\frac{g_1'}{2g_1})' = \frac{g_1''}{2g_1} - \frac{(g_1')^2}{2\,(g_1)^2},$$
$$\partial_2\widehat{L}^1_{\ 22} = (-\frac{g_2^\bullet}{2g_1})' = -\frac{g_2^{\bullet\prime}}{2g_1} + \frac{g_2^\bullet g_1'}{2\,(g_1)^2},$$
$$\partial_2\widehat{L}^2_{\ 11} = (-\frac{g_1'}{2g_2})' = -\frac{g_1''}{2g_2} + \frac{g_1^\bullet g_1'}{2\,(g_2)^2},$$
$$\partial_2\widehat{L}^2_{\ 12} = (\frac{g_2^\bullet}{2g_2})' = \frac{g_2^{\bullet\prime}}{2g_2} - \frac{g_2^\bullet g_2'}{2\,(g_2)^2},$$
$$\partial_2\widehat{L}^2_{\ 22} = (\frac{g_2'}{2g_2})' = \frac{g_2''}{2g_2} - \frac{(g_2')^2}{2\,(g_2)^2}.$$

Introducing these values in (39), we obtain two types of nontrivial components:

$$\widehat{R}^1_{\ 212} = \frac{g_2^{\bullet\bullet}}{2g_1} - \frac{g_1^\bullet g_2^\bullet}{4\,(g_1)^2} - \frac{(g_2^\bullet)^2}{4g_1g_2} + \frac{g_1''}{2g_1} - \frac{g_1'g_2'}{4g_1g_2} - \frac{(g_1')^2}{4\,(g_1)^2},$$
$$\widehat{R}^2_{\ 112} = -\frac{g_2^{\bullet\bullet}}{2g_2} + \frac{g_1^\bullet g_2^\bullet}{4g_1g_2} + \frac{(g_2^\bullet)^2}{4(g_2)^2} - \frac{g_1''}{2g_2} + \frac{g_1'g_2'}{4(g_2)^2} + \frac{(g_1')^2}{4g_1g_2}.$$

Considering that $\widehat{R}_{11} = -\widehat{R}^2_{\ 112}$ and $\widehat{R}_{22} = \widehat{R}^1_{\ 212}$, for $g^i = 1/g_i$ and $\widehat{R}^j_j = g^j\widehat{R}_{jj}$ (in these formulas, there is no summarizing on repeating indices), we compute

$$\widehat{R}^1_1 = \widehat{R}^2_2 = -\frac{1}{2g_1g_2}$$
$$\left[g_2^{\bullet\bullet} - \frac{g_1^{\bullet}g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1'g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}\right]. \quad (40)$$

At the next step, we compute N-adapted coefficients with mixed h- and v-indices of the canonical Ricci d-tensor. We consider the third formula in (26),

$$\widehat{R}^c{}_{bka} = \frac{\partial \widehat{L}^c_{bk}}{\partial y^a} - \widehat{C}^c{}_{ba|k} + \widehat{C}^c{}_{bd}\widehat{T}^d{}_{ka} = \frac{\partial \widehat{L}^c_{bk}}{\partial y^a}$$
$$-\left(\frac{\partial \widehat{C}^c_{ba}}{\partial x^k} + \widehat{L}^c_{dk}\widehat{C}^d_{ba} - \widehat{L}^d_{bk}\widehat{C}^c_{da} - \widehat{L}^d_{ak}\widehat{C}^c_{bd}\right) + \widehat{C}^c_{bd}\widehat{T}^d_{ka}.$$

Contracting respectively the indices, we obtain

$$\widehat{R}_{bk} = \widehat{R}^a{}_{bka} = \frac{\partial L^a_{bk}}{\partial y^a} - \widehat{C}^a_{ba|k} + \widehat{C}^a_{bd}\widehat{T}^d_{ka},$$

where for $\widehat{C}_b := \widehat{C}^c_{ba}$ (35),

$$\widehat{C}_{b|k} = \mathbf{e}_k\widehat{C}_b - \widehat{L}^d{}_{bk}\widehat{C}_d$$
$$= \partial_k\widehat{C}_b - N^e_k\partial_e\widehat{C}_b - \widehat{L}^d{}_{bk}\widehat{C}_d$$
$$= \partial_k\widehat{C}_b - w_k\widehat{C}^*_b - n_k\widehat{C}^{\diamond}_b - \widehat{L}^d{}_{bk}\widehat{C}_d.$$

We consider a conventional splitting $\widehat{R}_{bk} = {}_{[1]}R_{bk} + {}_{[2]}R_{bk} + {}_{[3]}R_{bk}$, where

$${}_{[1]}R_{bk} = (\widehat{L}^3_{bk})^* + (\widehat{L}^4_{bk})^{\diamond},\ {}_{[2]}R_{bk}$$
$$= -\partial_k\widehat{C}_b + w_k\widehat{C}^*_b + n_k\widehat{C}^{\diamond}_b + \widehat{L}^d{}_{bk}\widehat{C}_d,$$
$${}_{[3]}R_{bk} = \widehat{C}^a_{bd}\widehat{T}^d_{ka} = \widehat{C}^3_{b3}\widehat{T}^3_{k3} + \widehat{C}^3_{b4}\widehat{T}^4_{k3} + \widehat{C}^4_{b3}\widehat{T}^3_{k4} + \widehat{C}^4_{b4}\widehat{T}^4_{k4}.$$

Using formulas (34), (37) and (35), we compute

$${}_{[1]}R_{3k} = \left(\widehat{L}^3_{3k}\right)^* + \left(\widehat{L}^4_{3k}\right)^{\diamond} = \left(\frac{\mathbf{\partial}_k h_3}{2h_3} - w_k\frac{h^*_3}{2h_3}\right)^*$$
$$= -w^*_k\frac{h^*_3}{2h_3} - w_k\left(\frac{h^*_3}{2h_3}\right)^* + \frac{1}{2}\left(\frac{\mathbf{\partial}_k h_3}{h_3}\right)^*,$$
$${}_{[2]}R_{3k} = -\partial_k\widehat{C}_3 + w_k\widehat{C}^*_3 + n_k\widehat{C}^{\diamond}_3 + \widehat{L}^3{}_{3k}\widehat{C}_3 + \widehat{L}^4{}_{3k}\widehat{C}_4 =$$
$$= w_k\left[\frac{h^{**}_3}{2h_3} - \frac{3}{4}\frac{(h^*_3)^2}{(h_3)^2} + \frac{h^{**}_4}{2h_4} - \frac{1}{2}\frac{(h^*_4)^2}{(h_4)^2} - \frac{1}{4}\frac{h^*_3}{h_3}\frac{h^*_4}{h_4}\right]$$
$$+\frac{\mathbf{\partial}_k h_3}{2h_3}\left(\frac{h^*_3}{2h_3} + \frac{h^*_4}{2h_4}\right) - \frac{1}{2}\partial_k\left(\frac{h^*_3}{h_3} + \frac{h^*_4}{h_4}\right),$$
$${}_{[3]}R_{3k} = \widehat{C}^3_{33}\widehat{T}^3_{k3} + \widehat{C}^3_{34}\widehat{T}^4_{k3} + \widehat{C}^4_{33}\widehat{T}^3_{k4} + \widehat{C}^4_{34}\widehat{T}^4_{k4}$$
$$= w_k\left(\frac{(h^*_3)^2}{4(h_3)^2} + \frac{(h^*_4)^2}{4(h_4)^2}\right) + w^*_k\frac{h^*_3}{2h_3}$$
$$-\frac{h^*_3}{2h_3}\frac{\mathbf{\partial}_k h_3}{2h_3} - \frac{h^*_4}{2h_4}\frac{\mathbf{\partial}_k h_4}{2h_4}.$$

Putting together above formulas (originally such computations were provided in [26]), we find that

$$\widehat{R}_{3k} = w_k\left[\frac{h^{**}_4}{2h_4} - \frac{1}{4}\frac{(h^*_4)^2}{(h_4)^2} - \frac{1}{4}\frac{h^*_3}{h_3}\frac{h^*_4}{h_4}\right] + \frac{h^*_4}{2h_4}\frac{\mathbf{\partial}_k h_3}{2h_3}$$
$$-\frac{1}{2}\frac{\partial_k h^*_4}{h_4} + \frac{1}{4}\frac{h^*_4\partial_k h_4}{(h_4)^2}$$
$$= \frac{w_k}{2h_4}\left[h^{**}_4 - \frac{(h^*_4)^2}{2h_4} - \frac{h^*_3h^*_4}{2h_3}\right] + \frac{h^*_4}{4h_4}\left(\frac{\mathbf{\partial}_k h_3}{h_3} + \frac{\partial_k h^*_4}{h_4}\right)$$
$$-\frac{1}{2}\frac{\partial_k h^*_4}{h_4}. \quad (41)$$

The N-adapted coefficients $\widehat{R}_{4k} = {}_{[1]}R_{4k} + {}_{[2]}R_{4k} + {}_{[3]}R_{4k}$, are defined by

$${}_{[1]}R_{4k} = (\widehat{L}^3_{4k})^* + (\widehat{L}^4_{4k})^{\diamond},\ {}_{[2]}R_{4k} = -\partial_k\widehat{C}_4 + w_k\widehat{C}^*_4$$
$$+n_k\widehat{C}^{\diamond}_4 + \widehat{L}^3{}_{4k}\widehat{C}_3 + \widehat{L}^4{}_{4k}\widehat{C}_4,$$
$${}_{[3]}R_{4k} = \widehat{C}^a_{4d}\widehat{T}^d_{ka} = \widehat{C}^3_{43}\widehat{T}^3_{k3} + \widehat{C}^3_{44}\widehat{T}^4_{k3} + \widehat{C}^4_{43}\widehat{T}^3_{k4} + \widehat{C}^4_{44}\widehat{T}^4_{k4}.$$

Introducing in these formulas $\widehat{L}^3_{4k}$ and $\widehat{L}^4_{4k}$ from (34), we compute

$${}_{[1]}R_{4k} = (\widehat{L}^3_{4k})^* + (\widehat{L}^4_{4k})^{\diamond} = \left(-\frac{h_4}{2h_3}n^*_k\right)^*$$
$$= -n^{**}_k\frac{h_4}{2h_3} - n^*_k\frac{h^*_4h_3 - h_4h^*_3}{2(h_3)^2}.$$

The second term follows from $\widehat{C}_3$ and $\widehat{C}_4$, see (35), and for $\widehat{L}^3_{4k}$ and $\widehat{L}^4_{4k}$, (34),

$${}_{[2]}R_{4k} = -\partial_k\widehat{C}_4 + w_k\widehat{C}^*_4 + n_k\widehat{C}^{\diamond}_4 + \widehat{L}^3{}_{4k}\widehat{C}_3 + \widehat{L}^4{}_{4k}$$
$$\widehat{C}_4 = -n^*_k\frac{h_4}{2h_3}\left(\frac{h^*_3}{2h_3} + \frac{h^*_4}{2h_4}\right).$$

Then we use $\widehat{C}^3_{43}, \widehat{C}^3_{44}, \widehat{C}^4_{43}, \widehat{C}^4_{44}$ from (34) and $\widehat{T}^3_{k3}, \widehat{T}^4_{k3}, \widehat{T}^3_{k4}, \widehat{T}^4_{k4}$ from (37) to compute the third term,

$${}_{[3]}R_{4k} = \widehat{C}^3_{43}\widehat{T}^3_{k3} + \widehat{C}^3_{44}\widehat{T}^4_{k3} + \widehat{C}^4_{43}\widehat{T}^3_{k4} + \widehat{C}^4_{44}\widehat{T}^4_{k4} = 0.$$

Summarizing above three terms,

$$\widehat{R}_{4k} = -n^{**}_k\frac{h_4}{2h_3} + n^*_k$$
$$\times\left(-\frac{h^*_4}{2h_3} + \frac{h^*_4h^*_3}{2(h_3)^*} - \frac{h^*_4h^*_3}{4(h_3)^*} - \frac{h^*_4}{4h_3}\right). \quad (42)$$

For the N-adapted coefficients

$$\widehat{R}^i{}_{jka} = \frac{\partial\widehat{L}^i{}_{jk}}{\partial y^k}$$
$$-\left(\frac{\partial\widehat{C}^i_{ja}}{\partial x^k} + \widehat{L}^i_{lk}\widehat{C}^l_{ja} - \widehat{L}^l_{jk}\widehat{C}^i_{la} - \widehat{L}^c_{ak}\widehat{C}^i_{jc}\right) + \widehat{C}^i_{jb}\widehat{T}^b_{ka}$$

from (26), we obtain zero values because $\widehat{C}^i_{jb} = 0$ and $\widehat{L}^i_{jk}$ do not depend on y^k. So, we have $\widehat{R}_{ja} = \widehat{R}^i{}_{jia} = 0$.

Contracting the indices in $\widehat{R}^a{}_{bcd}$ from (26), we compute the Ricci v-coefficients,

$$\widehat{R}_{bc} = \frac{\partial\widehat{C}^d_{bc}}{\partial y^d} - \frac{\partial\widehat{C}^d_{bd}}{\partial y^c} + \widehat{C}^e_{bc}\widehat{C}_e - \widehat{C}^e_{bd}\widehat{C}^d_{ec}.$$

Summarizing on indices, we express

$$\widehat{R}_{bc} = (\widehat{C}^3_{bc})^* + (\widehat{C}^4_{bc})^\diamond - \partial_c \widehat{C}_b + \widehat{C}^3_{bc}\widehat{C}_3 + \widehat{C}^4_{bc}\widehat{C}_4 \\ -\widehat{C}^3_{b3}\widehat{C}^3_{3c} - \widehat{C}^3_{b4}\widehat{C}^4_{3c} - \widehat{C}^4_{b3}\widehat{C}^3_{4c} - \widehat{C}^4_{b4}\widehat{C}^4_{4c}.$$

There are nontrivial values,

$$\widehat{R}_{33} = \left(\widehat{C}^3_{33}\right)^* + \left(\widehat{C}^4_{33}\right)^\diamond - \widehat{C}^*_3 + \widehat{C}^3_{33}\widehat{C}_3 \\ +\widehat{C}^4_{33}\widehat{C}_4 - \widehat{C}^3_{33}\widehat{C}^3_{33} - 2\widehat{C}^3_{34}\widehat{C}^4_{33} - \widehat{C}^4_{34}\widehat{C}^4_{43} \\ = -\frac{1}{2}\frac{h_4^{**}}{h_4} + \frac{1}{4}\frac{(h_4^*)^2}{(h_4)^2} + \frac{1}{4}\frac{h_3^*}{h_3}\frac{h_4^*}{h_4},$$

$$\widehat{R}_{44} = \left(\widehat{C}^3_{44}\right)^* + \left(\widehat{C}^4_{44}\right)^\diamond - \partial_4\widehat{C}_4 + \widehat{C}^3_{44}\widehat{C}_3 \\ +\widehat{C}^4_{44}\widehat{C}_4 - \widehat{C}^3_{43}\widehat{C}^3_{34} - 2\widehat{C}^3_{44}\widehat{C}^4_{34} - \widehat{C}^4_{44}\widehat{C}^4_{44} \\ = -\frac{1}{2}\frac{h_4^{**}}{h_3} + \frac{1}{4}\frac{h_3^* h_4^*}{(h_3)^2} + \frac{1}{4}\frac{h_4^*}{h_3}\frac{h_4^*}{h_4}.$$

These formulas can be rewritten in the form

$$\widehat{R}^3{}_3 = \frac{1}{h_3}\widehat{R}_{33} = \frac{1}{2h_3h_4}\left(- h_4^{**} + \frac{(h_4^*)^2}{2h_4} + \frac{h_3^*h_4^*}{2h_3}\right),$$
$$\widehat{R}^4{}_4 = \frac{1}{h_4}\widehat{R}_{44} = \frac{1}{2h_3h_4}\left(- h_4^{**} + \frac{(h_4^*)^2}{2h_4} + \frac{h_3^*h_4^*}{2h_3}\right). \quad (43)$$

A quasi-stationary d-metric ansatz (32) is characterized by nontrivial N-adapted coefficients of the canonical d-connection $\widehat{R}^1_1 = \widehat{R}^2_2$ (40), $\widehat{R}_{3k}$ (41), $\widehat{R}_{4k}$ (42) and $\widehat{R}^3{}_3 = \widehat{R}^4{}_4$ (43). Here we note that for such ansatz other classes of coefficients are trivial in N-adapted frames, i.e. $\widehat{R}_{ka} \equiv 0$ for any $k = 1, 2$ and $a = 3, 4$. Such values may be not zero in other systems of reference/coordinates.

The canonical Ricci d-scalar is computed using above N-adapted nontrivial coefficients of the canonical Ricci d-tensor for formulas (30),

$$\widehat{R}sc := \widehat{\mathbf{g}}^{\alpha\beta}\widehat{\mathbf{R}}_{\alpha\beta} = \widehat{g}^{ij}\widehat{R}_{ij} + \widehat{g}^{ab}\widehat{R}_{ab} \\ = \widehat{R}^i{}_i + \widehat{R}^a{}_a = 2(\widehat{R}^2_2 + \widehat{R}^4{}_4),$$

for nontrivial (40) and (43). As a result we can compute the nontrivial components of the canonical Einstein d-tensor (21),

$$\widehat{\mathbf{E}}n := \{\widehat{\mathbf{R}}^\beta{}_\gamma - \frac{1}{2}\delta^\beta_\gamma \widehat{R}sc\} \\ = \{-\widehat{R}^4{}_4, -\widehat{R}^4{}_4;\ \widehat{R}_{ak};\ \widehat{R}_{ka} \equiv 0; -\widehat{R}^2_2, -\widehat{R}^2_2\}.$$

So, in N-adapted form, the canonical Ricci and Einstein d-tensors for quasi-stationary d-metrics posses similar but inverted symmetries for the h- and v-components.

3.1.5 *Explicit decoupling of the modified Einstein equations for canonical d-connections*

Let us define such N-adapted canonical parameterizations of the effective sources (23) for quasi-stationary configurations

$$\widehat{\mathbf{\Upsilon}}^\alpha{}_\beta = [\ {}^h\Upsilon\delta^i{}_j,\ {}^v\Upsilon\delta^a{}_b] \\ = [\ {}^h\Upsilon = -\,{}_1\widehat{\Upsilon}(x^k),\ {}^v\Upsilon = -\,{}_2\widehat{\Upsilon}(x^k, y^3)]. \quad (44)$$

As a result, the canonical distorted Einstein equations (24) for the ansatz (32) (using formulas (40), (41), (42) and (43) and can be written in the form

$$\widehat{R}^1_1 = \widehat{R}^2_2 = \frac{1}{2g_1g_2}[\frac{g_1^\bullet g_2^\bullet}{2g_1} + \frac{(g_2^\bullet)^2}{2g_2} - g_2^{\bullet\bullet} \\ + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1''] = -\,{}_1\widehat{\Upsilon},$$
$$\widehat{R}^3_3 = \widehat{R}^4_4 = \frac{1}{2h_3h_4}\left[\frac{(h_4^*)^2}{2h_4} + \frac{h_3^*h_4^*}{2h_3} - h_4^{**}\right] = -\,{}_2\widehat{\Upsilon},$$
$$\widehat{R}_{3k} = \frac{w_k}{2h_4}\left[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{(h_3^*)(h_4^*)}{2h_3}\right] \\ + \frac{h_4^*}{4h_4}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k(h_3^*)}{2h_3} = 0;$$
$$\widehat{R}_{4k} = \frac{h_4}{2h_3}n_k^{**} + \left(\frac{3}{2}h_4^* - \frac{h_4}{h_3}h_3^*\right)\frac{n_k^*}{2h_3} = 0. \quad (45)$$

Expressing $g_i = e^{\psi(x^k)}$; introducing coefficients

$$\alpha_i = h_4^*\partial_i(\varpi),\ \beta = h_4^*(\varpi)^* \text{ and } \gamma = (\ln\frac{|h_4|^{3/2}}{|h_3|})^*, \quad (46)$$

for $\varpi = \ln|h_4^*/\sqrt{|h_3h_4|}|$; and considering $\Psi = \exp(\varpi)$ as a **generating function**, we simplify the system of nonlinear PDEs (45) in the form:

$$\psi^{\bullet\bullet} + \psi'' = 2\,{}_1\widehat{\Upsilon}, \quad (47)$$
$$(\varpi)^* h_4^* = 2h_3h_4\,{}_2\widehat{\Upsilon}, \quad (48)$$
$$\beta w_j - \alpha_j = 0, \quad (49)$$
$$n_k^{**} + \gamma n_k^* = 0. \quad (50)$$

Any solution of this system of nonlinear equations is a solution of (24) parameterized a respective quasi-stationary d-metric ansatz (32) for a canonically parameterized effective source (44), where ${}_1\widehat{\Upsilon}(x^k)$ and ${}_2\widehat{\Upsilon}(x^k, y^3)$ are **generating sources**.

Let us explain the general decoupling property of above systems of equations for quasi-stationary configurations. The Eq. (47) is a standard 2-d Poisson equation with source $2\,{}_1\widehat{\Upsilon}$. We note that instead of a nonholonomic 2+2 splitting, we can consider a 5-d spacetime of signature $(++++-)$ with nonholonomic 3+2 splitting. In such a case, following tedious computations which similar to above 4-d nonholonomic geometric constructions, we obtain (47) as a standard

2-d Poisson equation. We omit such considerations in this work. If we prescribe any data $(h_3, {}_2\widehat{\Upsilon})$, we can search h_4 as a solution of a second order (on derivative ∂_3) nonlinear PDE (48). We can consider an inverse problem with prescribed data $(h_4, {}_2\widehat{\Upsilon})$ when h_3 as a solution of a first order nonlinear PDE. Introducing a generating function Ψ, such equation can be integrated in explicit form (we shall prove in next subsection). Having defined in some general forms $h_3(x^k, y^3)$ and $h_4(x^k, y^3)$, we can compute respective coefficients α_i and β for (49), which are linear equations for $w_j(x^k, y^3)$. This mean that such equations and respective unknown functions decoupled from the rest of the system of nonlinear PDEs. Respectively, we have (50) as a decoupled system of PDEs which allows to find $n_k(x^k, y^3)$ (after two general integrations on y^3) for any $\gamma(x^k, y^3)$ determined by $h_3(x^k, y^3)$ and $h_4(x^k, y^3)$ as in above formulas. This way we proved that the (modified) Einstein equations written in certain canonical d-connection variables can be decoupled in general forms for quasi-stationary generic off-diagonal metric ansatz determined by d-metric ansatz (32) and respective generating sources $({}_1\widehat{\Upsilon}, {}_2\widehat{\Upsilon})$ (44).

3.2 General solutions for quasi-stationary configurations

3.2.1 Integrating decoupled nonholonomic gravitational field equations

Let us show how the system of nonlinear PDEs (47)–(50) can be integrated step by step in certain general forms:

The coefficients $g_i = e^{\psi(x^k)}$ for the h-components of d-metric (32) are defined by solutions of the corresponding 2-d Poisson equation (47) for any given source ${}_1\widehat{\Upsilon}(x^k)$.

Introducing explicit values of coefficients (46) in (48)–(50), we obtain such a nonlinear system:

$$\Psi^* h_4^* = 2 h_3 h_4 \, {}_2\widehat{\Upsilon} \Psi, \qquad (51)$$

$$\sqrt{|h_3 h_4|}\Psi = h_4^*, \qquad (52)$$

$$\Psi^* w_i - \partial_i \Psi = 0, \qquad (53)$$

$$n_i^{**} + \left(\ln \frac{|h_4|^{3/2}}{|h_3|}\right)^* n_i^* = 0. \qquad (54)$$

Prescribing a generating function, Ψ, and a generating source, $\widehat{\Upsilon}$, we can integrate recurrently these equations if $h_4^* \neq 0$ and ${}_2\widehat{\Upsilon} \neq 0$. If such conditions are not satisfied, there are necessary more special analytic methods considered, for instance, in Sect. 5.5 of [29] and references therein. We introduce

$$\rho^2 := -h_3 h_4 \qquad (55)$$

and re-write (51) and (52), respectively, in the form

$$\Psi^* h_4^* = -2\rho^2 \, {}_2\widehat{\Upsilon} \, \Psi \text{ and } h_4^* = \rho \, \Psi. \qquad (56)$$

Substituting the value of h_4^* from the second equation into the first equation, we express

$$\rho = -\Psi^*/2 \, {}_2\widehat{\Upsilon}. \qquad (57)$$

Then we introduce this ρ into the second equation in (56) and integrate on y^3,

$$h_4 = h_4^{[0]}(x^k) - \int dy^3 [\Psi^2]^*/4({}_2\widehat{\Upsilon}). \qquad (58)$$

Using this coefficient and formulas in (55) and (57), we compute

$$h_3 = -\frac{1}{4h_4}\left(\frac{\Psi^*}{{}_2\widehat{\Upsilon}}\right)^2 = -\left(\frac{\Psi^*}{2\,{}_2\widehat{\Upsilon}}\right)^2 \left(h_4^{[0]}(x^k) - \int dy^3 \frac{[\Psi^2]^*}{4\,{}_2\widehat{\Upsilon}}\right)^{-1}. \qquad (59)$$

At the next step, we define the N-connection coefficients. Using h_3 (59) and h_4 (58), we can integrate two times on y^3 and find general solutions of the equation (54):

$$\begin{aligned} n_k(x^k, y^3) &= {}_1n_k + {}_2n_k \int dy^3 \frac{h_3}{|h_4|^{3/2}} \\ &= {}_1n_k + {}_2n_k \int dy^3 \left(\frac{\Psi^*}{2\,{}_2\widehat{\Upsilon}}\right)^2 |h_4|^{-5/2} \\ &= {}_1n_k + {}_2n_k \int dy^3 \left(\frac{\Psi^*}{2\,{}_2\widehat{\Upsilon}}\right)^2 \\ &\quad \times \left| h_4^{[0]}(x^k) - \int dy^3 [\Psi^2]^*/4({}_2\widehat{\Upsilon}) \right|^{-5/2}. \end{aligned} \qquad (60)$$

In these formulas, there are considered two integration functions ${}_1n_k = {}_1n_k(x^i)$ and (we may re-define by introducing certain coefficients) ${}_2n_k = {}_2n_k(x^i)$. Finally, solving the algebraic system (53) for w_i, we find

$$w_i = \partial_i \, \Psi/(\Psi)^*. \qquad (61)$$

Putting together above values for the coefficients of the d-metric and N-connection, we define general solutions of the Einstein equations in canonical nonholonomic variables.

3.2.2 Quadratic elements for quasi-stationary off-diagonal solutions

Using N-adapted h-coefficients $g_i = e^{\psi(x^k)}$, determined by solutions of 2-d Poisson equations (47); v-coefficients h_3 (59) and h_4 (58); and N-connection coefficients w_i (61) and n_k (60), for a quasi-stationary d-metric (32), we construct a quasi-stationary nonlinear quadratic element,

$$ds^2 = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + \frac{[\Psi^*]^2}{4({}_2\widehat{\Upsilon})^2\{g_4^{[0]} - \int dy^3[\Psi^2]^*/4({}_2\widehat{\Upsilon})\}}$$

$$\times(dy^3+\frac{\partial_i \Psi}{\Psi^*}dx^i)^2+\{g_4^{[0]} - \int dy^3 \frac{[\Psi^2]^*}{4(\,_2\widehat{\Upsilon})}\}\{dt+[\,_1n_k+\,_2n_k \times \int dy^3 \frac{[(\Psi)^2]^*}{4(\,_2\widehat{\Upsilon})^2|g_4^{[0]}-\int dy^3[\Psi^2]^*/4(\,_2\widehat{\Upsilon})|^{5/2}}]dx^k\}. \quad (62)$$

With respect to coordinate dual frames, such a quadratic element can be represented equivalently in the form (15), $\hat{\mathbf{g}} = \underline{\widehat{g}}_{\alpha\beta}(u)du^\alpha \otimes du^\beta$, with

$$\underline{\widehat{g}}_{\alpha\beta} = \begin{bmatrix} g_1+(N_1^3)^2h_3+(N_1^4)^2h_4 & N_1^3N_2^3h_3+N_1^4N_2^4h_4 & N_1^3h_3 & N_1^4h_4 \\ N_2^3N_1^3h_3+N_2^4N_1^4h_4 & g_2+(N_2^3)^2h_3+(N_2^4)^2h_4 & N_2^3h_3 & N_2^4h_4 \\ N_1^3h_3 & N_2^3h_3 & h_3 & 0 \\ N_1^4h_4 & N_2^4h_4 & 0 & h_4 \end{bmatrix} = \begin{bmatrix} e^\psi+(w_1)^2h_3+(n_1)^2h_4 & w_1w_2h_3+n_1n_2h_4 & w_1h_3 & n_1h_4 \\ w_1w_2h_3+n_1n_2h_4 & e^\psi+(w_2)^2h_3+(n_2)^2h_4 & w_2h_3 & n_2h_4 \\ w_1h_3 & w_2h_3 & h_3 & 0 \\ n_1h_4 & n_2h_4 & 0 & h_4 \end{bmatrix}, \quad (63)$$

when the coefficients are respective functions as stated by formulas $g_i = e^{\psi(x^k)}$ and (59)–(60), and/or (62). Such a metric/solution is generic off-diagonal if there are some non-zero anholonomy coefficients (13). In general, it is also characterized by a nontrivial nonholonomically induced canonical d-torsion (37) if additional LC-conditions (38) are not satisfied.

Above generic off-diagonal quasi-stationary solutions are general in the sense that they are determined by a generating function $\Psi(x^k, y^3)$, two generating effective sources $\,_1\widehat{\Upsilon}(x^k)$ and $\,_2\widehat{\Upsilon}(x^k, y^3)$, and integration functions $\,_1n_k(x^k)$, $\,_2n_k(x^k)$ and $h_4^{[0]}(x^k)$. Such values have to be found from certain boundary/asymptotic conditions and other physical considerations (for instance, additional linear and nonlinear symmetry conditions, causality problems, generating quasi-periodic structure, avoiding singularities etc.). We shall analyze explicit physically important examples in Sect. 4. Such quasi-stationary solutions are different from the former ones considered in [1–4]. In the case of diagonal metric ansatz, for instance, for generating BHs (2) and LC-configurations, we obtain certain systems of nonlinear second order ODE when the solutions are determined by two integration constants. One of the constants is stated to be zero in order to get at asymptotic the Minkowski spacetime and the second integration constant is identified with the BH mass. For quasi-stationary solutions (62), equivalently (63), we have (in general) six independent coefficients for the off-diagonal metric (four of them transforms into N-connection coefficients in N-adapted frame) which describes more "rich" gravitational configurations.

3.2.3 Nonlinear symmetries of quasi-stationary off-diagonal solutions

The solutions (62) posses important nonlinear shell symmetries which allow to transform generating functions and effective sources into other types of generating functions and effective cosmological constants. We change the generating data, $(\Psi, \,_2\widehat{\Upsilon}) \leftrightarrow (\Phi, \,_2\Lambda = const \neq 0)$, following formulas

$$\frac{[\Psi^2]^*}{\,_2\widehat{\Upsilon}} = \frac{[\Phi^2]^*}{\,_2\Lambda}, \text{ which can be integrated as} \quad (64)$$

$$\Phi^2 = \,_2\Lambda \int dy^3 (\,_2\widehat{\Upsilon})^{-1}[\Psi^2]^* \text{ and/or}$$
$$\Psi^2 = (\,_2\Lambda)^{-1} \int dy^3 (\,_2\widehat{\Upsilon})[\Phi^2]^*. \quad (65)$$

Using (64), we can simplify the formula (58) and write $h_4 = h_4^{[0]} - \frac{\Phi^2}{4\,_2\Lambda}$. To express formulas (59) and (60) in terms of new generating data, we have to write $(\Psi)^*/\,_2\widehat{\Upsilon}$ in terms of such $(\Phi, \,_2\Lambda)$. We re-write (64) and the second equation in (65) as

$$\frac{\Psi(\,\Psi)^*}{\,_2\widehat{\Upsilon}} = \frac{(\Phi^2)^*}{2(\,_2\Lambda)} \text{ and}$$
$$\Psi = |\,_2\Lambda|^{-1/2}\sqrt{|\int dy^3 \,_2\widehat{\Upsilon}\,(\Phi^2)^*|}.$$

Introducing Ψ from the second equation in the first equation, we re-define Ψ^* in terms of generating data $(\,_2\widehat{\Upsilon}, \Phi, \,_2\Lambda)$, when

$$\frac{\Psi^*}{\,_2\widehat{\Upsilon}} = \frac{[\Phi^2]^*}{2\sqrt{|\,_2\Lambda \int dy^3(\,_2\widehat{\Upsilon})[\Phi^2]^*|}}. \quad (66)$$

So, the solutions of nonlinear equations (48) determined by (59) and (58) can be written in two equivalent functional forms,

$$h_3[\Psi] = -\frac{[\Psi^*]^2}{4(\,_2\widehat{\Upsilon})^2 h_4[\Psi]} = h_3[\Phi]$$

$$= -\frac{1}{h_4[\Phi]}\frac{\Phi^2[\Phi^*]^2}{|\ {}_2\Lambda \int dy^3\ {}_2\widehat{\Upsilon}[\Phi^2]^*|}, \text{ where}$$

$$h_4[\Psi] = h_4^{[0]} - \int dy^3 \frac{[\Psi^2]^*}{4\ {}_2\widehat{\Upsilon}} = h_4[\Phi] = g_4^{[0]} - \frac{\Phi^2}{4\ {}_2\Lambda}.$$

In similar forms, the N-connection coefficients can be redefined for data $(\Phi,\ {}_2\Lambda)$ using respectively the nonlinear transforms ,

$$w_i(x^{k_1}, y^3) = \frac{\partial_i \Psi}{\Psi^*} = \frac{\partial_i [\Psi^2]}{[\Psi^2]^*} = \frac{\partial_i\ \int dy^3\ {}_2\widehat{\Upsilon}\ [\Phi^2]^*}{{}_2\widehat{\Upsilon}\ [\Phi^2]^*}; \text{ and}$$

$$n_k(x^{k_1}, y^3) = {}_1n_k + {}_2n_k \int dy^3\ \frac{h_3[\Phi]}{|\ h_4[\Phi]|^{3/2}}$$
$$= {}_1n_k + {}_2n_k \int dy^3 \left(\frac{\Psi^*}{2\ {}_2\widehat{\Upsilon}}\right)^2$$
$$\times \left| h_4^{[0]}(x^k) - \int dy^3 \frac{[\Psi^2]^*}{4\ {}_2\widehat{\Upsilon}} \right|^{-5/2}$$
$$= {}_1n_k + {}_2n_k \int dy^3 \frac{\Phi^2[\Phi^*]^2}{|\ {}_2\Lambda \int dy^3\ {}_2\widehat{\Upsilon}[\Phi^2]^*|}$$
$$\times \left| h_4^{[0]} - \frac{\Phi^2}{4\ {}_2\Lambda} \right|^{-5/2}.$$

We conclude that any quasi-stationary solution (62) possess important nonlinear symmetries of type (64) and (65). As a result, the nonlinear quadratic element for quasi-stationary solutions (62) can be written in the form

$$ds^2 = g_{\alpha_s\beta_s}(x^k, y^3, \Phi, {}_2\Lambda) du^\alpha du^\beta = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2]$$
$$- \frac{\Phi^2[\Phi^*]^2}{|\ {}_2\Lambda \int dy^3\ {}_2\widehat{\Upsilon}[\Phi^2]^*|[h_4^{[0]} - \Phi^2/4\ {}_2\Lambda]}$$
$$\times \{dy^3 + \frac{\partial_i\ \int dy^3\ {}_2\widehat{\Upsilon}\ [\Phi^2]^*}{{}_2\widehat{\Upsilon}\ [(\,{}_2^{|}\Phi)^2]^*} dx^i\}^2$$
$$-\{h_4^{[0]} - \frac{\Phi^2}{4\ {}_2\Lambda}\}\{dt + [\ {}_1n_k + {}_2n_k \int dy^3$$
$$\times \frac{\Phi^2[\Phi^*]^2}{|\ {}_2\Lambda \int dy^3\ {}_2\widehat{\Upsilon}[\Phi^2]^*|[h_4^{[0]} - \Phi^2/4\ {}_2\Lambda]^{5/2}}]\}, \quad (67)$$

for indices: $i, j, k, \ldots = 1, 2; a, b, c, \ldots = 3, 4$; generating functions $\psi(x^k)$ and $\Phi(x^{k_1} y^3)$; generating sources ${}_1\widehat{\Upsilon}(x^k)$ and ${}_2\widehat{\Upsilon}(x^{k_1}, y^3)$; effective cosmological constants ${}_1\Lambda$ and ${}_2\Lambda$; and integration functions ${}_1n_k(x^j)$, ${}_2n_k(x^j)$ and $g_4^{[0]}(x^k)$.

3.2.4 Using d-metric coefficients as generating functions

Taking the partial derivative on y^3 of formula (58), we obtain $h_4^* = -[\Psi^2]^*/4\ {}_2\widehat{\Upsilon}$. If we prescribe h_4 and ${}_2\widehat{\Upsilon}$, we can compute up to certain integration functions a Ψ using $[\Psi^2]^* = \int dy^3\ {}_2\widehat{\Upsilon} h_4^*$. This allows us to consider a generating data $(h_4,\ {}_2\widehat{\Upsilon})$ and re-write the quasi-stationary d-metric (62) in equivalent form,

$$d\widehat{s}^2 = \widehat{g}_{\alpha\beta}(x^k, y^3; h_4,\ {}_2\widehat{\Upsilon}) du^\alpha du^\beta$$
$$= e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2]$$
$$- \frac{(h_4^*)^2}{|\int dy^3[\ {}_2\widehat{\Upsilon} h_4]^*|\ h_4} \left\{dy^3 + \frac{\partial_i\left[\int dy^3(\ {}_2\widehat{\Upsilon})\ h_4^*\right]}{{}_2\widehat{\Upsilon}\ h_4^*} dx^i\right\}^2$$
$$+ h_4\{dt + \left[\ {}_1n_k + {}_2n_k \int dy^3 \frac{(h_4^*)^2}{|\int dy^3\left[\ {}_2\widehat{\Upsilon} h_4\right]^*|\ (h_4)^{5/2}}\right] dx^k\}.$$

(68)

The nonlinear symmetries (64) and (65) allow to perform similar computations and express $\Phi^2 = -4\ {}_2\Lambda h_4$. We can eliminate Φ from the nonlinear quadratic element in (67) and obtain a solution of type (68) determined by the generating data $(h_4;\ {}_2\Lambda,\ {}_2\widehat{\Upsilon})$.

3.2.5 Gravitational polarizations

Nonholonomic frame and connection deformations and nonlinear symmetries allow to perform another types of geometric constructions:

- We can consider deformations of a **prime** d-metric (it can be an arbitrary one)

$$\mathring{\mathbf{g}} = [\mathring{g}_\alpha,\ \mathring{N}_i^a] \quad (69)$$

into a **target** d-metric $\mathbf{g}$, for instance, being a quasi-stationary solutions of type (32), when

$$\mathring{\mathbf{g}} \to \mathbf{g} = [g_\alpha = \eta_\alpha \mathring{g}_\alpha, N_i^a = \eta_i^a\ \mathring{N}_i^a], \quad (70)$$

for $\eta_\alpha(x^k, y^3)$ and $\eta_i^a(x^k, y^3)$ called gravitational polarization (η-polarization) functions.

- For a target metric $\mathbf{g}$ defined as a solution of type (62), equivalently (67), we can consider nonholonomic deformations with respective generating functions/sources and effective cosmological constants when

$$(\Psi,\ {}_2\widehat{\Upsilon}) \leftrightarrow (\mathbf{g},\ {}_2\widehat{\Upsilon}) \leftrightarrow (\eta_\alpha\ \mathring{g}_\alpha \sim (\zeta_\alpha(1 + \kappa\chi_\alpha)\mathring{g}_\alpha,\ {}_2\widehat{\Upsilon}) \leftrightarrow$$
$$(\Phi,\ {}_2\Lambda) \leftrightarrow (\mathbf{g},\ {}_2\Lambda) \leftrightarrow (\eta_\alpha\ \mathring{g}_\alpha \sim (\zeta_\alpha(1 + \kappa\chi_\alpha)\mathring{g}_\alpha,\ {}_2\Lambda),$$

where ${}_2\Lambda$ is an effective cosmological constant in the v-subspace, κ is a small parameter $0 \le \kappa < 1$, with some $\zeta_\alpha(x^k, y^3)$ and $\chi_\alpha(x^k, y^3)$.

Using above η- and/or χ-polarizations, the nonlinear symmetries (65) are written in the form:

$$\partial_3[\Psi^2] = -\int dy^3\ {}_2\widehat{\Upsilon}\partial_3 h_4 \simeq -\int dy^3\ {}_2\widehat{\Upsilon}\partial_3(\eta_4\ \mathring{g}_4)$$
$$\simeq -\int dy^3\ {}_2\widehat{\Upsilon}\partial_3[\zeta_4(1 + \kappa\ \chi_4)\ \mathring{g}_4],$$
$$\Phi^2 = -4\ {}_2\Lambda h_4 \simeq -4\ {}_2\Lambda\eta_4\mathring{g}_4 \simeq -4\ {}_2\Lambda\ \zeta_4(1 + \kappa\chi_4)\ \mathring{g}_4. \quad (71)$$

Off-diagonal η-transforms of type (70) can be parameterized for η-polarizations,

$$\psi \simeq \psi(\kappa; x^k), \eta_4 \simeq \eta_4(x^k, y^3), \tag{72}$$

when the quasi-stationary nonlinear quadratic element (68) can be written in the form

where ψ and $\eta_2 = \eta_1$ are such way chosen to be related to the solutions of the 2-d Poisson equation $\partial^2_{11}\psi + \partial^2_{22}\psi = 2\ {}_1\widehat{\Upsilon}(x^k)$, see (47). For other type signatures of d-metrics, it can be a 2-d wave equation with respective source.

Using (74), we can compute κ-parametric deformations to quasi-stationary d-metrics with χ-generating functions:

$$d\,\widehat{s}^2 = \widehat{g}_{\alpha\beta}(x^k, y^3; \psi, g_4; {}_2\widehat{\Upsilon})du^\alpha du^\beta = e^{\psi_0}(1+\kappa\ {}^\psi\chi)[(dx^1)^2+(dx^2)^2] - \Big\{\frac{4[\partial_3(|\zeta_4\mathring{g}_4|^{1/2})]^2}{\mathring{g}_3|\int dy^3\{\ {}_2\widehat{\Upsilon}\partial_3(\zeta_4\mathring{g}_4)\}|} - \kappa[\frac{\partial_3(\chi_4|\zeta_4\mathring{g}_4|^{1/2})}{4\partial_3(|\zeta_4\mathring{g}_4|^{1/2})}$$
$$-\frac{\int dy^3\{\ {}_2\widehat{\Upsilon}\partial_3[(\zeta_4\mathring{g}_4)\chi_4]\}}{\int dy^3\{\ {}_2\widehat{\Upsilon}\partial_3(\zeta_4\mathring{g}_4)\}}]\Big\}\mathring{g}_3\{dy^3 + \Big[\frac{\partial_i\ \int dy^3\ {}_2\widehat{\Upsilon}\ \partial_3\zeta_4}{(\mathring{N}^3_i)\ {}_2\widehat{\Upsilon}\partial_3\zeta_4} + \kappa(\frac{\partial_i[\int dy^3\ {}_2\widehat{\Upsilon}\ \partial_3(\zeta_4\chi_4)]}{\partial_i\ [\int dy^3\ {}_2\widehat{\Upsilon}\partial_3\zeta_4]} - \frac{\partial_3(\zeta_4\chi_4)}{\partial_3\zeta_4})\Big]\mathring{N}^3_i dx^i\}^2$$
$$+\zeta_4(1+\kappa\ \chi_4)\ \mathring{g}_4\{dt + \Big[(\mathring{N}^4_k)^{-1}\Big[\ {}_1n_k + 16\ {}_2n_k\Big[\int dy^3 \frac{\Big(\partial_3\Big[(\zeta_4\mathring{g}_4)^{-1/4}\Big]\Big)^2}{|\int dy^3\partial_3\Big[\ {}_2\widehat{\Upsilon}(\zeta_4\mathring{g}_4)\Big]|}\Big]$$
$$+\kappa\frac{16\ {}_2n_k\int dy^3\frac{(\partial_3[(\zeta_4\mathring{g}_4)^{-1/4}])^2}{|\int dy^3\partial_3[\ {}_2\widehat{\Upsilon}(\zeta_4\mathring{g}_4)]|}(\frac{\partial_3[(\zeta_4\mathring{g}_4)^{-1/4}\chi_4)]}{2\partial_3[(\zeta_4\mathring{g}_4)^{-1/4}]} + \frac{\int dy^3\partial_3[\ {}_2\widehat{\Upsilon}(\zeta_4\chi_4\mathring{g}_4)]}{\int dy^3\partial_3[\ {}_2\widehat{\Upsilon}(\zeta_4\mathring{g}_4)]})}{{}_1n_k + 16\ {}_2n_k[\int dy^3\frac{(\partial_3[(\zeta_4\mathring{g}_4)^{-1/4}])^2}{|\int dy^3\partial_3[\ {}_2\widehat{\Upsilon}(\zeta_4\mathring{g}_4)]|}]}]\mathring{N}^4_k dx^k\}^2. \tag{75}$$

$$\begin{aligned} d\widehat{s}^2 &= \widehat{g}_{\alpha\beta}(x^k, y^3; \mathring{g}_\alpha; \psi, \eta_4;\ {}_2\Lambda,\ {}_2\widehat{\Upsilon})du^\alpha du^\beta \\ &= e^\psi[(dx^1)^2+(dx^2)^2] \\ &\quad -\frac{[\partial_3(\eta_4\ \mathring{g}_4)]^2}{|\int dy^3\ {}_2\widehat{\Upsilon}\partial_3(\eta_4\ \mathring{g}_4)|\ \eta_4\mathring{g}_4} \\ &\quad \times\{dy^3 + \frac{\partial_i[\int dy^3\ {}_2\widehat{\Upsilon}\ \partial_3(\eta_4\mathring{g}_4)]}{{}_2\widehat{\Upsilon}\partial_3(\eta_4\mathring{g}_4)}dx^i\}^2 \\ &\quad +\eta_4\mathring{g}_4\{dt + [\ {}_1n_k +\ {}_2n_k\int dy^3 \\ &\quad \times\frac{[\partial_3(\eta_4\mathring{g}_4)]^2}{|\int dy^3\ {}_2\widehat{\Upsilon}\partial_3(\eta_4\mathring{g}_4)|\ (\eta_4\mathring{g}_4)^{5/2}}]dx^k\}^2. \end{aligned} \tag{73}$$

For $\Phi^2 = -4\ {}_2\Lambda h_4$, we can transform (67) in a variant of (73) with η-polarizations determined by the generating data $(h_4;\ {}_2\Lambda,\ {}_2\widehat{\Upsilon})$.

3.2.6 *Generating solutions with small parametric off-diagonal decompositions*

Considering κ-linear functions for η-polarizations in (73), we can define small nonholonomic deformations of a prime d-metric $\mathring{\mathbf{g}}$ into so-called κ-parametric solutions with ζ- and χ-coefficients,

$$\begin{aligned} &\psi \simeq \psi(x^k) \simeq \psi_0(x^k)(1+\kappa\ {}_\psi\chi(x^k)),\ \text{for} \\ &\eta_2 \simeq \eta_2(x^{k_1}) \simeq \zeta_2(x^k)(1+\kappa\chi_2(x^k)),\ \text{we can consider}\ \eta_2 = \eta_1; \\ &\eta_4 \simeq \eta_4(x^k, y^3) \simeq \zeta_4(x^k, y^3)(1+\kappa\chi_4(x^k, y^3)), \end{aligned} \tag{74}$$

Such off-diagonal parametric solutions allow to define, for instance, ellipsoidal deformations of BH metrics into BE ones.

3.3 Space and time duality of generic off-diagonal solutions

We can repeat all computations presented for quasi-stationary metrics (32) with nontrivial partial derivatives ∂_3 presented above in this section for locally anisotropic cosmological solutions (33) with nontrivial partial derivatives $\partial_4 = \partial_t$. In abstract symbolic form, we can formulate a **principle of space and time duality** of such different generic off-diagonal configurations:

$$\begin{aligned} &y^3 \longleftrightarrow y^4 = t, h_3(x^k, y^3) \longleftrightarrow \underline{h}_4(x^k, t), \\ &h_4(x^k, y^3) \longleftrightarrow \underline{h}_3(x^k, t), \\ &N^3_i = w_i(x^k, y^3) \longleftrightarrow N^4_i = \underline{n}_i(x^k, t), \\ &N^4_i = n_i(x^k, y^3) \longleftrightarrow N^3_i = \underline{w}_i(x^k, t). \end{aligned}$$

Such duality conditions are considered also for prime d-metrics and respective generating functions/sources and gravitational polarization functions, when

$$\Upsilon_3^3 = \Upsilon_4^4 = {}^{v}\Upsilon(x^k, y^3) = {}_2\widehat{\Upsilon} \longleftrightarrow \underline{\Upsilon}_4^4 = \underline{\Upsilon}_3^3 = {}^{v}\underline{\Upsilon}(x^k, t) = {}_2\underline{\widehat{\Upsilon}}, \text{ see (44)};$$

$$\begin{array}{ccc} (\Psi,\ {}_2\widehat{\Upsilon}) \leftrightarrow (\mathbf{g},\ {}_2\widehat{\Upsilon}) \leftrightarrow & & (\underline{\Psi},\ {}_2\underline{\widehat{\Upsilon}}) \leftrightarrow (\underline{\mathbf{g}},\ {}_2\underline{\widehat{\Upsilon}}) \leftrightarrow \\ (\eta_\alpha\ \mathring{g}_\alpha \sim (\zeta_\alpha(1+\kappa\chi_\alpha)\mathring{g}_\alpha,\ {}_2\widehat{\Upsilon}) \leftrightarrow & \Longleftrightarrow & (\underline{\eta}_\alpha\ \underline{\mathring{g}}_\alpha \sim (\underline{\zeta}_\alpha(1+\kappa\underline{\chi}_\alpha)\underline{\mathring{g}}_\alpha,\ {}_2\underline{\widehat{\Upsilon}}) \leftrightarrow \\ (\Phi,\ {}_2\Lambda) \leftrightarrow (\mathbf{g},\ {}_2\Lambda) \leftrightarrow & & (\underline{\Phi},\ {}_2\underline{\Lambda}) \leftrightarrow (\underline{\mathbf{g}},\ {}_2\underline{\Lambda}) \leftrightarrow \\ (\eta_\alpha\ \mathring{g}_\alpha \sim (\zeta_\alpha(1+\kappa\chi_\alpha)\mathring{g}_\alpha,\ {}_2\Lambda), & \Longleftrightarrow & (\underline{\eta}_\alpha\ \underline{\mathring{g}}_\alpha \sim (\underline{\zeta}_\alpha(1+\kappa\underline{\chi}_\alpha)\underline{\mathring{g}}_\alpha,\ {}_2\underline{\Lambda}), \end{array}$$

and

$$\begin{array}{lcl} \Psi^* h_4^* = 2h_3h_4\,{}_2\widehat{\Upsilon}\Psi, & \longleftrightarrow & \sqrt{|\underline{h}_3\underline{h}_4|}\underline{\Psi} = \underline{h}_3^\diamond, \\ \sqrt{|h_3h_4|}\Psi = h_4^*, & \longleftrightarrow & \underline{\Psi}^\diamond \underline{h}_3^\diamond = 2\underline{h}_3\underline{h}_4\,{}_2\underline{\widehat{\Upsilon}}\underline{\Psi}, \\ \Psi^* w_i - \partial_i\Psi = 0, & \longleftrightarrow & \underline{n}_i^{\diamond\diamond} + \left(\ln\frac{|\underline{h}_3|^{3/2}}{|\underline{h}_4|}\right)^\diamond \underline{n}_i^\diamond = 0, \\ n_i^{**} + \left(\ln\frac{|h_4|^{3/2}}{|h_3|}\right)^* n_i^* = 0 & \longleftrightarrow & \underline{\Psi}^\diamond \underline{w}_i - \partial_i\underline{\Psi} = 0, \end{array}$$

see (51)−(54). (76)

For locally anisotropic cosmological configurations, the nonlinear symmetries (64) and (65) are written in respective dual forms,

$$\frac{[\underline{\Psi}^2]^\diamond}{{}_2\underline{\widehat{\Upsilon}}} = \frac{[\underline{\Phi}^2]^\diamond}{{}_2\underline{\Lambda}}, \text{ which can be integrated as}$$

$$\underline{\Phi}^2 = {}_2\underline{\Lambda}\int dt({}_2\underline{\widehat{\Upsilon}})^{-1}[\underline{\Psi}^2]^\diamond \text{ and/or}$$

$$\underline{\Psi}^2 = ({}_2\underline{\Lambda})^{-1}\int dt({}_2\underline{\widehat{\Upsilon}})[\underline{\Phi}^2]^\diamond.$$

As a result, there are similar duality properties of solutions determined by quasi-stationary d-metrics (62), (67), (68), (73) and (75) and their locally anisotropic cosmological analogs. For instance, the d-metric (62) transforms into

$$\begin{aligned} d\underline{s}^2 &= e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] \\ &+\left\{g_3^{[0]} - \int dt \frac{[\underline{\Psi}^2]^\diamond}{4({}_2\underline{\widehat{\Upsilon}})}\right\}\{dy^3 \\ &+\left[{}_1n_k + {}_2n_k\int dt \frac{[(\underline{\Psi})^2]^\diamond}{4({}_2\underline{\widehat{\Upsilon}})^2|g_3^{[0]} - \int dt[\underline{\Psi}^2]^\diamond/4({}_2\underline{\widehat{\Upsilon}})|^{5/2}}\right]dx^k\} \\ &+\frac{[\underline{\Psi}^\diamond]^2}{4({}_2\underline{\widehat{\Upsilon}})^2\{g_3^{[0]} - \int dt[\underline{\Psi}^2]^\diamond/4({}_2\underline{\widehat{\Upsilon}})\}}\left(dt + \frac{\partial_i\underline{\Psi}}{\underline{\Psi}^\diamond}dx^i\right)^2. \end{aligned} \quad (77)$$

Other d-metrics can be also derived in abstract dual form changing corresponding indices 3 into 4, 4 into 3, underlying the respective generating functions/effective sources/gravitational polarizations for dependencies on (x^i, t) and changing v-partial derivatives, $* \to \diamond$, i.e. $\partial_3 \to \partial_4$.

3.4 A toy 2+2 model with effective momentum variables

The geometric method for decoupling and integrating gravitational field equations can be re-defined for 4-d nonholonomic "phase" space ${}^{\shortmid}V$ gravitational equations with conventional 2+2 splitting. Such constructions are considered in so-called Hamilton–Finsler–Cartan geometry [7–10,29,30] when nonholonomic co-fibered structures on ${}^{\shortmid}V$, dim ${}^{\shortmid}V =$ 4, and such a pseudo-Riemanian manifold is enabled with a metric ${}^{\shortmid}\mathbf{g}$ with signature $(+-+-)$ and the geometric objects are adapted to certain "dual" fibration structures. A similar geometric formulation is possible for cotangent bundles $T^*V^{(2)} = hT^*V^{(2)} \oplus v\,T^*V^{(2)}$, dim $V^{(2)} = 2$, and the metrics on $V^{(2)}$ are of signature $(+-)$. In relativistic forms, such constructions and respective classes of off-diagonal solutions are performed on T^*V, with dim $T^*V = 8$. In this paper, we outline only the main ideas and methods for 4-d total phase spaces when the nonholonomic geometry with conventional 2+2 splitting is similar and dual as for $TV^{(2)}$ and $T^*V^{(2)}$. Let us explain the notations for this subsection, when the left abstract label "${}^{\shortmid}$" states that additionally to (9) it is considered a dual nonholonomic distribution

$${}^{\shortmid}\mathbf{N}:\ T^*V = hV \oplus cV, \quad (78)$$

where cV is stated as a conventional 2-d subspace which is dual to vV. Locally, we can consider on cV certain systems of coordinates $p_b(x^i, y^a)$. For the models of Lagrange–Hamilton mechanics on TV and, respectively, T^*V, we can consider $y^a = v^a$ and p_b as certain velocity and momentum like variables related via Legendre transforms etc. But for the purposes of this work, it is enough to see ${}^{\shortmid}p = p = (p_3, p_4 = E)$ as a part of ${}^{\shortmid}u = (x, p) = \{x^i, p_a\}$ defined as local coordinates on ${}^{\shortmid}V$.

In local dual coordinate form, a N-connection (78) can be written as ${}^{\shortmid}\mathbf{N} = {}^{\shortmid}N_{ia}(x,p)dx^i \otimes \partial/\partial p_a$, when the N-elongated (equivalently, N-adapted) local bases (partial derivatives), ${}^{\shortmid}\mathbf{e}_\nu$, and co-bases (differentials), ${}^{\shortmid}\mathbf{e}^\mu$, (compare, respectively, to (10) and (11)) are defined

$$\begin{aligned} {}^{\shortmid}\mathbf{e}_\nu &= ({}^{\shortmid}\mathbf{e}_i, {}^{\shortmid}e^a) = ({}^{\shortmid}\mathbf{e}_i = \partial/\partial x^i - {}^{\shortmid}N_{ib}({}^{\shortmid}u)\partial/\partial p_b, \\ {}^{\shortmid}e^a &= {}^{\shortmid}\partial^a = \partial/\partial p_a), \text{ and} \end{aligned} \quad (79)$$

$$\begin{aligned} {}^{\shortmid}\mathbf{e}^\mu &= (e^i, {}^{\shortmid}\mathbf{e}^a) = (e^i = dx^i, \\ {}^{\shortmid}\mathbf{e}_a &= dp_a + {}^{\shortmid}N_{ia}({}^{\shortmid}u)dx^i), \end{aligned} \quad (80)$$

Any phase space metric ${}^{\shortmid}\mathbf{g}$ on ${}^{\shortmid}V$ can be represented equivalently as a d-metric ${}^{\shortmid}\mathbf{g} = (h\,{}^{\shortmid}g, c\,{}^{\shortmid}g)$, when

$$\begin{aligned} {}^{\shortmid}\mathbf{g} &= {}^{\shortmid}g_{ij}(x,p)\, e^i \otimes e^j + {}^{\shortmid}g^{ab}(x,p)\ {}^{\shortmid}\mathbf{e}_a \otimes {}^{\shortmid}\mathbf{e}_b, \\ &= {}^{\shortmid}\underline{g}_{\alpha\beta}({}^{\shortmid}u)d\,{}^{\shortmid}u^\alpha \otimes d\,{}^{\shortmid}u^\beta, \end{aligned} \quad (81)$$

where $h\,{}^{\shortmid}g = \{{}^{\shortmid}g_{ij}\}$ and $c\,{}^{\shortmid}g = \{{}^{\shortmid}g^{ab}\}$.

In abstract geometric form, we can define on ${}^{\shortmid}V$ a **d-connection** structure ${}^{\shortmid}\mathbf{D} = (h\,{}^{\shortmid}D, c\,{}^{\shortmid}D)$ is a linear connection preserving under parallelism the N-connection splitting

(78),

$$ {}^{\shortmid}\mathbf{D} = \{ {}^{\shortmid}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta} = ({}^{\shortmid}L^{i}_{\ jk}, {}^{\shortmid}\acute{L}_{a\ k}^{\ b}; {}^{\shortmid}\acute{C}_{j}^{i\ c}, {}^{\shortmid}C_{a}^{\ bc})\}, \text{ where} $$
$$ h\,{}^{\shortmid}D = ({}^{\shortmid}L^{i}_{\ jk}, {}^{\shortmid}\acute{L}_{a\ k}^{\ b}) \text{ and } v\,{}^{\shortmid}D = ({}^{\shortmid}\acute{C}_{j}^{i\ c}, {}^{\shortmid}C_{a}^{\ bc}). \quad (82) $$

So, the c-indices in such N-adapted formulas are inverse to v-indices in N-adapted formulas for V. Using d-operator ${}^{\shortmid}\mathbf{D}$, we can define respective fundamental geometric d-objects as we considered on V, but with abstract symbolic definitions on ${}^{\shortmid}V$:

$$ {}^{\shortmid}\mathcal{T}({}^{\shortmid}\mathbf{X}, {}^{\shortmid}\mathbf{Y}) := {}^{\shortmid}\mathbf{D}_{{}^{\shortmid}\mathbf{X}} {}^{\shortmid}\mathbf{Y} - {}^{\shortmid}\mathbf{D}_{{}^{\shortmid}\mathbf{Y}} {}^{\shortmid}\mathbf{X} - [{}^{\shortmid}\mathbf{X}, {}^{\shortmid}\mathbf{Y}], $$
torsion d-tensor, d-torsion;
$$ {}^{\shortmid}\mathcal{R}({}^{\shortmid}\mathbf{X}, {}^{\shortmid}\mathbf{Y}) := {}^{\shortmid}\mathbf{D}_{{}^{\shortmid}\mathbf{X}} {}^{\shortmid}\mathbf{D}_{{}^{\shortmid}\mathbf{Y}} - {}^{\shortmid}\mathbf{D}_{{}^{\shortmid}\mathbf{Y}} {}^{\shortmid}\mathbf{D}_{{}^{\shortmid}\mathbf{X}} - {}^{\shortmid}\mathbf{D}_{[{}^{\shortmid}\mathbf{X}, {}^{\shortmid}\mathbf{Y}]}, $$
curvature d-tensor, d-curvature;
$$ {}^{\shortmid}\mathcal{Q}({}^{\shortmid}\mathbf{X}) := {}^{\shortmid}\mathbf{D}_{{}^{\shortmid}\mathbf{X}} {}^{\shortmid}\mathbf{g}, $$
nonmetricity d-fields, d-nonmetricity,

where d-vectors ${}^{\shortmid}\mathbf{X}$ and ${}^{\shortmid}\mathbf{Y}$, and their duals as 1-forms, can be decomposed respectively to N-linear frames (79) and (80).

Using geometric objects and formulas (78)–(82), we can re-define all geometric constructions and formulas for nonholonomic manifolds $\mathbf{V}$ and tangent bundles $T\mathbf{V}$ on ${}^{\shortmid}\mathbf{V}$ and $T\ {}^{\shortmid}\mathbf{V}$. In general abstract and N-adapted form, corresponding geometric, gravity and geometric flow models for (non) associative/commutative phase spaces are studies in [7–10,29,30], where various applications in MGTs and geometric and quantum information flow theories are elaborated.

The nonholonomic Einstein equations on 4-d phase spaces can be defined and proven using sympolic re-definitions of variables and geometric d-objects in (24) and (25),

$$ {}^{\shortmid}\widehat{\mathbf{R}}^{\alpha}_{\ \beta} = {}^{\shortmid}\mathbf{\Upsilon}^{\alpha}_{\ \beta}, \quad (83) $$
$$ {}^{\shortmid}\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta} = 0, \quad (84) $$

with effective generating sources ${}^{\shortmid}\widehat{\mathbf{\Upsilon}}^{\alpha}_{\ \beta} = [{}^{\shortmid}_{1}\Upsilon\delta^{i}_{\ j}, {}^{\shortmid}_{2}\Upsilon\delta^{a}_{\ b}]$.

The equations (83) and (84) can be solved by generic iff-diagonal ansatz with a Killing vector. For instance, the phase space analog of a quasi-stationary d-metric of type (14) is parameterized

$$ {}^{\shortmid}\hat{\mathbf{g}} = g_{i}(x^{k})dx^{i} \otimes dx^{i} + {}^{\shortmid}h^{3}(x^{k}, p_{3})\, {}^{\shortmid}\mathbf{e}_{3} \otimes {}^{\shortmid}\mathbf{e}_{3} $$
$$ + {}^{\shortmid}h^{4}(x^{k}, p_{3})\, {}^{\shortmid}\mathbf{e}_{4} \otimes {}^{\shortmid}\mathbf{e}_{4}, $$
$$ {}^{\shortmid}\mathbf{e}_{3} = dp_{3} + {}^{\shortmid}w_{i}(x^{k}, p_{3})dx^{i}, \ {}^{\shortmid} $$
$$ \mathbf{e}_{4} = dE + {}^{\shortmid}n_{i}(x^{k}, p_{3})dx^{i}, \quad (85) $$

with Killing symmetry on the time like coordinate ${}^{\shortmid}\partial^{4} = \partial^{E}$. For such ansatz, the N-connection coefficients ${}^{\shortmid}\widehat{N}^{3}_{i} = {}^{\shortmid}w_{i}(x^{k}, p_{3})$ and ${}^{\shortmid}\widehat{N}^{4}_{i} = {}^{\shortmid}n_{i}(x^{k}, p_{3})$ and N-adapted coefficients of d-metric ${}^{\shortmid}\widehat{\mathbf{g}}_{\alpha\beta} = [\widehat{g}_{ij}(x^{\kappa}), {}^{\shortmid}\widehat{g}^{ab}(x^{\kappa}, p_{3})]$ are functions of necessary smooth class.

The phase space analog of locally anisotropic cosmological d-metrics (33) is stated by formulas,

$$ {}^{\shortmid}\underline{\mathbf{g}} = g_{i}(x^{k})dx^{i} \otimes dx^{i} + {}^{\shortmid}\underline{h}^{3}(x^{k}, E)\, {}^{\shortmid}\underline{\mathbf{e}}_{3} \otimes {}^{\shortmid}\underline{\mathbf{e}}_{3} $$
$$ + {}^{\shortmid}\underline{h}^{4}(x^{k}, E)\, {}^{\shortmid}\underline{\mathbf{e}}_{4} \otimes {}^{\shortmid}\underline{\mathbf{e}}_{4}, $$
$$ {}^{\shortmid}\underline{\mathbf{e}}_{3} = dp_{3} + {}^{\shortmid}\underline{n}_{i}(x^{k}, E)dx^{i}, \ {}^{\shortmid} $$
$$ \underline{\mathbf{e}}_{4} = dE + {}^{\shortmid}\underline{w}_{i}(x^{k}, E)dx^{i}, \quad (86) $$

with Killing symmetry on the time like coordinate ${}^{\shortmid}\partial^{3}$. For such a d-metric, the N-connection coefficients ${}^{\shortmid}\underline{N}_{i3} = {}^{\shortmid}\underline{n}_{i}(x^{k}, E)$ and ${}^{\shortmid}\underline{N}_{i4} = {}^{\shortmid}\underline{w}_{i}(x^{k}, E)$ and the N-adapted coefficients of d-metrics are of type ${}^{\shortmid}\underline{\mathbf{g}}_{\alpha\beta} = [g_{ij}(x^{\kappa}), {}^{\shortmid}\underline{g}^{ab}(x^{\kappa}, E)]$.

In this subsection, we consider toy 2+2 phase space model with momentum like variables in order to show how the AFCDM can be extended on such spaces when in abstract geometric form we can generate exact and parametric solutions. For instance, the quasi-stationary solution (62) for a nonholonomic phase space can be represented in a form (85),

$$ d\,{}^{\shortmid}s^{2} = e^{\psi(x^{k})}[(dx^{1})^{2} + (dx^{2})^{2}] $$
$$ + \frac{[{}^{\shortmid}\partial^{3}\, {}^{\shortmid}\Psi]^{2}}{4({}^{\shortmid}_{2}\widehat{\Upsilon})^{2}\{ {}^{\shortmid}g^{4}_{[0]} - \int dp_{3}\, {}^{\shortmid}\partial^{3}[\Psi^{2}]/4({}_{2}\widehat{\Upsilon})\}} $$
$$ \times \left(dy^{3} + \frac{\partial_{i}\, {}^{\shortmid}\Psi}{ {}^{\shortmid}\partial^{3}\, {}^{\shortmid}\Psi} dx^{i} \right)^{2} $$
$$ + \{ {}^{\shortmid}g^{4}_{[0]} - \int dp_{3} \frac{ {}^{\shortmid}\partial^{3}[{}^{\shortmid}\Psi^{2}]}{4({}^{\shortmid}_{2}\widehat{\Upsilon})} \}\{dE + [{}_{1}n_{k} + {}_{2}n_{k} $$
$$ \times \int dp_{3} \frac{ {}^{\shortmid}\partial^{3}[({}^{\shortmid}\Psi)^{2}]}{4({}^{\shortmid}_{2}\widehat{\Upsilon})^{2}| {}^{\shortmid}g^{4}_{[0]} - \int dp_{3}\, {}^{\shortmid}\partial^{3}[{}^{\shortmid}\Psi^{2}]/4({}^{\shortmid}_{2}\widehat{\Upsilon})|^{5/2}}]dx^{k}\}^{2}. $$
$$ (87) $$

Such a d-metric is a phase space quasi-stationary one if we perform an pseudo-Euclidean rotation from 4-metrics of signature (+-+-) to (+++-).

For toy 2+2 phase space models, we can generate so-called "rainbow" d-metrics, see reviews in which are of type (86) and can be also of cosmological type if we fix, for instance, $x^{2} = t$. In abstract symbolic form, we can formulate a **principle of phase space, time, momentum and energy duality** of such different generic off-diagonal configurations:

$$ p_{3} \longleftrightarrow p_{4} = E, \ {}^{\shortmid}h^{3}(x^{k}, p_{3}) \longleftrightarrow {}^{\shortmid}\underline{h}^{4}(x^{k}, E), $$
$$ {}^{\shortmid}h^{4}(x^{k}, p_{3}) \longleftrightarrow {}^{\shortmid}\underline{h}^{3}(x^{k}, E), $$
$$ {}^{\shortmid}N_{i3} = {}^{\shortmid}w_{i}(x^{k}, p_{3}) \longleftrightarrow {}^{\shortmid}N_{i4} = {}^{\shortmid}\underline{n}_{i}(x^{k}, E), $$
$$ {}^{\shortmid}N_{i4} = {}^{\shortmid}n_{i}(x^{k}, p_{3}) \longleftrightarrow {}^{\shortmid}N_{i3} = {}^{\shortmid}\underline{w}_{i}(x^{k}, E). $$

We can also formulate duality conditions for phase prime d-metrics and respective generating functions/sources and gravitational polarization functions, when

$$
\begin{array}{c}
{}^{\shortmid}\Upsilon^3_3 = {}^{\shortmid}\Upsilon^4_4 = {}^{c}_{\shortmid}\Upsilon(x^k, p_3) = {}^{\shortmid}_{2}\widehat{\Upsilon} \longleftrightarrow {}^{\shortmid}\underline{\Upsilon}^4_4 = {}^{\shortmid}\underline{\Upsilon}^3_3 = {}^{c}_{\shortmid}\underline{\Upsilon}(x^k, E) = {}^{\shortmid}_{2}\underline{\widehat{\Upsilon}}, \text{ see (44)};\\
\begin{array}{c}({}^{\shortmid}\Psi, {}^{\shortmid}_{2}\widehat{\Upsilon}) \leftrightarrow ({}^{\shortmid}\mathbf{g}, {}^{\shortmid}_{2}\widehat{\Upsilon}) \leftrightarrow \\ ({}^{\shortmid}\eta_\alpha\ {}^{\shortmid}\mathring{g}_\alpha \sim ({}^{\shortmid}\zeta_\alpha(1 + {}^{\shortmid}\kappa\, {}^{\shortmid}\chi_\alpha)\mathring{g}_\alpha, {}^{\shortmid}_{2}\widehat{\Upsilon}) \leftrightarrow\end{array} \Longleftrightarrow \begin{array}{c}({}^{\shortmid}\underline{\Psi}, {}^{\shortmid}_{2}\underline{\widehat{\Upsilon}}) \leftrightarrow ({}^{\shortmid}\underline{\mathbf{g}}, {}^{\shortmid}_{2}\underline{\widehat{\Upsilon}}) \leftrightarrow \\ ({}^{\shortmid}\underline{\eta}_\alpha\ {}^{\shortmid}\underline{\mathring{g}}_\alpha \sim ({}^{\shortmid}\underline{\zeta}_\alpha(1 + {}^{\shortmid}\kappa\underline{\chi}_\alpha)\, {}^{\shortmid}\underline{\mathring{g}}_\alpha, {}_{2}\underline{\widehat{\Upsilon}}) \leftrightarrow\end{array}\\
\begin{array}{c}({}^{\shortmid}\Phi, {}^{\shortmid}_{2}\Lambda) \leftrightarrow (\mathbf{g}, {}^{\shortmid}_{2}\Lambda) \leftrightarrow \\ ({}^{\shortmid}\eta_\alpha\ {}^{\shortmid}\mathring{g}_\alpha \sim ({}^{\shortmid}\zeta_\alpha(1 + {}^{\shortmid}\kappa\, {}^{\shortmid}\chi_\alpha)\, {}^{\shortmid}\mathring{g}_\alpha, {}^{\shortmid}_{2}\Lambda),\end{array} \Longleftrightarrow \begin{array}{c}(\underline{\Phi}, {}^{\shortmid}_{2}\underline{\Lambda}) \leftrightarrow (\underline{\mathbf{g}}, {}^{\shortmid}_{2}\underline{\Lambda}) \leftrightarrow \\ ({}^{\shortmid}\underline{\eta}_\alpha\ {}^{\shortmid}\underline{\mathring{g}}_\alpha \sim ({}^{\shortmid}\underline{\zeta}_\alpha(1 + {}^{\shortmid}\kappa\, {}^{\shortmid}\underline{\chi}_\alpha)\, {}^{\shortmid}\underline{\mathring{g}}_\alpha, {}_{2}\underline{\Lambda}).\end{array}
\end{array}
$$

For locally anisotropic E-depending phase configurations, the nonlinear symmetries (64) and (65) are written in respective dual forms,

$$
\frac{{}^{\shortmid}\partial^4[{}^{\shortmid}\underline{\Psi}^2]}{{}^{\shortmid}_{2}\underline{\widehat{\Upsilon}}} = \frac{{}^{\shortmid}\partial^4[{}^{\shortmid}\underline{\Phi}^2]}{{}^{\shortmid}_{2}\underline{\Lambda}}, \text{ which can be integrated as}
$$

$$
\begin{aligned}
{}^{\shortmid}\underline{\Phi}^2 &= {}_{2}\underline{\Lambda} \int dE ({}^{\shortmid}_{2}\underline{\widehat{\Upsilon}})^{-1}\, {}^{\shortmid}\partial^4[{}^{\shortmid}\underline{\Psi}^2] \text{ and/or}\\
{}^{\shortmid}\underline{\Psi}^2 &= ({}^{\shortmid}_{2}\underline{\Lambda})^{-1} \int dE ({}^{\shortmid}_{2}\underline{\widehat{\Upsilon}})\, {}^{\shortmid}\partial^4[{}^{\shortmid}\underline{\Phi}^2].
\end{aligned} \tag{88}
$$

In a similar form, we can derive respective phase space formulas with labels " ${}^{\shortmid}$" for solutions determined by quasi-stationary d-metrics (62), (67), (68), (73) and (75) and their locally anisotropic cosmological analogs. For instance, the phase space analog of the locally anisotropic d-metric (77) transforms into

$$
\begin{aligned}
d\, {}^{\shortmid}\underline{s}^2 = {} & e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2]\\
& + \left\{ {}^{\shortmid}g^3_{[0]} - \int dE \frac{{}^{\shortmid}\partial^4[{}^{\shortmid}\underline{\Psi}^2]}{4({}^{\shortmid}_{2}\underline{\widehat{\Upsilon}})} \right\}\{dp_3 + \Big[{}_1n_k + {}_2n_k\\
& \times \int dE \frac{{}^{\shortmid}\partial^4[({}^{\shortmid}\underline{\Psi})^2]}{4({}^{\shortmid}_{2}\underline{\widehat{\Upsilon}})^2 |g^3_{[0]} - \int dE\, {}^{\shortmid}\partial^4[{}^{\shortmid}\underline{\Psi}^2]/4({}^{\shortmid}_{2}\underline{\widehat{\Upsilon}})|^{5/2}}]dx^k\}\\
& + \frac{\Big[{}^{\shortmid}\partial^4\, {}^{\shortmid}\underline{\Psi}\Big]^2}{4({}^{\shortmid}_{2}\underline{\widehat{\Upsilon}})^2 \{ {}^{\shortmid}g^3_{[0]} - \int dE\, {}^{\shortmid}\partial^4[{}^{\shortmid}\underline{\Psi}^2]/4({}^{\shortmid}_{2}\underline{\widehat{\Upsilon}})\}}\\
& \times \left(dE + \frac{\partial_i\, {}^{\shortmid}\underline{\Psi}}{{}^{\shortmid}\partial^4\, {}^{\shortmid}\underline{\Psi}} dx^i \right)^2.
\end{aligned} \tag{89}
$$

Other classes of phase space d-metrics with different types of generating functions, generating sources and effective cosmological constants can be also derived in abstract dual form changing corresponding indices 3 into 4, 4 into 3, underlying the respective generating functions/effective sources /gravitational polarizations for dependencies on (x^i, E) and changing v-partial derivatives ${}^{\shortmid}\partial^3 \to {}^{\shortmid}\partial^4$.

3.5 Extracting LC-configurations

The generic off-diagonal solutions constructed in previous subsections are constructed for a canonical d-connection $\widehat{\mathbf{D}}$ or corresponding phase space variants ${}^{\shortmid}\widehat{\mathbf{D}}$. In general, such solutions are characterized by nonholonomically induced d-torsion coefficients $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$ (27) completely defined by the N-connection and d-metric structures. We can extract zero torsion LC-configurations if we impose additionally the conditions (38). By straightforward computations for quasi-stationary configurations, we can verify that all d-torsion coefficients vanish if the coefficients of N-adapted frames and v-components of d-metrics are subjected to respective conditions,

$$
\begin{aligned}
& w_i^* = \mathbf{e}_i \ln \sqrt{|\, h_3|}, \mathbf{e}_i \ln \sqrt{|\, h_4|} = 0,\\
& \partial_i w_j = \partial_j w_i \text{ and } n_i^* = 0;\\
& n_k(x^i) = 0 \text{ and } \partial_i n_j(x^k) = \partial_j n_i(x^k).
\end{aligned} \tag{90}
$$

The solutions for necessary type of w- and n-functions depend on the class of vacuum or non-vacuum metrics which we attempt to construct. Such classes of generating functions and generating sources and N-connection coefficients can be constructed following such steps of finding solutions (90):

Prescribing a generating function $\Psi = \check{\Psi}(x^{i_1}, y^3)$, for which $[\partial_i({}_2\check{\Psi})]^* = \partial_i({}_2\check{\Psi})^*$,we solve the equations for w_j from (90) in explicit form if ${}_2\widehat{\Upsilon} = const$, or if such an effective source can be expressed as a functional ${}_2\widehat{\Upsilon}(x^i, y^3) = {}_2\widehat{\Upsilon}[{}_2\check{\Psi}]$. Then, the third conditions $\partial_i w_j = \partial_j w_i$, are solved by any generating function $\check{A} = \check{A}(x^k, y^3)$ for which

$$
w_i = \check{w}_i = \partial_i\, \check{\Psi}/(\check{\Psi})^* = \partial_i \check{A}.
$$

The equations for n-functions in (90) are solved for any $n_i = \partial_i[{}^2n(x^k)]$.

Putting together above formulas for respective classes of generating functions in , we construct a nonlinear quadratic element for quasi-stationary solutions with zero canonical d-torsions, (62),

$$
\begin{aligned}
d\check{s}^2 = {} & \check{g}_{\alpha\beta}(x^k, y^3) du^\alpha du^\beta\\
= {} & e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2]\\
& + \frac{[\check{\Psi}^*]^2}{4({}_2\widehat{\Upsilon}[\check{\Psi}])^2\{h_4^{[0]} - \int dy^3 [\check{\Psi}]^*/4\, {}_2\widehat{\Upsilon}[\check{\Psi}]\}}\\
& \{dy^3 + [\partial_i(\check{A})]dx^i\}^2\\
& + \left\{ h_4^{[0]} - \int dy^3 \frac{[\check{\Psi}^2]^*}{4({}_2\widehat{\Upsilon}[\check{\Psi}])} \right\} \{dt + \partial_i[{}^2n(x^k)]dx^i\}^2.
\end{aligned} \tag{91}
$$

In a similar form, we can extract LC-configurations for all classes of quasi-stationary, locally anisotropic cosmologic and toy phase space solutions considered in this section. This is always possible for generically off-diagonal metrics with nontrivial canonical d-torsion if we impose respective conditions for generating data of type $(\check{\Psi}(x^{i_1}, y^3),\ {}_2\widehat{\Upsilon}[{}_2\check{\Psi}], \check{A})$.

4 Examples of off-diagonal quasi-stationary or cosmological solutions

We show how choosing respective classes of generating and integration functions we can construct certain physically important examples of quasi-stationary generic off-diagonal solutions. They describe BHs, nonholonomic cylindrical systems, WHs, BT and BE configurations in MGTs modelled on 4-d Lorentz manifolds by effective sources determined by nonholonomic distortions of d-connections and off-diagonal deformations of metrics. In some dual on-the-time coordinates forms, the AFCDM allows us to construct off-diagonal cosmological solutions describing nonholonomic cosmological solitonic evolution scenarios and spheroid deformations involving 2-d vertices. Constraining the generating and integration functions to some subclasses of coefficients resulting in zero nonholonomic torsions, such generic off-diagonal solutions can be generated in the framework of GR theory.

4.1 New Kerr de Sitter solutions and nonholonomic deformations to spheroidal configurations

Nonholonomic off-diagonal deformations of the Kerr and Schwarzschild – (a) de Sitter, K(a)dS, and other types BH metrics determined by (effective) (non) associative/commutative sources in string theory, MGTs and geometric information flow were studied in a series of works [7,9,10,14,16,21–23,25,28,30,65]. For spherical rotating configurations of KdS, such metrics can be described by various families of rotating diagonal metrics involving, or not warping effects of curvature, see details in [66]. In this subsection, we analyse how rotating BHs can be nonholonomically deformed to parametric quasi-stationary d-metrics of type (75). There are also computed in explicit form spheroidal rotoid deformations.

4.1.1 Prime d-metrics as off-diagonal new KdS solutions

We consider a prime quadratic line element of type (69) for spherical coordinates parameterized in the form $x^1 = r, x^2 = \varphi, y^3 = \theta, y^4 = t$, when

$$d\breve{s}^2 = \breve{g}_\alpha(r,\varphi,\theta)(\breve{\mathbf{e}}^\alpha)^2, \quad (92)$$

with such nontrivial coefficients of the d-metric and N-connection:

$$\breve{g}_1 = \frac{\breve{\rho}^2}{\triangle_\Lambda}, \breve{g}_2 = \frac{\sin^2\theta}{\breve{\rho}^2}\left[\Sigma_\Lambda - \frac{(r^2+a^2-\triangle_\Lambda)^2}{a^2\sin^2\theta - \triangle_\Lambda}\right],$$

$$\breve{g}_3 = \breve{\rho}^2, \breve{g}_4 = \frac{a^2\sin^2\theta - \triangle_\Lambda}{\breve{\rho}^2}, \text{ and}$$

$$\breve{N}_2^4 = \breve{n}_2 = -a\sin\theta\frac{r^2+a^2-\triangle_\Lambda}{a^2\sin^2\theta - \triangle_\Lambda}.$$

If the functions and parameters are chosen in the form

$$\Sigma_\Lambda = (r^2+a^2)^2 - \triangle_\Lambda a^2\sin^2\theta,$$
$$\triangle_\Lambda = r^2 - 2Mr + a^2 - \frac{\Lambda_0}{3}r^4,$$
$$\breve{\rho}^2 = r^2 + a^2\cos^2\theta, \text{ for constants } a = J/M = const,$$

where J is the angular momentum and M is the total mass of the system, and the cosmological constant $\Lambda_0 > 0$, the d-metric (92) define a new KdS solution reported in [66] (see also relevant details and references in that work). We note that in this paper we use a different system of notations stated for another signature of the metrics. It is different from the standard KdS metrics, called also Λ-vacuum solutions, because the scalar curvature

$$R(r,\theta) = 4\widetilde{\Lambda}(r,\theta) = 4\Lambda_0\frac{r^2}{\breve{\rho}^2} \neq 4\Lambda_0.$$

Such a new KdS solution posses a warped effect when the curvature is warped every were excepting the equatorial plane. The effective polarization of the cosmological constant also shows a rotational effect on the vacuum energy. Mentioned effect disappear for $r \gg a$. A d-metric (92) can be considered as a rotating version of the Schwarzschild de Sitter metric and represents a new solution describing the exterior of a BH with cosmological constant. It contains certain bonds for $M(a,\Lambda_0)$ for existence of a BH solution. In explicit form, the respective upper, $M_{\max} := M_+$ and lower, $M_{\min} := M$, bounds, when

$$18\Lambda_0 M_\pm^2 = 1 + 12\Lambda_0 a^2 \pm (1-4\Lambda_0 a^2)^{3/2}. \quad (93)$$

Such a d-metric defines a LC-configuration for the standard Einstein equations (1) with fluid type energy momentum tensor (4), when

$$\breve{T}_{\alpha\beta}(r,\theta) = diag[p_r, p_\varphi = p_\theta, p_\theta = \rho - 2\Lambda_0 r^2/\mathring{\rho}^2,$$
$$\rho = -p_r = \widetilde{\Lambda}^2/\Lambda_0]. \quad (94)$$

So, we have two possibilities to interpret that such primary d-metrics: the first one is to consider that they are defined as solutions of some vacuum locally anisotropic polarizations on (r,θ) of the cosmological constant, $\Lambda_0 \to \widetilde{\Lambda}(r,\theta)$, or to consider that they consist a result of some locally anisotropic tensors of type $\breve{T}_{\alpha\beta}(r,\theta)$, or more general (effective) sources.

4.1.2 Nonholonomic quasi-stationary gravitational polarizations of KdS configurations

The goal of this subsection is to study more general off-diagonal deformations of the standard Kerr solution when there are involved vacuum polarizations both of effective cosmological constants and d-metric coefficients depending on all space coordinates (r, φ, θ) not only on (r, θ) as we considered for above prime d-metrics. Such new classes of quasi-stationary nonholonomic spacetimes posses nonlinear symmetries of type (64) and (65), defined by respective classes of nonholonomic quasi-stationary deformations and constraints. This class of target solutions of type (32), when $\hat{\mathbf{g}}(r, \varphi, \theta)$ is defined equivalently by respective generating sources of type (44)

$$\check{\mathbf{\Upsilon}}^{\alpha}{}_{\beta}(r, \varphi, \theta) = [\,{}^{h}\Upsilon\delta^{i}{}_{j},\ {}^{v}\Upsilon\delta^{a}{}_{b}] = [\,{}^{h}\Upsilon = -\,{}_{1}\check{\mathbf{\Upsilon}}(r, \varphi),\\ {}^{v}\Upsilon = -\,{}_{2}\check{\mathbf{\Upsilon}}(r, \varphi, \theta)].$$

Let us first consider solutions with η-polarization functions for d-metrics in the form (73) when

$$\begin{aligned} d\widehat{s}^{2} &= \widehat{g}_{\alpha\beta}(x^{k}, y^{3}; \breve{g}_{\alpha}; \psi, \eta_{4};\\ {}_{2}\Lambda &= \widetilde{\Lambda},\ {}_{2}\check{\mathbf{\Upsilon}})du^{\alpha}du^{\beta}\\ &= e^{\psi(r,\varphi)}[(dx^{1}(r,\varphi))^{2} + (dx^{2}(r,\varphi))^{2}]\\ &\quad -\frac{[\partial_{\theta}(\eta_{4}\ \breve{g}_{4})]^{2}}{|\int d\theta\ {}_{2}\check{\mathbf{\Upsilon}}\partial_{\theta}(\eta_{4}\ \breve{g}_{4})|\ \eta_{4}\breve{g}_{4}}\\ &\quad \times\{dy^{3} + \frac{\partial_{i}[\int d\theta\ {}_{2}\check{\mathbf{\Upsilon}}\ \partial_{3}(\eta_{4}\breve{g}_{4})]}{{}_{2}\check{\mathbf{\Upsilon}}\partial_{\theta}(\eta_{4}\breve{g}_{4})}dx^{i}\}^{2}\\ &\quad +\eta_{4}\breve{g}_{4}\{dt + [\,{}_{1}n_{k}(r,\varphi) + \,{}_{2}n_{k}(r,\varphi)\int d\theta\\ &\quad \times\frac{[\partial_{\theta}(\eta_{4}\breve{g}_{4})]^{2}}{|\int d\theta\ {}_{2}\check{\mathbf{\Upsilon}}\partial_{3}(\eta_{4}\breve{g}_{4})|\ (\eta_{4}\breve{g}_{4})^{5/2}}]dx^{k}\}^{2} \end{aligned} \tag{95}$$

is determined by a generating function $\eta_4 = \eta_4(r, \varphi, \theta)$ and respective integration functions like ${}_1n_k(r, \varphi)$ and ${}_2n_k(r, \varphi)$. The locally anisotropic vacuum effects in such a d-metric with anisotropic vertical coordinate θ is very complex and it is difficult to state conditions when it defines, for instance, BH configurations. Nevertheless, a quasi-stationary d-metric (95) can be characterized by nonlinear symmetries of type (71),

$$\begin{aligned} \partial_{\theta}[\Psi^{2}] &= -\int d\theta\ {}_{2}\check{\mathbf{\Upsilon}}\partial_{\theta}h_{4} \simeq -\int d\theta\ {}_{2}\check{\mathbf{\Upsilon}}\partial_{\theta}(\eta_{4}\ \breve{g}_{4})\\ &\simeq -\int d\theta\ {}_{2}\check{\mathbf{\Upsilon}}\partial_{\theta}[\zeta_{4}(1+\kappa\ \chi_{4})\ \breve{g}_{4}],\\ \Psi &= |\ \widetilde{\Lambda}|^{-1/2}\sqrt{|\int d\theta\ {}_{2}\check{\mathbf{\Upsilon}}\ (\Phi^{2})^{*}|},\ \Phi^{2}\\ &= -4\ \widetilde{\Lambda}h_{4} \simeq -4\ {}_{2}\Lambda\eta_{4}\breve{g}_{4}\\ &\simeq -4\ \widetilde{\Lambda}\ \zeta_{4}(1+\kappa\chi_{4})\ \breve{g}_{4}. \end{aligned} \tag{96}$$

We note that in a series of our former works [9,14,21–23,25,28,30] we constructed and studied physical properties of d-metrics when K(a)dS and other type BH solutions are nonholonomically deformed for $y^3 = \varphi$, when respective effective sources ${}_2\widehat{\Upsilon}$ are generated by certain extra dimension (super) string contributions, nonassociative and/or noncommutative terms, generalized Finsler and modified dispersion deformations, or other type MGTs. There were stated explicit conditions when off-diagonal φ-deformations may result in black ellipsoid, BE, configurations which can be quasi-stationary and for solutions which are different from the Kerr–Newmann-(a)dS configurations. It was proven that such locally anisotropic configurations can be stable, or stabilized by imposing corresponding nonholonomic constraints. In a similar form, we can construct gravitational η-polarizations when $y^3 = \theta$, which results in different classes of solutions constructed for other types of effective sources and, related via nonlinear symmetries, polarized or fixed values of some prescribed cosmological constants.

4.1.3 Off-diagonal quasi-stationary small parametric deformations of new KdS d-metrics

We can search for more clear physical interpretation of nonholonomic deformations of the class of prime metrics (95) and any class of similar BH ones if we consider small parametric decompositions with κ -linear terms as we considered in (74). To avoid possible singular off-diagonal frame/coordinate deformations we consider a new system of coordinates when there are nontrivial terms $\check{N}_i^a$ both for $a = 3$, with some $\check{N}_i^3 = \check{w}_i(r, \varphi, \theta)$, which can be zero in certain rotation frames, and, for $a = 4$, $\check{N}_i^4 = \check{n}_i(r, \varphi, \theta)$ which may be with a nontrivial $\check{n}_2 = -a\sin\theta(r^2 + a^2 - \triangle_\Lambda)/(a^2\sin^2\theta - \triangle_\Lambda)$ as we considered above. Applying the AFCDM, we construct a d-metric of type (75) determined by χ-generating functions:

$$
\begin{aligned}
d\,\widehat{s}^2 = {} & \widehat{g}_{\alpha\beta}(r,\varphi,\theta;\psi,g_4;\ {}_2\check{\Upsilon})du^\alpha du^\beta = e^{\psi_0}(1+\kappa\ {}^{\psi}\chi)[(dx^1(r,\varphi))^2+(dx^2(r,\varphi))^2] - \Big\{ \frac{4[\partial_\theta(|\zeta_4\breve{g}_4|^{1/2})]^2}{\breve{g}_3|\int d\theta[\ {}_2\check{\Upsilon}\partial_3(\zeta_4\breve{g}_4)]|} \\
& -\kappa\left[\frac{\partial_\theta(\chi_4|\zeta_4\breve{g}_4|^{1/2})}{4\partial_\theta(|\zeta_4\breve{g}_4|^{1/2})} - \frac{\int d\theta\{\ {}_2\check{\Upsilon}\partial_\theta[(\zeta_4\breve{g}_4)\chi_4]\}}{\int d\theta[\ {}_2\check{\Upsilon}\partial_\theta(\zeta_4\breve{g}_4)]}\right]\Big\}\breve{g}_3\Big\{d\theta + [\frac{\partial_i\ \int d\theta\ {}_2\check{\Upsilon}\ \partial_\theta\zeta_4}{(\check{N}_i^3)\ {}_2\check{\Upsilon}\partial_\theta\zeta_4} + \kappa(\frac{\partial_i[\int d\theta\ {}_2\check{\Upsilon}\ \partial_\theta(\zeta_4\chi_4)]}{\partial_i\ [\int d\theta\ {}_2\check{\Upsilon}\partial_\theta\zeta_4]} \\
& -\frac{\partial_\theta(\zeta_4\chi_4)}{\partial_\theta\zeta_4})]\check{N}_i^3 dx^i\Big\}^2 + \zeta_4(1+\kappa\ \chi_4)\ \breve{g}_4\{dt + [(\check{N}_k^4)^{-1}[\ {}_1n_k + 16\ {}_2n_k\left[\int d\theta \frac{\left(\partial_\theta[(\zeta_4\breve{g}_4)^{-1/4}]\right)^2}{|\int d\theta\partial_\theta[\ {}_2\check{\Upsilon}(\zeta_4\breve{g}_4)]|}\right] \\
& +\kappa\frac{16\ {}_2n_k\int d\theta\frac{\left(\partial_\theta[(\zeta_4\breve{g}_4)^{-1/4}]\right)^2}{|\int d\theta\partial_\theta[\ {}_2\check{\Upsilon}(\zeta_4\breve{g}_4)]|}(\frac{\partial_\theta[(\zeta_4\breve{g}_4)^{-1/4}\chi_4)]}{2\partial_\theta[(\zeta_4\breve{g}_4)^{-1/4}]} + \frac{\int d\theta\partial_\theta[\ {}_2\check{\Upsilon}(\zeta_4\chi_4\breve{g}_4)]}{\int d\theta\partial_\theta[\ {}_2\check{\Upsilon}(\zeta_4\breve{g}_4)]})}{{}_1n_k + 16\ {}_2n_k\left[\int d\theta\frac{\left(\partial_\theta[(\zeta_4\breve{g}_4)^{-1/4}]\right)^2}{|\int d\theta\partial_\theta[\ {}_2\check{\Upsilon}(\zeta_4\breve{g}_4)]|}\right]}\Big]\check{N}_k^4 dx^k\}^2. \qquad (97)
\end{aligned}
$$

We note that polarization functions $\zeta_4(r,\varphi,\theta)$ and $\chi_4(r,\varphi,\theta)$ in this d-metric can be prescribed to be a necessary smooth class form, when χ_4 is a generating function. The h-coordinates and generating h-function can be chosen in such a way that there are not introduced additional singularities. The d-metric (97) describes small κ-parametric deformations of the of new KdS d-metric (95) when the coefficients get additional anisotropy on φ-coordinate.

Solutions of type (97) can be generated with additional ellipsoidal deformations on θ if we chose

$$\chi_4(r,\varphi,\theta) = \underline{\chi}(r,\varphi)\sin(\omega_0\theta+\theta_0), \qquad (98)$$

where $\underline{\chi}(r,\varphi)$ is a smooth function and ω_0 and θ_0 are some constants. Really, for such generating polarizacion functions and $\zeta_4(r,\varphi,\theta)\neq 0$, we obtain that

$$(1+\kappa\ \chi_4)\ \breve{g}_4 \simeq a^2\sin^2\theta - \triangle_\Lambda + \kappa\ \chi_4 = 0$$

For small a and $\frac{\Lambda_0}{3}$, we can approximate

$$r = 2M/(1+\kappa\ \chi_4),$$

which is the parametric equation defining a rotoid configuration with κ being the eccentricity parameter and generating function (98).

In general, we can consider polarization functions when KdS BH are embedded into a nontrivial nonholonomic quasi-stationary background. The nonholonomic conditions can be imposed such way that the BH configuration is preserved as conventional h- and-distributions. For small ellipsoidal deformations of type (98), we model black ellipsoid, BE, objects. They can be stable [21–23] for certain classes of nonholonomic constraints. Imposing respective classes of generating and integration functions of type (90), we extract LC-configurations, when the scalar curvature is of type $R(r,\varphi,\theta)\simeq\Lambda(r,\varphi,\theta)$, which is determined by nonlinear symmetries (96). This modifies on κ the boundary conditions (93) for the effective mass M and cosmological constant Λ_0, when such values are with local anisotropic polarization because of vacuum gravitational background. The phenomenon of warped curvature described in [66] can be preserved for some subclasses of nonholonomic deformations but the gravitational vacuum became more complex and effective matter tensor (94) became of type $Y_{\alpha\beta}(r,\theta,\varphi)$.

4.2 Nonholonomic deformations of cylindrical systems in GR

The AFCDM can be applied for constructing exact/parametric solutions of (modified) Einstein equations describing off-diagonal deformations of cylindrical configurations. In this subsection, we study such an example when the solutions involve a nontrivial cosmological constant $\Lambda > 0$ and/or certain generating sources for (effective) matter.

4.2.1 Prime d-metrics for cylindrical systems

As a prime metric ansatz, we consider the cylindrical metric used for generating Linet–Tian (LT) families of solutions [67–69], for review of results see [70],

$$
\begin{aligned}
d\mathring{s}^2 = {} & dr^2 + [Q_1(r)]^{2(8\sigma^2-4\sigma-1)/3\varsigma}[Q_2(r)]^{2/3}dz^2 \\
& + \frac{[Q_2(r)]^{2/3}}{(c_0)^2[Q_1(r)]^{4(8\sigma^2-4\sigma-1)/3\varsigma}}d\varphi^2 \\
& - \frac{[Q_2(r)]^{2/3}}{(c_0)^2[Q_1(r)]^{2(8\sigma^2-4\sigma-1)/3\varsigma}}dt^2. \qquad (99)
\end{aligned}
$$

In this diagonal metric, there are considered cylindrical coordinates $u^\alpha=(r,z,\varphi,t)$, when $\varsigma = 4\sigma^2-2\sigma+1 = const$, for $0<\sigma<1/2$; c_0 is an integration constant, which can be fixed to be positive, and the functions $Q_1(r) = \frac{2}{\sqrt{3\Lambda}}\tan(\frac{\sqrt{3\Lambda}}{2}r)$ and $Q_2(r) = \frac{1}{\sqrt{3\Lambda}}\sin(\sqrt{3\Lambda}r)$ are defined in such form that this metric define a solution of (1) with $Tm=0$.

To avoid off-diagonal deformations with coordinate and frame coefficient singularities, we can consider frame transforms to a parametrization with trivial N-connection coefficients $\mathring{N}_i^a = \mathring{N}_i^a(u^\alpha(r,z,\varphi,t))$ and $\mathring{g}_\beta(u^j(r,z,\varphi),u^3(r,z,\varphi))$.

For instance, introducing new coordinates $u^1 = x^1 = r, u^2 = z$, and $u^3 = y^3 = \varphi + {}^3B(r,z), u^4 = y^4 = t + {}^4B(r,z)$, when

$$\mathring{\mathbf{e}}^3 = d\varphi = du^3 + \mathring{N}_i^3(r,z)dx^i = du^3 + \mathring{N}_1^3(r,z)dr + \mathring{N}_2^3(r,z)dz,$$
$$\mathring{\mathbf{e}}^4 = dt = du^4 + \mathring{N}_i^4(r,z)dx^i = du^4 + \mathring{N}_1^4(r,z)dr + \mathring{N}_2^4(r,z)dz,$$

for $\mathring{N}_i^3 = -\partial\, {}^3B/\partial x^i$ and $\mathring{N}_i^4 = -\partial\, {}^4B/\partial x^i$. We transform (99) into

$$d\mathring{s}^2 = \breve{g}_\alpha(r)[\mathring{\mathbf{e}}^\alpha(r,z)]^2 = dr^2 + [Q_1(r)]^{2(8\sigma^2-4\sigma-1)/3\varsigma}[Q_2(r)]^{2/3}dz^2 + \frac{[Q_2(r)]^{2/3}}{(c_0)^2[Q_1(r)]^{4(8\sigma^2-4\sigma-1)/3\varsigma}}(dy^3 + \mathring{N}_i^3 dx^i)^2 - \frac{[Q_2(r)]^{2/3}}{(c_0)^2[Q_1(r)]^{2(8\sigma^2-4\sigma-1)/3\varsigma}}(dy^4 + \mathring{N}_i^4 dx^i)^2. \quad (100)$$

Such a prime d-metric can be used for generating new classes of quasi-stationary solutions for corresponding types of η-polarization and/or χ-polarization functions of type (73) and/or (75). The explicit formulas for the target d-metrics depend on the type of coordinate transforms there are considered. We may keep a system of coordinates when $u^3(r,z,\varphi) \simeq \varphi$ and generate such quasi-stationary solutions with nontrivial derivatives on $\partial_3 = \partial_\varphi$ when the coefficients do not depend on $y^4 \simeq t$. To compute such target d-metrics, we can consider ${}_2\widehat{\Upsilon}(x^i, y^3) = \Lambda$, or to study any nontrivial (effective) matter field contributions encoded in a general ${}_2\widehat{\Upsilon}(r,z,\varphi) = {}^{cy}\Upsilon$. Such a generating source is stated in cylindric coordinates but may involve other type symmetries and contributions for various types of classical and quantum deformations, string symmetries etc.

4.2.2 Nonholonomic quasi-stationary gravitational polarizations of cylindrical configurations

Using η-polarization functions, we derived such target quasi-stationary metrics encoding primary d-metrics' data (100),

$$\widehat{ds}^2 = \widehat{g}_{\alpha\beta}(r,z,\varphi;\psi,\eta_4;\ {}_2\Lambda = \Lambda,\ {}^{cy}\Upsilon,\ \mathring{g}_4 = {}^{cy}g(r))du^\alpha du^\beta$$
$$= e^{\psi(r,z)}[(dx^1(r,z))^2 + (dx^2(r,z))^2] - \frac{[\partial_\varphi(\eta_4\ {}^{cy}g_4)]^2}{|\int d\varphi\ {}^{cy}\Upsilon\partial_\varphi(\eta_4\ \breve{g}_4)|\ \eta_4\, {}^{cy}g} \times\left\{dy^3 + \frac{\partial_i[\int d\varphi\ {}^{cy}\Upsilon\ \partial_\varphi(\eta_4\ {}^{cy}g)]}{{}^{cy}\Upsilon\partial_\varphi(\eta_4\ {}^{cy}g)}dx^i\right\}^2 + \eta_4\breve{g}_4$$
$$\times\left\{dt + [\ {}_1n_k(r,z) + {}_2n_k(r,z)\int d\varphi \times\frac{[\partial_\varphi(\eta_4\ {}^{cy}g)]^2}{|\int d\varphi\ {}^{cy}\Upsilon\partial_\varphi(\eta_4\ {}^{cy}g)|\ (\eta_4\ {}^{cy}g)^{5/2}}]dx^k\right\} \quad (101)$$

In these formulas $\mathring{g}_4 = {}^{cy}g(r) = -[Q_2(r)]^{2/3}/(c_0)^2 [Q_1(r)]^{2(8\sigma^2-4\sigma-1)/3\varsigma}$, when the families of solutions are determined by respective generating function $\eta_4 = \eta_4(r,z,\varphi)$ and integration functions ${}_1n_k(r,z)$ and ${}_2n_k(r,z)$. The function $\psi(r,z)$ is a solution of 2-d Poisson equation $\partial_{11}^2\psi + \partial_{22}^2\psi = 2\ {}_1\widehat{\Upsilon}(r,z)$, when

$$e^{\psi(r,z)}[(dx^1(r,z))^2 + (dx^2(r,z))^2] = \eta_1(r,z)dr^2 + \eta_2(r,z)[Q_1(r)]^{2(8\sigma^2-4\sigma-1)/3\varsigma}[Q_2(r)]^{2/3}dz^2.$$

The locally anisotropic vacuum effects described by a d-metric (101) with anisotropic vertical coordinate φ are very complex and it is difficult to state in general form explicit conditions when it defines, for instance, BH configurations, or result some generalized wormholes (to be studied in next subsection) etc. Any such quasi-stationary off-diagonal solution can be characterized by corresponding nonlinear symmetries of type (71),

$$\partial_\varphi[\Psi^2(r,z,\varphi)] = -\int d\varphi\, {}^{cy}\Upsilon\partial_\varphi g_4 \simeq -\int d\varphi\ {}^{cy}\Upsilon(r,z,\varphi)\partial_\varphi[\eta_4(r,z,\varphi)\ {}^{cy}g(r)] \simeq -\int d\varphi\ {}^{cy}\Upsilon(r,z,\varphi)\partial_\varphi[\zeta_4(r,z,\varphi) \times(1+\kappa\ \chi_4(r,z,\varphi))\ {}^{cy}g(r)],$$
$$\Psi(r,z,\varphi) = |\ \Lambda|^{-1/2}\sqrt{|\int d\varphi\ {}^{cy}\Upsilon(r,z,\varphi)\ \partial_\varphi(\Phi^2)|}, \times(\Phi(r,z,\varphi))^2 = -4\ \Lambda g_4(r,z,\varphi) \simeq -4\ {}_2\Lambda\eta_4(r,z,\varphi)\ {}^{cy}g(r) \simeq -4\Lambda\ \zeta_4(r,z,\varphi)(1+\kappa\chi_4(r,z,\varphi))\ {}^{cy}g(r).$$

We can consider that above class of nonholonomic deformations transform a cylindrical d-metric into a "spagetti" quasi-stationary configuration (with different sections, curved and waved, possible interruptions etc.) embedded into locally anisotropic gravitational vacuum media. The geometry of such objects is determined by prescribed generating functions and sources and integration functions.

4.2.3 Small parametric off-diagonal quasi-stationary deformations of cylindrical d-metrics

We can provide a more explicit physical interpretation for small parametric quasi-stationary deformation of cylindrical systems. In terms of χ -polarization functions, respective d-metrics can be written in the form

$$
\begin{aligned}
d\widehat{s}^2 &= \widehat{g}_{\alpha\beta}(r,z,\varphi;\psi,\ {}^{cy}\Upsilon)du^\alpha du^\beta = e^{\psi_0(r,z)}[1+\kappa\ {}^{\psi(r,z)}\chi(r,z)][(dx^1(r,z))^2+(dx^2(r,z))^2] \\
&-\left\{\frac{4[\partial_\varphi(|\zeta_4\ {}^{cy}g|^{1/2})]^2}{\breve{g}_3|\int d\varphi\{{}^{cy}\Upsilon\partial_\varphi(\zeta_4\ {}^{cy}g)\}|} - \kappa\left[\frac{\partial_\varphi(\chi_4|\zeta_4\ {}^{cy}g|^{1/2})}{4\partial_\varphi(|\zeta_4\ {}^{cy}g|^{1/2})} - \frac{\int d\varphi\{{}^{cy}\Upsilon\partial_\varphi[(\zeta_4\ {}^{cy}g)\chi_4\right]\}}{\int d\varphi\{{}^{cy}\Upsilon\partial_\varphi(\zeta_4\ {}^{cy}g)\}}]\}\ {}^{cy}g_3\left\{d\varphi\right. \\
&+\left[\frac{\partial_i\ \int d\varphi\ {}^{cy}\Upsilon\ \partial_\varphi\zeta_4}{(\mathring{N}^3_i)\ {}^{cy}\Upsilon\partial_\varphi\zeta_4} + \kappa\left(\frac{\partial_i[\int d\varphi\ {}^{cy}\Upsilon\ \partial_\varphi(\zeta_4\chi_4)]}{\partial_i\ [\int d\varphi\ {}^{cy}\Upsilon\partial_\varphi\zeta_4]} - \frac{\partial_\varphi(\zeta_4\chi_4)}{\partial_\varphi\zeta_4}\right)\right]\mathring{N}^3_i dx^i\Bigg\}^2 \\
&+\zeta_4(1+\kappa\ \chi_4)\ {}^{cy}g\{dt+[(\mathring{N}^4_k)^{-1}[\ {}_1n_k+16\ {}_2n_k\left[\int d\varphi\frac{\left(\partial_\varphi[(\zeta_4\ {}^{cy}g)^{-1/4}]\right)^2}{|\int d\varphi\partial_\varphi[\ {}^{cy}\Upsilon(\zeta_4\ {}^{cy}g)]|}\right] \\
&+\kappa\frac{16\ {}_2n_k\int d\varphi\frac{\left(\partial_\varphi[(\zeta_4\ {}^{cy}g)^{-1/4}]\right)^2}{|\int d\varphi\partial_\varphi[\ {}^{cy}\Upsilon(\zeta_4\ {}^{cy}g)]|}\left(\frac{\partial_\varphi[(\zeta_4\ {}^{cy}g)^{-1/4}\chi_4)]}{2\partial_\varphi[(\zeta_4\ {}^{cy}g)^{-1/4}]} + \frac{\int d\varphi\partial_\varphi[\ {}^{cy}\Upsilon(\zeta_4\chi_4\ {}^{cy}g)]}{\int d\varphi\partial_\varphi[\ {}^{cy}\Upsilon(\zeta_4\ {}^{cy}g)]}\right)}{{}_1n_k+16\ {}_2n_k[\int d\varphi\frac{\left(\partial_\varphi[(\zeta_4\ {}^{cy}g)^{-1/4}]\right)^2}{|\int d\varphi\partial_\varphi[\ {}^{cy}\Upsilon(\zeta_4\ {}^{cy}g)]|}]}]\mathring{N}^4_k dx^k\}^2.
\end{aligned}
\tag{102}
$$

We note that the polarization functions $\zeta_4(r,z,\varphi)$ and $\chi_4(r,z,\varphi)$ in this d-metric can be prescribed to be a necessary smooth class form, when χ_4 is a generating function. The trivial prime N-connection coefficients $\mathring{N}^a_i$ are taken from (100), when ${}^{cy}g_3 = [Q_2(r)]^{2/3}/(c_0)^2[Q_1(r)]^{4(8\sigma^2-4\sigma-1)/3\varsigma}$.

Quasi-stationary off-diagonal deformations of the metric (99) defines Λ-vacuum cylindrical models imbedded self-consistently in non-trivial off-diagonal gravitational vacuum. If such solutions are prescribed with certain deformed hypersurface/horizon configurations, they are different from radial cylindrical ones. We can generate, for instance, cylindrical configurations with certain elliptic deformations constructed as d-metrics of type (102) if we chose a generating function of type

$$\chi_4(r,z,\varphi) = \underline{\chi}(r,z)\sin(\omega_0\varphi+\varphi_0)$$

as in (98) but on a different angular coordinate. LC-configurations can be extracted by imposing additional nonholonomic constraints of type (90). Respective nonlinear symmetries (71) are defined in cylindrical coordinates and allow to introduce effective nontrivial sources ${}^{cy}\Upsilon(r,z,\varphi)$. In equivalent form, such solutions are characterized by polarizations of the cosmological constant with curvature scalar $R(r,z,\varphi) \simeq \Lambda(r,z,\varphi)$ which reflects a cylindrical type polarization of gravitational vacuum, in general, with local anisotropy and more degrees of freedom.

4.3 Locally anisotropic wormholes

Nonholonomic deformations of wormhole solutions to locally anisotropic were studied in [71,72]. Let us revise those constructions and generate new classes of off-diagonal quasi-stationary solutions derived for primary wormhole metrics.

4.3.1 *Prime metrics as Morris–Thorne and generalized Ellis–Bronnikov wormholes*

The generic Morris–Thorne wormhole solution [73] is defined by a quadratic line element

$$d\mathring{s}^2 = (1-\frac{b(r)}{r})^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2 - e^{2\Phi(r)}dt^2,$$

where $e^{2\Phi(r)}$ is a red-shift function and $b(r)$ is the shape function defined in spherically polar coordinates $u^\alpha = (r,\theta,\varphi,t)$. The usual Ellis–Bronnikov, EB, wormholes are defined for $\Phi(r)=0$ and $b(r) = {}_0b^2/r$ characterizing a zero tidal wormhole with ${}_0b$ the throat radius. We cite [74–76] for details and a recent review of results and approaches. A generalized EB is characterized additionally by even integers $2k$ (with $k=1,2,\ldots$) when $r(l) = (l^{2k}+{}_0b^{2k})^{1/2k}$ is a proper radial distance coordinate (tortoise) and the cylindrical angular coordinate $\phi \in [0,2\pi)$ is called parallel. In such coordinates, $-\infty < l < \infty$ which is diferent from the cylindrical radial coordinate ρ, when $0 \leq \rho < \infty$. This allows us to define a prime metric

$$d\mathring{s}^2 = dl^2 + r^2(l)d\theta^2 + r^2(l)\sin^2\theta d\varphi^2 - dt^2,$$

when

$$dl^2 = \left(1-\frac{b(r)}{r}\right)^{-1}dr^2 \text{ and}$$
$$b(r) = r - r^{3(1-k)}(r^{2k} - {}_0b^{2k})^{(2-1/k))}.$$

We can avoid off-diagonal deformations with coordinate and frame coefficient singularities, we can consider frame transforms to a parametrization with trivial N-connection coefficients $\check{N}^a_i = \check{N}^a_i(u^\alpha(l,\theta,\varphi,t))$ and $\check{g}_\beta(u^j(l,\theta,\varphi),u^3(l,\theta,\varphi))$. For instance, introducing new coordinates $u^1 = x^1 = l, u^2 = \theta$, and $u^3 = y^3 = \varphi + {}^3B(l,\theta), u^4 = y^4 = t + {}^4B(l,\theta)$, when

$$\breve{\mathbf{e}}^3 = d\varphi = du^3 + \breve{N}^3_i(l,\theta)dx^i$$
$$= du^3 + \breve{N}^3_1(l,\theta)dl + \breve{N}^3_2(l,\theta)d\theta,$$
$$\breve{\mathbf{e}}^4 = dt = du^4 + \breve{N}^4_i(l,\theta)dx^i$$
$$= du^4 + \breve{N}^4_1(l,\theta)dl + \breve{N}^4_2(l,\theta)d\theta,$$

for $\mathring{N}^3_i = -\partial\ {}^3B/\partial x^i$ and $\mathring{N}^4_i = -\partial\ {}^4B/\partial x^i$. Using such nonlinear coordinates, the quadratic elements for above wormhole solutions can be parameterized as a prime d-metric,

$$d\mathring{s}^2 = \breve{g}_\alpha(l,\theta,\varphi)[\breve{\mathbf{e}}^\alpha(l,\theta,\varphi)]^2, \tag{103}$$

where $\breve{g}_1 = 1$, $\breve{g}_2 = r^2(l)$, $\breve{g}_3 = r^2(l)\sin^2\theta$ and $\breve{g}_4 = -1$.

This class of solutions are determined by respective generating function $\eta_4 = \eta_4(l,\theta,\varphi)$ and integration functions ${}_1n_k(l,\theta)$ and ${}_2n_k(l,\theta)$. The function $\psi(l,\theta)$ is a solution of 2-d Poisson equation $\partial^2_{11}\psi + \partial^2_{22}\psi = 2\ {}_1\widehat{\Upsilon}(l,\theta)$.

The target d-metrics (104) do not describe wormhole like locally anisotropic object for general classes of generating and integrating data.

4.3.3 *Small parametric off-diagonal quasi-stationary deformations of wormhole d-metrics*

We can define locally anisotropic wormholes if we consider for small parametric quasi-stationary deformation of prime metrics of type (103). In terms of χ-polarization functions, the quadratic linear elements are computed

$$d\widehat{s}^2 = \widehat{g}_{\alpha\beta}(l,\theta,\varphi;\psi,\eta_4;\ {}_2\Lambda = \Lambda,\ {}^{wh}\Upsilon,\ \breve{g}_\alpha)du^\alpha du^\beta = e^{\psi_0(l,\theta)}[1+\kappa\ {}^{\psi(l,\theta)}\chi(l,\theta)][(dx^1(l,\theta))^2 + (dx^2(l,\theta))^2]$$
$$-\left\{\frac{4[\partial_\varphi(|\zeta_4\ \breve{g}_4|^{1/2})]^2}{\breve{g}_3|\int d\varphi\{\ {}^{wh}\Upsilon\partial_\varphi(\zeta_4\ \breve{g}_4)\}|} - \kappa\left[\frac{\partial_\varphi(\chi_4|\zeta_4\breve{g}_4|^{1/2})}{4\partial_\varphi(|\zeta_4\ \breve{g}_4|^{1/2})} - \frac{\int d\varphi\{\ {}^{wh}\Upsilon\partial_\varphi[(\zeta_4\ \breve{g}_4)\chi_4]\}}{\int d\varphi\{\ {}^{wh}\Upsilon\partial_\varphi(\zeta_4\ \breve{g}_4)\}}\right]\right\}\breve{g}_3$$
$$\times\left\{d\varphi + \left[\frac{\partial_i\ \int d\varphi\ {}^{wh}\Upsilon\ \partial_\varphi\zeta_4}{(\breve{N}^3_i)\ {}^{wh}\Upsilon\partial_\varphi\zeta_4} + \kappa\left(\frac{\partial_i[\int d\varphi\ \ {}^{wh}\Upsilon\ \partial_\varphi(\zeta_4\chi_4)]}{\partial_i\ [\int d\varphi\ \ {}^{wh}\Upsilon\partial_\varphi\zeta_4]} - \frac{\partial_\varphi(\zeta_4\chi_4)}{\partial_\varphi\zeta_4}\right)\right]\breve{N}^3_i dx^i\right\}^2 + \zeta_4(1+\kappa\ \chi_4)\ \breve{g}_4$$
$$\times\{dt + [(\breve{N}^4_k)^{-1}[\ {}_1n_k + 16\ {}_2n_k\left[\int d\varphi\frac{\left(\partial_\varphi[(\zeta_4\ \breve{g}_4)^{-1/4}]\right)^2}{|\int d\varphi\partial_\varphi[\ {}^{wh}\Upsilon(\zeta_4\ \breve{g}_4)]|}\right]$$
$$+\kappa\frac{16\ {}_2n_k\int d\varphi\frac{\left(\partial_\varphi[(\zeta_4\ \breve{g}_4)^{-1/4}]\right)^2}{|\int d\varphi\partial_\varphi[\ {}^{wh}\Upsilon(\zeta_4\ \breve{g}_4)]|}\left(\frac{\partial_\varphi[(\zeta_4\ \breve{g}_4)^{-1/4}\chi_4)]}{2\partial_\varphi[(\zeta_4\ {}^{cy}g)^{-1/4}]} + \frac{\int d\varphi\partial_\varphi[\ {}^{wh}\Upsilon(\zeta_4\chi_4\ \breve{g}_4)]}{\int d\varphi\partial_\varphi[\ {}^{wh}\Upsilon(\zeta_4\ \breve{g}_4)]}\right)}{{}_1n_k + 16\ {}_2n_k[\int d\varphi\frac{\left(\partial_\varphi[(\zeta_4\ \breve{g}_4)^{-1/4}]\right)^2}{|\int d\varphi\partial_\varphi[\ {}^{wh}\Upsilon(\zeta_4\ \breve{g}_4)]|}]}]\breve{N}^4_k dx^k\}^2. \tag{105}$$

4.3.2 *Nonholonomic quasi-stationary gravitational polarizations of wormholes*

Off-diagonal quasi-stationary deformations of wormhole (103) are generated by introducing nontrivial sources ${}_1\widehat{\Upsilon}(l,\theta)$ and ${}_2\widehat{\Upsilon}(l,\theta,\varphi) = {}^{wh}\Upsilon$ related to nonlinear symmetries of type (71) to a nonzero (effective) cosmological constant Λ. Using η -polarization functions, we derived such target quasi-stationary metrics

$$d\widehat{s}^2 = \widehat{g}_{\alpha\beta}(l,\theta,\varphi;\psi,\eta_4;\ {}_2\Lambda = \Lambda,\ \ {}^{wh}\Upsilon,\ \breve{g}_\alpha)du^\alpha du^\beta$$
$$= e^{\psi l,\theta)}[(dx^1(l,\theta))^2 + (dx^2(l,\theta))^2]$$
$$-\frac{[\partial_\varphi(\eta_4\ \breve{g}_4)]^2}{|\int d\varphi\ \ {}^{wh}\Upsilon\partial_\varphi(\eta_4\ \breve{g}_4)|\ \eta_4\ \breve{g}_4}$$
$$\times\left\{dy^3 + \frac{\partial_i[\int d\varphi\ \ {}^{wh}\Upsilon\ \partial_\varphi(\eta_4\ \breve{g}_4)]}{{}^{wh}\Upsilon\partial_\varphi(\eta_4\ \breve{g})}dx^i\right\}^2 + \eta_4\breve{g}_4$$
$$\times\left\{dt + [\ {}_1n_k(r,z) + \ {}_2n_k(r,z)\int d\varphi\right.$$
$$\left.\times\frac{[\partial_\varphi(\eta_4\ \breve{g}_4)]^2}{|\int d\varphi\ \ {}^{wh}\Upsilon\partial_\varphi(\eta_4\ \breve{g}_4)|\ (\eta_4\ \breve{g}_4)^{5/2}}]dx^k\right\}. \tag{104}$$

We can can model elliptic deformations as a particular case of d-metrics of type (105) if we chose a generating function of type

$$\chi_4(l,\theta,\varphi) = \underline{\chi}(l,\theta)\sin(\omega_0\varphi + \varphi_0)$$

as for cylindric configurations with φ-anisotropic deformations. Here we note that a different family of solutions of type (104) and/or (105) can be constructed if we change the order of angular coordinates in the primary and target d-metrics, $\theta\leftrightarrow\varphi$. We omit details on such applications of the AFCDM.

4.4 Nonholonomic toroid configurations and black torus, BT

Nonholonomic deformations of toroidal BHs were studied in [20,77]. They provided examples of generic off-diagonal deformations of black torus, BT, and black ring generalizations in GR and MGTs [78–81], see [82] for a recent review of results. Such different classes of new solutions can be generated for various types of nonholonomic distributions and nonlinear transforms. In this subsection, we study an example when the AFCDM is applied for generating quasi-stationary locally anisotropic solutions using prime BT metrics analyzed in [82]. For simplicity, we shall consider only small parametric deformations when the physical interpretation of new classes of solutions is very similar to the holonomic/diagonalizable metric ansatz.

4.4.1 Prime metrics for AdS BH with toroidal horizon

Let us consider a quadratic line element (see details in section 3.1 of [82])

$$d\tilde{s}^2 = f^{-1}(\tilde{r})d\tilde{r}^2 + \tilde{r}^2(\tilde{k}_1^2 dx^2 + \tilde{k}_2^2 dy^2) - f(\tilde{r})d\tilde{t}^2 = \tilde{g}_\alpha(\tilde{x}^1)(d\tilde{u}^\alpha)^2, \text{ for } f(\tilde{r}) = -\epsilon^2 b^2 - \tilde{\mu}/\tilde{r} - \Lambda\tilde{r}^2/3. \quad (106)$$

The coordinates in this metric $\tilde{g} = \{\tilde{g}_\alpha\}$ are are related via rescaling parameter ϵ to standard toroidal "normalized" coordinates, when r is a radial coordinate, with $\theta = 2\pi k_1 x$ and $\varphi = 2\pi k_2 y$ (when $x, y \in [0,1]$) and rescaling

$$k_1 = \epsilon\tilde{k}_1, k_2 = \epsilon\tilde{k}_2, \mu \rightarrow \frac{\mu}{(2\pi)^3} = \tilde{\mu}/\epsilon^3;$$
$$r \rightarrow \frac{r}{2\pi} = \tilde{r}/\epsilon, t \rightarrow 2\pi t = \epsilon\tilde{r}.$$

In above formulas, the parameter b is a coupling constant for the energy momentum tensor for the nonlinear SU(2) sigma model, which is parameterized

$$T_{\mu\nu} = \frac{b^2\epsilon^2}{8\pi G\tilde{r}^2}\left[f(\tilde{r})\delta^4_\mu\delta^4_\nu - f^{-1}(\tilde{r})\delta^1_\mu\delta^1_\nu\right], \quad (107)$$

and μ being an integration constant which can be fixed as a mass parameter. The value $\epsilon = 0$ allows to recover in a formal way certain toroidal vacuum solution, for instance, from [78,79]. The toroidal metric (106) is an exact static solution of the Einstein equations (1) for the LC-connection and energy-momentum tensor (107). It define an AdS BH with a toroidal horizon in 4-d Einstein gravity and nonlinear σ-model.

We consider frame transforms to an off-diagonal parametrization of (106) to a form with trivial N-connection coefficients $\tilde{N}^a_i = \tilde{N}^a_i(u^\alpha(\tilde{r},x,y,t))$ and $\tilde{g}_{\alpha\beta}(u^j(\tilde{r},x,y), u^3(\tilde{r},x,y))$ which are defined an any form which do not involve singular frame transforms and off-diagonal deformations. Let us introduce new coordinates $u^1 = x^1 = \tilde{r}, u^2 = x$, and $u^3 = y^3 = y + {}^3B(\tilde{r},x), u^4 = y^4 = t + {}^4B(\tilde{r},x)$, when

$$\tilde{\mathbf{e}}^3 = dy = du^3 + \tilde{N}^3_i(\tilde{r},x)dx^i = du^3 + \tilde{N}^3_1(\tilde{r},x)dr + \tilde{N}^3_2(\tilde{r},x)dz,$$
$$\tilde{\mathbf{e}}^4 = dt = du^4 + \tilde{N}^4_i(\tilde{r},x)dx^i = du^4 + \tilde{N}^4_1(\tilde{r},x)dr + \tilde{N}^4_2(\tilde{r},x)dz,$$

for $\tilde{N}^3_i = -\partial\ {}^3B/\partial x^i$ and $\tilde{N}^4_i = -\partial\ {}^4B/\partial x^i$. In such nonlinear coordinates, the diagonal metric (106) transforms into a toroidal d-metric

$$d\tilde{s}^2 = \tilde{g}_\alpha(\tilde{r},x,y)[\tilde{\mathbf{e}}^\alpha(\tilde{r},x,y)]^2, \quad (108)$$

where $\tilde{g}_1 = f^{-1}(x^1), \tilde{g}_2 = (x^1)^2\tilde{k}_1^2, \tilde{g}_3 = (x^2)^2\tilde{k}_2^2$ and $\tilde{g}_4 = f(x^1)$.

4.4.2 Small parametric off-diagonal quasi-stationary deformations of toroidal d-metrics

We can study locally anisotropic toroidal configurations if we construct small parametric quasi-stationary deformations of prime metrics of type (108) defined by an effective source ${}^{tor}\mathbf{Y}[\mathbf{g},\widehat{\mathbf{D}}] \simeq \{-\Lambda\mathbf{g}_{\alpha\beta} + \mathbf{T}_{\alpha\beta}\}$, for (107), see also (22). The left label "tor" will be used for toroidal configurations. Respective generating sources (44) are parameterized where ${}^{tor}_1\Upsilon(\tilde{r},x)$ and ${}^{tar}_2\Upsilon(\tilde{r},x,y)$.

Any quasi-stationary off-diagonal deformation (108) to a class of solutions of type of (62), (67), (68), (73) or (75) can be characterized by corresponding nonlinear symmetries of type (71),

$$\begin{aligned}
\partial_y[\Psi^2(\tilde{r},x,y)] &= -\int dy\ {}^{tor}\Upsilon\partial_y g_4 \\
&\simeq -\int dy\ {}^{tor}\Upsilon(\tilde{r},x,y)\partial_y[\eta_4(\tilde{r},x,y)\ \tilde{g}_4(\tilde{r})] \\
&\simeq -\int dy\ {}^{tor}\Upsilon(\tilde{r},x,y)\partial_y[\zeta_4(\tilde{r},x,y) \\
&\quad\times(1+\kappa\,\chi_4(\tilde{r},x,y))\ \tilde{g}_4(\tilde{r})], \\
\Psi(\tilde{r},x,y) &= |\ \Lambda + {}^{tor}\Lambda|^{-1/2} \\
&\quad\times\sqrt{|\int dy\ {}^{tor}\Upsilon(\tilde{r},x,y)\ \partial_y(\Phi^2)|}, (\Phi(\tilde{r},x,y))^2 \\
&= -4\ \Lambda\tilde{g}_4(\tilde{r},x,y) \\
&\simeq -4\ (\ \Lambda + {}^{tor}\Lambda)\eta_4(\tilde{r},x,y) \\
\tilde{g}_4(\tilde{r}) &\simeq -4(\ \Lambda + {}^{tor}\Lambda)\ \zeta_4(\tilde{r},x,y)(1+\kappa\chi_4(\tilde{r},x,y))\ \tilde{g}_4(\tilde{r}).
\end{aligned} \quad (109)$$

In these formulas, we use ${}^{tor}\Lambda$ as an effective cosmological constant to which the energy-momentum tensor (107) encoding nonlinear sigma interactions can be related via nonlinear symmetries. In general, such a ${}^{tor}\Lambda$ is different from a prescribed cosmological constant associated to other types of gravitational and matter interactions. It is possible to elaborate on models with nonlinear functionals $\widetilde{\Lambda}(\Lambda, {}^{tor}\Lambda)$, with in this subsection is approximated to a as $\widetilde{\Lambda} = \Lambda + {}^{tor}\Lambda$.

For parametric deformations in terms of χ-polarization functions, the quadratic linear elements for nonholonomic toroidal solutions are computed

$$
\begin{aligned}
d\widehat{s}^2 = {} & \widehat{g}_{\alpha\beta}(\tilde{r},x,y;\psi,\eta_4;\ {}_2\Lambda = \Lambda + {}^{tor}\Lambda,\ {}^{tor}\Upsilon,\ \tilde{g}_\alpha)du^\alpha du^\beta = e^{\psi_0(\tilde{r},x)}[1+\kappa\ ^{\psi(\tilde{r},x)}\chi(\tilde{r},x)][(dx^1(\tilde{r},x))^2+(dx^2(\tilde{r},x))^2] \\
& -\left\{\frac{4[\partial_y(|\zeta_4\ \tilde{g}_4|^{1/2})]^2}{\tilde{g}_3|\int dy\{{}^{tor}\Upsilon\partial_y(\zeta_4\ \tilde{g}_4)\}|} - \kappa\left[\frac{\partial_y(\chi_4|\zeta_4\tilde{g}_4|^{1/2})}{4\partial_y(|\zeta_4\tilde{g}_4|^{1/2})} - \frac{\int dy\{{}^{tor}\Upsilon\partial_y[(\zeta_4\ \tilde{g}_4)\chi_4]\}}{\int dy\{{}^{tor}\Upsilon\partial_y(\zeta_4\ \tilde{g}_4)\}}\right]\right\}\tilde{g}_3\{dy + \left[\frac{\partial_i\ \int dy\ {}^{tor}\Upsilon\ \partial_y\zeta_4}{(\tilde{N}_i^3)\ {}^{tor}\Upsilon\partial_y\zeta_4}\right. \\
& \left. +\kappa(\frac{\partial_i[\int dy\ \ {}^{tor}\Upsilon\ \partial_y(\zeta_4\chi_4)]}{\partial_i\ [\int dy\ \ {}^{tor}\Upsilon\partial_y\zeta_4]} - \frac{\partial_y(\zeta_4\chi_4)}{\partial_y\zeta_4})\right]\tilde{N}_i^3 dx^i\Big\}^2 + \zeta_4(1+\kappa\ \chi_4)\ \tilde{g}_4 \\
& \times\Bigg\{dt + [(\tilde{N}_k^4)^{-1}[\ {}_1n_k + 16\ {}_2n_k\left[\int dy\frac{\left(\partial_y[(\zeta_4\ \tilde{g}_4)^{-1/4}]\right)^2}{|\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\ \tilde{g}_4)]|}\right] \\
& +\kappa\frac{16\ {}_2n_k\int dy\frac{\left(\partial_y[(\zeta_4\ \tilde{g}_4)^{-1/4}]\right)^2}{|\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\ \tilde{g}_4)]|}(\frac{\partial_y[(\zeta_4\ \tilde{g}_4)^{-1/4}\chi_4)]}{2\partial_y[(\zeta_4\ \tilde{g}_4)^{-1/4}]} + \frac{\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\chi_4\ \tilde{g}_4)]}{\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\ \tilde{g}_4)]})}{{}_1n_k + 16\ {}_2n_k[\int dy\frac{\left(\partial_y[(\zeta_4\ \tilde{g}_4)^{-1/4}]\right)^2}{|\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\ \tilde{g}_4)]|}]}]\tilde{N}_k^4 dx^k\Bigg\}^2.
\end{aligned} \tag{110}
$$

We can use this formula in order to model elliptic deformations if we chose a generating function of type

$$\chi_4(\tilde{r},x,y) = \underline{\chi}(\tilde{r},x)\sin(\omega_0 y + y_0).$$

This defines a family of toroid configurations with ellipsoidal deformations on y coordinate. Above formulas can be re-defined for a different family of quasi-stationary solutions with off-diagonal deformations on x -coordinate if we change the order of coordinates $x \leftrightarrow y$. In such cases, the formulas of type (109) and (110) involve derivatives and integrals on x, with a different order of spacetime coordinates, $u^\alpha(\tilde{r},y,x,t)$, when y is considered as a h-coordinate and x as a v-coordinates.

Applying the AFCDM to (108) we can construct more general classes of off-diagonal quasi-stationary deformations of the prime toroid configurations, which are determined by certain η-deformations and d-metrics of type (73). For instance, we can consider a generating function $\eta_4(\tilde{r},x,y)$ and take instead of nonlinear sigma energy-momentum tensor (107) more general types of effective matter sources ${}_1\Upsilon(\tilde{r},x)$ and ${}_2\Upsilon(\tilde{r},x,y)$ parameterized in toroid coordinates. The physical interpretation of such more general classes of generic off-diagonal solutions depend of the type of generating functions and generating sources we consider. Nevertheless, they are always characterized by nonlinear symmetries of type (109).

Finally, we note that we can consider that various classes of nonholonomic deformations transform a toroid prime d-metric into "spagetti" quasi-stationary configurations (with different sections, curved and waved, possible interruptions, singularities etc.) embedded into locally anisotropic gravitational vacuum media. The geometry of such d-objects is determined by respective prime metrics and prescribed generating functions and sources and integration functions and assumptions on nonlinear symmetries of off-diagonal gravitational and matter field interactions. The physical meaning of such models should be determined/analyzed for corresponding types of geometric data and boundary/asymptotic conditions we state for a corresponding family of solutions.

4.5 Nonholonomic BT and BE configurations

Using the AFCDM, generic off-diagonal quasi-stationary solutions describing systems of black torus, BT, and black ellipsoid, BE, configurations we constructed in 2001 [83]. Similar classes of solutions constructed by different methods and describing so-called "Black Saturn" were constructed beginning 2006 [84–86]. The goal of this subsection is to show how the toroidal d-metric (110) can be generalized in such a form that for well defined conditions describes families of BT-BE configurations derived from a primary d-metric stating a Schwarzschild – (anti) de Sitter, A(d)S, metric imbedded into interior of a BT.

4.5.1 Prime metrics for systems of AdS BH with toroidal horizon & Schwarzschild – (a)dS BH

Let us consider a primary metric

$$
\begin{aligned}
d\grave{s}^2 &= \grave{f}^{-1}(\tilde{r})d\tilde{r}^2 + \tilde{r}^2(\tilde{k}_1^2dx^2 + \tilde{k}_2^2dy^2) - \grave{f}(\tilde{r})d\tilde{t}^2 \\
&= \grave{g}_\alpha(\tilde{x}^1)(d\tilde{u}^\alpha)^2, \text{ for } \grave{f}(\tilde{r}) \\
&= 1 - \tilde{\mu}_s/\tilde{r} - \epsilon^2b^2 - \tilde{\mu}/\tilde{r} - \Lambda\tilde{r}^2/3,
\end{aligned} \tag{111}
$$

where the local coordinates a labeled as in the toroidal metric (106). In this formula, $\grave{f}(\tilde{r})$ is different from $f(\tilde{r})$ because it contains an additional term, $1-\tilde{\mu}_s/\tilde{r}$, when $\tilde{\mu}_s < \tilde{\mu}$ is chosen in such a form that $1-\tilde{\mu}_s/\tilde{r} = 0$ describes a conventional horizon in the interior of a torus configurations when both the spherical and toroidal objects have the same planar and axial symmetry. In principle, a metric (111) may be not a solution of Einstein equations in GR but we shall search for quasi-stationary off-diagonal deformations, $\grave{g}_\alpha(\tilde{x}^1) \to g_{\alpha\beta}(u^\gamma(\tilde{u}^\delta))$, which are exact/parametric solutions of modified gravitational equations (47)–(50).

We re-write (111) in curved coordinates in a form with trivial N-connection coefficients $\grave{N}_i^a = \grave{N}_i^a(u^\alpha(\tilde{r},x,y,t))$ and $\tilde{g}_{\alpha\beta}(u^j(\tilde{r},x,y),u^3(\tilde{r},x,y))$ which are defined an any form which do not involve singular frame transforms and off-diagonal deformations. Such new coordinates are defined $u^1 = x^1 = \tilde{r}, u^2 = x$, and $u^3 =$

$y^3 = y + {}^3B(\tilde{r}, x), u^4 = y^4 = t + {}^4B(\tilde{r}, x)$, when

$$\begin{aligned}\grave{\mathbf{e}}^3 &= dy = du^3 + \grave{N}_i^3(\tilde{r}, x)dx^i \\ &= du^3 + \grave{N}_1^3(\tilde{r}, x)dr + \grave{N}_2^3(\tilde{r}, x)dz, \\ \grave{\mathbf{e}}^4 &= dt = du^4 + \grave{N}_i^4(\tilde{r}, x)dx^i \\ &= du^4 + \grave{N}_1^4(\tilde{r}, x)dr + \grave{N}_2^4(\tilde{r}, x)dz,\end{aligned}$$

for $\grave{N}_i^3 = -\partial\ {}^3B/\partial x^i$ and $\grave{N}_i^4 = -\partial\ {}^4B/\partial x^i$. In such nonlinear coordinates, we obtain an off-diagonal toroid-spheroid type metric, equivalently a respective d-metric,

$$d\grave{s}^2 = \grave{g}_\alpha(\tilde{r}, x, y)[\grave{\mathbf{e}}^\alpha(\tilde{r}, x, y)]^2, \tag{112}$$

$$\begin{aligned}\Psi(\tilde{r}, x, y) = &|\ \Lambda + {}^{tor}\Lambda|^{-1/2} \\ &\times\sqrt{|\int dy\ \ {}^{tor}\Upsilon(\tilde{r}, x, y)\ \partial_y(\Phi^2)|}, \\ &\times(\Phi(\tilde{r}, x, y))^2 = -4\ \Lambda\grave{g}_4(\tilde{r}, x, y) \\ \simeq &-4\ (\ \Lambda + {}^{tor}\Lambda)\eta_4(\tilde{r}, x, y)\ \ \tilde{g}_4(\tilde{r}) \\ \simeq &-4(\ \Lambda + {}^{tor}\Lambda)\ \zeta_4(\tilde{r}, x, y) \\ &\times(1 + \kappa\chi_4(\tilde{r}, x, y))\ \grave{g}_4(\tilde{r}).\end{aligned} \tag{113}$$

These formulas are similar to (109) but with different prime metrics which means that $\grave{g}_4(\tilde{r})$ is different from $\tilde{g}_4(\tilde{r})$.

New classes of quasi-stationary off-diagonal toroid- rotoid solutions can be generated by χ-polarization functions when the quadratic linear elements are computed

$$\begin{aligned}d\,\widehat{s}^2 =\ & \widehat{g}_{\alpha\beta}(\tilde{r}, x, y; \psi, \eta_4;\ {}_2\Lambda = \Lambda + {}^{tor}\Lambda,\ {}^{tor}\Upsilon,\ \grave{g}_\alpha)du^\alpha du^\beta = e^{\psi_0(\tilde{r},x)}[1 + \kappa\ ^{\psi(\tilde{r},x)}\chi(\tilde{r}, x)][(dx^1(\tilde{r}, x))^2 + (dx^2(\tilde{r}, x))^2] \\ & -\left\{\frac{4[\partial_y(|\zeta_4\ \grave{g}_4|^{1/2})]^2}{\grave{g}_3|\int dy\{\ {}^{tor}\Upsilon\partial_y(\zeta_4\ \grave{g}_4)\}|} - \kappa\left[\frac{\partial_y(\chi_4|\zeta_4\grave{g}_4|^{1/2})}{4\partial_y(|\zeta_4\grave{g}_4|^{1/2})} - \frac{\int dy\{\ {}^{tor}\Upsilon\partial_y[(\zeta_4\ \grave{g}_4)\chi_4]\}}{\int dy\{\ {}^{tor}\Upsilon\partial_y(\zeta_4\ \tilde{g}_4)\}}\right]\right\}\grave{g}_3 \\ & \times\{dy + \left[\frac{\partial_i\ \int dy\ {}^{tor}\Upsilon\ \partial_y\zeta_4}{(\grave{N}_i^3)\ \ {}^{tor}\Upsilon\partial_y\zeta_4} + \kappa\left(\frac{\partial_i[\int dy\ \ {}^{tor}\Upsilon\ \partial_y(\zeta_4\chi_4)]}{\partial_i\ [\int dy\ \ {}^{tor}\Upsilon\partial_y\zeta_4]} - \frac{\partial_y(\zeta_4\chi_4)}{\partial_y\zeta_4}\right)\right]\grave{N}_i^3 dx^i\}^2 + \zeta_4(1 + \kappa\ \chi_4)\ \grave{g}_4 \\ & \times\{dt + [(\grave{N}_k^4)^{-1}[\ {}_1n_k + 16\ {}_2n_k[\int dy\frac{\left(\partial_y[(\zeta_4\ \grave{g}_4)^{-1/4}]\right)^2}{|\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\ \grave{g}_4)]|}] \\ & + \kappa\frac{16\ {}_2n_k\int dy\frac{\left(\partial_y[(\zeta_4\ \grave{g}_4)^{-1/4}]\right)^2}{|\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\ \grave{g}_4)]|}\left(\frac{\partial_y[(\zeta_4\ \grave{g}_4)^{-1/4}\chi_4)]}{2\partial_y[(\zeta_4\ \grave{g}_4)^{-1/4}]} + \frac{\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\chi_4\ \grave{g}_4)]}{\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\ \grave{g}_4)]}\right)}{{}_1n_k + 16\ {}_2n_k[\int dy\frac{\left(\partial_y[(\zeta_4\ \grave{g}_4)^{-1/4}]\right)^2}{|\int dy\partial_y[\ {}^{tor}\Upsilon(\zeta_4\ \grave{g}_4)]|}]}]\grave{N}_k^4 dx^k\}^2.\end{aligned} \tag{114}$$

where $\grave{g}_1 = \grave{f}^{-1}(x^1), \grave{g}_2 = (x^1)^2\tilde{k}_1^2, \grave{g}_3 = (x^2)^2\tilde{k}_2^2$ and $\grave{g}_4 = \grave{f}(x^1)$.

4.5.2 *Small parametric off-diagonal quasi-stationary deformations of toroidal-rotoid d-metrics*

We show how to generate locally anisotropic toroid - rotoid configurations if we construct small parametric quasi-stationary deformations of a prime d-metric (108) defined by an effective source ${}^{tor}\mathbf{Y}[\mathbf{g}, \widehat{\mathbf{D}}] \simeq \{-\Lambda\mathbf{g}_{\alpha\beta} + \mathbf{T}_{\alpha\beta}\}$. The left label "tor" will be used for toroidal configurations of (effective) matter, when, for simplicity, the constant Λ is associated to Schwarzschild - (a) dS BH. Respective generating sources (44) are parameterized as in previous subsection, ${}_1^{tor}\Upsilon(\tilde{r}, x)$ and ${}_2^{tar}\Upsilon(\tilde{r}, x, y)$.

For quasi-stationary off-diagonal deformations of (112), we search for solutions described by nonlinear symmetries of type (71),

$$\begin{aligned}\partial_y[\Psi^2(\tilde{r}, x, y)] = &-\int dy\ \ {}^{tor}\Upsilon\partial_y g_4 \\ \simeq &-\int dy\ \ {}^{tor}\Upsilon(\tilde{r}, x, y)\partial_y[\eta_4(\tilde{r}, x, y)\ \grave{g}_4(\tilde{r})] \\ \simeq &-\int dy\ \ {}^{tor}\Upsilon(\tilde{r}, x, y)\partial_y[\zeta_4(\tilde{r}, x, y) \\ &\times(1 + \kappa\ \chi_4(\tilde{r}, x, y))\ \grave{g}_4(\tilde{r})],\end{aligned}$$

These formulas are similar to (110) but involve an additional spheroid configuration centered inside a toroid one. In general, they are with local anisotropic and deformed from the "perfect" torus-spherical structure. We can use (114) in order to model different types of elliptic deformations. For instance, if we chose a generating function of type

$$\chi_4(\tilde{r}, x, y) = \underline{\chi}(\tilde{r}, x)\sin(\omega_0 y + y_0)$$

when it performs rotoid deformations for the torus part. Another one, with other type effective constants and $\underline{\chi}$ can be used rotoid deformations of the Schwarzschild – (a) dS BH. For more sophisticate constructions, we can generate prolate/oblate deformations etc. It depends on the type of generating and integration functions we prescribe. We can construct exact/parametric solutions with rotoid deformations on y coordinate, and another type of deformations on x coordinate.

4.6 Nonholonomic cosmological solitonic and spheroid deformations involving 2-d vertices

In this subsection, we provide some explicit examples of locally anisotropic cosmological solutions (77) and their equivalents with gravitational η- and χ-polarizations depending on a time like coordinate. Such solutions can be generic off-diagonal and characterized by respective nonlinear symmetries.

4.6.1 Prime cosmological models with spheroidal symmetry and voids

The Minkowski spacetime can be written in **prolate** spheroidal coordinates $u^\alpha = (r, \theta, \phi, t)$, when the usual Cartesian coordinates $u^\alpha = (x, y, z, t)$ are defined

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = \sqrt{r^2 + r_\diamondsuit^2} \cos\theta,$$

where the constant parameter $r_\diamondsuit$ has the meaning of the distance of the foci from the origin of the coordinate system. For any fixed $r = {}_0r$, such coordinates define a prolate spheroid (rotoid/ellipsoid) with the foci along the z-axis, when

$$\frac{x^2 + y^2}{({}_0r)^2} + \frac{z^2}{({}_0r)^2 + r_\diamondsuit^2} = 1,$$

where ${}_0r$ correspond to the length of its minor radius and the size of its major radius is $\sqrt{({}_0r)^2 + r_\diamondsuit^2}$. The flat Minkowski spacetime metric can be written in such prolate coordinates:

$$ds^2 = (r^2 + r_\diamondsuit^2 \sin^2\theta)\left(\frac{dr^2}{r^2 + r_\diamondsuit^2} + d\theta^2\right) + r^2 \sin^2\theta d\phi - dt^2.$$

In a similar form, we can introduce **oblate** coordinates, when

$$x = \sqrt{r^2 + r_\diamondsuit^2} \sin\theta \cos\phi,$$
$$y = \sqrt{r^2 + r_\diamondsuit^2} \sin\theta \sin\phi,$$
$$z = r \cos\theta,$$

which for fixed $r = {}_0r$, there is defined an oblate spheroid with a z symmetric axis

$$\frac{x^2 + y^2}{({}_0r)^2 + r_\diamondsuit^2} + \frac{z^2}{({}_0r)^2} = 1.$$

In this hypersurface formula, the value $\sqrt{r^2 + r_\diamondsuit^2}$ corresponds to the major radius and ${}_0r$ is the minor one. In oblate coordinates, the flat Minkowski spacetime metric is written

$$ds^2 = (r^2 + r_\diamondsuit^2 \cos^2\theta)\left(\frac{dr^2}{r^2 + r_\diamondsuit^2} + d\theta^2\right) + r^2 \sin^2\theta d\phi - dt^2.$$

In [87], it was proposed that the cosmology of voids in 4-d gravity theories can be described by such quadratic line elements (we underline certain symbols and follow our system of notations in order to emphasize that we study locally anisotropic cosmological configurations):

$$d\underline{s}^2 = \frac{a^2(t)}{[1 + \frac{\varsigma}{4}(r^2 + r_\diamondsuit^2 \cos^2\theta)]^2}\Big[(r^2 + r_\diamondsuit^2 \sin^2\theta) \left(\frac{dr^2}{r^2 - \frac{M(r)}{r}(r^2 + r_\diamondsuit^2 \sin^2\theta) + r_\diamondsuit^2} + d\theta^2\right) + r^2 \sin^2\theta d\phi\Big] - B(r)dt^2,$$

with prolate spheroidal symmetry;

$$d\underline{s}^2 = \frac{a^2(t)}{[1 + \frac{\varsigma}{4}(r^2 + r_\diamondsuit^2 \sin^2\theta)]^2}\Big[(r^2 + r_\diamondsuit^2 \sin^2\theta) \left(\frac{dr^2}{r^2 - \frac{M(r)}{r}(r^2 + r_\diamondsuit^2 \cos^2\theta) + r_\diamondsuit^2} + d\theta^2\right) + (r^2 + r_\diamondsuit^2) \sin^2\theta d\phi\Big] - B(r)dt^2,$$

with oblate spheroidal symmetry. (115)

For $B(r) = 1$ and $M(r) = 0$, these formulas define FLRW cosmological quadratic line elements (in respective prolate/oblate coordinates), where $\varsigma = 1, 0, -1$ refer respectively to a positive curved, flat, hyperbolic spacial geometry. We discussed some details with respect to formulas (3) and (4).

The mass profile function $M(r)$ from (115) can be specified as in [88] (in a simple choice, one states $B(r) = 1$),

$$M(r) = \begin{cases} \frac{4\pi}{3}\rho_{int} r^3, & \text{for } r < {}_vr; \\ M({}_vr) + \frac{4\pi}{3}\rho_{bor}(r^3 - {}_vr^3), & \text{for } {}_vr \le r < {}_vr + {}_wr; \\ 0 & \text{for } {}_vr + {}_wr \le r. \end{cases}$$

In these formulas, ${}_vr$ is associated with the radius of the void, and the parameter ${}_wr$ is related to the size of wall. For spherical symmetry, such a profile is modelled in a form that the border compensates for the amount missing in the void (i.e. it models a compensated void). The respective internal density of the matter, ρ_{int}, and border density of matter, ρ_{bor}, are related to the mean density outside the void, ρ_0, using formulas

$$\rho_{int} = -\rho_0\xi \text{ and } \rho_{bor} = \rho_0\xi/[(1 + {}_wr/{}_vr)^3 - 1], \quad (116)$$

for a constant parameter $\xi < 1$. A cosmological metric (115) is a solution of the Einstein equations in GR if $a(t)$ is a solution of the Friedman equations

$$\frac{3}{a^2(t)}\left[\frac{da}{dt} + \varsigma\right] = 8\pi\rho_0.$$

To describe the characteristic phenomenology observed in astrophysical systems with dark matter [89], the function $B(r)$ can be parameterized in the form

$$B(r) = B_0\left[B_1 + \ln\left(\frac{r}{r_\diamondsuit}\right)\right]^2,$$

for some constants B_0 and B_1. The value of B_1 can be fixed in a form that the component $T^r_r = T^1_1$ of the energy momentum tensor remains of the same order as ρ_0 (they fix $B_1 = 10^7$). Other phenomenological parameters are typically stated ${}_w r = 0.3\ {}_v r$, $\xi = 0.1$, $r_\Diamond = 0.1\ {}_v r$ when a radius ${}_v r$ corresponds to a physical size of 22 Mpc.

To apply the AFCDM we have to re-write (115) in curved coordinates in a form with trivial N-connection coefficients $\underline{\mathring{N}}^a_i = \underline{\mathring{N}}^a_i(u^\alpha(r,\theta,\phi,t))$ and $\underline{\mathring{g}}_{\alpha\beta}(u^j(r,\theta,\phi,t), u^4(r,\theta,\phi,t))$ which are defined an any form which do not involve singular frame transforms and off-diagonal deformations. Such new coordinates are defined $u^1 = x^1 = r, u^2 = \theta$, and $u^3 = y^3 = y^3(r,\theta,\phi)$ and $u^4 = y^4 = t + {}^4B(r,\theta)$, when

$$\begin{aligned}\underline{\mathring{\mathbf{e}}}^3 &= du^3 + \underline{\mathring{N}}^3_i(r,\theta)dx^i\\ &= du^3 + \underline{\mathring{N}}^3_1(r,\theta)dr + \underline{\mathring{N}}^3_2(r,\theta)d\theta,\\ \underline{\mathring{\mathbf{e}}}^4 &= du^4 + \underline{\mathring{N}}^4_i(r,\theta)dx^i\\ &= du^4 + \underline{\mathring{N}}^4_1(r,\theta)dr + \underline{\mathring{N}}^4_2(r,\theta)dz,\end{aligned}$$

for $\underline{\mathring{N}}^3_i = -\partial\, y^3/\partial x^i$ and $\underline{\mathring{N}}^4_i = -\partial\, {}^4B/\partial x^i$. In such nonlinear coordinates, we obtain an off-diagonal spheroid type cosmological metric parameterized as a d-metric,

$$d\underline{\mathring{s}}^2 = \underline{\mathring{g}}_\alpha(r,\theta,t)[\underline{\mathring{\mathbf{e}}}^\alpha(r,\theta,t)]^2, \text{ where for } \begin{cases}\text{prolate}:\\ \text{oblate}:\end{cases}$$

$$\underline{\mathring{g}}_1(r,\theta,t) = \begin{cases}\frac{a^2(t)(r^2+r^2_\Diamond\sin^2\theta)}{[1+\frac{\varsigma}{4}(r^2+r^2_\Diamond\cos^2\theta)]^2[r^2-\frac{M(r)}{r}(r^2+r^2_\Diamond\sin^2\theta)+r^2_\Diamond]}\\ \frac{a^2(t)(r^2+r^2_\Diamond\sin^2\theta)}{[1+\frac{\varsigma}{4}(r^2+r^2_\Diamond\sin^2\theta)]^2[r^2-\frac{M(r)}{r}(r^2+r^2_\Diamond\cos^2\theta)+r^2_\Diamond]}\end{cases},$$

$$\underline{\mathring{g}}_2(r,\theta,t) = \begin{cases}\frac{a^2(t)}{[1+\frac{\varsigma}{4}(r^2+r^2_\Diamond\cos^2\theta)]^2}\\ \frac{a^2(t)}{[1+\frac{\varsigma}{4}(r^2+r^2_\Diamond\sin^2\theta)]^2}\end{cases},$$

$$\underline{\mathring{g}}_3(r,\theta,t) = \begin{cases}\frac{a^2(t)r^2\sin^2\theta}{[1+\frac{\varsigma}{4}(r^2+r^2_\Diamond\cos^2\theta)]^2}\\ \frac{a^2(t)(r^2+r^2_\Diamond)\sin^2\theta}{[1+\frac{\varsigma}{4}(r^2+r^2_\Diamond\sin^2\theta)]^2}\end{cases}, \underline{\mathring{g}}_4(r) = -B(r). \quad (117)$$

Such a prime cosmological metric can be nonholonomically deformed using gravitational η-polarization functions in order to generate other classes of exact and parametric solutions of nonholonomic Einstein equations (24) constructed as locally anisotropic cosmological d-metrics (33).

4.6.2 Off-diagonal cosmological solitonic evolution encoding 2-d vertices

We consider nonholonomic deformations of data

$$(\underline{\mathring{g}}_\alpha, \underline{\mathring{N}}^a_i) \to (\underline{g}_\alpha = \underline{\eta}_\alpha \underline{\mathring{g}}_\alpha, \underline{N}^a_i = \underline{\eta}^a_i \underline{\mathring{N}}^a_i),$$

where $\underline{\eta}_i(r,\theta,t) = a^{-2}(t)\eta_i(r,\theta)$, $\underline{\eta}_3(r,\theta,t) = a^{-2}(t)\underline{\eta}(r,\theta,t)$ and $\underline{\eta}_4(r,\theta,t)$ will be prescribed/computed in such forms that

$$\begin{aligned}\underline{\mathbf{g}} &= (g_i, g_b, \underline{N}^3_i = \underline{n}_i, \underline{N}^4_i = \underline{w}_i)\\ &= g_i(r,\theta)dx^i \otimes dx^i + \underline{h}_3(r,\theta,t)\underline{\mathbf{e}}^3 \otimes \underline{\mathbf{e}}^3\\ &\quad + \underline{h}_4(r,\theta,t)\underline{\mathbf{e}}^4 \otimes \underline{\mathbf{e}}^4,\\ \underline{\mathbf{e}}^3 &= d\phi + \underline{n}_i(r,\theta,t)dx^i,\ \underline{\mathbf{e}}^4 = dt + \underline{w}_i(r,\theta,t)dx^i, \quad (118)\end{aligned}$$

with Killing symmetry on the angular coordinate φ, when ∂_φ transforms into zero the N-adapted coefficients of such a d-metric.

In terms of η-polarization functions, a (24) can be written in a t-dual form to (73) as we explain in section 3.3, when

$$\begin{aligned}d\widehat{s}^2 &= \widehat{g}_{\alpha\beta}(r,\theta,t;\underline{\mathring{g}}_\alpha;\psi,\eta_3;\ {}_2\underline{\Lambda},\ {}_2\underline{\Upsilon})du^\alpha du^\beta\\ &= e^\psi[(dx^1)^2+(dx^2)^2]\\ &\quad +(\underline{\eta}\underline{\mathring{g}}_3)\Big\{d\phi + [\ {}_1n_k + {}_2n_k\int dt\\ &\quad \times\frac{[\partial_t(\underline{\eta}\underline{\mathring{g}}_3)]^2}{|\int dt\ {}_2\underline{\Upsilon}\partial_t(\underline{\eta}\underline{\mathring{g}}_3)|\ (\underline{\eta}\underline{\mathring{g}}_3)^{5/2}}]dx^k\Big\}^2\\ &\quad -\frac{[\partial_t(\underline{\eta}\,\underline{\mathring{g}}_3)]^2}{|\int dt\ {}_2\underline{\Upsilon}\partial_t(\underline{\eta}\underline{\mathring{g}}_3)|\ \underline{\eta}\mathring{g}_3}\Big\{dt + \frac{\partial_i[\int dt\ {}_2\underline{\Upsilon}\ \partial_t(\underline{\eta}\underline{\mathring{g}}_3)]}{{}_2\underline{\Upsilon}\partial_t(\underline{\eta}\underline{\mathring{g}}_3)}dx^i\Big\}^2.\end{aligned} \quad (119)$$

For $\underline{\Phi}^2 = -4\ {}_2\underline{\Lambda}\underline{g}_4$, we can transform (119) in a variant of (77) with η-polarizations determined by the generating data $(\underline{g}_4;\ {}_2\underline{\Lambda},\ {}_2\underline{\Upsilon})$. The effective cosmological constant ${}_2\underline{\Lambda}$ is chosen as effective ones which correspond via nonlinear symmetries (88) to a energy-momentum tensor (116) in a fluid type form (4) (when respective data $(\ {}_1\underline{\Upsilon},\ {}_2\underline{\Upsilon})$ are related to a $T_{\alpha\beta}$ via respective frame/coordinate transforms). We can model certain locally anisotropic cosmological scenarios which can evolve from a primary void configuration (117) being determined by generating polarization function,

$\psi \simeq \psi(x^k)$ and $\underline{\eta} \simeq \underline{\eta}(x^k, t)$.

In explicit form, we consider such a variant:

The h-part of the d-metric (119) may be prescribed to satisfy instead of 2-d Poisson equation the generalized Taubes equation for vortices on a curved background 2-d surface,

$${}_h\nabla^2\psi = \Omega_0(C_0 - C_1 e^{2\psi}), \quad (120)$$

where the position-dependent confromal factor Ω_0 and effective source $(C_0 - C_1 e^{2\psi})$ are prescribed as respective generating h-function and generating h-source ${}_1\underline{\Upsilon}(x^k)$. By rescaling, both constants C_0 and C_1 take standard values -1, 0, or 1, but there are only five combinations of these values allow vortex solutions $\psi[vortex]$ without singularities [90].

The v-part of (119) can be constructed if, for instance,

$$\underline{\eta} \simeq \begin{cases} {}^{sol}_{r}\underline{\eta}(r,t) & \text{as a solution of the modified KdV equation } \frac{\partial \eta}{\partial t} - 6\underline{\eta}^2 \frac{\partial \eta}{\partial r} + \frac{\partial^3 \eta}{\partial r^3} = 0, \text{ for radial solitons;} \\ {}^{sol}_{\theta}\underline{\eta}(\theta,t) & \text{as a solution of the modified KdV equation,} \\ \frac{\partial \eta}{\partial \theta} - 6\underline{\eta}^2 \frac{\partial \eta}{\partial \theta} + \frac{\partial^3 \eta}{\partial^3} = 0 & \text{for angular solitons.} \end{cases} \quad (121)$$

We cite [91] and reference therein on such types of solitonic wave equations.

Generic off-diagonal and locally anistoropic metrics of type (119) describe cosmological evolution scenarios with nontrivial nonholonomic structure with conventional h- and v-splitting. Under geometric evolution with gravitational polarizations and for respective generating sources, a primary metric with prolate/oblate rotoid void transforms into a vertex h-configuration (120) and, the v-part, into solitonic wave evolution of type (121), which results also in solitonic configurations for the N-connection coefficients. Such solitonic waves on t -variable can be with a radial space variable, r, or with an angular variable, θ. In a series of our and co-authors works, there were constructed more general classes of generic off-diagonal cosmological and quasi-stationary solutions with 3-d solitonic waves and solitoninc hierarchies in GR and MGTs [11–13,16,17,19,20] and with quasi-periodic and pattern forming structures [14,28], see a review of results in appendix B to [7].

4.6.3 *Small parametric off-diagonal cosmological deformations with solitonic vacua for voids*

Using t-symmetries defined in Sect. 3.3, we can construct locally anisotropic cosmological solutions with off-diagonal small κ-parametric deformations of (117). In terms of χ-polarization functions, respective d-metrics can be written in the form

$$\begin{aligned} d\widehat{s}^2 = {} & \widehat{g}_{\alpha\beta}(r,\theta,t;\psi,\ {}_2\underline{\Lambda},\ {}_2\underline{\Upsilon})du^\alpha du^\beta = e^{\psi_0(r,\theta)}[1+\kappa\ {}^{\psi}\chi(r,\theta)][(dx^1(r,\theta))^2+(dx^2(r,\theta))^2] \\ & +\zeta_3(1+\kappa\ \underline{\chi})\underline{\mathring{g}}_3\{d\phi + [(\underline{\mathring{N}}_k^3)^{-1}[\ {}_1n_k + 16\ {}_2n_k\Big[\int dt \frac{\left(\partial_t[(\underline{\zeta}_3\underline{\mathring{g}}_3)^{-1/4}]\right)^2}{|\int d\varphi\partial_\varphi[\ {}_2\underline{\Upsilon}(\zeta_4\ {}^{cy}g)]|}\Big] \\ & +\kappa \frac{16\ {}_2n_k \int dt \frac{\left(\partial_t[(\underline{\zeta}_3\ \underline{\mathring{g}}_3)^{-1/4}]\right)^2}{|\int dt\partial_t[\ {}_2\underline{\Upsilon}(\underline{\zeta}_3\ \underline{\mathring{g}}_3)]|}\left(\frac{\partial_t[(\underline{\zeta}_3\ \underline{\mathring{g}}_3)^{-1/4}\underline{\chi})]}{2\partial_t[(\underline{\zeta}_3\ \underline{\mathring{g}}_3)^{-1/4}]} + \frac{\int dt\partial_t[\ {}_2\underline{\Upsilon}(\underline{\zeta}_3\underline{\chi}\ \underline{\mathring{g}}_3)]}{\int dt\partial_t[\ {}_2\underline{\Upsilon}(\underline{\zeta}_3\ \underline{\mathring{g}}_3)]}\right)}{{}_1n_k + 16\ {}_2n_k[\int dt \frac{\left(\partial_t[(\underline{\zeta}_3\ \underline{\mathring{g}}_3)^{-1/4}]\right)^2}{|\int dt\partial_t[\ {}_2\underline{\Upsilon}(\underline{\zeta}_3\ \underline{\mathring{g}}_3)]|}]}]\underline{\mathring{N}}_k^3 dx^k\}^2. - \left\{\frac{4[\partial_t(|\underline{\zeta}_3\ \underline{\mathring{g}}_3|^{1/2})]^2}{\underline{\mathring{g}}_4|\int dt\{\ {}_2\underline{\Upsilon}\partial_t(\underline{\zeta}_3\ \underline{\mathring{g}}_3)\}|}\right. \\ & -\kappa\Big[\frac{\partial_t(\underline{\chi}|\underline{\zeta}_3\ \underline{\mathring{g}}_3|^{1/2})}{4\partial_t(|\underline{\zeta}_3\ \underline{\mathring{g}}_3|^{1/2})} - \frac{\int dt\{\ {}_2\underline{\Upsilon}\partial_t[(\underline{\zeta}_3\ \underline{\mathring{g}}_3)\underline{\chi}\Big]\Big\}}{\int dt\{\ {}_2\underline{\Upsilon}\partial_t(\underline{\zeta}_3\ \underline{\mathring{g}}_3)\}}]\}\ \underline{\mathring{g}}_4\Big\{dt + \Big[\frac{\partial_i\ \int dt\ {}_2\underline{\Upsilon}\ \partial_t\underline{\zeta}_3}{(\underline{\mathring{N}}_i^3)\ {}_2\underline{\Upsilon}\partial_t\underline{\zeta}_3} \\ & +\kappa\left(\frac{\partial_i[\int dt\ {}_2\underline{\Upsilon}\ \partial_t(\underline{\zeta}_3\ \underline{\mathring{g}}_3)]}{\partial_i\ [\int dt\ {}_2\underline{\Upsilon}\partial_t\zeta_4]} - \frac{\partial_t(\underline{\zeta}_3\ \underline{\mathring{g}}_3)}{\partial_t\underline{\zeta}_3}\right)\Big]\underline{\mathring{N}}_i^4 dx^i\Big\}^2 \end{aligned} \quad (122)$$

In above formulas, $\psi_0(r,\theta)$ and ${}^{\psi}\chi(r,\theta)$ are chosen in such a form that they define solutions of 2-d Poisson equations, or certain κ-parametric solutions of (120) with some small parametric generated vortices. The generating function $\underline{\chi} = \underline{\chi}_3(r,\theta,t)$ can be taken as a solution of solitonic wave equation (121), $\underline{\eta} \longleftrightarrow \underline{\chi}$, when $\underline{\zeta}_3(r,\theta,t)$ is also prescribed in a form for κ^0. Such d-metrics define a v-solitonic gravitational structure of voids with κ-parametric and t-evolution. For certain explicit configurations, such parametric gravitational void vacuum posses a nontrivial solitonic energy. The solutions can be characterized by nonlinear symmetries relating the effective generating source ${}_2\underline{\Upsilon}$ to a respective cosmological constant ${}_2\underline{\Lambda}$

The vertex-solitonic wave locally anisotropic cosmological d-metrics with respective prime prolate/oblate symmetry encode a nonholonomic vacuum structure with nontrivial canonical d-torsion. Imposing additional constraints, we can extract LC-configuration cosmological models if we follow the procedure described in Sect. 3.5.

Part II Nonassociative phase space and Finsler–Lagrange–Hamilton MGTs

In this part, we show how the AFCDM can be generalized for 8-d phase spaces modelled on (co) tangent Lorentz bundles and study explicit examples of quasi-stationary solutions. Such phase spaces present natural geometric arenas for nonassociative gravity theories determined by star products deformations and elaborating relativistic physically important models of Finsler–Lagrange–Hamilton geometries and theories of nonholonomic geometric flows of nonholonomic geometric objects depending on spacetime and velocity/momentum coordinates and on temperature like τ-parameter [7,8,29,29,30,63]. There are four important motivations to study such theories:

1. Modified dispersion relations, MDRs, result in nonholonomic generalized Finsler structures on phase spaces [92–98]. Star product R-flux deformations can be also characterized by MDRs encoding nonassociative and noncommutative data.
2. Non-geometric star product R-flux deformations in string theory [10,40,41] can be geometrized in nonassociative and nonholonomic forms on 8-d phase spaces involving complex or real momentum variables [5,6,38,39]. To prove general decoupling and integration properties of physically important systems of nonlinear PDEs in such theories, we have to consider nonholonomic dyadic decompositions and certain classes of generalized metrics and linear connections adapted to N-connection structures.
3. In (nonassociative) MGTs, we can define N-connection structure determined by semi-spray equations, i.e. nonlinear geodesic equations, which are equivalent to the Euler–Lagrange and/or Hamilton equations. This provides new ideas and methods for formulating generally integrable classical and quantum gravity theories when nonperturbative quantization methods are related to generic off-diagonal exact and parametric solutions in phase space gravity theories. Such geometric and quantum information formalisms can't be developed in the framework of the well-known approaches [59–62].
4. New classes of generic off-diagonal solutions in such (nonassociative) phase spaces are characterized by G. Perelman statistical and geometric thermodynamic models which are generalized for nonassociative Finsler–Lagrange–Hamilton geometric flow and nonholonomic Ricci soliton theories. In Part II, we show how such theories can be described equivalently in canonical dyadic variables (to derive important decoupling and integrating properties) and in generalized Finsler–Hamilton variables which can be used in our future works for elaborating quantum models encoding nonassociative geometric data and elaborating on new methods of quantization of gravity and matter field theories.

5 N-connections and Finsler–Lagrange–Hamilton phase space geometry

A series of important works on nonassociative geometry and physics [32–39] are based on the concept of nonassociative star product with R-flux considered in string theory. Such twisted algebraic and geometric structures result in nonassociative modifications of GR to nonholonomic geometries involving extra-dimension coordinates considered as momentum-like variables. This is similar and, for some well-defined conditions, equivalent to certain versions of nonassociative and noncommutative Finsler–Lagrange–Hamilton, FLH, geometries studied in details our former works [7,8,24,29,30,42]. So, in Part II of this review, nonassociative FLH gravitational theories are defined as minimal nonholonomic modifications of GR because of the nonassociative star product used in [38,39].

In a series of our partner works [5,6,9,10,40,41,151–154], we elaborated on new nonholonomic geometric methods of constructing exact and parametric solutions in nonassociative gravity. To decouple and solve in some general forms certain nonassociative generalizations of the Einstein equations using only the formalism elaborated in [37–39] was not possible. So, we had to perform a new research program on constructing off-diagonal solutions in MGTs by applying and developing our former results on (noncommutative/supersymmetric/string) generalized Finsler geometry [7,8,14,16,24,43,44] as a generalization of the AFCDM in GR (reviewed in Part I).

In this section, we summarize the necessary definitions and methods from the nonholonomic geometry of associative and commutative phase space geometry and relativistic models of Finsler–Lagrange–Hamilton geometry. Such geometric and physical models are elaborated on an 8-d phase space modelled as a cotangent Lorentz bundle ${}^{\shortmid}\mathcal{M} = T^*V$ on a spacetime manifold V of signature $(+++-)$; such a phase space is dual to $\mathcal{M} = TV$. In this approach, the GR theory on V is generalized on total phase spaces with conventional extra dimension velocity/momentum type coordinates.

5.1 Nonlinear connections and canonical nonholonomic (2+2)+(2+2) splitting

A nonlinear connection, N-connection, structure defining a 4+4 splitting is by definition a Whitney sum

$$
{}^{\shortmid}\mathbf{N}: \ \ \mathbf{T}\mathbf{T}^*\mathbf{V} = hT^*V \oplus cT^*V, \text{ which is dual to } \\
\mathbf{N}: \ \mathbf{T}\mathbf{T}\mathbf{V} = hTV \oplus vTV. \quad (123)
$$

These nonholonomic distributions provide a phase space extension of formulas (9), and related formulas and derived geometric and physical equations, when ${}^{\shortmid}\mathbf{N} = \{{}^{\shortmid}N_{ia}({}^{\shortmid}u)\}$, for ${}^{\shortmid}u = (x, p) = \{{}^{\shortmid}u^{\alpha} = (x^i, p_a)\}$; and, respectively, $\mathbf{N} = \{N^a_i(u)\}$, for $u = (x, y = v) = \{u^{\alpha} = (x^i, y^a = v^a)\}$. The 8-d indices split into 4+4 ones when, for instance, $\alpha, \beta, \ldots = 1, 2, \ldots 8$; $i, j, \ldots = 1, 2, 3, 4$ and $a, b, \ldots = 5, 6, 7, 8$.

To generalize and apply the AFCDM we have to consider conventional (2+2)+(2+2) splitting on respective phase spaces stated as a nonholonomic (equivalently, anholonomic-/non-integrable) dyadic, 2-d, decomposition into four oriented shells $s = 1, 2, 3, 4$. In brief, we shall say that this is a s-decomposition and use respective s-labels in abstract form, or for indices and coordinates when it will be necessary. The nonholonomic s-splitting is defined by respective N-connection (equivalently, s-connection), structure:

$$
\begin{aligned}
{}^{\shortmid}_{s}\mathbf{N}: \ {}_{s}T\mathbf{T}^{*}\mathbf{V} &= {}^{1}hT^{*}V \oplus {}^{2}vT^{*}V \\
&\oplus {}^{3}cT^{*}V \oplus {}^{4}cT^{*}V, \text{ which is dual to} \\
{}_{s}\mathbf{N}: \ {}_{s}\mathbf{T}\mathbf{T}\mathbf{V} &= {}^{1}hTV \oplus {}^{2}vTV \\
&\oplus {}^{3}vTV \oplus {}^{4}vTV, \text{ for } s = 1, 2, 3, 4.
\end{aligned} \tag{124}
$$

In these formulas, we write use ${}^{1}h$ for a conventional 2-d shell (dyadic) splitting on (co) tangent bundle, with x^{i_1} local coordinates and ${}^{2}v$ for a 2-d vertical like splitting with y^{a_2} coordinates on the shell $s = 2$. On the (co) fiber shell $s = 3$, the splitting is conventional (co) vertical, when we write ${}^{3}v$ (or ${}^{3}c$) and use local coordinates v^{a_3} (or p_{a_3}). Similarly, on the 4th shell $s = 4$, the respective symbols are ${}^{4}v$ and v^{a_4} (or ${}^{4}c$ and p_{a_4}). Hereafter, we shall write typically the formulas of s-geometric objects on ${}^{\shortmid}\mathcal{M} = T^{*}V$, when the formulas for similar ones on $\mathcal{M} = TV$ can be formulated to encode velocity type coordinates with necessary shell indices.

Using a set of N-connection coefficients, we can construct N-elongated bases (N-/s-adapted bases) as linear N-operators:

$$
\begin{aligned}
{}^{\shortmid}\mathbf{e}_{\alpha_s}[{}^{\shortmid}N_{i_s a_s}] &= \Big({}^{\shortmid}\mathbf{e}_{i_s} = \frac{\partial}{\partial x^{i_s}} - {}^{\shortmid}N_{i_s a_s}\frac{\partial}{\partial p_{a_s}}, \ {}^{\shortmid}e^{b_s} \\
&= \frac{\partial}{\partial p_{b_s}} \Big) \text{ on } {}_{s}\mathbf{T}\mathbf{T}^{*}_{\shortmid}\mathbf{V}, \\
{}^{\shortmid}\mathbf{e}_{\alpha}[{}^{\shortmid}N_{ia}] &= \Big({}^{\shortmid}\mathbf{e}_{i} = \frac{\partial}{\partial x^{i_s}} - {}^{\shortmid}N_{ia}\frac{\partial}{\partial p_{a}}, \ {}^{\shortmid}e^{b} \\
&= \frac{\partial}{\partial p_{b}} \Big) \text{ on } \mathbf{T}\mathbf{T}^{*}_{\shortmid}\mathbf{V},
\end{aligned} \tag{125}
$$

and, dual s-adapted bases, s-cobases,

$$
\begin{aligned}
{}^{\shortmid}\mathbf{e}^{\alpha_s}[{}^{\shortmid}N_{i_s a_s}] &= \Big({}^{\shortmid}\mathbf{e}^{i_s} = dx^{i_s}, \ {}^{\shortmid}\mathbf{e}_{a_s} \\
&= d\ p_{a_s} + {}^{\shortmid}N_{i_s a_s}dx^{i_s} \Big) \text{ on } {}_{s}T^{*}\mathbf{T}^{*}_{\shortmid}\mathbf{V}, \\
{}^{\shortmid}\mathbf{e}^{\alpha}[{}^{\shortmid}N_{ia}] &= \Big({}^{\shortmid}\mathbf{e}^{i} = dx^{i}, \ {}^{\shortmid}\mathbf{e}_{a} \\
&= d\ p_{a} + {}^{\shortmid}N_{ia}dx^{i} \Big) \text{ on } T^{*}\mathbf{T}^{*}_{\shortmid}\mathbf{V}.
\end{aligned} \tag{126}
$$

Such s-frames are not integrable, i.e. nonholonomic (equivalently, anholonomic) because, in general, they satisfy certain anholonomy conditions,

$$
{}^{\shortmid}\mathbf{e}_{\beta_s}\, {}^{\shortmid}\mathbf{e}_{\gamma_s} - {}^{\shortmid}\mathbf{e}_{\gamma_s}\, {}^{\shortmid}\mathbf{e}_{\beta_s} = {}^{\shortmid}w^{\tau_s}_{\beta_s\gamma_s}\, {}^{\shortmid}\mathbf{e}_{\tau_s}, \tag{127}
$$

see details in [5–8].

The geometric s-objects and respective formulas (9)–(126) can be generalized for additional running on a geometric flow evolution parameter τ, which is used in geometric flow theories, see details and references in [10,40,41]. In our works, τ can be considered as a temperature like parameter (as in G. Perelman's geometric flow thermodynamics [57]). In such cases, we write, for instance, ${}^{\shortmid}\mathbf{N}(\tau) \simeq {}^{\shortmid}\mathbf{N}(\tau, {}^{\shortmid}u) = \{ {}^{\shortmid}N_{ia}(\tau) \simeq {}^{\shortmid}N_{ia}(\tau, x^{j}, p_{b})\}$ and, respectively, ${}^{\shortmid}\mathbf{e}_{\alpha_s}(\tau)$, ${}^{\shortmid}\mathbf{e}^{\alpha_s}(\tau)$, etc., which will be used in next sections. For τ-running of geometric/physical objects, we shall write only the τ-dependence if that will not result in ambiguities. Here, we note that in a similar form we can introduce and write formulas for geometric objects on ${}_{s}\mathbf{T}\mathbf{T}\mathbf{V}$, i.e. when the total space coordinates are of spacetime-velocity type. In such case, we omit the labels "$\shortmid$" and write, for instance, $\mathbf{e}_{\alpha_s}(\tau)$ and $\mathbf{e}^{\alpha_s}(\tau)$. In general, the local coordinates are not just dual like fiber and co-fiber ones but may include certain Legendre transforms and symplectomorphisms [30]. We work on nonassociative phase spaces as in [38,39] and [5,6,9,10,40,41] using labels "$\shortmid$" in order to follow an unified system of notations which will allow in next section works to elaborate on nonassociative models of Finsler–Lagrange spaces, which are important in quantum information theory.

A metric field in a phase space ${}^{\shortmid}\mathcal{M}$ is a second rank symmetric tensor ${}^{\shortmid}g = \{{}^{\shortmid}g_{\alpha\beta}\} \in TT^{*}V \otimes TT^{*}V$ of local signature $(+,+,+,-;+,+,+,-)$. It can be written in equivalent form as a s-metric ${}^{\shortmid}_{s}\mathbf{g} = \{{}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}\}$ for ${}^{\shortmid}_{s}\mathcal{M}$ which is a 8-d phase space generalization of (14). For τ-families of phase space metrics (d-metrics for 4+4 splitting) and s-metrics, we shall use notations of type ${}^{\shortmid}g(\tau) = \{{}^{\shortmid}g_{\alpha\beta}(\tau)\}$ and, respectively, ${}^{\shortmid}_{s}\mathbf{g}(\tau) = \{{}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau)\}$.

Another important geometric concept is that of s-connection with a (2+2)+(2+2) splitting (the term distinguished connection, d-connection, is considered for a (4+4)-splitting). Such linear connections preserve respective shell or h-c structures under parallel transports a corresponding s-/N-connection splitting (124), or (123):

$$
\begin{aligned}
{}^{\shortmid}_{s}\mathbf{D} &= (h_1\, {}^{\shortmid}\mathbf{D},\ v_2\, {}^{\shortmid}\mathbf{D},\ c_3\, {}^{\shortmid}\mathbf{D},\ c_4\, {}^{\shortmid}\mathbf{D}) = \{{}^{\shortmid}\Gamma^{\alpha_s}{}_{\beta_s\gamma_s}\}, \text{ or} \\
{}^{\shortmid}\mathbf{D} &= (h\, {}^{\shortmid}\mathbf{D},\ c\, {}^{\shortmid}\mathbf{D}) = \{{}^{\shortmid}\Gamma^{\alpha}{}_{\beta\gamma}\}.
\end{aligned} \tag{128}
$$

Hereafter we shall provide only s-adapted or N-adapted formula not dubbing them using typical s-labels if that will not result in ambiguities.

Using standard definitions from differential geometry, we can introduce in abstract form and compute the coefficient formulas for any s-connection ${}_{s}^{|}\mathbf{D}$ and for such fundamental geometric s-objects:

$$
\begin{aligned}
&{}_{s}^{|}\mathcal{T} = \{{}^{|}\mathbf{T}^{\alpha_s}_{\ \beta_s\gamma_s}\}, \text{ the s-torsion ; } {}_{s}^{|}\mathcal{R} = \{{}^{|}\mathbf{R}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}, \\
&\text{the Riemannian s-curvature ;} \\
&{}_{s}^{|}\mathcal{R}ic = \{{}^{|}\mathbf{R}_{\ \beta_s\gamma_s} := {}^{|}\mathbf{R}^{\alpha_s}_{\ \beta_s\gamma_s\alpha_s} \neq {}^{|}\mathbf{R}_{\ \gamma_s\beta_s}\}, \text{ the Ricci s-tensor;} \\
&{}_{s}^{|}\mathcal{R}sc = \{{}^{|}\mathbf{g}^{\beta_s\gamma_s}\ {}^{|}\mathbf{R}_{\ \beta_s\gamma_s}\}, \text{the Riemannian scalar .}
\end{aligned} \tag{129}
$$

Geometric data $({}_{s}^{|}\mathbf{g}, {}_{s}^{|}\mathbf{D})$ enable a ${}_{s}\mathcal{M}$ with a dyadic metric-affine s-structure which is a s-adapted phase space version of metric-affine geometry [1,2,7,8]. In general, such nonholonomic phase spaces are characterized by a respective nonmetricity s-tensor, ${}_{s}^{|}\mathcal{Q} = \{{}^{|}\mathbf{Q}_{\gamma_s\alpha_s\beta_s} = {}^{|}\mathbf{D}_{\gamma_s}\ {}^{|}\mathbf{g}_{\alpha_s\beta_s}\}$. In Appendix, we provide additional abstract and s-adapted formulas (A.8)–(A.12) explaining how such values can be computed in explicit form.

Using a s-metric ${}^{|}g = {}_{s}^{|}\mathbf{g}$, we can define and compute in abstract and component forms 8-d generalizations of the formulas (19) for two important linear connection structures (the Levi–Civita, LC, connection and the canonical s-connection):

$$
({}_{s}^{|}\mathbf{g}, {}_{s}^{|}N) \rightarrow \left\{ \begin{array}{ll} {}^{|}\nabla : & {}^{|}\nabla\ {}_{s}^{|}\mathbf{g} = 0;\ {}_{\nabla}^{|}\mathcal{T} = 0, \quad \text{LC–connection ;} \\ {}_{s}^{|}\widehat{\mathbf{D}} : & {}_{s}^{|}\widehat{\mathbf{Q}} = 0;\ h_1\ {}^{|}\widehat{\mathcal{T}} = 0, v_2\ {}^{|}\widehat{\mathcal{T}} = 0, c_3\ {}^{|}\widehat{\mathcal{T}} = 0, c_4\ {}^{|}\widehat{\mathcal{T}} = 0, \quad \text{canonical} \\ & h_1 v_2\ {}^{|}\widehat{\mathcal{T}} \neq 0, h_1 c_s\ {}^{|}\widehat{\mathcal{T}} \neq 0, v_2 c_s\ {}^{|}\widehat{\mathcal{T}} \neq 0, c_3 c_4\ {}^{|}\widehat{\mathcal{T}} \neq 0, \quad \text{s-connection .} \end{array} \right. \tag{130}
$$

So, for higher dimensions, we can also use "hat" labels for geometric s-objects written in canonical form, for instance, ${}_{s}^{|}\widehat{\mathbf{D}}$, ${}_{s}^{|}\widehat{\mathcal{R}} = \{{}^{|}\widehat{\mathbf{R}}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}$ etc. In similar forms we can define and computed the canonical distortion relations for linear connections (of type ${}_{s}^{|}\widehat{\mathbf{D}} = {}^{|}\nabla + {}_{s}^{|}\widehat{\mathbf{Z}}$, with a distortions s-tensor ${}_{s}^{|}\widehat{\mathbf{Z}}$ defined by N-coefficients) which allow to compute canonical distortions of fundamental geometric objects (129). For instance, we can consider distortions of curvature tensors and s-tensors, for instance, ${}_{\nabla}^{|}\mathcal{R} = \{{}_{\nabla}^{|}R^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}$ and ${}_{s}^{|}\widehat{\mathcal{R}} = \{{}^{|}\widehat{\mathbf{R}}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}$; ${}_{\nabla}^{|}\mathcal{R}ic$ and ${}_{s}^{|}\widehat{\mathcal{R}}ic$ etc. For τ-families such formulas can written, for instance, ${}^{|}\nabla(\tau)$, ${}_{s}^{|}\widehat{\mathbf{D}}(\tau)$, ${}_{s}^{|}\widehat{\mathcal{R}}(\tau) = \{{}^{|}\widehat{\mathbf{R}}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}(\tau)\}$, ${}_{\nabla}^{|}\mathcal{R}ic(\tau)$, etc.

The modified Einstein equations for ${}_{s}^{|}\widehat{\mathbf{D}}$ (130) can be derived in abstract geometric form as in GR [2] but on phase space ${}_{s}\mathcal{M}$ and following respective conventions on s-adapted indices,

$$
{}^{|}\widehat{\mathbf{R}}ic_{\alpha_s\beta_s} = {}^{|}\Upsilon_{\alpha_s\beta_s}. \tag{131}
$$

In such formulas, the s-tensor for effective and/or matter field sources ${}^{|}\Upsilon_{\alpha_s\beta_s}$ can be postulated (or derived following a conventional s-variational calculus extending the constructions in GR or certain MGTs) in the forms

$$
{}^{|}\Upsilon_{\beta_s\gamma_s} = \left\{ \begin{array}{l} {}_{s}^{|}\Lambda_0\ {}^{|}\mathbf{g}_{\alpha_s\beta_s} = \frac{1}{2}\ {}^{|}\mathbf{g}_{\alpha_s\beta_s}\ {}_{s}^{|}\widehat{\mathbf{R}}sc + {}_{s}^{|}\lambda\ \ {}^{|}\mathbf{g}_{\alpha_s\beta_s}, \text{ vacuum with shell cosmological constants } {}_{s}^{|}\Lambda_0 \text{ or } {}_{s}^{|}\lambda; \\ {}_{s}^{|}\Lambda(\tau,\ {}^{|}u)\ {}^{|}\mathbf{g}_{\alpha_s\beta_s}, \text{ for polarized constants from geometric flow/string/quantum theories;} \\ {}^{|}\mathbf{Y}_{\beta_s\gamma_s}, \text{ from variational/geometric principles of interactions on } {}_{s}\mathcal{M}; \\ {}^{|}\mathbf{K}_{\beta_s\gamma_s}\lceil\hbar, \kappa s\rceil, \text{ for effective parametric star R-flux corrections, in this work and [6,9,10] .} \end{array} \right. \tag{132}
$$

The phase space gravitational field equations (131) can be written in terms of the LC-connection ${}^{|}\nabla_{\alpha}$ if we consider distortion relations. Imposing additional zero s-torsion conditions,

$$
{}_{s}^{|}\widehat{\mathbf{Z}} = 0, \text{ which is equivalent to } {}_{s}^{|}\widehat{\mathbf{D}}_{|\ {}_{s}^{|}\widehat{\mathcal{T}}=0} = {}^{|}\nabla, \tag{133}
$$

we can extract LC-configurations from canonical nonholonomic classes of solutions. Here we note that various conservation laws can be formulated by extending in s-adapted form the formulas from GR using ${}^{|}\nabla$ on $\mathcal{M}$, for instance, ${}^{|}\nabla({}_{\nabla}^{|}\mathcal{R}ic_{\alpha_s\beta_s} - \frac{1}{2}\ {}^{|}\mathbf{g}_{\alpha_s\beta_s}\ {}_{\nabla}^{|}Rsc) = 0$, but such laws are written in more cumbersome forms if we distort the geometrical objects and this equations in terms of ${}_{s}^{|}\widehat{\mathbf{D}}$. This is a typical property of nonholonomic systems in geometric mechanics and gravity theories. Here we note that notations for nonholonomic constraints of type ${}_{s}^{|}\widehat{\mathbf{D}}_{|\ {}_{s}^{|}\widehat{\mathcal{T}}=0}$ (7).

5.2 Modified dispersion relations and phase space Finsler–Lagrange–Hamilton geometry

For semi-classical commutative MGTs and nonassociative/noncommutative models and in QG, modified dispersion relations, MDRs, can be parameterized locally in the form

$$
c^2\overrightarrow{\mathbf{p}}^2 - E^2 + c^4 m^2 = \varpi(E, \overrightarrow{\mathbf{p}}, m; \ell_P, \kappa, \ldots). \tag{134}
$$

An indicator $\varpi(...)$ encodes in a functional form possible contributions of MGTs which, in general, can be with local Lorentz symmetry violation etc. Such MDRs can be extended to dependencies on 4-d spacetime coordinates $x^i = (x^1, x^2, x^3, x^4 = ct)$ and extended to higher dimensions and for various phase space models. In explicit form, certain classes of $\varpi(...)$ are prescribed following theoretical/ phenomenological/arguments, or determined experimentally. We can compute such values in the framework of certain classical/quantum theories of gravity and matter field interactions. If $\varpi = 0$, the Eq. (134) transforms into a standard quadratic dispersion relation for a relativistic point particle with mass m, energy E, and momentum $p_{\acute{\imath}}$ (for $\acute{\imath} = 1, 2, 3$), when such a particle propagates in a 4-d, flat Minkowski spacetime. A ϖ (134) may involve dependencies on a conventional energy-momentum $p_a = (p_{\acute{\imath}}, p_4 = E)$, $\overrightarrow{\mathbf{p}} = \{p_{\acute{\imath}}\}$, (for $a = 1, 2, 3, 4$), at the Planck scale $\ell_P := \sqrt{\hbar G/c^3} \sim 10^{-33} cm$ and $\kappa := \ell_s^3/6\hbar$ being a string constant, were ℓ_s is a length parameter. In this work, the light velocity is fixed $c = 1$ for a respective system of physical units. Different types of ϖ are considered in various approaches to QG and (non) commutative MGTs, supergravity and (super) string models etc. Here we note that MGTs with MDRs are studied also as candidates for explaining acceleration cosmology and dark energy, DE, and dark matter, DM, physics, see [92–97,99–103] and references therein.

We follow the Assumption 2.1 from [7,8] that the standard gravity and particle physics theories based on the special relativity and Einstein gravity principles and axioms can be generalized from a 4-d Lorentz spacetime manifold V on phase spaces TV or T^*V for total phase space metrics with signature $(+++-;+++-)$,

$$
\begin{aligned}
ds^2 &= g_{\alpha\beta}(x^k)du^\alpha du^\beta \\
&= g_{ij}(x^k)dx^i dx^j + \eta_{ab}dy^a dy^b, \\
&\quad \text{for } y^a \sim dx^a/d\zeta; \text{ and/or} \qquad (135)\\
d\,{}^{\shortmid}s^2 &= {}^{\shortmid}g_{\alpha\beta}(x^k)d\,{}^{\shortmid}u^\alpha d\,{}^{\shortmid}u^\beta \\
&= g_{ij}(x^k)dx^i dx^j + \eta^{ab}dp_a dp_b, \\
&\quad \text{for } p_a \sim dx_a/d\zeta, \qquad (136)
\end{aligned}
$$

when certain curves $x^a(\zeta)$ on V are parameterized by a positive parameter ζ. A pseudo-Riemannian spacetime metric $g = \{g_{ij}(x)\}$ can be a solution of the Einstein equations for the Levi–Civita connection ∇ as we considered in Part I. In diagonal form, the vertical metric η_{ab} and its dual η^{ab} are standard Minkowski metrics, $\eta_{ab} = diag[1, 1, 1, -1]$. The geometric and physical phase space models are elaborated for general frame/coordinate transforms on the base spacetime and in total spaces when the metric structures can be parameterized equivalently by the same h-components of $g_{\alpha\beta}(x^k)$ and ${}^{\shortmid}g_{\alpha\beta}(x^k) = g_{\alpha\beta}(x^k)$, respectively, in quadratic elements (135) and (136).

We suppose that the M-theory and string gravity related MGTs, and quasi-classical limits of QG, can be characterized by MDRs (134) can be modelled with (small) values of and indicator ϖ are described by basic Lorentzian and non-Riemannian total phase space geometries determined by nonlinear quadratic line elements for Lagrange–Hamilton spaces:

$$ds_L^2 = L(x, y), \text{ for models on } TV; \qquad (137)$$
$$d\,{}^{\shortmid}s_H^2 = H(x, p), \text{ for models on } T^*V. \qquad (138)$$

For localized $\varpi = 0$, the nonlinear quadratic line elements (137) and (138) transform correspondingly into linear quadratic elements (135) and (136). For any MDR (134), we can model a Hamilton space $H^{3,1}$ with an Hamilton function $H(p) := E = \pm(c^2\overrightarrow{\mathbf{p}}^2 + c^4m^2 - \varpi(E, \overrightarrow{\mathbf{p}}, m; \ell_P))^{1/2}$. Changing the system of frames/coordinates on total space, we obtain generating functions $H(x, p)$ depending also on spacetime coordinates. We can use for phase space geometric modeling certain general generating functions $H(x, p)$ on T^*V (for simplicity, we can work with regular configurations for nonzero Hessians of H). Here, we note that there are Legendre transforms $L \to H$, with $H(x, p) := p_a y^a - L(x, y)$ and y^a determining solutions of the equations $p_a = \partial L(x, y)/\partial y^a$. In a similar manner, the inverse Legendre transforms can be introduced, $H \to L$, for $L(x, y) := p_a y^a - H(x, p)$ and p_a determining solutions of the equations $y^a = \partial H(x, p)/\partial p_a$. For regular configurations, we can work equivalently both with Lagrange and/or Hamilton spaces. In this section, we provide the formulas for Hamilton type models on phase spaces which admit straightforward generalizations to nonassociative geometry determined by star products and R-flux deformations.

A relativistic 4-d model of Lagrange space $L^{3,1} = (TV, L(x, y))$ on a 8-d phase space with velocity type conventional coordinates $y \approx v$ is defined by a fundamental function (equivalently, generating function) $TV \ni (x, y) \to L(x, y) \in \mathbb{R}$, which is a real valued function, differentiable on $\widetilde{TV} := TV/\{0\}$, for $\{0\}$ being the null section of TV, and continuous on the null section of $\pi : TV \to V$. Such a model is regular if the Hessian (v-metric)

$$\widetilde{g}_{ab}(x, y) := \frac{1}{2}\frac{\partial^2 L}{\partial y^a \partial y^b} \qquad (139)$$

is non-degenerate, i.e. $\det|\widetilde{g}_{ab}| \neq 0$, and of constant signature.

In a similar form, a 4-d relativistic model of Hamilton space $H^{3,1} = (T^*V, H(x, p))$ can be constructed for a fundamental function (equivalently, generating Hamilton function) on a Lorentz manifold V; when $T^*V \ni (x, p) \to H(x, p) \in \mathbb{R}$ is defines by a real valued function being differentiable on $\widetilde{T^*V} := T^*V/\{0^*\}$, for $\{0^*\}$ being the null section of T^*V, and continuous on the null section of $\pi^* : T^*V \to V$. We say that such a model is regular if the Hessian (cv-metric)

$${}^{\shortmid}\widetilde{g}^{ab}(x, p) := \frac{1}{2}\frac{\partial^2 H}{\partial p_a \partial p_b} \qquad (140)$$

is non-degenerate, i.e. $\det|\,{}^{\shortmid}\widetilde{g}^{ab}| \neq 0$, and of constant signature.

The v-metric $\widetilde{g}_{ab}$ and c-metric ${}^{\shortmid}\widetilde{g}^{ab}$ are labeled by tilde " ~ " in order to emphasize that such conventional v-metrics are defined canonically by respective Lagrange and Hamilton generating functions. In general, such functions encode various types of MDRs and contributions for MGTs on phase spaces. General frame/coordinate transforms on TV and/or T^*V allow us to express any "tilde" Hessian in a general quadratic form, respectively as a vertical metric (v-metric), $g_{ab}(x,y)$, and/or co-vertical metric (cv-metric), ${}^{\shortmid}g^{ab}(x,p)$. We can works also with inverse transforms by prescribing any v-metric (cv-metric). In general, a g_{ab} is different from the inverse of ${}^{\shortmid}g^{ab}$, i.e. from ${}^{\shortmid}g_{ab}$. Lagrange and/or Hamilton models on corresponding $\mathcal{M}$ and/or ${}^{\shortmid}\mathcal{M}$ can be always constructed by prescribing certain generating functions $L(x,y)$ and/or $H(x,p)$. We shall omit tildes on geometrical/physical objects if certain formulas hold in general (not only canonical) forms and that will not result in ambiguities.

Let us consider an important geometric example: a relativistic 4-d model of Finsler space is defined as a particular case of Lagrange space when a regular $L=F^2$ is defined by a fundamental (generating) Finsler function subjected to such three conditions: (1) a generating F is a real positive valued function which is differential on $\widetilde{TV}$ and continuous on the null section of the projection $\pi: TV\rightarrow V$; (2) it satisfies also the homogeneity condition $F(x,\lambda y)=|\lambda|F(x,y)$, for a nonzero real value λ; and (3) for such a fundamental function, the Hessian (139) is defined by F^2 in such a form that in any point $(x_{(0)},y_{(0)})$ the v-metric is of signature $(+++-)$. In a similar form, we can define relativistic 4-d Cartan spaces $C^{3,1}=(V,C(x,p))$, when $H=C^2(x,p)$ is 1-homogeneous on co-fiber coordinates p_a. This is a 4-d Finsler space but with momentum like variables. Here we note that general MDRs encoding data for certain general MGTs do not involve certain homogeneity conditions. For the purposes of this work, we shall not use standard examples of Lagrange–Finsler spaces but elaborate on generalized Finsler like models with generating functions $H(x,p)$ which transform (for general nonholonomic frame transforms and distortions of connections) into certain metric-affine theories on ${}^{\shortmid}\mathcal{M}$.

On a Hamiltonian phase space $\widetilde{H}$, we can define canonical symplectic structure $\theta := dp_i\wedge dx^i$ and a unique vector filed $\widetilde{X}_H := \frac{\partial\widetilde{H}}{\partial p_i}\frac{\partial}{\partial x^i}-\frac{\partial\widetilde{H}}{\partial x^i}\frac{\partial}{\partial p_i}$ determined by the equation $i_{\widetilde{X}_H}\theta=-d\widetilde{H}$. In these formulas, $\wedge$ is the antisymmetric product and $i_{\widetilde{X}_H}$ denotes the interior produce defined by $\widetilde{X}_H$. This allows to formulate explicit Hamilton calculus for any functions ${}^1f(x,p)$ and ${}^2f(x,p)$ and respective canonical Poisson structure $\{{}^1f,{}^2f\}:=\theta(\widetilde{X}_{{}^1f},\widetilde{X}_{{}^2f})$. Let us consider how such a structure is related to respective Hamilton–Jacobi configurations. Any regular curve $c(\zeta)$, when $c:\zeta\in[0,1]\rightarrow x^i(\zeta)\subset U\subset V$, for a real parameter ζ, can be lifted to $\pi^{-1}(U)\subset\widetilde{TV}$ defining a curve in the total space, when $\widetilde{c}(\zeta):\zeta\in[0,1]\rightarrow\left(x^i(\zeta),y^i(\zeta)=dx^i/d\zeta\right)$ with a non-vanishing v-vector field $dx^i/d\zeta$. For any effective Hamilton phase space model, one holds the canonical Hamilton–Jacobi equations,

$$\frac{dx^i}{d\zeta}=\{\widetilde{H},x^i\} \text{ and } \frac{dp_a}{d\zeta}=\{\widetilde{H},p_a\}.$$

Equivalent Lagrange and Hamilton models of relativistic phase spaces can be formulated as L-dual effective phase spaces $\widetilde{H}^{3,1}$ and $\widetilde{L}^{3,1}$ described by fundamental generating functions $\widetilde{H}$ and $\widetilde{L}$ which satisfy respectively: the Hamilton–Jacobi equations written equivalently as

$$\frac{dx^i}{d\zeta}=\frac{\partial\widetilde{H}}{\partial p_i} \text{ and } \frac{dp_i}{d\zeta}=-\frac{\partial\widetilde{H}}{\partial x^i},$$

or as the Euler–Lagrange equations,

$$\frac{d}{d\zeta}\frac{\partial\widetilde{L}}{\partial y^i}-\frac{\partial\widetilde{L}}{\partial x^i}=0.$$

The last system of equations, in their turn, are equivalent to the nonlinear geodesic (semi-spray) equations

$$\frac{d^2x^i}{d\zeta^2}+2\widetilde{G}^i(x,p)=0, \text{ for } \widetilde{G}^i=\frac{1}{2}\widetilde{g}^{ij}(\frac{\partial^2\widetilde{L}}{\partial y^i}y^k-\frac{\partial\widetilde{L}}{\partial x^i}), \quad (141)$$

with $\widetilde{g}^{ij}$ being inverse to $\widetilde{g}_{ij}$ (139). These equations state that point like probing particles move not along usual geodesics as on Lorentz manifolds but follow some nonlinear geodesic equations.

Using (141), we can define a canonical N-connection in L-dual form following formulas

$$\begin{aligned}{}^{\shortmid}\widetilde{N} &= \left\{{}^{\shortmid}\widetilde{N}_{ij}:=\frac{1}{2}\left[\{\,{}^{\shortmid}\widetilde{g}_{ij},\widetilde{H}\}-\frac{\partial^2\widetilde{H}}{\partial p_k\partial x^i}\,{}^{\shortmid}\widetilde{g}_{jk}-\frac{\partial^2\widetilde{H}}{\partial p_k\partial x^j}\,{}^{\shortmid}\widetilde{g}_{ik}\right]\right\}\\ &\text{and } \widetilde{N}=\left\{\widetilde{N}^a_i:=\frac{\partial\widetilde{G}}{\partial y^i}\right\}. \end{aligned} \quad (142)$$

For general frame transforms on ${}^{\shortmid}\mathcal{M}$, ${}^{\shortmid}\widetilde{\mathbf{N}}\rightarrow{}^{\shortmid}\mathbf{N}$ (123) and for dyadic constructions, ${}^{\shortmid}\widetilde{\mathbf{N}}\rightarrow{}^{\shortmid}_{s}\mathbf{N}$ (124). Any "tilde" N-connection allows to define respective systems of N–adapted (co) frames of type (126), when

$$\begin{aligned}{}^{\shortmid}\widetilde{\mathbf{e}}_\alpha &= \left({}^{\shortmid}\widetilde{\mathbf{e}}_i=\frac{\partial}{\partial x^i}-{}^{\shortmid}\widetilde{N}_{ia}(x,p)\frac{\partial}{\partial p_a},\ {}^{\shortmid}e^b=\frac{\partial}{\partial p_b}\right), \text{ on } T^*V;\\ {}^{\shortmid}\widetilde{\mathbf{e}}^\alpha &= \left({}^{\shortmid}e^i=dx^i,\ {}^{\shortmid}\mathbf{e}_a=dp_a+{}^{\shortmid}\widetilde{N}_{ia}(x,p)dx^i\right) \text{ on } (T^*V)^*. \end{aligned} \quad (143)$$

There are canonical d-metric structures (d-metrics) $\widetilde{\mathbf{g}}$ and ${}^{\shortmid}\widetilde{\mathbf{g}}$ completely determined by respective data $(\widetilde{L},\ \widetilde{\mathbf{N}};\widetilde{\mathbf{e}}_\alpha,\widetilde{\mathbf{e}}^\alpha;\ \widetilde{g}_{jk},\widetilde{g}^{jk})$ and/or $(\widetilde{H},\ {}^{\shortmid}\widetilde{\mathbf{N}};\ {}^{\shortmid}\widetilde{\mathbf{e}}_\alpha,\ {}^{\shortmid}\widetilde{\mathbf{e}}^\alpha;\ {}^{\shortmid}\widetilde{g}^{ab},\ {}^{\shortmid}\widetilde{g}_{ab})$,

$$\begin{aligned}\widetilde{\mathbf{g}} &= \widetilde{\mathbf{g}}_{\alpha\beta}(x,y)\widetilde{\mathbf{e}}^\alpha\otimes\widetilde{\mathbf{e}}^\beta=\widetilde{g}_{ij}(x,y)e^i\otimes e^j\\ &+\widetilde{g}_{ab}(x,y)\widetilde{\mathbf{e}}^a\otimes\widetilde{\mathbf{e}}^a \text{ and/or} \end{aligned} \quad (144)$$

$$\begin{aligned}{}^{\shortmid}\widetilde{\mathbf{g}} &= {}^{\shortmid}\widetilde{\mathbf{g}}_{\alpha\beta}(x,p)\,{}^{\shortmid}\widetilde{\mathbf{e}}^\alpha\otimes{}^{\shortmid}\widetilde{\mathbf{e}}^\beta={}^{\shortmid}\widetilde{g}_{ij}(x,p)e^i\otimes e^j\\ &+{}^{\shortmid}\widetilde{g}^{ab}(x,p)\,{}^{\shortmid}\widetilde{\mathbf{e}}_a\otimes{}^{\shortmid}\widetilde{\mathbf{e}}_b. \end{aligned} \quad (145)$$

Using frame transforms, the d-metric structures [with tildes] (144) and (145) can be written, respectively, in general d-metric forms without tildes. General vierbein transforms can be parameterized respectively as $e_\alpha = e^{\underline{\alpha}}_{\ \alpha}(u)\partial/\partial u^{\underline{\alpha}}$ and $e^\beta = e^\beta_{\ \underline{\beta}}(u)du^{\underline{\beta}}$, where the local coordinate indices are underlined in order to distinguish them from arbitrary abstract ones. In such formulas, the matrix $e^\beta_{\ \underline{\beta}}$ is inverse to $e^{\underline{\alpha}}_{\ \alpha}$ for orthonormalized bases. For Hamilton like configurations, one writes ${}^{\shortmid}e_\alpha = {}^{\shortmid}e^{\underline{\alpha}}_{\ \alpha}({}^{\shortmid}u)\partial/\partial\, {}^{\shortmid}u^{\underline{\alpha}}$ and ${}^{\shortmid}e^\beta = {}^{\shortmid}e^\beta_{\ \underline{\beta}}({}^{\shortmid}u)d\, {}^{\shortmid}u^{\underline{\beta}}$. It should be noted that there are not used boldface symbols for such transforms because we can consider arbitrary decompositions. In particular, we can consider diadic 2+2+2+2 splitting which is important for decoupling of nonlinear systems of physically important PDEs. With respect to local coordinate frames, any d–metric structures on $\mathbf{TV}$ and/or $\mathbf{T^*V}$ (in particular, we can consider Lagrange and/or Hamilton models) can be written in the form

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}(x,y)\mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{\underline{\alpha}\underline{\beta}}(x,y)du^{\underline{\alpha}} \otimes du^{\underline{\beta}} \text{ and/or}$$
$$ {}^{\shortmid}\mathbf{g} = {}^{\shortmid}\mathbf{g}_{\alpha\beta}(x,p)\, {}^{\shortmid}\mathbf{e}^\alpha \otimes {}^{\shortmid}\mathbf{e}^\beta = {}^{\shortmid}g_{\underline{\alpha}\underline{\beta}}(x,p)d\, {}^{\shortmid}u^{\underline{\alpha}} \otimes d\, {}^{\shortmid}u^{\underline{\beta}},$$

when for respective frame transforms, $\mathbf{g}_{\alpha\beta} = e^{\underline{\alpha}}_{\ \alpha} e^{\underline{\beta}}_{\ \beta} g_{\underline{\alpha}\underline{\beta}}$ and ${}^{\shortmid}\mathbf{g}_{\alpha\beta} = {}^{\shortmid}e^{\underline{\alpha}}_{\ \alpha}\, {}^{\shortmid}e^{\underline{\beta}}_{\ \beta}\, {}^{\shortmid}g_{\underline{\alpha}\underline{\beta}}$, there are obtained corresponding off-diagonal forms:

$$g_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} g_{ij}(x) + g_{ab}(x,y)N^a_i(x,y)N^b_j(x,y) & g_{ae}(x,y)N^e_j(x,y) \\ g_{be}(x,y)N^e_i(x,y) & g_{ab}(x,y) \end{bmatrix} \text{ and/or}$$
$${}^{\shortmid}g_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} {}^{\shortmid}g_{ij}(x) + {}^{\shortmid}g^{ab}(x,p)\, {}^{\shortmid}N_{ia}(x,p)\, {}^{\shortmid}N_{jb}(x,p) & {}^{\shortmid}g^{ae}\, {}^{\shortmid}N_{je}(x,p) \\ {}^{\shortmid}g^{be}\, {}^{\shortmid}N_{ie}(x,p) & {}^{\shortmid}g^{ab}(x,p) \end{bmatrix}. \tag{146}$$

Such formulas get respective labels s for the abstract geometric objects or indices if we work in s-adapted variables. Parameterizations of type (146) are considered, for instance, in the Kaluza–Klein theory and various string theories with extra dimension coordinates. Such metrics are generic off-diagonal if the corresponding N-adapted structure is not integrable, see (127).

The formulas (146) can be written in "tilde" nonholonomic variables which allows us to define, for instance, ${}^{\shortmid}\widetilde{g}_{\underline{\alpha}\underline{\beta}}$ as off-diagonal coefficients of (145) with respective Hessian (140) and semi-spray N-connection (142) on ${}^{\shortmid}\widetilde{\mathcal{M}}$. We can consider arbitrary frame transforms, ${}^{\shortmid}\widetilde{\mathbf{g}}_{\alpha\beta} = {}^{\shortmid}e^{\underline{\alpha}}_{\ \alpha}\, {}^{\shortmid}e^{\underline{\beta}}_{\ \beta}\, {}^{\shortmid}\widetilde{g}_{\underline{\alpha}\underline{\beta}}$ and/or s-adapted ones, ${}^{\shortmid}\widetilde{\mathbf{g}}_{\alpha_s\beta_s} = {}^{\shortmid}e^{\underline{\alpha}}_{\ \alpha_s}\, {}^{\shortmid}e^{\underline{\beta}}_{\ \beta_s}\, {}^{\shortmid}\widetilde{g}_{\underline{\alpha}\underline{\beta}}$, and elaborate a Finsler–Hamilton model of phase space with geometric data $({}^{\shortmid}\widetilde{\mathcal{M}} : {}^{\shortmid}\widetilde{\mathbf{g}}, {}^{\shortmid}\widetilde{\mathbf{N}})$, which corresponding versions of generalized Einstein–Finsler–Hamilton equations can be integrated in general forms for corresponding nonholonomic dyadic configurations $({}^{\shortmid}_{s}\widetilde{\mathcal{M}} : {}^{\shortmid}_{s}\widetilde{\mathbf{g}}, {}^{\shortmid}_{s}\widetilde{\mathbf{N}})$. In the next sections, we shall define corresponding canonical s-connections and related Cartan–Finsler–Hamilton connections in next sections.

5.3 Almost Kähler Lagrange–Hamilton structures on phase spaces

Spacetime models encoding MDRs and formulated as almost Kähler geometries for relativistic phase space Lagrange–Hamilton configurations were reviewed and studied in [7, 8,24]. In those works, further developments for almost symplectic (algebroid, commutative and noncommutative) models of deformation and geometric quantization and various geometric flow theories are reviewed. Originally, the main ideas on almost Kähler realisation of Finsler and Lagrange geometry were proposed in [104–106]. We cite [107] for a review of standard approaches to modern geometric mechanics. In our approach, fundamental (generating) Lagrange–Hamilton functions and MDRs determine canonical models of almost Kähler geometry. Such nonholonomic variables can be introduced in classical and quantum MGTs on (co) tangent bundles.

Let us explain how MDRs (134) and related canonical N-connections $\widetilde{\mathbf{N}}$ and ${}^{\shortmid}\widetilde{\mathbf{N}}$ define respectively canonical almost complex structures $\widetilde{\mathbf{J}}$, on $\mathbf{TV}$, and ${}^{\shortmid}\widetilde{\mathbf{J}}$, on $\mathbf{T^*V}$. We introduce the linear operator $\widetilde{\mathbf{J}}$ acting on $\widetilde{\mathbf{e}}_\alpha = (\widetilde{\mathbf{e}}_i, e_b)$ in the form: $\widetilde{\mathbf{J}}(\mathbf{e}_i) = -\widetilde{\mathbf{e}}_{n+i}$ and $\widetilde{\mathbf{J}}(e_{n+i}) = \widetilde{\mathbf{e}}_i$. This defines globally an almost complex structure ($\widetilde{\mathbf{J}} \circ \widetilde{\mathbf{J}} = -\mathbf{I}$ for $\mathbf{I}$ being the unity matrix) on $\mathbf{TV}$ completely determined by a generating function $\widetilde{L}(x,y)$. Similarly, on $\mathbf{T^*V}$, we can consider a linear operator ${}^{\shortmid}\widetilde{\mathbf{J}}$ acting on ${}^{\shortmid}\mathbf{e}_\alpha = ({}^{\shortmid}\mathbf{e}_i, {}^{\shortmid}e^b)$ (143) when ${}^{\shortmid}\widetilde{\mathbf{J}}({}^{\shortmid}\mathbf{e}_i) = -{}^{\shortmid}e^{n+i}$ and ${}^{\shortmid}\widetilde{\mathbf{J}}({}^{\shortmid}e^{n+i}) = {}^{\shortmid}\mathbf{e}_i$. Such a ${}^{\shortmid}\widetilde{\mathbf{J}}$ defines globally an almost complex structure (${}^{\shortmid}\widetilde{\mathbf{J}} \circ {}^{\shortmid}\widetilde{\mathbf{J}} = -{}^{\shortmid}I$ for ${}^{\shortmid}\mathbf{I}$ being the unity matrix) on $\mathbf{T^*V}$ completely determined by a $\widetilde{H}(x,p)$. We note that $\widetilde{\mathbf{J}}$ and ${}^{\shortmid}\widetilde{\mathbf{J}}$ are standard almost complex structures only for the Euclidean signatures, respectively, on $\mathbf{TV}$ and $\mathbf{T^*V}$. For the pseudo-Euclidean signature, we define such operators in abstract geometric forms which needs additional assumptions and results in a different type of relativistic classical and quantum models. Considering arbitrary frame/coordinate transforms, we write $\mathbf{J}$ and ${}^{\shortmid}\mathbf{J}$.

The canonical Neijenhuis tensor fields determined by MDRs and respective Lagrange and Hamilton phase space structures and canonical almost complex structures $\widetilde{\mathbf{J}}$ on $\mathbf{TV}$ and/or ${}^{\shortmid}\widetilde{\mathbf{J}}$ on $\mathbf{T^*V}$, are considered as curvatures of respective N-connections:

$$\begin{aligned} \widetilde{\mathbf{\Omega}}(\widetilde{X},\widetilde{Y}) &:= -[\widetilde{\mathbf{X}},\widetilde{\mathbf{Y}}] + [\widetilde{\mathbf{J}}\widetilde{\mathbf{X}},\widetilde{\mathbf{J}}\widetilde{\mathbf{Y}}] - \widetilde{\mathbf{J}}[\widetilde{\mathbf{J}}\widetilde{\mathbf{X}},\widetilde{\mathbf{Y}}] - \widetilde{\mathbf{J}}[\widetilde{\mathbf{X}},\widetilde{\mathbf{J}}\widetilde{\mathbf{Y}}] \text{ and/or} \\ {}^{\shortmid}\widetilde{\mathbf{\Omega}}({}^{\shortmid}\widetilde{X},{}^{\shortmid}\widetilde{Y}) &:= -[{}^{\shortmid}\widetilde{\mathbf{X}},{}^{\shortmid}\widetilde{\mathbf{Y}}] + [{}^{\shortmid}\widetilde{\mathbf{J}}\,{}^{\shortmid}\widetilde{\mathbf{X}},{}^{\shortmid}\widetilde{\mathbf{J}}\,{}^{\shortmid}\widetilde{\mathbf{Y}}] - {}^{\shortmid}\widetilde{\mathbf{J}}[{}^{\shortmid}\widetilde{\mathbf{J}}\,{}^{\shortmid}\widetilde{\mathbf{X}},{}^{\shortmid}\widetilde{\mathbf{Y}}] \\ &\quad - {}^{\shortmid}\widetilde{\mathbf{J}}[{}^{\shortmid}\widetilde{\mathbf{X}},{}^{\shortmid}\widetilde{\mathbf{J}}\,{}^{\shortmid}\widetilde{\mathbf{Y}}], \end{aligned} \tag{147}$$

for any d-vectors $\mathbf{X}, \mathbf{Y}$ and ${}^{\shortmid}\mathbf{X}, {}^{\shortmid}\mathbf{Y}$. In general frame/coordinates, the curvatures (147) can be written in general form without tilde values and/or in index form:

$$\Omega^a_{ij} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}, \text{ or}$$

$$ {}^{\shortmid}\Omega_{ija} = \frac{\partial\ {}^{\shortmid}N_{ia}}{\partial x^j} - \frac{\partial\ {}^{\shortmid}N_{ja}}{\partial x^i} + {}^{\shortmid}N_{ib}\frac{\partial\ {}^{\shortmid}N_{ja}}{\partial p_b} - {}^{\shortmid}N_{jb}\frac{\partial\ {}^{\shortmid}N_{ia}}{\partial p_b}. $$

We obtain almost complex structures $\mathbf{J}$ and ${}^{\shortmid}\mathbf{J}$ transform into standard complex structures for Euclidean signatures if $\mathbf{\Omega} = 0$ and/or ${}^{\shortmid}\mathbf{\Omega} = 0$.

The priority of the Lagrange–Hamilton models is that they allow certain equivalent descriptions as almost symplectic geometries which can be used, for instance, for performing deformation quantization. Almost symplectic structures on $\mathbf{T}\mathbf{V}$ and $\mathbf{T}^*\mathbf{V}$ are defined by respective nondegenerate N-adapted 2-forms

$$ \theta = \frac{1}{2}\,\theta_{\alpha\beta}(u)\,\mathbf{e}^\alpha \wedge \mathbf{e}^\beta \text{ and } {}^{\shortmid}\theta = \frac{1}{2}\ {}^{\shortmid}\theta_{\alpha\beta}({}^{\shortmid}u)\ {}^{\shortmid}\mathbf{e}^\alpha \wedge\ {}^{\shortmid}\mathbf{e}^\beta. $$

For instance, in h-c components,

$$ {}^{\shortmid}\theta = \frac{1}{2}\ {}^{\shortmid}\theta_{ij}({}^{\shortmid}u)e^i \wedge e^j + \frac{1}{2}\ {}^{\shortmid}\theta^{ab}({}^{\shortmid}u)\,{}^{\shortmid}\mathbf{e}_a \wedge\ {}^{\shortmid}\mathbf{e}_b. \quad (148) $$

We emphasize that a N-connection ${}^{\shortmid}\mathbf{N}$ defines a unique decomposition of a d-vector ${}^{\shortmid}\mathbf{X} = X^h + {}^{\shortmid}X^{cv}$ on $T^*\mathbf{V}$, for $X^h = h\,{}^{\shortmid}\mathbf{X}$ and ${}^{\shortmid}X^{cv} = cv\ {}^{\shortmid}\mathbf{X}$. Respective projectors h and cv are related to a dual distribution ${}^{\shortmid}\mathbf{N}$ on $\mathbf{V}$, when the properties $h + cv = \mathbf{I}, h^2 = h, (cv)^2 = cv, h \circ cv = cv \circ h = 0$ are satisfied. The almost product operator ${}^{\shortmid}\mathbf{P} := I - 2cv = 2h - I$ acting on ${}^{\shortmid}\mathbf{e}_\alpha = ({}^{\shortmid}\mathbf{e}_i,\ {}^{\shortmid}e^b)$ is defined by formulas

$$ {}^{\shortmid}\mathbf{P}({}^{\shortmid}\mathbf{e}_i) = {}^{\shortmid}\mathbf{e}_i \text{ and } {}^{\shortmid}\mathbf{P}({}^{\shortmid}e^b) = -\ {}^{\shortmid}e^b. $$

In a similar form, a N-connection $\mathbf{N}$ induces an almost product structure $\mathbf{P}$ on $T\mathbf{V}$.

Another important d-geometric operators are the almost tangent (co) ones constructed to satisfy the conditions

$$ \mathbb{J}(\mathbf{e}_i) = e_{4+i} \text{ and } \mathbb{J}(e_a) = 0, \text{ or } \mathbb{J} = \frac{\partial}{\partial y^i} \otimes dx^i; $$
$$ {}^{\shortmid}\mathbb{J}({}^{\shortmid}\mathbf{e}_i) = {}^{\shortmid}g_{ib}\,{}^{\shortmid}e^b \text{ and } {}^{\shortmid}\mathbb{J}\left({}^{\shortmid}e^b\right) = 0, \text{ or } $$
$$ {}^{\shortmid}\mathbb{J} = {}^{\shortmid}g_{ia}\frac{\partial}{\partial p_a} \otimes dx^i. $$

Above introduced d-operators ${}^{\shortmid}\mathbf{P}$, ${}^{\shortmid}\mathbf{J}$ and ${}^{\shortmid}\mathbb{J}$ are respectively $\mathcal{L}$-dual to $\mathbf{P}$, $\mathbf{J}$ and $\mathbb{J}$ if and only if ${}^{\shortmid}\mathbf{N}$ and $\mathbf{N}$ are $\mathcal{L}$-dual and there are constructed respective (co) frame transforms to canonical values $[\,{}^{\shortmid}\widetilde{\mathbf{P}},\ {}^{\shortmid}\widetilde{\mathbf{J}},\ {}^{\shortmid}\widetilde{\mathbb{J}}]$ and $[\widetilde{\mathbf{P}},\widetilde{\mathbf{J}},\widetilde{\mathbb{J}}]$. We can verify by straightforward computations that there are satisfied for pairs of $\mathcal{L}$-dual N-connections $(\mathbf{N},\ {}^{\shortmid}\mathbf{N})$ the properties:

$$ \mathbf{J} = -\delta^a_i e_a \otimes e^i + \delta^i_a \mathbf{e}_i \otimes \mathbf{e}^a, $$
$$ {}^{\shortmid}\mathbf{J} = -\ {}^{\shortmid}g_{ia}\ {}^{\shortmid}e^a \otimes\ {}^{\shortmid}e^i + {}^{\shortmid}g^{ia}\ {}^{\shortmid}\mathbf{e}_i \otimes\ {}^{\shortmid}\mathbf{e}_a $$

correspond to a $\mathcal{L}$-dual pair of almost complex structures $(\mathbf{J},\ {}^{\shortmid}\mathbf{J})$;

$$ \mathbf{P} = \mathbf{e}_i \otimes e^i - e_a \otimes \mathbf{e}^a,\ \ {}^{\shortmid}\mathbf{P} = {}^{\shortmid}\mathbf{e}_i \otimes\ {}^{\shortmid}e^i -\ {}^{\shortmid}e^a \otimes\ {}^{\shortmid}\mathbf{e}_a $$

correspond to a $\mathcal{L}$-dual pair of almost product structures $(\mathbf{P},\ {}^{\shortmid}\mathbf{P})$, and respective almost symplectic structures

$$ \theta = g_{aj}(x,y)\mathbf{e}^a \wedge e^i \text{ and } {}^{\shortmid}\theta = \delta^a_i\ {}^{\shortmid}\mathbf{e}_a \wedge\ {}^{\shortmid}e^i \quad (149) $$

Such operators can be re-written in canonical form by considering canonical N-adapted bases with tilde, for instance, we can write (149) (using frame transforms) as $\widetilde{\theta} = \widetilde{g}_{aj}(x,y)\widetilde{\mathbf{e}}^a \wedge e^i$ and ${}^{\shortmid}\widetilde{\theta} = \delta^a_i\ {}^{\shortmid}\widetilde{\mathbf{e}}_a \wedge\ {}^{\shortmid}e^i$. For instance, a N-connection ${}^{\shortmid}\widetilde{\mathbf{N}}$ and/or ${}^{\shortmid}_{s}\widetilde{\mathbf{N}}$ can be used to N-elongate corresponding frames and define tilde data $\left({}^{\shortmid}\widetilde{\mathbf{J}},\ {}^{\shortmid}\widetilde{\mathbb{J}},\ {}^{\shortmid}\widetilde{\mathbf{P}}, {}^{\shortmid}\widetilde{\theta}\right)$ and re-define them for nonholonomic dyadic splitting of type $\left({}^{\shortmid}_{s}\widetilde{\mathbf{J}},\ {}^{\shortmid}_{s}\widetilde{\mathbb{J}},\ {}^{\shortmid}_{s}\widetilde{\mathbf{P}}, {}^{\shortmid}_{s}\widetilde{\theta}\right)$.

For modeling of (co) tangent bundle N-connection and almost symplectic geometries on (co) tangent bundles of total dimension 8, we can formulate the phase space nonholonomic geometry as an almost Hermitian model of a tangent Lorentz bundle $T\mathbf{V}$ equipped with a N-connection structure $\mathbf{N}$ is defined by a triple $\mathbf{H}^8 = (T\mathbf{V}, \theta, \mathbf{J})$, where $\theta(\mathbf{X},\mathbf{Y}) := \mathbf{g}(\mathbf{J}\mathbf{X},\mathbf{Y})$. On a cotangent Lorentz bundle $T^*\mathbf{V}$ with a (or ${}^{\shortmid}\mathbf{N}$), we can define a triple ${}^{\shortmid}\mathbf{H}^8 = (T^*\mathbf{V},\ {}^{\shortmid}\theta,\ {}^{\shortmid}\mathbf{J})$, where ${}^{\shortmid}\theta({}^{\shortmid}\mathbf{X},\ {}^{\shortmid}\mathbf{Y}) := {}^{\shortmid}\mathbf{g}\left({}^{\shortmid}\mathbf{J}\ {}^{\shortmid}\mathbf{X},\ {}^{\shortmid}\mathbf{Y}\right))$. A space $\mathbf{H}^8$ (or ${}^{\shortmid}\mathbf{H}^8$) is almost Kähler and denoted $\mathbf{K}^8$ if $d\,\theta = 0$ (or ${}^{\shortmid}\mathbf{K}^8$ if $d\,{}^{\shortmid}\theta = 0$). In tilde variables with 1-forms, respectively, defined by a regular Lagrangian L and Hamiltonian H (related by a Legendre transform), we have $\widetilde{\omega} = \frac{\partial L}{\partial y^i}e^i$ and ${}^{\shortmid}\widetilde{\omega} = p_i dx^i$, for which $\widetilde{\theta} = d\widetilde{\omega}$ and ${}^{\shortmid}\widetilde{\theta} = d\,{}^{\shortmid}\widetilde{\omega}$. As a result, we get that $d\widetilde{\theta} = 0$ and $d\ {}^{\shortmid}\widetilde{\theta} = 0$. If such conditions are satisfied, for instance, for ${}^{\shortmid}\widetilde{\mathbf{N}}$, we can consider arbitrary or nonholonomic dyadic structures with ${}^{\shortmid}_{s}\widetilde{\mathbf{N}}$ and $d\ {}_{s}\widetilde{\theta} = 0$ and $d\ {}^{\shortmid}_{s}\widetilde{\theta} = 0$. But such properties do not hold true for arbitrary ${}^{\shortmid}\mathbf{N}$ and ${}^{\shortmid}\theta$, when, in general, $d\ {}^{\shortmid}\theta \neq 0$. We have to introduce a special distribution ${}^{\shortmid}\widetilde{\mathbf{N}}$ determined by a $\widetilde{H}$. Such geometric and physical objects may have certain geometric or physical motivation but can be also prescribed to define a subclass of N-elongated frames ${}^{\shortmid}\widetilde{\mathbf{e}}_\alpha = ({}^{\shortmid}\widetilde{\mathbf{e}}_i,\ {}^{\shortmid}e^b)$.

5.4 Canonical and Lagrange–Hamilton connections and curvatures

We are not able to motivate and elaborate on self-consistent phase space generalizations of the Einstein gravity if we work only with Finsler like metrics determined by nonlinear quadratic forms $L(x,y)$ (137) and/or $H(x,p)$ (138) (or with arbitrary nonholonomic fibered 4+4 structures) and (co) vector/tangent bundles. Viable Lagrange–Hamilton theories encoding (134) can be formulated on $\mathbf{T}\mathbf{V}$ and $\mathbf{T}^*\mathbf{V}$ for additional assumptions on choosing certain types of N-connection and d-connection structures. This is different from the geometry of a (pseudo) Riemannian spacetime $(V, \{g_{ij}(x)\})$, which is completely determined by its metric structure $\{g_{is}\}$; and when the Levi–Civita (LC) connection, ∇, is uniquely defined by $\{g_{is}\}$. In non-relativistic form, there were developed certain approaches related to Finsler geometry and semi-spray configurations [108,109], where the priority was given to the Chern–Rund connection for Finsler spaces. In another class of Finsler gravity models, the priority was given to the Berwald connection in Finsler geometry [110–112]. Such a d-connection is not compatible with the metric structure on the total bundle. This creates a number of ambiguities related to elaborating metric non-compatible Finsler gravity theories (including definition of spinors, definition of compatible motion equations and conservation laws), see explicit results, critics and discussions in Refs. [5–8,24].

5.4.1 N-adapted distortions of s-connections, and s-curvatures

On phase spaces, we can elaborate on geometric models with affine (linear) connections and respective covariant derivatives in certain forms which are, or not, adapted to a chosen N-connection structure. Such constructions can be defined for a general nonholonomic 4+4 splitting.

A distinguished connection (d-connection) can be defined as a linear connection $\mathbf{D}$ on $\mathbf{TV}$ (or ${}^{\shortmid}\mathbf{D}$ on $\mathbf{T}^{*}\mathbf{V}$) which is compatible with the almost product structure $\mathbf{DP} = 0$ (or ${}^{\shortmid}\mathbf{D}\,{}^{\shortmid}\mathbf{P} = 0$). In equivalent form, such a d-connection can be defined to preserve under parallelism a respective N-connection splitting (123), which can be prescribed to be a more special N-connection ${}^{\shortmid}\tilde{\mathbf{N}}$ and then related to a non-holonomic dyadic decomposition (124).

For instance, the coefficients of d-connection ${}^{\shortmid}\mathbf{D}$ can be defined with respect to N-adapted frames (126) using equations

$$
{}^{\shortmid}\mathbf{D}_{\,{}^{\shortmid}\mathbf{e}_{k}}\,{}^{\shortmid}\mathbf{e}_{j} := {}^{\shortmid}L^{i}{}_{jk}\,{}^{\shortmid}\mathbf{e}_{i},\ {}^{\shortmid}\mathbf{D}_{\mathbf{e}_{k}}\,{}^{\shortmid}e^{b} := -\,{}^{\shortmid}\acute{L}_{a}{}^{b}{}_{k}\,{}^{\shortmid}e^{a},\ {}^{\shortmid}\mathbf{D}_{\,{}^{\shortmid}e^{c}}\,{}^{\shortmid}\mathbf{e}_{j} := {}^{\shortmid}\acute{C}^{i}{}_{j}{}^{c}\,{}^{\shortmid}\mathbf{e}_{i},\ {}^{\shortmid}\mathbf{D}_{\,{}^{\shortmid}e^{c}}\,{}^{\shortmid}e^{b} := -\,{}^{\shortmid}C_{a}{}^{bc}\,{}^{\shortmid}e^{a}.
$$

Using respective labeling of h- and v-indices, such equations can be considered for $\mathbf{D}$. In brief, the N-adapted coefficients of d-connections on a cotangent Lorentz bundles can be respectively parameterized

$$
\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \{L^{i}{}_{jk}, \acute{L}^{a}{}_{bk}, \acute{C}^{i}{}_{jc}, C^{a}{}_{bc}\} \text{ and }
{}^{\shortmid}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \{{}^{\shortmid}L^{i}{}_{jk}, {}^{\shortmid}\acute{L}_{a}{}^{b}{}_{k}, {}^{\shortmid}\acute{C}^{i}{}_{j}{}^{c}, {}^{\shortmid}C_{a}{}^{bc}\}, \tag{150}
$$

which allows to define h- and c-splitting of covariant derivatives ${}^{\shortmid}\mathbf{D} = \left({}_{h}^{\shortmid}\mathbf{D}, {}_{v}^{\shortmid}\mathbf{D}\right)$, where ${}_{h}\mathbf{D} = \{L^{i}{}_{jk}, \acute{L}^{a}{}_{bk}\}$, and ${}_{c}^{\shortmid}\mathbf{D} = \{{}^{\shortmid}\acute{C}^{i}{}_{j}{}^{c}, {}^{\shortmid}C_{a}{}^{bc}\}$. For dyadic decompositions, the symbols of geometric objects and/or indices of such objects are labelled additionally with a shell label, for instance, ${}_{cs}^{\shortmid}\mathbf{D} = \{{}^{\shortmid}\acute{C}^{i_{s}}{}_{j_{s}}{}^{c_{s}}, {}^{\shortmid}C_{a_{s}}{}^{b_{s}c_{s}}\}$, when, for instance, $j_{2} = 1, 2, 3, 4$ and $a_{3} = 5, 6$. In such case, we use the terms s-connection instead of d-connection (respectively, s-tensor instead of d-tensor). In result, we formulate 8-d s-adapted phase space variants of (4-d) formulas (17), (18) and, respectively, (26), (27) and (28). All higher dimension formulas on $\mathcal{M}$ and/or ${}^{\shortmid}\mathcal{M}$ can be proven in abstract geometric and s-adapted forms. We omit such details in this work.

5.4.2 Physically important Filsler–Lagrange–Hamilton and canonical d-connections

For elaborating classical and quantum MGTs, and alternative geometrization of mechanics and nonholonomic geometric flow theories [24], we can consider more specials classes of d-connections which can be defined completely by a d-metric/almost symplectic structure determined by a respective Lagrange–Finsler and/or Hamilton–Cartan fundamental form.

The almost Kähler–Lagrange and/or almost Kähler–Hamilton phase spaces (determined, or not, by respective MDRs (134) and a possible $\mathcal{L}$-duality) are characterized respectively by such geometric and physically important linear connections and canonical/almost symplectic connections:

$$
[\mathbf{g}, \mathbf{N}] \simeq [\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}] \simeq [\,\widetilde{\theta} := \widetilde{\mathbf{g}}(\widetilde{\mathbf{J}}\cdot,\cdot), \widetilde{\mathbf{P}}, \widetilde{\mathbf{J}}, \widetilde{\mathbb{J}}] \Longrightarrow
\begin{cases}
\nabla: & \nabla\mathbf{g} = 0;\ \mathbf{T}[\nabla] = 0, & \text{Lagrange LC-connection;}\\
\widehat{\mathbf{D}}: & \widehat{\mathbf{D}}\,\mathbf{g} = 0;\ h\widehat{\mathbf{T}} = 0,\ v\widehat{\mathbf{T}} = 0. & \text{canonical Lagrange d-connection;}\\
\widetilde{\mathbf{D}}: & \widetilde{\mathbf{D}}\widetilde{\theta} = 0, \widetilde{\mathbf{D}}\widetilde{\theta} = 0 & \text{almost symplectic Lagrange d-connection.;}
\end{cases} \tag{151}
$$

and/or

$$
[\,{}^{\shortmid}\mathbf{g}, {}^{\shortmid}\mathbf{N}] \simeq [\,{}^{\shortmid}\widetilde{\mathbf{g}}, {}^{\shortmid}\widetilde{\mathbf{N}}] \simeq [\,{}^{\shortmid}\widetilde{\theta} := {}^{\shortmid}\widetilde{\mathbf{g}}({}^{\shortmid}\widetilde{\mathbf{J}}\cdot,\cdot), {}^{\shortmid}\widetilde{\mathbf{P}}, {}^{\shortmid}\widetilde{\mathbf{J}}, {}^{\shortmid}\widetilde{\mathbb{J}}] \Longrightarrow
\begin{cases}
{}^{\shortmid}\nabla: & {}^{\shortmid}\nabla\,{}^{\shortmid}\mathbf{g} = 0;\ {}^{\shortmid}\mathbf{T}[\,{}^{\shortmid}\nabla] = 0, & \text{Hamilton LC-connection;}\\
{}^{\shortmid}\widehat{\mathbf{D}}: & {}^{\shortmid}\widehat{\mathbf{D}}\,\mathbf{g} = 0;\ h\,{}^{\shortmid}\widehat{\mathbf{T}} = 0,\ cv\,{}^{\shortmid}\widehat{\mathbf{T}} = 0. & \text{canonical Hamilton d-connection;}\\
{}^{\shortmid}\widetilde{\mathbf{D}}: & {}^{\shortmid}\widetilde{\mathbf{D}}\,{}^{\shortmid}\widetilde{\theta} = 0,\ {}^{\shortmid}\widetilde{\mathbf{D}}\,{}^{\shortmid}\widetilde{\theta} = 0 & \text{almost symplectic Hamilton d-connection.}
\end{cases} \tag{152}
$$

The formulas (151) and (152) consist 8-d Lagrange–Hamilton analogs of the 4-d canonical d-connection and LC-connection structure (19). We can use different "tilde, hat, shell, duality" and other type labels for corresponding s-connections $\,^{\shortmid}\widetilde{\mathbf{D}}$, $\,^{\shortmid}\widehat{\mathbf{D}}$, or $\,^{\shortmid}\nabla$. Respective fundamental s-tensor objects are labeled with respective "tilde, hat,...", for instance $_{\nabla}\mathcal{R} = \{ _{\nabla}R^{\alpha}{}_{\beta\gamma\delta}\}$, $\,^{\shortmid}\widehat{\mathcal{R}} = \{\,^{\shortmid}\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta}\}$, $\,^{\shortmid}\widetilde{\mathcal{R}} = \{\,^{\shortmid}\widetilde{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta}\}$ etc. Such s-tensors can be related via distortion relations and nonholonomic frame transforms. To derive exact and parametric solutions in certain geometric flow and MGTs is important to transform, for instance, a Ricci tensor $_{\nabla}R_{\ \beta\gamma}$, or d-tensor $\,^{\shortmid}\widetilde{\mathbf{R}}_{\ \beta\gamma}$, into a $\,^{\shortmid}\widehat{\mathbf{R}}_{\ \beta\gamma}$. The priority of the canonical d-connection $\,^{\shortmid}\widehat{\mathbf{D}}$ is that it can be written in s-form as a $\,^{\shortmid}_{s}\widehat{\mathbf{D}}$ when $\,^{\shortmid}\widehat{\mathbf{R}}_{\ \beta\gamma} \rightarrow \,^{\shortmid}\widehat{\mathbf{R}}_{\ \beta_s\gamma_s}$. In dyadic form, corresponding physically important equations can be decoupled for certain general off-diagonal metrics (quasi-stationary, or locally anisotropic ones). Details on such transforms are provided in (nonassociative) Finsler–Lagrange–Hamilton forms in Refs. [5–8,24] for respective generalizations of the Finsler geometry (in relativistic Lagrange–Hamilton forms and dropping the condition of homogeneity used in Finsler geometry).

It is well-known Chern's definition [113,114] that Finsler geometry is an example of geometry when the assumption on quadratic linear elements is dropped. But this is not enough for constructing physically viable Finser generalizations of the Einstein gravity theory. We need certain additional assumptions for elaborating self-consistent geometric constructions determined by nonlinear quadratic line elements. Here we note that the first self-consistent model of Finsler geometry (with local geometric constructions with generalized metric, N-connection and d-connection structures, and associated N-frames) was elaborated by E.Cartan [115], see and citations therein. In those works, there were defined thee coordinate transforms of nonlinear and linear connections. The original constructions with nonlinear quadratic elements were elaborated in the famous habilitation thesis of B. Riemann [116], but E. Cartan introduced the term of Finsler geometry using the original work [117] and completing the Finsler geometry with the concepts of N-connection and Cartan d-connection. Conventionally, that model of Finsler–Cartan geometry, which is metric compatible, can be described on tangent bundles (or on manifolds with fibred structure) by a triple of fundamental geometric structures $(F : \widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{D}})$.

Such constructions can be performed in relativistic forms on a phase space $\widetilde{\mathcal{M}}$ and re-defined in equivalent dual form on $\,^{\shortmid}\widetilde{\mathcal{M}}$ if we use momentum like coordinates. J. Kern [106] defined the Lagrange geometry as a model Finsler geometry without homogeneity conditions, which also can be redefined (using more sophisticate geometric constructions) on $\,^{\shortmid}\mathcal{M}$ as respective Hamilton and Cartan (phase) spaces.

5.4.3 Distortion s-tensors and curvature and Ricci s-tensors

There are unique distortion relations for any type of prescribed canonical d-connection, or the Cartan d-connection, and LC-connection:

$$\begin{aligned}
&\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}, \widetilde{\mathbf{D}} = \nabla + \widetilde{\mathbf{Z}}, \text{ and } \widehat{\mathbf{D}} = \widetilde{\mathbf{D}} + \mathbf{Z}, \text{ determined by } (\mathbf{g}, \mathbf{N});\\
&\quad \text{and } {}_{s}\widehat{\mathbf{D}} = \widetilde{\mathbf{D}} + {}_{s}\mathbf{Z}, \text{ determined by } ({}_{s}\mathbf{g}, {}_{s}\mathbf{N});\\
&\,^{\shortmid}\widehat{\mathbf{D}} = \,^{\shortmid}\nabla + \,^{\shortmid}\widehat{\mathbf{Z}}, \,^{\shortmid}\widetilde{\mathbf{D}} = \,^{\shortmid}\nabla + \,^{\shortmid}\widetilde{\mathbf{Z}}, \text{ and } \,^{\shortmid}\widehat{\mathbf{D}} = \,^{\shortmid}\widetilde{\mathbf{D}} + \,^{\shortmid}\mathbf{Z},\\
&\quad \text{determined by } (\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\mathbf{N});\\
&\quad \text{and } \,^{\shortmid}_{s}\widehat{\mathbf{D}} = \,^{\shortmid}\widetilde{\mathbf{D}} + \,^{\shortmid}_{s}\mathbf{Z}, \text{ determined by } (\,^{\shortmid}_{s}\mathbf{g}, \,^{\shortmid}_{s}\mathbf{N}),
\end{aligned} \tag{153}$$

for distortion s- and d-tensors $\widehat{\mathbf{Z}}$, $\widetilde{\mathbf{Z}}$, and $\mathbf{Z}$; and $\,^{\shortmid}\widehat{\mathbf{Z}}$, $\,^{\shortmid}\widetilde{\mathbf{Z}}$, and $\,^{\shortmid}\mathbf{Z}$ etc. For such formulas, we can associate some MDRs (134) (this is important for elaborating physical models, but not obligatory for geometric constructions) are characterized by respective canonical and/or almost symplectic distortion d-tensors $\widehat{\mathbf{Z}}[\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}]$, $\widetilde{\mathbf{Z}}[\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}]$, and $\mathbf{Z}[\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}]$, for (almost symplectic) Lagrange models, and $\,^{\shortmid}\widehat{\mathbf{Z}}[\,^{\shortmid}\widetilde{\mathbf{g}}, \,^{\shortmid}\widetilde{\mathbf{N}}]$, $\,^{\shortmid}\widetilde{\mathbf{Z}}[\,^{\shortmid}\widetilde{\mathbf{g}}, \,^{\shortmid}\widetilde{\mathbf{N}}]$, and $\,^{\shortmid}\mathbf{Z}[\,^{\shortmid}\widetilde{\mathbf{g}}, \,^{\shortmid}\widetilde{\mathbf{N}}]$, for (almost symplectic) Hamilton models.

Using distortions relations (153), the phase space geometry can be described in different equivalent forms (up to respective nonholonomic deformations of the linear connection and s-connection structures and nonholonomic frame transforms) by such data

$$\begin{array}{llll}
\text{MDRs} \nearrow & (\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}}) \leftrightarrows (L : \widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{D}}) & \leftrightarrow (\widetilde{\theta}, \widetilde{\mathbf{P}}, \widetilde{\mathbf{J}}, \widetilde{\mathbb{J}}, \widetilde{\mathbf{D}}) & \leftrightarrow [(\mathbf{g}[N], \nabla)], \text{ on } \mathbf{TV}\\
\text{indicator } \varpi & \quad \updownarrow \text{ possible } \mathcal{L}\text{-duality } \& & \text{symplectomorphisms} & \updownarrow \text{ not N-adapted}\\
\text{see (134)} \searrow & (\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\mathbf{N}, \,^{\shortmid}\widehat{\mathbf{D}}) \leftrightarrows (H : \,^{\shortmid}\widetilde{\mathbf{g}}, \,^{\shortmid}\widetilde{\mathbf{N}}, \,^{\shortmid}\widetilde{\mathbf{D}}) & \leftrightarrow (\,^{\shortmid}\widetilde{\theta}, \,^{\shortmid}\widetilde{\mathbf{P}}, \,^{\shortmid}\widetilde{\mathbf{J}}, \,^{\shortmid}\widetilde{\mathbb{J}}, \,^{\shortmid}\widetilde{\mathbf{D}}) & \leftrightarrow [(\,^{\shortmid}\mathbf{g}[\,^{\shortmid}N], \,^{\shortmid}\nabla)], \text{on} \mathbf{T}^{*}\mathbf{V}.
\end{array} \tag{154}$$

We can prove in abstract and N-adapted forms that there are canonical distortion relations for respective Lagrange–Finsler nonholonomic variables:

$$\begin{aligned}
&\widehat{\mathcal{R}}[\mathbf{g}, \widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}] = \mathcal{R}[\mathbf{g}, \nabla] + \widehat{\mathcal{Z}}[\mathbf{g}, \widehat{\mathbf{Z}}],\\
&\,^{\shortmid}\widehat{\mathcal{R}}[\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\widehat{\mathbf{D}} = \,^{\shortmid}\nabla + \,^{\shortmid}\widehat{\mathbf{Z}}] = \,^{\shortmid}\mathcal{R}[\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\nabla] + \,^{\shortmid}\widehat{\mathcal{Z}}[\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\widehat{\mathbf{Z}}],\\
&\widehat{R}ic[\mathbf{g}, \widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}] = Ric[\mathbf{g}, \nabla] + \widehat{Z}ic[\mathbf{g}, \widehat{\mathbf{Z}}],\\
&\,^{\shortmid}\widehat{R}ic[\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\widehat{\mathbf{D}} = \,^{\shortmid}\nabla + \,^{\shortmid}\widehat{\mathbf{Z}}] = \,^{\shortmid}Ric[\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\nabla] + \,^{\shortmid}\widehat{Z}ic[\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\widehat{\mathbf{Z}}],\\
&{}_{s}\widehat{R}[\mathbf{g}, \widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}] = \mathcal{R}[\mathbf{g}, \nabla] + {}_{s}\widehat{Z}[\mathbf{g}, \widehat{\mathbf{Z}}],\\
&\,^{\shortmid}_{s}\widehat{R}[\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\widehat{\mathbf{D}} = \,^{\shortmid}\nabla + \,^{\shortmid}\widehat{\mathbf{Z}}] = \,^{\shortmid}_{s}R[\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\nabla] + \,^{\shortmid}_{s}\widehat{Z}[\,^{\shortmid}\mathbf{g}, \,^{\shortmid}\widehat{\mathbf{Z}}],
\end{aligned}$$

Such distortion formulas can be considered for the almost symplectic Lagrange, or Finsler, d-connections,

$$\widetilde{\mathcal{R}}[\widetilde{\mathbf{g}} \simeq \widetilde{\theta}, \widetilde{\mathbf{D}} = \nabla + \widetilde{\mathbf{Z}}] = \mathcal{R}[\widetilde{\mathbf{g}} \simeq \widetilde{\theta}, \nabla] + \widetilde{\mathcal{Z}}[\widetilde{\mathbf{g}} \simeq \widetilde{\theta}, \widetilde{\mathbf{Z}}],$$
$${}^{\shortmid}\widetilde{\mathcal{R}}[{}^{\shortmid}\widetilde{\mathbf{g}} \simeq {}^{\shortmid}\widetilde{\theta}, {}^{\shortmid}\widetilde{\mathbf{D}} = {}^{\shortmid}\nabla + {}^{\shortmid}\widetilde{\mathbf{Z}}] = {}^{\shortmid}\mathcal{R}[{}^{\shortmid}\widetilde{\mathbf{g}} \simeq {}^{\shortmid}\widetilde{\theta}, {}^{\shortmid}\nabla] + {}^{\shortmid}\widetilde{\mathcal{Z}}[{}^{\shortmid}\mathbf{g} \simeq {}^{\shortmid}\widetilde{\theta}, {}^{\shortmid}\widetilde{\mathbf{Z}}],$$

and any geometric d-objects with "tilde" symbols.

Finally, we note that similar distortions can be defined and computed, for instance, for the Chern d-connection, Berwald d-connection (which are not metric compatible) and any d-connection structure considered in Finsler geometry [113,114] and can be introduced via corresponding nonholonomic deformations of 4-d Lorentz manifold [24]. The physical importance of such d- and s-connections is not clear (for instance, to it is an unsolved problem how to define in a unique and self-consistent form the Dirac equations on nonmetric curved spaces [7,8]) and how to solve respective geometric flow and modified Chern–/Berwald–Einstein equations is a very difficult technical problem. Re-defining the construction in nonholonomic canonical variables with "hat" distortions, we can prove the decoupling property of modified/generalized Einstein equations on $\mathbf{TV}$ and $\mathbf{T}^*\mathbf{V}$.

6 Nonassociative star product deformed Finsler–Hamilton phase space geometric flows

The goal of this section is to extend definition of nonassociative star product introduced in [118,119] for nonassociative phase spaces enabled with canonical nonholonomic and/or Finsler–Lagrange–Hamilton variables. We follow the approach with s-adapted frames [5,6] modifying for nontrivial N-connection structures the constructions from Sect. 2 of [38] and Sect. 2 of [39]. Such constructions provide also nonassociative generalizations of the models of noncommutative gauge gravity and generalized Finsler geometry and their deformation quantization (using N-adapted Moyal–Weyl star products) were considered in [43,120–122].

6.1 Nonassociative star products and nonsymmetric metrics

6.1.1 Definition of star products with Finsler–Hamilton and dyadic N-adapted frames

Using the canonical frame structure ${}^{\shortmid}\widetilde{\mathbf{e}}_\alpha = ({}^{\shortmid}\widetilde{\mathbf{e}}_i, {}^{\shortmid}e^b)$ (143), we can define a nonassociative star product $\widetilde{\star}$ on the phase space $\mathcal{M}$ modelled as Hamilton space

$$f \,\widetilde{\star} q := \cdot[\exp(-\frac{1}{2} i\hbar({}^{\shortmid}\widetilde{\mathbf{e}}_i \otimes {}^{\shortmid}e^i - {}^{\shortmid}e^i \otimes {}^{\shortmid}\widetilde{\mathbf{e}}_i) + \frac{i\ell^4}{12\hbar}\tilde{R}^{ija}(p_a\, {}^{\shortmid}\widetilde{\mathbf{e}}_i \otimes {}^{\shortmid}\widetilde{\mathbf{e}}_j - {}^{\shortmid}\widetilde{\mathbf{e}}_j \otimes p_a\, {}^{\shortmid}\widetilde{\mathbf{e}}_i))] f \otimes q$$
$$= f \cdot q - \frac{i}{2}\hbar[({}^{\shortmid}\widetilde{\mathbf{e}}_i f)({}^{\shortmid}e^i q) - ({}^{\shortmid}e^i f)({}^{\shortmid}\widetilde{\mathbf{e}}_i q)] + \frac{i\ell^4}{6\hbar} R^{ija} p_a({}^{\shortmid}\widetilde{\mathbf{e}}_i f)({}^{\shortmid}\widetilde{\mathbf{e}}_j q) + \cdots, \quad (155)$$

and/or, for ${}^{\shortmid}\widetilde{\mathbf{e}}_\alpha \rightarrow {}^{\shortmid}\mathbf{e}_\alpha = e_\alpha^{\ \beta}({}^{\shortmid}u){}^{\shortmid}\widetilde{\mathbf{e}}_\beta$,

$$f \star q := \cdot[\exp(-\frac{1}{2} i\hbar({}^{\shortmid}\mathbf{e}_i \otimes {}^{\shortmid}e^i - {}^{\shortmid}e^i \otimes {}^{\shortmid}\mathbf{e}_i) + \frac{i\ell^4}{12\hbar} R^{ija}(p_a\, {}^{\shortmid}\mathbf{e}_i \otimes {}^{\shortmid}\mathbf{e}_j - {}^{\shortmid}\mathbf{e}_j \otimes p_a\, {}^{\shortmid}\mathbf{e}_i))] f \otimes q$$
$$= f \cdot q - \frac{i}{2}\hbar[({}^{\shortmid}\mathbf{e}_i f)({}^{\shortmid}e^i q) - ({}^{\shortmid}e^i f)({}^{\shortmid}\mathbf{e}_i q)] + \frac{i\ell^4}{6\hbar} R^{ija} p_a({}^{\shortmid}\widetilde{\mathbf{e}}_i f)({}^{\shortmid}\widetilde{\mathbf{e}}_j q) + \cdots, \quad (156)$$

where $f(x,p)$ and $q(x,p)$ are functions on phase space coordinates; the constant ℓ characterizes the R-flux contributions determined by an antisymmetric $\tilde{R}^{ija}$, or R^{ija}, background in string theory; where $\otimes$ is the tensor product. For small parametric decompositions on $\hbar$ and $\kappa = \ell_s^3/6\hbar$, the tensor products turn into usual multiplications as in the second line (155).

A phase Hamilton space $\widetilde{\mathcal{M}}$ enabled with a star product (156) transforms into a nonassociative Hamilton one $\widetilde{\mathcal{M}}^\star$ (a star label can be used any form "up/low and or left/right", for instance, ${}^\star\widetilde{\mathcal{M}}$). We can re-define $\widetilde{\star}$ in s-adapted form considering frame transforms ${}^{\shortmid}\widetilde{\mathbf{e}}_\alpha \rightarrow {}^{\shortmid}\mathbf{e}_{\alpha_s} = e_{\alpha_s}^{\ \beta}({}^{\shortmid}_s u)\, {}^{\shortmid}\widetilde{\mathbf{e}}_\beta$ for an s-adapted basis ${}^{\shortmid}\mathbf{e}_{\alpha_s}$ (125) used instead of ${}^{\shortmid}\mathbf{e}_\alpha$, with $\widetilde{\star} \rightarrow \star_s$, which allows to work with a s-adapted star product $\star_s$ on ${}_s^\star\mathcal{M}$ as in [5,6]. For coordinate frames, ${}^{\shortmid}\mathbf{e}_\alpha = {}^{\shortmid}\boldsymbol{\partial}_\alpha$, such star products transform into that considered in [38,39]. The priority of $\star_s$is that such a dyadic star product structure allows to decouple and solve in general forms physically important systems of nonlinear PDEs. Nonassociative deformations $\widetilde{\star}$ can be used for elaborating nonassociative generalizations of almost Kaehler–Hamilton geometry (154). To elaborate on models of geometric flows on a (temperature like) parameter τ we can consider families of s-frames ${}^{\shortmid}\widetilde{\mathbf{e}}_{i_s}(\tau)$ and/or ${}^{\shortmid}\widetilde{\mathbf{e}}_i(\tau)$, we obtain define respective flow families s-adapted star products (with respective $\widetilde{\star}(\tau)$, $\star(\tau)$, $\star_s(\tau)$(even the functions f and q may not depend on evolution parameter. Similar τ-dependencies of geometric/physical s-objects and structures have to be defined for evolution on nonassociative and associative geometric models.

In s-adapted form, the star product is written

$$f \star_s q := \cdot\left[\exp\left(-\frac{1}{2} i\hbar({}^{\shortmid}\mathbf{e}_{i_s} \otimes {}^{\shortmid}e^{i_s} - {}^{\shortmid}e^{i_s} \otimes {}^{\shortmid}\mathbf{e}_{i_s}) + \frac{i\ell_s^4}{12\hbar} R^{i_s j_s a_s}(p_{a_s}\, {}^{\shortmid}\mathbf{e}_{i_s} \otimes {}^{\shortmid}\mathbf{e}_{j_a} - {}^{\shortmid}\mathbf{e}_{j_s} \otimes p_{a_s}\, {}^{\shortmid}\mathbf{e}_{i_s})\right)\right] f \otimes q$$
$$= f \cdot q - \frac{i}{2}\hbar[({}^{\shortmid}\mathbf{e}_{i_s} f)({}^{\shortmid}e^{i_s} q) - ({}^{\shortmid}e^{i_s} f)({}^{\shortmid}\mathbf{e}_{i_s} q)] + \frac{i\ell_s^4}{6\hbar} R^{i_s j_s a_s} p_{a_s}({}^{\shortmid}\mathbf{e}_{i_s} f)({}^{\shortmid}\mathbf{e}_{j_s} q) + \cdots. \quad (157)$$

Such R-flux deformations are computed in s-adapted from and allow to us develop nonassociative versions of the AFCDM [6,10,40,41]. Using corresponding adapted frame transforms, we can define equivalent nonassociative star product operations (structures) of type (155), (156) and (157), when $\widetilde{\star} \approx \star \approx \star_s$. We shall use tilde and/or s-labels to emphasize that we work with canonical Lagrange-Hamilton structures and/or shell configurations (which can be defined as double geometric structures with corresponding purposes).

6.1.2 Nonassociative star product symmetric and nonsymmetric d-metrics

For $\widetilde{\mathcal{M}} \rightarrow \widetilde{\mathcal{M}}^\star$, a $\widetilde{\star}$-structure transforms a symmetric metric ${}^{\shortmid}\widetilde{\mathbf{g}}$ (145) into a nonsymmetric one with respective symmetric, ${}^{\shortparallel}_{\star}\widetilde{\mathbf{g}}$, and nonsymmetric, ${}^{\shortparallel}_{\star}\widetilde{\mathfrak{g}}$, components. We can use labels ${}^{\shortparallel}$ instead of ${}^{\shortmid}$ because such nonassociative Finsler–Hamilton metrics may contain complex terms. Nevertheless, the nonholonomic s-structure can be always prescribed in such a form which allow to work with real terms and with quasi-Hopf s-structure determined by a nonassociative algebra $\mathcal{A}_s^\star$ (generalizing the constructions from [39,123]). Such nonassociative d-objects can be represented in the forms

$$
\begin{aligned}
{}^{\shortmid}_{\star}\widetilde{\mathbf{g}} &= {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\alpha\beta}\widetilde{\star}({}^{\shortmid}\widetilde{\mathbf{e}}^{\alpha} \otimes {}^{\shortmid}\widetilde{\mathbf{e}}^{\beta}), \text{ where} \\
&\quad {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}({}^{\shortmid}\widetilde{\mathbf{e}}_{\alpha}, {}^{\shortmid}\widetilde{\mathbf{e}}_{\beta}) = {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\alpha\beta} = {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\beta\alpha} \in \mathcal{A}_s^\star \\
{}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta} &= {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\alpha\beta} - \kappa \mathcal{R}^{\tau\xi}{}_{\alpha}\ {}^{\shortmid}\widetilde{\mathbf{e}}_{\xi}\ {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\beta\tau} \\
&= {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}^{[0]}_{\alpha\beta} + {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}^{[1]}_{\alpha\beta}(\kappa) = {}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}_{\alpha\beta} + {}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}_{\alpha\beta}.
\end{aligned}
$$

In these formulas, we consider that ${}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}_{\alpha\beta}$ (we use a inverse hat and tilde labels to emphasize that we encoded certain Finsler–Hamilton structures and symmetrization after R-flux deformation) is the symmetric part and ${}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}_{\alpha\beta}$ is the anti-symmetric part computed respectively:

$$
\begin{aligned}
{}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}_{\alpha\beta} &:= \frac{1}{2}({}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta} + {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\beta\alpha}) \\
&= {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\alpha\beta} - \frac{\kappa}{2}\left(\mathcal{R}^{\tau\xi}{}_{\beta}\ {}^{\shortmid}\widetilde{\mathbf{e}}_{\xi}\ {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\tau\alpha} + \mathcal{R}^{\tau\xi}{}_{\alpha}\ {}^{\shortmid}\widetilde{\mathbf{e}}_{\xi}\ {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\beta\tau}\right) \qquad (158) \\
&= {}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}^{[0]}_{\alpha\beta} + {}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}^{[1]}_{\alpha\beta}(\kappa), \\
&\quad \text{for } {}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}^{[0]}_{\alpha\beta} = {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\alpha\beta} \text{ and} \\
{}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}^{[1]}_{\alpha\beta}(\kappa) &= -\frac{\kappa}{2}\left(\mathcal{R}^{\tau\xi}{}_{\beta}\ {}^{\shortmid}\widetilde{\mathbf{e}}_{\xi}\ {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\tau\alpha} + \mathcal{R}^{\tau\xi}{}_{\alpha}\ {}^{\shortmid}\widetilde{\mathbf{e}}_{\xi}\ {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\beta\tau}\right); \\
{}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}_{\alpha\beta} &:= \frac{1}{2}({}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta} - {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\beta\alpha}) \\
&= \frac{\kappa}{2}\left(\mathcal{R}^{\tau\xi}{}_{\beta}\ {}^{\shortmid}\widetilde{\mathbf{e}}_{\xi}\ {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\tau\alpha} - \mathcal{R}^{\tau\xi}{}_{\alpha}\ {}^{\shortmid}\widetilde{\mathbf{e}}_{\xi}\ {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\beta\tau}\right) \\
&= {}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}^{[1]}_{\alpha\beta}(\kappa) = \frac{1}{2}({}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}^{[1]}_{\alpha\beta}(\kappa) - {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}^{[1]}_{\beta\alpha}(\kappa)). \qquad (159)
\end{aligned}
$$

We emphasize that choosing a primary nonholonomic distribution when ${}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}^{[0]}_{\alpha\beta} = {}^{\shortmid}\widetilde{\mathbf{g}}_{\alpha\beta}$ (145), we can work on effective nonassociative spaces with symmetric s- and d-metrics, when the nonsymmetric coefficients are induced (and can be recurrently computed on higher orders of κ) by data $(\mathcal{R}^{\tau\xi}{}_{\beta,\xi}, {}^{\shortmid}\widetilde{\mathbf{e}}_{\xi}, {}^{\shortmid}\widetilde{\mathbf{g}}_{\tau\alpha})$ (159). Here we note also that nonsymmetric inverse d-metrics can be parameterized in the form ${}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}^{\alpha\beta} = {}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}^{\alpha\beta} + {}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}^{\alpha\beta}$, but in nonassociative geometry the procedure of computing inverse matrices and metrics is more sophisticate than in the commutative and noncommutative cases, see details in [5,6,38,39]. For nonassociative constructions, ${}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}^{\alpha\beta}$ is not the inverse to ${}^{\shortmid}_{\star}\check{\mathfrak{g}}_{\alpha\beta}$ and ${}^{\shortmid}_{\star}\mathfrak{a}^{\alpha\beta}$ is not inverse to ${}^{\shortmid}_{\star}\mathfrak{a}_{\alpha\beta}$. To model nonassociative geometric flow evolution of symmetric and nonsymmetric components of star product deformed Finsler–Hamilton d-metrics, we have to consider respective families of d-objects and their d-adapted components which can be written, for instance, ${}^{\shortmid}_{\star}\widetilde{\mathbf{g}}(\tau)$, ${}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\beta\alpha}(\tau)$, ${}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta}(\tau) = {}^{\shortmid}_{\star}\check{\widetilde{\mathfrak{g}}}_{\alpha\beta}(\tau) + {}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}_{\alpha_s\beta_s}(\tau)$ etc.

6.2 Star product deformations of geometric s-objects on Finsler–Hamilton phase spaces

In [5,6,10,41], we used the Convention 2 for nonassociative star product deformations of geometric objects on ${}^{\shortmid}\mathcal{M}$ into respective s-objects on ${}^{\shortmid}_{s}\mathcal{M}^\star$. In this subsection, the Convention 2 is generalized in such forms when nonassociative s-objects and be generated by star product deformations of d-objects for Finsler–Hamilton phase spaces (for Finsler–Lagrange configurations the constructions are dual). All such nonassociative and nonholonomic geometric constructions can be performed in abstract geometric form when coefficient formulas are derived with respective d- and/or s-adapted frames.

6.2.1 Canonical and Finsler–Hamilton d-connections and LC-configurations

Using corresponding nonassociative star product (155), (156) and (157) and d-metric (145) , any linear/d-connection/s-connection structure from (153) can be deformed into nonassociative d-/s-geometric objects:

$$
\begin{aligned}
{}^{\shortmid}\widetilde{\mathbf{D}} &= {}^{\shortmid}\nabla + {}^{\shortmid}\widetilde{\mathbf{Z}} \rightarrow {}^{\shortmid}\widetilde{\mathbf{D}}^\star = {}^{\shortmid}\nabla^\star + {}^{\shortmid}\widetilde{\mathbf{Z}}^\star, {}^{\shortmid} \\
\widehat{\mathbf{D}} &= {}^{\shortmid}\widetilde{\mathbf{D}} + {}^{\shortmid}\mathbf{Z} \rightarrow {}^{\shortmid}\widehat{\mathbf{D}}^\star = {}^{\shortmid}\widetilde{\mathbf{D}}^\star + {}^{\shortmid}\mathbf{Z}^\star \quad \text{and} \\
{}^{\shortmid}_{s}\widehat{\mathbf{D}} &= {}^{\shortmid}\widetilde{\mathbf{D}} + {}^{\shortmid}_{s}\mathbf{Z} \rightarrow {}^{\shortmid}_{s}\widehat{\mathbf{D}}^\star = {}^{\shortmid}\widetilde{\mathbf{D}}^\star + {}^{\shortmid}_{s}\mathbf{Z}^\star. \qquad (160)
\end{aligned}
$$

We write conventionally ${}^{\shortmid}_{s}\widetilde{\mathbf{D}} = {}^{\shortmid}_{s}\widehat{\widetilde{\mathbf{D}}}$ (to avoid using double hat and tilde labels if that will not result in ambiguities). Here we note that, in principle, we can re-define the geometric constructions in any convenient "tilde" or "hat" variables, considering that ${}^{\shortmid}\widetilde{\mathbf{D}}$ can be transformed into s-adapted configurations for some nonholonomic s-adapted frame transforms when $({}_{s}^{|}\mathbf{g}, {}_{s}^{|}\mathbf{N}) \approx ({}^{|}\mathbf{g}, {}^{|}\mathbf{N}) \approx ({}^{|}g, {}^{|}\tilde{\tilde{N}})$. Such constructions are important if we want to generate nonassociative Lagrange–Hamilton phase configurations subjected to the conditions that certain models may elaborated for some classes of exact/ parametric solutions.

With respect to N-/s-adapted bases, ${}^{|}\widetilde{\mathbf{e}}_{\alpha}$, ${}^{|}\mathbf{e}_{\alpha}$ or ${}^{|}\mathbf{e}_{\alpha_s}$, we can compute corresponding coefficient formulas for (160). For instance, we can write in coefficient forms

$$ {}^{|}\widetilde{\mathbf{D}} = \{{}^{|}\widetilde{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}({}^{|}u)\} \rightarrow {}^{|}\widetilde{\mathbf{D}}^{\star} = \{{}^{|}\widetilde{\mathbf{\Gamma}}^{\alpha}_{\star}{}_{\beta\gamma}({}^{|}u)\}, \text{ after } $$
$$ {}^{|}\widetilde{\mathbf{e}}_{\alpha} \rightarrow {}^{|}\mathbf{e}_{\alpha_s} = e^{\beta}_{\alpha_s}({}_{s}^{|}u){}^{|}\widetilde{\mathbf{e}}_{\beta}, $$
$$ {}_{s}^{|}\widetilde{\mathbf{D}} = \{{}^{|}\widetilde{\mathbf{\Gamma}}^{\alpha_s}{}_{\beta_s\gamma_s}({}_{s}^{|}u)\} \rightarrow {}_{s}^{|}\widetilde{\mathbf{D}}^{\star} = \{{}^{|}\widetilde{\mathbf{\Gamma}}^{\alpha_s}_{\star}{}_{\beta_s\gamma_s}({}_{s}^{|}u)\}, $$

see details in [5,6] redefined for Finsler–Hamilton configurations in [7,8,24].

The N- and s-adapted metric affine data on phase spaces are transformed under star product transforms as

$$ ({}^{|}\widetilde{\mathbf{N}}, {}^{|}\widetilde{\mathbf{g}}, {}^{|}\widetilde{\mathbf{D}}) \approx ({}_{s}^{|}\widetilde{\mathbf{N}}, {}_{s}^{|}\widetilde{\mathbf{g}}, {}_{s}^{|}\widetilde{\mathbf{D}}) \rightarrow ({}_{s}^{|}\widetilde{\mathbf{N}}, {}_{\star}^{|}\widetilde{\mathfrak{g}} = ({}_{\star}^{|}\breve{\widetilde{\mathfrak{g}}}, {}_{\star}^{|}\widetilde{\mathfrak{a}}), {}^{|}\widetilde{\mathbf{D}}^{\star}) $$
$$ \approx ({}_{s}^{|}\widetilde{\mathbf{N}}, {}_{s}^{|}\widetilde{\mathfrak{g}}^{\star} = ({}_{s}^{|}\breve{\widetilde{\mathfrak{g}}}^{\star}, {}_{s}^{|}\widetilde{\mathfrak{a}}^{\star}), {}_{s}^{|}\widetilde{\mathbf{D}}^{\star}). $$

Such relations define respective nonholonomic and nonassociative Finsler–Hamilton spaces depending on the type of generating functions and star product deformations. For any ${}_{s}^{|}\widetilde{\mathbf{D}}$ and ${}_{s}^{|}\widetilde{\mathbf{D}}^{\star}$, we can define and compute in abstract/coefficient forms the corresponding star deformations of torsion s-tensors, ${}_{s}^{|}\widetilde{\mathcal{T}} \rightarrow {}_{s}^{|}\widetilde{\mathcal{T}}^{\star}$, and Riemann curvature s-tensors, ${}_{s}^{|}\widetilde{\mathcal{R}} \rightarrow {}_{s}^{|}\widetilde{\mathcal{R}}^{\star}$. Originally, such computations in coordinate frames were performed in [38,39] there were considered star product deformations without N-connection structure and for (pseudo) Riemannian data. Corresponding nonassociative geometric objects were defined for $({}^{|}g, {}^{|}\nabla) \rightarrow ({}_{\star}^{|}\mathfrak{g} = ({}_{\star}^{|}\breve{\mathfrak{g}}, {}_{\star}^{|}\mathfrak{a}), {}^{|}\nabla^{\star})$, where ${}^{|}\nabla$ and ${}^{|}\nabla^{\star}$ are respective Levi–Civita, LC, connections. Unfortunately, it is not possible to decouple nonholonomic and nonassociative (modified) Einstein equations using ${}^{|}\nabla$, ${}^{|}\nabla^{\star}$, and (for Finsler–Hamilton gravity theories) for ${}^{|}\widetilde{\mathbf{D}}$, or ${}^{|}\widetilde{\mathbf{D}}^{\star}$.

In [43], there were studied noncommutative black hole solutions for the Cartan–Finsler d-connection but those classes of solutions can't be generalized in a direct form for nonassociative models because R-flux terms induce additional coupling. The main result of [6] consisted in a proof that we can decouple and integrate such important systems of nonlinear PDEs if we use the canonical s-connections, ${}_{s}^{|}\widehat{\mathbf{D}}$ and ${}_{s}^{|}\widehat{\mathbf{D}}^{\star}$. The main idea of the Part II is that we consider certain classes of nonholonomic distributions and frame transforms when ${}^{|}\widetilde{\mathbf{D}}^{\star} \rightarrow {}_{s}^{|}\widehat{\mathbf{D}}^{\star}$, which allows us to construct more general classes of generic off-diagonal solutions. Then we can impose additional nonholonomic constraints on distortion s-tensors ${}_{s}^{|}\widetilde{\mathbf{Z}}$, when ${}_{s}^{|}\widehat{\mathbf{D}}_{|{}_{s}^{|}\widetilde{Z}=0} = {}^{|}\widetilde{\mathbf{D}}^{\star}$, and extract Finsler–Hamilton configurations. Typically, general Finsler geometries are characterized by certain Finsler d-connection structures which involve nontrivial d-torsion and nonmetricity d-tensor fields. LC-configurations are not considered for such models even they can be extracted by constraining respective distortion d-tensors. Star product deformations preserve such properties.

For any nonassociative geometric data which include corresponding frame transforms (we write $\approx$) when $({}^{|}\widetilde{\mathbf{N}} \approx {}_{s}^{|}\mathbf{N}, {}^{|}\widetilde{\mathbf{g}}^{\star} \approx {}_{s}^{|}\mathbf{g}^{\star}, {}_{s}^{|}\widehat{\mathbf{D}}^{\star} = {}_{s}^{|}\widetilde{\mathbf{D}}^{\star} + {}_{s}^{|}\widetilde{\mathbf{Z}}^{\star})$, we can define such canonical s-connection, Cartan–Finsler–Hamilton d-connection, and LC-connection structures,

$$ {}_{s}^{|}\widehat{\mathbf{D}}^{\star} = (h_1{}^{|}\widehat{\mathbf{D}}^{\star}, v_2{}^{|}\widehat{\mathbf{D}}^{\star}, c_3{}^{|}\widehat{\mathbf{D}}^{\star}, c_4{}^{|}\widehat{\mathbf{D}}^{\star}) $$
$$ = {}^{|}\nabla^{\star} + {}_{s}^{|}\widehat{\mathbf{Z}}^{\star} = {}_{s}^{|}\widetilde{\mathbf{D}}^{\star} - {}_{s}^{|}\widetilde{\mathbf{Z}}^{\star}, {}^{|}\widetilde{\mathbf{D}}^{\star} = {}^{|}\widetilde{\nabla}^{\star} + {}^{|}\widetilde{\mathbf{Z}}^{\star}, \quad (161) $$

where the canonical distortion s-tensors ${}_{s}^{|}\widehat{\mathbf{Z}}^{\star}[{}_{s}^{|}\widehat{\mathcal{T}}^{\star}[{}_{s}^{|}\mathbf{N}, {}_{s}^{|}\mathbf{g}^{\star}]]$ is an algebraic functional of the canonical s-torsion ${}_{s}^{|}\widehat{\mathcal{T}}^{\star}$, and this involves additional distortion relations with ${}^{|}\widetilde{\mathbf{Z}}^{\star}$ and ${}^{|}\widetilde{\mathcal{T}}^{\star}$. The corresponding linear connections are defined by the conditions

$$ \begin{array}{c} ({}^{|}\widetilde{\mathbf{g}}^{\star}, {}^{|}\widetilde{\mathbf{N}}) \\ \updownarrow \\ ({}_{s}^{|}\mathbf{g}^{\star}, {}_{s}^{|}\mathbf{N}) \end{array} \rightarrow \left\{ \begin{array}{lll} {}_{\star}^{|}\nabla: & \boxed{{}^{|}\nabla^{\star}\,{}^{|}\widetilde{\mathbf{g}}^{\star} = 0;\ {}_{\nabla}^{|}\mathcal{T}^{\star} = 0} & \text{star LC-connection;} \\ {}^{|}\widetilde{\mathbf{D}}^{\star}: & \boxed{\begin{array}{c}{}^{|}\widetilde{\mathbf{D}}^{\star}\,{}^{|}\widetilde{\mathbf{g}}^{\star} = 0;\ h\,{}^{|}\widetilde{\mathcal{T}}^{\star} = 0, c\,{}^{|}\widetilde{\mathcal{T}}^{\star} = 0, hc\,{}^{|}\widetilde{\mathcal{T}}^{\star} \neq 0 \\ \text{Cartan–Finsler–Hamilton d-connection;}\end{array}} & \\ {}_{s}^{|}\widehat{\mathbf{D}}^{\star}: & \boxed{\begin{array}{c}{}_{s}^{|}\widehat{\mathbf{D}}^{\star}\,{}_{s}^{|}\mathbf{g}^{\star} = 0;\ h_1\,{}^{|}\widehat{\mathcal{T}}^{\star} = 0, v_2\,{}^{|}\widehat{\mathcal{T}}^{\star} = 0, c_3\,{}^{|}\widehat{\mathcal{T}}^{\star} = 0, c_4\,{}^{|}\widehat{\mathcal{T}}^{\star} = 0, \\ h_1v_2\,{}^{|}\widehat{\mathcal{T}}^{\star} \neq 0, h_1c_s\,{}^{|}\widehat{\mathcal{T}}^{\star} \neq 0, v_2c_s\,{}^{|}\widehat{\mathcal{T}}^{\star} \neq 0, c_3c_4\,{}^{|}\widehat{\mathcal{T}}^{\star} \neq 0,\end{array}} & \text{can. s-connect.} \end{array} \right. \quad (162) $$

We note that in the definition of linear connections ${}_{\star}^{|}\nabla, {}^{|}\widetilde{\mathbf{D}}^{\star}$ and ${}_{s}^{|}\widehat{\mathbf{D}}^{\star}$ we use the s-tensor ${}^{|}\widetilde{\mathbf{g}}^{\star} \approx {}_{s}^{|}\mathbf{g}^{\star}$. The coefficient s-adapted formulas are provided in [5,6]. For defining linear connections (162), we can use also a nonsymmetric metric ${}_{\star}^{|}\mathfrak{g}_{\alpha_s\beta_s}$ which can be nonholonomically transformed and constrained to a ${}^{|}\widetilde{\mathbf{g}}^{\star}$ but such a choice results in a more strong coupling of tensor s-objects which does not allow decoupling of physically important systems of nonlinear PDEs.

6.2.2 Convention 2 for Finsler–Hamilton structures

To define and compute geometric and physical objects on a nonassociative phase space ${}^{\star}_{s}\mathcal{M}$ we formulated the Convention 2 (see details in [5,6,10,41]). Here we reformulate that convention in a form including "tilde" variables.

Convention 2FH (for Finsler–Hamiton variables) : The commutative and nonassociative geometric data derived for corresponding star products (155), (156) and (157), when $\widetilde{\star} \approx \star \approx \star_s$ can be expressed in such abstract/symbolic s-adapted forms:

$$\begin{array}{ccc}(\widetilde{\star},\ \widetilde{\mathcal{A}}^{\star},\ {}^{|}\widetilde{\mathbf{g}}^{\star},\ {}^{|}\widetilde{\mathfrak{g}}^{\star},\ {}^{|}\widetilde{\mathbf{N}},\ {}^{|}\widetilde{\mathbf{e}}_{\alpha},\ {}^{|}\widetilde{\mathbf{D}}^{\star}) & \Leftrightarrow & (\widetilde{\star}_s,\ \widetilde{\mathcal{A}}^{\star},\ {}^{|}_{s}\widetilde{\mathbf{g}}^{\star},\ {}_{s}^{|}\widetilde{\mathfrak{g}}^{\star},\ {}_{s}^{|}\widetilde{\mathbf{N}},\ {}^{|}\widetilde{\mathbf{e}}_{\alpha_s},\ {}_{s}^{|}\widetilde{\mathbf{D}}^{\star})\\ \updownarrow & & \updownarrow\\ (\star,\ \mathcal{A}^{\star},\ {}^{|}\mathbf{g}^{\star},\ {}^{|}\mathfrak{g}^{\star},\ {}^{|}\mathbf{N},\ {}^{|}\mathbf{e}_{\alpha},\ {}^{|}\widehat{\mathbf{D}}^{\star})) & & (\star_s,\ \mathcal{A}^{\star}_s,\ {}^{|}_{s}\mathbf{g}^{\star},\ {}_{s}^{|}\mathfrak{g}^{\star},\ {}_{s}^{|}\mathbf{N},\ {}^{|}\mathbf{e}_{\alpha_s},\ {}_{s}^{|}\widehat{\mathbf{D}}^{\star})\\ & \Uparrow & \\ (\widetilde{\mathcal{A}},\ {}^{|}\widetilde{\mathbf{g}},\ {}^{|}\widetilde{\mathfrak{g}},\ {}^{|}\widetilde{\mathbf{N}},\ {}^{|}\widetilde{\mathbf{e}}_{\alpha},\ {}^{|}\widetilde{\mathbf{D}}) & \Leftrightarrow & (\widetilde{\mathcal{A}}^{\star},\ {}_{s}^{|}\widetilde{\mathbf{g}},\ {}_{s}^{|}\widetilde{\mathfrak{g}},\ {}_{s}^{|}\widetilde{\mathbf{N}},\ {}^{|}\widetilde{\mathbf{e}}_{\alpha_s},\ {}_{s}^{|}\widetilde{\mathbf{D}})\\ \updownarrow & & \updownarrow\\ (\mathcal{A},\ {}^{|}\mathbf{g},\ {}^{|}\mathfrak{g},\ {}^{|}\mathbf{N},\ {}^{|}\mathbf{e}_{\alpha},\ {}^{|}\widehat{\mathbf{D}}) & & (\mathcal{A}^{\star}_s,\ {}_{s}^{|}\mathbf{g},\ {}_{s}^{|}\mathfrak{g},\ {}_{s}^{|}\mathbf{N},\ {}^{|}\mathbf{e}_{\alpha_s},\ {}_{s}^{|}\widehat{\mathbf{D}})\end{array} \quad (163)$$

for certain canonical distortions (161).

Following the Convention 2FH, we can define and compute star product deformations of fundamental geometric s-objects,

$$\begin{aligned}
&{}^{|}_{s}\mathcal{T} \rightarrow {}^{|}_{s}\widehat{\mathcal{T}}^{\star} = \{{}^{|}\widehat{\mathbf{T}}^{\alpha_s}_{\star\beta_s\gamma_s}\} \text{ and } {}^{|}_{s}\widetilde{\mathcal{T}} \rightarrow {}^{|}_{s}\widetilde{\mathcal{T}}^{\star} = \{{}^{|}\widetilde{\mathbf{T}}^{\alpha_s}_{\star\beta_s\gamma_s}\},\\
&\quad \text{nonassociative canonical s-torsion};\\
&{}^{|}_{s}\mathcal{R} \rightarrow {}^{|}_{s}\widehat{\mathcal{R}}^{\star} = \{{}^{|}\widehat{\mathbf{R}}^{\star\alpha_s}_{\ \beta_s\gamma_s\delta_s}\} \text{ and },\\
&\quad \text{nonassociative canonical Riemannian s-curvature};\\
&{}^{|}_{s}\mathcal{R}ic \rightarrow {}^{|}_{s}\widehat{\mathcal{R}}ic^{\star} = \{{}^{|}\widehat{\mathbf{R}}^{\star}_{\ \beta_s\gamma_s} := {}^{|}\widehat{\mathbf{R}}^{\star\alpha_s}_{\ \beta_s\gamma_s\alpha_s} \neq {}^{|}\widehat{\mathbf{R}}^{\star}_{\ \gamma_s\beta_s}\} \text{ and}\\
&{}^{|}_{s}\widetilde{\mathcal{R}}ic \rightarrow {}^{|}_{s}\widetilde{\mathcal{R}}ic^{\star} = \{{}^{|}\widetilde{\mathbf{R}}^{\star}_{\ \beta_s\gamma_s} := {}^{|}\widetilde{\mathbf{R}}^{\star\alpha_s}_{\ \beta_s\gamma_s\alpha_s} \neq {}^{|}\widetilde{\mathbf{R}}^{\star}_{\ \gamma_s\beta_s}\},\\
&\quad \text{nonassociative canonical Ricci s-tensor};\\
&{}^{|}_{s}\mathcal{R}sc \rightarrow {}^{|}_{s}\widehat{\mathcal{R}}sc^{\star} = \{{}^{|}\mathbf{g}^{\beta_s\gamma_s}\ {}^{|}\widehat{\mathbf{R}}^{\star}_{\ \beta_s\gamma_s}\} \text{ and}\\
&{}^{|}_{s}\widetilde{\mathcal{R}}sc \rightarrow {}^{|}_{s}\widetilde{\mathcal{R}}sc^{\star} = \{{}^{|}\mathbf{g}^{\beta_s\gamma_s}\ {}^{|}\widetilde{\mathbf{R}}^{\star}_{\ \beta_s\gamma_s}\}\\
&\quad \text{nonassociative canonical Riemannian scalar};\\
&{}^{|}_{s}\mathcal{Q} \rightarrow {}^{|}_{s}\mathcal{Q}^{\star} = \{{}^{|}\widehat{\mathbf{Q}}^{\star}_{\gamma_s\alpha_s\beta_s} = {}^{|}\widehat{\mathbf{D}}^{\star}_{\gamma_s}\ {}^{|}\mathbf{g}^{\star}_{\alpha_s\beta_s}\} \text{ and}\\
&{}^{|}_{s}\widetilde{\mathcal{Q}} = 0 \rightarrow {}^{|}_{s}\widetilde{\mathcal{Q}}^{\star} = \{{}^{|}\widetilde{\mathbf{Q}}^{\star}_{\gamma_s\alpha_s\beta_s} = {}^{|}\widetilde{\mathbf{D}}^{\star}_{\gamma_s}\ {}^{|}\widetilde{\mathbf{g}}^{\star}_{\alpha_s\beta_s}\} = 0\\
&\quad \text{zero nonassociative canonical nonmetricity s-tensor}.
\end{aligned} \quad (164)$$

For instance, the nonassociative Riemann s-tensor for Finsler–Hamilton phase geometry ${}^{|}_{s}\widetilde{\mathfrak{R}}^{\star} = \{{}^{|}\widetilde{\mathfrak{R}}^{\star\mu_s}_{\ \alpha_s\beta_s\gamma_s}\}$ from (164) can be defined and computed for the data $({}^{|}_{s}\mathbf{g}^{\star} = \{{}^{|}\mathbf{g}^{\star}_{\alpha_s\beta_s}\}, {}^{|}_{s}\widetilde{\mathbf{D}}^{\star} = \{{}^{|}\widetilde{\Gamma}^{\gamma_s}_{\star\alpha_s\beta_s}\})$ and written in a form with κ-linear decomposition,

$$\begin{aligned}
{}^{|}\widetilde{\mathbf{R}}^{\star\mu_s}_{\ \alpha_s\beta_s\gamma_s} &= {}^{|}_{1}\widetilde{\mathbf{R}}^{\star\mu_s}_{\ \alpha_s\beta_s\gamma_s} + {}^{|}_{2}\widetilde{\mathbf{R}}^{\star\mu_s}_{\ \alpha_s\beta_s\gamma_s}, \text{ where}\\
{}^{|}_{1}\widetilde{\mathbf{R}}^{\star\mu_s}_{\ \alpha_s\beta_s\gamma_s} &= {}^{|}\mathbf{e}_{\gamma_s}\ {}^{|}\widetilde{\Gamma}^{\mu_s}_{\star\alpha_s\beta_s} - {}^{|}\mathbf{e}_{\beta_s}\ {}^{|}\widetilde{\Gamma}^{\mu}_{\star\alpha_s\gamma_s}\\
&\quad + {}^{|}\widetilde{\Gamma}^{\mu_s}_{\star\nu_s\tau_s} \star_s (\delta^{\tau_s}_{\gamma_s}\ {}^{|}\widetilde{\Gamma}^{\nu_s}_{\star\alpha_s\beta_s} - \delta^{\tau_s}_{\beta_s}\ {}^{|}\widetilde{\Gamma}^{\nu_s}_{\star\alpha_s\gamma_s})\\
&\quad + {}^{|}w^{\tau_s}_{\beta_s\gamma_s} \star_s\ {}^{|}\widetilde{\Gamma}^{\mu_s}_{\star\alpha_s\tau_s},\\
{}^{|}_{2}\widetilde{\mathbf{R}}^{\star\mu_s}_{\ \alpha_s\beta_s\gamma_s} &= i\kappa\ {}^{|}\widetilde{\Gamma}^{\mu_s}_{\star\nu_s\tau_s} \star_s (\mathcal{R}^{\tau_s\xi_s}_{\quad\gamma_s}\ {}^{|}\mathbf{e}_{\xi_s}\ {}^{|}\widetilde{\Gamma}^{\nu_s}_{\star\alpha_s\beta_s}\\
&\quad - \mathcal{R}^{\tau_s\xi_s}_{\quad\beta_s}\ {}^{|}\mathbf{e}_{\xi_s}\ {}^{|}\widetilde{\Gamma}^{\nu_s}_{\star\alpha_s\gamma_s}).
\end{aligned} \quad (165)$$

Such formulas are provided in abstract from for LC-configurations in [38,39] and generalized for nonholonomic canonical s-connections in [5,6,10,40,41]. The abstract and s-adapted formulas from those papers can be redefined for "tilde" s-objects and when there is dependence on a geometric/ information flow τ-parameter.

6.2.3 Parametric decomposition of fundamental d-objects on Finsler–Hamilton phase spaces

Hereafter, for simplicity, we shall omit s-labels and s-indices for the geometric objects with tilde considering that s-adapted constructions can be always performed using corresponding s-frames and canonical distortions. In this subsection, we explain how κ-parametric decompositions of fundamental geometric d-objects in Finsler–Hamilton phase space geometry can be derived from respective decompositions of d-metrics and canonical d-connections.

We can consider a parametric decomposition of the star Cartan–Hamilton d-connection ${}^{|}\widetilde{\mathbf{D}}^{\star}$ (152)

$$\begin{aligned}
{}^{|}\widetilde{\Gamma}^{\gamma}_{\star\alpha\beta} &= {}^{|}_{[0]}\widetilde{\Gamma}^{\gamma}_{\star\alpha\beta} + i\kappa\ {}^{|}_{[1]}\widetilde{\Gamma}^{\gamma}_{\star\alpha\beta} = {}^{|}_{[00]}\widetilde{\Gamma}^{\gamma}_{\star\alpha\beta} + {}^{|}_{[01]}\widetilde{\Gamma}^{\gamma}_{\star\alpha\beta}(\hbar)\\
&\quad + {}^{|}_{[10]}\widetilde{\Gamma}^{\gamma}_{\star\alpha\beta}(\kappa) + {}^{|}_{[11]}\widetilde{\Gamma}^{\gamma}_{\star\alpha\beta}(\hbar\kappa) + O(\hbar^2, \kappa^2 ...).
\end{aligned}$$

Introducing such parametric d-coefficients in (165), we can compute respective parametric decompositions of the nonassociative tilde curvature tensor,

$$\begin{aligned}
{}^{|}\widetilde{\mathbf{R}}^{\mu}_{\star\alpha\beta\gamma} &= {}^{|}_{[00]}\widetilde{\mathbf{R}}^{\mu}_{\star\alpha\beta\gamma} + {}^{|}_{[01]}\widetilde{\mathbf{R}}^{\mu}_{\star\alpha\beta\gamma}(\hbar)\\
&\quad + {}^{|}_{[10]}\widetilde{\mathbf{R}}^{\mu}_{\star\alpha\beta\gamma}(\kappa) + {}^{|}_{[11]}\widetilde{\mathbf{R}}^{\mu}_{\star\alpha\beta\gamma}(\hbar\kappa)\\
&\quad + O(\hbar^2, \kappa^2, ...).
\end{aligned} \quad (166)$$

Contracting the first and forth indices in (166), we define the nonassociative canonical Ricci s-tensor,

$$\begin{aligned}
{}^{|}\widetilde{\mathfrak{R}}ic^{\star} &= {}^{|}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta}\widetilde{\star}({}^{|}\widetilde{\mathbf{e}}^{\alpha_s} \otimes {}^{|}\widetilde{\mathbf{e}}^{\beta_s}), \text{ where}\\
{}^{|}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta} &:= {}^{|}\widetilde{\mathfrak{R}}ic^{\star}({}^{|}\widetilde{\mathbf{e}}_{\alpha}, {}^{|}\widetilde{\mathbf{e}}_{\beta})\\
&= \langle {}^{|}\widetilde{\mathbf{R}}ic^{\star}_{\mu\nu}\widetilde{\star}({}^{|}\widetilde{\mathbf{e}}^{\mu} \otimes {}^{|}\widetilde{\mathbf{e}}^{\nu}), {}^{|}\widetilde{\mathbf{e}}_{\alpha} \otimes {}^{|}\widetilde{\mathbf{e}}_{\beta}\rangle_{\widetilde{\star}},
\end{aligned}$$

when the coefficients can be computed in parametric form:

$$\begin{aligned}
{}^{|}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta} &:= {}^{|}\widetilde{\mathfrak{R}}^{\mu}_{\star\alpha\beta\mu} = {}^{|}_{[00]}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta} + {}^{|}_{[01]}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta}(\hbar)\\
&\quad + {}^{|}_{[10]}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta}(\kappa)\\
&\quad + {}^{|}_{[11]}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta}(\hbar\kappa) + O(\hbar^2, \kappa^2, ...),\\
&\quad \text{where } {}^{|}_{[00]}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta}\\
&= {}^{|}_{[00]}\ {}^{|}\widetilde{\mathfrak{R}}^{\mu}_{\star\alpha\beta\mu},\ {}^{|}_{[01]}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta} = {}^{|}_{[01]}\ {}^{|}\widetilde{\mathfrak{R}}^{\mu}_{\star\alpha\beta\mu},\\
{}^{|}_{[10]}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta} &= {}^{|}_{[10]}\widetilde{\mathfrak{R}}^{\star\mu}_{\ \alpha\beta\mu},\ {}^{|}_{[11]}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta} = {}^{|}_{[11]}\widetilde{\mathfrak{R}}^{\star\mu}_{\ \alpha\beta\mu}.
\end{aligned} \quad (167)$$

Such Ricci d-tensors for are not symmetric for general nonassociative cases even for terms proportional to $\hbar^0$ and/or κ^0.

This is a typical property of nonholonomic configurations and deformations even in GR.

6.2.4 Nonassociative Finsler–Hamilton generalization of the Einstein equations

Considering the inverse d-metric ${}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}^{\mu\nu}$ (it is tedious procedure similar to that first introduced in [39], see also s-adapted formulas in [5,6]) we can contact the indices with ${}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta}$ (167) to define and compute the nonassociative Finsler–Hamilton Ricci scalar curvature:

$$
\begin{aligned}
{}^{\shortmid}\widetilde{\mathbf{R}}sc^{\star} &:= {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}^{\mu\nu}\ {}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{\mu\nu} = \left({}^{\shortmid}_{\star}\breve{\mathfrak{g}}^{\mu\nu} + {}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}^{\mu\nu}\right) \\
&\times \left({}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{(\mu\nu)} + {}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{[\mu\nu]}\right) = {}^{\shortmid}\widetilde{\mathbf{R}}ss^{\star} + {}^{\shortmid}\widetilde{\mathbf{R}}sa^{\star}, \\
&\text{where } {}^{\shortmid}\widetilde{\mathbf{R}}ss^{\star} =: {}^{\shortmid}_{\star}\breve{\mathfrak{g}}^{\mu\nu}\ {}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{(\mu\nu)} \text{ and } {}^{\shortmid}\widetilde{\mathbf{R}}sa^{\star} \\
&:= {}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}^{\mu\nu}\ {}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{[\mu\nu]},
\end{aligned} \tag{168}
$$

where respective symmetric (...) and anti-symmetric [...] operators are defined using the multiple 1/2. For instance, we define and compute ${}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{\mu\nu} = {}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{(\mu\nu)} + {}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{[\mu\nu]}$.

We can follow abstract geometric principles formulated in [2] but generalizing the constructions for nonholonomic phase spaces. A. Einstein postulated his gravitational field equations in a similar geometric form on pseudo-Riemannian spaces. This allows us to postulate for the Finsler–Cartan–Hamilton phase space data $({}^{\star}\widetilde{\mathcal{M}}, {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}, {}^{\shortmid}\widetilde{\mathbf{D}}^{\star})$ the nonassociative and noncommutative modified vacuum Einstein equations,

$$
{}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{\alpha\beta} - \frac{1}{2}\,{}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta}\ {}^{\shortmid}\widetilde{\mathbf{R}}sc^{\star} = {}^{\shortmid}\lambda\ {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta}. \tag{169}
$$

Such systems of nonlinear of PDEs can't be decoupled and integrated in certain general off-diagonal forms with 4+4 splitting for nonassociative star product deformations. In [43], we constructed nonassociative Finsler BH solutions for the Cartan d-connection. R-flux modifications introduce additional coupling into modified Einstein equations which makes more cumbersome the procedure of finding exact and parametric solutions. Here we note that (169) allows a formulation in (nonassociative and noncommutative) almost Kaehler variables which can be used for deformation quantization of such phase space theories, see [122] and references therein.

We can generate solutions of (169) if we introduce additional dyadic decompositions (${}^{\shortmid}\widetilde{\mathbf{e}}_{\alpha} \to {}^{\shortmid}\mathbf{e}_{\alpha_s} = e^{\beta}_{\ \alpha_s}({}^{\shortmid}_{s}u){}^{\shortmid}\widetilde{\mathbf{e}}_{\beta}$ with ${}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta} \to {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha_s\beta_s}$) and canonical s-distortions, ${}^{\shortmid}\widetilde{\mathbf{D}}^{\star} \to {}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star}$. For such nonholonomic transforms, distortions of d- and s-connections (160) result in distortion relations for respective nonassociative Ricci tensors/d-tensors/s-tensor, which in abstract index form can be expressed as

$$
\begin{aligned}
{}^{\shortmid}\widetilde{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s} &= {}^{\shortmid}\widehat{\widetilde{\mathbf{R}}}ic^{\star}_{\alpha_s\beta_s}[\,{}^{\shortmid}\widetilde{\mathbf{D}}^{\star},\ {}^{\shortmid}_{s}\mathbf{Z}^{\star},\ {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta}] \\
&+ {}^{\shortmid}\widetilde{\mathbf{Z}}ic^{\star}_{\alpha_s\beta_s}[\,{}^{\shortmid}\widetilde{\mathbf{D}}^{\star},\ {}^{\shortmid}_{s}\mathbf{Z}^{\star},\ {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta}],
\end{aligned}
$$

where [...] are used to emphasize that such value are determined as functionals of certain geometric d- and/or s-objects. Both the hat and tilde labels are kept in order to emphasize that we shall use a canonical Ricci s-tensor generated from a Finsler–Hamilton structure. As a result, we can re-write (169) in an equivalent form:

$$
\begin{aligned}
{}^{\shortmid}\widehat{\widetilde{\mathbf{R}}}ic^{\star}_{\alpha_s\beta_s} &= {}^{\shortmid}\widetilde{\Upsilon}^{\star}_{\alpha_s\beta_s}, \text{ where} \\
{}^{\shortmid}\widetilde{\Upsilon}^{\star}_{\alpha_s\beta_s} &= {}^{\shortmid}\lambda\ {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha_s\beta_s} + \frac{1}{2}\,{}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta}\ {}^{\shortmid}\widetilde{\mathbf{R}}sc^{\star} - {}^{\shortmid}\widetilde{\mathbf{Z}}ic^{\star}_{\alpha_s\beta_s}
\end{aligned} \tag{170}
$$

where the effective sources ${}^{\shortmid}\widetilde{\Upsilon}^{\star}_{\alpha_s\beta_s}$ involves a star - scalar functional ${}^{\shortmid}\widetilde{\mathbf{R}}sc^{\star}[\,{}^{\shortmid}\widetilde{\mathbf{D}}^{\star},\ {}^{\shortmid}_{s}\mathbf{Z}^{\star},\ {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta}]$. We do not provide explicit formulas for ${}^{\shortmid}\widetilde{\Upsilon}^{\star}_{\alpha_s\beta_s}$ because we prove in section 7.1.3 that such s-tensors are related via certain nonlinear symmetries to certain effective cosmological constants, ${}_{s}\widetilde{\Lambda}$, when ${}^{\shortmid}\widetilde{\Upsilon}^{\star}_{\alpha_s\beta_s} \to {}_{s}\widetilde{\Lambda}\ {}^{\shortmid}\widetilde{\mathbf{g}}_{\alpha_s\beta_s}$.

The system of nonlinear PDEs (170) can be generalized for nonassociative geometric flows and decoupled and integrated in general parametric form using the AFCDM as it was proven in nonassociative form in [5,6,10,40,41], where we considered a different type of effective and matter field sources. The main assumptions to get explicit off-diagonals decoupling are that the nonholonomic s-adapted frame structure is chosen in such a form that

$$
\begin{aligned}
&{}^{\shortmid}_{\star}\breve{\mathfrak{g}}^{[0]}_{\alpha_s\beta_s} = {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\alpha_s\beta_s} = {}^{\shortmid}\widetilde{\mathbf{g}}_{\alpha_s\beta_s};\ {}^{\shortmid}_{\star}\breve{\mathfrak{g}}^{[1]}_{\beta_s\gamma_s}(\kappa) = 0,\ {}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}^{[0]}_{\alpha_s\beta_s} = 0, \\
&{}^{\shortmid}_{\star}\widetilde{\mathfrak{a}}^{[1]}_{\alpha_s\beta_s} = i\kappa\mathcal{R}^{\tau_s\xi_s}_{\quad\ [\alpha_s}\ {}^{\shortmid}\widetilde{\mathbf{e}}_{|\xi_s}\ {}^{\shortmid}\widetilde{\mathbf{g}}_{\tau_s|\beta_s]}, \text{ and} \\
&{}^{\shortmid}\widetilde{\Upsilon}^{\alpha_s}_{\star\ \beta_s} = [\,{}^{\shortmid}_{1}\widetilde{\Upsilon}(\hbar,\kappa,x^{k_1})\delta^{j_1}_{i_1},\ {}^{\shortmid}_{2}\widetilde{\Upsilon}(\hbar,\kappa,x^{k_1},x^{3})\delta^{a_2}_{b_2},\ {}^{\shortmid}_{3} \\
&\qquad \widetilde{\Upsilon}(\hbar,\kappa,x^{k_2},\ {}^{\shortmid}p_6)\delta^{b_3}_{a_3},\ {}^{\shortmid}_{4}\widetilde{\Upsilon}(\hbar,\kappa,x^{k_3},\ {}^{\shortmid}p_8)\delta^{b_4}_{a_4}].
\end{aligned} \tag{171}
$$

The effective s-sources ${}^{\shortmid}_{s}\widetilde{\Upsilon}$ can be prescribed as generating sources for some classes of off-diagonal solutions or considered in recurrent form for effective parametric sources with coefficients proportional to $\hbar, \kappa$ and $\hbar\kappa$, when ${}^{\shortmid}\widetilde{\Upsilon}_{\beta_s\gamma_s} = {}^{\shortmid}_{[0]}\widetilde{\Upsilon}_{\beta_s\gamma_s} + {}^{\shortmid}_{[1]}\widetilde{\Upsilon}_{\beta_s\gamma_s}\lceil\hbar,\kappa\rceil$ as in [39]. General frame transforms on ${}^{\shortmid}_{s}\widetilde{\mathcal{M}}^{\star}$ transforms (171) into off-diagonal sources encoding nonassociative Finsler–Hamilton data which also are contained in nontrivial N-connection coefficients and respective coefficients of s-metrics.

The system of nonassociative modified Einstein equations on phase spaces does not have a variational proof for general twist product (this problem exists in nonassociative and noncommutative theories when we do not fix a unique differential and integral calculus). In parametric form, we can fix a N-adapted or s-adapted variational calculus on ${}^{\shortmid}_{s}\widetilde{\mathcal{M}}^{\star}$ and then perform a star product deformation procedure (for $\hbar, \kappa$ and $\hbar\kappa$ R-flux deformations). The general decoupling and integration properties of parametric (170) with effective sources of type (171) can be proven for 8-d phase spaces with nonholonomic canonical 2+2+2+2 decompositions as we considered in Sect. 3 for nonholonomic 2+2 splitting on Lorentz spacetime manifolds. In abstract geometric form, such nonassociative and Finsler–Hamilton parametric constructions and generation of higher dimension classes of solution can be performed by geometric analogy and extending with momentum shell variable the dependence of generating and integration function and sources.

6.3 Nonassociative Finsler–Lagrange–Hamilton geometric flows

The theory of nonassociative geometric flows on phase spaces ${}^{\shortmid}\mathcal{M}^{\star}$ and ${}^{\shortmid}_{s}\mathcal{M}^{\star}$ was elaborated in off-diagonally integrable form using canonical s-variables in [10,40,41]. The goal of this subsection is to study models of nonassociative Finsler–Cartan–Hamilton flows in tilde variables for ${}^{\shortmid}_{s}\widetilde{\mathcal{M}}^{\star}$. Such a formulation is important for connecting the AFCDM to theories of deformation quantization and other models of quantum phase space. In tilde variables, the nonassociative Finsler–Hamilton flow theory allows an equivalent formulation in almost symplectic variables (149) when exists deformation quantization procedure outlined in [122]. We show how the abstract geometric and N-adapted formalism can be applied to define cotangent Lorentz bundle generalizations of the Hamilton [124] and Friedan [125]) geometric flow equations and G. Perelman [57] thermodynamics for Ricci flows. Comprehensive mathematical reviews of results on geometric flows of Riemannian and Kahler metrics and related issues on Thurston–Poincaré conjecture (i.e. theorem, after Perelman's proof) are presented in [126–128]. For applications in modern mathematical particle physics, cosmology and quantum information flows, we cite [129–138].

6.3.1 Finsler variables for Perelman's functionals and nonassociative geometric flows

Let us consider a nonassociative star product R-flux deformed phase space ${}^{\shortmid}\widetilde{\mathcal{M}}^{\star}$ enabled with d-objects $[\ {}^{\shortmid}\widetilde{\mathfrak{g}}^{\star},\ {}^{\shortmid}\widetilde{\mathbf{D}}^{\star}]$ for a star product $\star$ structure (155) N-adapted to a nonholonomic (4+4) decomposition which can be also associated to a necessary nonholonomic shell (2+2)+(2+2) decomposition. We follow the **Convention 2FH** (163) with a κ-linear parametric decomposition of nonholonomic structures and geometric d-objects when ${}^{\shortmid}_{\star}\breve{\mathfrak{g}}^{[0]}_{\alpha\beta} = {}^{\shortmid}_{\star}\widetilde{\mathbf{g}}_{\alpha\beta} = {}^{\shortmid}\widetilde{\mathbf{g}}_{\alpha\beta}$ as in (171).

Nonassociative Finsler–Cartan–Hamilton will be modelled for as flows on temperature like parameter τ (when $0 \leq \tau \leq \tau_0$) of d-objects on ${}^{\shortmid}\widetilde{\mathcal{M}}^{\star}$ when in the [0]-approximation, i.e. zero power on κ, are defined flows of volume elements

$$d\ {}^{\shortmid}\widetilde{\mathcal{V}}ol(\tau) = \sqrt{|\ {}^{\shortmid}\widetilde{\mathbf{g}}_{\alpha\beta}\ (\tau)|}\ \widetilde{\delta}^{8}\ {}^{\shortmid}u^{\gamma_s}(\tau). \quad (172)$$

Such a value is computed using N-elongated s-differentials $\widetilde{\delta}^{8}\ {}^{\shortmid}u^{\gamma_s}(\tau)$ which are linear on ${}^{\shortmid}\widetilde{N}_{i_s a_s}(\tau)$ as in ${}^{\shortmid}\widetilde{\mathbf{e}}_i(\tau)$. The nonassociative geometric flow constructions from [10,40,41] can be reformulated for the geometric data $[\ {}^{\shortmid}\widetilde{\mathbf{g}}^{\star}(\tau),\ {}^{\shortmid}\widetilde{\mathbf{D}}^{\star}(\tau)]$, when the Perelman type functionals are postulated:

$$ {}^{\shortmid}\widetilde{\mathcal{F}}^{\star}(\tau) = \int_{{}^{\shortmid}\widetilde{\Xi}} (\ {}^{\shortmid}\widetilde{\mathbf{R}}sc^{\star} + |\ {}^{\shortmid}\widetilde{\mathbf{D}}^{\star}\ {}^{\shortmid}\widetilde{f}|^2)\widetilde{\star} e^{-\ {}^{\shortmid}\widetilde{f}}\ d\ {}^{\shortmid}\widetilde{\mathcal{V}}ol(\tau),\ \text{and} \quad (173)$$

$$ {}^{\shortmid}\widetilde{\mathcal{W}}^{\star}(\tau) = \int_{{}^{\shortmid}\widetilde{\Xi}} (4\pi\tau)^{-4}\ [\tau(\ {}^{\shortmid}\widetilde{\mathbf{R}}sc^{\star} + \sum_{s} |\ {}^{\shortmid}\widetilde{\mathbf{D}}^{\star}\widetilde{\star}\ {}^{\shortmid}\widetilde{f}|)^2 + {}^{\shortmid}\widetilde{f} - 8]\widetilde{\star} e^{-\ {}^{\shortmid}\widetilde{f}}\ d\ {}^{\shortmid}\widetilde{\mathcal{V}}ol(\tau). \quad (174)$$

The 8-d hypersurface integrals for such F- and W-functionals are determined by a volume element (172) and the h-c-normalizing functions ${}^{\shortmid}\widetilde{f}(\tau,\ {}^{\shortmid}u)$ can be stated to satisfy the condition

$$\int_{{}^{\shortmid}\widetilde{\Xi}} {}^{\shortmid}\widetilde{\nu}\ d\ {}^{\shortmid}\widetilde{\mathcal{V}}ol(\tau) := \int_{t_1}^{t_2}\int_{{}^{\shortmid}\widetilde{\Xi}_t}\int_{{}^{\shortmid}\widetilde{\Xi}_E} {}^{\shortmid}\widetilde{\nu}\ d\ {}^{\shortmid}\widetilde{\mathcal{V}}ol(\tau) = 1. \quad (175)$$

In these formulas, where the integration measures ${}^{\shortmid}\widetilde{\nu} = (4\pi\tau)^{-4}\,e^{-\ {}^{\shortmid}\widetilde{f}}$ are parameterized for the h- and c-components, with shell further parameterizations if necessary. For general topological considerations, such conditions may be not considered. We can consider also star-deformations of the volume form when

$$\begin{aligned} e^{-\ {}^{\shortmid}\widetilde{f}} d\ {}^{\shortmid}\widetilde{\mathcal{V}}ol(\tau) &\rightarrow e^{-\ {}^{\shortmid}_{s}\widehat{f}} d\ {}^{\shortmid}\mathcal{V}ol(\tau) \rightarrow e^{-\ {}^{\shortmid}_{s}\widehat{f}} d\ {}^{\shortmid}\mathcal{V}ol^{\star}(\tau) \\ &= e^{-\ {}^{\shortmid}_{s}\widehat{f}}\sqrt{|\ {}^{\shortmid}_{\star}\mathbf{g}_{\alpha_s\beta_s}\ (\tau)|}\delta\ {}^{\shortmid}u^{\gamma_s}(\tau) \\ &\rightarrow e^{-\ {}^{\shortmid}_{s}\widehat{f}} d\ {}^{\shortmid}\mathcal{V}ol^{\star}(\tau) \\ &= e^{-\ {}^{\shortmid}\widetilde{f}}\sqrt{|\ {}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha_s\beta_s}\ (\tau)|}\widetilde{\delta}\ {}^{\shortmid}u^{\gamma_s}(\tau). \end{aligned}$$

Other types of adapted integration measures and nonholonomic s-shells, for instance, involving ${}^{\shortmid}_{s}\mathfrak{g}^{\star} \approx {}^{\shortmid}\widetilde{\mathfrak{g}}^{\star}$. Such transforms can be encoded into respective normalizing functions and adapted to a respective separation of nonsymmetric components of s-metrics for κ-linear parameterizations. .

The nonassociative geometric flow evolution equation of the Finsler–Hamilton data $[\ {}^{\shortmid}\widetilde{\mathbf{g}}^{\star}(\tau),\ {}^{\shortmid}\widetilde{\mathbf{D}}^{\star}(\tau),\ {}^{\shortmid}\widetilde{f}(\tau)]$ are postulated in the form

$$\begin{aligned} \partial_\tau\ {}^{\shortmid}\widetilde{\mathfrak{g}}^{\star}_{\alpha\beta}(\tau) &= -2\ {}^{\shortmid}\widetilde{\mathbf{R}}^{\star}{}_{\alpha\beta}(\tau), \\ \partial_\tau\ {}^{\shortmid}\widetilde{f}(\tau) &= {}^{\shortmid}\widetilde{\mathbf{R}}sc^{\star}(\tau) - {}^{\star}\widetilde{\triangle}(\tau)\widetilde{\star}\ {}^{\shortmid}\widetilde{f}(\tau) \\ &\quad + (\ {}^{\shortmid}\widetilde{\mathbf{D}}^{\star}(\tau)\widetilde{\star}\ {}^{\shortmid}\widetilde{f}(\tau))^2. \end{aligned} \quad (176)$$

In (176), ${}^{\star}\widetilde{\triangle}(\tau) = [\ {}^{\shortmid}\widetilde{\mathbf{D}}^{\star}(\tau)]^2$ are families of the Laplace d-operators and the nonsymmetric components of ${}^{\shortmid}_{\star}\widetilde{\mathfrak{g}}_{\alpha_s\beta_s}(\tau)$ are computed using κ-linear parameterizations (158)–(159). In commutative versions, this system of such nonlinear PDEs can be derived in variational forms from the F- and W-potentials, respectively, (173) and (174) generalizing the proofs provided in [57], see details in monographs [126–128] and, for various nonassociative, nonholonomic non-Riemannian generalizations, [10,40,41]. Applying abstract geometric methods, we can derive (176) as a generalization of the relativistic canonical evolution equations following the **Convention 2FH** (163). Here we note that for κ-linear parametric decompositions such nonlinear geometric evolution equations can be derived in a variational form from κ-linear parameterizations of (173) and (174). Nonassociative geometric flow equations can be also motivated as star product R-flux deformations of a two-dimensional sigma model with beta functions and dilaton field as it was stated by Eqs. (79) and (80) in [134]. Nevertheless, variational proofs are not possible for general twist products as we discussed in details in [10,40]. when abstract geometric and N-adapted methods became very important.

Nonassociative Ricci solitons for the Finsler–Hamilton d-connection ${}^{|}\widetilde{\mathbf{D}}^{\star}$ are defined as self-similar configurations of gradient geometric flows (176) for a fixed parameter τ_0. On ${}^{|}\widetilde{\mathcal{M}}^{\star}$, the Ricci soliton d-equations are of type

$$ {}^{|}\widetilde{\mathbf{R}}^{\star}{}_{\alpha\beta} + {}^{|}\widetilde{\mathbf{D}}^{\star}_{\alpha}\ {}^{|}\widetilde{\mathbf{D}}^{\star}_{\beta}\ {}^{|}_{s}\widetilde{\varpi}({}^{|}u) = {}^{|}_{s}\lambda\ \ {}^{|}_{\star}\widetilde{\mathfrak{g}}_{\alpha\beta}. \quad (177)$$

where ${}^{|}_{s}\widetilde{\varpi}$ is a smooth potential function on every shell $s = 1, 2$ and $\lambda = const$. The the nonassociative Finsler–Hamilton modified Einstein equations (169) consist an example of nonassociative Ricci soliton ones (177).

We emphasize that the nonlinear systems of PDEs (176) and (177) can be decoupled and integrated in general form, applying a generalized AFCDM, after introducing a double nonholonomic splitting, with N-connections and nonholonomic dyadic structures, writing, for instance,

$$\begin{aligned} \partial_{\tau}\ {}^{|}\widetilde{\mathfrak{g}}^{\star}_{\alpha_s\beta_s}(\tau) &= -2\ {}^{|}\widetilde{\mathbf{R}}^{\star}{}_{\alpha_s\beta_s}(\tau), \\ \partial_{\tau}\ {}^{|}\widehat{f}(\tau) &= {}^{|}_{s}\widetilde{\mathbf{R}}sc^{\star}(\tau) - {}^{\star}_{s}\widetilde{\triangle}(\tau)\widetilde{\star}\ {}^{|}\widehat{f}(\tau) \\ &\quad +({}^{|}_{s}\widetilde{\mathbf{D}}^{\star}(\tau)\widetilde{\star}\ {}^{|}\widehat{f}(\tau))^2. \end{aligned} \quad (178)$$

with re-definition of normalizing functions for s-shells, ${}^{|}\widetilde{f}(\tau) \to {}^{|}\widehat{f}(\tau)$.

6.3.2 *Thermodynamic models for nonassociative Finsler–Hamilton flow thermodynamics*

Let us consider such geometric data: a family of d-metrics ${}^{|}\widetilde{\mathbf{g}}_{\alpha\beta}(\tau)$ used for nonassociative star product deformations; a closed hypersurface ${}^{|}\widetilde{\Xi}$ in the nonassociative phase space ${}^{|}\widetilde{M}^{\star} \subset {}^{|}\widetilde{\mathcal{M}}^{\star}$; and the volume form $d\ {}^{|}\widetilde{\mathcal{V}}ol(\tau)$ (172). We can introduce the partition function for nonassociative Finsler–Hamilton phase spaces of dimension $n = 8$,

$$ {}^{|}\widetilde{\mathcal{Z}}^{\star}(\tau) = \exp[\int_{{}^{|}\widetilde{\Xi}}[-\ {}^{|}\widetilde{f} + 4]\ (4\pi\tau)^{-4}\, e^{-\ {}^{|}\widetilde{f}}\ d\ {}^{|}\widetilde{\mathcal{V}}ol(\tau), \quad (179)$$

for which using standard statistical and geometric mechanics computations [10,40,41,57] we can define and compute such thermodynamic variables:

$$\begin{aligned} &\text{average energy},\ {}^{|}\widetilde{\mathcal{E}}^{\star}(\tau) = -\tau^2\int_{{}^{|}\widetilde{\Xi}}(4\pi\tau)^{-4} \\ &\left({}^{|}\widetilde{\mathbf{R}}sc^{\star} + |\ {}^{|}\widetilde{\mathbf{D}}^{\star}\ {}^{|}\widetilde{f}|^2 - \frac{4}{\tau}\right)\widetilde{\star}e^{-\ {}^{|}\widetilde{f}}\ d\ {}^{|}\widetilde{\mathcal{V}}ol(\tau); \\ &\text{entropy},\ {}^{|}\widetilde{\mathcal{S}}^{\star}(\tau) = -\int_{{}^{|}\widetilde{\Xi}}(4\pi\tau)^{-4} \\ &\left(\tau({}^{|}\widetilde{\mathbf{R}}sc^{\star} + |\ {}^{|}\widetilde{\mathbf{D}}^{\star}\ {}^{|}\widetilde{f}|^2) + {}^{|}\widetilde{f} - 8\right)\widetilde{\star}e^{-\ {}^{|}\widetilde{f}}\ d\ {}^{|}\widetilde{\mathcal{V}}ol(\tau); \\ &\text{fluctuation},\ {}^{|}_{s}\widehat{\sigma}^{\star}(\tau) = 2\,\tau^4\int_{{}^{|}\widetilde{\Xi}}(4\pi\tau)^{-4}|\ {}^{|}\widetilde{\mathbf{R}}^{\star}_{\alpha\beta} \\ &+ {}^{|}\widetilde{\mathbf{D}}^{\star}_{\alpha}\ {}^{|}\widetilde{\mathbf{D}}^{\star}_{\beta}\ {}^{|}\widetilde{f} - \frac{1}{2\tau}\ {}^{|}\widetilde{\mathfrak{g}}^{\star}_{\alpha\beta}|^2\widetilde{\star}e^{-\ {}^{|}\widetilde{f}}\ d\ {}^{|}\widetilde{\mathcal{V}}ol(\tau). \end{aligned} \quad (180)$$

These formulas can be derived generalizing in s-adapted form the commutative variational procedure with further twisted product deformations [57,126–128] using ${}^{|}\widetilde{\mathcal{W}}^{\star}(\tau) = -\ {}^{|}\widetilde{\mathcal{S}}^{\star}(\tau)$ (174) and ${}^{|}\widetilde{\mathcal{Z}}^{\star}(\tau)$ (179).

For applications in modern physics, we can consider κ-linear parametric decompositions of nonassociative geometric thermodynamic variables (180). Corresponding formulas can be derived in variational or abstract geometric form using the F- and W-functionals (173) and (174),

$$\begin{aligned} {}^{|}\widetilde{\mathcal{F}}^{\star}_{\kappa}(\tau) &= \int_{{}^{|}\widetilde{\Xi}}({}^{|}\widetilde{\mathbf{R}}sc + {}^{|}\widetilde{\mathbf{K}}sc + |\ {}^{|}\widetilde{\mathbf{D}}\ {}^{|}\widetilde{f}|^2)e^{-\ {}^{|}\widetilde{f}}\ d\ {}^{|}\widetilde{\mathcal{V}}ol(\tau),\ \text{and} \\ {}^{|}\widetilde{\mathcal{W}}^{\star}_{\kappa}(\tau) &= \int_{{}^{|}\widetilde{\Xi}}(4\pi\tau)^{-4}\,[\tau({}^{|}\widetilde{\mathbf{R}}sc + {}^{|}\widetilde{\mathbf{K}}sc + |\ {}^{|}\widetilde{\mathbf{D}}\ {}^{|}\widetilde{f}|^2)^2 \\ &\quad + {}^{|}\widetilde{f} - 8]e^{-\ {}^{|}\widetilde{f}}\ d\ {}^{|}\widetilde{\mathcal{V}}ol(\tau). \end{aligned} \quad (181)$$

In (181), the tilde Ricci scalar splits into two components, ${}^{|}\widetilde{\mathbf{R}}sc^{\star} = {}^{|}\widetilde{\mathbf{R}}sc + {}^{|}\widetilde{\mathbf{K}}sc$, where ${}^{|}\widetilde{\mathbf{K}}sc := {}^{|}_{\star}\widetilde{\mathfrak{g}}^{\mu\nu}\,{}^{|}\widetilde{\mathbf{K}}_{\beta\gamma}\lceil\hbar,\kappa\rceil$ contains the coefficients proportional to $\hbar$ and κ. The normalizing function ${}^{|}\widetilde{f}$ is re-defined to include $\lceil\hbar,\kappa\rceil$-terms from ${}^{|}\widetilde{\mathbf{D}}^{\star} \to {}^{|}\widetilde{\mathbf{D}}$ and other terms including κ-parametric decompositions.

Using (181), we derive such phase geometric flow equations with κ-terms encoding star product R-flux deformations,

$$\begin{aligned} \partial_{\tau}\ {}^{|}\widetilde{\mathbf{g}}_{\alpha\beta}(\tau) &= -2({}^{|}\widetilde{\mathbf{R}}_{\alpha\beta}(\tau) + {}^{|}\widetilde{\mathbf{K}}_{\alpha\beta}(\tau,\lceil\hbar,\kappa\rceil)), \\ \partial_{\tau}\ {}^{|}\widetilde{f}(\tau) &= {}^{|}\widetilde{\mathbf{R}}sc(\tau) + {}^{|}\widetilde{\mathbf{K}}sc(\tau) - \widetilde{\triangle}(\tau)\ {}^{|}\widetilde{f}(\tau) \\ &\quad +({}^{|}\widetilde{\mathbf{D}}(\tau)\ {}^{|}\widetilde{f}(\tau))^2(\tau), \end{aligned} \quad (182)$$

where the Laplace operator $\widetilde{\triangle}$ is constructed from the canonical s-connection ${}^{|}\widetilde{\mathbf{D}}$. For self-similar configurations with $\tau = \tau_0$, the parametric Finsler–Hamilton equations (182) transform into a system of nonlinear PDEs for κ-parametric generalized Finsler–Ricci solitons.

Let us express the parametric Finsler–Hamilton flow equations as a τ-family which similar to modified Einstein equations. We introduce τ -depending sources ${}^{|}\widetilde{\Im}^{\star}_{\alpha\beta}(\tau) =$

$-({}^{|}\widetilde{\mathbf{K}}_{\alpha_s\beta_s}(\tau) + \frac{1}{2}\partial_\tau\ {}^{|}\widetilde{\mathbf{g}}_{\alpha\beta}(\tau))$ and write (182) in the form

$$ {}^{|}\widetilde{\mathbf{R}}ic_{\alpha\beta}(\tau) = {}^{|}\widetilde{\Im}^{\star}_{\alpha\beta}(\tau), \tag{183}$$

which are similar to the modified Einstein equations (169). Using frame transforms ${}^{|}\widetilde{\Im}^{\star}_{\alpha'\beta'} = e^{\alpha_s}_{\alpha'} e^{\beta_s}_{\beta'}\ {}^{|}\widetilde{\Im}^{\star}_{\alpha_s\beta_s}$, the effective sources can be parameterized in the form

$$ {}^{|}\widetilde{\Im}^{\alpha_s}_{\star\ \beta_s}(\tau, {}^{|}u^{\gamma_s}) = [\ {}^{|}_{1}\widetilde{\Im}^{\star}(\kappa,\tau,x^{k_1})\delta^{j_1}_{i_1},\ {}^{|}_{2}\widetilde{\Im}^{\star}(\kappa,\tau,x^{k_1},y^{c_2}) \delta^{a_2}_{b_2},\ {}^{|}_{3}\widetilde{\Im}^{\star}(\kappa,\tau,x^{k_2},p_{c_3})\delta^{b_3}_{a_3},\ {}^{|}_{4}\widetilde{\Im}^{\star}(\kappa,\tau,{}^{|}x^{k_3},p_{c_4})\delta^{b_4}_{a_4}], \tag{184}$$

i.e. ${}^{|}\widetilde{\Im}^{\star}_{\beta_s\gamma_s}(\tau) = diag\{{}^{|}_{s}\widetilde{\Im}^{\star}(\tau)\}$. Prescribed values ${}^{|}_{s}\widetilde{\Im}^{\star}(\tau, {}^{|}u^{\gamma_s})$ imposes a s-shell nonholonomic constraint for τ-derivatives of the metrics s-coefficients $\partial_\tau\ {}^{|}\widetilde{\mathbf{g}}_{\alpha_s\beta_s}(\tau)$. For a fixed τ_0 , the system of nonlinear PDEs transform into a nonholonomic Ricci solution equation (177).

For small parametric deformations and a fixed τ_0, such constraints can be solved in explicit general forms, or allow recurrent parametric computations of the coefficients of s-metrics and s-connection for a corresponding class of solutions. Using effective sources (184), the κ-linear parametric geometric flow equations (183) can be written equivalently as a τ-family of R-flux deformed Einstein equations written in canonical s-variables,

$$ {}^{|}\widetilde{\mathbf{R}}^{\beta_s}_{\ \gamma_s}(\tau) = \delta^{\beta_s}_{\ \gamma_s}\ {}^{|}_{s}\widetilde{\Im}^{\star}(\tau). \tag{185}$$

The data ${}^{|}_{s}\widetilde{\Im}(\tau)^{\star}$ are considered as generating sources for certain re-defined effective sources including distortions , ${}^{|}_{s}\widetilde{\mathbf{D}}^{\star} = {}^{|}_{s}\widehat{\mathbf{D}}^{\star} + {}^{|}_{s}\widetilde{\mathbf{Z}}^{\star}$ when the system (185) can be decoupled and integrated in certain off-diagonal forms using the AFCDM. For 4-d configurations, the proofs were provided in details in the Part I when the constructions can be extended in abstract geometric form on nonassociative 8-d phase spaces.

7 General off-diagonal solutions for nonassociative Finsler–Hamilton flows and examples

The AFCDM allows us to construct general classes of off-diagonal solutions with dependencies on all spacetime and phase space coordinates in various geometric flow and MGTs (which nonassociative, generalized Finsler and other types). Most general the formulas became very cumbersome and we omit such considerations in this work. If a Killing symmetry exists, the procedure of generating exact/parametric solutions became more simple when the extensions of formulas to 8-d phase spaces and 10-d spacetimes consist abstract geometric and s-adapted generalizations of the 4-d proofs and solutions presented in the Part I. The main geometric constructions and formulas are summarized in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 the Appendix.

In this section, we provide the necessary off-diagonal ansatz which allow to generate parametric solutions of nonassociative geometric flow equations (185). We analyze the a class of nonlinear symmetries of such τ-families of nonassociative Finsler–Hamilton phase spaces. Then we study three physically important examples showing how the solutions from when the off-diagonal solutions from [10,40,41] can be modified to encode nonassociative Finsler like geometric data. Such off-diagonal metrics describe respective nonassociative Finsler–Hamilton black ellipsoids, BEs; nonassociative generalized WHs; or locally anisotropic cosmological solutions for τ-evolution and modified Finsler configurations.

7.1 Off-diagonal ansatz and nonlinear symmetries of nonassociative generalized Finsler flows

Technically, it is not possible to decouple in general form the nonassociative Finsler–Hamilton flows (183) with nonholonomic 4+4 splitting. Introducing an additional 2+2+2+2 splitting, when ${}^{|}\widetilde{\mathbf{g}}_{\alpha'\beta'} = e^{\alpha_s}_{\alpha'} e^{\beta_s}_{\beta'}\ {}^{|}\widetilde{\mathbf{g}}_{\alpha_s\beta_s}$, and the distortions ${}^{|}_{s}\widetilde{\mathbf{Z}}^{\star}$ from ${}^{|}_{s}\widetilde{\mathbf{D}}^{\star} = {}^{|}_{s}\widehat{\mathbf{D}}^{\star} + {}^{|}_{s}\widetilde{\mathbf{Z}}^{\star}$ are encoded into ${}^{|}\widetilde{\Im}^{\star}_{\alpha\beta}$ being parameterized in a form $diag\{{}^{|}_{s}\widetilde{\Im}^{\star}(\tau)\}$ (184), we transform the system of nonlinear PDEs in a form (185). This allows us to integrate such equations for respective ansatz with one Killing symmetry.

7.1.1 Off-diagonal ansatz with one Killing symmetry

Quasi-stationary solutions can be constructed using such a s-adapted ansatz

$$\begin{aligned} d\ {}^{|}\widetilde{s}^{2}(\tau) =\ & \widetilde{g}_{i_1}(\tau,x^{k_1})(dx^{i_1})^2 + \widetilde{g}_{a_2}(\tau,x^{i_1},y^3)(\widetilde{\mathbf{e}}^{a_2}(\tau))^2 \\ & + {}^{|}\widetilde{g}^{a_3}(\tau,x^{i_2},p_6)({}^{|}\widetilde{\mathbf{e}}_{a_3}(\tau))^2 \\ & + {}^{|}\widetilde{g}^{a_4}(\tau,{}^{|}x^{i_3},p_7)({}^{|}\widetilde{\mathbf{e}}_{a_4}(\tau))^2, \text{ where} \\ \widetilde{\mathbf{e}}^{a_2}(\tau) =\ & dy^{a_2} + \widetilde{N}^{a_2}_{k_1}(\tau,x^{i_1},y^3)dx^{k_1}, \\ {}^{|}\widetilde{\mathbf{e}}_{a_3}(\tau) =\ & dp_{a_3} + {}^{|}\widetilde{N}_{a_3k_2}(\tau,x^{i_2},p_6)dx^{k_2}, \\ {}^{|}\widetilde{\mathbf{e}}_{a_4}(\tau) =\ & dp_{a_4} + {}^{|}\widetilde{N}_{a_4k_3}(\tau,{}^{|}x^{i_3},p_7)d\ {}^{|}x^{k_3}. \end{aligned} \tag{186}$$

Such a s-metric ${}^{|}\widetilde{\mathbf{g}}_{\alpha_s\beta_s}$ redefined in a coordinate base can't be diagonalized in a finite phase space region by coordinate frame transforms and posses symmetry on a time like Killing vector, i.e. on $\partial_4 = \partial_t$, at least on the shells $s = 1$ and 2. We can prescribe a nonholonomic frame structure, when the coefficients for respective s-metric and d-metrics, ${}^{|}\widetilde{\mathbf{g}}_{\alpha\beta} \approx {}^{|}\widetilde{\mathbf{g}}_{\alpha_s\beta_s}$, and N-connections do not depend on $y^4 = t$. The s-metric (186) contains certain shell Killing symmetries: on ∂_4 for the shell $s = 2$; on ∂_5 for the shell $s = 3$; and on ∂_8 for the shell $s = 4$.

Fixing τ_0 and chosing self-similar configurations, we can generate parametric solutions for nonassociative Finsler–Hamilton modified Einstein equations (169) or nonassociative Ricci soliton equations (177). We can generate various classes of type (186) quasi-stationary, or their time dual locally anisotropic cosmological solutions, so-called rainbow metrics with Killing symmetry on ∂_7 and explicit dependence on an energy type coordinate $p_8 = E$.

A general ansatz for generating off-diagonal solutions of (185) with a phase space Killing symmetry on a ${}^{\shortmid}\partial^{a_4}$, for $a_4 = 7$, or 8; and a spacetime Killing symmetry on ∂_{a_2}, for $a_2 = 3$, or 4, (we distinguish two classes depending on a Killing symmetry on $\partial_4 = \partial_t$, or ∂_3, at least on the shells $s = 1$ and 2) can be written in the form

$$
\begin{aligned}
d\widetilde{s}^2(\tau) &= \widetilde{g}_{i_1}(\tau)(dx^{i_1})^2 + \widetilde{g}_{a_2}(\tau)(\widetilde{\mathbf{e}}^{a_2}(\tau))^2 + {}^{\shortmid}\widetilde{g}^{a_3}(\tau) \\
&\qquad ({}^{\shortmid}\widetilde{\mathbf{e}}_{a_3}(\tau))^2 + {}^{\shortmid}\widetilde{g}^{a_4}(\tau)({}^{\shortmid}\widetilde{\mathbf{e}}_{a_4}(\tau))^2, \text{ where} \\
\widetilde{\mathbf{e}}^{a_2}(\tau) &= dy^{a_2} + \widetilde{N}^{a_2}_{k_1}(\tau)dx^{k_1}, \\
{}^{\shortmid}\widetilde{\mathbf{e}}_{a_3}(\tau) &= dp_{a_3} + {}^{\shortmid}\widetilde{N}_{a_3 k_2}(\tau)dx^{k_2}, \\
{}^{\shortmid}\widetilde{\mathbf{e}}_{a_4}(\tau) &= dp_{a_4} + {}^{\shortmid}\widetilde{N}_{a_4 k_3}(\tau)d\,{}^{\shortmid}x^{k_3},
\end{aligned}
\tag{187}
$$

where s-metric and N-connection coefficients (related to tilde Finsler–Hamilton variables) are parameterized in the form

$\widetilde{g}_{i_1}(\tau, x^{k_1})$ $= e^{\widetilde{\psi}(\hbar,\kappa;\tau,x^{k_1})}$	$\widetilde{g}_{a_2}(\tau, x^{i_1}, y^3)$ $\widetilde{N}^{a_2}_{k_1}(\tau, x^{i_1}, y^3)$ $\widetilde{g}_{a_2}(\tau, x^{i_1}, t)$ $\widetilde{N}^{a_2}_{k_1}(\tau, x^{i_1}, t)$	quasi-stationary locally anisotropic cosmology	${}^{\shortmid}\widetilde{g}^{a_3}(\tau, x^{i_2}, p_5)$ ${}^{\shortmid}\widetilde{N}_{a_3 k_2}(\tau, x^{i_2}, p_5)$ ${}^{\shortmid}\widetilde{g}^{a_3}(\tau, x^{i_2}, p_6)$ ${}^{\shortmid}\widetilde{N}_{a_3 k_2}(\tau, x^{i_2}, p_6)$	${}^{\shortmid}\widetilde{g}^{a_4}(\tau, {}^{\shortmid}x^{i_3}, p_7)$ ${}^{\shortmid}\widetilde{N}_{a_4 k_3}(\tau, {}^{\shortmid}x^{i_3}, p_7)$	fixed $p_8 = E_0$
τ-flows of 2-d Poisson eqs $\partial^2_{11}\widetilde{\psi} + \partial^2_{22}\widetilde{\psi} =$ $2\ {}_1\widetilde{\Im}^{\star}(\hbar, \kappa; \tau, x^{k_1})$	$\widetilde{g}_{a_2}(\tau, x^{i_1}, y^3)$ $\widetilde{N}^{a_2}_{k_1}(\tau, x^{i_1}, y^3)$ $\widetilde{g}_{a_2}(\tau, x^{i_1}, t)$ $\widetilde{N}^{a_2}_{k_1}(\tau, x^{i_1}, t)$	quasi-stationary locally anisotropic cosmology	${}^{\shortmid}\widetilde{g}^{a_3}(\tau, x^{i_2}, p_5)$ ${}^{\shortmid}\widetilde{N}_{a_3 k_2}(\tau, x^{i_2}, p_5)$ ${}^{\shortmid}\widetilde{g}^{a_3}(\tau, x^{i_2}, p_6)$ ${}^{\shortmid}\widetilde{N}_{a_3 k_2}(\tau, x^{i_2}, p_6)$	${}^{\shortmid}\widetilde{g}^{a_4}(\tau, {}^{\shortmid}x^{i_3}, E)$ ${}^{\shortmid}\widetilde{N}_{a_4 k_3}(\tau, {}^{\shortmid}x^{i_3}, E)$	rainbow s-metrics variable $p_8 = E$

The AFCDM method for ansat of type (187) on commutative phase spaces is summarized in Tabels 13, 14, 15, 16 from Appendix. Those formulas are extended on nonassociative Finsler–Hamilton phase spaces by introducing respective nonassociative sources,

$$
{}^{\shortmid}_{s}\widetilde{\Im}(\tau) \approx {}^{\shortmid}_{s}\mathbf{\Upsilon}(\tau) \rightarrow {}^{\shortmid}_{s}\widetilde{\Im}^{\star}(\tau).
$$

For various general classes and explicit examples of nonassociative 4d-8d phase space solutions, s-adapted frame proofs are provided in [6,9,10,40,41]. In abstract geometric form, we can consider s-adapted generalizations of all 4-d formulas proven in Sects. 3 and 4. Because in this Part II we work on ${}^{\shortmid}\widetilde{\mathcal{M}}^{\star}$, the nonholonomic structures must be adapted to certain τ-running nonassociative geometric data (${}^{\shortmid}\widetilde{\mathbf{N}} \approx {}^{\shortmid}_{s}\mathbf{N}$, ${}^{\shortmid}\widetilde{\mathbf{g}}^{\star} \approx {}^{\shortmid}_{s}\mathbf{g}^{\star}$, ${}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star} = {}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star} + {}^{\shortmid}_{s}\widehat{\mathbf{Z}}^{\star}$) even for constructing solutions we introduce canonical s-variables (with hat labels and re-defined effective sources).

7.1.2 *Nonassociative quasi-stationary Finsler–Hamilton evolution*

One of the main purposes of Part II of this paper is to elaborate on geometric methods on finding physically important solutions (black holes, BH; wormholes, WH; cosmological solutions) in nonassociative gravity and string theory. Various nonassociative and noncommutative can be encoded into generic off-diagonal terms of metrics and modified nonlinear and linear connection structures. Such methods of finding solutions were developed Finsler-like MGTs and in our works on off-diagonal solutions in 4-d GR and MGTs (massive, quadratic, nonmetric etc, see [11–13,16–18,21–23,25–28] and Part I of this paper.

We extend the 4-d quasi-stationary quadratic elements (62) in abstract geometric form to quasi-stationary 8-d phase space configurations under nonassociative τ-evolution if ${}_{s}\widehat{\Upsilon}(x^{i_s}, y^{a_s})$ (for $s = 1, 2$) are respectively shell by shell substituted by ${}^{\shortmid}_{s}\widetilde{\Im}^{\star}(\tau, x^{i_1}, y^3, p_{a_3}, p_{a_4})$, (for $s = 1, 2, 3, 4$), in (184) as κ-parametric geometric flow off-diagonal solutions (187) if

$$
\begin{aligned}
d\widetilde{s}^2(\tau) &= {}^{\shortmid}\widetilde{\mathbf{g}}^{\star}_{\alpha\beta}(\tau)\widetilde{\mathbf{e}}^{\alpha}(\tau)\widetilde{\mathbf{e}}^{\beta}(\tau) \\
&= {}^{\shortmid}\widetilde{\mathbf{g}}^{\star}_{\alpha_s\beta_s}(\tau)\widetilde{\mathbf{e}}^{\alpha_s}(\tau)\widetilde{\mathbf{e}}^{\beta_s}(\tau) \\
&= \widetilde{\mathbf{g}}^{\star}_1(\tau)(dx^1)^2 + \widetilde{\mathbf{g}}^{\star}_2(\tau)(dx^2)^2 + \widetilde{\mathbf{g}}^{\star}_3(\tau)(\widetilde{\mathbf{e}}^3(\tau))^2 \\
&\quad + \widetilde{\mathbf{g}}^{\star}_4(\tau)(\widetilde{\mathbf{e}}^4(\tau))^2 + {}^{\shortmid}\widetilde{\mathbf{g}}^5_{\star}(\tau)(\widetilde{\mathbf{e}}_5(\tau))^2 \\
&\quad + {}^{\shortmid}\widetilde{\mathbf{g}}^6_{\star}(\tau)(\widetilde{\mathbf{e}}_6(\tau))^2 \\
&\quad + {}^{\shortmid}\widetilde{\mathbf{g}}^7_{\star}(\tau)(\widetilde{\mathbf{e}}_5(\tau))^2 + {}^{\shortmid}\widetilde{\mathbf{g}}^8_{\star}(\tau)(\widetilde{\mathbf{e}}_6(\tau))^2 \\
&= e^{\psi(\hbar,\kappa;\tau,x^{k_1})}[(dx^1)^2 + (dx^2)^2] \\
&\quad + \frac{[\partial_3({}_2\widetilde{\Psi}(\tau))]^2}{4({}^{\shortmid}_2\widetilde{\Im}^{\star}(\tau))^2\{g_4^{[0]}(\tau) - \int dy^3 \frac{\partial_3[({}_2\widetilde{\Psi}(\tau))^2]}{4({}^{\shortmid}_2\widetilde{\Im}^{\star}(\tau))}\}} \\
&\quad \times(\widetilde{\mathbf{e}}^3(\tau))^2 + (g_4^{[0]}(\tau) - \int dy^3 \frac{\partial_3[({}_2\widetilde{\Psi}(\tau))^2]}{4({}^{\shortmid}_2\widetilde{\Im}^{\star}(\tau))})(\widetilde{\mathbf{e}}^4(\tau))^2 \\
&\quad + \frac{[{}^{\shortmid}\partial^5({}^{\shortmid}_3\widetilde{\Psi}(\tau))]^2}{4({}^{\shortmid}_3\widetilde{\Im}^{\star})^2\{{}^{\shortmid}g^6_{[0]}(\tau) - \int dp_5 \frac{{}^{\shortmid}\partial^6[({}^{\shortmid}_3\widetilde{\Psi}(\tau))^2]}{4({}^{\shortmid}_3\widetilde{\Im}^{\star}(\tau))}\}} \\
&\quad \times({}^{\shortmid}\widetilde{\mathbf{e}}_5(\tau))^2 + ({}^{\shortmid}g^6_{[0]}(\tau) - \int dp_5 \frac{{}^{\shortmid}\partial^5[({}^{\shortmid}_3\widetilde{\Psi}(\tau))^2]}{4({}^{\shortmid}_3\widetilde{\Im}^{\star})})({}^{\shortmid}\widetilde{\mathbf{e}}_6(\tau))^2 \\
&\quad + \frac{[{}^{\shortmid}\partial^7({}^{\shortmid}_4\widetilde{\Psi}(\tau))]^2}{4({}^{\shortmid}_4\widetilde{\Im}^{\star}(\tau))^2\{{}^{\shortmid}g^8_{[0]}(\tau) - \int dp_7 \frac{{}^{\shortmid}\partial^7[({}^{\shortmid}_4\widetilde{\Psi}(\tau))^2]}{4({}^{\shortmid}_4\Im(\tau))}\}}({}^{\shortmid}\widetilde{\mathbf{e}}_7(\tau))^2 \\
&\quad + ({}^{\shortmid}g^8_{[0]}(\tau) - \int dp_7 \frac{{}^{\shortmid}\partial^7[({}^{\shortmid}_4\widetilde{\Psi}(\tau))^2]}{4({}^{\shortmid}_4\widetilde{\Im}^{\star}(\tau))})({}^{\shortmid}\widetilde{\mathbf{e}}_8(\tau))^2,
\end{aligned}
\tag{188}
$$

where $\widetilde{\mathbf{g}}_1^{\star}(\tau) = \widetilde{\mathbf{g}}_2^{\star}(\tau) = e^{\widetilde{\psi}(\hbar,\kappa;\tau,x^{k_1})}$, $\widetilde{\mathbf{g}}_3^{\star}(\tau) = \frac{[\partial_3(\,_2\widetilde{\Psi}(\tau))]^2}{4(\,^{\shortmid}_2\widetilde{\Im}^{\star}(\tau))^2\{g_4^{[0]}(\tau) - \int dy^3 \frac{\partial_3[(\,_2\widetilde{\Psi}(\tau))^2]}{4(\,^{\shortmid}_2\widetilde{\Im}^{\star}(\tau))}\}}, \ldots$, as above.

The nonholonomic s-frames in (188) are computed as

$$
\begin{aligned}
\widetilde{\mathbf{e}}^3(\tau) &= dy^3 + \widetilde{w}_{k_1}(\hbar,\kappa,\tau,x^{i_1},y^3)dx^{k_1} \\
&= dy^3 + \frac{\partial_{k_1}(\,_2\widetilde{\Psi}(\tau))}{\partial_3(\,_2\Psi(\tau))}dx^{k_1}, \\
\widetilde{\mathbf{e}}^4(\tau) &= dt + \widetilde{n}_{k_1}(\hbar,\kappa,\tau,x^{i_1},y^3)dx^{k_1} \\
&= dy^4 + (\,_1\widetilde{n}_{k_1}(\tau) + \,_2\widetilde{n}_{k_1}(\tau) \\
&\quad \times \int dy^3 \frac{\partial_3[(\,_2\widetilde{\Psi}(\tau))^2]}{4(\,^{\shortmid}_2\widetilde{\Im}^{\star}(\tau))^2|\widetilde{g}_4^{[0]}(\tau) - \int dy^3 \frac{\partial_3[(\,_2\widetilde{\Psi}(\tau))^2]}{4(\,^{\shortmid}_2\widetilde{\Im}^{\star}(\tau))}|^{5/2}})dx^{k_1}, \\
{}^{\shortmid}\widetilde{\mathbf{e}}_5(\tau) &= dp_5 + {}^{\shortmid}\widetilde{w}_{k_2}(\hbar,\kappa,\tau,x^{i_2},p_5)dx^{k_2} \\
&= dp_5 + \frac{\partial_{k_2}(\,^{\shortmid}_3\widetilde{\Psi}(\tau))}{{}^{\shortmid}\partial^5(\,^{\shortmid}_3\widetilde{\Psi}(\tau))}dx^{k_2}, \\
{}^{\shortmid}\widetilde{\mathbf{e}}_6(\tau) &= dp_6 + {}^{\shortmid}\widetilde{n}_{k_2}(\hbar,\kappa,\tau,x^{i_2},p_5)dx^{k_2} \\
&= dp_6 + (\,^{\shortmid}_1\widetilde{n}_{k_2}(\tau) + \,^{\shortmid}_2\widetilde{n}_{k_2}(\tau) \\
&\quad \times \int dp_5 \frac{{}^{\shortmid}\partial^5[(\,^{\shortmid}_3\widetilde{\Psi}(\tau))^2]}{4(\,^{\shortmid}_3\widetilde{\Im}^{\star}(\tau))^2|\,{}^{\shortmid}\widetilde{g}^6_{[0]}(\tau) - \int dp_5 \frac{{}^{\shortmid}\partial^5[(\,^{\shortmid}_3\widetilde{\Psi}(\tau))^2]}{4(\,^{\shortmid}_3\widetilde{\Im}^{\star}(\tau))}|^{5/2}})dx^{k_2}, \\
{}^{\shortmid}\widetilde{\mathbf{e}}_7(\tau) &= dp_7 + {}^{\shortmid}\widetilde{w}_{k_3}(\hbar,\kappa,\tau,x^{i_2},p_5,p_7)d\,{}^{\shortmid}x^{k_3} \\
&= dp_7 + \frac{{}^{\shortmid}\partial_{k_3}(\,^{\shortmid}_4\widetilde{\Psi}(\tau))}{{}^{\shortmid}\partial^7(\,^{\shortmid}_4\widetilde{\Psi}(\tau))}d\,{}^{\shortmid}x^{k_3}, \\
{}^{\shortmid}\widetilde{\mathbf{e}}_8(\tau) &= dp_8 + {}^{\shortmid}\widetilde{n}_{k_3}(\hbar,\kappa,\tau,x^{i_2},p_5,p_7)d\,{}^{\shortmid}x^{k_3} \\
&= dp_8 + (\,^{\shortmid}_1\widetilde{n}_{k_3}(\tau) + \,^{\shortmid}_2\widetilde{n}_{k_3}(\tau) \\
&\quad \times \int dp_7 \frac{{}^{\shortmid}\partial^7[(\,^{\shortmid}_4\widetilde{\Psi}(\tau))^2]}{4(\,^{\shortmid}_4\widetilde{\Im}^{\star}(\tau))^2|\,{}^{\shortmid}\widetilde{g}^8_{[0]}(\tau) - \int dp_7 \frac{{}^{\shortmid}\partial^7[(\,^{\shortmid}_4\widetilde{\Psi}(\tau))^2]}{4(\,^{\shortmid}_4\widetilde{\Im}^{\star}(\tau))}|^{5/2}})\,d\,{}^{\shortmid}x^{k_3}.
\end{aligned}
\tag{189}
$$

Such values are similar to those studied in Sect. 4 of [40] but for different generating sources when the nonholonomic frame transforms are constrained to relate certain tilde geometric data with 4+4 splitting to canonical data with s-shells. In (188) and (189), there are considered:

$$
\begin{aligned}
&\text{generating functions: } \widetilde{\psi}(\tau) \simeq \widetilde{\psi}(\hbar,\kappa;\tau,x^{k_1});\ _2\widetilde{\Psi}(\tau) \\
&\simeq \,_2\widetilde{\Psi}(\hbar,\kappa;\tau,x^{k_1},y^3);\ ^{\shortmid}_3\widetilde{\Psi}(\tau) \\
&\simeq \,^{\shortmid}_3\widetilde{\Psi}(\hbar,\kappa;\tau,x^{k_2},p_5);\ ^{\shortmid}_4\widetilde{\Psi}(\tau) \\
&\simeq \,^{\shortmid}_4\widetilde{\Psi}(\hbar,\kappa;\tau,\,{}^{\shortmid}x^{k_3},p_7); \\
&\text{generating sources: } \,^{\shortmid}_1\widetilde{\Im}^{\star}(\tau) \simeq \,^{\shortmid}_1\widetilde{\Im}^{\star}(\hbar,\kappa;\tau,x^{k_1});\ ^{\shortmid}_2\widetilde{\Im}^{\star}(\tau) \\
&\simeq \,^{\shortmid}_2\widetilde{\Im}^{\star}(\hbar,\kappa;\tau,x^{k_1},y^3);\ ^{\shortmid}_3\widetilde{\Im}^{\star}(\tau) \\
&\simeq \,^{\shortmid}_3\widetilde{\Im}^{\star}(\hbar,\kappa;\tau,x^{k_2},p_5);\ ^{\shortmid}_4\widetilde{\Im}^{\star}(\tau) \\
&\simeq \,^{\shortmid}_4\widetilde{\Im}^{\star}(\hbar,\kappa;\tau,\,{}^{\shortmid}x^{k_3},p_7); \\
&\text{integrating functions: } \widetilde{g}_4^{[0]}(\tau) \\
&\simeq \widetilde{g}_4^{[0]}(\hbar,\kappa;\tau,x^{k_1}),\ _1\widetilde{n}_{k_1}(\tau) \\
&\simeq \,_1\widetilde{n}_{k_1}(\hbar,\kappa;\tau,x^{j_1}),\ _2\widetilde{n}_{k_1}(\tau) \simeq \,_2\widetilde{n}_{k_1}(\hbar,\kappa;\tau,x^{j_1});\ ^{\shortmid}\widetilde{g}^6_{[0]}\tau) \\
&\simeq \,^{\shortmid}\widetilde{g}^6_{[0]}(\hbar,\kappa;\tau,x^{k_2}),\ _1\widetilde{n}_{k_2}(\tau) \\
&\simeq \,_1\widetilde{n}_{k_2}(\hbar,\kappa;\tau,x^{j_2}),\ _2\widetilde{n}_{k_2}(\tau) \simeq \,_2\widetilde{n}_{k_2}(\hbar,\kappa;\tau,x^{j_2});\ ^{\shortmid}\widetilde{g}^8_{[0]}(\tau) \\
&\simeq \,^{\shortmid}\widetilde{g}^8_{[0]}(\hbar,\kappa;\tau,\,{}^{\shortmid}x^{j_3}),\ _1\widetilde{n}_{k_3}(\tau) \\
&\simeq \,^{\shortmid}_1\widetilde{n}_{k_3}(\hbar,\kappa;\tau,\,{}^{\shortmid}x^{j_3}),\ _2\widetilde{n}_{k_3}(\tau) \simeq \,^{\shortmid}_2\widetilde{n}_{k_3}(\hbar,\kappa;\tau,\,{}^{\shortmid}x^{j_3}).
\end{aligned}
\tag{190}
$$

The functions $\widetilde{\psi}(\tau)$ are solutions of a respective family of 2-d Poisson equations,

$$\partial^2_{11}\widetilde{\psi}(\hbar,\kappa;\tau,x^{k_1}) + \partial^2_{22}\widetilde{\psi}(\hbar,\kappa;\tau,x^{k_1}) = 2\,_1\widetilde{\Im}^{\star}(\hbar,\kappa;\tau,x^{k_1}).$$

Nonassociative geometric parametric evolution of quasi-stationary Finsler–Hamilton configurations defined above are characterized by four types of additional geometric and thermodynamic flow variables:

1. The nonassociative geometric evolution of nonsymmetric metrics $\,^{\shortmid}_{\star}\mathfrak{a}_{\alpha_s\beta_s}(\tau) = \,^{\shortmid}_{\star}\mathfrak{a}_{\alpha_s\beta_s}(\hbar,\kappa;\tau,\,{}^{\shortmid}u^{\gamma_s})$ induced by Finsler–Hamilton configurations is computed in explicit form by introducing in (159) using the coefficients of the s-metric and the N-connection, respectively, (188) and (189).
2. Above classes of such solutions are with nontrivial geometric flows of nonholonomic torsion which is not zero for tilde variables. We can define certain classes of nonholonomic frame transforms and distortions to canonical s-variables when the nonassociative geometric evoution is described by families of LC-connections $\,^{\shortmid}_s\widetilde{\nabla}^{\star}(\tau)$ and $\,^{\shortmid}_s\widetilde{\nabla}(\tau)$.
3. We can compute necessary thermodynamic variables (180) associated to nonassociative quasi-stationary solutions, or their time dual ones defined as nonassociative locally anisotropic cosmological solutions with additional cosmological flow. In next subsections, we shall provide such examples for nonassociative BH and WH configurations.
4. The solutions for nonassociative Finsler–Lagrange–Ricci soliton equations (177) consist self-similar configurations of (188) and (189) with. $\tau = \tau_0$. We can construct such quasi-stationary solutions directly or after a class of generic off-diagonal solutions has be constructed for nonassociative geometric evolution flows. Such Ricci soliton configurations can be generated equivalently by solutions constructed using the AFCDM as it is outlined in Appendix B to [40].
5. Finally, we note that τ-families of nonassociative quasi-stationary can be generated using Tables 13 and 14 (see respectively ansatz (A.17) and (A.19)) when the s- and N-coefficients are considered with additional τ-dependence and the generating sources are correspondingly redefined for nonassociative Finsler–Hamilton distortions, $\,^{\shortmid}_s\widehat{\Upsilon} \to \,^{\shortmid}_s\widetilde{\Im}^{\star}$.

7.1.3 Nonlinear symmetries and space and time duality encoding nonassociative generalized Finsler data

Nonassociative quasi-stationary solutions encoding Finsler–Hamilton configurations very important nonlinear symmetries. We study nonholonomic geometric flow deformations of some families of **prime** s-metrics $\,^{\shortmid}_s\mathring{\mathbf{g}}(\tau)$. They can be arbitrary ones, i.e. not solutions of some (modified) Einstein equations, but for understanding nonlinear off-diagonal interaction and evolution models we can subject them to be certain trivial phase space extensions of some physically important solutions in GR. A corresponding family of **target** s-metrics ${}_{s}^{|}\mathbf{g}(\tau)$ defining a nonassociative Finsler–Hamilton flow evolution scenarios of quasi-stationary metrics on ${}_{s}^{\star}\widetilde{\mathcal{M}}$ can be modelled by R-flux deformations

$$
{}_{s}^{|}\mathring{\mathbf{g}}(\tau) \to {}_{s}^{|}\widetilde{\mathbf{g}}(\tau) = [\, {}^{|}\widetilde{g}_{\alpha_s}(\tau) = {}^{|}\widetilde{\eta}_{\alpha_s}(\tau)\, {}^{|}\mathring{g}_{\alpha_s}(\tau),\ {}^{|}\widetilde{N}^{a_s}_{i_{s-1}}(\tau) = {}^{|}\widetilde{\eta}^{a_s}_{i_{s-1}}(\tau)\, {}^{|}\mathring{N}^{a_s}_{i_{s-1}}(\tau)], \quad (191)
$$

with phase space gravitational η-polarizations generalized for τ-dependencies of respective s-and N-coefficients in (188) and (189). Such constructions consist 8-d phase space generalizations of formulas related to (69). Correspondingly, the generating functions and effective sources can be related to certain effective shell τ-running cosmological constants,

$$
\begin{aligned}
({}_{s}\widetilde{\Psi}(\tau),\ {}_{s}^{|}\widetilde{\Im}^{\star}(\tau)) &\leftrightarrow ({}_{s}^{|}\widetilde{\mathbf{g}}(\tau),\ {}_{s}^{|}\widetilde{\Im}^{\star}(\tau)) \leftrightarrow ({}_{s}^{|}\widetilde{\eta}(\tau)\, {}^{|}\mathring{g}_{\alpha_s}(\tau) \\
&\sim ({}^{|}\widetilde{\zeta}_{\alpha_s}(\tau)(1+\kappa\, {}^{|}\widetilde{\chi}_{\alpha_s}(\tau))\, {}^{|}\mathring{g}_{\alpha_s}(\tau),\ {}_{s}^{|}\widetilde{\Im}^{\star}(\tau)) \leftrightarrow \\
({}_{s}\widetilde{\Phi}(\tau),\ {}_{s}^{|}\widetilde{\Lambda}(\tau)) &\leftrightarrow ({}_{s}^{|}\widetilde{\mathbf{g}},\ {}_{s}^{|}\widetilde{\Lambda}(\tau)) \leftrightarrow ({}_{s}^{|}\widetilde{\eta}(\tau)\, {}^{|}\mathring{g}_{\alpha_s}(\tau) \\
&\sim ({}^{|}\widetilde{\zeta}_{\alpha_s}(\tau)(1+\kappa\, {}^{|}\widetilde{\chi}_{\alpha_s}(\tau))\, {}^{|}\mathring{g}_{\alpha_s}(\tau),\ {}_{s}^{|}\widetilde{\Lambda}(\tau)),
\end{aligned} \quad (192)
$$

where ${}_{s}^{|}\widetilde{\Lambda}_0 = {}_{s}^{|}\widetilde{\Lambda}(\tau_0) = const$ for deriving nonassociative Ricci soliton symmetries.

In explicit form, the nonlinear symmetries of quasi-stationary solutions (188) and (189) are defined by formulas

$$
\begin{aligned}
\partial_3[({}_{2}\widetilde{\Psi}(\tau))^2] &= -\int dy^3({}_{2}^{|}\widetilde{\Im}^{\star}(\tau))\partial_3\widetilde{g}_4(\tau) \\
&\simeq -\int dy^3({}_{2}^{|}\widetilde{\Im}^{\star}(\tau))\partial_3({}^{|}\widetilde{\eta}_4(\tau)\, \mathring{g}_4(\tau)) \\
&\simeq -\int dy^3({}_{2}^{|}\widetilde{\Im}^{\star}(\tau)) \\
&\quad \partial_3[\, {}^{|}\widetilde{\zeta}_4(\tau)(1+\kappa\, {}^{|}\widetilde{\chi}_4(\tau))\, \mathring{g}_4(\tau)], \\
({}_{2}\widetilde{\Phi}(\tau))^2 &= -4\, {}_{2}\widetilde{\Lambda}(\tau)\widetilde{g}_4(\tau) \simeq -4\, {}_{2}\widetilde{\Lambda}(\tau)\, {}^{|}\widetilde{\eta}_4(\tau)\, \mathring{g}_4(\tau) \\
&\simeq -4\, {}_{2}\widetilde{\Lambda}(\tau)\, {}^{|}\widetilde{\zeta}_4(\tau)(1+\kappa\, {}^{|}\widetilde{\chi}_4(\tau))\, \mathring{g}_4(\tau); \\
{}^{|}\partial^5[({}_{3}^{|}\widetilde{\Psi}(\tau))^2] &= -\int dp_5({}_{3}^{|}\widetilde{\Im}^{\star}(\tau))\, {}^{|}\partial^5\, {}^{|}\widetilde{g}^6(\tau) \\
&\simeq -\int dp_5({}_{3}^{|}\widetilde{\Im}^{\star}(\tau))\, {}^{|}\partial^5({}^{|}\widetilde{\eta}^6(\tau)\, {}^{|}\mathring{g}^6(\tau)) \\
&\simeq -\int dp_5({}_{3}^{|}\widetilde{\Im}^{\star}(\tau))\, {}^{|}\partial^5[\, {}^{|}\widetilde{\zeta}^6(\tau)(1+\kappa\, {}^{|}\widetilde{\chi}^6(\tau))\, \mathring{g}^6(\tau)], \\
({}_{3}^{|}\widetilde{\Phi}(\tau))^2 &= -4\, {}_{3}^{|}\widetilde{\Lambda}(\tau)\, {}^{|}\widetilde{g}^6(\tau) \\
&\simeq -4\, {}_{3}^{|}\widetilde{\Lambda}(\tau)\, {}^{|}\widetilde{\eta}^6(\tau)\, {}^{|}\mathring{g}^6(\tau) \\
&\simeq -4\, {}_{3}^{|}\widetilde{\Lambda}(\tau)\, {}^{|}\widetilde{\zeta}^6(\tau)(1+\kappa\, {}^{|}\widetilde{\chi}^6(\tau))\, {}^{|}\mathring{g}^6(\tau); \\
{}^{|}\partial^7[({}_{4}^{|}\widetilde{\Psi}(\tau))^2] &= -\int dp_7({}_{4}^{|}\widetilde{\Im}^{\star}(\tau))\, {}^{|}\partial^7\, {}^{|}\widetilde{g}^8(\tau) \\
&\simeq -\int dp_7({}_{4}^{|}\widetilde{\Im}^{\star}(\tau))\, {}^{|}\partial^7({}^{|}\widetilde{\eta}^8(\tau)\, {}^{|}\mathring{g}^8(\tau)) \\
&\simeq -\int dp_7({}_{4}^{|}\widetilde{\Im}^{\star}(\tau))\, {}^{|}\partial^7[\, {}^{|}\widetilde{\zeta}^8(\tau)(1+\kappa\, {}^{|}\widetilde{\chi}^8(\tau))\, \mathring{g}^8(\tau)], \\
({}_{4}^{|}\widetilde{\Phi}(\tau))^2 &= -4\, {}_{4}^{|}\widetilde{\Lambda}(\tau)\, {}^{|}\widetilde{g}^8(\tau) \\
&\simeq -4\, {}_{4}^{|}\widetilde{\Lambda}(\tau)\, {}^{|}\widetilde{\eta}^8(\tau)\, {}^{|}\mathring{g}^8(\tau) \\
&\simeq -4\, {}_{4}^{|}\widetilde{\Lambda}(\tau)\, {}^{|}\widetilde{\zeta}^8(\tau)(1+\kappa\, {}^{|}\widetilde{\chi}^8(\tau))\, {}^{|}\mathring{g}^8(\tau).
\end{aligned} \quad (193)
$$

For instance, nonlinear transforms (193), when

$$
\begin{aligned}
&{}_{s}^{|}\widetilde{\mathbf{g}}[\hbar,\kappa,\tau,\psi(\tau),\ {}_{s}\widetilde{\Psi}(\tau),\ {}_{s}^{|}\widetilde{\Im}^{\star}(\tau)] \\
&\to {}_{s}^{|}\widetilde{\mathbf{g}}[\hbar,\kappa,\tau,\psi(\tau),\ {}_{s}\widetilde{\Phi}(\tau),\ {}_{s}^{|}\widetilde{\Lambda}(\tau)],
\end{aligned}
$$

transform (185) into an equivalent system of nonlinear PDEs with effective τ-running cosmological constants,

$$
{}^{|}\widetilde{\mathbf{R}}^{\beta_s}{}_{\gamma_s}(\tau,\ {}_{s}\widetilde{\Phi}(\tau),\ {}_{s}^{|}\widetilde{\Im}^{\star}(\tau)) = \delta^{\beta_s}{}_{\gamma_s}\, {}_{s}^{|}\widetilde{\Lambda}(\tau). \quad (194)
$$

The solutions of such equations, in various η- and χ-variables, are presented in Sects. 4.3, 4.4, and 4.5 of [40]. For the Part II of this work, those proofs and formulas can be redefined in abstract geometric form for tilde variables encoding Finsler–Hamilton structures.

On shells $s = 1, 2$, the quasi-stationary solutions (185) and (194) can be transformed into local anisotropic cosmological solutions as we explained in Sect. 3.3 for formulas (77) with underlined s- and N-coefficients to emphasize dependencies on a time like variable $y^4 = 0$. The co-fiber variables p_{a_s} can be added in different forms which generates different classes of off-diagonal cosmological solutions with τ-dependence. For a toy 2+2 model, the main ideas and solutions were provided in section (87). In another turn, possible classes of nonassociative geometric flow equations (quasi-stationary or locally anisotropic ones) can be generated in abstract geometric form using Tables 12, 13, 14, 15, 16 from Appendix A.4.

7.2 Nonassociative Finsler–Hamilton evolution of phase space black holes

In this subsection, we study explicit examples how using the AFCDM we can construct nonassociative BH solutions and study their Finsler–Hamilton flow evolution and respective geometric thermodynamics properties. We reformulate in modified Finsler variables the constructions from Sect. 5.3 of [40] for nonassociative flows of phase space Reisner–Nordström-anti-de Sitter (in brief, RN AdS) BHs. The priority of the Finsler–Hamilton configurations is that we can extend naturally the approach to quantum deformations (which may include, or not, nonassociative data). We generate and study physical properties of nonassociative generalized Finsler flow deformations of RN BHs, when the effective cosmological constants are determined by negative cosmological constants and respective prime metric configurations.

7.2.1 Prime metrics for defining commutative phase spaces and RN-AdS BHs

We consider a set of prime metric coefficients $\breve{g}_1 = \breve{f}(\breve{r})^{-1}$, ${}^{|}\breve{g}_2 = {}^{|}\breve{g}_3 = {}^{|}\breve{g}^5 = \breve{r}^2$, $\breve{g}_4 = -\breve{f}(\breve{r})$, ${}^{|}\breve{g}^6 = {}^{|}\breve{g}^7 = -{}^{|}\breve{g}^8 = 1$ and ${}^{|}\breve{g}^{a_s}_{i_{s-1}}(\breve{r}, t, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_6, p_7, E) = 0$ for a quadratic line element

$$d\,\breve{s}^2_{[5+3]} = {}^{|}\breve{g}_{\alpha_s}({}^{|}u^{\gamma_s})(\breve{\mathbf{e}}^{\alpha_s})^2 = \frac{d\breve{r}^2}{\breve{f}(\breve{r})} - \breve{f}(\breve{r})dt^2 + \breve{r}^2[(d^2\hat{x}^2)^2 + (d\hat{x}^3)^2 + (dp_5)^5] + (dp_6)^2 + (dp_7)^2 - dE^2. \quad (195)$$

The coordinates are defined in natural units $\hat{x}^1 = \breve{r} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (p_5)^2}$ and $\hat{x}^2 = \hat{x}^2(x^2, x^3, p_5)$, $\hat{x}^3 = \hat{x}^3(x^2, x^3, p_5)$ and $\hat{x}^5 = \hat{x}^5(x^2, x^3, p_5)$ chosen as coordinates for a diagonal metric on an effective 3-d Einstein phase space $V_{[3]}$ of constant scalar curvature (let say, $6\hat{k}$, for $\hat{k} = 1$). The metric function $\breve{f}(\breve{r}) = 1 - \frac{\hat{m}}{\breve{r}^2} + \frac{\breve{r}^2}{l^2_{[5]}} + \frac{\hat{q}^2}{\breve{r}^4}$ in (195) is defined by integration constant $\hat{m}$ determined by the mass of a BH, $\hat{M} = 3\omega_{[3]}\hat{m}/16\pi G_{[5]}$, for $\omega_{[3]}$ denoting the volume of $V_{[3]}$; and the parameter $\hat{q}$ is related to the physical charge $\hat{Q}$ of the RN-AdS BH via formula $\hat{q} = 4\pi G_{[5]}\hat{Q}/\sqrt{3}\omega_{[3]}$, see [139,140]. In such a model a negative constant $\Lambda_{[5]} = -6/l^2_{[5]}$ is related to the AdS radius $l_{[5]}$ which can be naturally viewed as an effective truncation of the IIB supergravity on a 5-d sphere, $\mathbb{S}^5$. The 5-d part of the 8-d metric (195) can be uplifted to 10-d. In such a case, it can be viewed as a near horizon geometry of $\check{N}$ rotating black D3-branes in type IIB supergravity when $l^4_{[10]} = 2\check{N}\ell_p^4/\pi^2 \equiv \alpha^2\check{N}$, where ℓ_p is the 10-d Planck length.

To apply the AFCDM is convenient to consider certain nontrivial nonlinear coordinates when $\breve{\mathbf{e}}^{\alpha_s}$ are N-elongated by certain nontrivial $\breve{N}^3_{i_1}({}^{|}u^{\gamma_s})$ and ${}^{|}\breve{N}_{5i_2}({}^{|}u^{\gamma_s})$, see formulas below. This allow to construct off-diagonal nonassociative Finsler–Hamilton deformations in explicit form and without coordinate singularities.

7.2.2 Nonassociative Finsler–Hamilton κ-linear evolution of phase space RN-AdS BHs

We consider nonassociative generic off-diagonal generalizations of s-metric ${}^{|}\breve{\mathbf{g}}_{\alpha_s}$ (195) to certain quasi-stationary ${}^{|}\widetilde{\mathbf{g}}_{\alpha_s}$ under κ-linear geometric flow evolution with fixed 8-d phases space cosmological constant ${}_s\widetilde{\Lambda}(\tau_0) = \widetilde{\Lambda}_{[5]} < 0$, for $s = 1,2,3,4$, when $\widetilde{\Lambda}_{[5]}$ can be different from $\Lambda_{[5]}$ as a result of τ-evolution. The s-coefficients ${}^{|}\breve{g}_{\alpha_s}$ are taken for a prime s-metric (instead of ${}^{|}_{s}\mathring{\mathbf{g}}(\tau)$ in (191)) for respective effective sources ${}^{|}_{s}\widetilde{\Im}^{\star}(\tau)$ related via nonlinear symmetries (193) to $\widetilde{\Lambda}_{[5]}$ extended on all phase space. Applying the AFCDM we generate a τ-family of quasi-stationary solutions of nonassociative Finsler–Hamilton flow equations (185), ${}^{|}\breve{g}_{\alpha_s} \rightarrow {}^{|}_{s}\mathring{\mathbf{g}}(\tau)$,

$$\begin{aligned}
d\widetilde{s}^2(\tau) &= e^{\widetilde{\psi}(\hbar,\kappa;\tau,\breve{r},\hat{x}^2,\widetilde{\Lambda}_{[5]})}[(d\breve{r})^2 + (d\hat{x}^2)^2] \\
&- \frac{1}{\widetilde{g}^{[0]}_4(\tau) - \frac{({}_2\widetilde{\Phi}(\tau))^2}{4\,\widetilde{\Lambda}_{[5]}}} \frac{({}_2\widetilde{\Phi}(\tau))^2[\partial_3({}_2\widetilde{\Phi}(\tau))]^2}{|\,\widetilde{\Lambda}_{[5]}\int dy^3({}^{|}_2\widetilde{\Im}^{\star}(\tau))[\partial_3({}_2\widetilde{\Phi}(\tau))^2]|} \\
&\times(\widetilde{\mathbf{e}}^3(\tau))^2 + \left(\widetilde{g}^{[0]}_4(\tau) - \frac{({}_2\widetilde{\Phi}(\tau))^2}{4\,\widetilde{\Lambda}_{[5]}}\right)(\widetilde{\mathbf{e}}^4(\tau))^2 \\
&- \frac{1}{{}^{|}\widetilde{g}^6_{[0]}(\tau) - \frac{({}^{|}_3\widetilde{\Phi}(\tau))^2}{4\,\widetilde{\Lambda}_{[5]}}} \frac{({}^{|}_3\widetilde{\Phi}(\tau))^2[{}^{|}\partial^5({}^{|}_3\widetilde{\Phi}(\tau))]^2}{|\,\widetilde{\Lambda}_{[5]}\int dp_5({}^{|}_3\widetilde{\Im}^{\star}(\tau))\ {}^{|}\partial^5[({}^{|}_3\Phi(\tau))^2]|} \\
&\times({}^{|}\widetilde{\mathbf{e}}_5(\tau))^2 + \left({}^{|}\widetilde{g}^6_{[0]}(\tau) - \frac{({}^{|}_3\widetilde{\Phi}(\tau))^2}{4\,\widetilde{\Lambda}_{[5]}}\right)({}^{|}\widetilde{\mathbf{e}}_6(\tau))^2 \\
&- \frac{1}{{}^{|}\widetilde{g}^8_{[0]}(\tau) - \frac{({}^{|}_4\widetilde{\Phi}(\tau))^2}{4\,\widetilde{\Lambda}_{[5]}}} \frac{({}^{|}_4\widetilde{\Phi}(\tau))^2[{}^{|}\partial^7({}^{|}_4\widetilde{\Phi}(\tau))]^2}{|\,\widetilde{\Lambda}_{[5]}\int dp_7({}^{|}_4\widetilde{\Im}^{\star})\ {}^{|}\partial^7[({}^{|}_4\widetilde{\Phi}(\tau))^2]|} \\
&\times({}^{|}\widetilde{\mathbf{e}}_7(\tau))^2 + \left({}^{|}\widetilde{g}^8_{[0]}(\tau) - \frac{({}^{|}_4\widetilde{\Phi}(\tau))^2}{4\,\widetilde{\Lambda}_{[5]}}\right)({}^{|}\widetilde{\mathbf{e}}_8(\tau))^2.
\end{aligned} \quad (196)$$

In (196), there local coordinates are defined as ${}^{|}u^{\gamma_s} = (\breve{r}, t, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_6, p_7, E)$ and the s-adapted frames are computed:

$$\begin{aligned}
\widetilde{\mathbf{e}}^3(\tau) &= d\hat{x}^3 + \frac{\partial_{k_1}\int d\hat{x}^3({}^{|}_2\widetilde{\Im}^{\star}(\tau))\hat{\partial}_3[({}_2\widetilde{\Phi}(\tau))^2]}{({}^{|}_2\widetilde{\Im}^{\star}(\tau))\ \hat{\partial}_3[({}_2\widetilde{\Phi}(\tau))^2]}dx^{k_1}, \\
\widetilde{\mathbf{e}}^4(\tau) &= dt + ({}_1\widetilde{n}_{k_1}(\tau) + {}_2\widetilde{n}_{k_1}(\tau) \\
&\times \frac{\int d\hat{x}^3 \frac{({}_2\widetilde{\Phi}(\tau))^2[\hat{\partial}_3({}_2\widetilde{\Phi}(\tau))]^2}{|\Lambda_5(\tau)\int d\hat{x}^3({}^{|}_2\widetilde{\Im}^{\star}(\tau))[\hat{\partial}_3({}_2\widetilde{\Phi}(\tau))^2]|}}{\left|\widetilde{g}^{[0]}_4(\tau) - \frac{({}_2\widetilde{\Phi}(\tau))^2}{4\,\widetilde{\Lambda}_5(\tau)}\right|^{5/2}})dx^{k_1}, \\
{}^{|}\widetilde{\mathbf{e}}_5(\tau) &= d\hat{x}^5 + \frac{\partial_{k_2}\int d\hat{x}^5({}^{|}_3\widetilde{\Im}^{\star}(\tau))\ \hat{\partial}[({}^{|}_3\widetilde{\Phi}(\tau))^2]}{({}^{|}_3\widetilde{\Im}^{\star}(\tau))\ \ \hat{\partial}_5[({}^{|}_3\widetilde{\Phi}(\tau))^2]}dx^{k_2}, \\
{}^{|}\widetilde{\mathbf{e}}_6(\tau) &= dp_6 + ({}^{|}_1\widetilde{n}_{k_2}(\tau) + {}^{|}_2\widetilde{n}_{k_2}(\tau) \\
&\times \frac{\int dp_5 \frac{({}^{|}_3\widetilde{\Phi}(\tau))^2[{}^{|}\partial^5({}^{|}_3\widetilde{\Phi}(\tau))]^2}{|\Lambda_{[5]}\int dp_5({}^{|}_3\widetilde{\Im}^{\star}(\tau))[{}^{|}\partial^5({}^{|}_3\widetilde{\Phi}(\tau))^2]|}}{\left|{}^{|}\widetilde{g}^6_{[0]}(\tau) - \frac{({}^{|}_3\widetilde{\Phi}(\tau))^2}{4\widetilde{\Lambda}_{[5]}}\right|^{5/2}})dx^{k_2}, \\
{}^{|}\widetilde{\mathbf{e}}_7(\tau) &= dp_7 + \frac{\partial_{k_3}\int dp_7({}^{|}_4\widetilde{\Im}^{\star}(\tau))\ \ {}^{|}\partial^7[({}^{|}_4\widetilde{\Phi}(\tau))^2]}{({}^{|}_4\widetilde{\Im}^{\star}(\tau))\ \ {}^{|}\partial^7[({}^{|}_4\widetilde{\Phi}(\tau))^2]}d\,{}^{|}x^{k_3}, \\
{}^{|}\widetilde{\mathbf{e}}_8(\tau) &= dE + ({}^{|}_1\widetilde{n}_{k_3}(\tau) + {}^{|}_2\widetilde{n}_{k_3}(\tau) \\
&\times \frac{\int dp_7 \frac{({}^{|}_4\widetilde{\Phi}(\tau))^2[{}^{|}\partial^7({}^{|}_4\widetilde{\Phi}(\tau))]^2}{|\,\widetilde{\Lambda}_{[5]}\int dp_7({}^{|}_4\widetilde{\Im}^{\star}(\tau))[{}^{|}\partial^7({}^{|}_4\widetilde{\Phi}(\tau))^2]|}}{\left|{}^{|}\widetilde{g}^8_{[0]}(\tau) - \frac{({}^{|}_4\widetilde{\Phi}(\tau))^2}{4\,\widetilde{\Lambda}_{[5]}}\right|^{5/2}})d\,{}^{|}x^{k_3}.
\end{aligned}$$

The integration functions from the formulas above are certain parametric functions

$$\widetilde{g}^{[0]}_4(\hbar,\kappa,\tau,\breve{r},\hat{x}^2),\ {}_1\widetilde{n}_{k_1}(\hbar,\kappa,\tau,\breve{r},\hat{x}^2),\ {}_2\widetilde{n}_{k_1}(\hbar,\kappa,\tau,\breve{r},\hat{x}^2);$$

$$
{}^{\shortmid}\widetilde{g}^{5}_{[0]}(\hbar,\kappa,\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5}),\\
{}_{1}\widetilde{n}_{k_2}(\hbar,\kappa,\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5}),\\
{}_{2}\widetilde{n}_{k_2}(\hbar,\kappa,\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5});\\
{}^{\shortmid}\widetilde{g}^{7}_{[0]}(\hbar,\kappa,\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5},p_7),\\
{}^{\shortmid}_{1}\widetilde{n}_{k_3}(\hbar,\kappa,\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5},p_7),\\
{}^{\shortmid}_{2}\widetilde{n}_{k_3}(\hbar,\kappa,\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5},p_7).
$$

Using nonlinear symmetries (193), the quasi-stationary solutions (196) can be written in terms of of different other type generating functions and η-/χ-polarization functions, when

$$
\begin{aligned}
{}_{2}\widetilde{\Phi}(\tau) &= 2\sqrt{|\,\widetilde{\Lambda}_{[5]}\,\widetilde{g}_4(\tau)|} = 2\sqrt{|\,\widetilde{\Lambda}_{[5]}\,\widetilde{\eta}_4(\tau)\breve{g}_4)|}\\
&\simeq 2\sqrt{|\,\widetilde{\Lambda}_{[5]}\,\widetilde{\zeta}_4(\tau)\breve{g}_4|}[1-\frac{\kappa}{2}\widetilde{\chi}_4(\tau)],\\
{}^{\shortmid}_{3}\widetilde{\Phi}(\tau) &= 2\sqrt{|\,\widetilde{\Lambda}_{[5]}\,{}^{\shortmid}\widetilde{g}^{6}(\tau)|} = 2\sqrt{|\,\widetilde{\Lambda}_{[5]}\,{}^{\shortmid}\widetilde{\eta}^{6}(\tau)\,{}^{\shortmid}\breve{g}^{6}|}\\
&\simeq 2\sqrt{|\,\widetilde{\Lambda}_{[5]}\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\,{}^{\shortmid}\breve{g}^{6}|}[1-\frac{\kappa}{2}\,{}^{\shortmid}\widetilde{\chi}^{6}(\tau)],\\
{}^{\shortmid}_{4}\widetilde{\Phi}(\tau) &= 2\sqrt{|\,\widetilde{\Lambda}_{[5]}\,{}^{\shortmid}\widetilde{g}^{8}(\tau)|} = 2\sqrt{|\,\widetilde{\Lambda}_{[5]}\,{}^{\shortmid}\widetilde{\eta}^{8}(\tau)\,{}^{\shortmid}\breve{g}^{8}|}\\
&\simeq 2\sqrt{|\,\widetilde{\Lambda}_{[5]}\,{}^{\shortmid}\widetilde{\zeta}^{8}(\tau)\,{}^{\shortmid}\breve{g}^{8}|}[1-\frac{\kappa}{2}\,{}^{\shortmid}\widetilde{\chi}^{8}(\tau)].
\end{aligned}
\tag{197}
$$

For a subclass of solutions, the generating and integration functions are written in κ-linearized form as in (191),

$$
\begin{aligned}
\widetilde{\psi}(\tau) &\simeq \widetilde{\psi}(\hbar,\kappa;\tau,\breve{r},\hat{x}^{2})\\
&\simeq \widetilde{\psi}_0(\hbar,\tau,\breve{r},\hat{x}^{2})(1+\kappa\,{}_{\psi}\widetilde{\chi}(\hbar,\tau,\breve{r},\hat{x}^{2})),\ \text{for}\\
\widetilde{\eta}_2(\tau) &\simeq \widetilde{\eta}_2(\hbar,\kappa;\tau,\breve{r},\hat{x}^{2})\\
&\simeq \widetilde{\zeta}_2(\hbar,\tau,\breve{r},\hat{x}^{2})(1+\kappa\widetilde{\chi}_2(\hbar,\tau,\breve{r},\hat{x}^{2})),\\
&\quad \text{we can consider } \widetilde{\eta}_2(\tau)=\widetilde{\eta}_1(\tau);\\
\widetilde{\eta}_4(\tau) &\simeq \widetilde{\eta}_4(\hbar,\kappa;\tau,\breve{r},\hat{x}^{2},\hat{x}^{3}) \simeq \widetilde{\zeta}_4(\hbar,\tau,\breve{r},\hat{x}^{2},\hat{x}^{3})\\
&\quad\times(1+\kappa\,\widetilde{\chi}_4(\hbar,\tau,\breve{r},\hat{x}^{2},\hat{x}^{3})),\\
{}^{\shortmid}\widetilde{\eta}^{6}(\tau) &\simeq {}^{\shortmid}\widetilde{\eta}^{6}(\hbar,\kappa;\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5})\\
&\simeq {}^{\shortmid}\widetilde{\zeta}^{6}(\hbar,\kappa;\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5})\\
&\quad\times(1+\kappa\,{}^{\shortmid}\widetilde{\chi}^{6}(\hbar,\kappa;\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5})),\\
{}^{\shortmid}\widetilde{\eta}^{8}(\tau) &\simeq {}^{\shortmid}\widetilde{\eta}^{8}(\hbar,\kappa;\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5},p_7)\\
&\simeq {}^{\shortmid}\widetilde{\zeta}^{8}(\hbar,\kappa;\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5},p_7)\\
&\quad\times(1+\kappa\,{}^{\shortmid}\widetilde{\chi}^{8}(\hbar,\kappa;\tau,\breve{r},\hat{x}^{2},\hat{x}^{3},\hat{x}^{5},p_7)).
\end{aligned}
$$

Using formulas (197), we can define solutions with different type configurations. For instance, we can extract solutions with rotoid spacetime configurations determined by nonassociative star product R-flux deformation (considering χ-polarizations), or to compute volume forms (172) for η-polarizations defined by tilde variables.

7.2.3 The Bekenstein–Hawking entropy of τ-running Finsler Hamilton phase space RN-AdS BEs configurations

The nonholonomic configurations, the s-metrics (196) define higher dimension BH and/or BE configurations with conventional horizons which can be used for formulating models of generalized Bekenstein–Hawking thermodynamics [53–56]. For simplicity, we generate a family of solutions for 6-d τ-running quasi-stationary configurations evolving in a 8-d phase space when ${}^{\shortmid}_{1}\widetilde{n}_{k_s}=0$ and ${}^{\shortmid}_{2}\widetilde{n}_{k_s}=0$. Such s-metrics are parameterized in the form:

$$
\begin{aligned}
d\,{}^{\chi}_{\shortmid}\widetilde{s}^{2}_{[6\subset 8d]}(\tau) &= e^{\widetilde{\psi}_0}(1+\kappa\,{}^{\widetilde{\psi}(\tau)}\,{}^{\shortmid}\widetilde{\chi}(\tau))\\
&\times[\,\breve{g}_1(\breve{r})d\breve{r}^{2}+\breve{g}_2(\breve{r})(d\hat{x}^{2})]\\
&-\{\frac{4[\hat{\partial}_3(|\widetilde{\zeta}_4(\tau)\breve{g}_4(\breve{r})|^{1/2})]^{2}}{\breve{g}_4(\breve{r})|\int d\hat{x}^{3}\{{}^{\shortmid}_{2}\widetilde{\Im}^{\star}(\tau)\hat{\partial}_3(\widetilde{\zeta}_4(\tau)\,\breve{g}_4(\breve{r}))\}|}\\
&-\kappa[\frac{\hat{\partial}_3(\widetilde{\chi}_4(\tau)|\widetilde{\zeta}_4(\tau)\,\breve{g}_4(\breve{r})|^{1/2})}{4\hat{\partial}_3(|\widetilde{\zeta}_4(\tau)\,\breve{g}_4(\breve{r})|^{1/2})}\\
&-\frac{\int d\hat{x}^{3}\{{}^{\shortmid}_{2}\widetilde{\Im}^{\star}(\tau)\hat{\partial}_3[(\widetilde{\zeta}_4(\tau)\,\breve{g}_4(\breve{r}))\widetilde{\chi}_4(\tau)]\}}{\int d\hat{x}^{3}\{{}^{\shortmid}_{2}\widetilde{\Im}^{\star}(\tau)\hat{\partial}_3(\widetilde{\zeta}_4(\tau)\,\breve{g}_4(\breve{r}))\}}]\}\breve{g}_3(\mathbf{e}^{3}(\tau))^{2}\\
&+\widetilde{\zeta}_4(\tau)(1+\kappa\,\widetilde{\chi}_4(\tau))\breve{g}_4(\breve{r})dt^{2}\\
&-\{\frac{4[\hat{\partial}_5(|\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\ \breve{g}^{6}|^{1/2})]^{2}}{\breve{g}_5(\breve{r})|\int d\hat{x}^{5}\{{}^{\shortmid}_{3}\widetilde{\Im}^{\star}(\tau)\,{}^{\shortmid}\partial^{7}(\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\ \breve{g}^{6})\}|}\\
&-\kappa[\frac{\hat{\partial}_5(\,{}^{\shortmid}\widetilde{\chi}^{6}(\tau)|\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\ \breve{g}^{6}|^{1/2})}{4\hat{\partial}_5(|\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\ \breve{g}^{6}|^{1/2})}\\
&-\frac{\int d\hat{x}^{5}\{{}^{\shortmid}_{3}\widetilde{\Im}^{\star}(\tau)\ \hat{\partial}_5[(\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\breve{g}^{6})\,{}^{\shortmid}\widetilde{\chi}^{8}(\tau)]\}}{\int d\hat{x}^{5}\{{}^{\shortmid}_{3}\widetilde{\Im}^{\star}(\tau)\ \hat{\partial}_5[(\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\breve{g}^{6})]\}}]\}\\
&\breve{g}_5(\breve{r})(\,{}^{\shortmid}\mathbf{e}^{5}(\tau))^{2}\\
&+{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\ (1+\kappa\,{}^{\shortmid}\widetilde{\chi}^{6}(\tau))(dp_6)^{2}+(dp_7)^{2}-dE^{2},
\end{aligned}
\tag{198}
$$

where

$$
\begin{aligned}
\mathbf{e}^{3}(\tau) &= d\hat{x}^{3}+[\frac{\hat{\partial}_{i_1}\int d\hat{x}^{3}\ {}^{\shortmid}_{2}\widetilde{\Im}^{\star}(\tau)\ \hat{\partial}_3\widetilde{\zeta}_4(\tau)}{\breve{N}^{3}_{i_1}\ {}^{\shortmid}_{2}\widetilde{\Im}^{\star}(\tau)\hat{\partial}_3\widetilde{\zeta}_4(\tau)}\\
&\quad+\kappa(\frac{\hat{\partial}_{i_1}[\int d\hat{x}^{3}\ {}^{\shortmid}_{2}\widetilde{\Im}^{\star}(\tau)\hat{\partial}_3(\widetilde{\zeta}_4(\tau)\widetilde{\chi}_4(\tau))]}{\hat{\partial}_{i_1}\ [\int d\hat{x}^{3}\ {}^{\shortmid}_{2}\widetilde{\Im}^{\star}(\tau)\hat{\partial}_3\widetilde{\zeta}_4(\tau)]}\\
&\quad-\frac{\hat{\partial}_3(\widetilde{\zeta}_4(\tau)\widetilde{\chi}_4(\tau))}{\hat{\partial}_3\widetilde{\zeta}_4(\tau)})]\ \breve{N}^{3}_{i_1}dx^{i_1},\\
{}^{\shortmid}\mathbf{e}^{5}(\tau) &= d\hat{x}^{5}+[\frac{\hat{\partial}_{i_2}\ \int d\hat{x}^{5}\ {}^{\shortmid}_{3}\widetilde{\Im}^{\star}(\tau)\ \ \hat{\partial}_5(\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau))}{{}^{\shortmid}\breve{N}^{5}_{i_2}\ {}^{\shortmid}_{3}\widetilde{\Im}^{\star}(\tau)\ \ \hat{\partial}_5(\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau))}\\
&\quad+\kappa(\frac{\hat{\partial}_{i_2}[\int d\hat{x}^{5}\ {}^{\shortmid}_{3}\widetilde{\Im}^{\star}(\tau)\ \ \hat{\partial}_5(\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\ \ \breve{g}^{6})]}{\hat{\partial}_{i_2}\ [\int d\hat{x}^{5}\ {}^{\shortmid}_{3}\widetilde{\Im}^{\star}(\tau)\ \ \hat{\partial}_5(\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau))]}\\
&\quad-\frac{\hat{\partial}_5(\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau)\ \ \breve{g}^{6})}{\hat{\partial}_5(\,{}^{\shortmid}\widetilde{\zeta}^{6}(\tau))})]\ \ {}^{\shortmid}\breve{N}_{5 i_2}d\,{}^{\shortmid}x^{i_2}.
\end{aligned}
$$

The Finsler–Hamilton s-metrics (198) generate τ-families of rotoid configurations in coordinates $(\breve{r}, \hat{x}^2, \hat{x}^3)$ (as nonholonomic deformations of the phase BH solution (195)) if we chose such generating functions:

$$\widetilde{\chi}_4(\tau) = \widetilde{\chi}_4(\tau, \breve{r}, \hat{x}^2, \hat{x}^3) = 2\underline{\chi}(\tau, \breve{r}, \hat{x}^2)\sin(\omega_0 \hat{x}^3 + \hat{x}^3_0), \quad (199)$$

where $\underline{\chi}(\tau, \breve{r}, \hat{x}^2)$ are smooth functions (which can be approximated to some constants) and $(\omega_0, \hat{x}^3_0)$ is a couple of constants. In a conventional 5-d phase space on the shells $s = 1, 2, 3$, trivially imbedded into a 8-d phase space, such a solution posses a distinct ellipsoidal type horizon with respective eccentricity κ stated by the equations

$$\widetilde{\zeta}_4(\tau)(1 + \kappa\, \widetilde{\chi}_4(\tau))\breve{g}_4(\breve{r}) = 0 \text{ i.e. } (1 + \kappa\, \widetilde{\chi}_4)\breve{f}(\breve{r}) = \left(1 - \frac{\hat{m}}{\breve{r}^2} - \frac{\widetilde{\Lambda}_{[5]}}{6}\breve{r}^2 + \frac{\hat{q}^2}{\breve{r}^4} + \kappa\, \widetilde{\chi}_4\right) = 0,$$

for $\widetilde{\zeta}_4 \neq 0$. For small parametric deformations and configurations with $-\frac{\Lambda_{[5]}}{6}\breve{r}^2 + \frac{\hat{q}^2}{\breve{r}^4} \approx 0$, we can approximate for a fixed τ_0, $\breve{r} \simeq \hat{m}^{1/2}/(1 - \frac{\kappa}{2}\, \widetilde{\chi}_4)$. Such parametric formulas define for a rotoid horizon encoding data for small gravitational R-flux polarizations and Finsler–Hamilton configurations. In the limits of zero eccentricity, such e BE configurations transform into a 5-d BH embedded into nonassociative 8-d phase space.

Extending the concept of Bekenstein–Hawking entropy for phase spaces determined by quadratic linear elements (195), we can define such thermodynamic values (computations are similar to those for formulas (8)–(15) in [140] but with different constants and using notations for Finsler–Hamilton spaces):

$$^0\breve{S} = \frac{^0\breve{A}}{4G_{[5]}} = \frac{\omega_{[3]}\breve{r}_h}{4G_{[5]}} \text{ and}$$
$$^0\breve{T} = \frac{1}{2\pi \breve{r}_h}(\epsilon + 2\frac{\breve{r}_h^2}{l_{[5]}^2}) - \frac{2G_{[10]}^2\hat{Q}^2}{3\pi^9 l_{[5]}^8 \breve{r}_h^5}, \text{ for}$$
$$\hat{M} = \frac{3\omega_{[3]}\hat{m}}{16\pi G_{[5]}}(\epsilon\breve{r}_h^2 + \frac{\breve{r}_h^4}{l_{[5]}^2} + \frac{4G_{[5]}\hat{Q}^2 l_{[5]}^2}{3\pi^2\breve{r}_h^2}), \quad (200)$$

where $\breve{r}_h$ and $^0\breve{A}$ are, respectively the horizon and area of horizon of 5-d BH, $G_{[5]} = G_{[10]}/(\pi^3 l_{[5]}^5)$ and $G_{[10]} = \ell_p^8$. Such formulas can be generalized for rotoid deformations $\breve{r}_h \rightarrow \hat{m}^{1/2}/(1 - \frac{\kappa}{2}\widetilde{\chi}_4)$ and $^0\breve{A} \rightarrow {}^{rot}\breve{A}$, with a tilde $\widetilde{\chi}_4(\tau)$ (199), when we compute for respective BE configurations:

$$\breve{S}(\tau) = {}^0\breve{S}(1 + \frac{\kappa}{2}\widetilde{\chi}_4(\tau)) \text{ and } \breve{T}(\tau) = {}^0\breve{T} + \kappa \times \left(-\frac{\epsilon}{4\pi\breve{r}_h} + \frac{\breve{r}_h}{2\pi l_{[5]}^2} - \frac{5G_{[10]}^2\hat{Q}^2}{3\pi^9 l_{[5]}^8\breve{r}_h^5}\right)\widetilde{\chi}_4(\tau). \quad (201)$$

As in section 5.3.3 of [40] (see formulas (97) and (98) in that work), the modified Hawking temperatures $\breve{T}(\tau)$ and $^0\breve{T}$ are stated by requiring the absence of the potential conical singularity of the Euclidean BH at the horizon in the phase space. Such conditions can be imposed on Finsler–Hamilton configurations possessing certain phase space horizons.

7.2.4 G. Perelman thermodynamics of nonassociative flows of Finsler–Hamilton phase RN-AdS BHs

The Bekenstein–Hawking thermodynamic paradigm does not allow to characterize and study physical properties of general classes of quasi-stationary solutions (196) or (198) if there are not imposed special conditions for nonassociative deformations, for instance, when there are generated BE configurations of type (199). In our partner works [10,40,41] we generalized in nonassociative form the G. Perelman approach for the geometric flows. This allows also to define and compute statistical thermodynamic variables (180) for Finsler–Hamilton configurations.

We explain how to compute such values for any data ${}^{\shortmid}\breve{\mathbf{g}}_{\alpha_s}$ (195), and ${}^{\shortmid}_{s}\widetilde{\Im}^{\star}(\tau)$ which are via nonlinear symmetries (193) to $\widetilde{\Lambda}_{[5]}$, and

$$|\, \widetilde{\Lambda}_{[5]}\, \widetilde{\eta}_4(\tau)\breve{g}_4)| = |\, \widetilde{\Lambda}_{[5]}\, \widetilde{\zeta}_4(\tau)\breve{g}_4|(1 - \kappa\widetilde{\chi}_4(\tau)), |$$
$$\widetilde{\Lambda}_{[5]}\, {}^{\shortmid}\widetilde{\eta}^6(\tau)\, {}^{\shortmid}\breve{g}^6| = |\, \widetilde{\Lambda}_{[5]}\, {}^{\shortmid}\widetilde{\zeta}^6(\tau)\, {}^{\shortmid}\breve{g}^6|(1 - \kappa\, {}^{\shortmid}\widetilde{\chi}^6(\tau))$$

defining a a subclass of s-metrics (198). We obtain such thermodynamic functionals:

$${}^{\shortmid}_{s}\widetilde{\mathcal{W}}^{\star}_{\kappa}(\tau) = \int_{\tau'}^{\tau} \frac{d\tau}{32(\pi\tau)^4}\frac{2\tau\widetilde{\Lambda}_{[5]}^2 - 1}{\widetilde{\Lambda}_{[5]}^2}\, {}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau),$$
$${}^{\shortmid}_{s}\widetilde{\mathcal{Z}}^{\star}_{\kappa}(\tau) = \exp\left[\int_{\tau'}^{\tau}\frac{d\tau}{(2\pi\tau)^4}\frac{1}{\widetilde{\Lambda}_{[5]}^2}\, {}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau)\right],$$
$${}^{\shortmid}_{s}\widetilde{\mathcal{E}}^{\star}_{\kappa}(\tau) = -\int_{\tau'}^{\tau}\frac{d\tau}{64\pi^4\tau^3}\frac{\tau\widetilde{\Lambda}_{[5]} - 1}{\widetilde{\Lambda}_{[5]}^2}\, {}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau),$$
$${}^{\shortmid}_{s}\widetilde{\mathcal{S}}^{\star}_{\kappa}(\tau) = -\int_{\tau'}^{\tau}\frac{d\tau}{64(\pi\tau)^4}\frac{\tau\widetilde{\Lambda}_{[5]} - 2}{\widetilde{\Lambda}_{[5]}^2}\, {}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau),$$

for a running Finsler–Hamilton phase space volume functional

$${}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau) = \int_{{}^{\shortmid}_{s}\widehat{\Xi}} {}^{\shortmid}\delta\, {}^{\shortmid}_{\eta}\widetilde{\mathcal{V}}(\, {}^{\shortmid}_{s}\widetilde{\Im}^{\star}(\tau),\ {}^{\shortmid}\breve{g}_{\alpha_s}), \text{ for } s = 1, 2, 3.$$

The thermodynamic properties of such nonassociative geometric flow systems are studied in details in Sect. 5.3.4 of [40]. In this subsection, we use the tilde variables for Finsler–Hamilton configurations. They characterize different classes of nonassociative BH solutions and their τ-evolution.

8 Conclusions and perspectives

This is a status report on the anholonomic frames and connection deformation method, AFCDM, and some important results for constructing generic off-diagonal solutions in 4-d gravity theories and higher dimension generalizations. The approach includes new methods and original solutions for nonassociative star product deformed gravity and related 8-d Finsler–Lagrange–Hamilton phase space structures when an abstract geometric and s-adapted formalism for MGTs and nonassociative and nonholonomic flows is summarized in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 from Appendix. We reviewed and presented new classes of exact/parametric solutions constructed by using the AFCDM for constructing exact and parametric solutions in general relativity, GR, and modified gravity theories, MGTs [6–10,15–18,25–29,40,41].

The main results solving the objectives of this work are as follow:

The first objective, **Obj1**, was to outline the geometry of nonholonomic Lorentz manifolds with conventional (2+2)-splitting and distortion of connections. In such an approach, the fundamental geometric distinguished objects, d-objects, (for instance, the curvature and Ricci d-tensors) are derived as distorted from the Levi–Civita, LC, connection to a canonical distinguished connection, d-connection, structure and when the geometric constructions are adapted to a prescribed nonlinear connection, N-connection, structure. The difference between our nonholonomic dyadic formulation and other geometric and analytic methods involving dyadic variables (in most general forms, there are considered complex dyads, for instance, the Newman–Penrose formalism, outlined in [2–4]) is that we define and work with N-adapted distortions and canonical d-connections. Such canonical nonholonomic variables are important for proving general decoupling and integration properties of (modified) Einstein equations (which was the **Obj2**). Such proof is impossible for other dyadic/tetradic formalisms involving only the LC-connection structure if canonical d-connections are not considered.

In Sect. 4, we provided and studied explicit examples of new classes of generic off-diagonal solutions in GR and MGTs constructed following the AFCDM (stated by **Obj3**). We analyzed, in brief the physical, properties of new Kerr de Sitter solutions and their deformations to spheroidal configurations. Then, the geometric formalism was developed for nonholonomic off-diagonal deformations of cylindrical systems and applied for generating solutions describing locally anisotropic wormholes, black torus and black ellipsoid systems. We proved that corresponding types of nonlinear symmetries and time-space duality properties allow the use of AFCDM and formulas for quasi-stationary off-diagonal configurations to construct and analyze new classes of locally anisotropic cosmological solitonic and spheroidal deformations, study vacuum gravitational 2-d vertices and solitonic vacua for voids. Here we note that all examples of 16 classes of generic off-diagonal solutions constructed in explicit geometric and analytic forms in this article are different from similar ones (derived by more special parameterizations and AFCDM) in previous works [16,17,19,20,24,71,72,77,83].

The nonassociative geometric flow theory on 8-d phase spaces can be formulated in an equivalent form in nonholonomic canonical (hat) variables and in Finsler–Hamilton variables as we defined in 5 for **Obj4**. The first type of formulation allows to decouple and integrate in general off-diagonal form such systems of nonlinear PDEs when a generalized Finsler approach involve metric and affine structures derived from Lagrange and Hamilton generating functions. This provides a possibility to connect in future research the methods of generating off-diagonal solutions in MGTs to general quantum deformations (of gravitational and matter field interactions and geometric evolution scenarios, in general including nonassociative and noncommutative data) determined by conventional Lagrangians or Hamiltonians.

The **Obj5** was achieved by considering canonical distortion relations between hat and tilde connections and defining F- and W-functionals for nonassociative Finsler–Hamilton variables. This allowed to define in abstract geometric form the nonassociative Finsler–Hamilton geometric flow equations which consisted a solution of **Obj6** (for parametric solutions, it is possible also a N-adapted variational proofs using F- and W-functionals). Then, formulating the generalized G. Perelman thermodynamics for nonassociative Finsler–Hamilton geometric flows, provide a solutions of the **Obj7**. Such a statistical and geometric thermodynamics is very important because it allows to characterize very general classes of off-diagonal solutions in MGTs and nonassociative geometric flows which can't be considered in the framework of the Bekenstein–Hawking thermodynamic paradigm (which can be applied only for some subclasses of solutions involving hypersurface horizons, duality and holographic conditions).

The **Obj8**, for providing explicit examples of constructing quasi-stationary solutions encoding 8-d Finsler–Hamilton data as nonassociative geometric flows of higher dimension generalized solutions from Part 1, was solved by applying abstract geometric methods and re-defining the effective sources (to encode nonassociative data) from Tables 12, 13, 14 from Appendix. Correspondingly, that allowed us to define and use corresponding linear symmetries to construct an example of temperature-like evolution of nonassociative black holes and black ellipsoids and respective off-diagonal deformations, i.e. to solve the **Obj9**. We characterize such off-diagonal solutions in different forms by computing respective Bekenstein-Hawking and G. Perelman thermodynamic variables.

Recent progress in elaborating (non) associative/commutative geometric and quantum flow theories [29,98,131–133] with applications in modern MGTs, accelerating cosmology and dark matter and dark energy physics [13,24,63,141,143–146] is reviewed in a series of works [7,8,28,65,147]. Such constructions are performed on nonholonomic Lorentz manifolds of 4–10 dimensions and (co) tangent Lorentz bundles (i.e. phase spaces with respective velocity and/or momentum-like coordinates). Even detailed proofs and solutions are provided in the main part of this article only for 4-d nonholonomic dyadic geometries, our methods, main formulas and exact/parametric solutions can be extended in straightforward forms using a corresponding abstract geometric formalism to higher dimension spacetimes and phase spaces with conventional $(2+2)+(2+2)+2+\cdots$ splitting. For pedagogical purposes (to propose to interested researchers a complete classification and summary of most important, general, and typical solutions), all such constructions for higher dimensions are summarized in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 provided in Appendix (a solution of **Obj10**). Those tables and related typical ansatz for generic off-diagonal quasi-stationary/locally anisotropic/velocity/momentum phase space solutions provide a general solution for the **Obj4** of this work. Nevertheless, we emphasize here that for constructing explicit examples of exact and parametric solutions of nonlinear systems of PDEs describing geometric and information flows and with applications of the AFCDM in modern cosmology and astrophysics one should elaborate other original papers and status reports.

This review covers the Obj1–Obj4 as they were stated for SV's Fulbright scholar program (2022–2023, USA) and the Obj5–Obj10 for his CAS LMU "scholar at risk" fellowship in Munich, Germany (2024). A part of those programs consisted in elaborating on a series of lectures and supplementary material (as this status report) for a summary of results elaborated by his research groups during the last 25 years in Eastern Europe and collaboration with researchers from Western Countries, Romania and Turkey. To outline such original and new methods and results is necessary to cite and discuss certain tenths of works related to the AFCDM, further developments, and applications. Papers by other authors are cited and discussed only if they contain relevant former and important results.

As we proved in Sect. 4, the AFCDM can be applied for constructing and investigating nonholonomic off-diagonal deformations of black hole, BH, solutions into other types of quasi-stationary and/or locally anisotropic cosmological solutions in GR and MGTs. It is well known that there are fundamental and rigorous proofs of mathematical theorems on the uniqueness of smooth BH theorems in vacuum (for instance, for the Kerr solution), rigidity of stationary BHs, on the stability of the Minkowski space and various BH solutions, on the formation of trapped surfaces, cosmic censorship theorems etc. Such results are outlined in many monographs containing hundreds of pages with tedious mathematical proofs (see [148–150] and references therein). It is not clear if and how those rigorous mathematical theorems have connections to MGTs, quantum gravity models, and modern accelerating cosmology. This is a task for a new generation of mathematicians and theoretical physics and quantum information theory researchers.

The AFCDM can be considered not only as a general geometric method for decoupling nonlinear systems of PDEs in mathematical relativity, MGTs, and certain geometric flow evolution equations related to gravity and geometric thermodynamics; when such a general nonlinear decoupling allows formal explicit integrations and generating/ finding off-diagonal solutions for PDEs not reducing them to ODEs by some special diagonal ansatz. In principle, the AFCDM provides a new methodology for constructing different types of new classes of exact/parametric solutions using symbolic and formal geometric techniques, when the rigorous mathematical and physical properties of generated solutions have to be stated by further assumptions. We can model nonsingular and physically viable (at least for certain small parameters) off-diagonal solutions for necessary smooth classes of generating functions and generating sources, choosing specific types of integration functions/constants. They can define quasi-stationary gravitational and (effective) matter field configurations, or describe certain locally anisotropic cosmological scenarios. For certain types of (cosmological) evolution problems, we can put and solve respective Cauchy problems etc., analyze stability properties etc. (for instance, in [21–23] we studied if black ellipsoids, BE, can be stable in GR and (non) commutative MGTs).

In general, the new classes of constructed and studied in our works solutions, involving generic off-diagonal metrics and modified LC-connections, are not subjected to the conditions of theorems proven in mathematical relativity for spacetimes with higher symmetries and metrics with coefficients to be of some required smooth class of functions. It took more than 50 years till physicists and mathematicians understood the fundamental properties of the Schwarzschild and Kerr solutions in 4-d gravity. The approach with transforms of systems of nonlinear PDEs into nonlinear ODEs under various special assumptions on symmetries of interactions, smooth classes of solutions, asymptotic/boundary conditions etc. had in the past a strong motivation being used less exact observations in cosmology and astrophysics. The Universe was considered as an almost spherical one being isotropic and homogeneous; and BH solutions were found also for spherical configurations, with certain observed stability and asymptotic conditions. The "high symmetry" paradigm, including certain "fluctuations" with quantum anisotropies and structure formation, was changed in modern accelerating cosmology and related dark matter and dark energy physics. The Einstein gravity seems to be modified and new classes of solutions may be generic off-diagonal, with coefficients of metrics depending on all spacetime and possible phase space coordinates. Such solutions provide more rich opportunities to elaborate on global and local quasi-periodic structure formation scenarios, local anisotropic configurations, and provide new types of geometric evolution of nonholonomic gravitational systems and characterized by respective geometric thermodynamic models studied in [13,24,29,63,98,131–133,141,143–146]. Another important property is that such solutions can be generated in such forms when they define realistic, viable and important physical models even if they can be unstable, with singularities, nontrivial nonlinear anisotropic cosmology, encoding quantum and extra dimension contributions etc. In explicit form, this is possible by selecting respective classes of generating and integration functions and making certain assumptions on the type of effective sources, variation of physical constants and their polarization etc.

Finally, we note that the AFCDM and solutions reviewed and/or constructed in this work provide a commutative and associative background for developing a research program on nonassociative geometric methods and exact/parametric solutions applied in mathematical particle physics, string and M-theory, quantum information and gravity [5,6,9,10,40,41].

Acknowledgements This status report article consists of a bridge between SV's Fulbright in the USA and CAS LMU in Germany fellowships and research programs. The reviewed geometric methods in physics with new results and applications reflect also 40 years of research and collaborations between scholars and young researchers from Eastern Europe, Türkey, and Western Countries (more than 10 long-term NATO fellowships; visiting programs at CERN, Geneva; Perimeter and Fields Institutes, Canada, etc.). During the reviewing process of this work, a series of recent partner papers developing the approaches on nonassociative/noncommutative and nonmetric modifications to Einstein–Yang–Mills–Higgs–Dirac theories, with applications in quantum gravity and modern cosmology and astrophysics were published [151–156].

Data Availability Statement My manuscript has no associated data. [Authors' comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.]

Code Availability Statement My manuscript has no associated code/software. [Authors' comment: Code/Software sharing not applicable to this article as no code/software was generated or analysed during the current study.]

Funded by SCOAP[3].

A Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 for the AFCDM and 4–10-d spacetime or 8-d phase space solutions

In this Appendix, we summarize the results on the AFCDM for constructing exact/parametric solutions for various 4-d and extra dimensions, and/or with additional velocity/momentum type coordinates in respective GR and MGTs. In respective subsections, we provide some basic formulas and main references on rigorous geometric proofs and existing applications published in modern literature on mathematics and physics.

The main steps on certain general decoupling and integrating of generalized Einstein equations with generic off-diagonal quasi-stationary and locally anisotropic cosmological metrics in 4-d gravity theories outlined below in Tables 1, 2, 3. The geometric proofs on finding solutions and various examples for Lorentz manifolds $\mathbf{V}$, $dim\mathbf{V} = 4$ with nonholonomic 2+2 splitting and canonical deformation of the LC-connection were sketched in previous sections. We use geometric data $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}}[\mathbf{g}, \mathbf{N}] = \nabla[\mathbf{g}] + \widehat{\mathbf{Z}}[\mathbf{g}, \mathbf{N}])$ stated for a N-connection structure $\mathbf{N}$, when the canonical d-connection $\widehat{\mathbf{D}}$ is defined in such a way that corresponding modified Einstein equations can be decoupled and integrated in general off-diagonal form for a metrics $\mathbf{g}[u]$, with coefficients depending on all spacetime coordinates $u^{\alpha} = (x^i, y^a)$. The new classes of solutions are determined by respective generating and integration functions and generating (effective) sources $\widehat{\mathbf{Y}}$, all related via nonlinear symmetries to some effective cosmological constants $\Lambda = ({}_1\Lambda, {}_2\Lambda)$. We can extract LC-configurations by imposing additional nonholonomic constraints when $\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}\rightarrow 0} = \nabla$ and the related canonical both distortion, $\widehat{\mathbf{Z}}$, and torsion, $\widehat{\mathcal{T}}$, d-tensors vanish. The generating source $\widehat{\mathbf{Y}} = ({}_1\Upsilon, {}_2\Upsilon)$ encodes respectively the information on energy–momentum tensors for matter fields and various possible contributions from other classical and/or quantum gravity and matter field interactions. Geometric constructions and detailed proofs are provided in [25–28] and review [7].

In abstract geometric and N-adapted coordinate forms, the AFCDM can be generalized for constructing solutions in various higher dimension gravity theories. Such extensions and modifications can be performed in a formal abstract geometric form, for instance, for $\dim \mathbf{V} = 5, 6, \ldots, 10, 11, \ldots$ (which is applicable in (super) string/gravity theories), with a nonholonomic diadic splitting determined by a conventional $2+2+2+\cdots$ splitting of dimensions for some 2-d shells $s=1,2,3,4$ etc. The local signature of such metrics ${}^{s}\mathbf{g}$ is chosen to be $(+++-+++\cdots+)$ on higher dimension Lorentz manifolds (this in order to simplify the geometric formulas). For a corresponding set of geometric and physical data $({}^{s}\mathbf{g}, {}^{s}\mathbf{N}, {}^{s}\widehat{\mathbf{D}}[{}^{s}\mathbf{g}, {}^{s}\mathbf{N}] = \nabla[{}^{s}\mathbf{g}] + {}^{s}\widehat{\mathbf{Z}}[{}^{s}\mathbf{g}, {}^{s}\mathbf{N}], {}_{s}\Upsilon, {}_{s}\Lambda)$, the gravitational field equations can be decoupled and integrated in very general forms for generic off-diagonal metrics depending on all higher dimension spacetime coordinates $u^{\alpha_s}=(x^{i_1}, y^{a_2}, y^{a_3}, y^{a_4}, \ldots)$, when $i_1=1,2; a_2=3,4; a_3=5,6; a_4=7,8;$ ... A respective nonholonomic dyadic shell decomposition structure can be defined for all necessary geometric and physical objects and computed in general off-diagonal form for coordinate frames. The solutions may involve nontrivial torsion and/or nonmetricity fields structures, and various contributions of extra-dimensions, for instance, from string and M-theory theories, gauge gravity models, noncommutative and nonassociative models with data encoded into certain nontrivial effective sources ${}_{s}\Upsilon$. For detailed proofs and various examples and applications, we cite [6,9,10,12–14,17–20,23,24,65,71,72,77,83], see a recent review of results in Appendix B and references to [7].

Various relativistic phase space theories can be elaborated on tangent bundle, $T\mathbf{V}$, and contangent bundle, $T^{*}\mathbf{V}$, where $\mathbf{V}$ is a Lorentz manifold [7,8,24,29,30]. Here we study models with $\dim \mathbf{V}=4$, when $\dim T\mathbf{V}=8$ and $\dim T^{*}\mathbf{V}=8$. Theories on $T\mathbf{V}$ are relativistic generalizations of the so-called Finsler–Lagrange geometry when the total space metrics (and the coefficients of fundamental geometric objects, like the (non) linear connections, curvature/torsion/Ricci etc. tensors) depend both on spacetime coordinates x^{i} and on velocity type coordinates, $u^{a}=v^{a}$, for $u=(x,v)=\{u^{\alpha}=(x^{i},v^{a})\}$, where $i=1,2,3,4$ on the base manifold $\mathbf{V}$ and $a=5,6,7,8$ for a typical fiber in the phase space. The nonholonomic geometric constructions can be performed in shell dyadic form with conventional (2+2)+(2+2) splitting of dimensions and local coordinates $u^{\alpha_s}=(x^{i_1}, y^{a_2}, v^{a_3}, v^{a_4})$, when the canonical geometric data are defined by distortions

$$({}_{s}\mathbf{g}(x,v), {}_{s}\mathbf{N}(x,v), {}_{s}\widehat{\mathbf{D}}[{}_{s}\mathbf{g}, {}_{s}\mathbf{N}] = \nabla[{}_{s}\mathbf{g}] + {}_{s}\widehat{\mathbf{Z}}[{}_{s}\mathbf{g}, {}_{s}\mathbf{N}], {}_{s}\Upsilon, {}_{s}\Lambda).$$

Standard phase space models ale elaborated on cotangent bundles $\mathcal{M}=T^{*}\mathbf{V}$, when the geometric/physical objects depend on spacetime and momentum like variables ${}^{\shortmid}u=(x,p)=\{x^{i},p_{a}\}$, where ${}^{\shortmid}p=p=(p_3, p_4=E)$ are cofiber coordinates. Such dual coordinates can be related to velocity type ones v^{a} using, for instance, Legendre transforms, and subjected to additional conditions to define almost symplectic systems. On $\mathcal{M}$, we can construct various types of phase space kinetic, geometric thermodynamic, or (nonholonomic) gravitational models. The nonholonomic phase space constructions can be performed in shell dyadic form with conventional (2+2)+(2+2) splitting of dimensions and local coordinates $u^{\alpha_s}=(x^{i_1}, y^{a_2}, p_{a_3}, p_{a4})$. The canonical geometric data are defined as

$$({}_{s}^{\shortmid}\mathbf{g}(x,p), {}_{s}^{\shortmid}\mathbf{N}(x,v), {}_{s}^{\shortmid}\widehat{\mathbf{D}}[{}_{s}^{\shortmid}\mathbf{g}, {}_{s}^{\shortmid}\mathbf{N}] = {}^{\shortmid}\nabla[{}_{s}^{\shortmid}\mathbf{g}] + {}_{s}^{\shortmid}\widehat{\mathbf{Z}}[{}_{s}\mathbf{g}, {}_{s}^{\shortmid}\mathbf{N}], {}_{s}^{\shortmid}\Upsilon, {}_{s}^{\shortmid}\Lambda),$$

which allows us to decouple and integrate in certain general forms respective phase space modified gravitational equations. Generating sources ${}_{s}^{\shortmid}\Upsilon$ may encode, for instance, contributions from nonassociative/noncommutative terms in string theory, various quasi-classical and quantum deformations etc. when certain momentum like variables are introduced for respective geometric/physical models. It is possible to define certain nonholonomic variables when such phase space geometries are models as Hamilton type relativistic spaces (which are dual to respective Lagrange–Finsler geometries). Such geometric models can be generalized in supersymmetric forms, for nonassociative and noncommutative geometries; metric-affine theories with nonmetricity fields and torsion; nonsymmetric metrics and generalized connections; subjected to deformation quantization and generalized, for instance, to geometric and quantum information flow theories [6–10,15,24,29,30].

All above mentioned MGTs can be formulated in abstract form and in nonholonomic canonical variables which allow to apply the AFCDM in order to prove general decoupling and integration properties and construct exact and parametric solutions defined by generic off-diagonal metrics and generalized connections. If necessary, LC-configurations can be extracted by imposing additional nonholonomic constraints. Proofs of such properties and explicit generating of necessary types of quasi-stationary and/or locally anisotropic cosmological solutions are formal geometric symbolic generalizations and with higher dimension extensions of the constructions provided in the sections of the main part of the paper, for 4-d Lorentz manifolds.

A.1 4-d off-diagonal quasi-stationary and cosmological solutions, Tables 1, 2, 3

The first three tables summarize the main steps on how to use 2+2 nonholonomic variables and corresponding ansatz for metrics which allow us to construct quasi-stationary and, for respective t-dual symmetries, locally anisotropic cosmological solutions.

A.1.1 Metric ansatz and systems of nonlinear ODEs and PDEs

Parameterizations of frames/coordinates for Lorentz manifolds with N-connection h- and v-splitting of geometric objects and generating of (effective) sources are provided in Table 1. There are stated two types of generic off-diagonal metric ansatz. The first one is for generating quasi-stationary metrics with dependence only on space coordinates and the second one, for so-called locally anisotropic cosmological solutions, is with dependence on the time-like coordinate and possible dependencies on two other space like coordinates.

General decoupling properties can be proven in explicit form for generic off-diagonal ansatz with Killing symmetry on ∂_4, for quasi-stationary configurations, or on ∂_3, for locally anisotropic cosmological models, see respectively (32) and (33). All formulas derived for (33) are certain t-dual to those for quasi-stationary configurations, but with a change of local signature. To emphasize this, we underline respective symbols of geometric objects.

A.1.2 Decoupling and integration of (modified) Einstein equations & quasi-stationary configurations

The key steps for applying the AFCDM for generating stationary off-diagonal exact solutions of (modified) Einstein equations are outlined in Table 2. Such solutions are, in general, with nontrivial nonholonomically induced torsion (27). They can be re-defined equivalently in terms of generating functions $\Psi(x^k, y^3)$ or $\Phi(x^k, y^3)$ using nonlinear symmetries (64) and (65), see also (71). Considering η–polarization functions, respective d-metrics and N-connections can be parameterized to describe nonholonomic deformations of a primary (for instance, BH) d-metric $\mathring{\mathbf{g}}$ into target generic off diagonal stationary solutions $\widehat{\mathbf{g}}$ (32) (see also (62)) as $\mathring{\mathbf{g}} \to \widehat{\mathbf{g}}(x^k, y^3) = [g_\alpha(x^k, y^3) = \eta_\alpha(x^k, y^3)\mathring{g}_\alpha,\ \eta^a_i(x^k, y^3)\mathring{N}^a_i]$. Zero torsion LC-configurations in GR can be extracted for additional nonholonomic constraints which are satisfied for a more special class of "integrable" generating functions ($\check{h}_4(x^k, y^3)$, or $\check{\Psi}(x^k, y^3)$ and/or $\check{\Phi}(x^k, y^3)$) for respective sources ${}_2\check{\Upsilon}(x^k, y^3)$ and ${}_2\Lambda$ (90).

The main assumption on (effective) generating sources is that in N-adapted form they can be parameterized in the form $\widehat{\Upsilon}^\alpha_\beta = [\,{}^h\Upsilon\delta^i{}_j,\ {}^v\Upsilon\delta^a{}_b]$ (23), when certain relations to an energy–momentum tensor for matter (22) can be established in algebraic form (choosing respectively the coefficients of frame transforms). This imposes some nonholonomic constraints on the h- and v-dynamics of matter fields, determined by respective distributions of matter fields, and related effective cosmological constants. For such assumptions, we can prove general decoupling and integration properties of the nonholonomic canonical deformed Einstein equations (24). For quasi-stationary configurations, we need additional assumption on two generating sources, for instance, that ${}^h\Upsilon = {}_1\widehat{\Upsilon}(x^i)$ and ${}^v\Upsilon = {}_2\widehat{\Upsilon}(x^i, y^3)$. If such conditions are not satisfied, we can not integrate in explicit form the gravitational field equations for a respectively chosen ansatz. To construct generic off-diagonal solutions in explicit form, we prescribe such generating sources and then show how to decouple and find exact/parametric solutions for an ansatz (32).

If a $\widehat{\Upsilon}^\alpha{}_\beta$ involves certain small parameters encoding nonholonomic deformations and distortions and contributions, for instance, from some effective classical and/or quantum interactions, extra dimension etc., we can formulate a a physical interpretation which is similar to "not-deformed" models. In general, for different types of parameterizations of generating sources, it is not clear if a solutions may have importance for physical theories. Nevertheless, using the AFCDM we are able to investigate off-diagonal nonlinear gravitational and (effective) matter field interactions and construct respective classes of solutions in explicit form. This is more general than in the case when the (modified) Einstein equations are transformed in systems of nonlinear ODEs.

Different parameterizations of quasi-stationary metrics involving respective generating functions, effective sources and cosmological constants, and defining nonholonomic deformations of certain prime d-metrics into target d-metrics are stated by respective formulas (62), (67), (68), (73) and (75). Examples of physically important quasi-stationary solutions are given by explicit models of off-diagonal deformations of new KdS metrics; for cylindrical systems and their deformations; locally anisotropic black holes; BH and BT deformed systems etc.

A.1.3 Decoupling and integration of gravitational PDEs generating cosmological metrics

In Table 3, we summarize the main steps for generating off-diagonal locally anisotropic solutions of (modified) Einstein equations using the AFCDM.

Applying the nonholonomic deformation procedure (for simplicity, we consider metrics determined by a generating function $\underline{h}_4(x^k, t)$, we construct a class of generic off-diagonal cosmological solutions with Killing symmetry on ∂_3 determined by effective sources, ${}_1\underline{\Upsilon}$ and ${}_2\underline{\Upsilon}$, and a nontrivial cosmological constant, $\underline{\Lambda}$,

$$ds^2 = e^{\,\psi(x^k, {}_1\underline{\Upsilon})}[(dx^1)^2 + (dx^2)^2] + \underline{h}_3[dy^3 + (\,{}_1n_k + 4\,{}_2n_k \int dt \frac{(\underline{h}_3{}^\diamond)^2}{|\int dt\ {}_2\underline{\Upsilon}\underline{h}_3{}^\diamond|(\underline{h}_3)^{5/2}})dx^k] - \frac{(\underline{h}_3{}^\diamond)^2}{|\int dt\ {}_2\underline{\Upsilon}\underline{h}_3{}^\diamond|\ \overline{h}_3}[dt + \frac{\partial_i(\int dt\ {}_2\underline{\Upsilon}\ \underline{h}_3{}^\diamond])}{{}_2\underline{\Upsilon}\ \underline{h}_3{}^\diamond}dx^i], \quad (A.1)$$

Table 1 Diagonal and off-diagonal ansatz resulting in systems of nonlinear ODEs and PDEs applying the anholonomic frame and connection deformation method, **AFCDM**, for constructing generic off-diagonal exact and parametric solutions

diagonal ansatz: PDEs $\rightarrow$ **ODE**s		AFCDM: **PDEs with decoupling; generating functions**
radial coordinates $u^{\alpha}=(r,\theta,\varphi,t)$	$u=(x,y)$:	nonholonomic 2+2 splitting, $u^{\alpha}=(x^1,x^2,y^3,y^4=t)$ $\mathbf{N}: T\mathbf{V}=hT\mathbf{V}\oplus vT\mathbf{V}$, locally $\mathbf{N}=\{N_i^a(x,y)\}$
LC-connection $\mathring{\nabla}$	[connections]	canonical connection distortion $\widehat{\mathbf{D}}=\nabla+\widehat{\mathbf{Z}}; \widehat{\mathbf{D}}\mathbf{g}=\mathbf{0}$, $\widehat{\mathcal{T}}[\mathbf{g},\mathbf{N},\widehat{\mathbf{D}}]$ canonical d-torsion
diagonal ansatz $g_{\alpha\beta}(u)$ $=\begin{pmatrix} \mathring{g}_1 & & & \\ & \mathring{g}_2 & & \\ & & \mathring{g}_3 & \\ & & & \mathring{g}_4 \end{pmatrix}$	$\mathbf{g}\Leftrightarrow$	$g_{\alpha\beta}(x^i,y^a)$ general frames / coordinates $g_{\alpha\beta}=\begin{bmatrix} g_{ij}+N_i^a N_j^b h_{ab} & N_i^b h_{cb} \\ N_j^a h_{ab} & h_{ac}\end{bmatrix}$, 2 x 2 blocks $\mathbf{g}_{\alpha\beta}=[g_{ij},h_{ab}], \mathbf{g}=\mathbf{g}_i(x^k)dx^i\otimes dx^i+\mathbf{g}_a(x^k,y^b)\mathbf{e}^a\otimes\mathbf{e}^b$
$\mathring{g}_{\alpha\beta}=\begin{cases}\mathring{g}_{\alpha}(r) & \text{for BHs}\\ \mathring{g}_{\alpha}(t) & \text{for FLRW}\end{cases}$	[coord.frames]	$g_{\alpha\beta}=\begin{cases} g_{\alpha\beta}(x^i,y^3) & \text{quasi-stationary configurations}\\ \underline{g}_{\alpha\beta}(x^i,y^4=t) & \text{locally anisotropic cosmology}\end{cases}$
coord. transf. $e_{\alpha}=e^{\alpha'}_{\ \alpha}\partial_{\alpha'}$, $e^{\beta}=e^{\ \beta}_{\beta'}du^{\beta'}, \check{g}_{\alpha\beta}=\check{g}_{\alpha'\beta'}e^{\alpha'}_{\ \alpha}e^{\beta'}_{\ \beta}$ $\mathring{\mathbf{g}}_{\alpha}(x^k,y^a)\rightarrow\mathring{g}_{\alpha}(r)$, or $\mathring{g}_{\alpha}(t)$, $\mathring{N}_i^a(x^k,y^a)\rightarrow 0$.	[N-adapt. fr.]	$\begin{cases} \mathbf{g}_i(x^k),\mathbf{g}_a(x^k,y^3), \\ \text{or } \mathbf{g}_i(x^k),\underline{\mathbf{g}}_a(x^k,t), \\ N_i^3=w_i(x^k,y^3), N_i^4=n_i(x^k,y^3), \\ \text{or } \underline{N}_i^3=\underline{n}_i(x^k,t), \underline{N}_i^4=\underline{w}_i(x^k,t), \end{cases}$ d-metrics
$\mathring{\nabla}$, $Ric=\{\mathring{R}_{\ \beta\gamma}\}$	Ricci tensors	$\widehat{\mathbf{D}}$, $\widehat{R}ic=\{\widehat{\mathbf{R}}_{\ \beta\gamma}\}$
${}^m\mathcal{L}[\phi]\rightarrow{}^m\mathbf{T}_{\alpha\beta}[\phi]$	generating sources	$\widehat{\mathbf{\Upsilon}}^{\mu}_{\ \nu}=\mathbf{e}^{\mu}_{\ \mu'}\mathbf{e}_{\nu}^{\ \nu'}\mathbf{\Upsilon}^{\mu'}_{\ \nu'}[\,{}^m\mathcal{L}(\varphi),T_{\mu\nu},\Lambda]$ $=diag[\,{}_1\Upsilon(x^i)\delta^i_j,\ {}_2\Upsilon(x^i,y^3)\delta^a_b]$, quasi-stationary configurations $=diag[\,{}_1\Upsilon(x^i)\delta^i_j,\ {}_2\underline{\Upsilon}(x^i,t)\delta^a_b]$, locally anisotropic cosmology
trivial equations for $\mathring{\nabla}$-torsion	LC-conditions	$\widehat{\mathbf{D}}_{\mid\widehat{\mathcal{T}}\rightarrow 0}=\nabla$ extracting new classes of solutions in GR

Such a d-metric is equivalent to (77) like (68) is equivalent to (62). The d-metric (A.1) can be written in terms of gravitational η-polarization and/or χ-polarization functions.

A.2 Off-diagonal higher dimension quasi-stationary and cosmological solutions, Tables 4, 5, 6

The procedure of generating off-diagonal solutions and basic formulas from Tables 1, 2, 3 can be extended in abstract geometric form for higher dimension Lorentz manifolds. Such examples and applications in modern cosmology and astrophysics are provided in [14,65,142]. In explicit form, we can consider 10-d spacetime models with 9 space coordinates, which can be derived in the framework of string gravity theories. The AFCDM is extend and applied in a geometric abstract form using nonholonomic shell dyadic decompositions of dimensions. In this subsection, we consider five shells labelled as $s=1,2,3,4,5$, where the first two ones are stated for 4-d Lorentz manifolds as we considered in the main part of the paper and in the previous subsection of the appendix. There are used nonholonomic $2+2+2+2+2$ splitting of dimensions when the N-adapted coordinates and indices of geometric objects are labeled to describe increasing on order "anion" shells. We shall not provide in further subsections of the appendix the equations and formulas defining higher dimension LC-configurations using 10-d variants of zero torsion conditions (38). The constructions are incremental when the higher dimension generating and integrations functions are nonholonomically constrained for higher dimensions. Such solutions can be found by a respective extra dimension generalization of the system of Eq. (90) and corresponding d-metrics (91).

A.2.1 *Diagonal and off-diagonal ansatz for higher dimensions*

In this subsection, we parameterize the higher dimension coordinates in any form for a space like 9-d hypersurface when the time like coordinate is stated as $u^4=y^4=t$. The signature of the metrics is of type $(+++-++\cdots+)$. In principle, the geometric constructions and the procedure of generating solutions do not depend on signature. Adding, or substituting certain dyadic shells, the main formulas and solutions can be re-defined for higher/lower dimensions, for instance, for 12-d, 8-d, or 6-d nonholonomic Lorentz manifolds with the same parameterizations for the first two shells (used for describing the 4-d spacetimes).

In [14,65], we elaborated on string gravity models with almost symplectic structures in higher dimensions and provided examples of explicit classes of generic off-diagonal solutions. Such configurations can be with non-compactified extra dimensions even decompositions on a string constant parameter are necessary for deriving modified Einstein equations. The physical interpretation of solutions in such higher dimension gravity theories is different from those defined on (co) tangent bundles if the metrics are with different signatures and the extra dimension coordinates are not considered of velocity/momentum type.

Table 2 Off-diagonal quasi-stationary configurations Exact solutions of $\widehat{\mathbf{R}}_{\mu\nu} = \Upsilon_{\mu\nu}$ (24) transformed into a system of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry $\partial_4 = \partial_t$ general or spherical coordinates	$ds^2 = g_i(x^k)(dx^i)^2 + g_a(x^k,y^3)(dy^a + N_i^a(x^k,y^3)dx^i)^2$, for $g_i = e^{\psi(x^k)}, g_a = h_a(x^k,y^3), N_i^3 = w_i(x^k,y^3), N_i^4 = n_i(x^k,y^3)$; $g_i = e^{\psi(r,\theta)}, \ g_a = h_a(r,\theta,\varphi), \ N_i^3 = w_i(r,\theta,\varphi), \ N_i^4 = n_i(r,\theta,\varphi)$,
Effective matter sources	$\boldsymbol{\Upsilon}^{\mu}{}_{\nu} = [\ {}_1\widehat{\Upsilon}(r,\theta)\delta^i_j,\ {}_2\widehat{\Upsilon}(r,\theta,\varphi)\delta^a_b]$, if $x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t$
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ {}_1\widehat{\Upsilon}$; $\quad \varpi = \ln\|\partial_3 h_4/\sqrt{\|h_3 h_4\|}\|$, $\varpi^* \ h_4^* = 2h_3h_4\ {}_2\widehat{\Upsilon}$; for $\alpha_i = (\partial_3 h_4)\ (\partial_i\varpi), \beta = (\partial_3 h_4)\ (\partial_3\varpi)$, $\beta w_i - \alpha_i = 0$; $\quad \gamma = \partial_3\left(\ln\|h_4\|^{3/2}/\|h_3\|\right)$, $n_k^{**} + \gamma n_k^* = 0$; $\quad \partial_1 q = q^\bullet, \partial_2 q = q', \partial_3 q = \partial q/\partial\varphi = q^*$
Generating functions: $h_3(x^k,y^3)$, $\Psi(x^k,y^3) = e^{\varpi}, \Phi((x^k,y^3))$; integration functions: $h_4^{[0]}(x^k)$, ${}_1n_k(x^i),\ {}_2n_k(x^i)$; & nonlinear symmetries	$(\Psi^2)^* = -\int dy^3\ {}_2\widehat{\Upsilon} h_4{}^*$, $\Phi^2 = -4\ {}_2\Lambda h_4$, see (71); $h_4 = h_4^{[0]} - \Phi^2/4\ {}_2\Lambda, h_4^* \neq 0,\ {}_2\Lambda \neq 0 = const$
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ {}_1\widehat{\Upsilon}$; $h_3 = -(\Psi^*)^2/4\ {}_2\widehat{\Upsilon}^2 h_4$, see (59), (58); $h_4 = h_4^{[0]} - \int dy^3(\Psi^2)^*/4\ {}_2\widehat{\Upsilon} = h_4^{[0]} - \Phi^2/4\ {}_2\Lambda$; $w_i = \partial_i\ \Psi/\ \partial_3\Psi = \partial_i\ \Psi^2/\ \partial_3\Psi^2\|$; $n_k = {}_1n_k + {}_2n_k\int dy^3(\Psi^*)^2/\ {}_2\widehat{\Upsilon}^2\|h_4^{[0]} - \int dy^3(\Psi^2)^*/4\ {}_2\widehat{\Upsilon}^2\|^{5/2}$.
LC-configurations (90)	$\partial_\varphi w_i = (\partial_i - w_i\partial_3)\ln\sqrt{\|h_3\|}, (\partial_i - w_i\partial_3)\ln\sqrt{\|h_4\|} = 0$, $\partial_k w_i = \partial_i w_k, \partial_3 n_i = 0, \partial_i n_k = \partial_k n_i$; see d-metric (91) for $\Psi = \check{\Psi}(x^i,y^3), (\partial_i\check{\Psi})^* = \partial_i(\check{\Psi}^*)$ and $\Upsilon(x^i,\varphi) = \Upsilon[\check{\Psi}] = \check{\Upsilon}$, or $\Upsilon = const$.
N-connections, zero torsion	$w_i = \partial_i\check{A} = \begin{cases} \partial_i(\int dy^3\ \check{\Upsilon}\ \check{h}_4{}^*])/\check{\Upsilon}\ \check{h}_4{}^*; \\ \partial_i\check{\Psi}/\check{\Psi}^*; \\ \partial_i(\int dy^3\ \check{\Upsilon}(\check{\Phi}^2)^*)/(\check{\Phi})^*\check{\Upsilon}; \end{cases}$ and $n_k = \check{n}_k = \partial_k n(x^i)$.
polarization functions $\mathbf{\mathring{g}} \rightarrow \widehat{\mathbf{g}}$=$[g_\alpha = \eta_\alpha\mathring{g}_\alpha,\ \eta_i^a\mathring{N}_i^a]$	$ds^2 = \eta_1(r,\theta)\mathring{g}_1(r,\theta)[dx^1(r,\theta)]^2 + \eta_2(r,\theta)\mathring{g}_2(r,\theta)[dx^2(r,\theta)]^2 +$ $\eta_3(r,\theta,\varphi)\mathring{g}_3(r,\theta)[d\varphi + \eta_i^3(r,\theta,\varphi)\mathring{N}_i^3(r,\theta)dx^i(r,\theta)]^2 +$ $\eta_4(r,\theta,\varphi)\mathring{g}_4(r,\theta)[dt + \eta_i^4(r,\theta,\varphi)\mathring{N}_i^4(r,\theta)dx^i(r,\theta)]^2$,
Prime metric defines a BH	$[\mathring{g}_i(r,\theta), \mathring{g}_a = \mathring{h}_a(r,\theta); \mathring{N}_k^3 = \mathring{w}_k(r,\theta), \mathring{N}_k^4 = \mathring{n}_k(r,\theta)]$ diagonalizable by frame/ coordinate transforms.
Example of a prime metric	$\mathring{g}_1 = (1 - r_g/r)^{-1}, \mathring{g}_2 = r^2, \mathring{h}_3 = r^2\sin^2\theta, \mathring{h}_4 = (1 - r_g/r), r_g = const$ the Schwarzschild solution, or any BH solution. for new KdS solutions (92) with $\mathbf{\mathring{g}} \simeq \mathbf{\breve{g}}(x^i,y^3)$=$(\breve{g}_\alpha; \breve{N}_i^a)$;
Solutions for polarization funct.	$\eta_i = e^{\ \psi(x^k)}/\mathring{g}_i; \eta_3\mathring{h}_3 = -\frac{4[(\|\eta_4\mathring{h}_4\|^{1/2})^*]^2}{\|\int dy^3\ {}_2\widehat{\Upsilon}[(\eta_4\mathring{h}_4)]^*\|}$; $\eta_4 = \eta_4(x^k,y^3)$ as a generating function; $\eta_i^3\ \mathring{N}_i^3 = \frac{\partial_i\ \int dy^3\widehat{\Upsilon}(\eta_4\ \mathring{h}_4)^*}{\widehat{\Upsilon}\ (\eta_4\ \mathring{h}_4)^*}$; see (95) or (97); $\eta_k^4\ \mathring{N}_k^4 = {}_1n_k + 16\ {}_2n_k\int dy^3\frac{\left([(\eta_4\mathring{h}_4)^{-1/4}]^*\right)^2}{\|\int dy^3\widehat{\Upsilon}[(\eta_4\ \mathring{h}_4)]^*\|}$
Polariz. funct. with zero torsion	$\eta_i = e^{\ \psi(x^k)}/\mathring{g}_i; \eta_4 = \check{\eta}_4(x^k,y^3)$ as a generating function; $\eta_3 = -\frac{4[(\|\eta_4\mathring{h}_4\|^{1/2})^*]^2}{\mathring{g}_3\|\int dy^3\widehat{\Upsilon}[(\check{\eta}_4\mathring{h}_4)]^*\|}; \eta_i^3 = \partial_i\check{A}/\mathring{w}_k, \eta_k^4 = \frac{\partial_k n}{\mathring{n}_k}$.

Table 3 Off-diagonal locally anisotropic cosmological models Exact solutions of $\widehat{\mathbf{R}}_{\mu\nu} = \underline{\Upsilon}_{\mu\nu}$ (24) transformed into a system of nonlinear PDEs (76)

d-metric ansatz with Killing symmetry $\partial_3 = \partial_\varphi$	$d\underline{s}^2 = g_i(x^k)(dx^i)^2 + \underline{g}_a(x^k,y^4)(dy^a + \underline{N}_i^a(x^k,y^4)dx^i)^2,$ for $g_i = e^{\psi(x^k)}, \ \ \underline{g}_a = \underline{h}_a(x^k,t), \ \underline{N}_i^3 = \underline{n}_i(x^k,t), \ \ \underline{N}_i^4 = \underline{w}_i(x^k,t),$
Effective matter sources	$\underline{\mathbf{\Upsilon}}^\mu{}_\nu = [\ {}_h\underline{\Upsilon}(x^k)\delta^i_j, \ \ {}_v\underline{\Upsilon}(x^k,t)\delta^a_b]; x^1, x^2, y^3, y^4 = t$
Nonlinear PDEs	$\psi^{\bullet\bullet} + \psi'' = 2\ {}_1\underline{\Upsilon};$ $\underline{\varpi}^\diamond\ \underline{h}_3^\diamond = 2\underline{h}_3\underline{h}_4\ {}_2\underline{\Upsilon};$ $\underline{n}_k^{\diamond\diamond} + \underline{\gamma}\underline{n}_k^\diamond = 0;$ $\underline{\beta}\underline{w}_i - \underline{\alpha}_i = 0;$ for $\underline{\varpi} = \ln \vert\partial_t \underline{h}_3/\sqrt{\vert\underline{h}_3\underline{h}_4\vert}\vert,$ $\underline{\alpha}_i = (\partial_t\underline{h}_3)\ (\partial_i\underline{\varpi}), \ \underline{\beta} = (\partial_t\underline{h}_3)\ (\partial_t\underline{\varpi}),$ $\underline{\gamma} = \partial_t \left(\ln \vert\underline{h}_3\vert^{3/2}/\vert\underline{h}_4\vert\right),$ $\partial_1 q = q^\bullet, \partial_2 q = q', \partial_4 q = \partial q/\partial t = q^\diamond$
Generating functions: $\underline{h}_4(x^k,t),$ $\underline{\Psi}(x^k,t) = e^{\underline{\varpi}}, \underline{\Phi}(x^k,t);$ integr. functions: $h_4^{[0]}(x^k), \ {}_1n_k(x^i),$ ${}_2n_k(x^i);$ & nonlinear symmetries	$(\underline{\Psi}^2)^\diamond = -\int dt\ {}_2\underline{\Upsilon}\underline{h}_3^\diamond,$ $\underline{\Phi}^2 = -4\ {}_2\underline{\Lambda}\underline{h}_3;$ $\underline{h}_3 = \underline{h}_3^{[0]} - \underline{\Phi}^2/4\ {}_2\underline{\Lambda}, \underline{h}_3^\diamond \neq 0, \ {}_2\underline{\Lambda} \neq 0 = const$
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ {}_1\underline{\Upsilon};$ $\overline{h}_4 = -(\overline{\Psi}^2)^\diamond/4\ {}_2\underline{\Upsilon}^2\underline{h}_3;$ $\underline{h}_3 = h_3^{[0]} - \int dt(\underline{\Psi}^2)^\diamond/4\ {}_2\underline{\Upsilon} = h_3^{[0]} - \underline{\Phi}^2/4\ {}_2\underline{\Lambda};$ $\underline{n}_k = \ {}_1n_k + \ {}_2n_k \int dt(\underline{\Psi}^\diamond)^2/\ {}_2\underline{\Upsilon}^2\ \vert h_3^{[0]} - \int dt(\underline{\Psi}^2)^\diamond/4\ {}_2\underline{\Upsilon}\vert^{5/2};$ $\underline{w}_i = \partial_i\ \underline{\Psi}/\ \partial_t\underline{\Psi} = \partial_i\underline{\Psi}^2/\ \partial_t\underline{\Psi}^2.$
LC-configurations	$\partial_t\underline{w}_i = (\partial_i - \underline{w}_i\partial_t)\ln\sqrt{\vert\underline{h}_4\vert}, (\partial_i - \underline{w}_i\partial_4)\ln\sqrt{\vert\underline{h}_3\vert} = 0,$ $\partial_k\underline{w}_i = \partial_i\underline{w}_k, \partial_t\underline{n}_i = 0, \partial_i\underline{n}_k = \partial_k\underline{n}_i;$ $\underline{\Psi} = \check{\underline{\Psi}}(x^i,t), (\partial_i\check{\underline{\Psi}})^\diamond = \partial_i(\check{\underline{\Psi}}^\diamond)$ and ${}_2\underline{\Upsilon}(x^i,t) = \underline{\Upsilon}[\check{\underline{\Psi}}] = \check{\underline{\Upsilon}},$ or $\underline{\Upsilon} = const.$
N-connections, zero torsion	$\underline{n}_k = \check{\underline{n}}_k = \partial_k\underline{n}(x^i)$ and $\underline{w}_i = \partial_i\check{\underline{A}} = \begin{cases} \partial_i(\int dt\ \check{\underline{\Upsilon}}\ \check{\underline{h}}_3^\diamond])/\check{\underline{\Upsilon}}\ \check{\underline{h}}_3^\diamond; \\ \partial_i\check{\underline{\Psi}}/\check{\underline{\Psi}}^\diamond; \\ \partial_i(\int dt\ \check{\underline{\Upsilon}}(\check{\underline{\Phi}}^2)^\diamond)/\check{\underline{\Phi}}^\diamond\check{\underline{\Upsilon}}; \end{cases}.$
polarization functions $\mathring{\mathbf{g}} \rightarrow \widehat{\underline{\mathbf{g}}}$=$[\underline{g}_\alpha = \underline{\eta}_\alpha\mathring{\underline{g}}_\alpha, \underline{\eta}_i^a\mathring{\underline{N}}_i^a]$	$ds^2 = \underline{\eta}_i(x^k,t)\mathring{\underline{g}}_i(x^k,t)[dx^i]^2 + \underline{\eta}_3(x^k,t)\mathring{\underline{h}}_3(x^k,t)[dy^3 + \underline{\eta}_i^3(x^k,t)\mathring{\underline{N}}_i^3(x^k,t)dx^i]^2$ $+\underline{\eta}_4(x^k,t)\mathring{\underline{h}}_4(x^k,t)[dt + \underline{\eta}_i^4(x^k,t)\mathring{\underline{N}}_i^4(x^k,t)dx^i]^2,$
Prime metric defines a cosmological solution Example of a prime cosmological metric	$[\mathring{\underline{g}}_i(x^k,t), \mathring{\underline{g}}_a = \mathring{\underline{h}}_a(x^k,t); \mathring{\underline{N}}_k^3 = \mathring{\underline{w}}_k(x^k,t), \mathring{\underline{N}}_k^4 = \mathring{\underline{n}}_k(x^k,t)]$ diagonalizable by frame/ coordinate transforms. $\mathring{g}_1 = a^2(t)/(1-kr^2), \mathring{g}_2 = a^2(t)r^2,$ $\mathring{\underline{h}}_3 = a^2(t)r^2\sin^2\theta, \mathring{\underline{h}}_4 = c^2 = const, k = \pm 1, 0;$ any frame transform of a FLRW or a Bianchi metrics
Solutions for polarization funct.	$\eta_i = e^{\ \psi(x^k)}/\mathring{g}_i; \underline{\eta}_4\mathring{\underline{h}}_4 = -\frac{4[(\vert\underline{\eta}_3\mathring{\underline{h}}_3\vert^{1/2})^\diamond]^2}{\vert\int dt\ {}_2\underline{\Upsilon}[(\underline{\eta}_3\mathring{\underline{h}}_3)]^\diamond\vert};$ gener. funct. $\underline{\eta}_3 = \underline{\eta}_3(x^i,t);$ $\underline{\eta}_k^3\ \mathring{\underline{N}}_k^3 = \ {}_1n_k + 16\ \ {}_2n_k \int dt \frac{\left([(\underline{\eta}_3\mathring{\underline{h}}_3)^{-1/4}]^\diamond\right)^2}{\vert\int dt\ {}_2\underline{\Upsilon}[(\underline{\eta}_3\mathring{\underline{h}}_3)]^\diamond\vert};\ \underline{\eta}_i^4\ \mathring{\underline{N}}_i^4 = \frac{\partial_i\ \int dt\ {}_2\underline{\Upsilon}(\underline{\eta}_3\mathring{\underline{h}}_3)^\diamond}{{}_2\underline{\Upsilon}(\underline{\eta}_3\mathring{\underline{h}}_3)^\diamond},$
Polariz. funct. with zero torsion	$\eta_i = e^{\ \psi}/\mathring{g}_i; \underline{\eta}_4 = -\frac{4[(\vert\underline{\eta}_3\mathring{\underline{h}}_3\vert^{1/2})^\diamond]^2}{\mathring{\underline{g}}_4\vert\int dt\ {}_2\underline{\Upsilon}[(\underline{\eta}_3\mathring{\underline{h}}_3)]^\diamond\vert};$ gener. funct. $\underline{\eta}_3 = \check{\underline{\eta}}_3(x^i,t);$ $\underline{\eta}_k^4 = \partial_k\check{\underline{A}}/\mathring{w}_k; \underline{\eta}_k^3 = (\partial_k\underline{n})/\mathring{n}_k.$

The formulas from Table 4 can be generalized for polarizations η- and χ-polarization functions extending on higher dimensions formulas (70) and (75).

A.2.2 Quasi-stationary higher dimension solutions

The formulas for 4-d off-diagonal quasi-stationary solutions defined by nonlinear quadratic elements (62), (67), (68), (73) and (74) can be generalized in symbolic geometric form to 10-d spacetime Lorentz manifolds as we summarize in Table 5.

As an example of 10-d quasi-stationary quadratic element extending the 4-d formulas (68), we provide

$$
\begin{aligned}
d\widehat{s}^2_{[10d]} &= \widehat{g}_{\alpha_s\beta_s}(x^k, y^3, y^5, y^7, y^9; h_4, h_6, h_8, h_{10}; {}_s\widehat{\Upsilon};\ {}_s\Lambda)du^{\alpha_s}du^{\beta_s} \\
&= e^{\psi(x^k,\ {}_1\widehat{\Upsilon})}[(dx^1)^2 + (dx^2)^2] \\
&\quad - \frac{(h_4^*)^2}{\vert\int dy^3[\ {}_2\widehat{\Upsilon}h_4]^*\vert\ h_4} \\
&\quad \times\{dy^3 + \frac{\partial_{i_1}[\int dy^3(\ {}_2\widehat{\Upsilon})\ h_4^*]}{{}_2\widehat{\Upsilon}\ h_4^*}dx^{i_1}\}^2 \\
&\quad + h_4\{dt + [\ {}_1n_{k_1} + \ {}_2n_{k_1}\int dy^3 \\
&\quad \times \frac{(h_4^*)^2}{\vert\int dy^3[\ {}_2\widehat{\Upsilon}h_4]^*\vert\ (h_4)^{5/2}}]dx_1^k\} \\
&\quad + \frac{(\partial_5 h_6)^2}{\vert\int dy^5\partial_5[\ {}_3\widehat{\Upsilon}h_6]\vert\ h_6}
\end{aligned}
$$

Table 4 Diagonal and off-diagonal ansatz for 10-d Lorentz manifolds and the anholonomic frame and connection deformation method, **AFCDM**, *for constructing generic off-diagonal exact and parametric solutions*

diagonal ansatz: PDEs $\rightarrow$ **ODE**s		AFCDM: **PDEs with decoupling;**
coordinates $u^{\alpha_s}=(u^1,..,u^4=t,..u^{10})$	$^s u=(\ ^{s-1}x,\ ^s y)$	nonholonomic 2+2+2+2+2 splitting; shels $s=1,2,3,4,5$ $u^{\alpha_s}=(x^1,x^2,y^3,y^4=t,y^5,y^6,...,y^9,y^{10})$; $u^{\alpha_s}=(x^{i_1},y^{a_2},y^{a_3},y^{a_4},y^{a_5}); i_1=1,2; a_2=3,4;...,a_5=9,10$; $u^{\alpha_s}=(x^{i_{s-1}},y^{a_s});\ ^s u=(\ ^{s-1}x,\ ^s y)=(x,y),\ s=1,2,3,4,5$;
LC-connection $\mathring{\nabla}$	N-connection; canonical d-connection	$^s\mathbf{N}: T\ ^s\mathbf{V}=hT\mathbf{V}\oplus\ ^2vT\mathbf{V}\oplus\ ^3v\mathbf{TV}\oplus\ ^4v\mathbf{TV}\oplus\ ^5v\mathbf{TV}$, locally $\ ^s\mathbf{N}=\{N^{a_s}_{i_{s-1}}(x,y)=N^{a_s}_{i_{s-1}}(\ ^{s-1}x,\ ^s y)=N^{a_s}_{i_{s-1}}(\ ^s u)\}$ $^s\widehat{\mathbf{D}}=(\ ^1h\widehat{\mathbf{D}},\ ^2v\widehat{\mathbf{D}},\ ^3v\widehat{\mathbf{D}},\ ^4v\widehat{\mathbf{D}},\ ^5v\widehat{\mathbf{D}})=\{\Gamma^{\alpha_s}_{\ \beta_s\gamma_s}\}$; canonical connection distortion $\ ^s\widehat{\mathbf{D}}=\nabla+\ ^s\widehat{\mathbf{Z}};\ ^s\widehat{\mathbf{D}}\ ^s\mathbf{g}=\mathbf{0}$, $^s\widehat{\mathcal{T}}[\ ^s\mathbf{g},\ ^s\mathbf{N},\ ^s\widehat{\mathbf{D}}]$ canonical d-torsion
diagonal ansatz $^2\mathring{g}=\mathring{g}_{\alpha_2\beta_2}(\ ^s u)=\begin{pmatrix}\mathring{g}_1 & & & \\ & \mathring{g}_2 & & \\ & & \mathring{g}_3 & \\ & & & \mathring{g}_4\end{pmatrix}$; $^s g=\mathring{g}_{\alpha_s\beta_s}(\ ^s u)=\begin{pmatrix}^2\mathring{g} & & & \\ & \mathring{g}_5 & & \\ & & \ddots & \\ & & & \mathring{g}_{10}\end{pmatrix}$	$\mathbf{g}\Leftrightarrow$	$g_{\alpha_2\beta_2}(x^{i_1},y^{a_2})$ general frames / coordinates $g_{\alpha_2\beta_2}=\begin{bmatrix} g_{i_1j_1}+N^{a_2}_{i_1}N^{b_2}_{j_1}h_{a_2b_2} & N^{b_2}_{i_1}h_{c_2b_2} \\ N^{a_2}_{j_1}h_{a_2b_2} & h_{a_2c_2}\end{bmatrix}$, 2 x 2 blocks $^2\mathbf{g}=\{\mathbf{g}_{\alpha_2\beta_2}=[g_{i_1j_1},h_{a_2b_2}]\}$, $^2\mathbf{g}=\mathbf{g}_{i_1}(x^{k_1})dx^{i_1}\otimes dx^{i_1}+\mathbf{g}_{a_2}(x^{k_2},y^{b_2})\mathbf{e}^{a_2}\otimes\mathbf{e}^{b_2}$ $\vdots$ $g_{\alpha_s\beta_s}(x^{i_{s-1}},y^{a_s})$ general frames / coordinates $g_{\alpha_s\beta_s}=\begin{bmatrix} g_{i_sj_s}+N^{a_s}_{i_{s-1}}N^{b_s}_{j_{s-1}}h_{a_sb_s} & N^{b_s}_{i_{s-1}}h_{c_sb_s} \\ N^{a_s}_{j_{s-1}}h_{a_sb_s} & h_{a_sc_s}\end{bmatrix}$, $^s\mathbf{g}=\{\mathbf{g}_{\alpha_s\beta_s}=[g_{i_{s-1}j_{s-1}},h_{a_sb_s}]=[g_{i_1j_1},h_{a_2b_2},...,h_{a_5b_5}]\}$, $^s\mathbf{g}=\mathbf{g}_{i_{s-1}}(x^{k_{s-1}})dx^{i_{s-1}}\otimes dx^{i_{s-1}}+\mathbf{g}_{a_s}(x^{k_{s-1}},y^{b_s})\mathbf{e}^{a_s}\otimes\mathbf{e}^{b_s}$ $=\mathbf{g}_{i_1}(x^{k_1})dx^{i_1}\otimes dx^{i_1}+\mathbf{g}_{a_2}(x^{k_1},y^{b_2})\mathbf{e}^{a_2}\otimes\mathbf{e}^{b_2}+...$ $+\mathbf{g}_{a_5}(x^{k_1},y^{b_2},y^{b_3},y^{b_4},y^{b_5})\mathbf{e}^{a_5}\otimes\mathbf{e}^{b_5}$;
$\mathring{g}_{\alpha_2\beta_2}=\left\{\begin{array}{ll}\mathring{g}_{\alpha_2}(\ ^2r) & \text{for BHs} \\ \mathring{g}_{\alpha_2}(t) & \text{for FLRW}\end{array}\right.$ $\mathring{g}_{\alpha_s\beta_s}=\left\{\begin{array}{ll}\mathring{g}_{\alpha_s}(\ ^s r) & \text{for BHs} \\ \mathring{g}_{\alpha_s}(t) & \text{for FLRW}\end{array}\right.$	[coord.frames]	$g_{\alpha_2\beta_2}=\left\{\begin{array}{ll} g_{\alpha_2\beta_2}(x^i,y^3) & \text{quasi-stationary config.} \\ \underline{g}_{\alpha_2\beta_2}(x^i,y^4=t) & \text{locally anisotropic cosmology}\end{array}\right.$ $g_{\alpha_5\beta_5}=\left\{\begin{array}{l} g_{\alpha_5\beta_5}(x^{i_4},y^9) \\ \underline{g}_{\alpha_s\beta_s}(x^{i_4},y^{10})\end{array}\right.$
coord. transf. $e_{\alpha_s}=e^{\alpha'_s}_{\ \alpha_s}\partial_{\alpha'_s}$, $e^{\beta_s}=e^{\beta_s}_{\ \beta'_s}du^{\beta'_s}$, $\mathring{g}_{\alpha_s\beta_s}=\mathring{g}_{\alpha'_s\beta'_s}e^{\alpha'_s}_{\ \alpha_s}e^{\beta'_s}_{\ \beta_s}$ $\mathring{\mathbf{g}}_{\alpha_s}(x^{k_{s-1}},y^{a_s})\rightarrow\mathring{g}_{\alpha_s}(\ ^s r)$, or $\mathring{g}_{\alpha_s}(t),\mathring{N}^{a_s}_{i_{s-1}}(x^{k_{s-1}},y^{a_s})\rightarrow 0$.	[N-adapt. fr.]	$\left\{\begin{array}{l}\mathbf{g}_{i_1}(x^{k_1}),\mathbf{g}_{a_2}(x^{k_1},y^3), \\ \text{or } \mathbf{g}_{i_1}(x^{k_1}),\underline{\mathbf{g}}_{a_2}(x^{k_1},t), \\ N^3_{i_1}=w_{i_1}(x^k,y^3),N^4_{i_1}=n_{i_1}(x^k,y^3), \\ \text{or } \underline{N}^3_{i_1}=\underline{n}_{i_1}(x^{k_1},t),\underline{N}^4_{i_1}=\underline{w}_{i_1}(x^{k_1},t),\end{array}\right.$ d-metrics $\vdots$ $\left\{\begin{array}{l}\mathbf{g}_{i_4}(x^{k_4}),\mathbf{g}_{a_5}(x^{k_4},y^9), \\ \text{or } \mathbf{g}_{i_4}(x^{k_1}),\underline{\mathbf{g}}_{a_5}(x^{k_4},y^{10}), \\ N^9_{i_4}=w_{i_4}(x^{k_4},y^9),N^{10}_{i_1}=n_{i_4}(x^{k_4},y^9), \\ \text{or } \underline{N}^9_{i_4}=\underline{n}_{i_4}(x^{k_4},y^{10}),\underline{N}^{10}_{i_4}=\underline{w}_{i_4}(x^{k_4},y^{10}),\end{array}\right.$
$^s\mathring{\nabla},\ ^sRic=\{\mathring{R}_{\ \beta_s\gamma_s}\}$	Ricci tensors	$^s\widehat{\mathbf{D}},\ ^s\widehat{\mathcal{R}}ic=\{\widehat{\mathbf{R}}_{\ \beta_s\gamma_s}\}$
$^m\mathcal{L}[\phi]\rightarrow\ ^m\mathbf{T}_{\alpha_s\beta_s}[\phi]$	generating sources	$\widehat{\mathbf{\Upsilon}}^{\mu_s}_{\ \nu_s}=\mathbf{e}^{\mu_s}_{\ \mu'_s}\mathbf{e}^{\nu'_s}_{\ \nu_s}\mathbf{\Upsilon}^{\mu'_s}_{\ \nu'_s}[\ ^m\mathcal{L}(\varphi),T_{\mu_s\nu_s},\ ^s\Lambda]$ $=diag[\ _1\Upsilon(x^{i_1})\delta^{i_1}_{j_1},\ _2\Upsilon(x^{i_1},y^3)\delta^{a_2}_{b_2},$ $_3\Upsilon(x^{i_2},y^5)\delta^{a_3}_{b_3},\ _4\Upsilon(x^{i_3},y^7)\delta^{a_4}_{b_4},\ _5\Upsilon(x^{i_4},y^9)\delta^{a_5}_{b_5}]$, quasi-stationary configurations; $=diag[\ _1\Upsilon(x^{i_1})\delta^{i_1}_{j_1},\ _2\underline{\Upsilon}(x^{i_1},t)\delta^{a_2}_{b_2},$ $_3\underline{\Upsilon}(x^{i_2},y^6)\delta^{a_3}_{b_3},\ _4\underline{\Upsilon}(x^{i_3},y^8)\delta^{a_4}_{b_4},\ _5\underline{\Upsilon}(x^{i_4},y^{10})\delta^{a_5}_{b_5}]$, locally anisotropic cosmology;
trivial eqs for $\ ^s\mathring{\nabla}$-torsion	LC-conditions	$^s\widehat{\mathbf{D}}_{\mid\ ^s\widehat{\mathcal{T}}\rightarrow 0}=\ ^s\nabla.$

$$\times\{dy^5+\frac{\partial_{i_2}[\int dy^5(\ _3\widehat{\Upsilon})\ \partial_5 h_6]}{_3\widehat{\Upsilon}\ \partial_5 h_6}dx^{i_2}\}^2$$
$$+h_6\{dy^6+[\ _1n_{k_2}+\ _2n_{k_2}\int dy^5$$
$$\times\frac{(\partial_5 h_6)^2}{|\int dy^5\partial_5[\ _3\widehat{\Upsilon}h_6]|\ (h_6)^{5/2}}]dx^{k_2}\}$$
$$+\frac{(\partial_7 h_8)^2}{|\int dy^7\partial_7[\ _4\widehat{\Upsilon}h_8]|\ h_8}$$

$$\times\{dy^7+\frac{\partial_{i_3}[\int dy^7(\ _4\widehat{\Upsilon})\ \partial_7 h_8]}{_4\widehat{\Upsilon}\ \partial_7 h_8}dx^{i_3}\}^2$$
$$+h_8\{dy^8+[\ _1n_{k_3}+\ _2n_{k_3}\int dy^7$$
$$\times\frac{(\partial_7 h_8)^2}{|\int dy^7\partial_7[\ _4\widehat{\Upsilon}h_8]|\ (h_8)^{5/2}}]dx^{k_3}\}$$
$$+\frac{(\partial_9 h_{10})^2}{|\int dy^9\partial_9[\ _5\widehat{\Upsilon}h_{10}]|\ h_{10}}$$

Table 5 Higher dimension off-diagonal quasi-stationary configurations exact solutions of $\widehat{\mathbf{R}}_{\mu\nu} = \Upsilon_{\mu\nu}$ (24) transformed into a shall system of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry $\partial_4 = \partial_t$ Effective matter sources	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + g_{a_2}(x^{k_1}, y^3)(dy^{a_2} + N^{a_2}_{i_1}(x^{k_1}, y^3)dx^{i_1})^2$ $+g_{a_3}(x^{k_2}, y^5)(dy^{a_3} + N^{a_3}_{i_2}(x^{k_2}, y^5)dx^{i_2})^2$ $+g_{a_4}(x^{k_3}, y^7)(dy^{a_4} + N^{a_4}_{i_3}(x^{k_3}, y^7)dx^{i_3})^2$ $+g_{a_5}(x^{k_4}, y^9)(dy^{a_5} + N^{a_5}_{i_4}(x^{k_4}, y^9)dx^{i_4})^2$, for $g_{i_1} = e^{\psi(x^{k_1})}$, $g_{a_2} = h_{a_2}(x^{k_1}, y^3), N^3_{i_1} = {}^2w_{i_1} = w_{i_1}(x^{k_1}, y^3), N^4_{i_1} = {}^2n_{i_1} = n_{i_1}(x^{k_1}, y^3)$, $g_{a_3} = h_{a_3}(x^{k_2}, y^5), N^5_{i_2} = {}^3w_{i_2} = w_{i_2}(x^{k_2}, y^5), N^6_{i_2} = {}^3n_{i_2} = n_{i_2}(x^{k_2}, y^5)$, $g_{a_4} = h_{a_4}(x^{k_3}, y^7), N^7_{i_3} = {}^4w_{i_3} = w_{i_3}(x^{k_3}, y^7), N^8_{i_3} = {}^4n_{i_3} = n_{i_3}(x^{k_3}, y^7)$, $g_{a_5} = h_{a_5}(x^{k_4}, y^9), N^9_{i_4} = {}^5w_{i_4} = w_{i_4}(x^{k_4}, y^9), N^{10}_{i_4} = {}^5n_{i_4} = n_{i_4}(x^{k_4}, y^9)$, $\mathbf{\Upsilon}^{\mu_s}_{\nu_s} = [\ {}_1\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_2\widehat{\Upsilon}(x^{k_1}, y^3)\delta^{a_2}_{b_2},\ {}_3\widehat{\Upsilon}(x^{k_2}, y^5)\delta^{a_3}_{b_3}$, ${}_4\widehat{\Upsilon}(x^{k_3}, y^7)\delta^{a_4}_{b_4},\ {}_5\widehat{\Upsilon}(x^{k_4}, y^9)\delta^{a_5}_{b_5}]$,	
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon}$; ${}^2\varpi^*\ h_4^* = 2h_3h_4\ {}_2\widehat{\Upsilon}$; ${}^2\beta\ {}^2w_{i_1} - {}^2\alpha_{i_1} = 0$; ${}^2n^{**}_{k_1} + {}^2\gamma\ {}^2n^*_{k_1} = 0$; $\partial_5({}^3\varpi)\ \partial_5 h_6 = 2h_5h_6\ {}_3\widehat{\Upsilon}$; ${}^3\beta\ {}^3w_{i_2} - {}^3\alpha_{i_2} = 0$; $\partial_5(\partial_5\ {}^3n_{k_2}) + {}^3\gamma\partial_5({}^3n_{k_2}) = 0$; ⋮ $\partial_9({}^5\varpi)\ \partial_9 h_{10} = 2h_9h_{10}\ {}_5\widehat{\Upsilon}$; ${}^5\beta\ {}^5w_{i_4} - {}^5\alpha_{i_4} = 0$; $\partial_9(\partial_9\ {}^5n_{k_4}) + {}^5\gamma\partial_9({}^3n_{k_3}) = 0$;	${}^2\varpi = \ln\lvert\partial_3 h_4/\sqrt{\lvert h_3h_4\rvert}\rvert$, ${}^2\alpha_{i_1} = (\partial_3 h_4)\ (\partial_{i_1}\ {}^2\varpi)$, ${}^2\beta = (\partial_3 h_4)\ (\partial_3\ {}^2\varpi)$, ${}^2\gamma = \partial_3\left(\ln\lvert h_4\rvert^{3/2}/\lvert h_3\rvert\right)$, $\partial_1 q = q^\bullet, \partial_2 q = q', \partial_3 q = q^*$ ${}^3\varpi = \ln\lvert\partial_5 h_6/\sqrt{\lvert h_5h_6\rvert}\rvert$, ${}^3\alpha_{i_2} = (\partial_5 h_6)\ (\partial_{i_2}\ {}^3\varpi)$, ${}^3\beta = (\partial_5 h_6)\ (\partial_5\ {}^3\varpi)$, ${}^3\gamma = \partial_5\left(\ln\lvert h_6\rvert^{3/2}/\lvert h_5\rvert\right)$, ⋮ ${}^5\varpi = \ln\lvert\partial_9 h_{10}/\sqrt{\lvert h_9h_{10}\rvert}\rvert$, ${}^5\alpha_i = (\partial_9 h_{10})\ (\partial_i\varpi)$, ${}^5\beta = (\partial_9 h_{10})\ (\partial_9\varpi)$, ${}^5\gamma = \partial_9\left(\ln\lvert h_{10}\rvert^{3/2}/\lvert h_9\rvert\right)$,
Gener. functs: $h_3(x^{k_1}, y^3)$, ${}^2\Psi(x^{k_1}, y^3) = e^{\ {}^2\varpi},\ {}^2\Phi(x^{k_1}, y^3)$, integr. functs: $h_4^{[0]}(x^{k_1})$, ${}_1n_{k_1}(x^{i_1}),\ {}_2n_{k_1}(x^{i_1})$; Gener. functs:$h_5(x^{k_2}, y^5)$, ${}^3\Psi(x^{k_2}, y^5) = e^{\ {}^3\varpi},\ {}^3\Phi(x^{k_2}, y^5)$, integr. functs: $h_6^{[0]}(x^{k_2})$, ${}^3_1n_{k_2}(x^{i_2}),\ {}^3_2n_{k_2}(x^{i_2})$; ... Gener. functs:$h_9(x^{k_4}, y^9)$, ${}^5\Psi(x^{k_3}, y^9) = e^{\ {}^5\varpi},\ {}^5\Phi(x^{k_4}, y^9)$, integr. functs: $h_{10}^{[0]}(x^{k_4})$, ${}^5_1n_{k_4}(x^{i_4}),\ {}^5_2n_{k_4}(x^{i_4})$; & nonlinear symmetries	$(({}^2\Psi)^2)^* = -\int dy^3\ {}_2\widehat{\Upsilon}h_4{}^*$, $({}^2\Phi)^2 = -4\ {}_2\Lambda h_4$, see (71), $h_4 = h_4^{[0]} - ({}^2\Phi)^2/4\ {}_2\Lambda, h_4^* \neq 0,\ {}_2\Lambda \neq 0 = const$; $\partial_5(({}^3\Psi)^2) = -\int dy^5\ {}_3\widehat{\Upsilon}\partial_5 h_6$, $({}^3\Phi)^2 = -4\ {}_3\Lambda h_6$, $h_6 = h_6^{[0]} - ({}^3\Phi)^2/4\ {}_3\Lambda, \partial_5 h_6 \neq 0,\ {}_3\Lambda \neq 0 = const$; ... $\partial_9(({}^5\Psi)^2) = -\int dy^9\ {}_5\widehat{\Upsilon}\partial_9 h_{10}$, $({}^5\Phi)^2 = -4\ {}_5\Lambda h_{10}$, $h_{10} = h_{10}^{[0]} - ({}^5\Phi)^2/4\ {}_5\Lambda, \partial_9 h_{10} \neq 0,\ {}_5\Lambda \neq 0 = const$;	
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon}$; $h_3 = -(\Psi^*)^2/4\ {}_2\widehat{\Upsilon}^2 h_4$, see (59), (58); $h_4 = h_4^{[0]} - \int dy^3(\Psi^2)^*/4\ {}_2\widehat{\Upsilon} = h_4^{[0]} - \Phi^2/4\ {}_2\Lambda$; $w_i = \partial_i\ \Psi/\ \partial_3\Psi = \partial_i\ \Psi^2/\ \partial_3\Psi^2\rvert$; $n_k = {}_1n_k + {}_2n_k\int dy^3(\Psi^*)^2/\ {}_2\widehat{\Upsilon}^2\lvert h_4^{[0]} - \int dy^3(\Psi^2)^*/4\ {}_2\widehat{\Upsilon}^2\rvert^{5/2}$; $h_5 = -(\partial_5\ {}^3\Psi)^2/4\ {}_3\widehat{\Upsilon}^2 h_6$; $h_6 = h_6^{[0]} - \int dy^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon} = h_6^{[0]} - ({}^3\Phi)^2/4\ {}_3\Lambda$; $w_{i_2} = \partial_{i_2}({}^3\Psi)/\ \partial_5({}^3\Psi) = \partial_{i_2}({}^3\Psi)^2/\ \partial_5({}^3\Psi)^2\rvert$; $n_{k_2} = {}_1n_{k_2} + {}_2n_{k_2}\int dy^5(\partial_5\ {}^3\Psi)^2/\ {}_2\widehat{\Upsilon}^2\lvert h_6^{[0]} - \int dy^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon}^2\rvert^{5/2}$; ... $h_9 = -(\partial_9\ {}^5\Psi)^2/4\ {}_5\widehat{\Upsilon}^2 h_{10}$; $h_{10} = h_{10}^{[0]} - \int dy^9\partial_9(({}^5\Psi)^2)/4\ {}_5\widehat{\Upsilon} = h_{10}^{[0]} - ({}^5\Phi)^2/4\ {}_5\Lambda$; $w_{i_4} = \partial_{i_4}({}^5\Psi)/\ \partial_9({}^5\Psi) = \partial_{i_4}({}^5\Psi)^2/\ \partial_9({}^5\Psi)^2\rvert$; $n_{k_2} = {}_1n_{k_2} + {}_2n_{k_2}\int dy^9(\partial_9\ {}^5\Psi)^2/\ {}_5\widehat{\Upsilon}^2\lvert h_{10}^{[0]} - \int dy^9\partial_9(({}^5\Psi)^2)/4\ {}_5\widehat{\Upsilon}^2\rvert^{5/2}$.	

$$
\begin{aligned}
&\times\{dy^9 + \frac{\partial_{i_4}[\int dy^9 ({}_5\widehat{\Upsilon})\, \partial_9 h_{10}]}{{}_5\widehat{\Upsilon}\, \partial_9 h_{10}} dx^{i_3}\}^2 \\
&+h_{10}\{dy^{10} + [\, {}_1n_{k_4} + {}_2n_{k_4} \int dy^{10} \\
&\times \frac{(\partial_9 h_{10})^2}{|\int dy^9 \partial_9[\, {}_5\widehat{\Upsilon} h_{10}]|\, (h_{10})^{5/2}}]dx^{k_4}\}.
\end{aligned} \tag{A.2}
$$

The nonlinear symmetries (64) and (65) allow to perform similar computations and express shell by shell $({}^s\Phi)^2 = -4\, {}_s\Lambda h_{a_s}$. In similar forms, we can generate s-adapted solutions of type (A.2) when, for instance, Killing symmetries on ∂_6 are changed into ∂_5 (we can consider any permutations with Killing symmetries on ∂_8 changed into ∂_7 and/or ∂_{10} changed into ∂_9). Using η- and/or χ-polarizations, above classes of quasi-stationary higher dimension solutions can be considered for transforming certain prime 10-d s-metrics into respective nonholonomic deformed to target s-metrics of the same or lower dimensions.

A.2.3 Locally anisotropic cosmological solutions with extra dimensions

The formulas for 4-d locally anisotropic cosmological solutions from Table 3 can be extended in geometric abstract forms for 10-d nonholonomic Lorentz manifolds, see Table 6. Such construction and applications in modern cosmology were provide in a series of our previous works [24,28] when off-diagonal cosmological metrics are derived as dual ones (for a time like coordinate) to quasi-stationary metrics. In 4-d, we explain the geometric principles with respect to generating the d-metric (77). A series of works [141–146] is devoted to locally anisotropic cosmological scenarios for MGTs with massive terms and/or contributions from string theories; ekpyrotic scenarios with quasi-periodic and pattern structure formation; cosmological accelerating and inflationary space-time quasicrystal structure. A recent work [132] the AFCDM is applied for generating solutions for the Kaluza–Kelin gravity and cosmological models emerging from geometric and quantum information flow generalizations of the Einstein equations. For simplicity, we consider only canonical d-connections with Killing symmetry on ∂_3 and ∂_7 when respective restrictions to shells $s = 3$ and/or $s = 4$, can be considered, in similar forms, for Killing symmetries on ∂_5, ∂_6, ∂_7 and ∂_8.

Let us consider a 10-d generalization of 4-d locally anisortopic cosmological solutions (A.1), see also (77),

$$
\begin{aligned}
d\widehat{s}^2_{[10d]} &= \widehat{g}_{\alpha_s\beta_s}(x^k, t, y^5, y^7, y^9; \underline{h}_3, h_6, h_8, h_{10}; \\
&\quad {}_s\widehat{\underline{\Upsilon}};\ {}_s\Lambda) du^{\alpha_s} du^{\beta_s} \\
&= e^{\psi(x^k,\, {}_s\widehat{\Upsilon})}[(dx^1)^2 + (dx^2)^2] \\
&\quad +\underline{h}_3[dy^3 + (\, {}_1n_{k_1} + 4\, {}_2n_{k_1} \int dt \\
&\quad \times \frac{(\underline{h}_3{}^{\diamond})^2}{|\int dt\, {}_2\underline{\Upsilon}\underline{h}_3{}^{\diamond}|(\underline{h}_3)^{5/2}})dx^{k_1}] \\
&\quad - \frac{(\underline{h}_3{}^{\diamond})^2}{|\int dt\, {}_2\underline{\Upsilon}\underline{h}_3{}^{\diamond}|\, \overline{h}_3}[dt + \frac{\partial_i(\int dt\, {}_2\underline{\Upsilon}\, \underline{h}_3{}^{\diamond}])}{{}_2\underline{\Upsilon}\, \underline{h}_3{}^{\diamond}} dx^i] \\
&\quad + \frac{(\partial_5 \underline{h}_6)^2}{|\int dy^5 \partial_5[\, {}_3\underline{\Upsilon}\underline{h}_6]|\underline{h}_6} \\
&\quad \times\{dy^5 + \frac{\partial_{i_2}[\int dy^5 ({}_3\underline{\Upsilon})\, \partial_5 \underline{h}_6]}{{}_3\underline{\Upsilon}\, \partial_5 \underline{h}_6} dx^{i_2}\}^2 \\
&\quad +\underline{h}_6\{dy^6 + [\, {}_1n_{k_2} + {}_2n_{k_2} \int dy^5 \\
&\quad \times \frac{(\partial_5 \underline{h}_6)^2}{|\int dy^5 \partial_5[\, {}_3\underline{\Upsilon}\underline{h}_6]|\, (\underline{h}_6)^{5/2}}]dx^{k_2}\} \\
&\quad + \frac{(\partial_7 \underline{h}_8)^2}{|\int dy^7 \partial_7[\, {}_4\underline{\Upsilon}\underline{h}_8]|\, \underline{h}_8}\{dy^7 \\
&\quad + \frac{\partial_{i_3}[\int dy^7 ({}_4\underline{\Upsilon})\, \partial_7 \underline{h}_8]}{{}_4\underline{\Upsilon}\, \partial_7 \underline{h}_8} dx^{i_3}\}^2 \\
&\quad +h_8\{dy^8 + [\, {}_1n_{k_3} + {}_2n_{k_3} \int dy^7 \\
&\quad \frac{(\partial_7 \underline{h}_8)^2}{|\int dy^7 \partial_7[\, {}_4\underline{\Upsilon}\underline{h}_8]|\, (\underline{h}_8)^{5/2}}]dx^{k_3}\} \\
&\quad + \frac{(\partial_9 \underline{h}_{10})^2}{|\int dy^9 \partial_9[\, {}_5\underline{\Upsilon}\underline{h}_{10}]|\, h_{10}}\{dy^9 \\
&\quad + \frac{\partial_{i_4}[\int dy^9 ({}_5\widehat{\Upsilon})\, \partial_9 \underline{h}_{10}]}{{}_5\underline{\Upsilon}\, \partial_9 \underline{h}_{10}} dx^{i_3}\}^2 \\
&\quad +\underline{h}_{10}\{dy^{10} + [\, {}_1n_{k_4} + {}_2n_{k_4} \int dy^{10} \\
&\quad \times \frac{(\partial_9 \underline{h}_{10})^2}{|\int dy^9 \partial_9[\, {}_5\underline{\Upsilon}\underline{h}_{10}]|\, (\underline{h}_{10})^{5/2}}]dx^{k_4}\}.
\end{aligned} \tag{A.3}
$$

The s-metric (A.3) possess the same extra shell Killing symmetries on higher dimension coordinates. Such generic off-diagonal extra dimension cosmological solutions are characterized by nonlinear symmetries of type (64) and (65), when (shell by shell) $({}^s\underline{\Phi})^2 = -4\, {}_s\underline{\Lambda h}_{a_s}$. We can generate s-adapted solutions of type (A.3) when, for instance, Killing symmetries on ∂_6 are changed into ∂_5 and we can consider any permutations with Killing symmetries on ∂_8 changed into ∂_7 and/or ∂_{10} changed into ∂_9. Using η- and/or χ-polarizations with generic dependence on a time like coordinate t, above classes of locally anisortopic higher dimension solutions can be considered for transforming certain prime cosmological 10-d s-metrics into respective nonholonomic deformed to target s-metrics of the same or lower dimensions.

A.3 Off-diagonal velocity depending quasi-stationary/ cosmological solutions, Tables 7, 8, 9, 10, 11

Such geometries and MGTs are modelled on tangent bundle $T\mathbf{V}$ to a nonholonomic Lorentz manifold $\mathbf{V}$. They include

Table 6 Higher dimension off-diagonal cosmological solutions exact solutions of $\widehat{\mathbf{R}}_{\mu\nu} = \Upsilon_{\mu\nu}$ (24) transformed into a shall system of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry ∂_3, ∂_9	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + \underline{g}_{a_2}(x^{k_1},t)(dy^{a_2} + \underline{N}^{a_2}_{i_1}(x^{k_1},t)dx^{i_1})^2$ $+g_{a_3}(x^{k_2},y^5)(dy^{a_3} + N^{a_3}_{i_2}(x^{k_2},y^5)dx^{i_2})^2$ $+g_{a_4}(x^{k_3},y^7)(dy^{a_4} + N^{a_4}_{i_3}(x^{k_3},y^7)dx^{i_3})^2$ $+g_{a_5}(x^{k_4},y^{10})(dy^{a_5} + N^{a_5}_{i_4}(x^{k_4},y^{10})dx^{i_4})^2,$ for $g_{i_1} = e^{\psi(x^{k_1})}$, $\underline{g}_{a_2} = \underline{h}_{a_2}(x^{k_1},t), \underline{N}^3_{i_1} = {}^2\underline{n}_{i_1} = \underline{n}_{i_1}(x^{k_1},t), \underline{N}^4_{i_1} = {}^2\underline{w}_{i_1} = \underline{w}_{i_1}(x^{k_1},t)$, $g_{a_3} = h_{a_3}(x^{k_2},y^6), N^5_{i_2} = {}^3w_{i_2} = w_{i_2}(x^{k_2},y^5), N^6_{i_2} = {}^3n_{i_2} = n_{i_2}(x^{k_2},y^5)$, $g_{a_4} = h_{a_4}(x^{k_3},y^7), N^7_{i_3} = {}^4w_{i_3} = w_{i_3}(x^{k_3},y^7), N^8_{i_3} = {}^4n_{i_3} = n_{i_3}(x^{k_3},y^7)$, $g_{a_5} = h_{a_5}(x^{k_4},y^9), N^9_{i_4} = {}^5w_{i_4} = w_{i_4}(x^{k_4},y^9), N^{10}_{i_4} = {}^5n_{i_4} = n_{i_4}(x^{k_4},y^9)$,	
Effective matter sources	$\mathbf{\Upsilon}^{\mu_s}_{\nu_s} = [\ {}_1\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_2\widehat{\underline{\Upsilon}}(x^{k_1},t)\delta^{a_2}_{b_2},\ {}_3\widehat{\Upsilon}(x^{k_2},y^5)\delta^{a_3}_{b_3},$ ${}_4\widehat{\Upsilon}(x^{k_3},y^7)\delta^{a_4}_{b_4},\ {}_5\widehat{\Upsilon}(x^{k_4},y^9)\delta^{a_5}_{b_5}],$	
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon}$; ${}^2\underline{\varpi}^{\diamond}\ \underline{h}^{\diamond}_3 = 2\underline{h}_3\underline{h}_4\ {}_2\widehat{\Upsilon}$; ${}^2\underline{n}^{\diamond\diamond}_{k_1} + {}^2\underline{\gamma}\ {}^2\underline{n}^{\diamond}_{k_1} = 0$; ${}^2\underline{\beta}\ {}^2\underline{w}_{i_1} - {}^2\underline{\alpha}_{i_1} = 0$;	${}^2\underline{\varpi} = \ln \lvert \partial_4 \underline{h}_4 / \sqrt{\lvert \underline{h}_3 \underline{h}_4 \rvert} \rvert$, ${}^2\underline{\alpha}_{i_1} = (\partial_4 \underline{h}_3)\ (\partial_{i_1}\ {}^2\underline{\varpi})$, ${}^2\underline{\beta} = (\partial_4 \underline{h}_4)\ (\partial_3\ {}^2\underline{\varpi})$, ${}^2\underline{\gamma} = \partial_4 \left(\ln \lvert \underline{h}_3\rvert^{3/2} / \lvert \underline{h}_4 \rvert\right)$, $\partial_1 q = q^{\bullet}, \partial_2 q = q', \partial_4 q = \partial_t q = q^{\diamond}$
	$\partial_5({}^3\varpi)\ \partial_5 h_6 = 2h_5h_6\ {}_3\widehat{\Upsilon}$; ${}^3\beta\ {}^3w_{i_2} - {}^3\alpha_{i_2} = 0$; $\partial_5(\partial_5\ {}^3n_{k_2}) + {}^3\gamma\partial_5({}^3n_{k_2}) = 0$;	${}^3\varpi = \ln \lvert \partial_5 h_6 / \sqrt{\lvert h_5 h_6 \rvert} \rvert$, ${}^3\alpha_{i_2} = (\partial_5 h_6)\ (\partial_{i_2}\ {}^3\varpi)$, ${}^3\beta = (\partial_5 h_6)\ (\partial_5\ {}^3\varpi)$, ${}^3\gamma = \partial_5 \left(\ln \lvert h_6 \rvert^{3/2} / \lvert h_5 \rvert\right)$,
	$\vdots$	$\vdots$
	$\partial_9({}^5\varpi)\ \partial_9 h_{10} = 2h_9 h_{10}\ {}_5\widehat{\Upsilon}$; ${}^5\beta\ {}^5w_{i_4} - {}^5\alpha_{i_4} = 0$; $\partial_9(\partial_9\ {}^5n_{k_4}) + {}^5\gamma\partial_9({}^3n_{k_3}) = 0$;	${}^5\varpi = \ln \lvert \partial_9 h_{10} / \sqrt{\lvert h_9 h_{10} \rvert} \rvert$, ${}^5\alpha_i = (\partial_9 h_{10})\ (\partial_i\ {}^5\varpi)$, ${}^5\beta = (\partial_9 h_{10})\ (\partial_9\ {}^5\varpi)$, ${}^5\gamma = \partial_9 \left(\ln \lvert h_{10} \rvert^{3/2} / \lvert h_9 \rvert\right)$,
Gener. functs: $\underline{h}_4(x^{k_1},t)$, ${}^2\underline{\Psi}(x^{k_1},t) = e^{\ {}^2\underline{\varpi}}$, ${}^2\underline{\Phi}(x^{k_1},t)$, integr. functs: $\underline{h}^{[0]}_3(x^{k_1})$, ${}_1\underline{n}_{k_1}(x^{i_1})$, ${}_2\underline{n}_{k_1}(x^{i_1})$; Gener. functs: $h_5(x^{k_2},y^5)$, ${}^3\Psi(x^{k_2},y^5) = e^{\ {}^3\varpi}$, ${}^3\Phi(x^{k_2},y^5)$, integr. functs: $h^{[0]}_6(x^{k_2})$, ${}^3_1n_{k_2}(x^{i_2})$, ${}^3_2n_{k_2}(x^{i_2})$; ... Gener. functs: $h_9(x^{k_4},y^9)$, ${}^5\Psi(x^{k_3},y^9) = e^{\ {}^5\varpi}$, ${}^5\Phi(x^{k_4},y^9)$, integr. functs: $h^{[0]}_{10}(x^{k_4})$, ${}^5_1n_{k_4}(x^{i_4})$, ${}^5_2n_{k_4}(x^{i_4})$; & nonlinear symmetries	$(({}^2\underline{\Psi})^2)^{\diamond} = -\int dt\ {}_2\widehat{\underline{\Upsilon}}\underline{h}_3{}^{\diamond}$, $({}^2\underline{\Phi})^2 = -4\ {}_2\underline{\Lambda}\underline{h}_3$, $h_3 = h^{[0]}_3 - ({}^2\underline{\Phi})^2/4\ {}_2\underline{\Lambda}, \underline{h}^{\diamond}_3 \neq 0,\ {}_2\underline{\Lambda} \neq 0 = const$; $\partial_5(({}^3\Psi)^2) = -\int dy^5\ {}_3\widehat{\Upsilon}\partial_5 h_6$, $({}^3\Phi)^2 = -4\ {}_3\Lambda h_6$, $h_6 = h^{[0]}_6 - ({}^3\Phi)^2/4\ {}_3\Lambda, \partial_5 h_6 \neq 0,\ {}_3\Lambda \neq 0 = const$; ... $\partial_9(({}^5\Psi)^2) = -\int dy^9\ {}_5\widehat{\Upsilon}\partial_9 h_{10}$, $({}^5\Phi)^2 = -4\ {}_5\Lambda h_{10}$, $h_{10} = h^{[0]}_{10} - ({}^5\Phi)^2/4\ {}_5\Lambda, \partial_9 h_{10} \neq 0,\ {}_5\Lambda \neq 0 = const$;	
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon}$; $\underline{h}_4 = -(\underline{\Psi}^{\diamond})^2/4\ {}_2\widehat{\underline{\Upsilon}}^2\underline{h}_3$; $\underline{h}_3 = \underline{h}^{[0]}_3 - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\widehat{\underline{\Upsilon}} = \underline{h}^{[0]}_3 - \underline{\Phi}^2/4\ {}_2\underline{\Lambda}$; $\underline{w}_{i_1} = \partial_{i_1}\ \underline{\Psi}/\ \partial\underline{\Psi}^{\diamond} = \partial_{i_1}\ \underline{\Psi}^2/\ \partial_t\underline{\Psi}^2\rvert$; $\underline{n}_{k_1} = {}_1n_{k_1} + {}_2n_{k_1}\int dt(\underline{\Psi}^{\diamond})^2/\ {}_2\widehat{\underline{\Upsilon}}^2\lvert\underline{h}^{[0]}_3 - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\widehat{\underline{\Upsilon}}^2\rvert^{5/2}$; $h_5 = -(\partial_5\ {}^3\Psi)^2/4\ {}_3\widehat{\Upsilon}^2 h_6$; $h_6 = h^{[0]}_6 - \int dy^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon} = h^{[0]}_6 - ({}^3\Phi)^2/4\ {}_3\Lambda$; $w_{i_2} = \partial_{i_2}({}^3\Psi)/\ \partial_5({}^3\Psi) = \partial_{i_2}({}^3\Psi)^2/\ \partial_5({}^3\Psi)^2\rvert$; $n_{k_2} = {}_1n_{k_2} + {}_2n_{k_2}\int dy^5(\partial_5\ {}^3\Psi)^2/\ {}_2\widehat{\Upsilon}^2\lvert h^{[0]}_6 - \int dy^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon}^2\rvert^{5/2}$; ... $h_9 = -(\partial_9\ {}^5\Psi)^2/4\ {}_5\widehat{\Upsilon}^2 h_{10}$; $h_{10} = h^{[0]}_{10} - \int dy^9\partial_9(({}^5\Psi)^2)/4\ {}_5\widehat{\Upsilon} = h^{[0]}_{10} - ({}^5\Phi)^2/4\ {}_5\Lambda$; $w_{i_4} = \partial_{i_4}({}^5\Psi)/\ \partial_9({}^5\Psi) = \partial_{i_4}({}^5\Psi)^2/\ \partial_9({}^5\Psi)^2\rvert$; $n_{k_2} = {}_1n_{k_2} + {}_2n_{k_2}\int dy^9(\partial_9\ {}^5\Psi)^2/\ {}_5\widehat{\Upsilon}^2\lvert h^{[0]}_{10} - \int dy^9\partial_9(({}^5\Psi)^2)/4\ {}_5\widehat{\Upsilon}^2\rvert^{5/2}$.	

as particular examples various relativistic generalizations of Finsler–Lagrange geometry and theories with modified dispersion relations, MDR, with respective dual symmetries for (co) fiber coordinates. The typical signature of total metrics is of type $(+++-;+++-)$ for a Lorentz base with signature $(+++-)$. So, the dimension of geometric constructions and signature on such velocity depending phase spaces is $\dim TV = 8$ is different from that considered above in 10-d gravity. To apply the AFCDM we need four shells of dyads (when $s = 1, 2, 3, 4$) with a corresponding (2+2)+(2+2) nonholonomic splitting of the total dimension. The formulas are quite similar to those provided in previous subsection when $y^{a_s} = v^{a_s}$, for $s = 3$ and 4. Nevertheless, the physical interpretation of such velocity phase space models and respective exact/parametric is different from those considered for the higher dimension gravity. This is because the signature of metrics is different, when v^8 is a time like coordinate on the typical fiber, but y^8 was a space like coordinate in the space of velocities. If the phase space solutions are with Killing symmetry on ∂_8, we can fix $v^8 = v^8_{[0]}$, and elaborate on phase space models with space like velocity hypersurfaces. Another class of solutions can be with variable v^8 but a fixed, for instance, velocity $v^7 = v^7_{[0]}$, which provide examples of "velocity-rainbow" metrics in phase gravity. Both types of s-metrics with mentioned behaviour in the velocity typical fiber may have a Killing symmetry on ∂_4 (for locally anisotropic cosmological solutions), or, for instance, on ∂_3, for quasi-stationary solutions. As results, we obtain 4 different types of velocity-phase s-metrics with typical quadratic elements and applications of the AFCDM stated in subsections below and respective Tables 8, 9, 10, 11.

A.3.1 *Diagonal and off-diagonal ansatz for velocity phase spaces*

The parametrization of local coordinates, N-connection and canonical d-connection structures and s-metrics for velocity-phase spaces are sated in Table 7.

Such parameterizations, with respective polarization functions and generating sources can be considered for generalized relativistic Finsler spaces encoding data for nonassociative/noncommutative/ supersymmetric theories etc. The generating and integration functions can be restricted to define LC-configurations.

A.3.2 *Qusi-stationary solutions with fixed light velocity parameter*

Such quasi-stationary solutions are nonholonomic generalizations and extensions on tangent Lorentz bundles with $v^8 = const$, when the velocity phase space involve space like hypersurfaces.

As an example of 8-d quasi-stationary quadratic element with $v^8 = const$ on $T\mathbf{V}$, we provide

$$
\begin{aligned}
d\widehat{s}^2_{[8d]} &= \widehat{g}_{\alpha_s\beta_s}(x^k, y^3, y^5, y^7; h_4, h_6, h_{8,};\\
&\quad {}_s\widehat{\Upsilon};\ {}_s\Lambda)du^{\alpha_s}du^{\beta_s}\\
&= e^{\psi(x^k,\ {}_s\widehat{\Upsilon})}[(dx^1)^2 + (dx^2)^2]\\
&\quad - \frac{(h_4^*)^2}{|\int dy^3[\ {}_2\widehat{\Upsilon} h_4]^*|\ h_4}\\
&\quad \times\{dy^3 + \frac{\partial_{i_1}[\int dy^3(\ {}_2\widehat{\Upsilon})\ h_4^*]}{{}_2\widehat{\Upsilon}\ h_4^*}dx^{i_1}\}^2\\
&\quad + h_4\{dt + [\ {}_1n_{k_1} + \ {}_2n_{k_1}\int dy^3\\
&\quad \times \frac{(h_4^*)^2}{|\int dy^3[\ {}_2\widehat{\Upsilon} h_4]^*|\ (h_4)^{5/2}}]dx^k_1\}\\
&\quad + \frac{(\partial_5 h_6)^2}{|\int dy^5\partial_5[\ {}_3\widehat{\Upsilon} h_6]|\ h_6}\\
&\quad \times\{dv^5 + \frac{\partial_{i_2}[\int dy^5(\ {}_3\widehat{\Upsilon})\ \partial_5 h_6]}{{}_3\widehat{\Upsilon}\ \partial_5 h_6}dx^{i_2}\}^2\\
&\quad + h_6\{dv^5 + [\ {}_1n_{k_2} + \ {}_2n_{k_2}\int dv^5\\
&\quad \times \frac{(\partial_5 h_6)^2}{|\int dy^5\partial_5[\ {}_3\widehat{\Upsilon} h_6]|\ (h_6)^{5/2}}]dx^{k_2}\}\\
&\quad + \frac{(\partial_7 h_8)^2}{|\int dv^7\partial_7[\ {}_4\widehat{\Upsilon} h_8]|\ h_8}\\
&\quad \times\{dv^7 + \frac{\partial_{i_3}[\int dv^7(\ {}_4\widehat{\Upsilon})\ \partial_7 h_8]}{{}_4\widehat{\Upsilon}\ \partial_7 h_8}dx^{i_3}\}^2\\
&\quad + h_8\{dv^8 + [\ {}_1n_{k_3} + \ {}_2n_{k_3}\int dv^7\\
&\quad \times \frac{(\partial_7 h_8)^2}{|\int dv^7\partial_7[\ {}_4\widehat{\Upsilon} h_8]|\ (h_8)^{5/2}}]dx^{k_3}\}.
\end{aligned}
\tag{A.4}
$$

Such s-metrics possess nonlinear symmetries which allow to re-define the generating functions and generating sources and related them to conventions cosmological constants. Solutions with gravitational η- and χ -polarizations can be defined for respective off-diagonal deformations of prime s-metrics into target ones. All formulas can be proven using abstract geometric methods and corresponding applications of the AFCDM.

A.3.3 *Quasi-stationary solutions with variable light velocity parameter*

Another class of quasi-stationary extensions of a Lorentz manifold, $\mathbf{V}$, metrics is for quadratic line elements with $v^7 = const$ which provide examples of velocity rainbow s-metrics on $T\mathbf{V}$. Considering a $v^8 \leftrightarrow v^7$ changing of velocity phase coordinates in (A.4), we construct an example of 8-d quasi-stationary quadratic element with $v^7 = const$ on

Table 7 Diagonal and off-diagonal ansatz for 8-d tangent Lorentz bundles and the anholonomic frame and connection deformation method, **AFCDM**, *for constructing generic off-diagonal exact and parametric solutions*

diagonal ansatz: PDEs $\to$ **ODE**s		AFCDM: **PDEs with decoupling;**
coordinates $u^{\alpha_s}=(x^1,x^2,y^3,y^4=t, v^5,v^6,v^7,v^8)$	${}^s u=({}^{s-1}x,\ {}^s y)$ $s=1,2,3,4;$	nonholonomic 2+2+2+2+2 splitting; shels $s=1,2,3,4$ $u^{\alpha_s}=(x^1,x^2,y^3,y^4=t,y^5,y^6,y^7,y^8)$; $u^{\alpha_s}=(x^{i_1},y^{a_2},y^{a_3},y^{a_4}); u^{\alpha_s}=(x^{i_{s-1}},y^{a_s})$; $i_1=1,2; a_2=3,4; a_3=5,6; a_4=7,8;$ ${}^s\mathbf{N}: T\ {}^s\mathbf{V}=hT\mathbf{V}\oplus\ {}^2hT\mathbf{V}\oplus\ {}^3v\mathbf{TV}\oplus\ {}^4v\mathbf{TV},$
LC-connection $\mathring{\nabla}$	N-connection; canonical d-connection	locally ${}^s\mathbf{N}=\{N^{a_s}_{i_{s-1}}(x,v)=N^{a_s}_{i_{s-1}}({}^{s-1}x,\ {}^s y)=N^{a_s}_{i_{s-1}}({}^s u)\}$ ${}^s\widehat{\mathbf{D}}=({}^1h\widehat{\mathbf{D}},\ {}^2v\widehat{\mathbf{D}},\ {}^3v\widehat{\mathbf{D}},\ {}^4v\widehat{\mathbf{D}})=\{\Gamma^{\alpha_s}_{\ \beta_s\gamma_s}\}$; canonical connection distortion ${}^s\widehat{\mathbf{D}}=\nabla+\ {}^s\widehat{\mathbf{Z}};\ {}^s\widehat{\mathbf{D}}\ {}^s\mathbf{g}=\mathbf{0},$ ${}^s\widehat{\mathcal{T}}[\ {}^s\mathbf{g},\ {}^s\mathbf{N},\ {}^s\widehat{\mathbf{D}}]$ canonical d-torsion
diagonal ansatz ${}^2\mathring{g}=\mathring{g}_{\alpha_2\beta_2}({}^s u)=\begin{pmatrix}\mathring{g}_1 & & & \\ & \mathring{g}_2 & & \\ & & \mathring{g}_3 & \\ & & & \mathring{g}_4\end{pmatrix};$ ${}^s g=\mathring{g}_{\alpha_s\beta_s}({}^s u)=\begin{pmatrix}{}^2\mathring{g} & & & \\ & \mathring{g}_5 & & \\ & & \ddots & \\ & & & \mathring{g}_8\end{pmatrix}$	$\mathbf{g}\Leftrightarrow$	$g_{\alpha_2\beta_2}(x^{i_1},y^{a_2})$ general frames / coordinates $g_{\alpha_2\beta_2}=\begin{bmatrix} g_{i_1j_1}+N^{a_2}_{i_1}N^{b_2}_{j_1}h_{a_2b_2} & N^{b_2}_{i_1}h_{c_2b_2}\\ N^{a_2}_{j_1}h_{a_2b_2} & h_{a_2c_2}\end{bmatrix},$ ${}^2\mathbf{g}=\{\mathbf{g}_{\alpha_2\beta_2}=[g_{i_1j_1},h_{a_2b_2}]\},$ ${}^2\mathbf{g}=\mathbf{g}_{i_1}(x^{k_1})dx^{i_1}\otimes dx^{i_1}+\mathbf{g}_{a_2}(x^{k_2},y^{b_2})\mathbf{e}^{a_2}\otimes\mathbf{e}^{b_2}$ $\vdots$ $g_{\alpha_s\beta_s}(x^{i_{s-1}},y^{a_s})$ general frames / coordinates $g_{\alpha_s\beta_s}=\begin{bmatrix} g_{i_sj_s}+N^{a_s}_{i_{s-1}}N^{b_s}_{j_{s-1}}h_{a_sb_s} & N^{b_s}_{i_{s-1}}h_{c_sb_s}\\ N^{a_s}_{j_{s-1}}h_{a_sb_s} & h_{a_sc_s}\end{bmatrix},$ ${}^s\mathbf{g}=\{\mathbf{g}_{\alpha_s\beta_s}=[g_{i_{s-1}j_{s-1}},h_{a_sb_s}]$ $=[g_{i_1j_1},h_{a_2b_2},h_{a_3b_3},h_{a_4b_4}]\}$ ${}^s\mathbf{g}=\mathbf{g}_{i_{s-1}}(x^{k_{s-1}})dx^{i_{s-1}}\otimes dx^{i_{s-1}}+$ $\mathbf{g}_{a_s}(x^{k_{s-1}},y^{b_s})\mathbf{e}^{a_s}\otimes\mathbf{e}^{b_s}$ $=\mathbf{g}_{i_1}(x^{k_1})dx^{i_1}\otimes dx^{i_1}+\mathbf{g}_{a_2}(x^{k_1},y^{b_2})\mathbf{e}^{a_2}\otimes\mathbf{e}^{b_2}+$ $\mathbf{g}_{a_3}(x^{k_1},y^{b_2},v^{b_3})\mathbf{e}^{a_3}\otimes\mathbf{e}^{b_3}+\mathbf{g}_{a_4}(x^{k_1},y^{b_2},v^{b_3},v^{b_4})\mathbf{e}^{a_4}\otimes\mathbf{e}^{b_4};$
$\mathring{g}_{\alpha_2\beta_2}=\begin{cases}\mathring{g}_{\alpha_2}({}^2r) & \text{for BHs}\\ \mathring{g}_{\alpha_2}(t) & \text{for FLRW}\end{cases}$ $\mathring{g}_{\alpha_s\beta_s}=\begin{cases}\mathring{g}_{\alpha_s}({}^s r) & \text{for BHs}\\ \mathring{g}_{\alpha_s}(t) & \text{for FLRW}\end{cases}$	[coord.frames]	$g_{\alpha_2\beta_2}=\begin{cases} g_{\alpha_2\beta_2}(x^i,y^3) & \text{quasi-stationary config.}\\ \underline{g}_{\alpha_2\beta_2}(x^i,y^4=t) & \text{locally anisotropic cosmology}\end{cases}$ $g_{\alpha_5\beta_5}=\begin{cases} g_{\alpha_5\beta_5}(x^{i_3},v^7)\\ \underline{g}_{\alpha_s\beta_s}(x^{i_3},y^8)\end{cases}$
coord. transf. $e_{\alpha_s}=e^{\alpha'_s}_{\ \alpha_s}\partial_{\alpha'_s},$ $e^{\beta_s}=e^{\beta_s}_{\ \beta'_s}du^{\beta'_s},$ $\mathring{g}_{\alpha_s\beta_s}=\mathring{g}_{\alpha'_s\beta'_s}e^{\alpha'_s}_{\ \alpha_s}e^{\beta'_s}_{\ \beta_s}$ $\mathring{\mathbf{g}}_{\alpha_s}(x^{k_{s-1}},y^{a_s})\to\mathring{g}_{\alpha_s}({}^s r),$ or $\mathring{g}_{\alpha_s}(t),\mathring{N}^{a_s}_{i_{s-1}}(x^{k_{s-1}},y^{a_s})\to 0.$	[N-adapt. fr.]	$\begin{cases}\mathbf{g}_{i_1}(x^{k_1}),\mathbf{g}_{a_2}(x^{k_1},y^3),\\ \text{or } \mathbf{g}_{i_1}(x^{k_1}),\underline{\mathbf{g}}_{a_2}(x^{k_1},t),\\ N^3_{i_1}=w_{i_1}(x^k,y^3),N^4_{i_1}=n_{i_1}(x^k,y^3),\\ \text{or } \underline{N}^3_{i_1}=\underline{n}_{i_1}(x^{k_1},t),\underline{N}^4_{i_1}=\underline{w}_{i_1}(x^{k_1},t),\end{cases}$ d-metrics $\vdots$ $\begin{cases}\mathbf{g}_{i_3}(x^{k_3}),\mathbf{g}_{a_4}(x^{k_3},v^7),\\ \text{or } \mathbf{g}_{i_3}(x^{k_1}),\underline{\mathbf{g}}_{a_4}(x^{k_3},v^8),\\ N^7_{i_3}=w_{i_3}(x^{k_3},v^7),N^8_{i_3}=n_{i_3}(x^{k_3},v^7),\\ \text{or } \underline{N}^8_{i_3}=\underline{n}_{i_3}(x^{k_3},v^8),\underline{N}^8_{i_3}=\underline{w}_{i_3}(x^{k_3},v^8),\end{cases}$
${}^s\mathring{\nabla},\ {}^sRic=\{\mathring{R}_{\ \beta_s\gamma_s}\}$	Ricci tensors	${}^s\widehat{\mathbf{D}},\ \ {}^s\widehat{\mathcal{R}}ic=\{\widehat{\mathbf{R}}_{\ \beta_s\gamma_s}\}$
${}^m\mathcal{L}[\phi]\to{}^m\mathbf{T}_{\alpha_s\beta_s}[\phi]$	generating sources	$\widehat{\Upsilon}^{\mu_s}_{\ \nu_s}=\mathbf{e}^{\mu_s}_{\ \mu'_s}\mathbf{e}^{\nu'_s}_{\ \nu_s}\Upsilon^{\mu'_s}_{\ \nu'_s}[\ {}^m\mathcal{L}(\varphi),T_{\mu_s\nu_s},\ {}^s\Lambda]$ $=diag[\ {}_1\Upsilon(x^{i_1})\delta^{i_1}_{j_1},\ {}_2\Upsilon(x^{i_1},y^3)\delta^{a_2}_{b_2},$ ${}_3\Upsilon(x^{i_2},v^5)\delta^{a_3}_{b_3},\ {}_4\Upsilon(x^{i_3},v^7)\delta^{a_4}_{b_4}],$ quasi-stationary configurations; $=diag[\ {}_1\Upsilon(x^{i_1})\delta^{i_1}_{j_1},\ {}_2\underline{\Upsilon}(x^{i_1},t)\delta^{a_2}_{b_2},$ ${}_3\underline{\Upsilon}(x^{i_2},v^6)\delta^{a_3}_{b_3},\ {}_4\underline{\Upsilon}(x^{i_3},v^8)\delta^{a_4}_{b_4}],$ locally anisotropic cosmology;
trivial eqs for ${}^s\mathring{\nabla}$-torsion	LC-conditions	${}^s\widehat{\mathbf{D}}_{\mid\ {}^s\widehat{\mathcal{T}}\to 0}=\ {}^s\nabla.$

$T\mathbf{V}$ defining an example of velocity rainbow s-metric,

$$
\begin{aligned}
d\widehat{s}^2_{[8d]} &= \widehat{g}_{\alpha_s\beta_s}(x^k,y^3,y^5,y^7;h_4,h_6,h_8,;\\
&\quad {}_s\widehat{\Upsilon};\ {}_s\Lambda)du^{\alpha_s}du^{\beta_s}\\
&= e^{\psi(x^k,\ {}_s\widehat{\Upsilon})}[(dx^1)^2+(dx^2)^2]\\
&\quad -\frac{(h_4^*)^2}{\mid\int dy^3[\ {}_2\widehat{\Upsilon}h_4]^*\mid\ h_4}\\
&\quad\times\{dy^3+\frac{\partial_{i_1}[\int dy^3(\ {}_2\widehat{\Upsilon})\ h_4^*]}{{}_2\widehat{\Upsilon}\ h_4^*}dx^{i_1}\}^2\\
&\quad +h_4\{dt+[\ {}_1n_{k_1}+\ {}_2n_{k_1}\int dy^3\\
&\quad\times\frac{(h_4^*)^2}{\mid\int dy^3[\ {}_2\widehat{\Upsilon}h_4]^*\mid\ (h_4)^{5/2}}]dx_1^k\}
\end{aligned}
$$

Table 8 Off-diagonal quasi-stationary spacetime and space velocity configurations exact solutions of $\widehat{\mathbf{R}}_{\mu_s\nu_s} = \Upsilon_{\mu_s\nu_s}$ (24) on TV transformed into a shall system of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry $\partial_4=\partial_t, \partial_8$	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + g_{a_2}(x^{k_1},y^3)(dy^{a_2} + N^{a_2}_{i_1}(x^{k_1},y^3)dx^{i_1})^2$ $+g_{a_3}(x^{k_2},v^5)(dy^{a_3} + N^{a_3}_{i_2}(x^{k_2},v^5)dx^{i_2})^2$ $+g_{a_4}(x^{k_3},v^7)(dy^{a_4} + N^{a_4}_{i_3}(x^{k_3},v^7)dx^{i_3})^2,$ for $g_{i_1} = e^{\psi(x^{k_1})},$ $g_{a_2} = h_{a_2}(x^{k_1},y^3), N^3_{i_1} = {}^2w_{i_1} = w_{i_1}(x^{k_1},y^3), N^4_{i_1} = {}^2n_{i_1} = n_{i_1}(x^{k_1},y^3),$ $g_{a_3} = h_{a_3}(x^{k_2},v^5), N^5_{i_2} = {}^3w_{i_2} = w_{i_2}(x^{k_2},v^5), N^6_{i_2} = {}^3n_{i_2} = n_{i_2}(x^{k_2},v^5),$ $g_{a_4} = h_{a_4}(x^{k_3},v^7), N^7_{i_3} = {}^4w_{i_3} = w_{i_3}(x^{k_3},v^7), N^8_{i_3} = {}^4n_{i_3} = n_{i_3}(x^{k_3},v^7),$	
Effective matter sources	$\boldsymbol{\Upsilon}^{\mu_s}_{\nu_s} = [\ {}_1\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_2\widehat{\Upsilon}(x^{k_1},y^3)\delta^{a_2}_{b_2},\ {}_3\widehat{\Upsilon}(x^{k_2},v^5)\delta^{a_3}_{b_3}, {}_4\widehat{\Upsilon}(x^{k_3},v^7)\delta^{a_4}_{b_4},],$	
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon};$ ${}^2\varpi^*\ h^*_4 = 2h_3h_4\ {}_2\widehat{\Upsilon};$ ${}^2\beta\ {}^2w_{i_1} -\ {}^2\alpha_{i_1} = 0;$ ${}^2n^{**}_{k_1} +\ {}^2\gamma\ {}^2n^*_{k_1} = 0;$	${}^2\varpi = \ln \vert\partial_3 h_4/\sqrt{\vert h_3h_4\vert}\vert,$ ${}^2\alpha_{i_1} = (\partial_3 h_4)\ (\partial_{i_1}\ {}^2\varpi),$ ${}^2\beta = (\partial_3 h_4)\ (\partial_3\ {}^2\varpi),$ ${}^2\gamma = \partial_3\left(\ln \vert h_4\vert^{3/2}/\vert h_3\vert\right),$ $\partial_1 q = q^\bullet, \partial_2 q = q', \partial_3 q = q^*$
	$\partial_5(\ {}^3\varpi)\ \partial_5 h_6 = 2h_5h_6\ {}_3\widehat{\Upsilon};$ ${}^3\beta\ {}^3w_{i_2} -\ {}^3\alpha_{i_2} = 0;$ $\partial_5(\partial_5\ {}^3n_{k_2}) +\ {}^3\gamma\partial_5(\ {}^3n_{k_2}) = 0;$	${}^3\varpi = \ln \vert\partial_5 h_6/\sqrt{\vert h_5h_6\vert}\vert,$ ${}^3\alpha_{i_2} = (\partial_5 h_6)\ (\partial_{i_2}\ {}^3\varpi),$ ${}^3\beta = (\partial_5 h_6)\ (\partial_5\ {}^3\varpi),$ ${}^3\gamma = \partial_5\left(\ln \vert h_6\vert^{3/2}/\vert h_5\vert\right),$
	$\partial_7(\ {}^4\varpi)\ \partial_7 h_8 = 2h_7h_8\ {}_4\widehat{\Upsilon};$ ${}^4\beta\ {}^4w_{i_3} -\ {}^4\alpha_{i_3} = 0;$ $\partial_7(\partial_7\ {}^4n_{k_3}) +\ {}^4\gamma\partial_7(\ {}^4n_{k_3}) = 0;$	${}^4\varpi = \ln \vert\partial_7 h_8/\sqrt{\vert h_7h_8\vert}\vert,$ ${}^4\alpha_i = (\partial_7 h_8)\ (\partial_i\ {}^4\varpi),$ ${}^4\beta = (\partial_7 h_8)\ (\partial_7\ {}^4\varpi),$ ${}^4\gamma = \partial_7\left(\ln \vert h_8\vert^{3/2}/\vert h_7\vert\right),$
Gener. functs: $h_3(x^{k_1},y^3)$, ${}^2\Psi(x^{k_1},y^3) = e^{\ {}^2\varpi},\ {}^2\Phi(x^{k_1},y^3),$ integr. functs: $h^{[0]}_4(x^{k_1}),$ ${}_1n_{k_1}(x^{i_1}),\ {}_2n_{k_1}(x^{i_1});$	$((\ {}^2\Psi)^2)^* = -\int dy^3\ {}_2\widehat{\Upsilon}h_4{}^*,$ $(\ {}^2\Phi)^2 = -4\ {}_2\Lambda h_4,$ see (71), $h_4 = h^{[0]}_4 - (\ {}^2\Phi)^2/4\ {}_2\Lambda, h^*_4 \neq 0,\ {}_2\Lambda \neq 0 = const;$	
Gener. functs:$h_5(x^{k_2},v^5)$, ${}^3\Psi(x^{k_2},v^5) = e^{\ {}^3\varpi},\ {}^3\Phi(x^{k_2},v^5),$ integr. functs: $h^{[0]}_6(x^{k_2}),$ ${}^3_1n_{k_2}(x^{i_2}),\ {}^3_2n_{k_2}(x^{i_2});$	$\partial_5((\ {}^3\Psi)^2) = -\int dv^5\ {}_3\widehat{\Upsilon}\partial_5 h_6,$ $(\ {}^3\Phi)^2 = -4\ {}_3\Lambda h_6,$ $h_6 = h^{[0]}_6 - (\ {}^3\Phi)^2/4\ {}_3\Lambda, \partial_5 h_6 \neq 0,\ {}_3\Lambda \neq 0 = const;$	
Gener. functs:$h_7(x^{k_3},v^7)$, ${}^5\Psi(x^{k_2},v^7) = e^{\ {}^4\varpi},\ {}^4\Phi(x^{k_3},v^7),$ integr. functs: $h^{[0]}_8(x^{k_3}),$ ${}^4_1n_{k_3}(x^{i_3}),\ {}^4_2n_{k_3}(x^{i_4});$ & nonlinear symmetries	$\partial_7((\ {}^4\Psi)^2) = -\int dv^7\ {}_4\widehat{\Upsilon}\partial_7 h_8,$ $(\ {}^4\Phi)^2 = -4\ {}_4\Lambda h_8,$ $h_8 = h^{[0]}_8 - (\ {}^4\Phi)^2/4\ {}_4\Lambda, \partial_7 h_8 \neq 0,\ {}_4\Lambda \neq 0 = const;$	
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon};$ $h_3 = -(\Psi^*)^2/4\ {}_2\widehat{\Upsilon}^2 h_4,$ see (59), (58); $h_4 = h^{[0]}_4 - \int dy^3(\Psi^2)^*/4\ {}_2\widehat{\Upsilon} = h^{[0]}_4 - \Phi^2/4\ {}_2\Lambda;$ $w_i = \partial_i\ \Psi/\ \partial_3\Psi = \partial_i\ \Psi^2/\ \partial_3\Psi^2\vert;$ $n_k =\ {}_1n_k +\ {}_2n_k\int dy^3(\Psi^*)^2/\ {}_2\widehat{\Upsilon}^2\vert h^{[0]}_4 - \int dy^3(\Psi^2)^*/4\ {}_2\widehat{\Upsilon}^2\vert^{5/2};$ $h_5 = -(\partial_5\ {}^3\Psi)^2/4\ {}_3\widehat{\Upsilon}^2 h_6;$ $h_6 = h^{[0]}_6 - \int dv^5\partial_5((\ {}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon} = h^{[0]}_6 - (\ {}^3\Phi)^2/4\ {}_3\Lambda;$ $w_{i_2} = \partial_{i_2}(\ {}^3\Psi)/\ \partial_5(\ {}^3\Psi) = \partial_{i_2}(\ {}^3\Psi)^2/\ \partial_5(\ {}^3\Psi)^2\vert;$ $n_{k_2} =\ {}_1n_{k_2} +\ {}_2n_{k_2}\int dv^5(\partial_5\ {}^3\Psi)^2/\ {}_2\widehat{\Upsilon}^2\vert h^{[0]}_6 - \int dv^5\partial_5((\ {}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon}^2\vert^{5/2};$ $h_7 = -(\partial_7\ {}^3\Psi)^2/4\ {}_4\widehat{\Upsilon}^2 h_8;$ $h_8 = h^{[0]}_8 - \int dv^7\partial_9((\ {}^4\Psi)^2)/4\ {}_4\widehat{\Upsilon} = h^{[0]}_8 - (\ {}^4\Phi)^2/4\ {}_4\Lambda;$ $w_{i_3} = \partial_{i_3}(\ {}^4\Psi)/\ \partial_7(\ {}^4\Psi) = \partial_{i_3}(\ {}^4\Psi)^2/\ \partial_7(\ {}^4\Psi)^2\vert;$ $n_{k_3} =\ {}_1n_{k_3} +\ {}_2n_{k_3}\int dv^7(\partial_7\ {}^4\Psi)^2/\ {}_4\widehat{\Upsilon}^2\vert h^{[0]}_8 - \int dv^7\partial_7((\ {}^4\Psi)^2)/4\ {}_4\widehat{\Upsilon}^2\vert^{5/2}.$	

$$+\frac{(\partial_5 h_6)^2}{|\int dy^5 \partial_5[\ _3\widehat{\Upsilon} h_6]|\ h_6}\{dv^5$$
$$+\frac{\partial_{i_2}[\int dy^5 (\ _3\widehat{\Upsilon})\ \partial_5 h_6]}{_3\widehat{\Upsilon}\ \partial_5 h_6} dx^{i_2}\}^2$$
$$+h_6\{dy^6 + [\ _1n_{k_2} +\ _2n_{k_2} \int dv^5$$
$$\times \frac{(\partial_5 h_6)^2}{|\int dv^5 \partial_5[\ _3\widehat{\Upsilon} h_6]|\ (h_6)^{5/2}}]dx^{k_2}\}$$
$$+\underline{h}_7\{dv^7 + [\ _1n_{k_3} +\ _2n_{k_3} \int dv^8$$
$$\times \frac{(\partial_8 \underline{h}_7)^2}{|\int dv^8 \partial_8[\ _4\underline{\widehat{\Upsilon}} h_7]|\ (\underline{h}_7)^{5/2}}]dx^{k_3}\} +$$
$$\times \frac{(\partial_8 \underline{h}_7)^2}{|\int dv^8 \partial_8[\ _4\widehat{\underline{\Upsilon}} \underline{h}_7]|\ \underline{h}_7}$$
$$\times\{dv^8 + \frac{\partial_{i_3}[\int dv^8 (\ _4\widehat{\underline{\Upsilon}})\ \partial_8 \underline{h}_7]}{_4\widehat{\underline{\Upsilon}}\ \partial_8 \underline{h}_7} dx^{i_3}\}^2. \quad (A.5)$$

The principles of generating such quasi-stationary and rainbow solutions are summarized in Table 9.

Other types of quasi-stationary and velocity rainbow solutions can be constructed using nonlinear transforms of generating functions, gravitational polarizations and constraints to LC-configurations. All nonholonomic geometric constructions involve respective abstract geometric proofs and modifications/generalizations of formulas.

A.3.4 Locally anisotropic cosmological solutions with phase space velocity configurations

Such cosmological models are 8-d versions of (A.3) derived following the AFCDM as in Table 6 but redefined on velocity phase spaces. Respective classes of generic off-diagonal s-metrics are constructed following the steps outlined below in Table 10.

As an example of 8-d quasi-stationary quadratic element with $v^8 = const$ on $T\mathbf{V}$, we provide

$$d\widehat{s}^2_{[8d]} = \widehat{g}_{\alpha_s\beta_s}(x^k, t, y^5, y^7; \underline{h}_3, h_6, h_8,;$$
$$_1\widehat{\Upsilon},\ _2\widehat{\underline{\Upsilon}},\ _3\widehat{\Upsilon},_4\widehat{\Upsilon};\ _1\Lambda,\ _2\underline{\Lambda},\ _3\Lambda,\ _4\Lambda) du^{\alpha_s} du^{\beta_s}$$
$$= e^{\psi(x^k,\ _s\widehat{\Upsilon})}[(dx^1)^2 + (dx^2)^2]$$
$$+\underline{h}_3[dy^3 + (\ _1n_{k_1} + 4\ _2n_{k_1}$$
$$\int dt \frac{(\underline{h}_3{}^\diamond)^2}{|\int dt\ _2\underline{\Upsilon} \underline{h}_3{}^\diamond|(\underline{h}_3)^{5/2}}) dx^{k_1}]$$
$$-\frac{(\underline{h}_3{}^\diamond)^2}{|\int dt\ _2\underline{\Upsilon} \underline{h}_3{}^\diamond|\ \overline{h}_3}[dt + \frac{\partial_i(\int dt\ _2\underline{\Upsilon}\ \underline{h}_3{}^\diamond])}{_2\underline{\Upsilon}\ \underline{h}_3{}^\diamond} dx^i]$$
$$+\frac{(\partial_5 h_6)^2}{|\int dy^5 \partial_5[\ _3\widehat{\Upsilon} h_6]|\ h_6}\{dv^5$$
$$+\frac{\partial_{i_2}[\int dv^5 (\ _3\widehat{\Upsilon})\ \partial_5 h_6]}{_3\widehat{\Upsilon}\ \partial_5 h_6} dx^{i_2}\}^2$$
$$+h_6\{dv^5 + [\ _1n_{k_2} +\ _2n_{k_2} \int dv^5$$
$$\times \frac{(\partial_5 h_6)^2}{|\int dy^5 \partial_5[\ _3\widehat{\Upsilon} h_6]|\ (h_6)^{5/2}}]dx^{k_2}\}$$
$$+\frac{(\partial_7 h_8)^2}{|\int dv^7 \partial_7[\ _4\widehat{\Upsilon} h_8]|\ h_8}$$
$$\times\{dv^7 + \frac{\partial_{i_3}[\int dv^7 (\ _4\widehat{\Upsilon})\ \partial_7 h_8]}{_4\widehat{\Upsilon}\ \partial_7 h_8} dx^{i_3}\}^2$$
$$+h_8\{dv^8 + [\ _1n_{k_3} +\ _2n_{k_3} \int dv^7$$
$$\times \frac{(\partial_7 h_8)^2}{|\int dv^7 \partial_7[\ _4\widehat{\Upsilon} h_8]|\ (h_8)^{5/2}}]dx^{k_3}\}. \quad (A.6)$$

Similar classes of locally cosmological phase velocity space solutions can be derived for the same Killing symmetries on ∂_3 and ∂_8 using respective nonlinear symmetries and generating and integration functions.

A.3.5 Locally anisotropic cosmological solutions with phase space rainbow symmetries

The locally anisotropic cosmological models from previous Table 10 can be re-defined by phase space rainbow symmetries with the shells $s = 3, 4$ part as in Table 9. The shells $s = 1, 2$ parts are as in Table 6 when the AFCDM is redefined on $T\mathbf{V}$. The procedure of constructing such classes of solutions with Killing symmetries on ∂_3 and ∂_7 is summarized below in Table 11. As an example of 8-d locally anisotropic cosmological quadratic element with $v^7 = const$ on $T\mathbf{V}$, and defining rainbow configurations as for $s = 3, 4$ in (A.5), we provide

$$d\widehat{s}^2_{[8d]} = \widehat{g}_{\alpha_s\beta_s}(x^k, t, v^5, v^8; \underline{h}_3, h_6, \underline{h}_{7,};$$
$$_1\widehat{\Upsilon},\ _2\widehat{\underline{\Upsilon}},\ _3\widehat{\Upsilon},\ _4\widehat{\underline{\Upsilon}};\ _1\Lambda,\ _2\underline{\Lambda},\ _3\Lambda,\ _4\underline{\Lambda}) du^{\alpha_s} du^{\beta_s}$$
$$= e^{\psi(x^k,\ _s\widehat{\Upsilon})}[(dx^1)^2 + (dx^2)^2]$$
$$+\underline{h}_3[dy^3 + (\ _1n_{k_1} + 4\ _2n_{k_1}$$
$$\int dt \frac{(\underline{h}_3{}^\diamond)^2}{|\int dt\ _2\underline{\Upsilon} \underline{h}_3{}^\diamond|(\underline{h}_3)^{5/2}}) dx^{k_1}]$$
$$-\frac{(\underline{h}_3{}^\diamond)^2}{|\int dt\ _2\underline{\Upsilon} \underline{h}_3{}^\diamond|\ \overline{h}_3}[dt + \frac{\partial_i(\int dt\ _2\underline{\Upsilon}\ \underline{h}_3{}^\diamond])}{_2\underline{\Upsilon}\ \underline{h}_3{}^\diamond} dx^i]$$
$$+\frac{(\partial_5 h_6)^2}{|\int dv^5 \partial_5[\ _3\widehat{\Upsilon} h_6]|\ h_6}\{dv^5$$
$$+\frac{\partial_{i_2}[\int dv^5 (\ _3\widehat{\Upsilon})\ \partial_5 h_6]}{_3\widehat{\Upsilon}\ \partial_5 h_6} dx^{i_2}\}^2$$
$$+h_6\{dv^5 + [\ _1n_{k_2} +\ _2n_{k_2}$$
$$\int dv^5 \frac{(\partial_5 h_6)^2}{|\int dv^5 \partial_5[\ _3\widehat{\Upsilon} h_6]|\ (h_6)^{5/2}}]dx^{k_2}\}$$
$$+\underline{h}_7\{dv^7 + [\ _1n_{k_3} +\ _2n_{k_3}$$

Table 9 Off-diagonal quasi-stationary spacetimes with velocity rainbows exact solutions of $\widehat{\mathbf{R}}_{\mu_s\nu_s} = \Upsilon_{\mu_s\nu_s}$ (24) on TV transformed into a shall system of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry $\partial_4 = \partial_t, \partial_7$	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + g_{a_2}(x^{k_1},y^3)(dy^{a_2} + N^{a_2}_{i_1}(x^{k_1},y^3)dx^{i_1})^2$ $+g_{a_3}(x^{k_2},v^5)(dy^{a_3} + N^{a_3}_{i_2}(x^{k_2},v^5)dx^{i_2})^2$ $+\underline{g}_{a_4}(x^{k_3},v^8)(dy^{a_4} + \underline{N}^{a_4}_{i_3}(x^{k_3},v^8)dx^{i_3})^2$, for $g_{i_1} = e^{\psi(x^{k_1})}$, $g_{a_2} = h_{a_2}(x^{k_1},y^3), N^3_{i_1} = {}^2w_{i_1} = w_{i_1}(x^{k_1},y^3), N^4_{i_1} = {}^2n_{i_1} = n_{i_1}(x^{k_1},y^3)$, $g_{a_3} = h_{a_3}(x^{k_2},v^5), N^5_{i_2} = {}^3w_{i_2} = w_{i_2}(x^{k_2},v^5), N^6_{i_2} = {}^3n_{i_2} = n_{i_2}(x^{k_2},v^5)$, $\underline{g}_{a_4} = \underline{h}_{a_4}(x^{k_3},v^8), \underline{N}^7_{i_3} = {}^4\underline{n}_{i_3} = \underline{n}_{i_3}(x^{k_3},v^8), \underline{N}^8_{i_3} = {}^4\underline{w}_{i_3} = \underline{w}_{i_3}(x^{k_3},v^8)$,	
Effective matter sources	$\mathbf{\Upsilon}^{\mu_s}_{\nu_s} = [\ {}_1\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_2\widehat{\Upsilon}(x^{k_1},y^3)\delta^{a_2}_{b_2},\ {}_3\widehat{\Upsilon}(x^{k_2},v^5)\delta^{a_3}_{b_3}, {}_4\underline{\widehat{\Upsilon}}(x^{k_3},v^8)\delta^{a_4}_{b_4},]$,	
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon}$; ${}^2\varpi^*\ h^*_4 = 2h_3h_4\ {}_2\widehat{\Upsilon}$; ${}^2\beta\ {}^2w_{i_1} - {}^2\alpha_{i_1} = 0$; ${}^2n^{**}_{k_1} + {}^2\gamma\ {}^2n^*_{k_1} = 0$;	${}^2\varpi = \ln\|\partial_3 h_4/\sqrt{\|h_3h_4\|}\|$, ${}^2\alpha_{i_1} = (\partial_3 h_4)\ (\partial_{i_1}\ {}^2\varpi)$, ${}^2\beta = (\partial_3 h_4)\ (\partial_3\ {}^2\varpi)$, ${}^2\gamma = \partial_3\left(\ln\|h_4\|^{3/2}/\|h_3\|\right)$, $\partial_1 q = q^\bullet, \partial_2 q = q', \partial_3 q = q^*$
	$\partial_5({}^3\varpi)\ \partial_5 h_6 = 2h_5h_6\ {}_3\widehat{\Upsilon}$; ${}^3\beta\ {}^3w_{i_2} - {}^3\alpha_{i_2} = 0$; $\partial_5(\partial_5\ {}^3n_{k_2}) + {}^3\gamma\partial_5({}^3n_{k_2}) = 0$;	${}^3\varpi = \ln\|\partial_5 h_6/\sqrt{\|h_5h_6\|}\|$, ${}^3\alpha_{i_2} = (\partial_5 h_6)\ (\partial_{i_2}\ {}^3\varpi)$, ${}^3\beta = (\partial_5 h_6)\ (\partial_5\ {}^3\varpi)$, ${}^3\gamma = \partial_5\left(\ln\|h_6\|^{3/2}/\|h_5\|\right)$,
	$\partial_8({}^4\underline{\varpi})\ \partial_8\underline{h}_7 = 2\underline{h}_7\underline{h}_8\ {}_4\underline{\widehat{\Upsilon}}$; $\partial_8(\partial_8\ {}^4\underline{n}_{k_3}) + {}^4\underline{\gamma}\partial_8({}^4\underline{n}_{k_3}) = 0$; ${}^4\underline{\beta}\ {}^4\underline{w}_{i_3} - {}^4\underline{\alpha}_{i_3} = 0$;	${}^4\underline{\varpi} = \ln\|\partial_8\underline{h}_7/\sqrt{\|\underline{h}_7\underline{h}_8\|}\|$, ${}^4\underline{\alpha}_i = (\partial_8\underline{h}_7)\ (\partial_i\ {}^4\underline{\varpi})$, ${}^4\underline{\beta} = (\partial_8\underline{h}_7)\ (\partial_8\ {}^4\underline{\varpi})$, ${}^4\underline{\gamma} = \partial_8\left(\ln\|\underline{h}_7\|^{3/2}/\|\underline{h}_8\|\right)$,
Gener. functs: $h_3(x^{k_1},y^3)$, ${}^2\Psi(x^{k_1},y^3) = e^{\ {}^2\varpi}$, ${}^2\Phi(x^{k_1},y^3)$, integr. functs: $h^{[0]}_4(x^{k_1})$, ${}_1n_{k_1}(x^{i_1})$, ${}_2n_{k_1}(x^{i_1})$;	$(({}^2\Psi)^2)^* = -\int dy^3\ {}_2\widehat{\Upsilon}h_4{}^*$, $({}^2\Phi)^2 = -4\ {}_2\Lambda h_4$, see (71), $h_4 = h^{[0]}_4 - ({}^2\Phi)^2/4\ {}_2\Lambda, h^*_4 \neq 0,\ {}_2\Lambda \neq 0 = const$;	
Gener. functs: $h_5(x^{k_2},v^5)$, ${}^3\Psi(x^{k_2},v^5) = e^{\ {}^3\varpi}$, ${}^3\Phi(x^{k_2},v^5)$, integr. functs: $h^{[0]}_6(x^{k_2})$, ${}^3_1n_{k_2}(x^{i_2})$, ${}^3_2n_{k_2}(x^{i_2})$;	$\partial_5(({}^3\Psi)^2) = -\int dv^5\ {}_3\widehat{\Upsilon}\partial_5 h_6$, $({}^3\Phi)^2 = -4\ {}_3\Lambda h_6$, $h_6 = h^{[0]}_6 - ({}^3\Phi)^2/4\ {}_3\Lambda, \partial_5 h_6 \neq 0,\ {}_3\Lambda \neq 0 = const$;	
Gener. functs: $\underline{h}_8(x^{k_3},v^8)$, ${}^4\underline{\Psi}(x^{k_2},v^8) = e^{\ {}^4\underline{\varpi}}$, ${}^4\underline{\Phi}(x^{k_3},v^8)$, integr. functs: $h^{[0]}_8(x^{k_3})$, ${}^4_1n_{k_3}(x^{i_3})$, ${}^4_2n_{k_3}(x^{i_4})$; & nonlinear symmetries	$\partial_8(({}^4\underline{\Psi})^2) = -\int dv^8\ {}_4\underline{\widehat{\Upsilon}}\partial_8\underline{h}_7$, $({}^4\underline{\Phi})^2 = -4\ {}_4\underline{\Lambda}\underline{h}_7$, $\underline{h}_7 = h^{[0]}_7 - ({}^4\underline{\Phi})^2/4\ {}_4\underline{\Lambda}, \partial_8\underline{h}_7 \neq 0,\ {}_4\underline{\Lambda} \neq 0 = const$;	
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon}$; $h_3 = -(\Psi^*)^2/4\ {}_2\widehat{\Upsilon}^2 h_4$, see (59), (58); $h_4 = h^{[0]}_4 - \int dy^3(\Psi^2)^*/4\ {}_2\widehat{\Upsilon} = h^{[0]}_4 - \Phi^2/4\ {}_2\Lambda$; $w_i = \partial_i\ \Psi/\ \partial_3\Psi = \partial_i\ \Psi^2/\ \partial_3\Psi^2\|$; $n_{k_1} = {}_1n_{k_1} + {}_2n_k\int dy^3(\Psi^*)^2/\ {}_2\widehat{\Upsilon}^2\|h^{[0]}_4 - \int dy^3(\Psi^2)^*/4\ {}_2\widehat{\Upsilon}^2\|^{5/2}$; $h_5 = -(\partial_5\ {}^3\Psi)^2/4\ {}_3\widehat{\Upsilon}^2 h_6$; $h_6 = h^{[0]}_6 - \int dv^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon} = h^{[0]}_6 - ({}^3\Phi)^2/4\ {}_3\Lambda$; $w_{i_2} = \partial_{i_2}({}^3\Psi)/\ \partial_5({}^3\Psi) = \partial_{i_2}({}^3\Psi)^2/\ \partial_5({}^3\Psi)^2\|$; $n_{k_2} = {}_1n_{k_2} + {}_2n_{k_2}\int dv^5(\partial_5\ {}^3\Psi)^2/\ {}_2\widehat{\Upsilon}^2\|h^{[0]}_6 - \int dv^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon}^2\|^{5/2}$; $\underline{h}_7 = \underline{h}^{[0]}_7 - \int dv^8\partial_8(({}^4\underline{\Psi})^2)/4\ {}_4\underline{\widehat{\Upsilon}} = \underline{h}^{[0]}_7 - ({}^4\underline{\Phi})^2/4\ {}_4\underline{\Lambda}$; $\underline{h}_8 = -(\partial_8\ {}^4\underline{\Psi})^2/4\ {}_4\underline{\widehat{\Upsilon}}^2\underline{h}_7$; $n_{k_3} = {}_1n_{k_3} + {}_2n_{k_3}\int dv^8(\partial_8\ {}^4\underline{\Psi})^2/\ {}_4\underline{\widehat{\Upsilon}}^2\|\underline{h}^{[0]}_7 - \int dv^8\partial_8(({}^4\underline{\Psi})^2)/4\ {}_4\underline{\widehat{\Upsilon}}^2\|^{5/2}$; $w_{i_3} = \partial_{i_3}({}^4\underline{\Psi})/\ \partial_8({}^4\underline{\Psi}) = \partial_{i_3}({}^4\underline{\Psi})^2/\ \partial_8({}^4\underline{\Psi})^2\|$.	

Table 10 Off-diagonal cosmological spacetimes with space velocity configurations exact solutions of $\widehat{\mathbf{R}}_{\mu_s \nu_s} = \Upsilon_{\mu_s \nu_s}$ (24) on TV transformed into a shall system of nonlinear PDEs (47)-(50)

d-metric ansatz with Killing symmetry $\partial_4 = \partial_t, \partial_8$	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + \underline{g}_{a_2}(x^{k_1},t)(dy^{a_2} + \underline{N}^{a_2}_{i_1}(x^{k_1},t)dx^{i_1})^2$ $+ g_{a_3}(x^{k_2},v^5)(dy^{a_3} + N^{a_3}_{i_2}(x^{k_2},v^5)dx^{i_2})^2$ $+ g_{a_4}(x^{k_3},v^7)(dy^{a_4} + N^{a_4}_{i_3}(x^{k_3},v^7)dx^{i_3})^2$, for $g_{i_1} = e^{\psi(x^{k_1})}$, $\underline{g}_{a_2} = \underline{h}_{a_2}(x^{k_1},t), \underline{N}^3_{i_1} = {}^2\underline{n}_{i_1} = \underline{n}_{i_1}(x^{k_1},t), \underline{N}^4_{i_1} = {}^2\underline{w}_{i_1} = \underline{w}_{i_1}(x^{k_1},t)$, $g_{a_3} = h_{a_3}(x^{k_2},v^5), N^5_{i_2} = {}^3w_{i_2} = w_{i_2}(x^{k_2},v^5), N^6_{i_2} = {}^3n_{i_2} = n_{i_2}(x^{k_2},v^5)$, $g_{a_4} = h_{a_4}(x^{k_3},v^7), N^7_{i_3} = {}^4w_{i_3} = w_{i_3}(x^{k_3},v^7), N^8_{i_3} = {}^4n_{i_3} = n_{i_3}(x^{k_3},v^7)$,	
Effective matter sources	$\boldsymbol{\Upsilon}^{\mu_s}_{\ \nu_s} = [\ {}_1\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_2\widehat{\underline{\Upsilon}}(x^{k_1},t)\delta^{a_2}_{b_2},\ {}_3\widehat{\Upsilon}(x^{k_2},v^5)\delta^{a_3}_{b_3}, {}_4\widehat{\Upsilon}(x^{k_3},v^7)\delta^{a_4}_{b_4},]$,	
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ {}_1\widehat{\Upsilon}$; ${}^2\underline{\varpi}^{\diamond}\ \underline{h}_3^{\diamond} = 2\underline{h}_3\underline{h}_4\ {}_2\widehat{\Upsilon}$; ${}^2\underline{n}^{\diamond\diamond}_{k_1} + {}^2\underline{\gamma}\ {}^2\underline{n}^{\diamond}_{k_1} = 0$; ${}^2\underline{\beta}\ {}^2\underline{w}_{i_1} - {}^2\underline{\alpha}_{i_1} = 0$;	${}^2\underline{\varpi} = \ln\lvert\partial_4\underline{h}_4/\sqrt{\lvert\underline{h}_3\underline{h}_4\rvert}\rvert$, ${}^2\underline{\alpha}_{i_1} = (\partial_4\underline{h}_3)\ (\partial_{i_1}\ {}^2\underline{\varpi})$, ${}^2\underline{\beta} = (\partial_4\underline{h}_4)\ (\partial_3\ {}^2\underline{\varpi})$, ${}^2\underline{\gamma} = \partial_4\left(\ln\lvert\underline{h}_3\rvert^{3/2}/\lvert\underline{h}_4\rvert\right)$, $\partial_1 q = q^{\bullet}, \partial_2 q = q', \partial_4 q = \partial_t q = q^{\diamond}$
	$\partial_5({}^3\varpi)\ \partial_5 h_6 = 2h_5h_6\ {}_3\widehat{\Upsilon}$; ${}^3\beta\ {}^3w_{i_2} - {}^3\alpha_{i_2} = 0$; $\partial_5(\partial_5\ {}^3n_{k_2}) + {}^3\gamma\partial_5({}^3n_{k_2}) = 0$;	${}^3\varpi = \ln\lvert\partial_5 h_6/\sqrt{\lvert h_5h_6\rvert}\rvert$, ${}^3\alpha_{i_2} = (\partial_5h_6)\ (\partial_{i_2}\ {}^3\varpi)$, ${}^3\beta = (\partial_5h_6)\ (\partial_5\ {}^3\varpi)$, ${}^3\gamma = \partial_5\left(\ln\lvert h_6\rvert^{3/2}/\lvert h_5\rvert\right)$,
	$\partial_7({}^4\varpi)\ \partial_7 h_8 = 2h_7h_8\ {}_4\widehat{\Upsilon}$; ${}^4\beta\ {}^4w_{i_3} - {}^4\alpha_{i_3} = 0$; $\partial_7(\partial_7\ {}^4n_{k_3}) + {}^4\gamma\partial_7({}^4n_{k_3}) = 0$;	${}^4\varpi = \ln\lvert\partial_7 h_8/\sqrt{\lvert h_7h_8\rvert}\rvert$, ${}^4\alpha_i = (\partial_7h_8)\ (\partial_i\ {}^4\varpi)$, ${}^4\beta = (\partial_7h_8)\ (\partial_7\ {}^4\varpi)$, ${}^4\gamma = \partial_7\left(\ln\lvert h_8\rvert^{3/2}/\lvert h_7\rvert\right)$,
Gener. functs: $\underline{h}_4(x^{k_1},t)$, ${}^2\underline{\Psi}(x^{k_1},t) = e^{\ {}^2\underline{\varpi}}$, ${}^2\underline{\Phi}(x^{k_1},t)$, integr. functs: $\underline{h}_3^{[0]}(x^{k_1})$, ${}_1\underline{n}_{k_1}(x^{i_1})$, ${}_2\underline{n}_{k_1}(x^{i_1})$;	$(({}^2\underline{\Psi})^2)^{\diamond} = -\int dt\ {}_2\widehat{\underline{\Upsilon}}\underline{h}_3{}^{\diamond}$, $({}^2\underline{\Phi})^2 = -4\ {}_2\underline{\Lambda}\underline{h}_3$, $h_3 = h_3^{[0]} - ({}^2\underline{\Phi})^2/4\ {}_2\underline{\Lambda}, \underline{h}_3^{\diamond} \neq 0,\ {}_2\underline{\Lambda} \neq 0 = const$;	
Gener. functs: $h_5(x^{k_2},v^5)$, ${}^3\Psi(x^{k_2},v^5) = e^{\ {}^3\varpi}$, ${}^3\Phi(x^{k_2},v^5)$, integr. functs: $h_6^{[0]}(x^{k_2})$, ${}^3_1n_{k_2}(x^{i_2})$, ${}^3_2n_{k_2}(x^{i_2})$;	$\partial_5(({}^3\Psi)^2) = -\int dv^5\ {}_3\widehat{\Upsilon}\partial_5 h_6$, $({}^3\Phi)^2 = -4\ {}_3\Lambda h_6$, $h_6 = h_6^{[0]} - ({}^3\Phi)^2/4\ {}_3\Lambda, \partial_5 h_6 \neq 0,\ {}_3\Lambda \neq 0 = const$;	
Gener. functs: $h_7(x^{k_3},v^7)$, ${}^4\Psi(x^{k_2},v^7) = e^{\ {}^4\varpi}$, ${}^4\Phi(x^{k_3},v^7)$, integr. functs: $h_8^{[0]}(x^{k_3})$, ${}^4_1n_{k_3}(x^{i_3})$, ${}^4_2n_{k_3}(x^{i_4})$; & nonlinear symmetries	$\partial_7(({}^4\Psi)^2) = -\int dv^7\ {}_4\widehat{\Upsilon}\partial_7 h_8$, $({}^4\Phi)^2 = -4\ {}_4\Lambda h_8$, $h_8 = h_8^{[0]} - ({}^4\Phi)^2/4\ {}_4\Lambda, \partial_7 h_8 \neq 0,\ {}_4\Lambda \neq 0 = const$;	
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ {}_1\widehat{\Upsilon}$; $\underline{h}_4 = -(\underline{\Psi}^{\diamond})^2/4\ {}_2\widehat{\underline{\Upsilon}}^2\underline{h}_3$; $\underline{h}_3 = \underline{h}_3^{[0]} - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\widehat{\underline{\Upsilon}} = \underline{h}_3^{[0]} - \underline{\Phi}^2/4\ {}_2\underline{\Lambda}$; $\underline{w}_{i_1} = \partial_{i_1}\ \underline{\Psi}/\ \partial\underline{\Psi}^{\diamond} = \partial_{i_1}\ \underline{\Psi}^2/\ \partial_t\underline{\Psi}^2\rvert$; $\underline{n}_{k_1} = {}_1n_{k_1} + {}_2n_{k_1}\int dt(\underline{\Psi}^{\diamond})^2/\ {}_2\widehat{\underline{\Upsilon}}^2\lvert\underline{h}_3^{[0]} - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\widehat{\underline{\Upsilon}}^2\rvert^{5/2}$; $h_5 = -(\partial_5\ {}^3\Psi)^2/4\ {}_3\widehat{\Upsilon}^2 h_6$; $h_6 = h_6^{[0]} - \int dv^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon} = h_6^{[0]} - ({}^3\Phi)^2/4\ {}_3\Lambda$; $w_{i_2} = \partial_{i_2}({}^3\Psi)/\ \partial_5({}^3\Psi) = \partial_{i_2}({}^3\Psi)^2/\ \partial_5({}^3\Psi)^2\rvert$; $n_{k_2} = {}_1n_{k_2} + {}_2n_{k_2}\int dv^5(\partial_5\ {}^3\Psi)^2/\ {}_2\widehat{\Upsilon}^2\lvert h_6^{[0]} - \int dv^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon}^2\rvert^{5/2}$; $h_7 = -(\partial_7\ {}^3\Psi)^2/4\ {}_4\widehat{\Upsilon}^2 h_8$; $h_8 = h_8^{[0]} - \int dv^7\partial_9(({}^4\Psi)^2)/4\ {}_4\widehat{\Upsilon} = h_8^{[0]} - ({}^4\Phi)^2/4\ {}_4\Lambda$; $w_{i_3} = \partial_{i_3}({}^4\Psi)/\ \partial_7({}^4\Psi) = \partial_{i_3}({}^4\Psi)^2/\ \partial_7({}^4\Psi)^2\rvert$; $n_{k_3} = {}_1n_{k_3} + {}_2n_{k_3}\int dv^7(\partial_7\ {}^4\Psi)^2/\ {}_4\widehat{\Upsilon}^2\lvert h_8^{[0]} - \int dv^7\partial_7(({}^4\Psi)^2)/4\ {}_4\widehat{\Upsilon}^2\rvert^{5/2}$.	

$$\int dv^8 \frac{(\partial_8 \underline{h}_7)^2}{|\int dv^8 \partial_8[\ _4\widehat{\underline{\Upsilon}} h_7]|\ (\underline{h}_7)^{5/2}}]dx^{k_3}\} + \frac{(\partial_8 \underline{h}_7)^2}{|\int dv^8 \partial_8[\ _4\widehat{\underline{\Upsilon}} h_7]|\ \underline{h}_7}\{dv^8 + \frac{\partial_{i_3}[\int dv^8 (\ _4\widehat{\underline{\Upsilon}})\ \partial_8 \underline{h}_7]}{_4\widehat{\underline{\Upsilon}}\ \partial_8 \underline{h}_7} dx^{i_3}\}^2. \quad (A.7)$$

The AFCDM modifications for generating such solutions are described as follow:

Velocity rainbow s-metrics (A.7) can be considered also for Finsler–Lagrange spaces for respective generating functions. We can impose homogeneity and other type conditions in order to define more special classes of relativistic generalized Finsler geometries. Such models can be redefined for momentum variables on cotangent Lorentz bundles as in next subsection.

A.4 Phase space momentum depending quasi-stationary and cosmological solutions, Tables 12, 13, 14, 15, 16

A series of recent works on nonassociative geometric and quantum information flows, nonassociative and noncommutative gravity and Hamilton–Cartan geometry and gravity are elaborated on nonholonomic phases spaces modeled on a cotangent Lorentz bundle, $\mathcal{M} = T^*\mathbf{V}$, see details and review of results in [5–10,24,29,30]. In section (3.4), we studied a 2+2 toy model with conventional 2-d cofiber coordinates. We generalize those constructions on 8-d phase spaces with conventional dyadic splitting (2+2)+(2+2), when the locall coordinates on shells $s = 3$ and $s = 4$ are momentum type p_a and the local coordinates on the total space are labeled ${}^{\shortmid}u = (x, p) = \{{}^{\shortmid}u^{\alpha} = (x^i, p_a)\} = \{{}^{\shortmid}u^{\alpha_s} = (x^{i_1}, y^{a_2}, p_{a_3}, p_{a_4})\}$ for ${}^{\shortmid}p = p = (p_{a_3}, p_7, p_8 = E)$, where E is an energy type variable. For mechanical like models on cotangent bundles, the momentum like variables (p_{a_3}, p_{a_4}) can be related to velocity type variables (v^{b_3}, v^{b_4}), considered in previous subsection, via certain Legendre transforms. Here it should be noted that theories with momentum like variables admit a respective almost symplectic formulation (in this work, we omit such considerations which are important, for instance, in deformation quantization).

The N-connection structure defining a nonholonomic dyadic splitting on $\mathcal{M}$ is defined as dual nonholonomic distribution (extending the definition (78) from 4-d to 8-d)

$$ {}^{\shortmid}_{s}\mathbf{N}:\ T^*_s\mathbf{V} = {}^1hV \oplus {}^2vV \oplus {}^3cV \oplus {}^4cV. \quad (A.8)$$

In local dual 8-d coordinate form, a N-connection (A.8) can be written as ${}^{\shortmid}_{s}\mathbf{N} = {}^{\shortmid}N_{i_{s-1}a_s}({}^{\shortmid}_{s}u)dx^{i_{s-1}} \otimes \partial/\partial p_{a_s}$. Such N-coefficients define N-elongated (equivalently, N-adapted) local bases (partial derivatives), ${}^{\shortmid}\mathbf{e}_{\nu_s}$, and co-bases (differentials), ${}^{\shortmid}\mathbf{e}^{\mu_s}$, when

$${}^{\shortmid}\mathbf{e}_{\nu} = ({}^{\shortmid}\mathbf{e}_i, {}^{\shortmid}e^a) = ({}^{\shortmid}\mathbf{e}_i = \partial/\partial x^i - {}^{\shortmid}N_{ib}({}^{\shortmid}u)\partial/\partial p_b,$$
$${}^{\shortmid}e^a = {}^{\shortmid}\partial^a = \partial/\partial p_a), \text{ and} \quad (A.9)$$

$${}^{\shortmid}\mathbf{e}^{\mu} = (e^i, {}^{\shortmid}\mathbf{e}^a) = (e^i = dx^i,$$
$${}^{\shortmid}\mathbf{e}_a = dp_a + {}^{\shortmid}N_{ia}({}^{\shortmid}u)dx^i), \quad (A.10)$$

Any phase space metric ${}^{\shortmid}\mathbf{g}$ on ${}^{\shortmid}V$ can be represented equivalently as a s-metric (s, from shell) ${}^{\shortmid}_{s}\mathbf{g} = ({}^1h\ g, {}^2v\ g, {}^3c\ {}^{\shortmid}g, {}^4c\ {}^{\shortmid}g)$, when

$${}^{\shortmid}\mathbf{g} = {}^{\shortmid}g_{i_{s-1}j_{s-1}}(x,p)\ e^{i_{s-1}} \otimes e^{j_{s-1}} + {}^{\shortmid}g^{a_s b_s}(x,p)\ {}^{\shortmid}\mathbf{e}_{a_s} \otimes {}^{\shortmid}\mathbf{e}_{b_s}, = {}^{\shortmid}\underline{g}_{\alpha_s\beta_s}({}^{\shortmid}_{s}u) d\ {}^{\shortmid}u^{\alpha_s} \otimes d\ {}^{\shortmid}u^{\beta_s}, \quad (A.11)$$

where ${}^1h\ {}^{\shortmid}g = \{{}^{\shortmid}g_{i_1j_1}\}$, ${}^2v\ {}^{\shortmid}g = \{{}^{\shortmid}g_{a_2b_2}\}$, ${}^3c\ {}^{\shortmid}g = \{{}^{\shortmid}g^{a_3b_3}\}$ and ${}^4c\ {}^{\shortmid}g = \{{}^{\shortmid}g^{a_4b_4}\}$.

We can define on $\mathcal{M}$ a **d-connection** structure ${}^{\shortmid}_{s}\mathbf{D} = ({}^{s-1}h\ {}^{\shortmid}D, {}^sc\ {}^{\shortmid}D)$ is a linear connection preserving under parallelism the N-connection splitting (A.8),

$${}^{\shortmid}_{s}\mathbf{D} = \{{}^{\shortmid}\Gamma^{\gamma_s}_{\alpha_s\beta_s} = ({}^{\shortmid}L^{i_{s-1}}_{j_{s-1}k_{s-1}}, {}^{\shortmid}\acute{L}_{a_s\ k_{s-1}}^{\ b_s}; {}^{\shortmid}\acute{C}^{i_{s-1}\ c_s}_{j_{s-1}}, {}^{\shortmid}C_{a_s}^{b_s c_s})\}, \text{ where}$$
$${}^{s-1}h\ {}^{\shortmid}D = ({}^{\shortmid}L^{i_{s-1}}_{j_{s-1}k_{s-1}}, {}^{\shortmid}\acute{L}_{a_s\ k_{s-1}}^{\ b_s})$$
$$\text{and } {}^sc\ {}^{\shortmid}D = ({}^{\shortmid}\acute{C}^{i_{s-1}\ c_s}_{j_{s-1}}, {}^{\shortmid}C_{a_s}^{b_s c_s}). \quad (A.12)$$

Tthe c-indices in such N-adapted formulas are inverse to v-indices in N-adapted formulas for $T\mathbf{V}$ considered for previous subsection. Using d-operator ${}^{\shortmid}_{s}\mathbf{D}$, we can define respective fundamental geometric s-objects as in higher dimension Lorentz manifolds, or on their tangent bundles but with abstract definitions on $T^*_s\mathbf{V}$:

$${}^{\shortmid}_{s}\mathcal{T}({}^{\shortmid}_{s}\mathbf{X}, {}^{\shortmid}_{s}\mathbf{Y}) := {}^{\shortmid}_{s}\mathbf{D}_{{}^{\shortmid}_{s}\mathbf{X}}\, {}^{\shortmid}_{s}\mathbf{Y} - {}^{\shortmid}_{s}\mathbf{D}_{{}^{\shortmid}_{s}\mathbf{Y}}\, {}^{\shortmid}_{s}\mathbf{X} - [{}^{\shortmid}_{s}\mathbf{X}, {}^{\shortmid}_{\mathbf{s}}\mathbf{Y}],$$
torsion s-tensor, s-torsion;

$${}^{\shortmid}_{s}\mathcal{R}({}^{\shortmid}_{s}\mathbf{X}, {}^{\shortmid}_{s}\mathbf{Y}) := {}^{\shortmid}_{s}\mathbf{D}_{{}^{\shortmid}_{s}\mathbf{X}}\, {}^{\shortmid}_{s}\mathbf{D}_{{}^{\shortmid}_{s}\mathbf{Y}} - {}^{\shortmid}_{s}\mathbf{D}_{{}^{\shortmid}_{s}\mathbf{Y}}\, {}^{\shortmid}_{s}\mathbf{D}_{{}^{\shortmid}_{s}\mathbf{X}} - {}^{\shortmid}_{s}\mathbf{D}_{[{}^{\shortmid}_{s}\mathbf{X}, {}^{\shortmid}_{s}\mathbf{Y}]},$$
curvature s-tensor, s-curvature;

$${}^{\shortmid}_{s}\mathcal{Q}({}^{\shortmid}\mathbf{X}) := {}^{\shortmid}_{s}\mathbf{D}_{{}^{\shortmid}_{s}\mathbf{X}}\, {}^{\shortmid}_{s}\mathbf{g},$$
nonmetricity s-fields, s-nonmetricity,

where d-vectors ${}^{\shortmid}_{s}\mathbf{X}$ and ${}^{\shortmid}_{s}\mathbf{Y}$, and their duals as 1-forms, can be decomposed respectively to N-linear frames (A.9) and (A.10).

Considering geometric s-objects and formulas (A.8)–(A.12), we can re-define all geometric constructions and formulas for nonholonomic manifolds $\mathbf{V}$ and tangent bundles $T\mathbf{V}$ on cotangent bundles $T^*_s\mathbf{V}$, with shell dyadic structure. For details and applications in MGTs and geometric and quantum information flow theories, we cite [7–10,29,30].

The nonholonomic canonical Einstein equations on 8-d phase spaces with momentum variables can be defined and proven using sympolic re-definitions of variables and geometric d-objects in (24) and (25),

$${}^{\shortmid}\widehat{\mathbf{R}}^{\alpha_s}_{\ \beta_s} = {}^{\shortmid}\widehat{\Upsilon}^{\alpha_s}_{\ \beta_s}, \quad (A.13)$$
$${}^{\shortmid}\widehat{\mathbf{T}}^{\gamma_s}_{\ \alpha_s\beta_s} = 0, \text{ for effective geneating sources}$$

Table 11 Off-diagonal cosmological spacetimes with velocity rainbow symmetries exact solutions of $\widehat{\mathbf{R}}_{\mu_s\nu_s} = \Upsilon_{\mu_s\nu_s}$ (24) on TV transformed into a shall system of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry $\partial_3 = \partial_t, \partial_8$	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + \underline{g}_{a_2}(x^{k_1},t)(dy^{a_2} + \underline{N}^{a_2}_{i_1}(x^{k_1},t)dx^{i_1})^2$ $+g_{a_3}(x^{k_2},v^5)(dy^{a_3} + N^{a_3}_{i_2}(x^{k_2},v^5)dx^{i_2})^2$ $+\underline{g}_{a_4}(x^{k_3},v^8)(dy^{a_4} + \underline{N}^{a_4}_{i_3}(x^{k_3},v^8)dx^{i_3})^2$, for $g_{i_1} = e^{\psi(x^{k_1})}$, $\underline{g}_{a_2} = \underline{h}_{a_2}(x^{k_1},t), \underline{N}^3_{i_1} = {}^2\underline{n}_{i_1} = \underline{n}_{i_1}(x^{k_1},t), \underline{N}^4_{i_1} = {}^2\underline{w}_{i_1} = \underline{w}_{i_1}(x^{k_1},t)$, $g_{a_3} = h_{a_3}(x^{k_2},v^5), N^5_{i_2} = {}^3w_{i_2} = w_{i_2}(x^{k_2},v^5), N^6_{i_2} = {}^3n_{i_2} = n_{i_2}(x^{k_2},v^5)$, $\underline{g}_{a_4} = \underline{h}_{a_4}(x^{k_3},v^8), \underline{N}^7_{i_3} = {}^4\underline{n}_{i_3} = \underline{n}_{i_3}(x^{k_3},v^8), \underline{N}^8_{i_3} = {}^4\underline{w}_{i_3} = \underline{w}_{i_3}(x^{k_3},v^8)$,	
Effective matter sources	$\mathbf{\Upsilon}^{\mu_s}_{\ \nu_s} = [\ {}_1\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_2\widehat{\underline{\Upsilon}}(x^{k_1},t)\delta^{a_2}_{b_2},\ {}_3\widehat{\Upsilon}(x^{k_2},v^5)\delta^{a_3}_{b_3},\ {}_4\widehat{\underline{\Upsilon}}(x^{k_3},v^8)\delta^{a_4}_{b_4},]$,	
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon}$; ${}^2\underline{\varpi}^{\diamond}\ \underline{h}_3^{\diamond} = 2\underline{h}_3\underline{h}_4\ {}_2\widehat{\Upsilon}$; ${}^2\underline{n}^{\diamond\diamond}_{k_1} + {}^2\underline{\gamma}\ {}^2\underline{n}^{\diamond}_{k_1} = 0$; ${}^2\underline{\beta}\ {}^2\underline{w}_{i_1} - {}^2\underline{\alpha}_{i_1} = 0$;	${}^2\underline{\varpi} = \ln\|\partial_4\underline{h}_4/\sqrt{\|\underline{h}_3\underline{h}_4\|}\|$, ${}^2\underline{\alpha}_{i_1} = (\partial_4\underline{h}_3)\ (\partial_{i_1}\ {}^2\underline{\varpi})$, ${}^2\underline{\beta} = (\partial_4\underline{h}_4)\ (\partial_3\ {}^2\underline{\varpi})$, ${}^2\underline{\gamma} = \partial_4\left(\ln\|\underline{h}_3\|^{3/2}/\|\underline{h}_4\|\right)$, $\partial_1 q = q^{\bullet}, \partial_2 q = q', \partial_4 q = \partial_t q = q^{\diamond}$
	$\partial_5({}^3\varpi)\ \partial_5 h_6 = 2h_5h_6\ {}_3\widehat{\Upsilon}$; ${}^3\beta\ {}^3w_{i_2} - {}^3\alpha_{i_2} = 0$; $\partial_5(\partial_5\ {}^3n_{k_2}) + {}^3\gamma\partial_5({}^3n_{k_2}) = 0$;	${}^3\varpi = \ln\|\partial_5 h_6/\sqrt{\|h_5h_6\|}\|$, ${}^3\alpha_{i_2} = (\partial_5 h_6)\ (\partial_{i_2}\ {}^3\varpi)$, ${}^3\beta = (\partial_5 h_6)\ (\partial_5\ {}^3\varpi)$, ${}^3\gamma = \partial_5\left(\ln\|h_6\|^{3/2}/\|h_5\|\right)$,
	$\partial_8({}^4\underline{\varpi})\ \partial_8\underline{h}_7 = 2\underline{h}_7\underline{h}_8\ {}_4\widehat{\underline{\Upsilon}}$; $\partial_8(\partial_8\ {}^4\underline{n}_{k_3}) + {}^4\underline{\gamma}\partial_8({}^4\underline{n}_{k_3}) = 0$; ${}^4\underline{\beta}\ {}^4\underline{w}_{i_3} - {}^4\underline{\alpha}_{i_3} = 0$;	${}^4\underline{\varpi} = \ln\|\partial_8\underline{h}_7/\sqrt{\|\underline{h}_7\underline{h}_8\|}\|$, ${}^4\underline{\alpha}_i = (\partial_8\underline{h}_7)\ (\partial_i\ {}^4\underline{\varpi})$, ${}^4\underline{\beta} = (\partial_8\underline{h}_7)\ (\partial_8\ {}^4\underline{\varpi})$, ${}^4\underline{\gamma} = \partial_8\left(\ln\|\underline{h}_7\|^{3/2}/\|\underline{h}_8\|\right)$,
Gener. functs: $\underline{h}_4(x^{k_1},t)$, ${}^2\underline{\Psi}(x^{k_1},t) = e^{\ {}^2\underline{\varpi}},\ {}^2\underline{\Phi}(x^{k_1},t)$, integr. functs: $\underline{h}_3^{[0]}(x^{k_1})$, ${}_1\underline{n}_{k_1}(x^{i_1}),\ {}_2\underline{n}_{k_1}(x^{i_1})$;	$(({}^2\underline{\Psi})^2)^{\diamond} = -\int dt\ {}_2\widehat{\underline{\Upsilon}}\underline{h}_3^{\diamond}$, $({}^2\underline{\Phi})^2 = -4\ {}_2\underline{\Lambda}\underline{h}_3$, $h_3 = h_3^{[0]} - ({}^2\underline{\Phi})^2/4\ {}_2\underline{\Lambda}, \underline{h}_3^{\diamond} \neq 0,\ {}_2\underline{\Lambda} \neq 0 = const$;	
Gener. functs: $h_5(x^{k_2},v^5)$, ${}^3\Psi(x^{k_2},v^5) = e^{\ {}^3\varpi},\ {}^3\Phi(x^{k_2},v^5)$, integr. functs: $h_6^{[0]}(x^{k_2})$, ${}^3_1n_{k_2}(x^{i_2}),\ {}^3_2n_{k_2}(x^{i_2})$;	$\partial_5(({}^3\Psi)^2) = -\int dv^5\ {}_3\widehat{\Upsilon}\partial_5 h_6$, $({}^3\Phi)^2 = -4\ {}_3\Lambda h_6$, $h_6 = h_6^{[0]} - ({}^3\Phi)^2/4\ {}_3\Lambda, \partial_5 h_6 \neq 0,\ {}_3\Lambda \neq 0 = const$;	
Gener. functs: $h_7(x^{k_3},v^7)$, ${}^4\underline{\Psi}(x^{k_2},v^8) = e^{\ {}^4\underline{\varpi}},\ {}^4\underline{\Phi}(x^{k_3},v^8)$, integr. functs: $h_8^{[0]}(x^{k_3})$, ${}^4_1n_{k_3}(x^{i_3}),\ {}^4_2n_{k_3}(x^{i_4})$; & nonlinear symmetries	$\partial_8(({}^4\underline{\Psi})^2) = -\int dv^8\ {}_4\widehat{\underline{\Upsilon}}\partial_8\underline{h}_7$, $({}^4\underline{\Phi})^2 = -4\ {}_4\underline{\Lambda}\underline{h}_7$, $\underline{h}_7 = h_7^{[0]} - ({}^4\underline{\Phi})^2/4\ {}_4\underline{\Lambda}, \partial_8\underline{h}_7 \neq 0,\ {}_4\underline{\Lambda} \neq 0 = const$;	
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon}$; $\underline{h}_4 = -(\underline{\Psi}^{\diamond})^2/4\ {}_2\widehat{\underline{\Upsilon}}^2\underline{h}_3$; $\underline{h}_3 = \underline{h}_3^{[0]} - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\widehat{\underline{\Upsilon}} = \underline{h}_3^{[0]} - \underline{\Phi}^2/4\ {}_2\underline{\Lambda}$; $\underline{w}_{i_1} = \partial_{i_1}\ \underline{\Psi}/\ \partial\underline{\Psi}^{\diamond} = \partial_{i_1}\ \underline{\Psi}^2/\ \partial_t\underline{\Psi}^2\|$; $\underline{n}_{k_1} = {}_1n_{k_1} + {}_2n_{k_1}\int dt(\underline{\Psi}^{\diamond})^2/\ {}_2\widehat{\underline{\Upsilon}}^2\|\underline{h}_3^{[0]} - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\widehat{\underline{\Upsilon}}^2\|^{5/2}$; $h_5 = -(\partial_5\ {}^3\Psi)^2/4\ {}_3\widehat{\Upsilon}^2 h_6$; $h_6 = h_6^{[0]} - \int dv^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon} = h_6^{[0]} - ({}^3\Phi)^2/4\ {}_3\Lambda$; $w_{i_2} = \partial_{i_2}({}^3\Psi)/\ \partial_5({}^3\Psi) = \partial_{i_2}({}^3\Psi)^2/\ \partial_5({}^3\Psi)^2\|$; $n_{k_2} = {}_1n_{k_2} + {}_2n_{k_2}\int dv^5(\partial_5\ {}^3\Psi)^2/\ {}_2\widehat{\Upsilon}^2\|h_6^{[0]} - \int dv^5\partial_5(({}^3\Psi)^2)/4\ {}_3\widehat{\Upsilon}^2\|^{5/2}$; $\underline{h}_7 = \underline{h}_7^{[0]} - \int dv^8\partial_8(({}^4\underline{\Psi})^2)/4\ {}_4\widehat{\underline{\Upsilon}} = \underline{h}_7^{[0]} - ({}^4\underline{\Phi})^2/4\ {}_4\underline{\Lambda}$; $\underline{h}_8 = -(\partial_8\ {}^4\underline{\Psi})^2/4\ {}_4\widehat{\underline{\Upsilon}}^2\underline{h}_7$; $n_{k_3} = {}_1n_{k_3} + {}_2n_{k_3}\int dv^8(\partial_8\ {}^4\underline{\Psi})^2/\ {}_4\widehat{\underline{\Upsilon}}^2\|\underline{h}_7^{[0]} - \int dv^8\partial_8(({}^4\underline{\Psi})^2)/4\ {}_4\widehat{\underline{\Upsilon}}^2\|^{5/2}$; $w_{i_3} = \partial_{i_3}({}^4\underline{\Psi})/\ \partial_8({}^4\underline{\Psi}) = \partial_{i_3}({}^4\underline{\Psi})^2/\ \partial_8({}^4\underline{\Psi})^2\|$.	

$$ {}^{\shortmid}\widehat{\Upsilon}^{\alpha_s}_{\ \beta_s} = [\ {}^{\shortmid}_1\Upsilon\delta^{i_1}_{\ j_1},\ {}^{\shortmid}_2\Upsilon\delta^{a_2}_{\ b_2},\ {}^{\shortmid}_3\Upsilon\delta^{a_3}_{\ b_3},\ {}^{\shortmid}_4\Upsilon\delta^{a_4}_{\ b_4}]. \quad (A.14)$$

The Eqs. (83) and (84) can be solved by generic off-diagonal ansatz with a Killing vector on respective shells. For instance, a phase space analog of a quasi-stationary s-metric of type (A.4) but with momentum space like hypersurface and fixed $p_8 = E_0$ is parameterized

$$\begin{aligned} {}^{\shortmid}\hat{\mathbf{g}} &= g_{i_1}(x^{k_1})dx^{i_1}\otimes dx^{i_1} + g_{a_2}(x^{k_1}, y^{b_2})\mathbf{e}^{a_2}\otimes\mathbf{e}^{a_2} \\ &\quad + {}^{\shortmid}h^{a_3}(x^{k_2}, p_{b_3})\ {}^{\shortmid}\mathbf{e}_{a_3}\otimes\ {}^{\shortmid}\mathbf{e}_{a_3} \\ &\quad + {}^{\shortmid}h^{7}(\ {}^{\shortmid}x^{k_3}, p_7)\ {}^{\shortmid}\mathbf{e}_7\otimes\ {}^{\shortmid}\mathbf{e}_7 \\ &\quad + {}^{\shortmid}h^{8}(\ {}^{\shortmid}x^{k_3}, p_7)\ {}^{\shortmid}\mathbf{e}_8\otimes\ {}^{\shortmid}\mathbf{e}_8, \\ \mathbf{e}^{a_2} &= dy^{a_2} + N^{a_2}_{i_1}(x^{k_1}, y^{b_2})dx^{i_1}, \\ {}^{\shortmid}\mathbf{e}_{a_3} &= dp_{a_3} + \ {}^{\shortmid}N_{i_2}(\ {}^{\shortmid}x^{k_2}, p_{b_2})d\ {}^{\shortmid}x^{i_2}, \\ {}^{\shortmid}\mathbf{e}_7 &= dp_7 + \ {}^{\shortmid}w_{i_3}(\ {}^{\shortmid}x^{k_3}, p_7)dx^{i_3}, \\ {}^{\shortmid}\mathbf{e}_8 &= dE + \ {}^{\shortmid}n_{i_3}(\ {}^{\shortmid}x^{k_3}, p_7)d\ {}^{\shortmid}x^{i_3}, \end{aligned} \quad (A.15)$$

with Killing symmetry on coordinate vector ${}^{\shortmid}\partial^8 = \partial^E$.

The phase space analog of locally anisotropic cosmological s-metrics (A.5) is stated by formulas,

$$\begin{aligned} {}^{\shortmid}\hat{\mathbf{g}} &= g_{i_1}(x^{k_1})dx^{i_1}\otimes dx^{i_1} + g_{a_2}(x^{k_1}, y^{b_2})\mathbf{e}^{a_2}\otimes\mathbf{e}^{a_2} \\ &\quad + {}^{\shortmid}h^{a_3}(x^{k_2}, p_{b_3})\ {}^{\shortmid}\mathbf{e}_{a_3}\otimes\ {}^{\shortmid}\mathbf{e}_{a_3} \\ &\quad + {}^{\shortmid}h^{7}(\ {}^{\shortmid}x^{k_3}, p_7)\ {}^{\shortmid}\mathbf{e}_7\otimes\ {}^{\shortmid}\mathbf{e}_7 \\ &\quad + {}^{\shortmid}h^{8}(\ {}^{\shortmid}x^{k_3}, p_7)\ {}^{\shortmid}\mathbf{e}_8\otimes\ {}^{\shortmid}\mathbf{e}_8, \\ \mathbf{e}^{a_2} &= dy^{a_2} + N^{a_2}_{i_1}(x^{k_1}, y^{b_2})dx^{i_1}, {}^{\shortmid}\mathbf{e}_{a_3} \\ &= dp_{a_3} + \ {}^{\shortmid}N_{i_2}(\ {}^{\shortmid}x^{k_2}, p_{b_2})d\ {}^{\shortmid}x^{i_2}, \\ {}^{\shortmid}\mathbf{e}_7 &= dp_7 + \ {}^{\shortmid}n_{i_3}(\ {}^{\shortmid}x^{k_3}, E)d\ {}^{\shortmid}x^{i_3}, {}^{\shortmid}\mathbf{e}_8 \\ &= dE + \ {}^{\shortmid}w_{i_3}(\ {}^{\shortmid}x^{k_3}, E)dx^{i_3}, \end{aligned} \quad (A.16)$$

with Killing symmetry on coordinate vector ${}^{\shortmid}\partial^7$.

A.4.1 Diagonal and off-diagonal ansatz for momentum phase spaces

The parametrization of local coordinates, N-connection and canonical d-connection structures and s-metrics for velocity-phase spaces are sated in Table 12.

Parameterizations of geometric s-objects on shells $s = 2, 3$ depend on the type of shell Killing symmetries we prescribe for such nonholonomic phase spaces with momentum like variables.

A.4.2 Quasi-stationary solutions with fixed energy parameter

Such quasi-stationary solutions are nonholonomic momentum type phase configurations modeled on cotangent Lorentz bundles with $p_8 = E = const$, when the momentum phase space involves space like hypersurfaces (Table 13).

As a $T^*\mathbf{V}$ analog of the nonlinear quadratic element (A.4), with $v^8 = const$, and data from Table 8 we provide example of 8-d quasi-stationary quadratic element (A.15) with $p_8 = E = const\mathbf{V}$,

$$\begin{aligned} d\widehat{s}^2_{[8d]} &= \widehat{g}_{\alpha_s\beta_s}(x^k, y^3, p_5, p_7; h_4,\ {}^{\shortmid}h^6,\ {}^{\shortmid}h^8;\ {}^{\shortmid}_s\widehat{\Upsilon}; \\ &\quad {}^{\shortmid}_s\Lambda)d\ {}^{\shortmid}u^{\alpha_s}d\ {}^{\shortmid}u^{\beta_s} \\ &= e^{\psi(x^k,\ _s\widehat{\Upsilon})}[(dx^1)^2 + (dx^2)^2] \\ &\quad - \frac{(h_4^*)^2}{|\int dy^3[\ _2\widehat{\Upsilon}h_4]^*|\ h_4}\{dy^3 \\ &\quad + \frac{\partial_{i_1}[\int dy^3(\ _2\widehat{\Upsilon})\ h_4^*]}{_2\widehat{\Upsilon}\ h_4^*}dx^{i_1}\}^2 \\ &\quad + h_4\{dt + [\ _1n_{k_1} + \ _2n_{k_1}\int dy^3 \\ &\quad \times \frac{(h_4^*)^2}{|\int dy^3[\ _2\widehat{\Upsilon}h_4]^*|\ (h_4)^{5/2}}]dx_1^k\} \\ &\quad + \frac{(\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)^2}{|\int dp_5\ {}^{\shortmid}\partial^5[\ {}^{\shortmid}_3\widehat{\Upsilon}\ {}^{\shortmid}h^6]|\ {}^{\shortmid}h^6} \\ &\quad \times\{dp_5 + \frac{\partial_{i_2}[\int dp_5(\ {}^{\shortmid}_3\widehat{\Upsilon})\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6]}{{}^{\shortmid}_3\widehat{\Upsilon}\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6}dx^{i_2}\}^2 \\ &\quad + {}^{\shortmid}h^6\{dp_5 + [\ _1n_{k_2} + \ _2n_{k_2}\int \\ &\quad \times dp_5\frac{(\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)^2}{|\int dp_5\ {}^{\shortmid}\partial^5[\ {}^{\shortmid}_3\widehat{\Upsilon}\ {}^{\shortmid}h^6]|\ (\ {}^{\shortmid}h^6)^{5/2}}]dx^{k_2}\} \\ &\quad + \frac{(\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8)^2}{|\int dp_7\ {}^{\shortmid}\partial^7[\ _4\widehat{\Upsilon}\ {}^{\shortmid}h^8]|\ {}^{\shortmid}h^8} \\ &\quad \times\{dp_7 + \frac{\partial_{i_3}[\int dp_7(\ _4\widehat{\Upsilon})\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8]}{{}^{\shortmid}_4\widehat{\Upsilon}\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8}d\ {}^{\shortmid}x^{i_3}\}^2 \\ &\quad + {}^{\shortmid}h^8\{dE + [\ {}^{\shortmid}_1n_{k_3} + \ {}^{\shortmid}_2n_{k_3}\int dp_7 \\ &\quad \times \frac{(\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8)^2}{|\int dp_7\ {}^{\shortmid}\partial^7[\ {}^{\shortmid}_4\widehat{\Upsilon}\ {}^{\shortmid}h^8]|\ (\ {}^{\shortmid}h^8)^{5/2}}]d\ {}^{\shortmid}x^{k_3}\}. \end{aligned} \quad (A.17)$$

Such s-metrics possess nonlinear symmetries in phase spaces which allow to re-define the generating functions and generating sources and related them to conventions cosmological constants ${}^{\shortmid}_s\Lambda$.

A.4.3 Quasi-stationary and rainbow phase space solutions

Another class of quasi-stationary momentum phase space solutions of type (A.15) can be generated if in the s-metric (A.17) we change the Killing symmetry on ${}^{\shortmid}\partial^8$ into ${}^{\shortmid}\partial^7$ and introduce in the shell $s = 4$ dependencies on E-variable (in literature called rainbow metrics). Corresponding coefficients of the geometric s-objects will be underlined. Respectively, for such phase space configurations, the constructions

Table 12 Diagonal and off-diagonal ansatz for 8-d cotangent Lorentz bundles and the anholonomic frame and connection deformation method, **AFCDM** *for constructing generic off-diagonal exact and parametric solutions*

diagonal ansatz: PDEs → **ODE**s		AFCDM: **PDE**s **with decoupling;**
coordinates ${}^{\shortmid}u^{\alpha_s}=(x^1,x^2,y^3,y^4=t,$ $p_5,p_6,p_7,p_8=E)$	${}_{s}^{\shortmid}u=({}_{s-1}x,\ {}_{s}^{\shortmid}y)$ $s=1,2,3,4;$	nonholonomic 2+2+2+2+2 splitting; shels $s=1,2,3,4$ ${}^{\shortmid}u^{\alpha_s}=(x^1,x^2,y^3,y^4=t,p_5,p_6,p_7,p_8=E);$ ${}^{\shortmid}u^{\alpha_s}=(x^{i_1},y^{a_2},p_{a_3},p_{a_4});\ {}^{\shortmid}u^{\alpha_s}=(x^{i_{s-1}},\ {}^{\shortmid}y^{a_s});$ $i_1=1,2;a_2=3,4;a_3=5,6;a_4=7,8;$ ${}_{s}^{\shortmid}\mathbf{N}:T\ {}_{s}^{*}\mathbf{V}=hT^{*}\mathbf{V}\oplus\ {}^{2}vT^{*}\mathbf{V}\oplus\ {}^{3}c\mathbf{T}^{*}\mathbf{V}\oplus\ {}^{4}c\mathbf{T}^{*}\mathbf{V},$ locally ${}_{s}^{\shortmid}\mathbf{N}=\{\ {}^{\shortmid}N^{a_s}_{i_{s-1}}(x,p)=$ ${}^{\shortmid}N^{a_s}_{i_{s-1}}({}_{s-1}x,\ {}_{s}^{\shortmid}y)=\ {}^{\shortmid}N^{a_s}_{i_{s-1}}({}_{s}^{\shortmid}u)\}$
LC-connection ${}^{\shortmid}\mathring{\nabla}$	N-connection; canonical d-connection	${}_{s}^{\shortmid}\widehat{\mathbf{D}}=(\ {}^{1}h\ {}^{\shortmid}\widehat{\mathbf{D}},\ {}^{2}v\ {}^{\shortmid}\widehat{\mathbf{D}},\ {}^{3}c\ {}^{\shortmid}\widehat{\mathbf{D}},\ {}^{4}c\ {}^{\shortmid}\widehat{\mathbf{D}})=\{\ {}^{\shortmid}\Gamma^{\alpha_s}_{\ \beta_s\gamma_s}\};$ canonical s-connection distortion ${}_{s}^{\shortmid}\widehat{\mathbf{D}}=\ {}^{\shortmid}\nabla+\ {}_{s}^{\shortmid}\widehat{\mathbf{Z}};\quad {}_{s}^{\shortmid}\widehat{\mathbf{D}}\ {}_{s}^{\shortmid}\mathbf{g}=\mathbf{0},$ ${}_{s}^{\shortmid}\widehat{\mathcal{T}}[\ {}_{s}^{\shortmid}\mathbf{g},\ {}_{s}^{\shortmid}\mathbf{N},\ {}_{s}^{\shortmid}\widehat{\mathbf{D}}]$ canonical d-torsion
diagonal ansatz ${}^{2}\mathring{g}=\mathring{g}_{\alpha_2\beta_2}(\ {}^{s}u)=$ $\begin{pmatrix}\mathring{g}_1 & & & \\ & \mathring{g}_2 & & \\ & & \mathring{g}_3 & \\ & & & \mathring{g}_4\end{pmatrix};$ ${}^{s}g=\mathring{g}_{\alpha_s\beta_s}(\ {}^{s}u)=$ $\begin{pmatrix}{}^{2}\mathring{g} & & & \\ & {}^{\shortmid}\mathring{g}^{5} & & \\ & & \ddots & \\ & & & {}^{\shortmid}\mathring{g}^{8}\end{pmatrix}$	${}^{\shortmid}\mathbf{g}\Leftrightarrow$	$g_{\alpha_2\beta_2}(x^{i_1},y^{a_2})$ general frames / coordinates $g_{\alpha_2\beta_2}=\begin{bmatrix} g_{i_1j_1}+N^{a_2}_{i_1}N^{b_2}_{j_1}h_{a_2b_2} & N^{b_2}_{i_1}h_{c_2b_2}\\ N^{a_2}_{j_1}h_{a_2b_2} & h_{a_2c_2}\end{bmatrix}$ ${}^{2}\mathbf{g}=\{\mathbf{g}_{\alpha_2\beta_2}=[g_{i_1j_1},h_{a_2b_2}]\},$ ${}^{2}\mathbf{g}=\mathbf{g}_{i_1}(x^{k_1})dx^{i_1}\otimes dx^{i_1}+\mathbf{g}_{a_2}(x^{k_2},y^{b_2})\mathbf{e}^{a_2}\otimes\mathbf{e}^{b_2}$ $\vdots$ ${}^{\shortmid}g_{\alpha_s\beta_s}=$ ${}^{\shortmid}g_{\alpha_s\beta_s}(x^{i_{s-1}},\ {}^{\shortmid}y^{a_s})$ general frames / coordinates $\begin{bmatrix} {}^{\shortmid}g_{i_sj_s}+\ {}^{\shortmid}N^{a_s}_{i_{s-1}}\ {}^{\shortmid}N^{b_s}_{j_{s-1}}\ {}^{\shortmid}h_{a_sb_s} & {}^{\shortmid}N^{b_s}_{i_{s-1}}\ {}^{\shortmid}h_{c_sb_s}\\ {}^{\shortmid}N^{a_s}_{j_{s-1}}\ {}^{\shortmid}h_{a_sb_s} & {}^{\shortmid}h_{a_sc_s}\end{bmatrix}$ ${}_{s}^{\shortmid}\mathbf{g}=\{\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}=[\ {}^{\shortmid}g_{i_{s-1}j_{s-1}},\ {}^{\shortmid}h_{a_sb_s}]$ $=[g_{i_1j_1},h_{a_2b_2},\ {}^{\shortmid}h^{a_3b_3},\ {}^{\shortmid}h^{a_4b_4}]\}$ ${}_{s}^{\shortmid}\mathbf{g}=\ {}_{s}^{\shortmid}\mathbf{g}_{i_{s-1}}(x^{k_{s-1}})dx^{i_{s-1}}\otimes dx^{i_{s-1}}+$ ${}_{s}^{\shortmid}\mathbf{g}_{a_s}(x^{k_{s-1}},y^{b_s})\mathbf{e}^{a_s}\otimes\mathbf{e}^{b_s}$ $=\ \mathbf{g}_{i_1}(x^{k_1})dx^{i_1}\otimes dx^{i_1}+\mathbf{g}_{a_2}(x^{k_1},y^{b_2})\mathbf{e}^{a_2}\otimes\mathbf{e}^{a_2}+$ ${}^{\shortmid}\mathbf{g}^{a_3}(x^{k_1},y^{b_2},p_{b_3})\ {}^{\shortmid}\mathbf{e}_{a_3}\otimes\ {}^{\shortmid}\mathbf{e}_{a_3}$ $+\ {}^{\shortmid}\mathbf{g}_{a_4}(x^{k_1},y^{b_2},p_{b_3},p_{b_4})\ {}^{\shortmid}\mathbf{e}_{a_4}\otimes\ {}^{\shortmid}\mathbf{e}_{a_4};$
$\mathring{g}_{\alpha_2\beta_2}=\begin{cases}\mathring{g}_{\alpha_2}(\ {}^{2}r) & \text{for BHs}\\ \mathring{g}_{\alpha_2}(t) & \text{for FLRW}\end{cases}$ ${}^{\shortmid}\mathring{g}_{\alpha_s\beta_s}=\begin{cases}{}^{\shortmid}\mathring{g}_{\alpha_s}(\ {}_{s}^{\shortmid}r) & \text{for BHs}\\ {}^{\shortmid}\mathring{g}_{\alpha_s}(t) & \text{for FLRW}\end{cases}$	[coord.frames]	$g_{\alpha_2\beta_2}=\begin{cases} g_{\alpha_2\beta_2}(x^i,y^3)\\ \underline{g}_{\alpha_2\beta_2}(x^i,y^4=t)\end{cases}$ ${}^{\shortmid}g_{\alpha_s\beta_s}=\begin{cases}{}^{\shortmid}g_{\alpha_s\beta_s}(x^{i_3},p_7)\\ {}^{\shortmid}\underline{g}_{\alpha_s\beta_s}(x^{i_3},E)\end{cases}$
coord. transf. ${}^{\shortmid}e_{\alpha_s}=\ {}^{\shortmid}e^{\alpha'_s}_{\ \alpha_s}\ {}^{\shortmid}\partial_{\alpha'_s},$ ${}^{\shortmid}e^{\beta_s}=\ {}^{\shortmid}e^{\ \beta_s}_{\beta'_s}d\ {}^{\shortmid}u^{\beta'_s},$ ${}^{\shortmid}\mathring{g}_{\alpha_s\beta_s}=\ {}^{\shortmid}\mathring{g}_{\alpha'_s\beta'_s}\ {}^{\shortmid}e^{\alpha'_s}_{\ \alpha_s}\ {}^{\shortmid}e^{\beta'_s}_{\ \beta_s}$ ${}^{\shortmid}\mathring{\mathbf{g}}_{\alpha_s}(\ {}^{\shortmid}x^{k_{s-1}},\ {}^{\shortmid}y^{a_s})\rightarrow\ {}^{\shortmid}\mathring{g}_{\alpha_s}(\ {}_{s}^{\shortmid}r),$ ${}^{\shortmid}\mathring{g}_{\alpha_s}(t),\ {}^{\shortmid}\mathring{N}^{a_s}_{i_{s-1}}(x^{k_{s-1}},\ {}^{\shortmid}y^{a_s})\rightarrow 0.$	[N-adapt. fr.]	$\begin{cases}\mathbf{g}_{i_1}(x^{k_1}),\mathbf{g}_{a_2}(x^{k_1},y^3),\\ \text{or } \mathbf{g}_{i_1}(x^{k_1}),\underline{\mathbf{g}}_{a_2}(x^{k_1},t),\\ N^3_{i_1}=w_{i_1}(x^k,y^3),N^4_{i_1}=n_{i_1}(x^k,y^3),\\ \text{or } \underline{N}^3_{i_1}=\underline{n}_{i_1}(x^{k_1},t),\underline{N}^4_{i_1}=\underline{w}_{i_1}(x^{k_1},t),\end{cases}$ $\vdots$ $\begin{cases}{}^{\shortmid}\mathbf{g}_{i_3}(\ {}^{\shortmid}x^{k_3}),\ {}^{\shortmid}\mathbf{g}_{a_4}(\ {}^{\shortmid}x^{k_3},p_7),\\ \text{or }\ {}^{\shortmid}\mathbf{g}_{i_3}(\ {}^{\shortmid}x^{k_1}),\ {}^{\shortmid}\underline{\mathbf{g}}_{a_4}(\ {}^{\shortmid}x^{k_3},E),\\ {}^{\shortmid}N_{i_37}=\ {}^{\shortmid}w_{i_3}(\ {}^{\shortmid}x^{k_3},p_7),\ {}^{\shortmid}N_{i_38}=\ {}^{\shortmid}n_{i_3}\ {}^{\shortmid}x^{k_3},p_7)\\ {}^{\shortmid}\underline{N}_{i_38}=\ {}^{\shortmid}\underline{n}_{i_3}\ {}^{\shortmid}x^{k_3},E),\ {}^{\shortmid}\underline{N}_{i_38}=\ {}^{\shortmid}\underline{w}_{i_3}\ {}^{\shortmid}x^{k_3},E)\end{cases}$
${}_{s}^{\shortmid}\mathring{\nabla},\quad {}_{s}^{\shortmid}Ric=\{\ {}^{\shortmid}\mathring{R}\ {}_{\beta_s\gamma_s}\}$	Ricci tensors	${}_{s}^{\shortmid}\widehat{\mathbf{D}},\quad {}_{s}^{\shortmid}\widehat{\mathcal{R}}ic=\{\ {}^{\shortmid}\widehat{\mathbf{R}}\ {}_{\beta_s\gamma_s}\}$
${}_{s}^{\shortmid m}\mathcal{L}[\phi]\rightarrow\ {}_{s}^{\shortmid m}\mathbf{T}_{\alpha_s\beta_s}[\phi]$	generating sources	${}^{\shortmid}\widehat{\Upsilon}^{\mu_s}_{\ \nu_s}=\ {}^{\shortmid}\mathbf{e}^{\mu_s}_{\ \mu'_s}\ {}^{\shortmid}\mathbf{e}^{\ \nu'_s}_{\nu_s}\ {}^{\shortmid}\Upsilon^{\mu'_s}_{\ \nu'_s}[\ {}^{m}\mathcal{L}(\varphi),\ {}^{\shortmid}T_{\mu_s\nu_s},\ {}_{s}^{\shortmid}\Lambda]$ $=diag[\ {}_{1}\Upsilon(x^{i_1})\delta^{i_1}_{j_1},\ {}_{2}\Upsilon(x^{i_1},y^3)\delta^{a_2}_{b_2},$ ${}_{3}^{\shortmid}\Upsilon(x^{i_2},p_5)\delta^{a_3}_{b_3},\ {}_{4}^{\shortmid}\Upsilon(x^{i_3},p_7)\delta^{a_4}_{b_4}],$ quasi-stationary configurations; $=diag[\ {}_{1}\Upsilon(x^{i_1})\delta^{i_1}_{j_1},\ {}_{2}\underline{\Upsilon}(x^{i_1},t)\delta^{a_2}_{b_2},$ ${}_{3}^{\shortmid}\underline{\Upsilon}(x^{i_2},p_6)\delta^{a_3}_{b_3},\ {}_{4}^{\shortmid}\underline{\Upsilon}(x^{i_3},E)\delta^{a_4}_{b_4}],$ locally anisotropic cosmology;
trivial eqs for ${}_{s}^{\shortmid}\mathring{\nabla}$-torsion	LC-conditions	${}_{s}^{\shortmid}\widehat{\mathbf{D}}_{\vert\ {}_{s}^{\shortmid}\widehat{\mathcal{T}}\rightarrow 0}=\ {}_{s}^{\shortmid}\nabla.$

Table 13 Off-diagonal quasi-stationary and pase space configurations with fixed energy exact solutions of ${}^{\shortmid}\widehat{\mathbf{R}}_{\mu_s \nu_s} = {}^{\shortmid}\Upsilon_{\mu_s \nu_s}$ (A.13) on T_s^*V transformed into a momentum version of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry $\partial_4 = \partial_t,\ {}^{\shortmid}\partial^8$	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + g_{a_2}(x^{k_1},y^3)(dy^{a_2} + N^{a_2}_{i_1}(x^{k_1},y^3)dx^{i_1})^2$ $+ {}^{\shortmid}g^{a_3}(x^{k_2},p_5)(dp_{a_3} + {}^{\shortmid}N_{i_2 a_3}(x^{k_2},p_5)dx^{i_2})^2$ $+ {}^{\shortmid}g^{a_4}({}^{\shortmid}x^{k_3},p_7)(dp_{a_4} + {}^{\shortmid}N_{i_3 a_4}({}^{\shortmid}x^{k_3},p_7)d\,{}^{\shortmid}x^{i_3})^2,$ for $g_{i_1} = e^{\psi(x^{k_1})},$ $g_{a_2} = h_{a_2}(x^{k_1},y^3), N^3_{i_1} = {}^{2}w_{i_1} = w_{i_1}(x^{k_1},y^3), N^4_{i_1} = {}^{2}n_{i_1} = n_{i_1}(x^{k_1},y^3),$ ${}^{\shortmid}g^{a_3} = {}^{\shortmid}h^{a_3}(x^{k_2},p_5),\ {}^{\shortmid}N_{i_2 5} = {}^{3}_{\shortmid}w_{i_2} = {}^{\shortmid}w_{i_2}(x^{k_2},p_5),$ ${}^{\shortmid}N_{i_2 6} = {}^{3}_{\shortmid}n_{i_2} = {}^{\shortmid}n_{i_2}(x^{k_2},p_5),$ ${}^{\shortmid}g^{a_4} = {}^{\shortmid}h^{a_4}({}^{\shortmid}x^{k_3},p_7),\ {}^{\shortmid}N_{i_3 7} = {}^{4}_{\shortmid}w_{i_3} = {}^{\shortmid}w_{i_3}(x^{k_3},p_7),$ ${}^{\shortmid}N_{i_3 8} = {}^{4}_{\shortmid}n_{i_3} = {}^{\shortmid}n_{i_3}(x^{k_3},p_7),$	
Effective matter sources	${}^{\shortmid}\boldsymbol{\Upsilon}^{\mu_s}_{\ \nu_s} = [\ {}_{1}\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_{2}\widehat{\Upsilon}(x^{k_1},y^3)\delta^{a_2}_{b_2},\ {}^{\shortmid}_{3}\widehat{\Upsilon}(x^{k_2},p_5)\delta^{a_3}_{b_3},\ {}^{\shortmid}_{4}\widehat{\Upsilon}(x^{k_3},p_7)\delta^{a_4}_{b_4},],$	
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_{1}\widehat{\Upsilon};$ ${}^{2}\varpi^{*}\ h^{*}_4 = 2h_3 h_4\ {}_{2}\widehat{\Upsilon};$ ${}^{2}\beta\ {}^{2}w_{i_1} - {}^{2}\alpha_{i_1} = 0;$ ${}^{2}n^{**}_{k_1} + {}^{2}\gamma\ {}^{2}n^{*}_{k_1} = 0;$ ${}^{\shortmid}\partial^5({}^{3}_{\shortmid}\varpi)\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6 = 2\,{}^{\shortmid}h^5\,{}^{\shortmid}h^6\ \ {}^{\shortmid}_{3}\widehat{\Upsilon};$ ${}^{3}_{\shortmid}\beta\ {}^{3}_{\shortmid}w_{i_2} - {}^{3}_{\shortmid}\alpha_{i_2} = 0;$ ${}^{\shortmid}\partial^5({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}n_{k_2}) + {}^{3}_{\shortmid}\gamma\ {}^{\shortmid}\partial^5({}^{3}_{\shortmid}n_{k_2}) = 0;$ ${}^{\shortmid}\partial^7({}^{4}_{\shortmid}\varpi)\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8 = 2\,{}^{\shortmid}h^7\,{}^{\shortmid}h^8\ \ {}^{\shortmid}_{4}\widehat{\Upsilon};$ ${}^{4}_{\shortmid}\beta\ {}^{4}_{\shortmid}w_{i_3} - {}^{4}_{\shortmid}\alpha_{i_3} = 0;$ ${}^{\shortmid}\partial^7({}^{\shortmid}\partial^7\ {}^{4}_{\shortmid}n_{k_3}) + {}^{4}_{\shortmid}\gamma\ {}^{\shortmid}\partial^7({}^{4}_{\shortmid}n_{k_3}) = 0;$	${}^{2}\varpi = \ln\vert\partial_3 h_4/\sqrt{\vert h_3 h_4\vert}\vert,$ ${}^{2}\alpha_{i_1} = (\partial_3 h_4)\ (\partial_{i_1}\ {}^{2}\varpi),$ ${}^{2}\beta = (\partial_3 h_4)\ (\partial_3\ {}^{2}\varpi),$ ${}^{2}\gamma = \partial_3\left(\ln\vert h_4\vert^{3/2}/\vert h_3\vert\right),$ $\partial_1 q = q^{\bullet}, \partial_2 q = q', \partial_3 q = q^{*}$ ${}^{3}_{\shortmid}\varpi = \ln\vert{}^{\shortmid}\partial^5\ {}^{\shortmid}h^6/\sqrt{\vert{}^{\shortmid}h^5\ {}^{\shortmid}h^6\vert}\vert,$ ${}^{3}_{\shortmid}\alpha_{i_2} = ({}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)\ (\partial_{i_2}\ {}^{3}_{\shortmid}\varpi),$ ${}^{3}_{\shortmid}\beta = ({}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)\ ({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}\varpi),$ ${}^{3}_{\shortmid}\gamma = {}^{\shortmid}\partial^5\left(\ln\vert{}^{\shortmid}h^6\vert^{3/2}/\vert{}^{\shortmid}h^5\vert\right),$ ${}^{4}_{\shortmid}\varpi = \ln\vert{}^{\shortmid}\partial^7\ {}^{\shortmid}h^8/\sqrt{\vert{}^{\shortmid}h^7\ {}^{\shortmid}h^8\vert}\vert,$ ${}^{4}_{\shortmid}\alpha_{i_3} = ({}^{\shortmid}\partial^7\ {}^{\shortmid}h^8)\ ({}^{\shortmid}\partial_{i_3}\ {}^{4}_{\shortmid}\varpi),$ ${}^{4}_{\shortmid}\beta = ({}^{\shortmid}\partial^7\ {}^{\shortmid}h^8)\ ({}^{\shortmid}\partial^7\ {}^{4}_{\shortmid}\varpi),$ ${}^{4}_{\shortmid}\gamma = {}^{\shortmid}\partial^7\left(\ln\vert{}^{\shortmid}h^8\vert^{3/2}/\vert{}^{\shortmid}h^7\vert\right),$
Gener. functs: $h_3(x^{k_1},y^3),$ ${}^{2}\Psi(x^{k_1},y^3) = e^{\ {}^{2}\varpi},\ {}^{2}\Phi(x^{k_1},y^3),$ integr. functs: $h^{[0]}_4(x^{k_1}),$ ${}_{1}n_{k_1}(x^{i_1}),\ {}_{2}n_{k_1}(x^{i_1});$ Gener. functs: ${}^{\shortmid}h^5(x^{k_2},p_5),$ ${}^{3}_{\shortmid}\Psi(x^{k_2},p_5) = e^{\ {}^{3}_{\shortmid}\varpi},\ {}^{3}_{\shortmid}\Phi(x^{k_2},p_5),$ integr. functs: $h^{[0]}_6(x^{k_2}),$ ${}^{3}_{1}n_{k_2}(x^{i_2}),\ {}^{3}_{2}n_{k_2}(x^{i_2});$ Gener. functs: ${}^{\shortmid}h^7({}^{\shortmid}x^{k_3},p_7),$ ${}^{4}_{\shortmid}\Psi(x^{k_2},p_7) = e^{\ {}^{4}_{\shortmid}\varpi},\ {}^{4}_{\shortmid}\Phi({}^{\shortmid}x^{k_3},p_7),$ integr. functs: $h^{[0]}_8({}^{\shortmid}x^{k_3}),$ ${}^{4}_{1}n_{k_3}({}^{\shortmid}x^{i_3}),\ {}^{4}_{2}n_{k_3}({}^{\shortmid}x^{i_3});$ & nonlinear symmetries	$(({}^{2}\Psi)^2)^{*} = -\int dy^3\ {}_{2}\widehat{\Upsilon}h_4{}^{*},$ $({}^{2}\Phi)^2 = -4\ {}_{2}\Lambda h_4,$ see (71), $h_4 = h^{[0]}_4 - ({}^{2}\Phi)^2/4\ {}_{2}\Lambda, h^{*}_4 \neq 0,\ {}_{2}\Lambda \neq 0 = const;$ ${}^{\shortmid}\partial^5(({}^{3}_{\shortmid}\Psi)^2) = -\int dp_5\ {}^{\shortmid}_{3}\widehat{\Upsilon}\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6,$ $({}^{3}_{\shortmid}\Phi)^2 = -4\ {}^{\shortmid}_{3}\Lambda\ {}^{\shortmid}h^6,$ ${}^{\shortmid}h^6 = {}^{\shortmid}h^6_{[0]} - ({}^{3}_{\shortmid}\Phi)^2/4\ {}_{3}\Lambda,\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6 \neq 0,\ {}^{\shortmid}_{3}\Lambda \neq 0 = const;$ ${}^{\shortmid}\partial^7(({}^{4}_{\shortmid}\Psi)^2) = -\int dp_7\ {}^{\shortmid}_{4}\widehat{\Upsilon}\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8,$ $({}^{4}_{\shortmid}\Phi)^2 = -4\ {}^{\shortmid}_{4}\Lambda\ {}^{\shortmid}h^8,$ ${}^{\shortmid}h^8 = {}^{\shortmid}h^8_{[0]} - ({}^{4}_{\shortmid}\Phi)^2/4\ {}^{\shortmid}_{4}\Lambda,\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8 \neq 0,\ {}^{\shortmid}_{4}\Lambda \neq 0 = const;$	
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_{1}\widehat{\Upsilon};$ $h_3 = -(\Psi^{*})^2/4\ {}_{2}\widehat{\Upsilon}^2 h_4,$ see (59), (58); $h_4 = h^{[0]}_4 - \int dy^3(\Psi^2)^{*}/4\ {}_{2}\widehat{\Upsilon} = h^{[0]}_4 - \Phi^2/4\ {}_{2}\Lambda;$ $w_i = \partial_i\ \Psi/\ \partial_3\Psi = \partial_i\ \Psi^2/\ \partial_3\Psi^2\vert;$ $n_k = {}_{1}n_k + {}_{2}n_k\int dy^3(\Psi^{*})^2/\ {}_{2}\widehat{\Upsilon}^2\vert h^{[0]}_4 - \int dy^3(\Psi^2)^{*}/4\ {}_{2}\widehat{\Upsilon}^2\vert^{5/2};$ ${}^{\shortmid}h^5 = -({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}\Psi)^2/4\ {}^{\shortmid}_{3}\widehat{\Upsilon}^2\ {}^{\shortmid}h^6;$ ${}^{\shortmid}h^6 = {}^{\shortmid}h^6_{[0]} - \int dp_5\ {}^{\shortmid}\partial^5(({}^{3}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{3}\widehat{\Upsilon} = {}^{\shortmid}h^6_{[0]} - ({}^{3}_{\shortmid}\Phi)^2/4\ {}^{\shortmid}_{3}\Lambda;$ $w_{i_2} = \partial_{i_2}({}^{3}_{\shortmid}\Psi)/\ {}^{\shortmid}\partial^5({}^{3}_{\shortmid}\Psi) = \partial_{i_2}({}^{3}_{\shortmid}\Psi)^2/\ {}^{\shortmid}\partial^5({}^{3}_{\shortmid}\Psi)^2\vert;$ $n_{k_2} = {}_{1}n_{k_2} + {}_{2}n_{k_2}\int dp_5({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}\Psi)^2/\ {}^{\shortmid}_{3}\widehat{\Upsilon}^2\vert\ {}^{\shortmid}h^6_{[0]} -$ $\int dp_5\ {}^{\shortmid}\partial^5(({}^{3}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{3}\widehat{\Upsilon}^2\vert^{5/2}$ ${}^{\shortmid}h^7 = -({}^{\shortmid}\partial^7\ {}^{4}_{\shortmid}\Psi)^2/4\ {}^{4}_{\shortmid}\widehat{\Upsilon}^2\ {}^{\shortmid}h^8;$ ${}^{\shortmid}h^8 = {}^{\shortmid}h^8_{[0]} - \int dp_7\ {}^{\shortmid}\partial^7(({}^{4}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{4}\widehat{\Upsilon} = h^{[0]}_8 - ({}^{4}_{\shortmid}\Phi)^2/4\ \ {}^{\shortmid}_{4}\Lambda;$ ${}^{\shortmid}w_{i_3} = {}^{\shortmid}\partial_{i_3}({}^{4}_{\shortmid}\Psi)/\ {}^{\shortmid}\partial^7({}^{4}_{\shortmid}\Psi) = {}^{\shortmid}\partial_{i_3}({}^{4}_{\shortmid}\Psi)^2/\ {}^{\shortmid}\partial^7({}^{4}_{\shortmid}\Psi)^2\vert;$ ${}^{\shortmid}n_{k_3} = {}^{\shortmid}_{1}n_{k_3} + {}^{\shortmid}_{2}n_{k_3}\int dp_7({}^{4}_{\shortmid}\Psi)^2/\ {}^{\shortmid}_{4}\widehat{\Upsilon}^2\vert h^{[0]}_8 - \int dp_7\ {}^{\shortmid}\partial^7(({}^{4}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{4}\widehat{\Upsilon}^2\vert^{5/2}$	

stated by Table 9 and s-metric (A.5) transform via duality transforms $v^{a_s} \rightarrow p_{a_s}$ into those for Table 14.

Chronologically, we note that rainbow s-metrics in generalized Finsler–Lagrange and dual Cartan–Hamilton forms were constructed following different nonholonomic parameterizations in [24,25], see further developments and review of results in [7,63,147]. The cosmological scenarios elaborated in [141–146] can be re-defined on $T_s^* V$. They can be exploited as some alternative models of dark matter and dark energy theories when the structure formation and phase space dynamics depend on certain E type variables/coordinates.

The rainbow type solutions (for toy models) (86), (87) and (89) can be re-defined into quasi-stationary, or t-depending, and/or E-depending s-metrics with effective shell cosmological constants, and related generating functions, or η- and χ-polarization functions.

A typical quasi-stationary rainbow metric on $T^*\mathbf{V}$ constructed for changing indices $7 \longleftrightarrow 8$ and respective dependencies on coordinates and Killing symmetry on $s = 4$ in (A.17) is defined by such a s-metric with explicit dependence on E-variable:

$$
\begin{aligned}
d\widehat{s}^2_{[8d]} &= \widehat{g}_{\alpha_s\beta_s}(x^k, y^3, p_5, E; h_4, {}^{\shortmid}h^6, {}^{\shortmid}\underline{h}^7; \\
&\quad {}_{1}^{\shortmid}\widehat{\Upsilon}, {}_{2}^{\shortmid}\widehat{\Upsilon}, {}_{3}^{\shortmid}\widehat{\Upsilon}, {}_{4}^{\shortmid}\underline{\widehat{\Upsilon}}; {}_{1}^{\shortmid}\Lambda, {}_{2}^{\shortmid}\Lambda, {}_{3}^{\shortmid}\Lambda, {}_{4}^{\shortmid}\underline{\Lambda}) d\,{}^{\shortmid}u^{\alpha_s} d\,{}^{\shortmid}u^{\beta_s} \\
&= e^{\psi(x^k,\, {}_s\widehat{\Upsilon})}[(dx^1)^2 + (dx^2)^2] \\
&\quad - \frac{(h_4^*)^2}{|\int dy^3 [{}_2\widehat{\Upsilon} h_4]^*|\, h_4} \{dy^3 \\
&\quad + \frac{\partial_{i_1}[\int dy^3 ({}_2\widehat{\Upsilon})\, h_4^*]}{{}_2\widehat{\Upsilon}\, h_4^*} dx^{i_1}\}^2 \\
&\quad + h_4\{dt + [{}_1n_{k_1} + {}_2n_{k_1} \\
&\quad \int dy^3 \frac{(h_4^*)^2}{|\int dy^3 [{}_2\widehat{\Upsilon} h_4]^*|\, (h_4)^{5/2}}] dx_1^k\} \\
&\quad + \frac{({}^{\shortmid}\partial^5\, {}^{\shortmid}h^6)^2}{|\int dp_5\, {}^{\shortmid}\partial^5[{}_{3}^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}h^6]|\, {}^{\shortmid}h^6} \{dp_5 \\
&\quad + \frac{\partial_{i_2}[\int dp_5 ({}_{3}^{\shortmid}\widehat{\Upsilon})\, {}^{\shortmid}\partial^5\, {}^{\shortmid}h^6]}{{}_{3}^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}\partial^5\, {}^{\shortmid}h^6} dx^{i_2}\}^2 \\
&\quad + {}^{\shortmid}h^6\{dp_5 + [{}_1n_{k_2} + {}_2n_{k_2} \int dp_5 \\
&\quad \times \frac{({}^{\shortmid}\partial^5\, {}^{\shortmid}h^6)^2}{|\int dp_5\, {}^{\shortmid}\partial^5[{}_{3}^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}h^6]|\, ({}^{\shortmid}h^6)^{5/2}}] dx^{k_2}\} \\
&\quad + {}^{\shortmid}\underline{h}^7\{dp_7 + [{}_{1}^{\shortmid}\underline{n}_{k_3} + {}_{2}^{\shortmid}\underline{n}_{k_3} \int dp_7 \\
&\quad \times \frac{({}^{\shortmid}\underline{\partial}^8\, {}^{\shortmid}\underline{h}^7)^2}{|\int dE\, {}^{\shortmid}\underline{\partial}^8[{}_{4}^{\shortmid}\underline{\widehat{\Upsilon}}\, {}^{\shortmid}\underline{h}^7]|\, ({}^{\shortmid}\underline{h}^7)^{5/2}}] d\,{}^{\shortmid}x^{k_3}\} \\
&\quad + \frac{({}^{\shortmid}\underline{\partial}^8\, {}^{\shortmid}\underline{h}^7)^2}{|\int dE\, {}^{\shortmid}\underline{\partial}^8[{}_4\underline{\widehat{\Upsilon}}\, {}^{\shortmid}\underline{h}^7]|\, {}^{\shortmid}\underline{h}^7} \\
&\quad \times \{dE + \frac{\partial_{i_3}[\int dE ({}_4\underline{\widehat{\Upsilon}})\, {}^{\shortmid}\underline{\partial}^8\, {}^{\shortmid}\underline{h}^7]}{{}_{4}^{\shortmid}\underline{\widehat{\Upsilon}}\, {}^{\shortmid}\underline{\partial}^8\, {}^{\shortmid}\underline{h}^7} d\,{}^{\shortmid}x^{i_3}\}^2.
\end{aligned} \tag{A.18}
$$

Such rainbow s-metrics can be re-parameterized for another types of generating functions and/or with gravitational polarization functions using respective nonlinear symmetries.

A.4.4 Locally anisotropic cosmological solutions with fixed energy parameter

For dual fiber to cofiber transforms, the procedure for constructing locally anisotropic cosmological phase space solutions described in Table 10 transforms into a method of generating such solutions with off-diagonal dependence on momentum like variables. Such generalizations and applications of the AFCDM are summarized in Table 15. As a $T^*\mathbf{V}$ analog of the nonlinear quadratic element (A.4), with $v^8 = const$, and data from Table 8 we provide example of 8-d quasi-stationary quadratic element (A.15) with $p_8 = E = const\mathbf{V}$,

$$
\begin{aligned}
d\widehat{s}^2_{[8d]} &= \widehat{g}_{\alpha_s\beta_s}(x^k, t, p_5, p_7; \underline{h}_3, {}^{\shortmid}h^6, {}^{\shortmid}h^8; \\
&\quad {}_{s}^{\shortmid}\widehat{\Upsilon}; {}_1\Lambda, {}_2\underline{\Lambda}, {}_{3}^{\shortmid}\Lambda, {}_{4}^{\shortmid}\Lambda) d\,{}^{\shortmid}u^{\alpha_s} d\,{}^{\shortmid}u^{\beta_s} \\
&= e^{\psi(x^k,\, {}_s\widehat{\Upsilon})}[(dx^1)^2 + (dx^2)^2] \\
&\quad + \underline{h}_3[dy^3 + ({}_1n_{k_1} + 4\, {}_2n_{k_1} \int dt \\
&\quad \times \frac{(\underline{h}_3{}^{\diamond})^2}{|\int dt\, {}_2\underline{\Upsilon}\underline{h}_3{}^{\diamond}|(\underline{h}_3)^{5/2}}) dx^{k_1}] \\
&\quad - \frac{(\underline{h}_3{}^{\diamond})^2}{|\int dt\, {}_2\underline{\Upsilon}\underline{h}_3{}^{\diamond}|\, \overline{h}_3} [dt + \frac{\partial_i(\int dt\, {}_2\underline{\Upsilon}\, \underline{h}_3{}^{\diamond}])}{{}_2\underline{\Upsilon}\, \underline{h}_3{}^{\diamond}} dx^i] \\
&\quad + \frac{({}^{\shortmid}\partial^5\, {}^{\shortmid}h^6)^2}{|\int dp_5\, {}^{\shortmid}\partial^5[{}_{3}^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}h^6]|\, {}^{\shortmid}h^6} \{dp_5 \\
&\quad + \frac{\partial_{i_2}[\int dp_5 ({}_{3}^{\shortmid}\widehat{\Upsilon})\, {}^{\shortmid}\partial^5\, {}^{\shortmid}h^6]}{{}_{3}^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}\partial^5\, {}^{\shortmid}h^6} dx^{i_2}\}^2 \\
&\quad + {}^{\shortmid}h^6\{dp_5 + [{}_1n_{k_2} + {}_2n_{k_2} \int dp_5 \\
&\quad \times \frac{({}^{\shortmid}\partial^5\, {}^{\shortmid}h^6)^2}{|\int dp_5\, {}^{\shortmid}\partial^5[{}_{3}^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}h^6]|\, ({}^{\shortmid}h^6)^{5/2}}] dx^{k_2}\} \\
&\quad + \frac{({}^{\shortmid}\partial^7\, {}^{\shortmid}h^8)^2}{|\int dp_7\, {}^{\shortmid}\partial^7[{}_4\widehat{\Upsilon}\, {}^{\shortmid}h^8]|\, {}^{\shortmid}h^8} \\
&\quad \times \{dp_7 + \frac{\partial_{i_3}[\int dp_7 ({}_4\widehat{\Upsilon})\, {}^{\shortmid}\partial^7\, {}^{\shortmid}h^8]}{{}_{4}^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}\partial^7\, {}^{\shortmid}h^8} d\,{}^{\shortmid}x^{i_3}\}^2 \\
&\quad + {}^{\shortmid}h^8\{dE + [{}_{1}^{\shortmid}n_{k_3} + {}_{2}^{\shortmid}n_{k_3} \int dp_7 \\
&\quad \times \frac{({}^{\shortmid}\partial^7\, {}^{\shortmid}h^8)^2}{|\int dp_7\, {}^{\shortmid}\partial^7[{}_{4}^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}h^8]|\, ({}^{\shortmid}h^8)^{5/2}}] d\,{}^{\shortmid}x^{k_3}\}.
\end{aligned} \tag{A.19}
$$

The procedure of generating such s-metrics is described as follow:

The spacetime part in (A.19) is equivalent to the spacetime part of (A.6) (in both cases, on shells $s = 1, 2$).

Table 14 Off-diagonal quasi-stationary and pase space configurations with variable energy exact solutions of ${}^{\shortmid}\widehat{\mathbf{R}}_{\mu_s\nu_s} = {}^{\shortmid}\Upsilon_{\mu_s\nu_s}$ (A.13) on $T^{*}_{s}V$ transformed into a momentum version of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry $\partial_4 = \partial_t,\ {}^{\shortmid}\partial^7$	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + g_{a_2}(x^{k_1},y^3)(dy^{a_2} + N^{a_2}_{i_1}(x^{k_1},y^3)dx^{i_1})^2$ $+\ {}^{\shortmid}g^{a_3}(x^{k_2},p_5)(dp_{a_3} + {}^{\shortmid}N_{i_2 a_3}(x^{k_2},p_5)dx^{i_2})^2$ $+\ {}^{\shortmid}g^{a_4}({}^{\shortmid}x^{k_3},p_7)(dp_{a_4} + {}^{\shortmid}N_{i_3 a_4}({}^{\shortmid}x^{k_3},p_7)d\,{}^{\shortmid}x^{i_3})^2,\ \text{for } g_{i_1} = e^{\psi(x^{k_1})},$ $g_{a_2} = h_{a_2}(x^{k_1},y^3), N^3_{i_1} = {}^{2}w_{i_1} = w_{i_1}(x^{k_1},y^3),$ $N^4_{i_1} = {}^{2}n_{i_1} = n_{i_1}(x^{k_1},y^3),$ ${}^{\shortmid}g^{a_3} = {}^{\shortmid}h^{a_3}(x^{k_2},p_5),\ {}^{\shortmid}N_{i_2 5} = {}^{3}_{\shortmid}w_{i_2} = {}^{\shortmid}w_{i_2}(x^{k_2},p_5),$ ${}^{\shortmid}N_{i_2 6} = {}^{3}_{\shortmid}n_{i_2} = {}^{\shortmid}n_{i_2}(x^{k_2},p_5),$ ${}^{\shortmid}\underline{g}^{a_4} = {}^{\shortmid}\underline{h}^{a_4}({}^{\shortmid}x^{k_3},E),\ {}^{\shortmid}\underline{N}_{i_3 7} = {}^{4}_{\shortmid}\underline{n}_{i_3} = {}^{\shortmid}\underline{n}_{i_3}(x^{k_3},E),$ ${}^{\shortmid}\underline{N}_{i_3 8} = {}^{4}_{\shortmid}\underline{w}_{i_3} = {}^{\shortmid}\underline{w}_{i_3}(x^{k_3},E),$	
Effective matter sources	${}^{\shortmid}\mathbf{\Upsilon}^{\mu_s}_{\ \nu_s} = [\ {}_{1}\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_{2}\widehat{\Upsilon}(x^{k_1},y^3)\delta^{a_2}_{b_2},\ {}^{\shortmid}_{3}\widehat{\Upsilon}(x^{k_2},p_5)\delta^{a_3}_{b_3},\ {}^{\shortmid}_{4}\underline{\widehat{\Upsilon}}(x^{k_3},E)\delta^{a_4}_{b_4},],$	
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_{1}\widehat{\Upsilon};$ ${}^{2}\varpi^{*}\ h^{*}_{4} = 2h_3 h_4\ {}_{2}\widehat{\Upsilon};$ ${}^{2}\beta\ {}^{2}w_{i_1} - {}^{2}\alpha_{i_1} = 0;$ ${}^{2}n^{**}_{k_1} + {}^{2}\gamma\ {}^{2}n^{*}_{k_1} = 0;$ ${}^{\shortmid}\partial^5({}^{3}_{\shortmid}\varpi)\ \ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6 = 2\ {}^{\shortmid}h^5\ {}^{\shortmid}h^6\ \ {}^{\shortmid}_{3}\widehat{\Upsilon};$ ${}^{3}_{\shortmid}\beta\ {}^{3}_{\shortmid}w_{i_2} - {}^{3}_{\shortmid}\alpha_{i_2} = 0;$ ${}^{\shortmid}\partial^5({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}n_{k_2}) + {}^{3}_{\shortmid}\gamma\ {}^{\shortmid}\partial^5({}^{3}_{\shortmid}n_{k_2}) = 0;$ ${}^{\shortmid}\underline{\partial}^8({}^{4}_{\shortmid}\underline{\varpi})\ \ {}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7 = 2\ {}^{\shortmid}\underline{h}^7\ {}^{\shortmid}\underline{h}^8\ {}^{\shortmid}_{4}\underline{\widehat{\Upsilon}};$ ${}^{\shortmid}\underline{\partial}^8({}^{\shortmid}\underline{\partial}^8\ {}^{4}_{\shortmid}\underline{n}_{k_3}) + {}^{4}_{\shortmid}\underline{\gamma}\ {}^{\shortmid}\underline{\partial}^8({}^{4}_{\shortmid}\underline{n}_{k_3}) = 0;$ ${}^{4}_{\shortmid}\underline{\beta}\ {}^{4}_{\shortmid}\underline{w}_{i_3} - {}^{4}_{\shortmid}\underline{\alpha}_{i_3} = 0;$	${}^{2}\varpi = \ln\lvert\partial_3 h_4/\sqrt{\lvert h_3 h_4\rvert}\rvert,$ ${}^{2}\alpha_{i_1} = (\partial_3 h_4)\ (\partial_{i_1}\ {}^{2}\varpi),$ ${}^{2}\beta = (\partial_3 h_4)\ (\partial_3\ {}^{2}\varpi),$ ${}^{2}\gamma = \partial_3\left(\ln\lvert h_4\rvert^{3/2}/\lvert h_3\rvert\right),$ $\partial_1 q = q^{\bullet}, \partial_2 q = q', \partial_3 q = q^{*}$ ${}^{3}_{\shortmid}\varpi = \ln\lvert\,{}^{\shortmid}\partial^5\ {}^{\shortmid}h^6/\sqrt{\lvert\,{}^{\shortmid}h^5\ {}^{\shortmid}h^6\rvert}\rvert,$ ${}^{3}_{\shortmid}\alpha_{i_2} = ({}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)\ (\partial_{i_2}\ {}^{3}_{\shortmid}\varpi),$ ${}^{3}_{\shortmid}\beta = ({}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)\ ({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}\varpi),$ ${}^{3}_{\shortmid}\gamma = {}^{\shortmid}\partial^5\left(\ln\lvert\,{}^{\shortmid}h^6\rvert^{3/2}/\lvert\,{}^{\shortmid}h^5\rvert\right),$ ${}^{4}_{\shortmid}\underline{\varpi} = \ln\lvert\,{}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7/\sqrt{\lvert\,{}^{\shortmid}\underline{h}^7\ {}^{\shortmid}\underline{h}^8\rvert}\rvert,$ ${}^{4}_{\shortmid}\underline{\alpha}_{i_3} = ({}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7)\ ({}^{\shortmid}\partial_{i_3}\ {}^{4}_{\shortmid}\underline{\varpi}),$ ${}^{4}_{\shortmid}\underline{\beta} = ({}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7)\ ({}^{\shortmid}\underline{\partial}^8\ {}^{4}_{\shortmid}\underline{\varpi}),$ ${}^{4}_{\shortmid}\underline{\gamma} = {}^{\shortmid}\underline{\partial}^8\left(\ln\lvert\,{}^{\shortmid}\underline{h}^7\rvert^{3/2}/\lvert\,{}^{\shortmid}\underline{h}^8\rvert\right),$
Gener. functs: $h_3(x^{k_1},y^3)$, ${}^{2}\Psi(x^{k_1},y^3) = e^{\ {}^{2}\varpi},\ {}^{2}\Phi(x^{k_1},y^3),$ integr. functs: $h^{[0]}_4(x^{k_1}),$ ${}_{1}n_{k_1}(x^{i_1}),\ {}_{2}n_{k_1}(x^{i_1});$ Gener. functs: ${}^{\shortmid}h^5(x^{k_2},p_5),$ ${}^{3}_{\shortmid}\Psi(x^{k_2},p_5) = e^{\ {}^{3}_{\shortmid}\varpi},\ \ {}^{3}_{\shortmid}\Phi(x^{k_2},p_5),$ integr. functs: $h^{[0]}_6(x^{k_2}),$ ${}^{3}_{1}n_{k_2}(x^{i_2}),\ {}^{3}_{2}n_{k_2}(x^{i_2});$ Gener. functs: ${}^{\shortmid}h^7({}^{\shortmid}x^{k_3},p_7),$ ${}^{4}_{\shortmid}\underline{\Psi}(x^{k_2},E) = e^{\ {}^{4}_{\shortmid}\underline{\varpi}},\ \ {}^{4}_{\shortmid}\underline{\Phi}({}^{\shortmid}x^{k_3},E),$ integr. functs: $\underline{h}^{[0]}_7({}^{\shortmid}x^{k_3}),$ ${}^{4}_{1}\underline{n}_{k_3}({}^{\shortmid}x^{i_3}),\ {}^{4}_{2}\underline{n}_{k_3}({}^{\shortmid}x^{i_3});$ & nonlinear symmetries	$(({}^{2}\Psi)^2)^{*} = -\int dy^3\ {}_{2}\widehat{\Upsilon}h_4{}^{*},$ $({}^{2}\Phi)^2 = -4\ {}_{2}\Lambda h_4,$ see (71), $h_4 = h^{[0]}_4 - ({}^{2}\Phi)^2/4\ {}_{2}\Lambda, h^{*}_4 \neq 0,\ {}_{2}\Lambda \neq 0 = const;$ ${}^{\shortmid}\partial^5(({}^{3}_{\shortmid}\Psi)^2) = -\int dp_5\ {}^{\shortmid}_{3}\widehat{\Upsilon}\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6,$ $({}^{3}_{\shortmid}\Phi)^2 = -4\ {}^{\shortmid}_{3}\Lambda\ {}^{\shortmid}h^6,$ ${}^{\shortmid}h^6 = {}^{\shortmid}h^6_{[0]} - ({}^{3}_{\shortmid}\Phi)^2/4\ {}_{3}\Lambda,\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6 \neq 0,\ {}^{\shortmid}_{3}\Lambda \neq 0 = const;$ ${}^{\shortmid}\underline{\partial}^8(({}^{4}_{\shortmid}\underline{\Psi})^2) = -\int dE\ {}^{\shortmid}_{4}\underline{\widehat{\Upsilon}}\ {}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7,$ $({}^{4}_{\shortmid}\underline{\Phi})^2 = -4\ {}^{\shortmid}_{4}\underline{\Lambda}\ {}^{\shortmid}\underline{h}^7,$ ${}^{\shortmid}\underline{h}^7 = {}^{\shortmid}\underline{h}^7_{[0]} - ({}^{4}_{\shortmid}\underline{\Phi})^2/4\ {}^{\shortmid}_{4}\underline{\Lambda},\ {}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7 \neq 0,\ {}^{\shortmid}_{4}\underline{\Lambda} \neq 0 = const;$	
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_{1}\widehat{\Upsilon};$ $h_3 = -(\Psi^{*})^2/4\ {}_{2}\widehat{\Upsilon}^2 h_4,$ see (59), (58); $h_4 = h^{[0]}_4 - \int dy^3(\Psi^2)^{*}/4\ {}_{2}\widehat{\Upsilon} = h^{[0]}_4 - \Phi^2/4\ {}_{2}\Lambda;$ $w_i = \partial_i\ \Psi/\ \partial_3\Psi = \partial_i\ \Psi^2/\ \partial_3\Psi^2\rvert;$ $n_k = {}_{1}n_k + {}_{2}n_k\int dy^3(\Psi^{*})^2/\ {}_{2}\widehat{\Upsilon}^2\lvert h^{[0]}_4 - \int dy^3(\Psi^2)^{*}/4\ {}_{2}\widehat{\Upsilon}^2\rvert^{5/2};$ ${}^{\shortmid}h^5 = -({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}\Psi)^2/4\ {}^{\shortmid}_{3}\widehat{\Upsilon}^2\ {}^{\shortmid}h^6;$ ${}^{\shortmid}h^6 = {}^{\shortmid}h^6_{[0]} - \int dp_5\ {}^{\shortmid}\partial^5(({}^{3}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{3}\widehat{\Upsilon} = {}^{\shortmid}h^6_{[0]} - ({}^{3}_{\shortmid}\Phi)^2/4\ {}^{\shortmid}_{3}\Lambda;$ $w_{i_2} = \partial_{i_2}({}^{3}_{\shortmid}\Psi)/\ {}^{\shortmid}\partial^5({}^{3}_{\shortmid}\Psi) = \partial_{i_2}({}^{3}_{\shortmid}\Psi)^2/\ {}^{\shortmid}\partial^5({}^{3}_{\shortmid}\Psi)^2\rvert;$ $n_{k_2} = {}_{1}n_{k_2} + {}_{2}n_{k_2}\int dp_5({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}\Psi)^2/\ {}^{\shortmid}_{3}\widehat{\Upsilon}^2\lvert\ {}^{\shortmid}h^6_{[0]} -$ $\int dp_5\ {}^{\shortmid}\partial^5(({}^{3}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{3}\widehat{\Upsilon}^2\rvert^{5/2}$ ${}^{\shortmid}\underline{h}^8 = -({}^{\shortmid}\underline{\partial}^8\ {}^{4}_{\shortmid}\underline{\Psi})^2/4\ {}^{4}_{\shortmid}\underline{\widehat{\Upsilon}}^2\ {}^{\shortmid}\underline{h}^7;$ ${}^{\shortmid}\underline{h}^7 = {}^{\shortmid}\underline{h}^7_{[0]} - \int dE\ {}^{\shortmid}\underline{\partial}^8(({}^{4}_{\shortmid}\underline{\Psi})^2)/4\ {}^{\shortmid}_{4}\underline{\widehat{\Upsilon}} = \underline{h}^7_{[0]} - ({}^{4}_{\shortmid}\underline{\Phi})^2/4\ \ {}^{\shortmid}_{4}\underline{\Lambda};$ ${}^{\shortmid}\underline{n}_{k_3} = {}^{\shortmid}_{1}\underline{n}_{k_3} + {}^{\shortmid}_{2}\underline{n}_{k_3}\int dE({}^{4}_{\shortmid}\underline{\Psi})^2/\ {}^{\shortmid}_{4}\underline{\widehat{\Upsilon}}^2\lvert\underline{h}^7_{[0]} -$ $\int dE\ {}^{\shortmid}\underline{\partial}^8(({}^{4}_{\shortmid}\underline{\Psi})^2)/4\ {}^{\shortmid}_{4}\underline{\widehat{\Upsilon}}^2\rvert^{5/2};$ ${}^{\shortmid}\underline{w}_{i_3} = {}^{\shortmid}\partial_{i_3}({}^{4}_{\shortmid}\underline{\Psi})/\ {}^{\shortmid}\partial^8({}^{4}_{\shortmid}\underline{\Psi}) = {}^{\shortmid}\partial_{i_3}({}^{4}_{\shortmid}\underline{\Psi})^2/\ {}^{\shortmid}\partial^8({}^{4}_{\shortmid}\underline{\Psi})^2\rvert.$	

Table 15 Off-diagonal cosmological and pase space configurations with fixed energy exact solutions of ${}^{\shortmid}\widehat{\mathbf{R}}_{\mu_s \nu_s} = {}^{\shortmid}\Upsilon_{\mu_s \nu_s}$ (A.13) on $T_s^* V$ transformed into a momentum version of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry $\partial_3 = \partial_t,\ {}^{\shortmid}\partial^8$	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + \underline{g}_{a_2}(x^{k_1},t)(dy^{a_2} + \underline{N}_{i_1}^{a_2}(x^{k_1},t)dx^{i_1})^2$ $+ {}^{\shortmid}g^{a_3}(x^{k_2},p_5)(dp_{a_3} + {}^{\shortmid}N_{i_2 a_3}(x^{k_2},p_5)dx^{i_2})^2$ $+ {}^{\shortmid}g^{a_4}({}^{\shortmid}x^{k_3},p_7)(dp_{a_4} + {}^{\shortmid}N_{i_3 a_4}({}^{\shortmid}x^{k_3},p_7)d\,{}^{\shortmid}x^{i_3})^2,$ for $g_{i_1} = e^{\psi(x^{k_1})},$ $\underline{g}_{a_2} = \underline{h}_{a_2}(x^{k_1},t), \underline{N}_{i_1}^3 = {}^2\underline{n}_{i_1} = \underline{n}_{i_1}(x^{k_1},t), N_{i_1}^4 = {}^2\underline{w}_{i_1} = \underline{w}_{i_1}(x^{k_1},t),$ ${}^{\shortmid}g^{a_3} = {}^{\shortmid}h^{a_3}(x^{k_2},p_5),\ {}^{\shortmid}N_{i_2 5} = {}^{3}_{\shortmid}w_{i_2} = {}^{\shortmid}w_{i_2}(x^{k_2},p_5),$ ${}^{\shortmid}N_{i_2 6} = {}^{3}_{\shortmid}n_{i_2} = {}^{\shortmid}n_{i_2}(x^{k_2},p_5),$ ${}^{\shortmid}g^{a_4} = {}^{\shortmid}h^{a_4}({}^{\shortmid}x^{k_3},p_7),\ {}^{\shortmid}N_{i_3 7} = {}^{4}_{\shortmid}w_{i_3} = {}^{\shortmid}w_{i_3}(x^{k_3},p_7),$ $N_{i_3 8} = {}^{4}_{\shortmid}n_{i_3} = {}^{\shortmid}n_{i_3}(x^{k_3},p_7),$
Effective matter sources	${}^{\shortmid}\boldsymbol{\Upsilon}^{\mu_s}_{\ \nu_s} = [\ {}_1\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_2\underline{\widehat{\Upsilon}}(x^{k_1},t)\delta^{a_2}_{b_2},\ {}^{\shortmid}_{3}\widehat{\Upsilon}(x^{k_2},p_5)\delta^{a_3}_{b_3},\ {}^{\shortmid}_{4}\widehat{\Upsilon}(x^{k_3},p_7)\delta^{a_4}_{b_4},],$
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon};$ ${}^2\underline{\varpi}^{\diamond}\ \underline{h}_3^{\diamond} = 2\underline{h}_3\underline{h}_4\ {}_2\widehat{\Upsilon};$ ${}^2\underline{n}_{k_1}^{\diamond\diamond} + {}^2\underline{\gamma}\ {}^2\underline{n}_{k_1}^{\diamond} = 0;$ ${}^2\underline{\beta}\ {}^2\underline{w}_{i_1} - {}^2\underline{\alpha}_{i_1} = 0;$ ${}^2\underline{\varpi} = \ln\|\partial_4\underline{h}_4/\sqrt{\|\underline{h}_3\underline{h}_4\|}\|,$ ${}^2\underline{\alpha}_{i_1} = (\partial_4\underline{h}_3)\ (\partial_{i_1}\ {}^2\underline{\varpi}),$ ${}^2\underline{\beta} = (\partial_4\underline{h}_4)\ (\partial_3\ {}^2\underline{\varpi}),$ ${}^2\underline{\gamma} = \partial_4\left(\ln\|\underline{h}_3\|^{3/2}/\|\underline{h}_4\|\right),$ $\partial_1 q = q^{\bullet}, \partial_2 q = q',$ $\partial_4 q = \partial_t q = q^{\diamond}$ ${}^{\shortmid}\partial^5({}^{3}_{\shortmid}\varpi)\ \ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6 = 2\ {}^{\shortmid}h^5\ {}^{\shortmid}h^6\ \ {}^{\shortmid}_{3}\widehat{\Upsilon};$ ${}^{3}_{\shortmid}\beta\ {}^{3}_{\shortmid}w_{i_2} - {}^{3}_{\shortmid}\alpha_{i_2} = 0;$ ${}^{\shortmid}\partial^5({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}n_{k_2}) + {}^{3}_{\shortmid}\gamma\ {}^{\shortmid}\partial^5({}^{3}_{\shortmid}n_{k_2}) = 0;$ ${}^{3}_{\shortmid}\varpi = \ln\|\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6/\sqrt{\|\ {}^{\shortmid}h^5\ {}^{\shortmid}h^6\|}\|,$ ${}^{3}_{\shortmid}\alpha_{i_2} = ({}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)\ (\partial_{i_2}\ {}^{3}_{\shortmid}\varpi),$ ${}^{3}_{\shortmid}\beta = ({}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)\ ({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}\varpi),$ ${}^{3}_{\shortmid}\gamma = {}^{\shortmid}\partial^5\left(\ln\|\ {}^{\shortmid}h^6\|^{3/2}/\|\ {}^{\shortmid}h^5\|\right),$ ${}^{\shortmid}\partial^7({}^{4}_{\shortmid}\varpi)\ \ {}^{\shortmid}\partial^7\ \ {}^{\shortmid}h^8 = 2\ {}^{\shortmid}h^7\ {}^{\shortmid}h^8\ \ {}^{\shortmid}_{4}\widehat{\Upsilon};$ ${}^{4}_{\shortmid}\beta\ {}^{4}_{\shortmid}w_{i_3} - {}^{4}_{\shortmid}\alpha_{i_3} = 0;$ ${}^{\shortmid}\partial^7({}^{\shortmid}\partial^7\ {}^{4}_{\shortmid}n_{k_3}) + {}^{4}_{\shortmid}\gamma\ {}^{\shortmid}\partial^7({}^{4}_{\shortmid}n_{k_3}) = 0;$ ${}^{4}_{\shortmid}\varpi = \ln\|\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8/\sqrt{\|\ {}^{\shortmid}h^7\ {}^{\shortmid}h^8\|}\|,$ ${}^{4}_{\shortmid}\alpha_{i_3} = ({}^{\shortmid}\partial^7\ {}^{\shortmid}h^8)\ ({}^{\shortmid}\partial_{i_3}\ {}^{4}_{\shortmid}\varpi),$ ${}^{4}_{\shortmid}\beta = ({}^{\shortmid}\partial^7\ {}^{\shortmid}h^8)\ ({}^{\shortmid}\partial^7\ {}^{4}_{\shortmid}\varpi),$ ${}^{4}_{\shortmid}\gamma = {}^{\shortmid}\partial^7\left(\ln\|\ {}^{\shortmid}h^8\|^{3/2}/\|\ {}^{\shortmid}h^7\|\right),$
Gener. functs: $\underline{h}_4(x^{k_1},t)$, ${}^2\underline{\Psi}(x^{k_1},t) = e^{\ {}^2\underline{\varpi}},\ {}^2\underline{\Phi}(x^{k_1},t)$ integr. functs: $h_4^{[0]}(x^{k_1})$, ${}_1n_{k_1}(x^{i_1}),\ {}_2n_{k_1}(x^{i_1});$ Gener. functs: ${}^{\shortmid}h^5(x^{k_2},p_5)$, ${}^{3}_{\shortmid}\Psi(x^{k_2},p_5) = e^{\ {}^{3}_{\shortmid}\varpi},\ {}^{3}_{\shortmid}\Phi(x^{k_2},p_5)$ integr. functs: $h_6^{[0]}(x^{k_2})$, ${}^3_1 n_{k_2}(x^{i_2}),\ {}^3_2 n_{k_2}(x^{i_2});$ Gener. functs: ${}^{\shortmid}h^7({}^{\shortmid}x^{k_3},p_7)$, ${}^{4}_{\shortmid}\Psi(x^{k_2},p_7) = e^{\ {}^{4}_{\shortmid}\varpi},\ \ {}^{4}_{\shortmid}\Phi({}^{\shortmid}x^{k_3},p_7)$, integr. functs: $h_8^{[0]}({}^{\shortmid}x^{k_3})$, ${}^4_1 n_{k_3}({}^{\shortmid}x^{i_3}),\ {}^4_2 n_{k_3}({}^{\shortmid}x^{i_3});$ & nonlinear symmetries	$(({}^2\underline{\Psi})^2)^{\diamond} = -\int dt\ {}_2\underline{\widehat{\Upsilon}h_3}^{\diamond},$ $({}^2\underline{\Phi})^2 = -4\ {}_2\underline{\Lambda h}_3,$ $h_3 = h_3^{[0]} - ({}^2\underline{\Phi})^2/4\ {}_2\underline{\Lambda}, \underline{h}_3^{\diamond} \neq 0,\ {}_2\underline{\Lambda} \neq 0 = const;$ ${}^{\shortmid}\partial^5(({}^{3}_{\shortmid}\Psi)^2) = -\int dp_5\ {}^{\shortmid}_{3}\widehat{\Upsilon}\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6,$ $({}^{3}_{\shortmid}\Phi)^2 = -4\ {}^{\shortmid}_{3}\Lambda\ {}^{\shortmid}h^6,$ ${}^{\shortmid}h^6 = {}^{\shortmid}h^6_{[0]} - ({}^{3}_{\shortmid}\Phi)^2/4\ {}_3\Lambda,\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6 \neq 0,\ {}^{\shortmid}_{3}\Lambda \neq 0 = const;$ ${}^{\shortmid}\partial^7(({}^{4}_{\shortmid}\Psi)^2) = -\int dp_7\ {}^{\shortmid}_{4}\widehat{\Upsilon}\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8,$ $({}^{4}_{\shortmid}\Phi)^2 = -4\ {}^{\shortmid}_{4}\Lambda\ {}^{\shortmid}h^8,$ ${}^{\shortmid}h^8 = {}^{\shortmid}h^8_{[0]} - ({}^{4}_{\shortmid}\Phi)^2/4\ {}^{\shortmid}_{4}\Lambda,\ {}^{\shortmid}\partial^7\ {}^{\shortmid}h^8 \neq 0,\ {}^{\shortmid}_{4}\Lambda \neq 0 = const;$
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon};$ $\underline{h}_4 = -(\underline{\Psi}^{\diamond})^2/4\ {}_2\underline{\widehat{\Upsilon}}^2\underline{h}_3;$ $\underline{h}_3 = \underline{h}_3^{[0]} - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\underline{\widehat{\Upsilon}} = \underline{h}_3^{[0]} - \underline{\Phi}^2/4\ {}_2\underline{\Lambda};$ $\underline{w}_{i_1} = \partial_{i_1}\ \underline{\Psi}/\ \partial\underline{\Psi}^{\diamond} = \partial_{i_1}\ \underline{\Psi}^2/\ \partial_t\underline{\Psi}^2\|;$ $\underline{n}_{k_1} = {}_1n_{k_1} + {}_2n_{k_1}\int dt(\underline{\Psi}^{\diamond})^2/\ {}_2\underline{\widehat{\Upsilon}}^2\|\underline{h}_3^{[0]} - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\underline{\widehat{\Upsilon}}^2\|^{5/2};$ ${}^{\shortmid}h^5 = -({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}\Psi)^2/4\ {}^{\shortmid}_{3}\widehat{\Upsilon}^2\ {}^{\shortmid}h^6;$ ${}^{\shortmid}h^6 = {}^{\shortmid}h^6_{[0]} - \int dp_5\ {}^{\shortmid}\partial^5(({}^{3}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{3}\widehat{\Upsilon} = {}^{\shortmid}h^6_{[0]} - ({}^{3}_{\shortmid}\Phi)^2/4\ {}^{\shortmid}_{3}\Lambda;$ $w_{i_2} = \partial_{i_2}({}^{3}_{\shortmid}\Psi)/\ {}^{\shortmid}\partial^5({}^{3}_{\shortmid}\Psi) = \partial_{i_2}({}^{3}_{\shortmid}\Psi)^2/\ {}^{\shortmid}\partial^5({}^{3}_{\shortmid}\Psi)^2\|;$ $n_{k_2} = {}_1n_{k_2} + {}_2n_{k_2}\int dp_5({}^{\shortmid}\partial^5\ {}^{3}_{\shortmid}\Psi)^2/\ {}^{\shortmid}_{3}\widehat{\Upsilon}^2\|\ {}^{\shortmid}h^6_{[0]} -$ $\int dp_5\ {}^{\shortmid}\partial^5(({}^{3}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{3}\widehat{\Upsilon}^2\|^{5/2};$ ${}^{\shortmid}h^7 = -({}^{\shortmid}\partial^7\ {}^{4}_{\shortmid}\Psi)^2/4\ {}^{4}_{\shortmid}\widehat{\Upsilon}^2\ {}^{\shortmid}h^8;$ ${}^{\shortmid}h^8 = {}^{\shortmid}h^8_{[0]} - \int dp_7\ {}^{\shortmid}\partial^7(({}^{4}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{4}\widehat{\Upsilon} = h_8^{[0]} - ({}^{4}_{\shortmid}\Phi)^2/4\ \ {}^{\shortmid}_{4}\Lambda;$ ${}^{\shortmid}w_{i_3} = {}^{\shortmid}\partial_{i_3}({}^{4}_{\shortmid}\Psi)/\ \ {}^{\shortmid}\partial^7({}^{4}_{\shortmid}\Psi) = {}^{\shortmid}\partial_{i_3}({}^{4}_{\shortmid}\Psi)^2/\ \ {}^{\shortmid}\partial^7({}^{4}_{\shortmid}\Psi)^2\|;$ ${}^{\shortmid}n_{k_3} = {}^{\shortmid}_{1}n_{k_3} + {}^{\shortmid}_{2}n_{k_3}\int dp_7({}^{4}_{\shortmid}\Psi)^2/\ {}^{\shortmid}_{4}\widehat{\Upsilon}^2\|h_8^{[0]} -$ $\int dp_7\ {}^{\shortmid}\partial^7(({}^{4}_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_{4}\widehat{\Upsilon}^2\|^{5/2}.$

Table 16 Off-diagonal cosmological and pase space configurations with variable energy exact solutions of ${}^{\shortmid}\widehat{\mathbf{R}}_{\mu_s\nu_s} = {}^{\shortmid}\Upsilon_{\mu_s\nu_s}$ (A.13) on $T^*_s V$ transformed into a momentum version of nonlinear PDEs (47)–(50)

d-metric ansatz with Killing symmetry $\partial_3 = \partial_t,\ {}^{\shortmid}\partial^7$	$ds^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + \underline{g}_{a_2}(x^{k_1},t)(dy^{a_2} + \underline{N}^{a_2}_{i_1}(x^{k_1},t)dx^{i_1})^2$ $+ {}^{\shortmid}g^{a_3}(x^{k_2},p_5)(dp_{a_3} + {}^{\shortmid}N_{i_2 a_3}(x^{k_2},p_5)dx^{i_2})^2$ $+ {}^{\shortmid}g^{a_4}({}^{\shortmid}x^{k_3},p_7)(dp_{a_4} + {}^{\shortmid}N_{i_3 a_4}({}^{\shortmid}x^{k_3},p_7)d\,{}^{\shortmid}x^{i_3})^2$, for $g_{i_1} = e^{\psi(x^{k_1})}$, $\underline{g}_{a_2} = \underline{h}_{a_2}(x^{k_1},t), \underline{N}^3_{i_1} = {}^2\underline{n}_{i_1} = \underline{n}_{i_1}(x^{k_1},t), \underline{N}^4_{i_1} = {}^2\underline{w}_{i_1} = \underline{w}_{i_1}(x^{k_1},t),$ ${}^{\shortmid}g^{a_3} = {}^{\shortmid}h^{a_3}(x^{k_2},p_5),\ {}^{\shortmid}N_{i_2 5} = {}^3_{\shortmid}w_{i_2} = {}^{\shortmid}w_{i_2}(x^{k_2},p_5),$ ${}^{\shortmid}N_{i_2 6} = {}^3_{\shortmid}n_{i_2} = {}^{\shortmid}n_{i_2}(x^{k_2},p_5),$ ${}^{\shortmid}\underline{g}^{a_4} = {}^{\shortmid}\underline{h}^{a_4}({}^{\shortmid}x^{k_3},E),\ {}^{\shortmid}\underline{N}_{i_3 7} = {}^4_{\shortmid}\underline{n}_{i_3} = {}^{\shortmid}\underline{n}_{i_3}(x^{k_3},E),$ ${}^{\shortmid}\underline{N}_{i_3 8} = {}^4_{\shortmid}\underline{w}_{i_3} = {}^{\shortmid}\underline{w}_{i_3}(x^{k_3},E),$	
Effective matter sources	${}^{\shortmid}\mathbf{\Upsilon}^{\mu_s}_{\ \nu_s} = [\ {}_1\widehat{\Upsilon}(x^{k_1})\delta^{i_1}_{j_1},\ {}_2\widehat{\Upsilon}(x^{k_1},y^3)\delta^{a_2}_{b_2},\ {}^{\shortmid}_3\widehat{\Upsilon}(x^{k_2},p_5)\delta^{a_3}_{b_3},\ {}^{\shortmid}_4\underline{\widehat{\Upsilon}}(x^{k_3},E)\delta^{a_4}_{b_4},],$	
Nonlinear PDEs (47)-(50)	$\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon};$ ${}^2\underline{\varpi}^{\diamond}\ \underline{h}^{\diamond}_3 = 2\underline{h}_3\underline{h}_4\ {}_2\widehat{\Upsilon};$ ${}^2\underline{n}^{\diamond\diamond}_{k_1} + {}^2\underline{\gamma}\ {}^2\underline{n}^{\diamond}_{k_1} = 0;$ ${}^2\underline{\beta}\ {}^2\underline{w}_{i_1} - {}^2\underline{\alpha}_{i_1} = 0;$	${}^2\underline{\varpi} = \ln\lvert\partial_4\underline{h}_4/\sqrt{\lvert\underline{h}_3\underline{h}_4\rvert}\rvert,$ ${}^2\underline{\alpha}_{i_1} = (\partial_4\underline{h}_3)\ (\partial_{i_1}\ {}^2\underline{\varpi}),$ ${}^2\underline{\beta} = (\partial_4\underline{h}_4)\ (\partial_3\ {}^2\underline{\varpi}),$ ${}^2\underline{\gamma} = \partial_4\left(\ln\lvert\underline{h}_3\rvert^{3/2}/\lvert\underline{h}_4\rvert\right),$ $\partial_1 q = q^{\bullet}, \partial_2 q = q',$ $\partial_4 q = \partial_t q = q^{\diamond}$
	${}^{\shortmid}\partial^5({}^3_{\shortmid}\varpi)\ \ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6 = 2\ {}^{\shortmid}h^5\ {}^{\shortmid}h^6\ \ {}^{\shortmid}_3\widehat{\Upsilon};$ ${}^3_{\shortmid}\beta\ {}^3_{\shortmid}w_{i_2} - {}^3_{\shortmid}\alpha_{i_2} = 0;$ ${}^{\shortmid}\partial^5({}^{\shortmid}\partial^5\ {}^3_{\shortmid}n_{k_2}) + {}^3_{\shortmid}\gamma\ {}^{\shortmid}\partial^5({}^3_{\shortmid}n_{k_2}) = 0;$	${}^3_{\shortmid}\varpi = \ln\lvert{}^{\shortmid}\partial^5\ {}^{\shortmid}h^6/\sqrt{\lvert{}^{\shortmid}h^5\ {}^{\shortmid}h^6\rvert}\rvert,$ ${}^3_{\shortmid}\alpha_{i_2} = ({}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)\ (\partial_{i_2}\ {}^3_{\shortmid}\varpi),$ ${}^3_{\shortmid}\beta = ({}^{\shortmid}\partial^5\ {}^{\shortmid}h^6)\ ({}^{\shortmid}\partial^5\ {}^3_{\shortmid}\varpi),$ ${}^3_{\shortmid}\gamma = {}^{\shortmid}\partial^5\left(\ln\lvert{}^{\shortmid}h^6\rvert^{3/2}/\lvert{}^{\shortmid}h^5\rvert\right),$
	${}^{\shortmid}\underline{\partial}^8({}^4_{\shortmid}\underline{\varpi})\ \ {}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7 = 2\ {}^{\shortmid}\underline{h}^7\ {}^{\shortmid}\underline{h}^8\ {}^{\shortmid}_4\underline{\widehat{\Upsilon}};$ ${}^{\shortmid}\underline{\partial}^8({}^{\shortmid}\underline{\partial}^8\ {}^4_{\shortmid}\underline{n}_{k_3}) + {}^4_{\shortmid}\underline{\gamma}\ {}^{\shortmid}\underline{\partial}^8({}^4_{\shortmid}\underline{n}_{k_3}) = 0;$ ${}^4_{\shortmid}\underline{\beta}\ {}^4_{\shortmid}\underline{w}_{i_3} - {}^4_{\shortmid}\underline{\alpha}_{i_3} = 0;$	${}^4_{\shortmid}\underline{\varpi} = \ln\lvert{}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7/\sqrt{\lvert{}^{\shortmid}\underline{h}^7\ {}^{\shortmid}\underline{h}^8\rvert}\rvert,$ ${}^4_{\shortmid}\underline{\alpha}_{i_3} = ({}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7)\ ({}^{\shortmid}\partial_{i_3}\ {}^4_{\shortmid}\underline{\varpi}),$ ${}^4_{\shortmid}\underline{\beta} = ({}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7)\ ({}^{\shortmid}\underline{\partial}^8\ {}^4_{\shortmid}\underline{\varpi}),$ ${}^4_{\shortmid}\underline{\gamma} = {}^{\shortmid}\underline{\partial}^8\left(\ln\lvert{}^{\shortmid}\underline{h}^7\rvert^{3/2}/\lvert{}^{\shortmid}\underline{h}^8\rvert\right),$
Gener. functs: $\underline{h}_4(x^{k_1},t)$, ${}^2\underline{\Psi}(x^{k_1},t) = e^{\ {}^2\underline{\varpi}},\ {}^2\underline{\Phi}(x^{k_1},t),$ integr. functs: $\underline{h}^{[0]}_3(x^{k_1}),$ ${}_1\underline{n}_{k_1}(x^{i_1}),\ {}_2\underline{n}_{k_1}(x^{i_1});$ Gener. functs: ${}^{\shortmid}h^5(x^{k_2},p_5),$ ${}^3_{\shortmid}\Psi(x^{k_2},p_5) = e^{\ {}^3_{\shortmid}\varpi},\ {}^3_{\shortmid}\Phi(x^{k_2},p_5)$ integr. functs: $h^{[0]}_6(x^{k_2}),$ ${}^3_1 n_{k_2}(x^{i_2}),\ {}^3_2 n_{k_2}(x^{i_2});$ Gener. functs: ${}^{\shortmid}h^7({}^{\shortmid}x^{k_3},p_7),$ ${}^4_{\shortmid}\underline{\Psi}(x^{k_2},E) = e^{\ {}^4_{\shortmid}\underline{\varpi}},\ {}^4_{\shortmid}\underline{\Phi}({}^{\shortmid}x^{k_3},E)$ integr. functs: $\underline{h}^{[0]}_7({}^{\shortmid}x^{k_3}),$ ${}^4_1\underline{n}_{k_3}({}^{\shortmid}x^{i_3}),\ {}^4_2\underline{n}_{k_3}({}^{\shortmid}x^{i_3});$ & nonlinear symmetries	$(({}^2\underline{\Psi})^2)^{\diamond} = -\int dt\ {}_2\underline{\widehat{\Upsilon}}\underline{h}_3{}^{\diamond},$ $({}^2\underline{\Phi})^2 = -4\ {}_2\underline{\Lambda}\underline{h}_3,$ $h_3 = h^{[0]}_3 - ({}^2\underline{\Phi})^2/4\ {}_2\underline{\Lambda}, \underline{h}^{\diamond}_3 \neq 0,\ {}_2\underline{\Lambda} \neq 0 = const;$ ${}^{\shortmid}\partial^5(({}^3_{\shortmid}\Psi)^2) = -\int dp_5\ {}^{\shortmid}_3\widehat{\Upsilon}\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6,$ $({}^3_{\shortmid}\Phi)^2 = -4\ {}^{\shortmid}_3\Lambda\ {}^{\shortmid}h^6,$ ${}^{\shortmid}h^6 = {}^{\shortmid}h^6_{[0]} - ({}^3_{\shortmid}\Phi)^2/4\ {}_3\Lambda,\ {}^{\shortmid}\partial^5\ {}^{\shortmid}h^6 \neq 0,\ {}^{\shortmid}_3\Lambda \neq 0 = const;$ ${}^{\shortmid}\underline{\partial}^8(({}^4_{\shortmid}\underline{\Psi})^2) = -\int dE\ {}^{\shortmid}_4\underline{\widehat{\Upsilon}}\ {}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7,$ $({}^4_{\shortmid}\underline{\Phi})^2 = -4\ {}^{\shortmid}_4\underline{\Lambda}\ {}^{\shortmid}\underline{h}^7,$ ${}^{\shortmid}\underline{h}^7 = {}^{\shortmid}\underline{h}^7_{[0]} - ({}^4_{\shortmid}\underline{\Phi})^2/4\ {}^{\shortmid}_4\underline{\Lambda},\ {}^{\shortmid}\underline{\partial}^8\ {}^{\shortmid}\underline{h}^7 \neq 0,\ {}^{\shortmid}_4\underline{\Lambda} \neq 0 = const;$	
Off-diag. solutions, d–metric N-connec.	$g_i = e^{\ \psi(x^k)}$ as a solution of 2-d Poisson eqs. $\psi^{\bullet\bullet} + \psi'' = 2\ \ {}_1\widehat{\Upsilon};$ $\underline{h}_4 = -(\underline{\Psi}^{\diamond})^2/4\ {}_2\underline{\widehat{\Upsilon}}^2\underline{h}_3;$ $\underline{h}_3 = \underline{h}^{[0]}_3 - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\underline{\widehat{\Upsilon}} = \underline{h}^{[0]}_3 - \underline{\Phi}^2/4\ {}_2\underline{\Lambda};$ $\underline{w}_{i_1} = \partial_{i_1}\ \underline{\Psi}/\ \partial\underline{\Psi}^{\diamond} = \partial_{i_1}\ \underline{\Psi}^2/\ \partial_t\underline{\Psi}^2\rvert;$ $\underline{n}_{k_1} = {}_1n_{k_1} + {}_2n_{k_1}\int dt(\underline{\Psi}^{\diamond})^2/\ {}_2\underline{\widehat{\Upsilon}}^2\lvert\underline{h}^{[0]}_3 - \int dt(\underline{\Psi}^2)^{\diamond}/4\ {}_2\underline{\widehat{\Upsilon}}^2\rvert^{5/2};$ ${}^{\shortmid}h^5 = -({}^{\shortmid}\partial^5\ {}^3_{\shortmid}\Psi)^2/4\ {}^{\shortmid}_3\widehat{\Upsilon}^2\ {}^{\shortmid}h^6;$ ${}^{\shortmid}h^6 = {}^{\shortmid}h^6_{[0]} - \int dp_5\ {}^{\shortmid}\partial^5(({}^3_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_3\widehat{\Upsilon} = {}^{\shortmid}h^6_{[0]} - ({}^3_{\shortmid}\Phi)^2/4\ {}^{\shortmid}_3\Lambda;$ $w_{i_2} = \partial_{i_2}({}^3_{\shortmid}\Psi)/\ {}^{\shortmid}\partial^5({}^3_{\shortmid}\Psi) = \partial_{i_2}({}^3_{\shortmid}\Psi)^2/\ {}^{\shortmid}\partial^5({}^3_{\shortmid}\Psi)^2\rvert;$ $n_{k_2} = {}_1n_{k_2} + {}_2n_{k_2}\int dp_5({}^{\shortmid}\partial^5\ {}^3_{\shortmid}\Psi)^2/\ {}^{\shortmid}_3\widehat{\Upsilon}^2\lvert\ {}^{\shortmid}h^6_{[0]} -$ $\int dp_5\ {}^{\shortmid}\partial^5(({}^3_{\shortmid}\Psi)^2)/4\ {}^{\shortmid}_3\widehat{\Upsilon}^2\rvert^{5/2};$ ${}^{\shortmid}\underline{h}^8 = -({}^{\shortmid}\underline{\partial}^8\ {}^4_{\shortmid}\underline{\Psi})^2/4\ {}^4_{\shortmid}\underline{\widehat{\Upsilon}}^2\ {}^{\shortmid}\underline{h}^7;$ ${}^{\shortmid}\underline{h}^7 = {}^{\shortmid}\underline{h}^7_{[0]} - \int dE\ {}^{\shortmid}\underline{\partial}^8(({}^4_{\shortmid}\underline{\Psi})^2)/4\ {}^{\shortmid}_4\underline{\widehat{\Upsilon}} = \underline{h}^7_{[0]} - ({}^4_{\shortmid}\underline{\Phi})^2/4\ \ {}^{\shortmid}_4\underline{\Lambda};$ ${}^{\shortmid}\underline{n}_{k_3} = {}^{\shortmid}_1\underline{n}_{k_3} + {}^{\shortmid}_2\underline{n}_{k_3}\int dE({}^4_{\shortmid}\underline{\Psi})^2/\ {}^{\shortmid}_4\underline{\widehat{\Upsilon}}^2\lvert\underline{h}^7_{[0]} -$ $\int dE\ {}^{\shortmid}\underline{\partial}^8(({}^4_{\shortmid}\underline{\Psi})^2)/4\ {}^{\shortmid}_4\underline{\widehat{\Upsilon}}^2\rvert^{5/2};$ ${}^{\shortmid}\underline{w}_{i_3} = {}^{\shortmid}\partial_{i_3}({}^4_{\shortmid}\underline{\Psi})/\ {}^{\shortmid}\partial^8({}^4_{\shortmid}\underline{\Psi}) = {}^{\shortmid}\partial_{i_3}({}^4_{\shortmid}\underline{\Psi})^2/\ {}^{\shortmid}\partial^8({}^4_{\shortmid}\underline{\Psi})^2\rvert.$	

A.4.5 Locally anisotropic cosmological solutions with variable energy parameter

The Table 16 is a momentum phase version of the Table 11. In this subsection, it is summarized the AFCDM for constructing locally anisotropic cosmological rainbow solutions.

As an example of such s-metrics we provide below a cosmological rainbow metric with the $s = 1, 2$ part being equivalent to

$$
\begin{aligned}
d\widehat{s}^2_{[8d]} &= \widehat{g}_{\alpha_s\beta_s}(x^k, t, p_5, E; \underline{h}_3, {}^{\shortmid}h^6, {}^{\shortmid}\underline{h}^7; \\
&\quad {}_1^{\shortmid}\widehat{\Upsilon}, {}_2^{\shortmid}\underline{\widehat{\Upsilon}}, {}_3^{\shortmid}\widehat{\Upsilon}, {}_4^{\shortmid}\underline{\widehat{\Upsilon}}; \\
&\quad {}_1^{\shortmid}\Lambda, {}_2^{\shortmid}\underline{\Lambda}, {}_3^{\shortmid}\Lambda, {}_4^{\shortmid}\underline{\Lambda}) d\, {}^{\shortmid}u^{\alpha_s} d\, {}^{\shortmid}u^{\beta_s} \\
&= e^{\psi(x^k, {}_s\widehat{\Upsilon})}[(dx^1)^2 + (dx^2)^2] + \underline{h}_3[dy^3 \\
&\quad + ({}_1n_{k_1} + 4\, {}_2n_{k_1} \int dt \frac{(\underline{h}_3{}^{\diamond})^2}{|\int dt\, {}_2\underline{\widehat{\Upsilon}}\underline{h}_3{}^{\diamond}|(\underline{h}_3)^{5/2}}) dx^{k_1}] \\
&\quad + \frac{(\underline{h}_3{}^{\diamond})^2}{|\int dt\, {}_2\underline{\widehat{\Upsilon}}\underline{h}_3{}^{\diamond}|\, \overline{h}_3}[dt + \frac{\partial_i(\int dt\, {}_2\underline{\Upsilon}\, \underline{h}_3{}^{\diamond}])}{{}_2\underline{\widehat{\Upsilon}}\, \underline{h}_3{}^{\diamond}} dx^i] \\
&\quad + \frac{({}^{\shortmid}\partial^5\, {}^{\shortmid}h^6)^2}{|\int dp_5\, {}^{\shortmid}\partial^5[{}_3^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}h^6]|\, {}^{\shortmid}h^6}\{dp_5 \\
&\quad + \frac{\partial_{i_2}[\int dp_5({}_3^{\shortmid}\widehat{\Upsilon})\, {}^{\shortmid}\partial^5\, {}^{\shortmid}h^6]}{{}_3^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}\partial^5\, {}^{\shortmid}h^6} dx^{i_2}\}^2 \\
&\quad + {}^{\shortmid}h^6\{dp_5 + [{}_1n_{k_2} + {}_2n_{k_2} \int dp_5 \\
&\quad \times \frac{({}^{\shortmid}\partial^5\, {}^{\shortmid}h^6)^2}{|\int dp_5\, {}^{\shortmid}\partial^5[{}_3^{\shortmid}\widehat{\Upsilon}\, {}^{\shortmid}h^6]|({}^{\shortmid}h^6)^{5/2}}] dx^{k_2}\} \\
&\quad + {}^{\shortmid}\underline{h}^7\{dp_7 + [{}_1^{\shortmid}\underline{n}_{k_3} + {}_2^{\shortmid}\underline{n}_{k_3} \int dp_7 \\
&\quad \times \frac{({}^{\shortmid}\underline{\partial}^8\, {}^{\shortmid}\underline{h}^7)^2}{|\int dE\, {}^{\shortmid}\underline{\partial}^8[{}_4^{\shortmid}\underline{\widehat{\Upsilon}}\, {}^{\shortmid}\underline{h}^7]|({}^{\shortmid}\underline{h}^7)^{5/2}}] d\, {}^{\shortmid}x^{k_3}\} \\
&\quad + \frac{({}^{\shortmid}\underline{\partial}^8\, {}^{\shortmid}\underline{h}^7)^2}{|\int dE\, {}^{\shortmid}\underline{\partial}^8[{}_4\underline{\widehat{\Upsilon}}\, {}^{\shortmid}\underline{h}^7]|\, {}^{\shortmid}\underline{h}^7} \\
&\quad \times \{dE + \frac{\partial_{i_3}[\int dE({}_4\underline{\widehat{\Upsilon}})\, {}^{\shortmid}\underline{\partial}^8\, {}^{\shortmid}\underline{h}^7]}{{}_4^{\shortmid}\underline{\widehat{\Upsilon}}\, {}^{\shortmid}\underline{\partial}^8\, {}^{\shortmid}\underline{h}^7} d\, {}^{\shortmid}x^{i_3}\}^2. \quad \text{(A.20)}
\end{aligned}
$$

The locally anisotropic cosmological s-metric (A.20) is an example of phase space rainbow s-metric (A.16) constructed on $T^*\mathbf{V}$.

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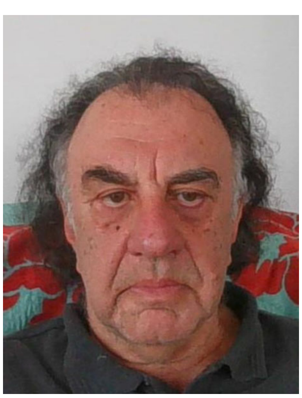

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