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Using wavelet bases to separate scales in quantum field theory

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USING WAVELET BASES TO SEPARATE SCALES IN QUANTUM FIELD THEORY

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ABSTRACT

This thesis investigates the use of Daubechies wavelets to separate scales in local quantum field theory. Field theories have an infinite number of degrees of freedom on all distance scales. Quantum field theories are believed to describe the physics of subatomic particles. These theories have no known mathematically convergent approximation methods. Daubechies wavelet bases can be used to separate degrees of freedom on different distance scales. Volume and resolution truncations lead to mathematically well-defined truncated theories that can be treated using established methods. This work demonstrates that flow equation methods can be used to block diagonalize truncated field theoretic Hamiltonians by scale. This eliminates the fine scale degrees of freedom. This may lead to approximation methods and provide an understanding of how to formulate well-defined fine resolution limits.

PUBLIC ABSTRACT

Wavelets are mathematical functions used in digital photography to create JPEG files from raw images. They significantly reduce the amount computer memory needed to store a photograph without much loss in resolution. Wavelets are fractal valued functions, making them different from most other mathematical functions. This means that they are like snowflakes, where the same structure is repeated on arbitrarily small distance scales. The fractal property makes wavelets ideal candidates for modeling a large class of problems that simultaneously involve structures on all distance scales. Photographs have this property. The class of problems with many scales is one of the most difficult to treat in science. Quantum field theory is one of these problems. Quantum field theories are believed to govern three of the four fundamental forces of nature but have defied mathematical solution for almost 100 years. The goal of this thesis is to use wavelets to decompose quantum theories into degrees of freedom on all scales and then to decouple the short and long distance degrees of freedom. The application of a technique called the flow equation is used to do this. The flow equation is designed to continuously decouple degrees of freedom in quantum field theory on different distance scales. This method is tested on a free field theory, since it is one of the few solvable field theories still involving degrees of freedom on all distance scales. This work is limited to decoupling two distance scales. In this case the flow equation successfully decoupled the two scale degrees of freedom both by resolution and energy scale. This is an important first step in understanding

how to decouple all distance scales.

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CHAPTER 1
INTRODUCTION

The purpose of this thesis is to use wavelet methods to decouple degrees of freedom on different distance scales in local quantum field theory. Several groups have applied wavelet methods to treat problems in quantum field theory.

There are many types of wavelets. Daubechies wavelets made famous by and named after Ingrid Daubechies for her work in the field were used. These functions have scaling properties and compact support that makes them useful to model systems, like quantum fields, that have structure on many scales [19][20].

Wavelets have important applications in signal processing and data compression. Digital photography uses wavelets in JPEG data compression. The wavelets ability to compress 2×2 matrices of digital photo data led to successful methods for reducing large linear systems that appear in scattering integral equations to sparse systems, allowing for efficient solutions with minimal loss of accuracy [17][30][31][32]. The ability to treat problems with many scales also makes them natural for treating problems in turbulence [22].

One of the earliest applications of wavelets to field theory was by C. Best and A. Schafer [10][11] who used a Daubechies wavelet expansion of fields to study the renormalization group flow in statistical field theories and in the Landau-Ginzburg model.

P. Federbush [23] used smooth wavelets to regularize fields in Yang-Mills theories. Unlike Daubechies wavelets, these wavelets are smooth, have nice localiztion

properties, but do not have compact support. The program of using these wavelets to treat problems in constructive field theory is discussed extensively in the book by G. Battle [9].

Evenbly and White [21] established a connection between discrete wavelet transforms and entanglement renormalization in the context of free particle systems. They used Daubechies wavelets to approximate the ground state of the critical Ising model.

Halliday and Suranyi [28] used Haar wavelet expansions as a new approach to simulations of ϕ^4 field theories. These are the simplest Daubechies wavelets where the resulting expansions replace fields at each point by averaging over the surrounding blocks. They compared the use of wavelets to the standard Metropolis algorithm in lattice field theory applied to a scalar $2D$ ϕ^4 field theory.

Brennen, Rohde, Sanders and Singh [16] used Daubechies wavelets to resolve physics into different length or energy scales to understand field theories and found that scalar bosonic quantum field theories can be simulated efficiently on a quantum computer using a wavelet basis.

M. Altaisky (and collaborators) has a sustained record of applications of various problems in quantum field theory. His approach is based on the continuous wavelet transform. He has advocated using wavelet methods to regularize local fields by introducing natural scale cutoffs in quantum field theory. He also proposed to use these methods in gauge theories [1][2][3][4][5][6].

This thesis is motivated by the works of F. Bulut and W. N. Polyzou [18]

who advocated using Daubechies wavelets to construct natural volume and resolution truncations of field theories to study the restoration of Poincaré symmetry and separation of scales. The work in this thesis is also discussed in [33].

What is new in this thesis is the application of flow equation methods to block diagonalize truncated field theory Hamiltonians by scale. Direct block diagonalization methods were used by Gloeckle and Muller to eliminate pion degrees of freedom in models of interacting nucleons [27].

Flow equations were introduced by Wegner [38]. They generate a set of unitary transformations that depend continuously on a parameter that evolves a matrix to a given form; typically diagonal or block diagonal. These methods were used by Wilson, Glazek, Perry and collaborators [25][26] to eliminate coupling of low and high-energy degrees of freedom in quantum chromodynamics. Furnstahl, Perry, Schwenk, Wendt and others [7][13][14] used them to transform nucleon-nucleon interactions to equivalent softer interactions for use in nuclear structure calculations. Bartlett [8] gave a discussion on the use of flow equations to block diagonalize the Hamiltonian.

The organization of this thesis follows. In chapter 2 quantum field theory, which is believed to be the underlying theoretical framework to model physics on sub-atomic distance scales, is discussed. In chapter 3 Daubechies' wavelets are described, including why their properties are useful for studying quantum field theory. In chapter 4 the wavelet truncations of quantum field theories are introduced. In chapter 5 flow equation methods and how they can be applied to the problem of separating distance scales in wavelet truncations of quantum field theories is discussed. In chapter 6

the project and the results are described in detail, with chapter 7 providing the conclusions and future outlook. There are two appendices at the end that discuss some of the calculations in more detail.

CHAPTER 2
CANONICAL QUANTUM FIELD THEORY

2.1 Why Field Theory

Of the four fundamental forces of nature it is thought that three, the strong, weak and electromagnetic forces, are governed by quantum field theory. While these theories are widely accepted it is not currently known how to formulate them mathematically.

To understand field theory start by defining what it is meant by a field. A typical field will have a value at each point in space and time. Temperature is one example with a scalar value, the temperature, at each point and time. Classical electric and magnetic fields are examples with a vector at each point and time.

The time evolution of the fields associated with the fundamental forces of nature are determined by equations of motion derived from a Lagrangian by the principle of stationary action. The fields associated with the fundamental forces of nature are quantum fields. These fields are formally constructed by applying a quantization procedure to the corresponding classical field. This is done either by a path integral or by transforming to a Hamiltonian system and replacing the classical fields and their generalized momenta with quantum operators, and by replacing the classical Poisson brackets by i times a commutator.

Quantum fields should satisfy some physical requirements. These include a spectral condition, relativistic transformation properties, cluster properties and lo-

cality. The mathematical difficulties come from simultaneously satisfying these requirements.

The spectral condition says that the Hamiltonian is a positive self-adjoint operator, so the energy of the system is bounded from below, and thus the theory is stable. Relativistic invariance requires that the probabilities are independent of the choice of inertial coordinate system (coordinate systems where free particles move with constant velocity). This implies that an experiment performed in one inertial coordinate system will have an equivalent result if it is performed in another inertial coordinate system. Cluster properties ensure that the principle relativistic invariance holds for isolated subsystems. And locality says that if all points in two regions are space-like separated, then experiments in each region (no matter the size of the region) will not influence each other.

2.2 General Field Theory

Below is outlined the key steps in formulating a quantum field theory by canonical quantization of classical Lagrangian field theory. The classical field momentum is the functional derivative of the Lagrangian with respect to the time derivative of the field

$$\Pi(\mathbf{x}, t) = \frac{\delta L}{\delta \dot{\Phi}(\mathbf{x}, t)} \tag{2.1}$$

The classical Lagrangian is transformed to a classical Hamiltonian with a Legendre transformation:

$$H = \int \dot{\Phi}(\mathbf{x}, t)\Pi(\mathbf{x}, t)d\mathbf{x} - L \tag{2.2}$$

The classical system is made into a quantum system

$$H_{Classical} \rightarrow H_{Quantum} \tag{2.3}$$

by replacing the classical fields $\Phi(\mathbf{x}, t)$ and $\Pi(\mathbf{x}, t)$ with quantum operator valued distributions

$$\Phi(\mathbf{x}, t) \rightarrow \hat{\Phi}(\mathbf{x}, t) \tag{2.4}$$

$$\Pi(\mathbf{x}, t) \rightarrow \hat{\Pi}(\mathbf{x}, t) \tag{2.5}$$

The Poisson brackets of the classical fields are replaced with equal-time commutators of the field operators. Thus,

$$\{\Phi(\mathbf{x}, 0), \Pi(\mathbf{y}, 0)\} = \delta(\mathbf{x} - \mathbf{y}) \tag{2.6}$$

becomes

$$[\hat{\Phi}(\mathbf{x}, 0), \hat{\Pi}(\mathbf{y}, 0)] = i\delta(\mathbf{x} - \mathbf{y}) \tag{2.7}$$

The classical Hamilton's equations of motion

$$\dot{\Phi}(\mathbf{x}, t) = \{H, \Phi(\mathbf{x}, t)\} \tag{2.8}$$

$$\dot{\Pi}(\mathbf{x}, t) = \{H, \Pi(\mathbf{x}, t)\} \tag{2.9}$$

become the Heisenberg equations of motion

$$\dot{\hat{\Phi}}(\mathbf{x}, t) = i[\hat{H}, \hat{\Phi}(\mathbf{x}, t)] \tag{2.10}$$

$$\dot{\hat{\Pi}}(\mathbf{x}, t) = i[\hat{H}, \hat{\Pi}(\mathbf{x}, t)] \tag{2.11}$$

where $\hat{H}$ is the quantum Hamiltonian, constructed from the classical Hamiltonian H by replacing the classical fields by quantum fields. The equal time commutation relations are initial conditions for the Heisenberg equations of motion.

This procedure leads to several mathematical difficulties. First the locality requirement implies that the quantum Hamiltonian involves products of field operators and their derivatives at the same point. Since the field operators are operator valued distributions, the Hamiltonian obtained by quantizing the classical Hamiltonian is not well defined.

Second, the quantum Hamiltonian and field operators act on a Hilbert space. The locality condition requires that the field theory is a system of infinite number of degrees of freedom. Unlike finite systems, it is known that there are an infinite number of inequivalent Hilbert space representations of the canonical equal time commutation relations that define the initial conditions of the field theory. The ground state of the Hamiltonian is a Poincaré invariant linear functional of the algebra of operators generated by canonical fields. A Hilbert space representation is generated by applying functions of fields integrated against test functions to the ground state using a construction due to Gelfand, Naimark and Segal [24][37]. Defining the Hamiltonian, determining the ground state and constructing the Hilbert space is an open problem.

One possible approach to defining a quantum field theory is to truncate the theory to a finite number of degrees of freedom. This can be done by expanding the field in an orthonormal basis and eliminating all but a finite number of basis functions. This makes the resulting truncated Hamiltonian into a well-defined operator.

The initial canonical fields can be replaced by free canonical fields acting on the Hilbert space generated from the free field ground state functional because of

Von Neumann's theorem on the unitary equivalence of finite dimensional representations of the canonical commutation relations. This allows the truncated theory to become a mathematically well-defined problem on the free field Hilbert space. It is important to note that basis truncations are not approximations in quantum field theory. Adding more basis functions will just recover the same ill-defined theory. A solution necessarily involves devising a new limiting procedure as the number of basis functions become infinite and verifying that the resulting theory is well-defined and has the expected properties of a quantum field theory. This is a delicate limit due to the distributional nature of the limiting fields. This is currently an unsolved problem and beyond the scope of this thesis.

The purpose of this thesis is to look at methods that have natural volume and resolution truncations and study and look at what happens as the number of degrees of freedom is increased. Experiments are performed in finite volumes, take a finite amount of time and involve finite energies. This means that for a locally compact Hamiltonian the number of relevant degrees of freedom are finite [35]. The finite energies limit the resolution of a measurement. The adjustable parameters of the truncated theory can be adjusted to be consistent with experiment. New adjustments to the parameters of the theory will be needed as more degrees of freedom are added in order to remain consistent with experiment. Even if there does not exist a limiting theory, it is expected that by adding additional small volume and resolution degrees of freedom that the predictive power of the theory can be improved for experiments on the same volume and resolution scale of the original measurement. The purpose of

this work is to develop methods to understand how fine resolution degrees of freedom affect the physics of coarser resolution degrees of freedom.

In what follows the "hats" from the quantum field operators are removed.

The first step is to discuss truncations of a quantum field theory based on truncations of an orthonormal basis to a finite dimensional subspace. Starting with an orthonormal basis of functions of spatial variables

$$\{\xi_n(\mathbf{x})\}_{n=1}^{\infty} \qquad \int \xi_m^*(\mathbf{x})\xi_n(\mathbf{x})d\mathbf{x} = \delta_{mn} \tag{2.12}$$

the completeness relation

$$\delta(\mathbf{x}-\mathbf{y}) = \sum_n \xi_n(\mathbf{x})\xi_n^*(\mathbf{y}) \tag{2.13}$$

is used to get an exact representation of the field operators of the form

$$\Phi(\mathbf{x},t) = \int \delta(\mathbf{x}-\mathbf{y})\Phi(\mathbf{y},t))d\mathbf{y} = \sum \int \xi_n(\mathbf{x})\xi_n^*(\mathbf{y})\Phi(\mathbf{y},t)d\mathbf{y} = \sum \xi_n(\mathbf{x})\Phi_n(t) \tag{2.14}$$

and

$$\Pi(\mathbf{x},t) = \int \delta(\mathbf{x}-\mathbf{y})\Pi(\mathbf{y},t))d\mathbf{y} = \sum \int \xi_n(\mathbf{x})\xi_n^*(\mathbf{y})\Pi(\mathbf{y},t)d\mathbf{y} = \sum \xi_n(\mathbf{x})\Pi_n(t) \tag{2.15}$$

where the discrete canonical fields are defined by

$$\Phi_n(t) = \int d\mathbf{x}\xi_n^*(\mathbf{x})\Phi(\mathbf{x},t) \tag{2.16}$$

$$\Pi_n(t) = \int d\mathbf{x}\xi_n^*(\mathbf{x})\Pi(\mathbf{x},t). \tag{2.17}$$

Because of the orthonormality of the basis in this representation, the equal-

time canonical commutation relations

$$[\Phi(\mathbf{x},t),\Phi(\mathbf{y},t)]=0 \tag{2.18}$$

$$[\Pi(\mathbf{x},t),\Pi(\mathbf{y},t)]=0 \tag{2.19}$$

and

$$[\Phi(\mathbf{x},t),\Pi(\mathbf{y},t)]=i\delta(\mathbf{x}-\mathbf{y}) \tag{2.20}$$

become

$$[\Phi_m(t),\Phi_n(t)]=0 \tag{2.21}$$

$$[\Pi_m(t),\Pi_n(t)]=0 \tag{2.22}$$

and

$$[\Phi_m(t),\Pi_n(t)]=i\delta_{mn} \tag{2.23}$$

This representation of the field is used in the Hamiltonian. While the operators $\Phi_m(t)$ and $\Pi_n(t)$ are well defined, the resulting Hamiltonian is not because the infinities are replaced by non-convergent sums. However, the infinite sums can be truncated to a finite number of terms. Then the exact expressions for the fields are replaced by the truncated operators

$$\Phi^t(\mathbf{x},t) = \sum_{n=0}^{N}\Phi_n(t)\xi_n(\mathbf{x}) \tag{2.24}$$

$$\Pi^t(\mathbf{x},t) = \sum_{n=0}^{N}\Pi_n(t)\xi_n(\mathbf{x}) \tag{2.25}$$

2.3 Interacting Fields

In this section the construction described in the previous section is applied to the example of a scalar field theory. The Hamiltonian for a scalar field theory is

$$H(t) = \int d\mathbf{x}\mathcal{H}(\mathbf{x}, t) \tag{2.26}$$

where the Hamiltonian density $\mathcal{H}(\mathbf{x}, t)$ is a local function of the fields and their derivatives at the point $(\mathbf{x}, t)$

$$\mathcal{H}(\mathbf{x}, t) = \frac{1}{2}\Pi^2(\mathbf{x}, t) + \frac{1}{2}\nabla\Phi(\mathbf{x}, t) \cdot \nabla\Phi(\mathbf{x}, t) + \frac{\mu^2}{2}\Phi^2(\mathbf{x}, t) + V(\Phi(\mathbf{x}, t)), \tag{2.27}$$

where $V(\Phi(\mathbf{x}, t))$ is a local interaction and μ is the mass.

The Heisenberg equations of motion

$$\frac{dH}{dt} = -i[H, H] = 0 \tag{2.28}$$

imply that the Hamiltonian is time independent which allows the Hamiltonian density to be expressed in terms of time 0 fields

$$H = \int d\mathbf{x}\mathcal{H}(\mathbf{x}, 0) \tag{2.29}$$

Replacing the fields in the Hamiltonian equation (2.29) by the truncated fields gives a truncated Hamiltonian H^t of the form

$$H^t = \frac{1}{2}\Big(\sum_{n=0}^{N}\Pi^t(n,0)\Pi^t(n,0) + \sum_{m,n=0}^{N} D_{mn}\Phi^t(n,0)\Phi^t(m,0) + \sum_{n=0}^{N}\mu^2\Phi^t(n,0)\Phi^t(n,0)\Big) + V(\Phi^t) \tag{2.30}$$

where

$$D_{mn} = \int \nabla\xi_m(\mathbf{x}) \cdot \nabla\xi_m(\mathbf{x})d\mathbf{x} \tag{2.31}$$

and for $V = \lambda\Phi^k(\mathbf{x}, 0)$

$$V(\Phi^t) = \lambda \sum_{n_1 \cdots n_k}^{N} \Gamma_{n_1 \cdots n_k} \Phi(n_1, 0) \cdots \Phi(n_k, 0)\mathbf{x} \tag{2.32}$$

with

$$\Gamma_{n_1 \cdots n_k} = \int \xi_{n_1}(\mathbf{x}) \cdots \xi_{n_k}(\mathbf{x}) d\mathbf{x} \tag{2.33}$$

Note that field modes with different n interact through both the derivative terms and the interaction term.

The diagonal (nn) part of the Hamiltonian without the potential has the structure of the Hamiltonian for N uncoupled harmonic oscillators. The discrete canonical fields can be expressed in terms of creation and annihilation operators, associated with N uncoupled harmonic oscillators, where the annihilation operators are defined to annihilate the oscillator ground state.

The truncated problem is to diagonalize the discrete truncated Hamiltonian on the Hilbert space generated by applying creation operators to the N-oscillator ground state. The truncated fields satisfy the discrete Heisenberg equations of motion

$$\frac{d\Phi^t(n,t)}{dt} = i[H^t, \Phi^t(n,t)] \tag{2.34}$$

and

$$\frac{d\Pi^t(n,t)}{dt} = i[H^t, \Pi^t(n,t)] \tag{2.35}$$

Because of the Von Neumann uniqueness theorem, the free canonical fields can be taken as initial conditions. Truncated correlation functions can be defined as

$$C^t(x_1, \cdots, x_k) = \langle g|\Phi^t(\mathbf{x}_1, t_1) \cdots \Phi^t(\mathbf{x}_k, t_k)|g\rangle \tag{2.36}$$

where $|g\rangle$ is the ground state of the truncated Hamiltonian.

Whether this procedure leads to meaningful predictions for a limited class of observables depends on the class of observables, the type of basis used to define the truncation and the truncation itself. The real problem in field theory is to find a way to construct an inductive limit out of a sequence of solutions of truncated problems.

2.4 Free Fields

Free time-zero fields are an irreducible set of operators that satisfy the same commutator relations as interacting fields at time zero. If the theory has a finite number of degrees of freedom they would be unitarily equivalent and they could be used as initial conditions in the theory. So initially at time $t = 0$, $\Phi(n, 0)$ and $\Pi(m, 0)$ are treated as free fields. The time evolved fields are not free, instead their time dependence is determined by the Heisenberg field equations

$$\frac{d\Phi(n,t)}{dt} = i[H, \Phi(n,t)] \tag{2.37}$$

and

$$\frac{d\Pi(m,t)}{dt} = i[H, \Pi(m,t)] \tag{2.38}$$

Because time evolution is unitary, the time evolved field will still satisfy equal-time commutation relations.

For interacting fields, these equations generate additional operators, making the calculations more cumbersome. Because of equation (2.28) and because the Hamiltonian is time independent, the truncated Hamiltonian, H^t, can be expressed in terms of time zero free fields.

CHAPTER 3
DAUBECHIES WAVELETS

3.1 Basis

Wavelets are a class of functions used to represent signals. The wavelets of interest in this thesis are constructed from the solution of a scaling equation by translations and dyadic dilations. The scaling equation is a linear renormalization group equation relating a given function on one scale to linear combinations of translates of that same function on a smaller scale. The scaling equation has some freedom that can be used to generate solutions with useful properties. In this thesis the solution of the Daubechies scaling equation is used to generate an orthonormal basis with useful properties.

For quantum field theory applications the basis should have

1. Functions of arbitrarily small support.
2. Some smoothness.
3. Orthonormality.
4. All functions with compact support.

3.2 Daubecies Wavelets

Daubechies wavelets are used in this work. They are a class of wavelets with the above properties. They also have the following properties:

1. They are a basis for $L^2(\mathbb{R})$.
2. They have compact support.

3. They have limited smoothness. This allows finite difference approximations to be replaced by derivatives of the basis function.

4. They have multiple resolutions. That is to say they are basis functions that represent structures on different distance scales.

5. They are related to the fixed points of a renormalization group leading to very simple scaling properties.

Linear combinations of translates of the Daubechies scaling functions reproduce exact locally finite representations of low-degree polynomials, while the corresponding wavelets are orthogonal to the same set of polynomials.

The general form of the scaling equation is

$$s(x) = \sum_{n=0}^{2K-1} h_n R T^n s(x) \tag{3.1}$$

where R is a unitary dyadic scale transformation

$$Rf(x) = \sqrt{2} f(2x) \tag{3.2}$$

and T is a unit translation

$$Tf(x) = f(x-1) \tag{3.3}$$

The h_n values are numerical constants that are determined by the type of wavelet. Table 3.1 shows the h_n values worked out for some Daubecies wavelets.

Equation (3.1) states that the scaling function is a linear combination of $2K-1$ translated copies of itself on a smaller scale by a factor of two. This work uses the $K = 3$ wavelet. This choice is motivated by the fact that the $K = 3$ wavelets are the simplest Daubechies' wavelets with at least one continuous derivative. Since this is a

homogeneous equation for $s(x)$, it needs to be normalized. The normalization is fixed by the scale fixing condition

$$\int s(x)dx = 1 \tag{3.4}$$

Table 3.1: Daubechies Wavelet Values of h_n

h_n	$K=1$	$K=2$	$K=3$
h_0	$1/\sqrt{2}$	$(1+\sqrt{3})/4\sqrt{2}$	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3+\sqrt{3})/4\sqrt{2}$	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

Generally if the scaling equation has a solution, it has support on the interval $[0, 2K-1]$. If K is finite the solution has compact support. The constants h_n in the Daubechies scaling equation are chosen such that the solutions have certain desirable properties.

Taking the Fourier transform of both sides of the scaling equation (3.1) leads to the following constraint, which is a necessary condition for the scaling equation to have a solution.

$$\sum_n h_n = \sqrt{2} \tag{3.5}$$

A second constraint is that the translates of the scaling functions by integers define

an orthonormal system of functions

$$\int s(x-m)s(x-n)dx = \delta_{mn} \tag{3.6}$$

Finally the special property that characterizes the Daubechies scaling function is that the orthonormal system of integer translates can be used to construct locally finite representations of low-degree polynomials.

$$\sum s(x-n)c_n = x^m \qquad\qquad m = 0, 1, \cdots K \tag{3.7}$$

with

$$c_n = \int s_n(x)x^m dx = \int s(x-n)x^m dx = \int s(x)(x+n)^m \tag{3.8}$$

These constraint equations can be solved to compute the constants h_n. For $K > 1$ there are two solutions related by $h'_n = h_{2K-1-n}$. One gives the mirror image of the scaling function associated with the other.

One compelling property of the wavelet representation is that it can be used to decompose the Hilbert space into an orthogonal direct sum of subspaces with different resolutions.

This is started by defining the scaling functions

$$s_n^k := R^k T^n s(x) \tag{3.9}$$

which involves translating the original function by n units to the right and unitarily shrinking the support by a factor of 2^k.

Fixed scale 2^{-k} linear combinations of $\{s_n^k(x)\}_{n=-\infty}^{\infty}$ functions with square summable coefficients span a subspace S_k of $L^2(\mathbb{R})$ which is called the resolution 2^{-k}

subspace. Applying R^k to the scaling equation implies that

$$S_{k+1} \supset S_k \tag{3.10}$$

These are different subspaces and there is a non-empty orthogonal compliment to S_k in S_{k+1}. This space is denoted by W_k, which leads to the decomposition

$$S_{k+1} = S_k \oplus W_k \tag{3.11}$$

of the resolution $2^{-(k+1)}$ subspace S_{k+1}, into a direct sum of the resolution 2^{-k} resolution subspace and it's orthogonal compliment. The subspace W_k includes functions of resolution $2^{-(k+1)}$ that are orthogonal to functions of resolution 2^{-k}.

This can be repeated to get the following decomposition

$$S_{k+m+1} = S_k \oplus W_k \oplus W_{k+1} \oplus \cdots \oplus W_{k+m} \tag{3.12}$$

which eventually leads to

$$L^2(\mathbb{R}) = S_k \oplus W_k \oplus W_{k+1} \oplus \cdots \tag{3.13}$$

The subspaces W_k are called the wavelet subspaces. To construct a basis for this subspace the "mother wavelet" is defined by

$$w(x) = \sum_{n=1}^{2K-1} RT^n g_n s(x) \tag{3.14}$$

where

$$g_n = (-1)^n h_{2K-1-n} \tag{3.15}$$

An orthonormal basis of functions for the wavelet subspace W_k is given by translations and dyadic dilations of the mother wavelet

$$w_n^k := R^k T^n w(x) \tag{3.16}$$

where $-\infty < n < \infty$

Equation (3.13) implies that since $s(x)$ and $w(x)$ have support on the interval $[0, 2K-1]$, for any fixed scale 2^{-k} the functions

$$\{s_n^k(x)\}_{n=-\infty}^{\infty} \cup \{w_n^m(x)\}_{n=-\infty, m=k}^{\infty} \tag{3.17}$$

are an orthonormal basis for $L^2(\mathbb{R})$

The basis fuctions $s_n^k(x)$ and $w_n^k(x)$ have compact support on the interval $[2^{-k}n, 2^{-k}(n+2K-1)]$. They also satisfy the orthonormality conditions

$$\int dx s_n^k(x) s_m^k(x) = \delta_{mn} \tag{3.18}$$

$$\int dx s_n^k(x) w_n^{k+l}(x) = 0 \qquad (l \geq 0) \tag{3.19}$$

and

$$\int dx w_n^k(x) w_m^l(x) = \delta_{mn}\delta_{kl} \tag{3.20}$$

This means that any square integrable function, $f(x)$ has an expansion of the form

$$f(x) = \sum_{n=-\infty}^{\infty} f_n s_n^k(x) + \sum_{n=-\infty}^{\infty} \sum_{l=k}^{\infty} f_{nl} w_n^l(x) \tag{3.21}$$

where

$$\sum_{n=-\infty}^{\infty} |f_n|^2 + \sum_{n=-\infty}^{\infty} \sum_{l=k}^{\infty} |f_{nl}|^2 < \infty \tag{3.22}$$

Note that scaling functions with different values of k are not orthogonal. Also $w_n^{k+l}(x)$ is not orthogonal to $s_n^k(x)$ for $l < 0$. The basis in equation (3.13) has scaling functions $s_n^k(x)$ of a fixed largest scale 2^{-k} and wavelets $w_n^{k+l}(x)$ that can represent the structure on all finer scales $2^{-(k+l)}$ with $l > 0$.

The scale $2^{-(k-1)}$ scaling functions and scale $2^{-(k-1)}$ wavelets are linear functions of the scale 2^{-k} scaling functions

$$s_n^{k-1}(x) = \sum_{l=0}^{2K-1} h_l s_{n+l}^k(x) \tag{3.23}$$

and

$$w_n^{k-1}(x) = \sum_{l=0}^{2K-1} g_l s_{n+l}^k(x) \tag{3.24}$$

The inverse of these relationships express the scale 2^{-k} scaling function as a linear combination of the scale $2^{-(k-1)}$ scaling functions and wavelets by

$$s_n^k(x) = \sum_m h_{n-2m} s_m^{k-1}(x) + \sum_m g_{n-2m} w_m^{k-1}(x) \tag{3.25}$$

These equations comprise the wavelet transform. It generates an orthogonal transformation that relates S_{k+1} to $S_k \oplus W_k$. It provides the decomposition of a fine-scale subspace into a coarse-scale subspace and a fine scale correction subspace.

The Daubechies basis is used to constuct the truncated fields. One of the quantities that appear in the basis truncated Hamiltonian are integrals of the form

$$D_{mn} = \int \frac{d\xi_m(x)}{dx} \frac{d\xi_n(x)}{dx} dx \tag{3.26}$$

and

$$\Gamma_{n_1 \cdots n_m} = \int \xi_{n_1}(x) \cdots \xi_{n_m}(x) dx \tag{3.27}$$

where the $\xi_n(x)$ are wavelet or scaling basis functions. It is difficult to compute these integrals numerically because the basis functions are fractal in nature. The fractal nature is a consequence of being a solution of the scaling equation. Fortunately the scaling equation and the normalization condition can be used to reduce the computation to linear algebra. This is illustrated in the appendix A.

CHAPTER 4
RESOLUTION AND VOLUME WAVELET TRUNCATED FIELD THEORIES

4.1 Field Decompositions

Physics experiments are normally done in finite regions of space-time involving systems with a finite amount of energy leading to a system with a finite number of degrees of freedom. In mathematics, a self-adjoint Hamiltonian is locally compact if the product of the projections on finite volume and finite energy subspaces is compact, i.e. the range of this product is approximately finite dimensional. In physics, Hamiltonians usually have this property. For example, while a system of free particles in a finite box has an infinite number of discrete energy eigenstates, the number of these states with energy less than some fixed value is finite.

While field theories necessarily have an infinite number of degrees of freedom, it is expected that when applied to study phenomena with finite energies in finite space-time volumes there are a finite number of dominant degrees of freedom. The uncertainty principle associates the maximum energy or momentum transferred in a system with minimal resolution.

With this in mind the basis consisting of Daubechies scaling functions on an initial scale and Daubechies wavelets on smaller scales is a perfect basis for decomposing a local field into parts associated with different volumes and spatial resolutions. These basis functions have two independent indices, one associated with the location of the degree of freedom and the other associated with the resolution.

The Daubechies basis is in one dimension. Products of three such basis functions can be used in three dimensional applications. While doing this is straightforward, it becomes complicated to keep track of the indices in each dimension. To keep the bookkeeping as simple as possible, the discussion in this thesis will be limited to one space and one time dimension.

Finally, note that while fields are expected to be operator valued distributions, the scaling-wavelet basis functions do not have an infinite number of derivatives, so they are not elements of a standard space of test functions. For the case of free fields, however, the operators obtained by integrating the product of these basis functions with the fields are well-defined Fock space operators. The situation can be compared to a simple delta function. While it is a distribution, it is also a well-defined linear functional on any continuous function, even functions that are not differentiable.

Once the interacting field theory is truncated to a finite number of degrees of freedom, the truncated Hamiltonian can be expressed in terms of the algebra free fields on the free field Fock space. In the limit of infinite number of degrees of freedom the infinities of the theory will appear as divergent sums of well-defined quantities. The problem of formulating a convergent finite limiting procedure is beyond the scope of this thesis. However, once the procedure is determined, it should result in a convergent sequence of truncated theories. More importantly, given the relevant scale and volume for an experiment, it is expected that a finite number of degrees of freedom should be sufficient to describe the experiments and the accuracy should be improved by finite increases in the resolution and the size of the volume. This is

independent of whether a limit that defines physics on all scales and volumes exists. For this reason it makes sense to study properties of volume and resolution truncated theories.

4.2 Discrete Formulation of Canonical Field Theory

The scaling-wavelet basis discussed in chapter 3 can be used to write the fields exactly as sums of products of basis functions with time-dependent operator coefficients,

$$\Phi(x,t) = \sum_n \Phi^k(n,t) s_n^k(x) + \sum_{n,l\geq k} \Phi^l(n,t) w_n^l(x) \tag{4.1}$$

$$\Pi(x,t) = \sum_n \Pi^k(n,t) s_n^k(x) + \sum_{n,l\geq k} \Pi^l(n,t) w_n^l(x) \tag{4.2}$$

where the discrete operator coefficients are projections of field operators on the orthonormal basis of scaling functions and wavelets

$$\Phi^k(s,n,t) = \int dx \Phi(x,t) s_n^k(x) \tag{4.3}$$

$$\Phi^l(w,n,t) = \int dx \Phi(x,t) w_n^l(x) \qquad (l \geq k) \tag{4.4}$$

$$\Pi^k(s,n,t) = \int dx \Pi(x,t) s_n^k(x) \tag{4.5}$$

$$\Pi^l(w,n,t) = \int dx \Pi(x,t) w_n^l(x) \qquad (l \geq k) \tag{4.6}$$

The scalar fields are assumed to satisfy the canonical equal-time commutation relations

$$[\Phi(x,t), \Pi(y,t)] = i\delta(x-y) \tag{4.7}$$

and

$$[\Phi(x,t), \Phi(y,t)] = [\Pi(x,t), \Pi(y,t)] = 0 \tag{4.8}$$

Discrete fields satisfying the discrete form of these commutation relations can be created by integrating the spatial coordinates of the fields against the scaling-wavelet basis functions.

Because of the orthonormality of the scaling-wavelet basis, the discrete fields defined in equations (4.1) and (4.2) satisfy discrete versions of the canonical commutation relations.

$$\Phi^k(s,n,t) = \int dx \Phi(x,t) s_n^k(x) \quad \Phi^l(w,n,t) = \int dx \Phi(x,t) w_n^l(x) \quad (l \geq k) \tag{4.9}$$

and

$$\Pi^k(s,n,t) = \int dx \Pi(x,t) s_n^k(x) \quad \Pi^l(w,n,t) = \int dx \Pi(x,t) w_n^l(x) \quad (l \geq k) \tag{4.10}$$

The discrete equal-time commutators are

$$[\Phi^k(s,n,t), \Phi^k(s,m,t)] = [\Pi^k(s,n,t), \Pi^k(s,m,t)] = 0 \tag{4.11}$$

$$[\Phi^k(s,n,t), \Pi^k(s,m,t)] = i\delta_{nm} \tag{4.12}$$

$$[\Phi^k(w,n,t), \Phi^l(w,m,t)] = [\Pi^k(w,n,t), \Pi^l(s,m,t)] = 0 \tag{4.13}$$

$$[\Phi^k(w,n,t), \Pi^l(w,m,t)] = i\delta_{kl}\delta_{mn} \tag{4.14}$$

$$[\Phi^k(s,n,t), \Phi^l(w,m,t)] = [\Pi^k(s,n,t), \Pi^l(w,m,t)] = 0 \tag{4.15}$$

and

$$[\Phi^k(s,n,t), \Pi^l(w,m,t)] = [\Pi^k(s,n,t), \Phi^l(w,m,t)] = 0 \tag{4.16}$$

The expansions (4.1) and (4.2) can be inserted into expressions for the free field Hamiltonian. The integral over the Hamiltonian density is replaced by integrals

over products of basis functions. The integrals over basis functions become multi-index constants with the indices summed over discrete fields. One advantage of the scaling-wavelet basis is that in general the computation of these overlap integrals reduces to finite linear algebra. Another is that integrals of products of functions on different scales are related by simple scaling relations. Taken together this means that all of the constant terms in the Hamiltonian can be expressed in terms of the solution of a small linear system of equations. This is shown for the case of a free scalar field of mass μ which has a Hamiltonian of the form:

$$H = \frac{1}{2}\int d\mathbf{x}(\Pi(\mathbf{x},0)\Pi(\mathbf{x},0) + \nabla\Phi(\mathbf{x},0)\cdot\nabla\Phi(\mathbf{x},0) + \mu^2\Phi(\mathbf{x},0)\Phi(\mathbf{x},0)) \tag{4.17}$$

The expansions of the fields in (4.1) and (4.2) can be inserted in equation (4.17). The result is a representation of the Hamiltonian as an infinite sum of products of discrete fields with constant coefficients. The discrete form of the Hamiltonian can be decomposed by scale. The decomposition is a sum of an operator with only scaling function fields, an operator with only wavelet fields and an operator with products of both types of fields.

$$H = H_s + H_w + H_{sw} \tag{4.18}$$

where

$$H_s = \frac{1}{2}(\sum_n \Pi^k(s,n,0)\Pi^k(s,n,0) + \sum_{nm}\Phi^k(s,n,0)D^k_{s,nm}\Phi^k(s,m,0) + \mu^2\sum_n \Phi^k(s,n,0)\Phi^k(s,n,0)) \tag{4.19}$$

$$H_w = \frac{1}{2}(\sum_n \Pi^l(w,n,0)\Pi^l(w,n,0) + \sum_{nm} \Phi^l(w,n,0)D^{lj}_{w,nm}\Phi^j(w,m,0) + \mu^2 \sum_n \Phi^l(w,n,0)\Phi^w(w,n,0)) \tag{4.20}$$

and

$$H_{sw} = \frac{1}{2}\sum_{nlm} \Phi^l(w,n,0)D^{lk}_{b,nm}\Phi^k(s,m,0) \tag{4.21}$$

where the coefficients

$$D^k_{s,nm} = \int dx \frac{d}{dx}s^k_n(x)\frac{d}{dx}s^k_m(x) \tag{4.22}$$

$$D^{lj}_{w,nm} = \int dx \frac{d}{dx}w^l_n(x)\frac{d}{dx}w^j_m(x) \tag{4.23}$$

and

$$D^{lk}_{b,nm} = \int dx \frac{d}{dx}w^l_n(x)\frac{d}{dx}s^k_m(x) \tag{4.24}$$

are constant matrices that couple near neighbor fields. $D^k_{s,nm}$ represents combining two scaling functions, $D^{lj}_{w,nm}$ represents combining two wavelet functions and $D^{lk}_{b,nm}$ represents combing a scaling and a wavelet function.

For $D^k_{s,nm}$ if $|n-m| \geq 2K-1$, the support properties of the basis functions imply that the matrices will vanish. Thus they have a structure similar to a finite difference approximation. There are similar locality constraints for the integrals that involve two wavelets and the integral that involves both a scaling function and a wavelet.

For $l \neq j$, the matrices $D^{lj}_{w,nm}$ and $D^{lk}_{b,nm}$ couple the physical degrees of freedom on different scales in a free field theory. However, additional couplings also arise from

local products of multiple fields in interacting theories. For example:

$$\int \phi^4(x,t)dx = \sum_{n_1 n_2 n_3 n_4} \Gamma_{s,n_1\cdots n_4} \Phi^k(s,n_1,t)\Phi^k(s,n_2,t)\Phi^k(s,n_3,t)\Phi^k(s,n_4,t) + \cdots \tag{4.25}$$

where

$$\Gamma_{s,n_1\cdots n_4} := \int s^k_{n_1}(x)s^k_{n_2}(x)s^k_{n_3}(x)s^k_{n_4}(x)dx \tag{4.26}$$

and the $\cdots$ represents additional terms involving the wavelet basis and fields.

Similar to the D coefficients, the Γ coefficients are almost local since they vanish unless all of the functions in the integral in equation (4.26) have overlapping support.

Also, an important feature of the fractal nature of the scaling-wavelet basis is that the D coefficients (and for that matter the Γ coefficients) have simple scaling properties. For example, the $D^k_{s,mn}$ coefficients can be simplified using

$$D^k_{s,mn} = 2^{2k} D^0_{s,mn} = 2^{2k} D_{s,0(n-m)} \tag{4.27}$$

All of these coefficients can be calculated exactly using equations (3.1), (3.2) and (3.3). This calculation is discussed in appendix A. The calculations in appendix A also demonstrate that while there are an infinite number of matrices $D^x_{yz,mn}$, they can all be expressed in terms of known quantities.

CHAPTER 5
FLOW EQUATIONS

5.1 Overview of Flow Equation Methods

The goal of this work is to transform the Hamiltonian so that the coupling between the scaling function and wavelet degrees of freedom are approximately eliminated. Flow equation methods were used to accomplish this goal. This will evolve the original Hamiltonian into a unitarily equivalent Hamiltonian of the desired form.

These flow equation methods are good for treating wavelet truncated field theories because the commutators of the operators that appear in the flow equation are easily computed. In the free field model discussed in equation (4.17) the flow equation will not generate new classes of operators. While the exact theory is solvable, the derivative terms provide a non-trivial coupling of degrees of freedom on different scales, leading to a non-trivial truncated test problem.

For an interacting theory the flow equations would generate more complicated interactions than the ones in the original Hamiltonian. Some of these may be unimportant and may not even contribute in some of the limits, while others may grow and dominate the dynamics. The scaling properties of the coupling provide tools for evaluating the relevance of many of the generated interactions. A careful treatment of this is beyond the scope of this thesis. The goal of this work is to test the flow equation method as a candidate for block diagonalizing the Hamiltonian according to scale.

The application of the flow equation used in this thesis differs because (1) it is formulated in a coordinate representation rather than a momentum representation and (2) the flow generator is designed to block diagonalize the Hamiltonian rather diagonalize it.

The flow equation performs the block diagonalization using a continuously parameterized unitary transformation, $U(\lambda)$, that defines a parameterized set of equivalent Hamiltonians by

$$H(\lambda) = U(\lambda)HU^{\dagger}(\lambda) \tag{5.1}$$

where $H = H(0)$ is the original Hamiltonian. The problem is to choose the operator valued function $U(\lambda)$ so $H(\lambda)$ will evolve to the desired form as λ increases.

To understand how to do this it is useful to define $U(\lambda)$ in terms of a suitable generator. The generator is denoted by $K(\lambda)$. To relate $U(\lambda)$ to $K(\lambda)$ note that unitarity implies that $U(\lambda)$ satisfies the differential equation

$$\frac{dU(\lambda)}{d\lambda} = \frac{dU(\lambda)}{d\lambda}U^{\dagger}(\lambda)U(\lambda) = K(\lambda)U(\lambda) \tag{5.2}$$

where

$$K(\lambda) = \frac{dU(\lambda)}{d\lambda}U^{\dagger}(\lambda) = -K^{\dagger}(\lambda) \tag{5.3}$$

defines the anti-Hermitian generator of the unitary transformation. Equation (5.3) is equivalent to

$$\frac{dU(\lambda)}{d\lambda} = K(\lambda)U(\lambda) \tag{5.4}$$

which can be solved to get $U(\lambda)$ in terms of $K(\lambda)$ given the initial condition $U(0) = I$.

Since $U(\lambda)$ is unitary and $K(\lambda)$ is anti-Hermetian, it also follows that

$$\frac{dU^\dagger(\lambda)}{d\lambda} = -U^\dagger(\lambda)K(\lambda) \tag{5.5}$$

It is possible to use these equations to get an equation directly for $H(\lambda)$ in terms of $K(\lambda)$ without explicitly constructing $U(\lambda)$. Differentiating equation (5.1) using equation (5.3) gives the following differential equation for $H(\lambda)$

$$\frac{dH(\lambda)}{d\lambda} = [K(\lambda), H(\lambda)] \tag{5.6}$$

It is necessary to choose a generator that will evolve the initial Hamiltonian to a block diagonal form. This can be accomplished by choosing

$$K(\lambda) = [H_b(\lambda), H(\lambda)] \tag{5.7}$$

This is a λ-dependent generator that is a function of the evolved Hamiltonian. It will have the desired behavior as long as $H_b(\lambda)$ is chosen appropriately.

Here $H(0)$ is the original wavelet-truncated Hamiltonian, $H_b(\lambda)$ is the part of the evolved wavelet-truncated Hamiltonian, $H(\lambda)$, with the operators that couple degrees of freedom in the different blocks set to zero. Since $H_b(\lambda) = H_b^\dagger(\lambda)$, $K(\lambda)$ is anti-Hermetian. Using equation (5.7) in equation (5.6) gives

$$\frac{dH(\lambda)}{d\lambda} = [K(\lambda), H(\lambda)] = [[H_b(\lambda), H(\lambda)], H(\lambda)] = [H(\lambda), [H(\lambda), H_b(\lambda)]] \tag{5.8}$$

Equation (5.8) is called the flow equation. With initial condition $H(0) = H$ this equation has a fixed points λ^* when

$$[H(\lambda^*), H_b(\lambda^*)] = 0 \quad \text{or} \quad [H(\lambda^*), [H(\lambda^*, H_b(\lambda^*)]] = 0 \tag{5.9}$$

To understand how this evolves to a block diagonal form it is useful to start with a matrix. Let M be a matrix let P be the projection on a subspace of the Hilbert space and let $Q = I - P$ be the projection on the complementary block. Then M can be decomposed into

$$M(\lambda) = PMP(\lambda) + QMQ(\lambda) + PMQ(\lambda) + QMP(\lambda) \tag{5.10}$$

where $M_b(\lambda) = PMP(\lambda) + QMQ(\lambda)$ is the block preserving part of M and $M_c(\lambda) = PMQ(\lambda) + QMP(\lambda)$ is the block coupling part. The flow equation can be expressed in a basis of eigenstates of $M_c(\lambda)$ (respectively $M_b(\lambda)$) that couples the subspace as

$$\begin{aligned} \frac{d}{d\lambda} \langle c_m | M_b(\lambda) | c_n \rangle &= \langle c_m | [M_c(\lambda)[M_c(\lambda), M_b(\lambda)]] | c_n \rangle \\ &= (e_{cm} - e_{cn})^2 \langle c_m | M_b(\lambda) | c_n \rangle \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} \frac{d}{d\lambda} \langle b_m | M_c(\lambda) | b_n \rangle &= - \langle b_m | [M_b(\lambda)[M_b(\lambda), M_c(\lambda)]] | b_n \rangle \\ &= -(e_{bm} - e_{bn})^2 \langle b_m | M_c(\lambda) | b_n \rangle \end{aligned} \tag{5.12}$$

These equations show that the block diagonal form of the matrix is approached exponentially, as long as there are no degeneracies and the eigenvalues do not cross as λ increases. The evolution will slow if the spacing between eigenvalues gets small.

5.2 Scale Separation in Field Theory

In the field theory case this construction is repeated by replacing the matrix M with the operator H. The field theoretic problem is a many-body problem. In

general there is a subspace with scaling function degrees of freedom, wavelet degrees of freedom and one that contains both types of degrees of freedom. After the Hamiltonian is truncated by volume and resolution, it can be expressed as a sum of three terms, one with only scaling function fields, one with wavelet fields and one with products of both kinds of fields. $H_b(\lambda)$ represents the sum of the terms with products of scaling function fields and products of wavelet fields, and $H_c(\lambda)$ represent terms with mixed products that couple the two scales. A fixed point of the flow equation will occur when $[H_b(\lambda), H(\lambda)] = 0$. In the general case it is not clear whether the equation will drive the system to a fixed point in the same way it does for projection methods.

The free field case is the simplest test model. For an interacting Hamiltonian, integrating the flow equation will generate complex many-body interaction terms with each iteration. For the free field, there still is a non-trivial coupling between scales, but solving the flow equation does not generate an infinite number of many-body interactions. In this case it is possible that the flow equation will evolve the Hamiltonian to the block diagonal form in the same way that is does using projectors.

To show that this choice of generators leads to block diagonalization of the Hamiltonian, the flow equation is separated into two equations where $H_b(\lambda)$, as above, is the part of the Hamiltonian that does not couple the scaling and wavelet degrees of freedom. The coupling term is defined by

$$H_c(\lambda) = H(\lambda) - H_b(\lambda) \tag{5.13}$$

which is the portion of the Hamiltonian that only includes operators that couple the

wavelet and scaling functions degrees of freedom.

The flow equation has the form

$$\begin{aligned}\frac{dH(\lambda)}{d\lambda} &= [H_b(\lambda) + H_c(\lambda), [H_b(\lambda) + H_c(\lambda), H_b(\lambda)]] \\ &= [H_c(\lambda), [H_c(\lambda), H_b(\lambda)]] + [H_b(\lambda), [H_c(\lambda), H_b(\lambda)]] \end{aligned} \tag{5.14}$$

Due to the quadratic nature of these terms and the commutation relations of the truncated free fields, the first term on the right hand side of this equation is block diagonal; it only contains products of scaling function fields or product of wavelet fields. The second term is a scale coupling operator, it only contains products of scaling function and wavelet fields. Because of this decomposition, the flow equation can be replaced by a pair of coupled equations, one for the block diagonal part and one for the coupling term:

$$\frac{dH_b(\lambda)}{d\lambda} = [H_c(\lambda), [H_c(\lambda), H_b(\lambda)]] \tag{5.15}$$

and

$$\frac{dH_c(\lambda)}{d\lambda} = [H_b(\lambda), [H_c(\lambda), H_b(\lambda)]] = -[H_b(\lambda), [H_b(\lambda), H_c(\lambda)]] \tag{5.16}$$

These equations are symmetric between $H_b(\lambda)$ and $H_c(\lambda)$ except for a sign.

Equation (5.15) is expressed in a basis of eigenstates of $H_c(\lambda)$ with eigenvalues e_{cn} and equation (5.16) in a basis of eigenstates of $H_b(\lambda)$ with eigenvalues e_{bn} to get

$$\frac{d}{d\lambda} \langle c_m|H_b(\lambda)|c_n\rangle = (e_{cm}(\lambda) - e_{cn}(\lambda))^2 H_{bmn}(\lambda) \langle c_m|H_b(\lambda)|c_n\rangle \tag{5.17}$$

and

$$\frac{d}{d\lambda} \langle b_m|H_c(\lambda)|b_n\rangle = -(e_{bm}(\lambda) - e_{bn}(\lambda))^2 \langle b_m|H_c(\lambda)|b_n\rangle \tag{5.18}$$

These equations have the same form as equations (5.11) and (5.12). They indicate that the matrix elements of H_b will increase while the elements of H_c will decrease. This can stall if there are degeneracies or near degeneracies. The interesting difference is that in this case everything is expressed directly in terms of the discrete field operators and there are no explicit projectors.

CHAPTER 6
PROJECT DESCRIPTION, RESULTS AND ANALYSIS

6.1 Process

The main goal of this project is to see if flow equation methods can be used with the multi-resolution wavelet decomposition of fields to separate scales in quantum field theory.

This is tested with a free scalar field theory that is truncated to a finite volume with two resolutions. There is a coarse scale resolution associated with the scaling function degrees of freedom and a fine scale resolution associated with the wavelet degrees of freedom. The truncated theory has a non-trivial coupling between the two scales generated by the spatial derivative terms that appear in the free field Hamiltonian. Both the exact and truncated theory are exactly solvable, which helps in analyzing the results. The advantage of the free field theory is that integrating the flow equation does not generate the infinite number of complicated many-body interactions that would be seen with an interacting field. Because of this it is possible to separate issues associated with convergence of the flow equation from dynamical complications.

The goal is to replace the truncated Hamiltonian by an equivalent Hamiltonian that has no coupling between the two scales. The test truncation uses 32 basis functions; 16 scaling function degrees of freedom and 16 wavelet degrees of freedom. For simplicity the calculation uses a 1 space and 1 time dimensional free field.

The exact Hamiltonian for a free scalar field is

$$H = \frac{1}{2}\int (\Pi(x,0)\Pi(x,0) + \nabla\Phi(x,0)\cdot\nabla\Phi(x,0) + \mu^2\Phi^2(x,0))dx \tag{6.1}$$

where μ is the mass.

The fields are expressed as a linear combinations of local operators associated with different volumes and resolutions. The exact linear combinations are truncated to a finite number of terms, which are inserted into the Hamiltonian.

The expansions of the fields lead to a decomposition of the Hamiltonian as a sum of operators with two different resolutions, as well as a coupling term.

$$H = H_s + H_w + H_{sw} \tag{6.2}$$

where

$$H_s = \frac{1}{2}\Big(\sum_n \Pi^k(s,n,0)\Pi^k(s,n,0) + \sum_{mn}\Phi^k(s,m,0)D^k_{s,mn}\Phi^k(s,n,0) + \mu^2\sum_n \Phi^k(n,0)\Phi^k(n,0)\Big) \tag{6.3}$$

is the scaling function (coarse scale resolution) portion of the Hamiltonian,

$$H_w = \frac{1}{2}\Big(\sum_{n,l} \Pi^l(w,n,0)\Pi^l(w,n,0) + \sum_{mn,lj\geq k}\Phi^l(w,m,0)D^{lj}_{w,mn}\Phi^j(w,n,0) + \mu^2\sum_{ln}\Phi^l(w,n,0)\Phi^l(w,n,0)\Big) \tag{6.4}$$

is the wavelet function (fine scale resolution) portion of the Hamiltonian, and

$$H_{sw} = \frac{1}{2}\Big(\sum_{mnk\geq l}\Phi^k(s,m,0)D^{kl}_{b,mn}\Phi^l(w,n,0)\Big) \tag{6.5}$$

is the coupling portion of the Hamiltonian. The constant matrices are integrals of products of the basis functions:

$$D^k_{s,mn} = \int dx \frac{d}{dx} s^k_m(x) \frac{d}{dx} s^k_n(x) \tag{6.6}$$

$$D^{lj}_{w,mn} = \int dx \frac{d}{dx} w^l_m(x) \frac{d}{dx} w^j_n(x) \tag{6.7}$$

$$D^{lk}_{b,mn} = \int dx \frac{d}{dx} w^l_m(x) \frac{d}{dx} s^k_n(x) \tag{6.8}$$

The expansion is exact for any fixed k. The truncated Hamiltonian is obtained by using $k = 0$ and replacing fields by expansions using 16 scaling functions and 16 wavelets.

The truncated Hamiltonian is evolved using the flow equation

$$\frac{dH}{d\lambda} = [H(\lambda), [H(\lambda), H_b(\lambda)]] \tag{6.9}$$

where $H_b(\lambda)$ is obtained from $H(\lambda)$ by setting the coupling terms to zero, so

$$H_b(\lambda) = H_s(\lambda) + H_w(\lambda) \tag{6.10}$$

The generator $[H(\lambda), H_b(\lambda)]$ of the flow equation is chosen to evolve the Hamiltonian to an equivalent Hamiltonian with block diagonal form.

This non-linear equation is solved using the Euler method. The commutators appearing in this equation are discussed in appendix B. The Euler iteration is computed using Matlab.

Truncated fields are given by

$$\Phi(x,t) = \sum_{n=0}^{15} s_n(x)\Phi(s,n,t) + \sum_{n=0}^{15} w_n(x)\Phi(w,n,t) \tag{6.11}$$

and

$$\Pi(x,t) = \sum_{n=0}^{15} s_n(x)\Pi(s,n,t) + \sum_{n=0}^{15} w_n(x)\Pi(w,n,t) \tag{6.12}$$

The truncated Hamiltonian can be expressed as a quadratic form in the time $t = 0$ fields, $\Phi(w,n,0)$, $\Pi(w,n,0)$, $\Phi(s,n,0)$ and $\Pi(s,n,0)$ field operators with coefficients that are functions of the flow parameter λ.

The truncated Hamiltonian is

$$\begin{aligned}
H =& \sum_{mn} a_{ssmn}(\lambda)\Pi(s,m,0)\Pi(s,n,0) + \sum_{mn} b_{ssmn}(\lambda)\Pi(s,m,0)\Phi(s,n,0) \\
&+ \sum_{mn} c_{ssmn}(\lambda)\Phi(s,m,0)\Pi(s,n,0) + \sum_{mn} d_{ssmn}(\lambda)\Phi(s,m,0)\Phi(s,n,0) \\
&+ \sum_{mn} a_{swmn}(\lambda)\Pi(s,m,0)\Pi(w,n,0) + \sum_{mn} b_{swmn}(\lambda)\Pi(s,m,0)\Phi(w,n,0) \\
&+ \sum_{mn} c_{swmn}(\lambda)\Phi(s,m,0)\Pi(w,n,0) + \sum_{mn} d_{swmn}(\lambda)\Phi(s,m,0)\Phi(w,n,0) \\
&+ \sum_{mn} a_{wsmn}(\lambda)\Pi(w,m,0)\Pi(s,n,0) + \sum_{mn} b_{wsmn}(\lambda)\Pi(w,m,0)\Phi(s,n,0) \\
&+ \sum_{mn} c_{wsmn}(\lambda)\Phi(w,m,0)\Pi(s,n,0) + \sum_{mn} d_{wsmn}(\lambda)\Phi(w,m,0)\Phi(s,n,0) \\
&+ \sum_{mn} a_{wwmn}(\lambda)\Pi(w,m,0)\Pi(w,n,0) + \sum_{mn} b_{wwmn}(\lambda)\Pi(w,m,0)\Phi(w,n,0) \\
&+ \sum_{mn} c_{wwmn}(\lambda)\Phi(w,m,0)\Pi(w,n,0) + \sum_{mn} d_{wwmn}(\lambda)\Phi(w,m,0)\Phi(w,n,0)
\end{aligned} \tag{6.13}$$

The initial values of the coefficients ($\lambda = 0$) are determined by the structure of the initial Hamiltonian. They are

$$a_{ssmn} = \frac{1}{2}\delta_{mn}, \quad a_{swmn} = a_{wsmn} = 0, \quad a_{wwmn} = \frac{1}{2}\delta_{mn}$$

$$b_{ssmn} = b_{swmn} = b_{wsmn} = b_{wwmn} = 0$$

$$c_{ssmn} = c_{swmn} = c_{wsmn} = c_{wwmn} = 0$$

$$d_{ssmn} = \frac{\mu^2}{2} + D_{ss,mn}, \quad d_{swmn} = D_{sw,mn},$$
$$d_{wsmn} = D_{ws,mn}, \quad d_{wwmn} = \frac{\mu^2}{2} + D_{ww,mn} \tag{6.14}$$

where the matrices D_{xxmn} are the integrals appearing in equations (6.6-6.8). The way these coefficients are evolved is demonstrated in appendix B.

The equation for $H_b(\lambda)$ is similar with e replacing a, f replacing b, g replacing c and h replacing d, and where $h_{swmn} = h_{wsmn} = 0$. The flow parameter, λ, has units of $1/(\text{energy})^2$, so the scale is set by the mass $\mu = 1$.

The evolution of the coefficients with the flow parameter was computed using the Euler method. This evaluates the coefficients using small steps such that the value of the coefficients from the current step are used as the initial data to compute the coefficients in the next step in the flow. While a higher order method would have allowed a larger step size, the simplicity of the equations allowed the treatment of each step very efficiently. A step size $\Delta\lambda = .001$ was small enough to ensure convergence. Matlab was used to perform these calculations.

6.2 Results

To understand the evolution of the flow equations, the Hilbert Schmidt norms of the coefficient matrices of the different terms in the Hamiltonian are computed as a function of the flow parameter. The norms of the coefficient matrices that involve both scale and wavelet terms should get small with increasing λ.

Figures 6.1-6.8 show how the Hilbert Schmidt norms evolve for all the non-zero

coefficient matrices, given by

$$\sqrt{\sum a^*_{xyij} a_{xyji}} \qquad \sqrt{\sum b^*_{xyij} b_{xyji}} \qquad \sqrt{\sum c^*_{xyij} c_{xyji}} \qquad \sqrt{\sum d^*_{xyij} d_{xyji}} \tag{6.15}$$

in the Hamiltonian as a function of λ for values of λ between 0 and 20. There are 16 types of quadratic expressions from the four operators, $\Pi(s,n,t)$, $\Phi(s,n,t)$, $\Pi(w,m,t)$ and $\Phi(w,m,t)$.

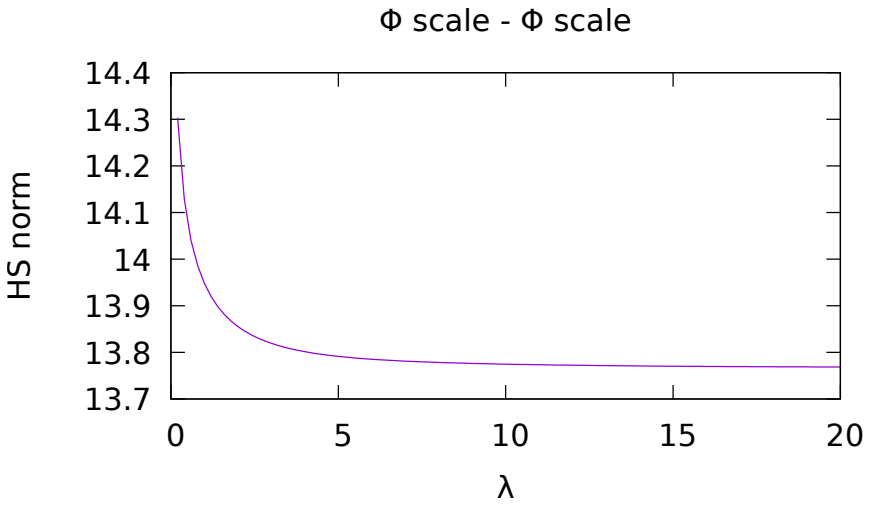

Figure 6.1: Hilbert-Schmidt norm: Φ scale - Φ scale

The Hilbert-Schmidt norms of the coefficient matrices involving only scaling function fields or only wavelet fields evolve to non-zero values. The norms of the coupling terms, however, evolve to zero as expected.

Figures 6.9-6.12 are a graphical representation of the coefficient matrices. They correspond to flow parameters $\lambda = 0$ (the initial coefficient matrix), $\lambda = 0.2$ (the

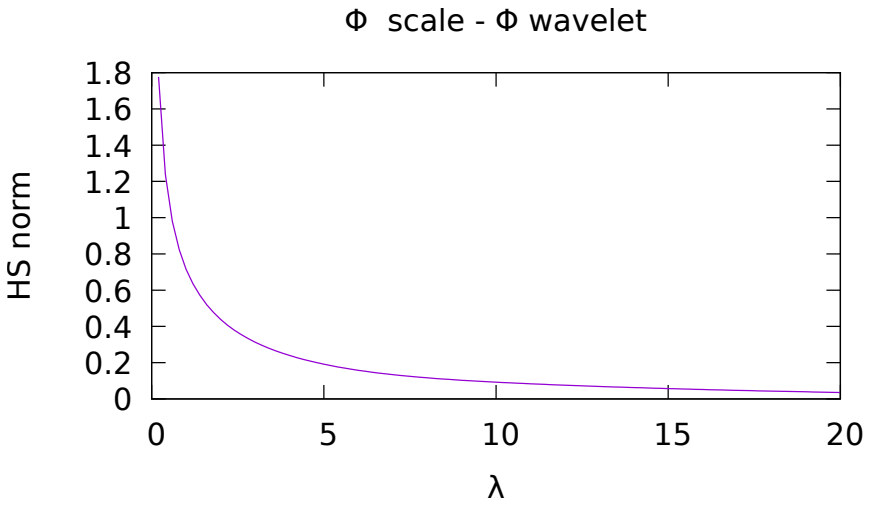

Figure 6.2: Hilbert-Schmidt norm: Φ wavelet - Φ wavelet

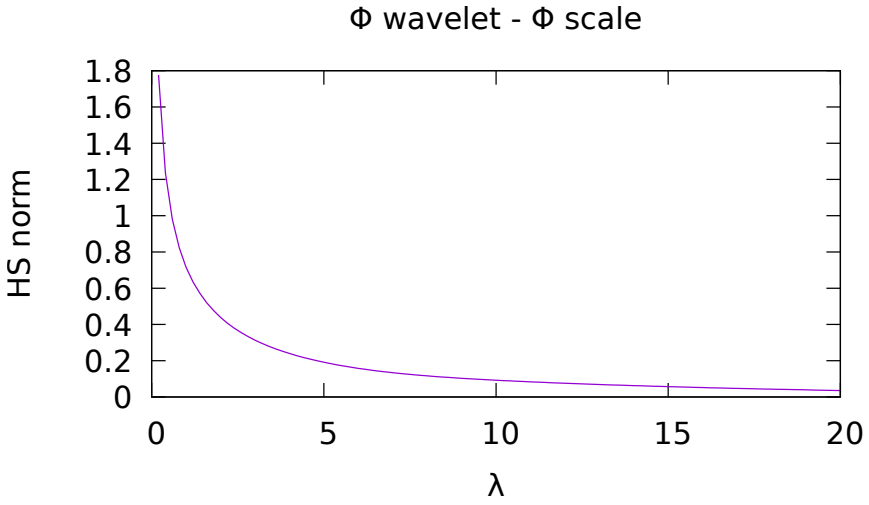

Figure 6.3: Hilbert-Schmidt norm: Φ wavelet - Φ scale

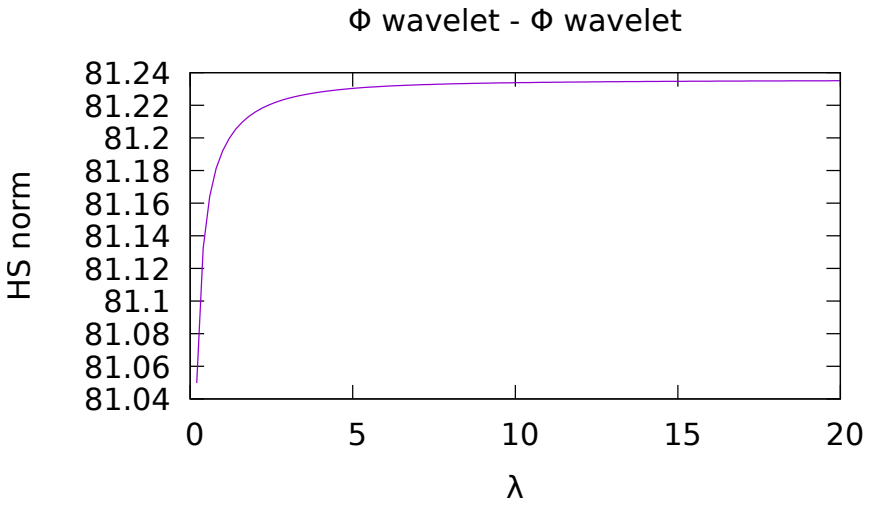

Figure 6.4: Hilbert-Schmidt norm: Φ scale - Φ wavelet

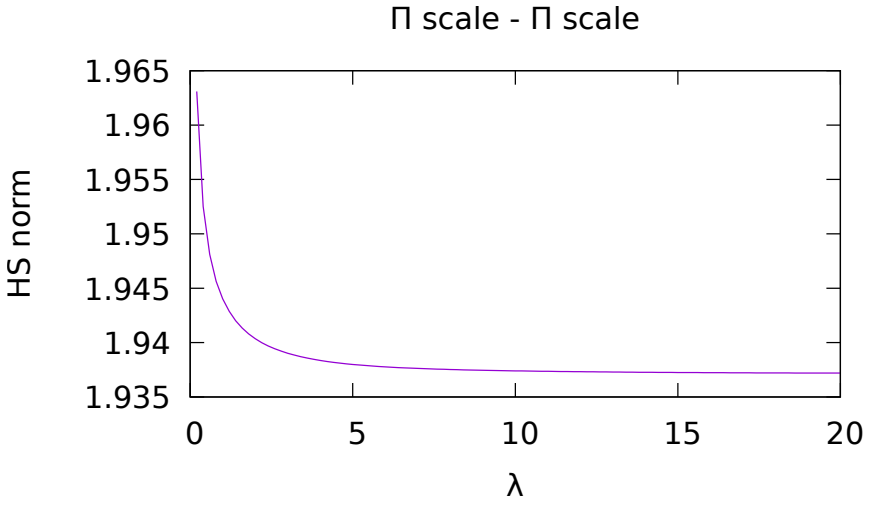

Figure 6.5: Hilbert-Schmidt norm: Π scale - Π scale

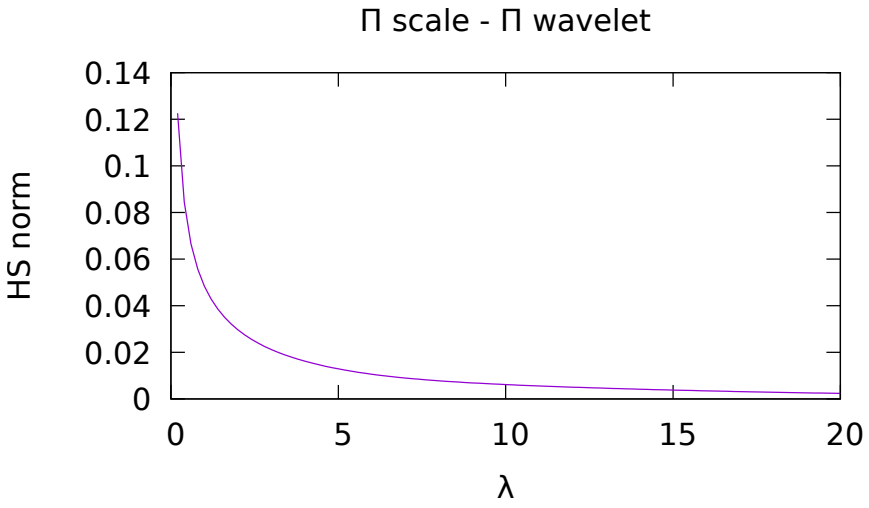

Figure 6.6: Hilbert-Schmidt norm: Π wavelet - Π wavelet

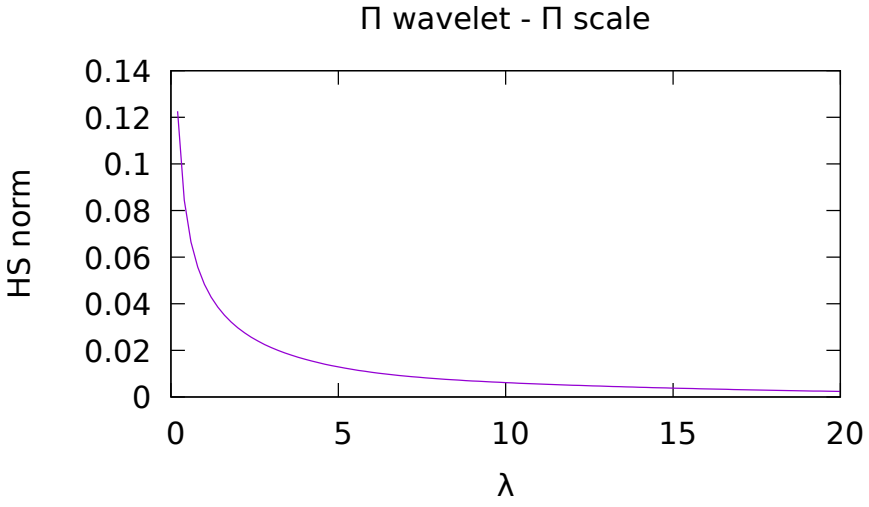

Figure 6.7: Hilbert-Schmidt norm: Π wavelet - Π scale

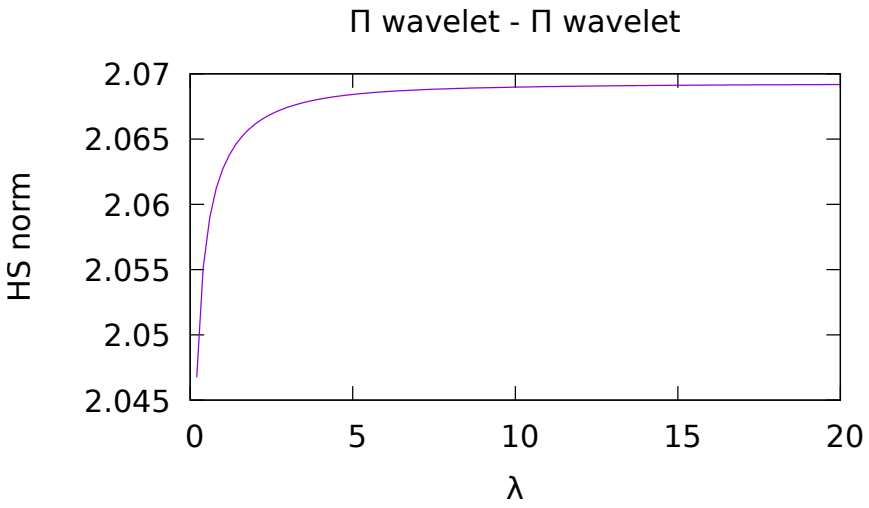

Figure 6.8: Hilbert-Schmidt norm: Π scale - Π wavelet

coefficient matrix after 200 iterations), $\lambda = 2.0$ (the coefficient matrix after 2000 iterations) and $\lambda = 20.0$ (the coefficient matrix after 20000 iterations).

These figures should be thought of as four by four block matrices, with each block consisting of a 16 by 16 matrix. The placement corresponds to

$$\begin{bmatrix} \Pi_s\Pi_s & \Pi_s\Phi_s & \Pi_s\Pi_w & \Pi_s\Phi_w \\ \Phi_s\Pi_s & \Phi_s\Phi_s & \Phi_s\Pi_w & \Phi_s\Phi_w \\ \Pi_w\Pi_s & \Pi_w\Phi_s & \Pi_w\Pi_w & \Pi_w\Phi_w \\ \Phi_w\Pi_s & \Phi_w\Phi_s & \Phi_w\Pi_w & \Phi_w\Phi_w \end{bmatrix} \tag{6.16}$$

Figure 6.10 shows that initially, the flow generates off-diagonal terms. As the flow evolves the size of these terms is suppressed. This is evident in figure 6.11 where the coupling terms have mostly disappeared. By the time $\lambda = 20$ in figure 6.12, the coupling terms are not visible at all.

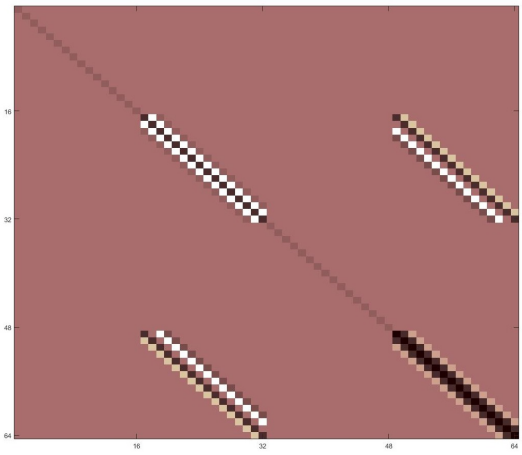

Figure 6.9: Initial Matrix, $\lambda = 0$

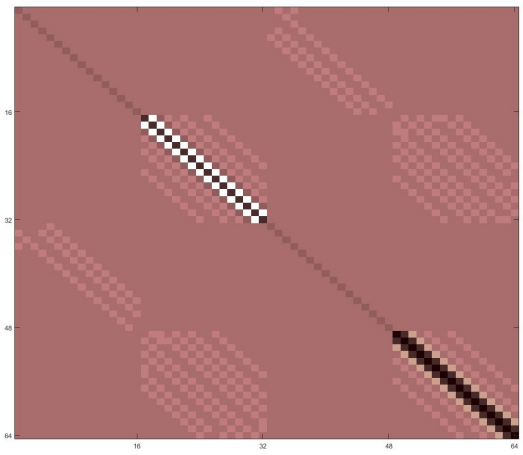

Figure 6.10: 200 iterations, $\lambda = 0.2$

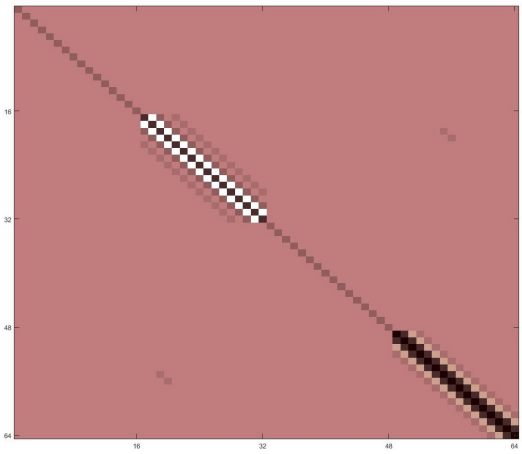

Figure 6.11: 2000 iterations, $\lambda = 2.0$

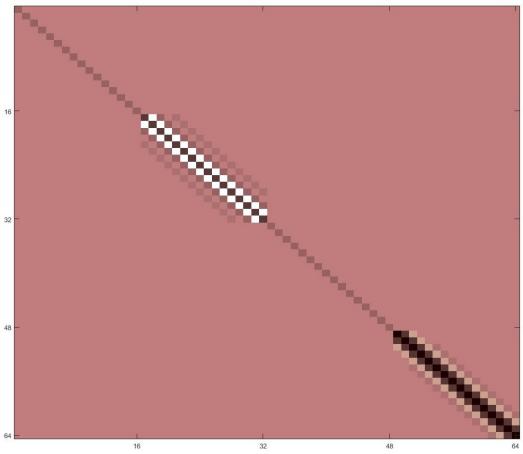

Figure 6.12: 20000 iterations, $\lambda = 20.0$

6.3 Analysis

The calculations in the previous section demonstrated that in the free field case the flow equation methods approximately block diagonalized the truncated Hamiltonian by scale. In order to understand how the solution of the block diagonal Hamiltonian responds to change in volume, resolution and mass the Hamiltonian is exactly diagonlized. This in turn determines how the solution of the block diagonal Hamiltonian responds to these changes.

This is begun by writing the truncated Hamiltonian in matrix form

$$H = \frac{1}{2}\Big[(\Pi^s, \Pi^w)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \Pi^s \\ \Pi^w \end{pmatrix} + (\Phi^s, \Phi^w)\begin{pmatrix} \mu^2 I + D_{ss} & D_{sw} \\ D_{ws} & \mu^2 I + D_{ww} \end{pmatrix}\begin{pmatrix} \Phi^s \\ \Phi^w \end{pmatrix}\Big] \quad (6.17)$$

where the upper components represent the scaling function fields and the lower represent the wavelet fields. The matrix

$$M := \begin{pmatrix} \mu^2 I + D_{ss} & D_{sw} \\ D_{ws} & \mu^2 I + D_{ww} \end{pmatrix} \quad (6.18)$$

is a real, symmetric matrix, which means that it can be diagonalized by a real orthogonal matrix O:

$$O^T M O = \begin{pmatrix} m^s & 0 \\ 0 & m^w \end{pmatrix} \quad (6.19)$$

where m^s and m^w are diagonal matrices consisting of the eigenvalues of the matrix M.

The transformed discrete fields are defined by

$$\begin{pmatrix} \Pi'^s \\ \Pi'^w \end{pmatrix} := O\begin{pmatrix} \Pi^s \\ \Pi^w \end{pmatrix} \quad (6.20)$$

and

$$\begin{pmatrix} \Phi'^s \\ \Phi'^w \end{pmatrix} := O\begin{pmatrix} \Phi^s \\ \Phi^w \end{pmatrix} \quad (6.21)$$

Since O is orthogonal, the transformed fields satisfy the canonical commutation relations. From the Stone-Von Neumann uniqueness theorem this transformation of field operators can be implemented by a unitary operator U:

$$U \begin{pmatrix} \Pi^s \\ \Pi^w \end{pmatrix} U^\dagger = \begin{pmatrix} \Pi'^s \\ \Pi'^w \end{pmatrix} = O \begin{pmatrix} \Pi^s \\ \Pi^w \end{pmatrix} \tag{6.22}$$

and

$$U \begin{pmatrix} \Phi^s \\ \Phi^w \end{pmatrix} U^\dagger = \begin{pmatrix} \Phi'^s \\ \Phi'^w \end{pmatrix} = O \begin{pmatrix} \Phi^s \\ \Phi^w \end{pmatrix} \tag{6.23}$$

Applying this transformation to the truncated Hamiltonian, transforms it into the sum of uncoupled harmonic oscillator Hamiltonians, where the squares of oscillator frequencies are exactly the eigenvalues of the matrix M:

$$H' = UHU^\dagger = \frac{1}{2}\Big[(\Pi^s, \Pi^w) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \Pi^s \\ \Pi^w \end{pmatrix} + (\Phi^s, \Phi^w) \begin{pmatrix} m^s & 0 \\ 0 & m^w \end{pmatrix} \begin{pmatrix} \Phi^s \\ \Phi^w \end{pmatrix} \Big] \tag{6.24}$$

or

$$H = \frac{1}{2}\sum \Pi^s_n \Pi^s_n + \frac{1}{2}\sum m^s_n \Phi^s_n \Phi^s_n + \frac{1}{2}\sum \Pi^w_n \Pi^w_n + \frac{1}{2}\sum m^w_n \Phi^w_n \Phi^w_n \tag{6.25}$$

The ground state of the truncated Hamiltonian is annihilated by the annihilation operators

$$a^s_n := \frac{1}{\sqrt{2}(m^s_n)^{1/4}} \sum_j (\sqrt{m^s_n}\Phi^s_n + i\Pi^s_n) \tag{6.26}$$

and

$$a^w_n := \frac{1}{\sqrt{2}(m^w_n)^{1/4}} \sum_j (\sqrt{m^w_n}\Phi^w_n + i\Pi^w_n) \tag{6.27}$$

where m^s_n and m^w_n are eigenvalues of the diagonal matrices m^s and m^w. It follows that the unitary operator U does a complete diagonalization of the truncated Hamiltonian.

The transformation O is not unique, since permutations of the columns of O permute the eigenvalues. So the identification of a given oscillator frequency with a wavelet or a scaling function degree of freedom depends on the choice of O.

Observing that the truncated system is equivalent to a system of uncoupled oscillators is important. This is because it is possible to understand how the solutions respond to changes in volume, scale and mass. Increasing the volume increases the number of oscillators. It should introduce additional modes with longer wavelength or lower oscillator frequencies. Changing the resolution changes the matrix M whose eigenvalues are squares of the normal mode frequencies. M has the form $M = \mu^2 I + D$. The D matrix is a positive symmetric matrix, and is the only part of M effected by a scale change. For every factor of 2 the scale is changed by, the D goes up by 4, i.e. $D \to 4^k D$. So the eigenvalues go from the form of $m_i \to \mu^2 + \lambda_i$ to $m_i \to \mu^2 + 4^k \lambda_i$. Thus doubling the resolution increases the separation of the square of the oscillator frequencies by a factor of 4. The mass factor sets the lower bound on the normal mode frequencies.

In the flow equation case for $\mu = 1$ and $\lambda = 20$ the coefficients d_{ssmn} and d_{wwmn} are within about 2% of their initial values. This means that the square of the normal mode frequencies of the scaling block diagonal Hamiltonian are approximately eigenvalues of the matrix $2d_{ssmn}(\lambda)$ for $\lambda = 20$. These are compared to the 32 squares of the normal mode frequencies of the full truncated Hamiltonian.

Table 6.1 shows that the scaling block at $\lambda = 20$ contains the 16 lowest frequency oscillators, though they are not coupled in this block. These can also be

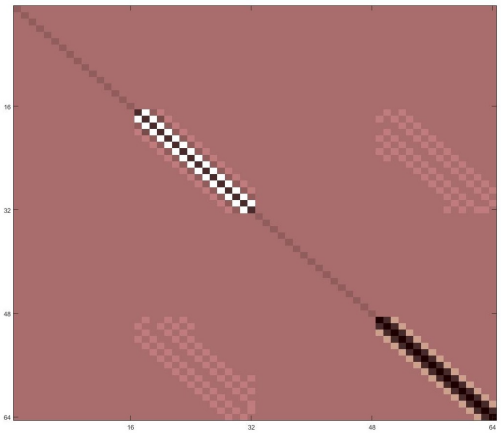

Figure 6.13: Matrix: $m = 0$, $\lambda = 0.2$

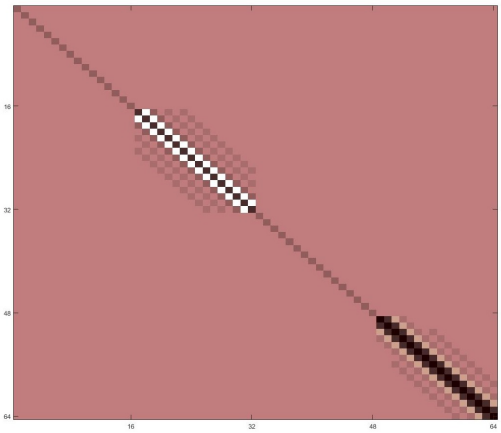

Figure 6.14: Matrix: $m = 0$, $\lambda = 20$

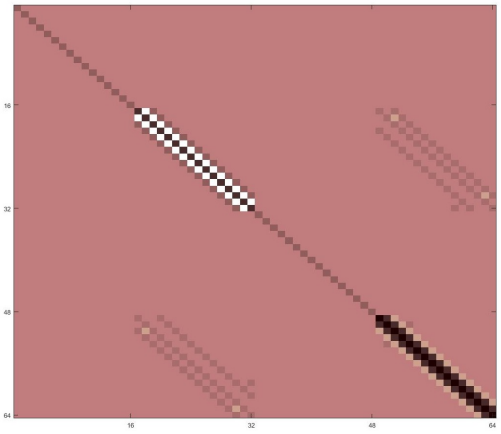

Figure 6.15: Matrix: $m = 4$, $\lambda = 0.2$

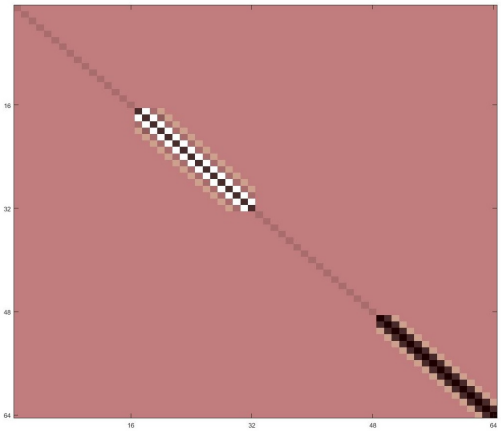

Figure 6.16: Matrix: $m = 4$, $\lambda = 20$

Table 6.1: Normal Mode Frequencies

$\lambda = 20, \mu = 1$	truncated	exact 1:16	exact 17:32
1.037e+00	1.037e+00	1.041e+00	1.665e+01
1.145e+00	1.146e+00	1.153e+00	1.925e+01
1.326e+00	1.333e+00	1.340e+00	2.208e+01
1.583e+00	1.609e+00	1.604e+00	2.512e+01
1.919e+00	1.995e+00	1.947e+00	2.834e+01
2.341e+00	2.525e+00	2.373e+00	3.167e+01
2.861e+00	3.236e+00	2.890e+00	3.507e+01
3.493e+00	4.161e+00	3.508e+00	3.846e+01
4.263e+00	5.317e+00	4.243e+00	4.178e+01
5.201e+00	6.689e+00	5.112e+00	4.495e+01
6.346e+00	8.232e+00	6.134e+00	4.789e+01
7.722e+00	9.859e+00	7.332e+00	5.053e+01
9.309e+00	1.145e+01	8.729e+00	5.279e+01
1.102e+01	1.289e+01	1.034e+01	5.462e+01
1.274e+01	1.403e+01	1.219e+01	5.597e+01
1.435e+01	1.476e+01	1.429e+01	5.679e+01

approximated by diagonalization M because the $\Pi - \Pi$ coefficients are approximated as $\frac{1}{2}\delta_{mn}$. The flow is unitary, which implies that the wavelet block has the 16 highest frequency oscillators. This shows that the block diagonalization achieved by the flow equation separates both distance and energy scales. It is worth noting that both blocks consist of unbounded operators; it is ground state energies that are separated by scale.

The figures in section 6.2 were all created using a mass value of $\mu = 1$. While the role of the mass is involved in the determination of the normal mode frequencies, it is unknown how the mass impacts the convergence of the flow equation. This was tested by solving the flow equation for a mass $\mu = 0$ and a mass $\mu = 4$ for $\lambda = 0.2$

and $\lambda = 20.0$. The results for $\lambda = 0.2$ are shown in figures 6.13 and 6.15 and look similar to figure 6.10. The results for $\lambda = 20.0$ are shown in figures 6.14 and 6.16 and look similar to figure 6.12.

CHAPTER 7
SUMMARY CONCLUSION AND OUTLOOK

7.1 Conclusions

In this thesis a basis of Daubechies scaling functions and wavelets was used to decompose quantum fields into an infinite linear combination of operators that are sensitive to physics on different scales. The expansion of the fields in this basis was used to define truncated field operators that included degrees of freedom on two different distance scales. The truncated fields were inserted in the free field Hamiltonian to construct a Hamiltonian with degrees of freedom on the two scales.

The theory was reformulated in terms of degrees of freedom determined by experiment by eliminating smaller scale degrees of freedom. The problem of removing all smaller scales is difficult, however it is expected that it will eventually be understood as a limit of theories that eliminate successively smaller degrees of freedom. A key step in formulating such a limit is to understand the problem of how to decouple degrees of freedom. The flow equation generates a family of unitary operators that depend on a continuous parameter λ. The generator of the flow is chosen to evolve the Hamiltonian to a block diagonal form. The free field example was used because it has a non-trivial coupling between scales but integrating the flow equation does not generate an infinite number of new operators. Also the free field is exactly solvable for the exact and truncated theory allowing for a precise understanding of how the flow equation separates scales.

The generator of the flow equation was taken to be $[H(\lambda), H_b(\lambda)]$ where $H(\lambda)$ is the evolved Hamiltonian and $H_b(\lambda)$ the block diagonal portion of $H(\lambda)$. The calculation demonstrated that this generator evolves the Hamiltonian to the desired block diagonal form. This was tested by numerically solving the flow equation. An initial attempt using perturbation theory failed to converge. Convergence was achieved using the Euler method, which uses the differential equation at the current step to proceed to the next step. While the step size is a dimensional parameter, with units fixed by the mass, it was determined that for a mass of $\mu = 1$ a step size of .001 was sufficient for convergence. The equation was integrated for flow values of λ between 0 and 20. The Hilbert-Schmidt norms of the matrices that coupled the different scales decreased as the flow parameter increased. All of the discrete field operators were examined for the following parameters of λ: 0, 0.2, 2.0 and 20.0. The graphical representation of the coefficients of the operators in the evolved Hamiltonian showed most of the coupling had vanished by $\lambda = 2.0$. And while each iteration of the flow equation formally delocalizes the Hamiltonian, the figures show the evolved Hamiltonian is close to diagonal. This means that the block diagonal Hamiltonian is still dominated by local terms.

The truncated Hamiltonian is equivalent to the Hamiltonian for a finite system of coupled harmonic oscillators. Those oscillators have a finite number of normal mode frequencies. In the block diagonal representation the normal modes are divided between the two blocks. The normal mode frequencies of the coarse scale block Hamiltonian were calculated at $\lambda = 20.0$. It was found that the flow equation put

the lowest normal mode frequencies in the coarse scale block and the highest normal mode frequencies in the fine scale block, which is the desired result of a Hamiltonian block diagonalized by scale. A general unitary transformation that block diagonalizes the truncated Hamiltonian can divide normal modes among the blocks in $(2n)!/(n!)^2$ different ways, where $2n$ is the total number of normal modes. It is reassuring that the flow equation distributes the normal modes by scale.

7.2 Outlook

This test simplified a lot of the physics. The test calculations were in one space and one time dimension. An extension to three dimensions is needed. It would consist of more complicated bookkeeping, but otherwise it should be straight forward.

The more serious difficulty is that realistic theories have additional interactions that were avoided by using free field theory. The difficulty with these interactions is that each iteration of the flow equation generates new classes of operators. For $\lambda = 20.0$, 20,000 iterations were used. It is not numerically feasible to solve the flow equation, retaining all of the generated interactions. Another difficulty is that it is not obvious that the generator $[H(\lambda), H_b\lambda)]$ used in the free field case will separate scales in the interacting case.

Future progress would require an investigation of both how to choose a suitable flow generator and how to control the infinite number of operators generated in the interacting case. The obvious next step would be to study solvable models in one space - one time dimension that have non-trivial interactions.

APPENDIX A
THE DERIVATIVE MATRICES

In this appendix the application of renormalization group methods to compute the integrals of products of derivatives that appear in equations (6.6-6.8) are discussed. These integrals were computed in [18]. In this appendix a different method due to Beylkin [12] is used.

The following identities are used in the equations below:

$$\int s_n^k(x)dx = \frac{1}{\sqrt{2^k}} \tag{A.1}$$

$$RT^{2k} = T^k R \tag{A.2}$$

$$\frac{d}{dx}R = 2R\frac{d}{dx} \tag{A.3}$$

$$Rx = 2xR \tag{A.4}$$

$$Tx = (x-1)T \tag{A.5}$$

Combining these with the scaling equation and the definition of the mother wavelet gives:

$$s_m^k(x) = \sum h_{m-2n} s_n^{k+1}(x) \tag{A.6}$$

$$w_m^k(x) = \sum g_{n-2m} s_n^{k+1}(x) \tag{A.7}$$

$$\frac{d}{dx}s_m^k(x) = 2\sum h_{m-2n}\frac{d}{dx}s_n^{k+1}(x) \tag{A.8}$$

$$\frac{d}{dx}w_m^k(x) = 2\sum g_{n-2m}\frac{d}{dx}s_n^{k+1}(x) \tag{A.9}$$

These equations can be used to express each derivative matrix, $D^k_{s,mn}$, $D^{kl}_{w,nm}$, $D^{kl}_{sw,mn}$, and $D^{lk}_{ws,mn}$ in terms of $D^0_{s,0n}$, where n is an integer. These calculations are done in detail in subsections A.1 through A.3.

For $K = 3$ the support properties imply that if n is not between -4 and 4, then $D^0_{s,0n}$ is zero. So there are nine non-zero coefficients

$$D^0_{s,0m} = \int s'(x)s'(x-m)dx \qquad -4 \le m \le 4 \tag{A.10}$$

Letting $x' = x - m$ gives $D^0_{s,0m} = D^0_{s,0(-m)}$ leading to only five integrals that need to be evaluated. These integrals can be calculated using the scaling equation

$$s(x) = \sum_{n=0}^{2K-1} h_n R T^n s(x) \tag{A.11}$$

Differentiating the above scaling equation gives the following renormalization group equation for the derivatives

$$s'(x-m) = 2\sqrt{2}\sum_l h_l s'(2x-2m-l) \tag{A.12}$$

Using (A.10) and (A.12) gives homogeneous equations relating the non-zero $D^0_{s,0m}$:

$$D^0_{s,0m} = 4\sum_{ln} h_l h_{l+n} D^0_{s,0(2m+n)} \tag{A.13}$$

The coefficients

$$a_n := 2\sum_l h_l h_{l+n} \tag{A.14}$$

are called the auto-correlation coefficients. The orthogonality constraints on translates of the scaling function can be used to show that $a_0 = 2$ and $a_{2n} = 0$ for all

$n \neq 0$. The auto-correlation coefficients for odd n are rational. The $K = 3$ values for h_l from table 3.1 are used to calculate the relevant auto-correlation coefficients.

$$a_0 = 2 \qquad a_1 = a_{-1} = \frac{75}{64} \qquad a_3 = a_{-3} = -\frac{25}{128} \qquad a_5 = a_{-5} = \frac{3}{128} \tag{A.15}$$

Using the auto-correlation coefficients, the homogeneous equation (A.13) can be expressed as:

$$D^0_{s,0m} = \sum 2a_n D_{s,0(2m+n)} \tag{A.16}$$

To solve for $D^0_{s,0m}$ an inhomogeneous equation is needed. To derive this equation the property that for $K = 3$, 1, x and x^2 can be pointwise expressed as locally finite expansions in the scaling functions is used. These equations have the form

$$1 = \sum_n s_n(x) \tag{A.17}$$

$$x = \sum_n (n + < x >) s_n(x) \tag{A.18}$$

and

$$x^2 = \sum_n (n^2 + 2n < x > + < x^2 >) s_n(x) \tag{A.19}$$

where $< x^n > = \int s(x) x^n dx$ are the moments of the scaling function. While the moments can also be calculated analytically, their values are not needed in what follows. Differentiating (A.18) using (A.17) gives

$$1 = \sum_n n s'_n(x) \tag{A.20}$$

Differentiating (A.19) using (A.18) gives

$$2x = \sum_n (n^2 + 2n < x >) s'_n(x) = \sum_n n^2 s'_n(x) + 2 < x > \tag{A.21}$$

Multiplying by $s'(x)$ and integrating gives

$$\int 2xs'(x) = -2 = \sum_n n^2 \int s'_n(x)s'(x)dx + 2 < x > \int s'(x)dx = \sum_n n^2 \int s'_n(x)s'(x)dx \tag{A.22}$$

This gives the inhomogeneous equation

$$\sum_n n^2 D^0_{s,n0} = -2 \tag{A.23}$$

The linear system of equations (A.16) and (A.23) has rational coefficients, so can be solved exactly for rational solutions in table A.1.

Table A.1: Starting Derivative Matrix Values

	Value
$D_{s,0(-4)}$	$-3/560$
$D_{s,0(-3)}$	$-4/35$
$D_{s,0(-2)}$	$92/105$
$D_{s,0(-1)}$	$-356/105$
$D_{s,00}$	$295/56$
$D_{s,01}$	$-356/105$
$D_{s,02}$	$92/105$
$D_{s,03}$	$-4/35$
$D_{s,04}$	$-3/560$

A.1 Scale Derivative Matrices

For scale 2^{-k} resolution scaling functions, the $D^k_{s,mn}$ values can be expressed in terms of the nine matrix elements in table A.1.

$$
\begin{aligned}
D^k_{s,mn} &= \int dx \frac{d}{dx} s^k_m(x) \frac{d}{dx} s^k_n(x) \\
&= \int dx \frac{d}{dx} s^k(x-m) \frac{d}{dx} s^k(x-n) \\
&= \int dx \frac{d}{dx} s^k(x) \frac{d}{dx} s^k(x-n+m) \\
&= \int dx \frac{d}{dx} R^k s(x) \frac{d}{dx} R^k s(x-n+m) \\
&= 2^{2k} \int dx R^k \frac{d}{dx} s(x) R^k \frac{d}{dx} s(x-n+m) \\
&= 2^{2k} D^0_{0,n-m}
\end{aligned} \tag{A.24}
$$

So if letting $l = n - m$ gives

$$
D^k_{s,mn} = 2^{2k} D_{s,0l} \tag{A.25}
$$

A.2 Wavelet Derivative Matrices

The wavelet coefficient matrix $D^{ll}_{w,mn}$ can also be expressed in terms of the scaling $D^0_{s,0n}$ using the scaling equation

$$
\frac{d}{dx} w^k_m(x) = \sum g_l \frac{d}{dx} s^{k+1}_{l+2m}(x) \tag{A.26}
$$

Then

$$
\begin{aligned}
D^{ll}_{w,mn} &= \int dx \sum_{l=0}^{2K-1} g_l 2^{k+1} R^{k+1} T^{l+2m} \frac{d}{dx} s(x) \sum_{j=0}^{2K-1} g_j 2^{k+1} R^{k+1} T^{j+2n} \frac{d}{dx} s(x) \\
&= 2^{2k+2} \sum_{l=0}^{2K-1} g_l \sum_{j=0}^{2K-1} g_j \int dx T^{l+2m} \frac{d}{dx} s(x) T^{j+2n} \frac{d}{dx} s(x) \\
&= 2^{2k+2} \sum_{l=0}^{2K-1} g_l \sum_{j=0}^{2K-1} g_j \int dx \frac{d}{dx} s(x-l-2m) \frac{d}{dx} s(x-j-2n)
\end{aligned}
$$

$$= 2^{2k+2} \sum_{l=0}^{2K-1} g_l \sum_{j=0}^{2K-1} g_j \int dx \frac{d}{dx} s(x) \frac{d}{dx} s(x - j - 2n + l + 2m)$$
$$= 2^{2k+2} \sum_{l=0}^{2K-1} g_l \sum_{j=0}^{2K-1} g_j D^0_{s,0(j+2n-l-2m)} \tag{A.27}$$

A.3 Combined Derivative Matrices

It is also possible to express the integrals of the scale coupling matrix, $D^{kk}_{b,mn}$, in terms of the scaling $D^0_{s,0n}$, giving

$$\begin{aligned} D^{kk}_{b,mn} &= 2 \int dx \frac{d}{dx} w^k_m(x) \frac{d}{dx} s^k_n(x) \\ &= 2 \int dx \sum_{l=0}^{2K-1} g_l 2^{k+1} R^{k+1} T^{l+2m} \frac{d}{dx} s(x) \sum_{j=0}^{2K-1} h_j 2^{k+1} R^{k+1} T^{j+2n} \frac{d}{dx} s(x) \\ &= 2^{2k+3} \sum_{l=0}^{2K-1} g_l \sum_{j=0}^{2K-1} h_j \int T^{l+2m} \frac{d}{dx} s(x) T^{j+2n} \frac{d}{dx} s(x) \\ &= 2^{2k+3} \sum_{l=0}^{2K-1} g_l \sum_{j=0}^{2K-1} h_j \int dx \frac{d}{dx} s(x - l - 2m) \frac{d}{dx} s(x - j - 2n) \\ &= 2^{2k+3} \sum_{l=0}^{2K-1} g_l \sum_{j=0}^{2K-1} h_j \int dx \frac{d}{dx} s(x) \frac{d}{dx} s(x - j - 2n + l + 2m) \\ &= 2^{2k+3} \sum_{l=0}^{2K-1} g_l \sum_{j=0}^{2K-1} h_j D^0_{s,0(j+2n-l-2m)} \end{aligned} \tag{A.28}$$

All together these equations can be used to express any combination of products of the derivatives of wavelets and scaling basis functions in terms of the nine numbers $D^0_{s,0n}$, for $-4 < n < 4$.

APPENDIX B
THE COMMUTATOR

In order to solve the flow equation it is necessary to compute the double commutator of operators that are quadratic in the discrete fields.

To compute the double commutator start with the two quadratic functions of the field operators:

$$A = a_{ij}\Pi_i\Pi_j + b_{ij}\Pi_i\Phi_j + c_{ij}\Phi_i\Pi_j + d_{ij}\Phi_i\Phi_j \tag{B.1}$$

and

$$B = e_{kl}\Pi_k\Pi_l + f_{kl}\Pi_k\Phi_l + g_{kl}\Phi_k\Pi_l + h_{kl}\Phi_k\Phi_l \tag{B.2}$$

where a_{ij}, b_{ij} etc for fixed i and j (or k and l) are constant 16×16 matrices that taken together compose a 64×64 matrix. The indices $i, j, k, l \in \{s, w\}$ where s represents the scaling-function terms, and w represents the wavelet terms.

The block matrices are represented either with Π's and Φ's or with the coefficients with so that

$$\begin{bmatrix} \Pi_s\Pi_s & \Pi_s\Phi_s & \Pi_s\Pi_w & \Pi_s\Phi_w \\ \Phi_s\Pi_s & \Phi_s\Phi_s & \Phi_s\Pi_w & \Phi_s\Phi_w \\ \Pi_w\Pi_s & \Pi_w\Phi_s & \Pi_w\Pi_w & \Pi_w\Phi_w \\ \Phi_w\Pi_s & \Phi_w\Phi_s & \Phi_w\Pi_w & \Phi_w\Phi_w \end{bmatrix} \tag{B.3}$$

corresponds to

$$\begin{bmatrix} a_{ss} & b_{ss} & a_{sw} & b_{sw} \\ c_{ss} & d_{ss} & c_{sw} & d_{sw} \\ a_{ws} & b_{ws} & a_{ww} & b_{ww} \\ c_{ws} & d_{ws} & c_{ww} & d_{ww} \end{bmatrix} \tag{B.4}$$

and

$$\begin{bmatrix} e_{ss} & f_{ss} & e_{sw} & f_{sw} \\ g_{ss} & h_{ss} & g_{sw} & h_{sw} \\ e_{ws} & f_{ws} & e_{ww} & f_{ww} \\ g_{ws} & h_{ws} & g_{ww} & h_{ww} \end{bmatrix} \tag{B.5}$$

The equation

$$[ij,kl] = i[j,k]l + [i,k]jl + ki[j,l] + k[i,l]j \tag{B.6}$$

is used with the following commutator relationships

$$[\Pi_i, \Pi_j] = 0 \qquad [\Phi_i, \Phi_j] = 0 \qquad [\Phi_i, \Pi_j] = i\delta_{ij} \tag{B.7}$$

to calculate $[A, B]$.

$$[A, B] = \sum (x_{ij}\Pi_i\Pi_j + y_{ij}\Pi_i\Phi_j + z_{ij}\Phi_i\Pi_j + w_{ij}\Phi_i\Phi_j) \tag{B.8}$$

There are 16 commutators to calculate. Two of them, $a_{ij}e_{kl}[\Pi_i\Pi_j, \Pi_k\Pi_l]$ and $d_{ij}h_{kl}[\Phi_i\Phi_j, \Phi_k\Phi_l]$ are trivial. Two of them create four terms. For example

$$a_{ij}h_{kl}[\Pi_i\Pi_j, \Phi_k\Phi_l] = -ia_{ij}h_{kl}(\Pi_i\Phi_l\delta_{jk} + \Pi_j\Phi_l\delta_{ik} + \Phi_k\Pi_i\delta_{jl} + \Phi_k\Pi_j\delta_{il}) \tag{B.9}$$

The other commutator that creates four terms is $d_{ij}e_{kl}[\Phi_i\Phi_j, \Pi_k\Pi_l]$. The other 12 create two terms, for example

$$a_{ij}f_{kl}[\Pi_i\Pi_j, \Pi_k\Phi_l] = -ia_{ij}f_{kl}(\Pi_i\Pi_l\delta_{jk} + \Pi_j\Pi_l\delta_{ik}) \tag{B.10}$$

Grouping like terms together and arranging the indices properly gives the following four expressions for the coefficients x_{ij}, y_{ij}, z_{ij} and w_{ij}:

$$x_{ij} = i \sum_{k\in\{s,w\}} (-f_{ik}a_{kj} - f_{ik}a_{kj} - a_{ik}g_{kj} - a_{ik}g_{kj} + b_{ik}e_{kj} + e_{ik}b_{kj} + c_{ik}e_{kj} + e_{ik}c_{kj}) \tag{B.11}$$

$$y_{ij} = i \sum_{k\in\{s,w\}} (-a_{ik}h_{kj} - a_{ik}h_{kj} + b_{ik}f_{kj} - f_{ik}b_{kj} + c_{ik}f_{kj} - f_{ik}c_{kj} + e_{ik}d_{kj} + e_{ik}d_{kj}) \tag{B.12}$$

$$z_{ij} = i \sum_{k\in\{s,w\}} (-h_{ik}a_{kj} - h_{ik}a_{kj} - b_{ik}g_{kj} + g_{ik}b_{kj} + g_{ik}c_{kj} - c_{ik}g_{kj} + d_{ik}e_{kj} + d_{ik}e_{kj}) \tag{B.13}$$

$$w_{ij} = i \sum_{k\in\{s,w\}} (-b_{ik}h_{jk} - b_{ik}h_{jk} - c_{ik}h_{k}j - c_{ik}h_{k}j + d_{ik}f_{kj} + d_{ik}f_{kj} + g_{ik}d_{kj} + g_{ik}d_{kj}) \quad \text{(B.14)}$$

Using these equations gives a commutator, i.e. $[A, B]$. Then e, f, g, h are replaced with x, y, z, w respectively to get the double commutator, i.e. $[A, [A, B]]$.

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