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Study of Anisotropic Fluid Distributed Hyperbolically in f(R,T,Q) Gravity

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Abstract: The core of this manuscript is to conduct a broad investigation into the features of static matter configurations with hyperbolical symmetry, which might possibly serve as formation of corresponding spacetime within the limits of $f(R,T,Q)$ gravity, where ($Q \equiv R_{\alpha\sigma}T^{\alpha\sigma}$). We recognize that such matter distributions can be anisotropic in pressure, with just two primary stresses unequal and a negative energy density. Usually, negative matter densities are suggested in extreme cosmological and astrophysical situations, particularly with regard to quantum occurrences that might occur within the horizon. Eventually, we construct a generic formalism that allows every static hyperbolically symmetric (*HS*) fluid solution to be expressed with respect to two generating functions (*GFs*).

Keywords: mathematical cosmology; gravitation; anisotropy; mathematical techniques

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1. Introduction

Nature has been experimenting with hyperbolic forms for hundreds of millions of years, though mathematicians have spent hundreds of years attempting to prove that such structures are basically impossible. However, as a result of these efforts, it was discovered that hyperbolic geometry is logically valid. This, in turn, sparked a revolution that resulted in the type of mathematics that now governs general relativity and, hence, the structure of our cosmos. Different geometries can also be understood in the context of their curvature. For instance, flat and spherical surfaces have zero and positive curvature, respectively, while the hyperbolic plane has a negative curvature [1]. Harrison was the first to propose a solution to Einstein equations of the sort specified by hyperbolic symmetry. Following that, several researchers embraced the concept, as one can see [2–4].

As a unique paradigm for topological states, Chen et al. [5] introduced and empirically demonstrated hyperbolic matter, which is composed of particles traveling in the hyperbolic plane with negative curvature. Richter [6] developed hyperbolic complex algebraic structures based on appropriately specified vector products and powers that enabled a number of formulations of the hyperbolic vector exponential function in a standard manner. By doing this, he altered arrow multiplication, which Feynman claimed to be essential for understanding quantum electrodynamics, and provided a geometrical justification for when it makes sense to create random vector products. Herrera [7] outlined the general characteristics of matter content that are dynamic, dissipative, and spherically symmetric. Prospective applications to astrophysical and cosmological settings are described along with a number of exact answers that are evaluated. Malik et al. [8] provided a few quantitative solutions that exhibit the distribution of hyperbolically symmetric (*HS*) matter along the cylindrical metric after solving the related differential equations.

Yousaf et al. [9] investigated the hyperbolically distributed anisotropic static and nonstatic fluids within the context of modified theories. They assessed a number of analytical solutions in terms of structure scalars. Assuming the nonstatic domain, Herrera et al. [10] performed a thorough analysis on the *HS* matter structure. For both dissipative and nondissipative systems, they examined a number of solutions utilizing quasi-homologous condition, the diminishing of complexity factor criterion, and additional restrictions. Miguel [11] looked into several characteristics of hyperbolically symmetrical spacetime, and for the conformal families of solutions, he discovered the nonconvex Cauchy temporal function.

Despite the fact that locally isotropic models are a relatively prevalent assumption in the study of compact objects, there is compelling evidence that a wide range of physical events that we would anticipate to occur in compact objects can actually generate local anisotropy for specific density ranges. These processes span from incredibly dense matter configurations, such as exotic phase transitions in the process of gravitational collapse, to very low ones, such as the processes that take place during stellar formation. According to Jeans' work [12], spherical galaxies with anisotropic velocity distributions can produce local anisotropies. In 1933, Lemaître [13] published an article in the general relativistic literature demonstrating how anisotropy might loosen the upper bound on the highest magnitude of the surface gravitational potential. In light of these pioneering publications, there has been extensive theoretical research on the impact of unequal stresses on Newtonian and relativistic systems [14–17].

For the anisotropic and homogeneous Kantowski–Sachs geometry, Leon and Paliathanasis [18] investigated the development of the cosmological field equations in the modified teleparallel $f(T,B)$ theory of gravity. They introduced a scalar field into the aforementioned fourth-order theory of gravity using a Lagrange multiplier so that we may formulate the field equations as second-order equations by enhancing the enumerates of dependent variables. Andrade [19] developed an entirely new static, spherical solution comprising an anisotropic matter content. He achieved this by taking into account a specific instance of the extended minimum geometric deformation within the context of gravitational decoupling. In order to solve the field equations consisting of anisotropic cosmic background geometries, Paliathanasis [20] used the Noether symmetry analysis on the $f(T,B)$ theory.

There are compelling theoretical arguments for carefully considering the idea that Einstein's general theory of relativity (GR) may not be the most accurate description of gravity. Initial efforts to renormalize GR in the 1960s and 1970s made it abundantly evident that counter terms must be included that fundamentally modify the theory and convert its second-order field equations to fourth order. Following this, $f(R)$ theories were developed. In these theories, the generic function of the Ricci scalar in the action function was used to include the higher-order invariants. Moreover, $f(R)$ theories show how nonminimal coupling connects geometry and matter. Some intriguing subjects have also been addressed in the limits of aforementioned gravity [21–24].

Yousaf et al. [25] studied the hyperbolic spacetime in the limit of $f(R)$ gravity. Bani et al. [26] analyzed the Lemaître–Tolman–Bondi (LTB) metric constituting dissipative matter content in the framework of Palatini gravity. Odintsov et al. [27] investigated the issue of the primordial gravitational wave systematically. They selected $f(R)$ gravity model in such a manner that permits the descriptions for the unification of inflation as well as dark energy. After that, the generalization in the $f(R)$ theories was constructed to account for the exotic imperfect fluids or quantum effects as well as matter–geometry coupling that is not minimal. Therefore, the $f(R,T)$ theory, in which T stands for the energy-momentum tensor's trace, was named by Harko et al. [28]. The assumption of $T=0$ in $f(R,T)$ theory yields $f(R)$, and hence the effects of strong minimal coupling vanishes. Gonçalves et al. [29] looked at the probability of abrupt singularities developing in Friedmann–Lemaître–Robertson–Walker (FLRW) comprisingisotropic fluid in the geometrical as well as the scalar-tensor representations within the limits of $f(R,T)$ gravity. Bhatti et al. [30] addressed dynamical instability constituting axially symmetric fluid content and discovered some significant effects of the astrophysical $f(R,T)$ model.

This encouraged Haghani et al. [31] in further generalization. Subsequently, the $f(R,T,Q)$ theory ($Q\equiv R_{\alpha\sigma}T^{\alpha\sigma}$), which connects the contraction of the stress energy-momentum tensor with Ricci tensor, came into being. Henceforth, it could be necessary to modify the gravity theory itself for explaining the extra gravity that is typically assigned to dark matter. Baffou et al. [32] conducted the stability analysis in this gravitational theory and derived the perturbation function for it. Moreover, by numerically calculating the resulting perturbation functions, they looked for applications that make the stability of two specific examples of the theoretical model possible. In f(R,T,Q) theory, Elizalde and Vacaru [33] examined the circumstances in which a large group can be effectively converted into off-diagonal Einstein spaces in the light of nonholonomic deformations and restrictions for the nonlinear dynamics of matter and gravity. Yousaf et al. [34] conducted an extensive study into the distinguishing characteristics of compact objects that are alternatives to black holes using cylindrical symmetry in the context of f(R,T,Q) gravity. They performed this work both for charged and noncharged compact objects.

The subject of this paper is to apply the methodology first described by Herrera et al. [3] to investigate the impact of f(R,T,Q) gravity on *HS* sources. After discussing the general formalism of the respective modified gravity in Section 2, the modified field equations are evaluated in Section 3. Sections 4 and 5 deal with the conformal scalar and Tolman mass, respectively. The computation of structural scalars from the Curvature tensor's decomposition is the focus of Section 6. Different *HS* solutions characterized with various models and constraints are investigated in Section 7. Our findings and discussions are summed up in the last section.

2. Basic Formalism of the f(R,T,Q) Gravity

In order to handle the major issues in cosmology that anticipate the fate of our cosmos, modified gravity has emerged as an effective candidate. Among the modified theories that are based on nonminimal connection between matter and geometry, the f(R,T,Q) gravity is an intriguing contender. This theory is motivated by the idea that it can be perceived as a useful mathematical exercise for the investigation of the present state of our enigmatic cosmos due to the involvement of comprehensive version of f(R,T) gravity. In this theory, the contribution from the contraction of the Ricci and the energy-momentum tensors is included in addition to the dependence of Lagrangian on R and T. Next, we provide the general formalism to derive field equations in the background of f(R,T,Q) gravity utilizing different line elements. The first step is to consider the modified Einstein–Hilbert action as

$$S = \frac{1}{2\kappa}\left(\int d^4x\,\mathtt{f}(\mathtt{R},\mathtt{T},\mathtt{R}_{\sigma\alpha}\mathtt{T}^{\sigma\alpha})\sqrt{-g} + \int d^4x\mathrm{L}_m\sqrt{-g}\right), \tag{1}$$

where $\mathtt{R}_{\sigma\alpha}$ and $\mathtt{T}^{\sigma\alpha}$ depict the Ricci tensor and the energy-momentum tensor, while the symbols R and T illustrate their respective traces. The terms κ, L_m, and $\sqrt{-g}$ are used to indicate coupling constant, matter Lagrangian, and the magnitude of metric tensor, respectively. By varying Equation (1) with respect to $g_{\sigma\alpha}$, one can achieve

$$\begin{aligned} &- \quad G_{\sigma\alpha}(f_Q L_m - f_R) - g_{\sigma\alpha}\left\{\frac{f}{2} - \Box f_R - \frac{R}{2}f_R - \frac{1}{2}\nabla_\beta\nabla_\rho(f_Q T^{\beta\rho}) - L_m f_T\right\} \\ &+ \quad 2f_Q R_{\beta(\sigma}T^{\beta}_{\alpha)} + \frac{1}{2}\Box(f_Q T_{\sigma\alpha}) - \nabla_\beta\nabla_{(\sigma}(T^{\beta}_{\alpha)}f_Q) - 2(f_T g^{\beta\rho} + f_Q R^{\beta\rho})\frac{\partial^2 L_m}{\partial g^{\sigma\alpha}\partial g^{\beta\rho}} \\ &- \quad T^{(m)}_{\sigma\alpha}(f_T + \frac{R}{2}f_Q + 1) - \nabla_\sigma\nabla_\alpha f_R = 0, \end{aligned} \tag{2}$$

where f_R, f_T, and f_Q represent the partial derivative of f(R,T,Q) with respect to R, T, and Q, respectively. The covariant derivative is illustrated with the symbol ∇_β, and $\Box = g^{\beta\rho}\nabla_\beta\nabla_\rho$ is the d'Alembert operator. In addition, $T^{(m)}_{\sigma\alpha}$ is the usual matter and is defined in terms of energy density μ, anisotropic pressure P, and anisotropic tensor $\Pi_{\sigma\alpha}$ as

$$T^{(m)}_{\sigma\alpha} = (\mu + P)V_\sigma V_\alpha - Pg_{\sigma\alpha} + \Pi_{\sigma\alpha}, \tag{3}$$

where V_σ is the four velocity. The anisotropic pressure and anisotropic tensor in terms of its radial P_r, tangential $P_\perp$ components, and the projection tensor $h_{\sigma\alpha}$ is delineated as

$$
\begin{aligned}
P &= \frac{P_r + 2P_\perp}{3}, \\
\Pi_{\sigma\alpha} &= \Pi\Big(K_\sigma K_\alpha + \frac{h_{\sigma\alpha}}{3}\Big), \\
h_{\sigma\alpha} &= g_{\sigma\alpha} - V_\alpha V_\sigma, \\
P_r &= \Pi + P_\perp.
\end{aligned}
$$

The orthonormal basis is indicated by K_σ. These are locally defined sets of vector fields that are orthonormal to each other and are linearly independent. If orthonormal bases do exist, it makes it easier to define a locally inertial frame and describe the tensor components as seen by an observer at rest in that frame. Contrary to the coordinate basis, these bases are significantly more advantageous and are used frequently in the literature [3]. The compact form of Equation (2) is

$$G_{\sigma\alpha} = \kappa T^{(M)}_{\sigma\alpha}, \tag{4}$$

where $T^{\sigma(M)}_{\alpha}$ shows the matter in the presence of modified terms and is delineated as

$$
\begin{aligned}
T^{(M)}_{\sigma\alpha} &= \tfrac{1}{f_R - L_m f_Q}\Big[\Big(f_T + \tfrac{1}{2}Rf_Q + 1\Big)T^{(m)}_{\sigma\alpha} + \Big\{\tfrac{f}{2} - \tfrac{f_R R}{2} - L_m f_T \\
&- \tfrac{1}{2}\nabla_\rho\nabla_\beta(f_Q T^{\rho\beta})\Big\}g_{\sigma\alpha} - \tfrac{1}{2}\Box(f_Q T_{\sigma\alpha}) - (g_{\sigma\alpha}g^{\rho\beta}\nabla_\rho\nabla_\beta - \nabla_\sigma\nabla_\alpha)f_R \\
&- 2f_Q R_{\rho(\sigma}T^{\rho}_{\alpha)} + \nabla_\rho\nabla_{(\sigma}[T^{\rho}_{\alpha)}f_Q] + 2(f_Q R^{\rho\beta} + f_T g^{\rho\beta})\tfrac{\partial^2 L_m}{\partial g^{\sigma\alpha}\partial g^{\rho\beta}}\Big].
\end{aligned} \tag{5}
$$

3. Modified Field Equations

We analyze *HS* anisotropic distributions of static fluid, which may or may not be constrained from the outside by a surface Σ^e and is represented with the help of the constraint $r = constant = r_{\Sigma^e}$. The matter content, however, may be unable to fill the center of fluid. In this situation, we consider that the central portion is equivalent to an empty space, which implies that the fluid content is identically confined from the inside by a surface Σ^i which is penned as $r = constant = r_{\Sigma^i}$. The metric is, therefore, defined as

$$\tilde{ds}^2 = -e^{\lambda(r)}dr^2 - r^2 d\theta^2 - r^2 sinh^2\theta d\phi^2 + e^{\nu(r)}dt^2. \tag{6}$$

The Greek letters λ and ν rely on radius only. The `f(R,T,Q)` field equations achieved by scrutinizing Equation (4) and metric (6) assume the following form:

$$\frac{8\pi\mu^{(M)}}{f_R + \mu f_Q} = \frac{\lambda' e^{-\lambda}}{r} - \frac{1 + e^{-\lambda}}{r^2}, \tag{7}$$

$$\frac{8\pi P_r^{(M)}}{f_R + \mu f_Q} = \frac{\nu' e^{-\lambda}}{r} + \frac{1 + e^{-\lambda}}{r^2}, \tag{8}$$

$$\frac{8\pi P_\perp^{(M)}}{f_R + \mu f_Q} = \frac{e^{-\lambda}}{2}\Big(-\frac{\lambda'\nu'}{2} + \nu'' + \frac{\nu'^2}{2} + \frac{\nu'}{r} - \frac{\lambda'}{r}\Big), \tag{9}$$

where $\mu^{(M)}$, $P_r^{(M)}$, and $P_\perp^{(M)}$ illustrate the energy density, radial pressure, and tangential pressure, respectively, under the effects of correction terms. The previously stated terms are defined in the Appendix A. The prime shows the derivative with respect to the radial coordinate. The law of conservation of the energy-momentum tensor can be obtained from the equation $\nabla_\alpha T^{\sigma\alpha} = 0$. The strong minimal coupling between matter and geometry, however, violates this conservation law in `f(R,T,Q)` gravity. Therefore, the nonconservation equation in the context of `f(R,T,Q)` is evaluated as

$$\frac{\partial}{\partial r}\left[\frac{P_r^{(M)}}{f_R+\mu f_Q}\right]+\frac{\nu'}{2}\left[\frac{\mu^{(M)}+P_r^{(M)}}{f_R+\mu f_Q}\right]+\frac{2\Pi^{(M)}}{r(f_R+\mu f_Q)}=e^{\lambda}Z, \tag{10}$$

where $\Pi^{(M)}+P_\perp^{(M)}=P_r^{(M)}$, and Z depicts the terms which appeared due to the nonconservation of the theory; it is described in the Appendix A. Equation (10) is referred to as the Tolman–Opphenheimer–Volkoff (TOV) formalism for anisotropic materials in the static *HS* system. The mass function $m=m(r)$ for the anisotropic matter configuration endorsed with hyperbolic symmetry is determined as

$$m(r)=e^{-\lambda}\frac{r}{2}+\frac{r}{2}. \tag{11}$$

The substitution of Equation (11) into Equation (7) yields

$$m(r)=-4\pi\int_0^r\frac{\mu^{(M)}}{f_R+\mu f_Q}r^2dr. \tag{12}$$

Equation (12) states that the energy density should undoubtedly be negative as mass cannot be taken as a negative quantity. This leads to the breakdown of weak energy constraint as well as the inclusion of quantum occurrences, which further demonstrates worthwhile results, which will be discussed in this article. In order to keep mass positive, we replace $\mu^{(M)}$ with $-|\mu^{(M)}|$ in Equation (12) as

$$m(r)=4\pi\int_{r_{min}}^r\frac{|\mu^{(M)}|}{f_R+\mu f_Q}r^2dr. \tag{13}$$

The ν' can be determined from Equation (8) as

$$\nu'=2\left\{\frac{4\pi P_r^{(M)}r^3-m(f_R+\mu f_Q)}{r(f_R+\mu f_Q)(2m-r)}\right\}. \tag{14}$$

Substituting back value of ν' in Equation (10), we achieve

$$\frac{\partial}{\partial r}\left[\frac{P_r^{(M)}}{f_R+\mu f_Q}\right]+\frac{4\pi P_r^{(M)}r^3-m(f_R+\mu f_Q)}{r(f_R+\mu f_Q)(2m-r)}\left[\frac{P_r^{(M)}-|\mu^{(M)}|}{f_R+\mu f_Q}\right]+\frac{2\Pi^{(M)}}{r(f_R+\mu f_Q)}=e^{\lambda}Z. \tag{15}$$

Equation (15) is termed as a hydrostatic equilibrium equation for our *HS* anisotropic matter configuration in f(R,T,Q) theory. The term $\frac{\partial}{\partial r}\left[\frac{P_r^{(M)}}{f_R+\mu f_Q}\right]$ shows the pressure gradient in the presence of modified terms and behaves as antigravity. The term $\frac{4\pi P_r^{(M)}r^3-m(f_R+\mu f_Q)}{r(f_R+\mu f_Q)(2m-r)}\left[\frac{P_r^{(M)}-|\mu^{(M)}|}{f_R+\mu f_Q}\right]$ tells us about the strength of the gravitational force. The first half, i.e., $\frac{4\pi P_r^{(M)}r^3-m(f_R+\mu f_Q)}{r(f_R+\mu f_Q)(2m-r)}$, is generally described as the passive gravitational mass density in the presence of modified terms. The second half, i.e., $\left[\frac{P_r^{(M)}-|\mu^{(M)}|}{f_R+\mu f_Q}\right]$, is interpreted as the active gravitational mass under the influence of modified terms.

4. Conformal Scalar

The conformal scalar is assessed using a conformal tensor in this section. The conformally invariant portion of the curvature tensor is known as the conformal tensor [35]. A body experiences a tidal force when traveling along a geodesic, which may be described with the support of a conformal tensor. The conformal tensor is designated in our case as

$$W^{(e)}_{\sigma\alpha} = \varepsilon\left(\chi_\sigma\chi_\alpha - \frac{h_{\alpha\sigma}}{3}\right).$$

In our scenario, the magnetic component of the conformal tensor disappears and just the electric component ($W^{(e)}_{\sigma\alpha}$) remains. The conformal scalar is represented by the symbol $\mathfrak{E}$ and is calculated using our static spacetime as

$$\mathfrak{E} = -\frac{\nu'' e^{-\lambda}}{4} - \frac{1}{2r^2} + \frac{\lambda'\nu' e^{-\lambda}}{8} + \frac{\nu' e^{-\lambda}}{4r} - \frac{\lambda' e^{-\lambda}}{4r} - \frac{\nu'^2 e^{-\lambda}}{8} - \frac{e^{-\lambda}}{2r^2}. \quad (16)$$

$$\frac{3m}{r^3} = 4\pi\left[\frac{|\mu^{(M)}| + \Pi^{(M)}}{f_R + \mu f_Q}\right] - \mathfrak{E}. \quad (17)$$

The use of Equation (12) along with the r derivative of Equation (17) produces

$$\mathfrak{E} = 4\pi\left[\frac{\Pi^{(M)}}{f_R + \mu f_Q}\right] + \frac{4\pi}{r^3}\int_0^r \frac{\partial}{\partial r}\left[\frac{|\mu^{(M)}|}{f_R + \mu f_Q}\right] r^3 dr. \quad (18)$$

Equation (18) represents the conformal scalar in terms of density inhomogeneity and the anisotropic tensor under the influence of correction terms. Inputting Equation (18) into (17), we obtain

$$m = \frac{4\pi r^3}{3}\left[\frac{|\mu^{(M)}|}{f_R + \mu f_Q}\right] - \frac{4\pi}{3}\int_0^r \frac{\partial}{\partial r}\left[\frac{|\mu^{(M)}|}{f_R + \mu f_Q}\right] r^3 dr. \quad (19)$$

Equation (19) claimed mass function as the sum of matter density and the changes caused in its distribution. These changes occur due to the involvement of the partial derivative of energy density in the presence of modified correction terms, as one can also witness from Equation (19).

5. Tolman Mass

In this section, to determine the amount of matter content in static relativistic compact bodies, we will use a different method. Two different types of boundary surfaces are likely to be found in *HS* sources. Devitt and Florides [36] achieved the modified Tolman mass-energy formula for spherically symmetric and time-independent systems. They came to the conclusion that surface discontinuity has no effect on the computed formula. We will thus discuss here a general formalism provided by Tolman many years ago. The formula for calculating the active gravitational mass is

$$m_T = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \tilde{r}^2 e^{\frac{\nu+\lambda}{2}} \sinh\theta (T_0^{0(M)} - T_1^{1(M)} - 2T_2^{2(M)}) d\tilde{r} d\theta d\phi, \quad (20)$$

where $T_0^{0(M)}$, $T_1^{1(M)}$ and $T_2^{2(M)}$ illustrate the stress-energy tensor in the presence of correction terms. Substituting its values into Equation (20), we achieve

$$m_T = 2\pi(\cosh\pi - 1)\int_0^r e^{\frac{\nu+\lambda}{2}} \tilde{r}^2 \left(\frac{-|\mu^{(M)}| + P_r^{(M)} + 2P_\perp^{(M)}}{f_R + \mu f_Q}\right) d\tilde{r}. \quad (21)$$

Taking into account modified field Equations (7)–(9) along with integrating Equation (20), we attain

$$m_T = \nu' r^2 e^{\frac{\nu-\lambda}{2}} \frac{\cosh\pi - 1}{4}. \quad (22)$$

Utilizing Equations (14) and (22), we obtain

$$m_T = \left\{\frac{4\pi P_r^{(M)} r^3 - m(f_R + \mu f_Q)}{(f_R + \mu f_Q)}\right\} e^{\frac{\nu+\lambda}{2}} \frac{\cosh\pi - 1}{2}. \quad (23)$$

Equation (23) reveals the physical significance that m_T is an effective inertial mass. Moreover, in the locally anisotropic static metric, it supplies the repulsive nature (if $4\pi P_r^{(M)} r^3 < m(f_R + \mu f_Q)$) caused by the gravitational force. The gravitational acceleration of a test particle for the static metric is determined as

$$\tilde{A} = \frac{\nu' e^{\frac{-\lambda}{2}}}{2} = \frac{2m_T e^{\frac{-\nu}{2}}}{r^2(\cosh\pi - 1)}. \tag{24}$$

Thus, from Equation (23), if $4\pi P_r^{(M)} r^3 < m(f_R + \mu f_Q)$, then m_T will be treated as a negative entity. Moreover, according to Equation (24), negative m_T results in negative $\tilde{A}$. Therefore, the gravitational acceleration $\tilde{A}$ points in the inward direction, radially. Eventually, the r derivative of Equation (20) with the combination of Equation (23) gives

$$m_T' = -\left(\frac{\cosh\pi - 1}{2}\right) r^2 e^{\frac{\nu+\lambda}{2}} \left(\mathfrak{E} + \frac{4\pi\Pi^{(M)}}{f_R + \mu f_Q}\right) + \frac{3m_T}{r}. \tag{25}$$

Integration of Equation (25) produces

$$m_T = \left(\frac{\cosh\pi - 1}{2}\right) r^3 \int_r^{r_{\Sigma^e}} \frac{e^{\frac{\nu+\lambda}{2}}}{\tilde{r}} \left(\mathfrak{E} + \frac{4\pi\Pi^{(M)}}{f_R + \mu f_Q}\right) d\tilde{r} + (m_T)_{\Sigma_e} \left(\frac{r^3}{r^3_{\Sigma^e}}\right). \tag{26}$$

Inserting Equation (18) into Equation (26), we obtain

$$\begin{aligned} m_T &= \left(\frac{-1+\cosh\pi}{2}\right) r^3 \int_r^{r_{\Sigma^e}} \frac{e^{\frac{\nu+\lambda}{2}}}{\tilde{r}} \left[\frac{4\pi}{\tilde{r}^3} \int_0^r \frac{\partial}{\partial r}\left[\frac{|\mu^{(M)}|}{f_R+\mu f_Q}\right] \tilde{r}^3 d\tilde{r} \right. \\ &\left. + \frac{8\pi\Pi^{(M)}}{f_R+\mu f_Q} d\tilde{r} + (m_T)_{\Sigma^e} \left(\frac{r^3}{r^3_{\Sigma^e}}\right)\right. . \end{aligned} \tag{27}$$

Equation (26) shows the effect of conformal scalar on the total energy budget in the presence of `f(R,T,Q)` correction terms, while Equation (27) relates the density inhomogeneity and anisotropic tensor with the Tolman mass in the presence of modified terms. Subsequently, one can deduce from Equations (24), (26) and (27) that Tolman mass contributes, as the active gravitational mass, inhomogeneities in energy density and local anisotropies. The term $(m_T)_{\Sigma^e}\left(\frac{r^3}{r^3_{\Sigma^e}}\right)$ indicates the dependence of Tolman mass on the gravitational mass of homogeneous, static system of radius r restricted to Σ.

6. Complexity Factor

The Riemann tensor may be split into distinct portions using its dual and the four velocity vectors, leading to a small number of tensor quantities and scalar functions termed as structure scalars. Such scalars are essential for comprehending the motion of celestial objects and seem to be linked with fluid variables. Here, our focus is to evaluate the structure scalar Y_{TF}, which is termed as the `complexity factor`. In order to achieve this, we introduce the following tensors in accordance with the orthogonal splitting method of curvature tensors [37].

$$Y_{\sigma\alpha} = R_{\sigma\mu\alpha\delta} V^\mu V^\delta, \tag{28}$$

$$Z_{\sigma\alpha} =^* R_{\sigma\mu\alpha\delta} V^\mu V^\delta = \frac{1}{2}\eta_{\sigma\mu\epsilon\rho} R^{\epsilon\rho}_{\alpha\delta} V^\mu V^\delta, \tag{29}$$

$$X_{\sigma\alpha} =^* R^*_{\sigma\mu\alpha\delta} u^\gamma V^\delta = \frac{1}{2}\eta^{\epsilon\rho}_{\sigma\mu} R^*_{\epsilon\rho\alpha\delta} V^\mu V^\delta, \tag{30}$$

where * represents the dual tensor and is formulated as $R^*_{\sigma\alpha\mu\delta} = \frac{1}{2}\eta_{\epsilon\omega\mu\delta} R^{\epsilon\omega}_{\sigma\alpha}$. The Riemann and Ricci tensor, along with the Ricci scalar, are used to illustrate the conformal tensor, which is given as follows:

$$R^{\mu}_{\eta\rho\sigma}=C^{\mu}_{\eta\rho\sigma}+\frac{1}{2}R^{\mu}_{\rho}g_{\eta\sigma}+\frac{1}{2}R_{\eta\rho}\delta^{\mu}_{\sigma}+\frac{1}{2}R_{\eta\sigma}\delta^{\mu}_{\rho}-\frac{1}{2}R^{\mu}_{\sigma}g_{\eta\rho}-\frac{1}{6}R\left(\delta^{\mu}_{\rho}g_{\eta\sigma}-g_{\eta\rho}\delta^{\mu}_{\sigma}\right). \quad (31)$$

Utilizing modified field equations in Equation (31), we obtain

$$R^{\alpha\sigma}_{\mu\gamma}=C^{\alpha\sigma}_{\mu\gamma}+16\pi\,T^{(M)}\,{}^{[\alpha}_{[\mu}\delta^{\sigma]}_{\gamma]}+8\pi\,T^{(M)}\left(\frac{1}{3}\delta^{\alpha}_{[\mu}\delta^{\sigma}_{\gamma]}-\delta^{[\alpha}_{[\mu}\delta^{\sigma]}_{\gamma]}\right). \quad (32)$$

Substituting the value of $T^{(M)}$ in Equation (32) and following the procedure of splitting of Riemann tensor, we reach

$$R^{\alpha\sigma}_{\mu\gamma}=R^{\alpha\sigma}_{(I)\mu\gamma}+R^{\alpha\sigma}_{(II)\mu\gamma}+R^{\alpha\sigma}_{(III)\mu\gamma},$$

where

$$\begin{aligned}R^{\sigma\gamma}_{(I)\alpha\delta}&=\tfrac{16\pi}{f_R+\mu f_Q}\left(f_T+\tfrac{1}{2}Rf_Q+1\right)\left[\mu V^{[\sigma}V_{[\alpha}\delta^{\gamma]}_{\delta]}-Ph^{[\sigma}_{[\alpha}\delta^{\gamma]}_{\delta]}+\Pi^{[\sigma}_{[\alpha}\delta^{\gamma]}_{\delta]}\right]-\tfrac{8\pi}{f_R+\mu f_Q}\\&\left[\left(f_T+\tfrac{1}{2}f_QR+1\right)(3P+|\mu|)+4\left\{\left(\tfrac{f}{2}-f_RR\right)+f_T\mu-\tfrac{1}{2}\nabla_\nu\nabla_\mu\left(f_QT^{\nu\mu}\right)\right\}+\right.\\&\left.\tfrac{1}{2}\Box\left\{f_Q(3P+|\mu|)\right\}-3\Box f_R-2f_QR_{\mu\sigma}T^{\mu\sigma}+\nabla_\mu\nabla_\sigma\left(f_QT^{\mu\sigma}\right)\left(\tfrac{1}{3}\delta^{\sigma}_{[\alpha}\delta^{\gamma}_{\delta]}-\delta^{[\sigma}_{[\alpha}\delta^{\gamma]}_{\delta]}\right)\right],\end{aligned} \quad (33)$$

$$\begin{aligned}R^{\sigma\gamma}_{(II)\alpha\delta}&=\tfrac{4\pi}{f_R+\mu f_Q}\left[2\left\{\tfrac{R}{2}\left(-f_R+\tfrac{f}{R}\right)+f_T\mu-\tfrac{1}{2}\nabla_\nu\nabla_\mu\left(f_QT^{\nu\mu}\right)\left(\delta^{\sigma}_{\alpha}\delta^{\gamma}_{\delta}-\delta^{\sigma}_{\delta}\delta^{\gamma}_{\alpha}\right)\right.\right.\\&-\tfrac{1}{2}\Box\left\{f_Q\left(T^{\sigma}_{\alpha}\delta^{\gamma}_{\delta}-T^{\sigma}_{\delta}\delta^{\gamma}_{\alpha}-T^{\gamma}_{\alpha}\delta^{\sigma}_{\delta}+T^{\gamma}_{\delta}\delta^{\sigma}_{\alpha}\right)\right\}-2\Box f_R\left(\delta^{\sigma}_{\alpha}\delta^{\gamma}_{\delta}-\delta^{\sigma}_{\delta}\delta^{\gamma}_{\alpha}\right)+\left(\delta^{\gamma}_{\delta}\nabla^{\sigma}\nabla_{\alpha}\right.\\&\left.-\delta^{\gamma}_{\alpha}\nabla^{\sigma}\nabla_{\delta}-\delta^{\sigma}_{\delta}\nabla^{\gamma}\nabla_{\alpha}+\delta^{\sigma}_{\alpha}\nabla^{\gamma}\nabla_{\delta}\right)f_R-f_Q\left(R^{\sigma}_{\mu}T^{\mu}_{\alpha}\delta^{\gamma}_{\delta}-R^{\sigma}_{\mu}T^{\mu}_{\delta}\delta^{\gamma}_{\alpha}-R^{\gamma}_{\mu}T^{\mu}_{\alpha}\delta^{\sigma}_{\delta}+\right.\\&\left.R^{\gamma}_{\mu}T^{\mu}_{\delta}\delta^{\sigma}_{\alpha}\right)-f_Q\left(R_{\mu\alpha}T^{\mu\sigma}\delta^{\gamma}_{\delta}-R_{\mu\delta}T^{\mu\sigma}\delta^{\gamma}_{\alpha}-R_{\mu\alpha}T^{\mu\gamma}\delta^{\sigma}_{\delta}+R_{\mu\delta}T^{\mu\gamma}\delta^{\sigma}_{\alpha}\right)+\tfrac{1}{2}\nabla_\mu\nabla^{\sigma}\times\\&\left\{f_Q\left(T^{\mu}_{\alpha}\delta^{\gamma}_{\delta}-T^{\mu}_{\delta}\delta^{\gamma}_{\alpha}\right)\right\}+\tfrac{1}{2}\nabla_\mu\nabla_\alpha\left\{f_Q\left(T^{\sigma\mu}\delta^{\gamma}_{\delta}-T^{\gamma\mu}\delta^{\sigma}_{\delta}\right)\right\}+\tfrac{1}{2}\nabla_\mu\nabla^{\gamma}\left\{f_Q\left(T^{\mu}_{\delta}\delta^{\sigma}_{\alpha}\right.\right.\\&\left.\left.\left.-T^{\mu}_{\alpha}\delta^{\sigma}_{\delta}\right)\right\}+\tfrac{1}{2}\nabla_\mu\nabla_\delta\left\{f_Q\left(T^{\gamma\mu}\delta^{\sigma}_{\alpha}-T^{\sigma\mu}\delta^{\gamma}_{\alpha}\right)\right\}\right],\end{aligned} \quad (34)$$

$$R^{\alpha\sigma}_{(III)\mu\gamma}=4V^{[\alpha}V_{[\mu}E^{\sigma]}_{\gamma]}-\epsilon^{\alpha\sigma}_{\delta}\epsilon_{\pi\mu\gamma}E^{\delta\pi}. \quad (35)$$

Equations (28)–(30) provide support to write three tensors $(Y_{\sigma\alpha},X_{\sigma\alpha},Z_{\sigma\alpha})$ in terms of significant parameters as

$$\begin{aligned}Y_{\sigma\alpha}&=E_{\sigma\alpha}+\tfrac{4\pi}{f_R+\mu f_Q}\Pi_{\sigma\alpha}\left(f_T+\tfrac{Rf_Q}{2}+1\right)+\tfrac{4\pi h_{\sigma\alpha}}{3(f_R+\mu f_Q)}\left[(|\mu|+3P)\left(f_T+\tfrac{Rf_Q}{2}+1\right)\right.\\&\left.+(Rf_R-6\mu f_T-f-\tfrac{5}{3}\nabla_\rho\nabla_\gamma(f_QT^{\rho\gamma})-\Box f_Q(|\mu|+3P)+f_QR_{\rho\gamma}T^{\rho\gamma}-2\nabla_\rho\nabla_\gamma(f_QT^{\rho\gamma})\right]\\&+\tfrac{4\pi}{f_R+\mu f_Q}\left[\nabla_\alpha\nabla_\sigma f_R-V_\sigma V^{\gamma}\nabla_\alpha\nabla_\gamma f_R-V_\rho V_\alpha\nabla_\sigma\nabla^{\rho}f_R+g_{\sigma\alpha}V_\rho V^{\gamma}\nabla^{\rho}\nabla_\gamma f_R-\tfrac{\Box f_Q}{2}(T_{\sigma\alpha}\right.\\&-T^{\rho}_{\sigma}V_\rho V_\alpha-T_{\gamma\alpha}V_\sigma V_\gamma+T^{\rho}_{\gamma}g_{\sigma\alpha}V_\rho V^{\gamma})-f_Q(R_{\delta\sigma}T^{\delta}_{\alpha}-R_{\delta\gamma}T^{\delta}_{\alpha}V_\sigma V^{\gamma}-R_{\delta\sigma}T^{\delta\rho}V_\rho V_\alpha\\&+R_{\delta\gamma}T^{\delta\rho}V_\rho V^{\gamma}g_{\sigma\alpha})+\nabla_\alpha\nabla_\delta(T^{\delta}_{\sigma}f_Q)-\nabla_\alpha\nabla_\delta(T^{\delta}_{\gamma}f_Q)V_\sigma V^{\gamma}-\nabla^{\rho}\nabla_\delta(T^{\delta}_{\sigma}f_Q)V_\rho V_\alpha\\&+g_{\sigma\alpha}V^{\gamma}V_\rho\nabla_\delta\nabla_\rho(T^{\delta}_{\gamma}f_Q)+\nabla_\delta\nabla_\sigma(T^{\delta}_{\alpha}f_Q)-\nabla_\delta\nabla_\gamma(T^{\delta}_{\alpha}f_Q)V^{\gamma}V_\sigma-\nabla_\delta\nabla_\sigma(T^{\delta\rho}f_Q)V_\rho V_\alpha\\&\left.+g_{\sigma\alpha}\nabla_\delta\nabla_\gamma(T^{\delta}_{\rho}f_Q)V_\rho V^{\gamma}\right],\end{aligned} \quad (36)$$

$$X_{\sigma\alpha} = -E_{\sigma\alpha} - \frac{1}{(f_R+\mu f_Q)}\left(\frac{8\pi}{3}|\mu|h_{\sigma\alpha} - 4\pi\Pi_{\sigma\alpha}\right)\left(f_T + \frac{1}{2}Rf_Q + 1\right) + \frac{4\pi}{(f_R+\mu f_Q)}\Bigg[\Bigg\{ -\frac{1}{2}\Box\left(f_Q T^{\rho}_{\epsilon}\right) + \nabla^{\rho}\nabla_{\gamma}f_R + \frac{1}{2}\nabla_{\mu}\nabla^{\rho}\left(f_Q T^{\mu}_{\gamma}\right) + \frac{1}{2}\nabla_{\mu}\nabla_{\gamma}\left(f_Q T^{\mu\rho}\right)\Bigg\}\epsilon^{\gamma\delta}_{\sigma}\epsilon_{\rho\delta\alpha} + f_Q R^{\rho}_{\mu}\times \left(P - \frac{\Pi}{3}\right)\epsilon^{\mu\delta}_{\sigma}\epsilon_{\rho\delta\alpha} + f_Q R_{\mu\gamma}\left(P - \frac{\Pi}{3}\right)\epsilon^{\gamma\delta}_{\sigma}\epsilon^{\mu}_{\delta\alpha}\Bigg] + \frac{8\pi}{3(f_R+\mu f_Q)}\Bigg[\Bigg\{\frac{R}{2}\left(\frac{f}{R} - f_R\right) + \mu f_T - \frac{1}{2}\nabla_{\mu}\nabla_{\beta}\left(f_Q T^{\mu\beta}\right)\Bigg\} + \frac{1}{2}\Box\Big\{f_Q(3P + |\mu|)\Big\} + 2f_Q R\left(-\frac{\Pi}{3} + P\right) + \nabla_{\mu}\nabla_{\rho}\left(f_Q T^{\mu\rho}\right)\Bigg]h_{\sigma\alpha}, \tag{37}$$

$$Z_{\sigma\alpha} = \frac{4\pi}{(f_R+\mu f_Q)}\Bigg[\frac{1}{2}V^{\delta}\Box\left(f_Q T^{\gamma}_{\delta}\right) - V^{\delta}\nabla^{\gamma}\nabla_{\delta}f_R + f_Q\mu R^{\gamma}_{\delta}V^{\delta} - f_Q P R^{\gamma}_{\delta}V^{\delta} + \frac{1}{3}f_Q\Pi R^{\gamma}_{\delta}V^{\delta} - \frac{1}{2}V^{\delta}\nabla_{\mu}\nabla^{\gamma}\left(f_Q T^{\mu}_{\delta}\right) - \frac{1}{2}V^{\delta}\nabla_{\mu}\nabla_{\delta}\left(f_Q T^{\mu\gamma}\right)\Bigg]\epsilon_{\gamma\alpha\sigma}. \tag{38}$$

The structural scalar Y_{TF}, which has been assigned as the `complexity factor` of matter configuration, is the subject of our discussion from now onwards [38–40]. It will eventually be possible to express the `complexity factor` in terms of metric coefficients and their derivatives, which will make it easier to accomplish various static solutions. Thus, the tensor $Y_{\sigma\alpha}$ in Equation (36) is decomposed into its trace and trace free parts as follows:

$$Y_T = \frac{4\pi}{f_R+\mu f_Q}\left[(-|\mu| + 3P)\left(\frac{Rf_Q}{2} + 1 + f_T\right)\right] + \frac{4\pi}{f_R+\mu f_Q}\Big[(Rf_R - f - 6\mu f_T) + g^{\sigma\alpha}\nabla_{\alpha}\nabla_{\sigma}f_R - g^{\sigma\alpha}V_{\sigma}V^{\gamma}\nabla_{\alpha}\nabla_{\gamma}f_R - g^{\sigma\alpha}V_{\rho}V_{\alpha}\nabla^{\rho}\nabla_{\sigma}f_R + 4V_{\rho}V^{\gamma}\nabla^{\rho}\nabla_{\gamma}f_R - \frac{5}{3}\nabla_{\rho}\nabla_{\gamma}(f_Q T^{\rho\gamma}) - \Box f_Q(|\mu| + 3P) + f_Q R_{\rho\gamma}T^{\rho\gamma} - 2\nabla_{\rho}\nabla_{\gamma}(f_Q T^{\rho\gamma})\Big] + \frac{4\pi g^{\sigma\alpha}}{f_R+\mu f_Q}\Big[-\frac{\Box f_Q}{2}(T_{\sigma\alpha} - T^{\rho}_{\sigma}V_{\rho}V_{\alpha} - T_{\gamma\alpha}V_{\sigma}V_{\gamma} + T^{\rho}_{\gamma}g_{\sigma\alpha}V_{\rho}V^{\gamma}) - f_Q(R_{\delta\sigma}T^{\delta}_{\alpha} - R_{\delta\gamma}T^{\delta}_{\alpha}V_{\sigma}V^{\gamma} - R_{\delta\sigma}T^{\delta\rho}V_{\rho}V_{\alpha} + R_{\delta\gamma}T^{\delta\rho}V_{\rho}V^{\gamma}g_{\sigma\alpha}) + \nabla_{\alpha}\nabla_{\delta}(T^{\delta}_{\sigma}f_Q) - \nabla_{\alpha}\nabla_{\delta}(T^{\delta}_{\gamma}f_Q)V_{\sigma}V^{\gamma} - \nabla^{\rho}\nabla_{\delta}(T^{\delta}_{\sigma}f_Q)V_{\rho}V_{\alpha} + \nabla_{\delta}\nabla_{\sigma}(T^{\delta}_{\alpha}f_Q) + g_{\sigma\alpha}V^{\gamma}V_{\rho}\nabla_{\delta}\nabla_{\rho}(T^{\delta}_{\gamma}f_Q) - \nabla_{\delta}\nabla_{\gamma}(T^{\delta}_{\alpha}f_Q)V^{\gamma}V_{\sigma} - \nabla_{\delta}\nabla_{\sigma}(T^{\delta\rho}f_Q)V_{\rho}V_{\alpha} + g_{\sigma\alpha}\nabla_{\delta}\nabla_{\gamma}(T^{\delta}_{\rho}f_Q)V_{\rho}V^{\gamma}\Big], \tag{39}$$

$$Y_{TF} = \mathfrak{E} + \frac{4\pi}{f_R+\mu f_Q}\Pi\left(f_T + \frac{Rf_Q}{2} + 1\right) + \psi_1. \tag{40}$$

Using Equation (18) in Equation (40), we obtain

$$Y_{TF} = 4\pi\left[\frac{\Pi^{(M)}}{f_R + \mu f_Q}\right] + \frac{4\pi}{r^3}\int_0^r \frac{\partial}{\partial r}\left[\frac{|\mu^{(M)}|}{f_R + \mu f_Q}\right]r^3 dr + \frac{4\pi}{f_R + \mu f_Q}\Pi\left(f_T + \frac{Rf_Q}{2} + 1\right) + \psi_1. \tag{41}$$

Equation (40) depicts that `complexity factor` can be determined by measuring the change in the direction of pressure as well as the inhomogeneity in the matter distribution produced as an effect of these changes in the presence of correction terms. Utilizing Equations (20), (27), and (40), we achieve

$$m_T = (m_T)_{\Sigma^e}\left(\frac{r}{r_{\Sigma^e}}\right)^3 + \left(\frac{\cosh\pi - 1}{2}\right)r^3\int_r^{r_{\Sigma^e}}\frac{e^{\frac{\nu+\lambda}{2}}}{r^2}\left(Y_{TF} - 4\pi\frac{\Pi^{(M)}}{f_R+\mu f_Q} - \frac{4\pi\Pi(1+f_T+\frac{Rf_Q}{2})}{f_R+\mu f_Q}\right) - \psi_1. \tag{42}$$

Equation (41) illustrates the relationship among the Tolman mass and the `complexity factor` in the presence of modified terms. Many researchers adopted the topic of complexity in various disciplines [41–43].

7. Static Solutions in Modified Gravity

This section is devoted to provide a general framework of support to obtain any anisotropic *HS* static solutions that rely on twogenerating functions. For this, using Equations (8) and (9), we achieve

$$\frac{1+e^{-\lambda}}{r^2}-\frac{e^{-\lambda}}{2}\left(\nu''+\frac{\nu'^2}{2}-\frac{\lambda'\nu'}{2}-\frac{\nu'}{r}-\frac{\lambda'}{r}\right)=\frac{8\pi}{f_R+\mu f_Q}(P_r^{(M)}-P_\perp^{(M)}). \tag{43}$$

In order to proceed further, the auxiliary functions, i.e., $\frac{\nu' r}{2}=zr-1$ and $y=\frac{1}{e^\lambda}$, are involved, which transform Equation (43) into

$$zr^2y'+y\left[4+2z^2r^2+2z'r^2-6zr\right]=2\left[1-\frac{8\pi r^2}{f_R+\mu f_Q}\Pi^{(M)}\right]. \tag{44}$$

Equation (44) upon integration gives

$$e^{\lambda(r)}=\frac{z^2e^{\int\left(2z+\frac{4}{zr^2}\right)dr}}{\left[2\int\left\{z\left(\frac{f_R+\mu f_Q-8\pi\Pi^{(M)}r^2}{(f_R+\mu f_Q)r^8}\right)e^{\int\left(2z+\frac{4}{zr^2}\right)dr}\right\}dr+A_1\right]r^6}, \tag{45}$$

where A_1 indicates the integration constant. Equation (45) accomplishes our requirement of representing any anisotropic static *HS* solutions that rely on two generating functions, which are z and $\Pi^{(M)}$. Utilizing Equations (7)–(9) and (12) and the auxiliary variables, we obtain

$$4\pi|\mu^{(M)}|=\frac{(f_R+\mu f_Q)m'}{r^2}, \tag{46}$$

$$4\pi P_r^{(M)}=f_R+\mu f_Q\left(\frac{z(2m-r)+1}{r^2}-\frac{m}{r^3}\right), \tag{47}$$

$$8\pi P_\perp^{(M)}=\frac{f_R+\mu f_Q}{r^2}\left\{(2m-r)\left[\frac{z'r^2+1+z^2r^2-zr}{r}\right]+z(rm'-m)\right\}. \tag{48}$$

Equations (46)–(48) represent the physical parameters in terms of modified terms and generating functions (*GF*'s), i.e., z and $\Pi^{(M)}$. Next, we will find the explicit solutions and their corresponding *GF*s.

7.1. Solution with Vanishing Weyl Scalar and Radial Pressure

In this subsection, we use the condition where the Weyl scalar vanishes and we obtain a solution. The vanishing of the Weyl scalar means that spacetime is conformally flat. In other words, we can say that a neighborhood conformal to an open subset of the Minkowski spacetime exists around every point. In this context, in order to obtain a solution, we assume that $\mathfrak{E}=0$. Therefore, Equation (16) becomes

$$\frac{\partial}{\partial r}\left[\frac{\nu'}{2r}(e^\nu+e^{-\lambda})\right]-e^{\nu+\lambda}\frac{\partial}{\partial r}\left[\frac{1+e^{-\lambda}}{r^2}\right]=0. \tag{49}$$

Equation (49), by utilizing new variables, i.e., $ye^\lambda=1$ and $\frac{\nu'}{2}=\frac{\mathfrak{a}'}{\mathfrak{a}}$, transforms into

$$\left[\mathfrak{a}'-\frac{\mathfrak{a}}{r}\right]y'+2\left[\mathfrak{a}''-\frac{\mathfrak{a}'}{r}+\frac{\mathfrak{a}}{r^2}\right]y+\frac{2\mathfrak{a}}{r^2}=0, \tag{50}$$

where the formal solution is accomplished upon integration of Equation (50) as

$$y=\left(\int e^{-\int\mathfrak{l}_1(r)dr}\mathfrak{l}_2(r)dr+\mathfrak{B}\right)e^{-\int\mathfrak{l}_1(r)dr}, \tag{51}$$

where

$$\mathfrak{l}_1(r)=2\frac{d}{dr}\left[ln\left(\mathfrak{a}'-\frac{\mathfrak{a}}{r}\right)\right],$$

$$\mathfrak{l}_2(r)=\frac{-2\mathfrak{a}}{(\mathfrak{a}'-\frac{\mathfrak{a}}{r})r^2},$$

and $\mathfrak{B}$ is an integration constant. The original values are obtained by substituting the new variables, and Equation (51) will take the form as

$$\frac{\nu'}{2} = \frac{1}{r} + e^{\lambda/2}\sqrt{e^{-\nu}\psi - \frac{1}{r^2}}, \tag{52}$$

where ψ is an integration constant. The matching conditions are determined using the Darmois regime [44,45] in f(R,T,Q) theory and Schwarzschild spacetime as

$$e^{\nu_{\Sigma^e}} = \frac{2M}{r_\Sigma} - 1, \qquad e^{\lambda_{\Sigma^e}} = \left(\frac{2M}{r_\Sigma} - 1\right)^{-1}, \qquad P_r^{(M)}(r_{\Sigma^e}) = 0. \tag{53}$$

The integration constant ψ that appeared in Equation (52) is then calculated using the matching conditions as

$$\psi = \frac{-4Mr_{\Sigma^e} + 9M^2}{r^4_{\Sigma^e}}.$$

The integration of Equation (52) produces

$$e^{\nu} = r^2\psi \sin^2\left(\int \frac{e^{\lambda/2}}{r}dr + \beta\right),$$

where the integration constant β is determined using matching conditions as follows:

$$\beta = \sin^{-1}\left[r_{\Sigma^e}\left(\frac{1 - \frac{2M}{r_{\Sigma^e}}}{4r_{\Sigma_e}M - 9M^2}\right)^{1/2}\right] - \left[\int \frac{e^{\lambda/2}}{r}dr\right]_{\Sigma^e}.$$

In order to accomplish our solution, we will apply another restriction, i.e., $P_r = 0$, which takes into account the diminishing radial pressure in the *HS* gravitational source. Equation (8) finally gives following solution against this context:

$$\nu' = \frac{\frac{8\pi r^2 e^{\lambda}\varphi_1}{f_R + \mu f_Q} - \frac{1+e^{\lambda}}{r}}{\varphi_2}. \tag{54}$$

where φ_1 and φ_2 are defined in the Appendix A. The following equation may be easily found by substituting Equation (54) into (16) and using $\mathfrak{C} = 0$.

$$\frac{(1+e^{\lambda})^2}{\varphi_2(r)} + 4(1+e^{\lambda})\left[1 + \frac{1}{\varphi_2(r)} + \frac{r\varphi_2'}{2\varphi_2^2}\right] + \lambda' r\left[2 + \frac{1}{\varphi_2(r)} + \frac{e^{\lambda}r^3\varphi_1}{\varphi_2(\mu f_Q + f_R)}\right] + D_0 = 0, \tag{55}$$

where D_0 indicates the correction terms of f(R,T,Q) theory and is defined in the Appendix A. Alternatively, Equation (55), using expression $\frac{1}{2e^{\lambda}} + \frac{1}{2} = g(r)$, can be written as

$$g'\left[-\frac{2g(g-1)}{r(2g-1)} + \varphi_4\right] + g\varphi_5 + \varphi_3 = 0, \tag{56}$$

where φ_3, φ_4, and φ_5 show the influence of correction terms in this modified gravity and are defined in the Appendix A. Upon integration, Equation (56) transforms into

$$g = \int \frac{r(2g-1)(g\varphi_5 + \varphi_3)}{-2g(g-1) + r(2g-1)\varphi_4}dr + \mathfrak{C}_1, \tag{57}$$

where $\mathfrak{C}_1$ is the integration constant. The utilization of Equations (52) and (54) produces

$$e^{\nu}=\frac{\psi r^2(2g-1)\varphi_2^2}{\left[-\frac{g}{r}+\frac{8\pi r^2\varphi_1}{f_R+\mu f_Q}-\varphi_2(2g-1)\right]^2+\varphi_2^2(2g-1)}. \quad (58)$$

The physical attributes for this particular model are accomplished as

$$|\mu^{(M)}|=f_R+\mu f_Q\left[-\frac{g}{4\pi r^2}-\frac{g}{8\pi\varphi_4(g(g-1)+r(2g-1))}\left\{\varphi_5-2g\varphi_5+\varphi_3+\frac{\varphi_3}{g}\right\}\right], \quad (59)$$

$$P_{\perp}^{(M)}=f_R+\mu f_Q\left[\frac{g^2}{8\pi r^2(2g-1)}+\frac{g^2\left\{\varphi_5-2g\varphi_5-2\varphi_3+\frac{\varphi_3}{g}\right\}}{16\pi\varphi_4(2g-1)[g(g-1)+r(2g-1)]}\right]. \quad (60)$$

The *GF* for this specific model is calculated as

$$\begin{aligned}\Pi^{(M)} &= f_R+\mu f_Q\left[-\frac{g^2}{8\pi r^2(2g-1)}+\frac{2g(f_R+\mu f_Q)\varphi_2+8\pi r^3\varphi_1-2g}{r^2(f_R+\mu f_Q)\varphi_2}\right.\\ &\left.-\frac{g^2}{16\pi\varphi_4(2g-1)[g(g-1)+r(2g-1)]}\left\{\varphi_5-2g\varphi_5-2\varphi_3+\frac{\varphi_3}{g}\right\}\right],\end{aligned} \quad (61)$$

$$z=\frac{-g+\frac{8\pi r^3\varphi_1}{f_R+\mu f_Q}+\varphi_2(2g-1)}{r\varphi_2(2g-1)}. \quad (62)$$

Equations (61) and (62) illustrate that the *GFs* depend on the physical characteristics of the fluid and the correction terms of `f(R,T,Q)` theory.

7.2. Solution with $Y_{TF}=0$

A model different from the conformally flat solution that satisfies the condition of diminishing `complexity factor` ($Y_{TF}=0$) would be interesting to find, as the scalar Y_{TF} is specified to be an appropriate evaluation of the complexity of the matter content. This is due to its property to define the density inhomogeneity and pressure anisotropy [46]. Since there are infinitely many possible solutions, we must apply a further limitation in order to produce a particular model. In this case, we will assume that $P_r=0$ along with the limitation of $Y_{TF}=0$. Utilizing the relation $e^{-\lambda}+1=g$ in Equation (52), we achieve

$$\nu'=\frac{\frac{-2g}{r}+\frac{8\pi r^2\varphi_1}{f_R+\mu f_Q}}{\varphi_2(2g-1)}. \quad (63)$$

Placing ν' from Equation (63) into Equation (40), we acquire

$$\begin{aligned}&g'\left[r\varphi_2(3g-2g^2-1)+\frac{4\Pi r^4\varphi_1\varphi_2(1-2g)}{f_R+\mu f_Q}\right]+g[g(10g-1-8\varphi_2+4r\varphi_2'(g-1))\\&+\varphi_6]+\varphi_7=0.\end{aligned} \quad (64)$$

Upon integration, Equation (64) turns out to be

$$g=-\int\frac{g[g(10g-1+4r\varphi_2'(g-1)-8\varphi_2)+\varphi_6]-\varphi_7}{r\varphi_2(3g-2g^2-1)+\frac{4\Pi r^4\varphi_1\varphi_2(1-2g)}{f_R+\mu f_Q}}dr+\mathfrak{C}_2.$$

The physical attributes for this peculiar model are calculated as

$$|\mu^{(M)}|=\frac{f_R+\mu f_Q}{4\pi r^2}\left[\frac{-g[g(10g-1-8\varphi_2+4r\varphi_2'(g-1))+\varphi_6]-\varphi_7}{\varphi_2(3g-2g^2-1)+\frac{4\Pi r^3\varphi_1\varphi_2(1-2g)}{f_R+\mu f_Q}}+g\right], \quad (65)$$

$$P_{\perp}^{(M)}=\frac{g(f_R+\mu f_Q)}{8\pi(2g-1)r^2}\left[\frac{-g[g(10g-1-8\varphi_2+4r\varphi_2'(g-1))+\varphi_6]-\varphi_7}{\varphi_2(3g-2g^2-1)+\frac{4\Pi r^3\varphi_1\varphi_2(1-2g)}{f_R+\mu f_Q}}+g\right]. \quad (66)$$

The *GFs* for this model are achieved as

$$\begin{aligned}
\Pi^{(M)} &= -\frac{g(f_R+\mu f_Q)}{8\pi(2g-1)r^2}\left[\frac{-g[g(10g-1-8\varphi_2+4r\varphi_2'(g-1))+\varphi_6]-\varphi_7}{\varphi_2(3g-2g^2-1)+\frac{4\Pi r^3\varphi_1\varphi_2(1-2g)}{f_R+\mu f_Q}}+g\right]+ \\
&\frac{2g(f_R+\mu f_Q)\varphi_2+8\pi r^3\varphi_1-2g}{r^2(f_R+\mu f_Q)\varphi_2}, \\
z &= \frac{-g+\frac{8\pi r^3\varphi_1}{f_R+\mu f_Q}+\varphi_2(2g-1)}{r\varphi_2(2g-1)}.
\end{aligned} \tag{67}$$

Equation (67) depicts that the *GFs* can be expressed in terms of the state determinants and the higher curvature terms.

7.3. Solution Using Stiff EoS

This subsection is specified to find the results of *HS* spacetime obeying a stiff equation of state (EoS). Therefore, to obey this state, the difference between modified radial pressure and modified matter density must be nil, i.e.,

$$|\mu^{(M)}| = P_r^{(M)}. \tag{68}$$

According to condition (68), Equation (15) takes the form

$$\frac{\partial}{\partial r}\left[\frac{P_r^{(M)}}{f_R+\mu f_Q}\right] + \frac{2\Pi^{(M)}}{r(f_R+\mu f_Q)} = e^{\lambda}Z. \tag{69}$$

Besides upholding the restriction of the stiff state equation, this aforementioned nonconserved equation can be applied to the ultradense matter spread throughout the region. To continue our study, we will consider two different conditions. The first one is to assume the modified tangential pressure is equal to zero, i.e.,

- $P_\perp^{(M)} = 0$

  Implementing the previously mentioned condition (68) on Equation (69), we acquire

$$\frac{\partial}{\partial r}\left[\frac{P_r^{(M)}}{f_R+\mu f_Q}\right] + \frac{2P_r^{(M)}}{r(f_R+\mu f_Q)} = e^{\lambda}Z. \tag{70}$$

  The solution of Equation (70) reads

$$\begin{aligned}
P_r^{(M)} &= \frac{\mathfrak{K}}{r^2} + \frac{f_R+\mu f_Q}{r^2}\int r^2 e^{\lambda} Z dr, \\
|\mu^{(M)}| &= \frac{\mathfrak{K}}{r^2} + \frac{f_R+\mu f_Q}{r^2}\int r^2 e^{\lambda} Z dr,
\end{aligned} \tag{71}$$

  where $\mathfrak{K}$ is the integration constant. The mass function and metric potentials are evaluated for this condition as

$$\begin{aligned}
&m = 4\pi\int_0^r \frac{\mathfrak{K}}{f_R+\mu f_Q} + 4\pi\int_0^r \int [r^2 e^{\lambda} Z] dr dr, \\
&e^{-\lambda} = \frac{8\pi}{r}\int_0^r \frac{\mathfrak{K}}{f_R+\mu f_Q} + \frac{8\pi}{r}\int_0^r \int [r^2 e^{\lambda} Z] dr dr - 1, \\
&\nu' = \frac{-8\pi\left[\int_0^r \frac{\mathfrak{K}}{f_R+\mu f_Q} + \int_0^r \int [r^2 e^{\lambda} Z] dr dr\right]}{r\left(2\left[4\pi\int_0^r \frac{\mathfrak{K}}{f_R+\mu f_Q} + 4\pi\int_0^r \int [r^2 e^{\lambda} Z] dr dr\right] - r\right)} + \\
&\frac{8\pi\left[\frac{\mathfrak{K}}{r^2} + \frac{f_R+\mu f_Q}{r^2}\int r^2 e^{\lambda} Z dr\right] r^3}{r(f_R+\mu f_Q)\left(2\left[4\pi\int_0^r \frac{\mathfrak{K}}{f_R+\mu f_Q} + 4\pi\int_0^r \int [r^2 e^{\lambda} Z] dr dr\right] - r\right)}.
\end{aligned} \tag{72}$$

The *GFs* for this specific model are determined as

$$\Pi^{(M)} = \frac{\mathfrak{K}}{r^2} + \frac{f_R+\mu f_Q}{r^2}\int(r^2e^{\lambda}Z)dr,$$
$$z = \frac{1}{r} + \frac{4\pi\left[\frac{\mathfrak{K}}{r^2}+\frac{f_R+\mu f_Q}{r^2}\int r^2e^{\lambda}Zdr\right]r^3}{r(f_R+\mu f_Q)\left(2\left[4\pi\int_0^r\frac{\mathfrak{K}}{f_R+\mu f_Q}+4\pi\int_0^r\int[r^2e^{\lambda}Z]drdr\right]-r\right)}$$
$$+\frac{-4\pi\left[\int_0^r\frac{\mathfrak{K}}{f_R+\mu f_Q}+\int_0^r\int[r^2e^{\lambda}Z]drdr\right]}{r\left(2\left[4\pi\int_0^r\frac{\mathfrak{K}}{f_R+\mu f_Q}+4\pi\int_0^r\int[r^2e^{\lambda}Z]drdr\right]-r\right)}. \quad (73)$$

Next, to witness the fluid configuration following the stiff EoS linked with the non-complex *HS* system, we assume the second condition as

- $Y_{TF}=0$
  The application of previously stated condition (68) on Equation (41) yields

$$\frac{\partial^2}{\partial r^2}\left[\frac{P_r^{(M)}}{f_R+\mu f_Q}\right]+\frac{2}{r}\frac{\partial}{\partial r}\left[\frac{P_r^{(M)}}{f_R+\mu f_Q}\right]=\frac{6\Pi}{r^2}\left[\frac{f_T+\frac{Rf_Q}{2}+1}{f_R+\mu f_Q}\right]+\frac{1}{r^4}\left[\frac{d(Ze^{\lambda}r^4)}{dr}\right]$$
$$+\frac{2}{r}\frac{\partial}{\partial r}\left[\frac{\Pi\left(f_T+\frac{Rf_Q}{2}+1\right)}{f_R+\mu f_Q}\right]+\frac{\frac{\partial(r^3\psi_1)}{\partial r}}{2\pi r^4}, \quad (74)$$

The formal solution of Equation (74) is

$$\frac{P_r^{(M)}}{f_R+\mu f_Q}=\int\frac{2\Pi}{r}\left[\frac{f_T+\frac{Rf_Q}{2}+1}{f_R+\mu f_Q}\right]+\int\frac{6}{r^2}\left[\int\frac{\Pi\left(f_T+\frac{Rf_Q}{2}+1\right)}{f_R+\mu f_Q}dr\right]+\frac{b}{r^2}-a$$
$$-\int\frac{1}{r^2}\int r^2\frac{\partial}{\partial r}\left[\frac{\Pi\left(f_T+\frac{Rf_Q}{2}+1\right)}{f_R+\mu f_Q}dr\right]+\int\frac{1}{r^2}\left[\int\frac{\frac{\partial(r^3\psi_1)}{\partial r}}{2\pi r^4}dr+Ze^{\lambda}r^4\right]dr. \quad (75)$$

The mass function for this specific model is evaluated as

$$m=4\pi\int\left\{\int\frac{2\Pi}{r}\left[\frac{f_T+\frac{Rf_Q}{2}+1}{f_R+\mu f_Q}\right]+\int\frac{6}{r^2}\left[\int\frac{\Pi\left(f_T+\frac{Rf_Q}{2}+1\right)}{f_R+\mu f_Q}dr\right]-\int\frac{1}{r^2}\int r^2\frac{\partial}{\partial r}\right.$$
$$\left.\left[\frac{\Pi\left(f_T+\frac{Rf_Q}{2}+1\right)}{f_R+\mu f_Q}dr\right]+\int\frac{1}{r^2}\left[\int\frac{\frac{\partial(r^3\psi_1)}{\partial r}}{2\pi r^4}dr+Ze^{\lambda}r^4\right]dr\right\}r^2dr+4\pi br-\frac{4\pi ar^3}{3}. \quad (76)$$

Equation (76) illustrates the mass function under the influence of extra curvature terms. One may be able to determine the values of the metric coefficients using this equation.

8. Conclusions

The more extended version of the matter Lagrangian L_m has been adopted in these types of theories to describe the strong nonminimal connection between matter and geometry with the help of the inclusion of the term Q. Such coupling can be seen in Einstein–Born–Infeld theories. According to Odintsov and Sáez-Gómez [47], under some circumstances, f(R,T,Q) theory would be able to illustrate some of the insights offered by Hořava-like gravity. Therefore, these theories could be seen as a theoretical link between modified theories of gravity and Hořava–Lifshitz gravity. The f(R,T,Q) theory is reduced to the gravity f(R,T) theory in the situation $Q=0$. The f(R) theory, however, results from the vacuum case of f(R,T,Q) theory. Even if we examine the traceless energy-momentum tensor, i.e., $T=0$, this theory will be able to explain the nonminimal coupling to the electromagnetic field owing to the inclusion of term Q.

These theories may illustrate late-time acceleration without relying on the cosmological constant or dark energy. If we consider $L_m = -\mu$, these theories include the consequences of the additional force exerted on the massive particle. The peculiarities of galaxy rotation curves might be described in the existence of this additional force. Based on various modified theories [48–50], the related effective matter geometry coupling favors the nongeodesic motion of test particles, which produces additional force. Moreover, this theory achieves complicated modified equations. Eventually, different forms of function of f(R,T,Q) can be used to produce distinct qualitative cosmological solutions.

The fundamental characteristics of the fluid are significantly correlated with the structure scalars. These scalars make it simple to manage complex systems and can be used as a tool to address various essential facets of the system, such as density inhomogeneity, shear and expansion evolution, complexity, etc. These scalars provide plenty of information about how the system evolves. The structure scalar Y_{TF} is termed as `complexity factor` as it contains the maximum information about the evolution of the system. Subsequently, we evaluated `complexity factor` Y_{TF} and gathered the following results about it.

- Under the effect of modified terms, the factor Y_{TF} incorporates inhomogeneous energy density and locally anisotropic pressure, in the context of the f(R,T,Q) theory.
- Taking into account additional higher-degree factors of modified theory, the quantity Y_{TF} assesses the Tolman mass in terms of inhomogeneous energy density and anisotropic pressure.
- In a nonstatic dissipative matter distribution, this scalar might hold dissipative fluxes with irregularities in pressure anisotropy and density with the involvement of modified terms.

The phenomena of core development can also be observed in the absence of expansion. The matter content develops without being compressed in this case. For example, during the growth of a spherical stellar gradient, changes in its volume cause a comparable expansion in the exterior hypersurface, which counteracts a similar expansion in the interior surface. As a result, the zero expansion scalar starts a special type of system evolution in which the innermost shell drags away from the central region, ending in the vacuum core. Based on this idea, expansion-free matter populations might be useful for explaining voids.

It is worthwhile to stress that in our modeling, the weak energy requirement is violated. This indicates that modified energy density is inevitably negative in the context of f(R,T,Q) gravity. One can also witness this from Equation (12). Usually, negative matter densities are suggested in extreme cosmological and astrophysical situations, particularly with regard to quantum occurrences that might occur within the horizon. The Tolman mass, also known as the active gravitational mass, is the mass function that describes the source's entire mass and energy content. The expression for Tolman mass is evaluated in Equation (23); it reveals the physical significance that m_T is an effective inertial mass. Moreover, in the locally anisotropic static metric, it supplies the repulsive nature (if $4\pi P_r^{(M)} r^3 < m(f_R + \mu f_Q)$ caused by the gravitational force.

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Appendix A

The terms $\mu^{(M)}, P_r^{(M)}$, and $P_\perp^{(M)}$ appearing in Equations (7)–(9) are described as follows:

$$\mu^{(M)}=\mu\bigg[2f_T+f_Q\bigg(-\frac{3\nu'^2}{8e^\lambda}-\frac{3\nu'}{2re^\lambda}+\frac{R}{2}+\frac{5\lambda'\nu'}{8e^\lambda}-\frac{3\nu''}{4e^\lambda}\bigg)+1+\frac{f''_Q}{2e^\lambda}+f'_Q\bigg(\frac{1}{re^\lambda}-\frac{\lambda'}{4e^\lambda}\bigg)\bigg]+$$
$$\mu'\bigg[\bigg(\frac{1}{r}-\frac{\lambda'}{4}\bigg)f_Qe^{-\lambda}+\frac{f'_Q}{e^\lambda}\bigg]+\frac{\mu''f_Q}{2e^\lambda}+P_r\bigg[f_Q\bigg(-\frac{1}{r^2e^\lambda}+\frac{\lambda'}{2re^\lambda}+\frac{\nu'^2}{8e^\lambda}-\frac{\lambda'\nu'}{8e^\lambda}+\frac{\nu''}{4e^\lambda}\bigg)+$$
$$\bigg(\frac{\lambda'}{4}-\frac{2}{r}\bigg)f'_Qe^{-\lambda}-\frac{f''_Q}{2e^\lambda}\bigg]+P'_r\bigg[f_Q\bigg(+\frac{e^{-\lambda}\lambda'}{4}-\frac{2e^{-\lambda}}{r}\bigg)-\frac{f'_Q}{e^\lambda}\bigg]+\frac{P'_\perp f_Q}{re^\lambda}-\frac{P''_rf_Q}{2e^\lambda}+P_\perp\bigg[f_Q$$
$$\bigg(\frac{1}{r^2e^\lambda}+\frac{\nu'}{2re^\lambda}-\frac{\lambda'}{2re^\lambda}\bigg)+\frac{f'_Q}{re^\lambda}\bigg]+\bigg(\frac{f}{2}-\frac{f_RR}{2}\bigg)+f'_R\bigg(\frac{2}{re^\lambda}-\frac{\lambda'}{2e^\lambda}\bigg)+\frac{f''_R}{e^\lambda},$$
$$P_r^{(M)}=\mu\bigg[f_Q-f_T\bigg(\frac{e^{-\lambda}\nu'^2}{8}+\frac{e^{-\lambda}\nu'}{2r}-\frac{\lambda'\nu'}{8e^\lambda}+\frac{\nu''}{4e^\lambda}\bigg)-\frac{f'_Q\nu'}{4e^\lambda}\bigg]-\frac{\mu'\nu'f_Q}{4}+P_r\bigg[f_T+1+$$
$$f_Q\bigg(-\frac{3\nu'^2}{8e^\lambda}+\frac{R}{2}+\frac{\nu'e^{-\lambda}}{r}+\frac{1}{r^2e^\lambda}+\frac{3\lambda'\nu'}{8e^\lambda}+\frac{3\lambda'}{2re^\lambda}-\frac{3\nu''}{4e^\lambda}\bigg)+f'_Q\bigg(\frac{1}{re^\lambda}-\frac{\nu'}{4e^\lambda}+\frac{\nu'}{2e^\lambda}\bigg)\bigg]+$$
$$f_QP'_r\bigg[\frac{e^{-\lambda}}{r}+\frac{e^{-\lambda}\nu'}{4}\bigg]+f_QP_\perp\bigg[\bigg(-\frac{\nu'}{2re^\lambda}-\frac{1}{r^2e^\lambda}+\frac{\lambda'}{2re^\lambda}\bigg)+\frac{f'_Qe^{-\lambda}}{r}\bigg]+\frac{f_QP'_\perp}{r}$$
$$-\bigg(\frac{f}{2}-\frac{f_RR}{2}\bigg)-\frac{f'_R}{2e^\lambda}\bigg(\nu'+\frac{4}{r}\bigg),$$
$$P_\perp^{(M)}=\mu\bigg[-f_T+f_Q\bigg(\frac{\nu'^2e^{-\lambda}}{8}+\frac{\nu'e^{-\lambda}}{2r}-\frac{\lambda'e^{-\lambda}\nu'}{8}+\frac{\nu''}{4e^\lambda}\bigg)+\frac{e^{-\lambda}f'_Q\nu'}{4}\bigg]+\frac{e^{-\lambda}\mu'f_Q\nu'}{4}$$
$$+P_r\bigg[f_Q\bigg(\frac{\nu'^2e^{-\lambda}}{8}+\frac{\nu'}{2re^\lambda}-\frac{\lambda'e^{-\lambda}\nu'}{8}+\frac{\nu''}{4e^\lambda}\bigg)+f'_Q\bigg(\frac{\nu'}{2e^\lambda}-\frac{e^{-\lambda}\lambda'}{4}+\frac{e^{-\lambda}}{r}\bigg)+\frac{f''_Q}{2e^\lambda}\bigg]+$$
$$P'_r\bigg[f_Q\bigg(\frac{\nu'}{2e^\lambda}+\frac{1}{re^\lambda}-\frac{\lambda'}{4e^\lambda}\bigg)+\frac{f'_Q}{e^\lambda}\bigg]+\frac{P''_rf_Q}{2e^\lambda}+P_\perp\bigg[f_T+1+f_Q\bigg(-\frac{2}{r^2e^\lambda}+\frac{R}{2}+\frac{\lambda'}{re^\lambda}-$$
$$\frac{\nu'}{re^\lambda}+\frac{2}{r^2}\bigg)+f'_Q\bigg(\frac{\nu'}{4e^\lambda}-\frac{\lambda'}{4e^\lambda}\bigg)+\frac{f''_Q}{2e^\lambda}\bigg]+P'_\perp\bigg[f_Q\bigg(\frac{\nu'}{4e^\lambda}-\frac{\lambda'}{4e^\lambda}+\frac{2}{re^\lambda}\bigg)+\frac{f'_Q}{e^\lambda}\bigg]+\frac{P''_\perp f_Q}{2e^\lambda}-$$
$$\frac{R}{2}\bigg(\frac{f}{R}-f_R\bigg)+f'_R\bigg(\frac{\lambda'}{2e^\lambda}-\frac{1}{re^\lambda}-\frac{\nu'}{2e^\lambda}\bigg)-\frac{f''_R}{e^\lambda}.$$

The term Z appearing in Equation (10) due to nonconservation of energy-momentum tensor is evaluated as follows:

$$Z=\frac{2}{\big(2+Rf_Q+2f_T\big)}\bigg[f'_Qe^{-\lambda}P_r\bigg(\frac{\nu'}{r}-\frac{e^\lambda}{r^2}+\frac{1}{r^2}\bigg)+f_Qe^{-\lambda}P_r\bigg(\frac{\nu''}{r}-\frac{\lambda'}{r^2}-\frac{\nu'\lambda'}{r}-\frac{\nu'}{r^2}$$
$$+\frac{2e^\lambda}{r^3}-\frac{2}{r^3}\bigg)+\frac{f_Qe^{-\lambda}}{2}P'_r\bigg(-\frac{\nu'^2}{4}+\frac{\nu'\lambda'}{4}-\frac{\nu''}{4}+\frac{\lambda'}{r}-\frac{f_T}{2}\bigg)-\mu f'_T-\mu'\bigg\{\frac{e^{-\lambda}f_Q}{8}$$
$$\times\bigg(-\lambda'\nu'+\nu'^2+2\nu''+4\nu'r^{-1}\bigg)+\frac{3f_T}{2}\bigg\}-f_QP'_\perp\bigg\{\frac{e^{-\lambda}}{r}\bigg(\frac{\lambda'}{2}-\frac{1}{r}-\frac{\nu'}{2}\frac{e^\lambda}{r}\bigg)-\frac{f_T}{f_Q}\bigg\}$$
$$-\bigg(\frac{1}{r^2}-\frac{e^{-\lambda}}{r^2}-\frac{\nu'}{re^\lambda}\bigg)\bigg(f_Q\mu'+f'_Q\mu\bigg)+P_rf'_T\bigg].$$

The value of D_0 appearing in Equation (55) is defined as follows:

$$D_0=-\frac{2r^4\varphi_1e^\lambda}{\varphi_2(f_R+\mu f_Q)}\bigg[\frac{1-\varphi_2}{r\varphi_2}+\frac{e^\lambda}{r\varphi_2}\bigg(1-\frac{r^3\varphi_1}{f_R+\mu f_Q}\bigg)+\frac{(f_R+\mu f_Q)'}{f_R+\mu f_Q}-\frac{\varphi'_1}{\varphi_1}+\frac{\varphi'_2}{\varphi_2}\bigg]$$

The terms φ_1 and φ_2 occurring in Equation (64) are calculated as follows:

$$\varphi_1 = -\mu f_T + \frac{P_\perp}{re^\lambda}\left(\frac{-f_Q}{r} + f'_Q + \frac{f_Q\lambda' e^\lambda}{2}\right) + \frac{P'_\perp f_Q}{re^\lambda} - \frac{f}{2} + \frac{f_R R}{2} - \frac{2f'_R e^{-\lambda}}{r}$$

$$\varphi_2 = 1 - \frac{8\pi re^\lambda}{f_R + \mu f_Q}\left[\frac{\mu}{e^\lambda}\left(\frac{f_Q}{2r} - \frac{f'_Q}{4} - \frac{f_Q\lambda'}{8}\right) - \frac{f_Q\mu'}{4} - \frac{P_\perp f_Q}{2re^\lambda} - \frac{f'_R e^{-\lambda}}{2}\right]$$

The terms φ_3, φ_4, and φ_5 appearing in Equation (56) are calculated as follows:

$$\varphi_3 = \frac{8\pi^2 r^4 \varphi_1^2}{(2g-1)\varphi_2^2(f_R+\mu f_Q)^2} + \frac{2\pi r^2\varphi_1}{\varphi_2(f_R+\mu f_Q)^2} - \frac{2\pi r(\varphi_2 - r\varphi'_1\varphi_2 + r\varphi_1\varphi'_2)}{\varphi_2^2(f_R+\mu f_Q)},$$

$$\varphi_4 = \frac{g(-2g+3)-(\varphi_2-1)}{2r(2g-1)\varphi_2} - \frac{2\pi r^2\varphi_1}{(f_R+\mu f_Q)\varphi_2},$$

$$\varphi_5 = \frac{g}{2r^2\varphi_2^2} + \frac{4\pi r\varphi_1(1-2g+4\pi r^3\varphi_1)}{(2g-1)\varphi_2^2(f_R+\mu f_Q)} - \frac{4\pi r^2\varphi_1(f_Q\mu' - \mu f_Q + f'_R)}{\varphi_2(f_R+\mu f_Q)^2} + \frac{4\pi r\varphi_1(2g-1)}{\varphi_2 r^2(f_R+\mu f_Q)}\left(1 + \frac{1}{\varphi_2} + \frac{\varphi'_2}{2r\varphi_2^2}\right).$$

The values of φ_6 and φ_7 occurring in Equation (64) are

$$\varphi_6 = 16\pi r^4\varphi_1\varphi'_2(g-1) + \frac{8\pi r^3\varphi_1}{f_R+\mu f_Q}\left[1 - 2g^2 + 2\varphi_2(g-1) + r\varphi'_2(-4g+3) + \frac{4\pi r^3\varphi_1 + 2rf_Q\varphi_2\mu'(g-1) + 2rf'_Q\varphi_2\mu(g-1) + 2rf'_R\varphi_2(g-1)}{f_R+\mu f_Q}\right]$$

$$\varphi_7 = \frac{4\pi r^3\varphi_1}{f_R+\mu f_Q}\left[\varphi_2 - r\varphi'_2 - \frac{4\pi r^3\varphi_1 - r\varphi_2\mu' f_Q - r\varphi_2\mu f'_Q - r\chi_2 f'_R}{f_R+\mu f_Q} + \psi_1\right]$$

The term ψ_1 occurring in Equation (40) is calculated as follows:

$$\psi_1 = -\frac{4\pi}{f_R(K_\sigma K_\alpha + \frac{h_{\sigma\alpha}}{3})}\Bigg(-\frac{2\pi}{f_R+\mu f_Q}\left[h^\rho_\sigma h^\pi_\alpha \Box(f_Q T_{\rho\pi}) - \Box(f_Q T_{\sigma\alpha}) - V_\sigma V_\alpha V_\gamma V^\nu \Box(f_Q T^\gamma_\nu)\right] + \frac{4\pi}{f_R+\mu f_Q}\Big[h^\rho_\sigma h^\pi_\alpha \nabla_\pi\nabla_\rho f_R - \nabla_\sigma\nabla_\alpha f_R - V_\sigma V_\alpha V_\gamma V^\nu \nabla^\gamma\nabla_\nu f_R + f_Q(h^\rho_\sigma h^\beta_\alpha R_{\rho\beta}P - h^\beta_\alpha R_{\sigma\beta}P - h^\rho_\sigma R_{\rho\beta}\Pi^\beta_\alpha + R_{\sigma\beta}\Pi^\beta_\alpha) + f_Q(h^\beta_\sigma h^\pi_\alpha R_{\beta\pi}P - h^\beta_\sigma R_{\beta\alpha}P - h^\pi_\alpha R_{\beta\pi}\Pi^\beta_\sigma + R_{\beta\alpha}\Pi^\beta_\sigma) + \frac{1}{2}\Big\{h^\rho_\sigma h^\pi_\alpha \nabla_\beta\nabla_\rho (f_Q T^\beta_\pi) + h^\rho_\sigma h^\pi_\alpha \nabla_\beta\nabla_\pi(f_Q T^\beta_\rho) - \nabla_\beta\nabla_\sigma(f_Q T^\beta_\alpha) - \nabla_\beta\nabla_\alpha(f_Q T^\beta_\sigma) - V_\sigma V_\alpha V_\gamma V^\nu \nabla_\beta\nabla^\gamma(f_Q T^\beta_\nu) - V_\sigma V_\alpha V_\gamma V^\nu \nabla_\beta\nabla_\nu(f_Q T^{\beta\gamma})\Big\}\Big]\Bigg)$$

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