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Non-singlet structure function $F_2^{NS}(x,t)$ in DGLAP approach

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Abstract. In this work the non-singlet structure function $F_2^{NS}(x,t)$ have been obtained by solving DGLAP evolution equation in leading order (LO) at the small x limit. Here we have used a Taylor series expansion and then the method of characteristics and Lagrange's method to solve the evolution equations. We make a detailed comparison of the predictions of the two methods with two different levels of approximations with experimental data as well as with numerical solutions.

1. Introduction

The structure functions of the nucleon are not calculable in QCD but their evolution in Q^2 are predicted by a set of integro-differential equations known as DGLAP equations [1]. Apart from the numerical solution [2], there is the alternative approach of studying analytically these equations at small x and there are many analytical solutions available in literature [3, 4] and the present authors have also pursued such an approach with reasonable phenomenological success [5–7]. In this paper we study analytical solutions of the non-singlet structure functions. We convert the LO DGLAP equation into a partial differential equations in the two variables (x, Q^2) by a Taylor series expansion, with two different levels of approximations, valid at low x. The resulting partial differential equations are then solved analytically by two different methods: Lagrange's auxiliary method [8] and the method of characteristics [9, 10]. The aim of the paper is to make a detailed comparison of the predictions of the two methods, with two different levels of approximations. In section 2 we give the formalism, section 3 is devoted to discussions of the solutions and in section 4 we give our conclusion.

2. Formalism

The evolution equation for non-singlet flavour dependent contribution, which evolve independently in the DGLAP approach can be written as [1],

$$\frac{\delta F_2^{NS}(x,t)}{\delta t} = \frac{A_f}{t}\Big[\{3+4\ln(1-x)\}F_2^{NS}(x,t) + 2\int_x^1 \frac{dz}{1-z}\left\{(1+z^2)F_2^{NS}\left(\frac{x}{z},t\right) - 2F_2^{NS}(x,t)\right\}\Big]. \quad (1)$$

Here $A_f = 4/(3\beta_0)$ and $\beta_0 = 11 - 2n_f/3$ is QCD beta function at LO. At small x, approximating $F_2^{NS}(\frac{x}{z}, t)$ on RHS of Eq.(1) and defining $u = 1 - z$ [6, 7] we get,

$$\frac{\delta F_2^{NS}(x,t)}{\delta t} = \frac{A_f}{t}\left[\{3 + 4\ln(1-x)\} F_2^{NS}(x,t) + 2\int_x^1 \frac{dz}{1-z}(z^2-1)F_2^{NS}(x,t) \right. \\ \left. +2\int_x^1 \frac{dz}{1-z}(1+z^2)(x\sum_{k=1}^{\infty} u^k)\frac{\delta F_2^{NS}(x,t)}{\delta x}\right]. \tag{2}$$

Carrying out the integration in z, we can write Eq.(2) as,

$$\frac{\delta F_2^{NS}(x,t)}{\delta t} - \frac{A_f x}{t}\frac{\delta F_2^{NS}(x,t)}{\delta x}\left[2\ln(\frac{1}{x}) + (1-x^2)\right] = \frac{A_f}{t}\left[3 + 4\ln(1-x) \right. \\ \left. + (x-1)(x+3)\right] F_2^{NS}(x,t). \tag{3}$$

Eq.(3) is a partial differential equation for the non-singlet structure function $F_2^{NS}(x,t)$ with respect to the variables x and t.

While performing the integration in z, neglecting terms $\mathcal{O}(x^2)$ and higher, we can also express Eq.(2) as,

$$\frac{\delta F_2^{NS}(x,t)}{\delta t} - \frac{8A_f}{3}\frac{x}{t}\frac{\delta F_2^{NS}(x,t)}{\delta x} = \frac{A_f\{4\ln(1-x) + 2x\}}{t}F_2^{NS}(x,t) \tag{4}$$

This we get by considering,

$$x\sum_{k=1}^{\infty} u^k = xu = x(1-z) \tag{5}$$

during integration.

We solve both the PDE Eq.(3) and Eq.(4) with the two formalisms, the Lagrange's method and method of characteristics.Though both these PDE are obtained from the same Eq.(1).

2.1. Solution by the method of characteristics

By adopting the method of characteristics [9, 10], one can express Eq.(3), as an ordinary derivative with respect to t and the equation becomes an ordinary differential equation:

$$\frac{dF_2^{NS}(x(t),t)}{dt} = c^{NS}(x(t),t)\, F_2^{NS}(x(t),t) \tag{6}$$

where

$$c^{NS}(x(t),t) = \frac{A_f\{4\ln(1-x(t)) + (x-1)(x+3)\}}{t} \tag{7}$$

Integrating Eq.(6) over t from t_0 to t along the characteristic curve, one gets the solution for the non-singlet as:

$$F_2^{NS}(x,t) = F_2^{NS}(\tau)\left(\frac{t}{t_0}\right)^{\alpha}, \alpha = \frac{8}{3\beta_0}\{2\ln x - x\} \tag{8}$$

Eq.(8) is the analytical solution of the Eq.(3) within the present formalism. Using the same formalism for the PDE Eq.(4), in a similar way we get a different form of solution for the non-singlet structure function F_2^{NS} as,

$$F_2^{NS}(x,t) = F_2^{NS}(\tau)\left(\frac{t}{t_0}\right)^{\beta}, \beta = \frac{1}{\ln\frac{t}{t_0}}\left[\frac{3}{4A_f}x\left(\left(\frac{t}{t_0}\right)^{\frac{8A_f}{3}} - 1\right) - \frac{3}{2A_f}\left\{\sum_{k=1}^{\infty}\frac{x^k}{k^2}\left(\left(\frac{t}{t_0}\right)^{\frac{8A_f k}{3}} - 1\right)\right\}\right] \tag{9}$$

Eq.(8) and Eq.(9) are the two analytical solutions of Eq.(3) and Eq.(4) respectively.

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2.2. Solution by the Lagrange's auxiliary method

To solve the equation Eq.(3) and Eq.(4) by the Lagrange's Auxiliary Method [8], we can write the Eq.(3) and Eq.(4) in the form:

$$Q(x,t)\frac{\delta F_2^{NS}(x,t)}{\delta t} + P(x,t)\frac{\delta F_2^{NS}(x,t)}{\delta x} = R(x,t,F_2^{NS}) \tag{10}$$

The general solution of the Eq.(10) is obtained by solving the following auxiliary system of ordinary differential equations,

$$\frac{dx}{P(x)} = \frac{dt}{Q(t)} = \frac{dF_2^{NS}(x,t)}{R(x,t,F_2^{NS}(x,t))} \tag{11}$$

If $u(x,t,F_2^{NS}) = C_1$ and $v(x,t,F_2^{NS}) = C_2$ are the two independent solutions of Eq.(15), then in general, the solution of Eq.(10) is,

$$F(u,v) = 0 \tag{12}$$

where F is an arbitrary function of u and v.

In this approach we try to find a specific solution that satisfies some physical conditions on the structure function. Such a solution can be extracted from the combination of u and v linear in F_2^{NS},

$$u + \alpha v = \beta \tag{13}$$

where α and β are two quantities to be determined from the boundary conditions on F_2^{NS}. Using the physically plausible boundary conditions, the solution of Eq.(10) takes the form,

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t}{t_0}\right) \tag{14}$$

In a similar way for the Eq.(4), the Lagrange's method leads us to a solution for the non-singlet structure function F_2^{NS} as given below,

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t}{t_0}\right)h(x,t) \tag{15}$$

where

$$h(x,t) = \frac{X^{NS}(1) - X^{NS}(x)]}{[(\frac{t}{t_0})X^{NS}(1) - X^{NS}(x)]} \tag{16}$$

with $h(x,t) \leq 1$ for $t \geq t_0$. Here $h(x,t)$ measures deviation of Eq.(15) from solution Eq.(14). We note that the apparent absence of $\log x$ dependence in the solution Eq.(15) is due to algebraic cancellation. Eq.(14) and Eq.(15) are the solutions of the Eq.(3) and Eq.(4) respectively.

3. Results and discussion

We test the validity and compatibility of the two sets of analytical solutions, by comparing them directly with the available data [11] and with the MSTW 08 numerical solutions [12], using the MSTW2008 [13]input. We plot our set of solutions Eq.(8), Eq.(14) and Eq.(9), Eq.(15) respectively with the BCDMS data in Fig.1, where we explore a relatively high x and Q^2 range, $(0.07 \leq x \leq 0.75)$ and $(13\,\mathrm{GeV}^2 \leq Q^2 \leq 63\,\mathrm{GeV}^2)$. From that we observe that though our both set of solutions predict the same behaviour at low value of x and agree with the numerical solutions towards low x range, as we approach the high x range our solutions overshoot both data and the numerical solutions. In case of the solutions obtained by Lagrange's method given by Eq.(14) and Eq.(15), while the Eq.(14) shows logarithmic growth with the increasing Q^2 values, the other solution Eq.(15) remains almost constant with increasing Q^2 for fixed x values. The solutions by Method of Characteristics Eq.(8) and Eq.(9), also show very slow growth with increasing Q^2 for fixed x values.

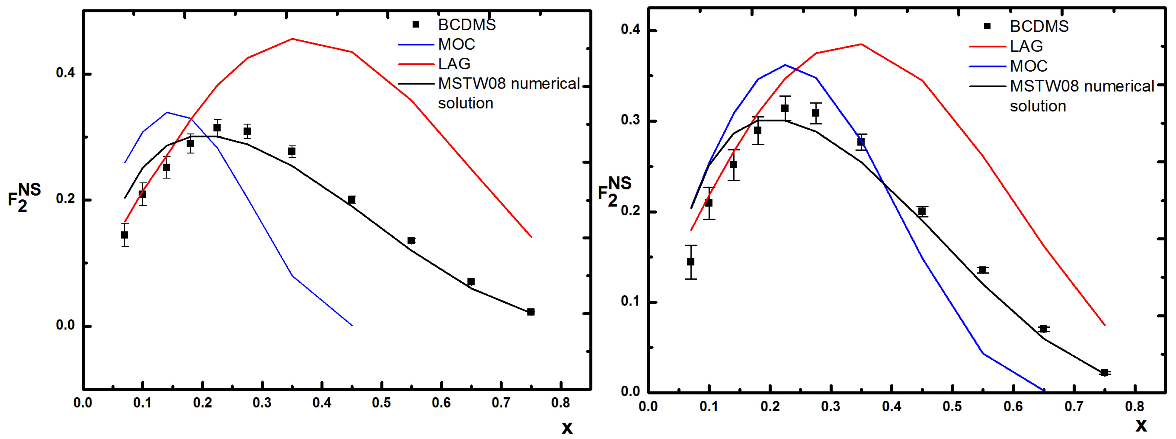

Figure 1. Non-singlet structure function $F_2^{NS}(x, Q^2)$ as function of x at different Q^2 values according to Eq.(8) and Eq.(14) (left) and values according to Eq.(9) and Eq.(15) (right). Data from refs [11].

4. Conclusion

The Taylor approximated DGLAP equation for the non-singlet structure function is solved analytically by two different methods: the Lagrange's auxiliary method and the Method of Characteristics. The quantitative and qualitative differences of the solutions are then discussed. Considering the solutions together, they are valid towards low x region. This demonstrated that two powerful methods of solving differential equations can be successfully applied in the DGLAP framework to obtain analytical solutions. Results of these methods to the polarised structure function $g_1^{NS}(x, t)$ has been reported elsewhere [14].

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