5 TWISTORS, GEOMETRY AND ALGEBRAIC ASPECTS OF SUPERSYMMETRY

An Analysis of $\mathcal{N} = 8$ Supergravity in Supertwistor Space

Yasuhiro Abe

Physics Department, City College of the CUNY,
New York, NY 10031, USA
abe@sci.ccny.cuny.edu

Abstract

By analogy with $\mathcal{N} = 4$ super Yang-Mills theory, the superspace constraint equations for $\mathcal{N} = 8$ supergravity are also solvable in a certain sector where the spinorial curvatures vanish. This sector can naturally be interpreted as an anti-self-dual part of $\mathcal{N} = 8$ supergravity. As in the Yang-Mills case, we find that the solvable part of these constraints arises from a Wess-Zumino-Witten (WZW) model whose target space is some extended superspace.

1 Introduction

It is well known that $\mathcal{N} = 8$ supergravity is closely related to the eleven-dimensional supergravity (for the foundation of this theory, see [1]; for the superspace formulation of it, see [2, 3]). It is shown by Howe [4] that the constraint equations for eleven-dimensional supergravity can be expressed as a simple supertorsion constraint by use of the so-called Weyl superspace. Dimensional reduction of this constraint leads to the $\mathcal{N} = 8$ supergravity constraints. This is analogous to how one obtains the constraints of $\mathcal{N} = 4$ super Yang-Mills theory [5] by those of ten-dimensional super Yang-Mills theory [6] via dimensional reduction. In [7], this analogous relation was utilized to investigate the geometrical meaning of superstring theory. Following these lines, in this paper we attempt to solve a subset of the constraint equations for $\mathcal{N} = 8$ supergravity. Our strategy is similar to the harmonic superspace approach which has been successful for $\mathcal{N} = 2$ super Yang-Mills theory [8] as well as for some supergravity theories [9, 10]. This paper can be considered as a natural extension of the previous work [11] on the maximally supersymmetric Yang-Mills theory to a theory of gravity.

2 Superspace constraints and dimensional reduction

It is known that the equations of motion for 10-dimensional super Yang-Mills theory are equivalent to its superspace constraints by use of Bianchi identities [6, 7]. The constraints can be expressed as a flatness condition, as we briefly review below. Ten-dimensional superspace is described by the coordinates (x^m, θ^α). The spinorial covariant derivative is given by

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\Gamma^m_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^m} \tag{1}$$

where Γ^m is a 10-dimensional gamma matrix $(m = 1, 2, \cdots, 10)$ and θ^α is the corresponding spinor $(\alpha = 1, 2, \cdots, 32)$. Gauged versions of the covariant derivatives are written as

$$\mathcal{D}_\alpha = D_\alpha + A_\alpha \ , \quad \mathcal{D}_m = \frac{\partial}{\partial x^m} + A_m \tag{2}$$

by which we can define the following field strengths on the superspace

$$\begin{aligned} F_{\alpha\beta} &= \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} + i2\Gamma^m_{\alpha\beta}\mathcal{D}_m && (3) \\ F_{\alpha m} &= [\mathcal{D}_\alpha, \mathcal{D}_m] && (4) \\ F_{mn} &= [\mathcal{D}_m, \mathcal{D}_n]. && (5) \end{aligned}$$

The constraint equations are simply expressed as

$$F_{\alpha\beta} = 0. \tag{6}$$

Under naive dimensional reduction, this constraint reduces to the superspace constraints of $\mathcal{N} = 4$ super Yang-Mills theory [5].

We would like to consider analogous constraint equations for 11-dimensional supergravity such that dimensional reduction to the $\mathcal{N} = 8$ theory is transparent. It is shown by Howe [4] that the equations of motion for 11-dimensional supergravity are described by the following (super)torsion constraints

$$T^m_{\alpha\beta} = -i2 \ \Gamma^m_{\alpha\beta} \tag{7}$$

where Γ^m is now a 11-dimensional gamma matrix $(m = 1, 2, \cdots, 11)$ and the torsion and curvature are defined by

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = T^M_{\alpha\beta}\mathcal{D}_M + R^{mn}_{\alpha\beta}\Sigma^{mn}. \tag{8}$$

Here $\mathcal{D}_M$ denote composite covariant derivatives $\mathcal{D}_M = (\mathcal{D}_m, \mathcal{D}_\alpha)$, while Σ^{mn} denotes the Lorentz generator on 10-dimensional (vectorial) space. Gauge potentials relevant to $\mathcal{D}_M$ can be defined as

$$A_M = e^N_M D_N + \Omega^{mn}_M \Sigma^{mn}. \tag{9}$$

(Note that one can impose $T^\gamma_{\alpha\beta} = 0$ by an analysis on the so-called Weyl superspace [4].)

Dimensional reduction of the constraint (7) can be carried out and we obtain

$$\begin{aligned} T^{\mu \ j}_{Ai\dot{B}} &= -i2 \ (-1)^{i(i-1)/2} \ \delta^j_i \ \sigma^\mu_{A\dot{B}} && (10) \\ T^\mu_{AiBj} &= T^{\mu \ ij}_{\dot{A}\dot{B}} = 0 && (11) \end{aligned}$$

where $\mu = 1, 2, 3, 4$ and $\sigma^\mu = (1, -\sigma^i)$ with σ^i being the Pauli matrices. Equations (10) and (11) can be considered as the superspace constraints of $\mathcal{N} = 8$ supergravity.

3 A sector of vanishing spinorial curvature and self-duality

Eleven-dimensional supergravity has three dynamical fields in x-space, *i.e.*, the 11-bein (graviton), the Rarita-Schwinger field (gravitino) and a totally antisymmetric tensor field X_{mnr} [1]. In the superspace formulation [3], the torsions and curvatures are all described by a single superfield H_{abcd} which is defined by $H_{abcd} = \partial_a \wedge X_{bcd}$ ($a, b, c, d = 1, 2, \cdots, 11$ are vectorial indices). In the previous section, we observe that there is a direct analogy between Yang-Mills theory and general relativity in a subspace where those terms that involve the Lorentz generator Σ^{mn} are negligible. This subspace can be identified with a condition that a spinorial curvature $R^{mn}_{\alpha\beta}$ vanishes in the definition (8). In terms of H_{abcd}, the spinorial curvature is expressed by

$$R^{ab}_{\alpha\beta} = \frac{1}{6}\left[(\Gamma^{cd})_{\alpha\beta}H_{abcd} + \frac{1}{3}(\Gamma_{abcdef})_{\alpha\beta}H^{cdef}\right] \tag{12}$$

where $\Gamma^{a_1 a_2 \cdots a_n} = \Gamma^{[a_1}\Gamma^{a_2}\cdots\Gamma^{a_n]}$ are the antisymmetrized product of 11-dimensional gamma matrices (up to normalization). Under dimensional reduction, the vanishing of (12) leads to the relation

$$(\gamma^{cd})_{\alpha\beta}H_{abcd} = -\frac{1}{2}\ \epsilon_{abcd}\ (\gamma^0\gamma_{ef})_{\alpha\beta}H^{cdef} \tag{13}$$

where γ's are the usual 4-dimensional gamma matrices and $\gamma_{abcd} = \epsilon_{abcd}\gamma^0$. Notice the indices are now reduced to $1, 2, 3, 4$. This relation can be further written as

$$W_{ab} = -\frac{1}{2}\ \epsilon_{abcd}\ \gamma^0 W^{cd} \tag{14}$$

with an introduction of a matrix field $W_{ab} = \gamma^{cd}H_{abcd}$. The relation (14) can be seen as an anti-self-dual condition for W_{ab}. In this sense, the sector of vanishing $R^{ab}_{\alpha\beta}$ can be considered as 'anti-self-dual' supergravity, although self-dual supergravity is generally defined in a different manner (see, for example, [12]).

The vectorial curvature in 11-dimensions are defined by [3]

$$R^{cd}_{ab} = (\Gamma^{cd})_{\alpha\beta}R^{\alpha\beta}_{ab} \tag{15}$$

$$R_{ab,\gamma\delta} = D_a T_{b\gamma\delta} - D_b T_{a\gamma\delta} + D_\gamma T_{ab\delta} + T^{\epsilon}_{a\gamma}T_{b\epsilon\delta} - T^{\epsilon}_{b\gamma}T_{a\epsilon\gamma} \tag{16}$$

$$T^{\gamma}_{a\beta} = -\frac{1}{36}\left[(\Gamma^{bcd})^{\gamma}_{\beta}H_{abcd} + \frac{1}{8}(\Gamma_{abcde})^{\gamma}_{\beta}H^{bcde}\right] \tag{17}$$

$$T_{ab\alpha} = -\frac{i}{42}(\Gamma^{cd})^{\beta}_{\alpha}D_\beta H_{abcd}. \tag{18}$$

Under dimensional reduction we find $T^{\gamma}_{a\beta} = 0$, using $\{\gamma^0, \gamma_a\} = 0$. The vectorial curvature then reduces to

$$\begin{aligned} R^{cd}_{ab} &= D_\alpha D_\beta(\gamma^{cd}\gamma^{ef})^{\alpha\beta}H_{abef} \\ &= D_\alpha D_\beta(\gamma^{cd}W_{ab})^{\alpha\beta} \\ &= -i(\gamma^\mu)_{\alpha\beta}\frac{\partial}{\partial x^\mu}(\gamma^{cd}W_{ab})^{\alpha\beta} \end{aligned} \tag{19}$$

where we use $\{D_\alpha, D_\beta\} = -i2(\Gamma^m)_{\alpha\beta}\frac{\partial}{\partial x^m}$ to obtain the last line (with $m \to \mu = 1,2,3,4$). This reduction means that the vectorial (non-supersymmetric) curvature *does* have a nontrivial value in the sector of our interest and that this sector is indeed physically sensible.

4 A solution to the constraints of $\mathcal{N} = 8$ 'anti-self-dual' supergravity

Under the 'anti-self-dual' condition (14), the constraints of $\mathcal{N} = 8$ supergravity (10), (11) reduce to the following forms

$$\begin{aligned} \{\mathcal{D}_{Ai}, \mathcal{D}_{Bj}\} &= \{\mathcal{D}^i_{\dot{A}}, \mathcal{D}^j_{\dot{B}}\} \;=\; 0, &(20)\\ \{\mathcal{D}_{Ai}, \mathcal{D}^j_{\dot{B}}\} &= -i2\,(-1)^{i(i-1)/2}\,\delta^j_i\,\mathcal{D}_{A\dot{B}}. &(21)\end{aligned}$$

In what follows, we will obtain a solution to these constraints. We introduce a complex two-component spinor u^A with scale invariance $u^A \to \lambda u^A$, with λ being a non-zero complex variable. This spinor is closely related to the harmonic variables introduced in the construction of harmonic superspace. In our case, since there is a scale invariance on u^A, the extended superspace corresponds to supertwistor space $\mathbf{CP}^{3|\mathcal{N}}$. (In the case of $\mathcal{N} = 4$, it is known that this space is a supersymmetric Calabi-Yau manifold and one can construct string theory on it.) Motivated by our previous work [11], we then introduce the following additional derivative operators

$$D^+_i = (-1)^{i(i-1)/2}\,u^A D_{Ai} \;\;,\quad D^-_i = -(-1)^{i(i-1)/2}\,\bar{\omega}^A D_{Ai} \qquad (22)$$

where $\bar{\omega}^A$ is another two-component spinor that can be related to u^A by $\bar{\omega}^A = K^{A\dot{A}}\bar{u}_{\dot{A}}$ where $\bar{u}_{\dot{A}}$ is a complex conjugate of u^A, $\bar{u}_{\dot{A}} = (u^A)^*$, and $K_{A\dot{A}}$ is an arbitrary frame vector.

Note that we can express the constraints (20) and (21) as flatness conditions; $F_{AiBj} = F^{ij}_{\dot{A}\dot{B}} = 0$ and $F^{\;j}_{Ai\dot{B}} = 0$, respectively. Let $\mathcal{D}^\pm_i$, $\mathcal{D}^i_{\dot{A}}$ be the gauged versions of spinorial derivatives. In terms of these, the constraints can be written as $F^{++}_{ij} = F^{+-}_{ij} = F^{-+}_{ij} = F^{--}_{ij} = 0$, $F^{ij}_{\dot{A}\dot{B}} = 0$ and $F^{\pm\;j}_{i\dot{B}} = 0$, or explicitly,

$$\begin{aligned} \{\mathcal{D}^+_i, \mathcal{D}^+_j\} &= \{\mathcal{D}^+_i, \mathcal{D}^-_j\} = \{\mathcal{D}^-_i, \mathcal{D}^+_j\} = \{\mathcal{D}^-_i, \mathcal{D}^-_j\} = 0 &(23)\\ \{\mathcal{D}^i_{\dot{A}}, \mathcal{D}^j_{\dot{B}}\} &= 0 &(24)\\ \{\mathcal{D}^+_i, \mathcal{D}^j_{\dot{A}}\} &= -i2(-1)^{i(i-1)/2}\,\delta^j_i\,u^A\mathcal{D}_{A\dot{A}} &(25)\\ \{\mathcal{D}^-_i, \mathcal{D}^j_{\dot{A}}\} &= i2(-1)^{i(i-1)/2}\,\delta^j_i\,\bar{\omega}^A\mathcal{D}_{A\dot{A}}\;. &(26)\end{aligned}$$

The derivative operators on our extended superspace are expressed by $D^\pm_i$, $D^i_{\dot{A}}$, $D_{A\dot{A}} = \sigma^\mu_{A\dot{A}}\frac{\partial}{\partial x^\mu}$ along with

$$\begin{aligned} D^{++} &= u^A\frac{\partial}{\partial\bar{\omega}^A}\;,\quad D^{--} \;=\; -\bar{\omega}^A\frac{\partial}{\partial u^A}\\ D^0 &= u^A\frac{\partial}{\partial u^A} - \bar{\omega}^A\frac{\partial}{\partial\bar{\omega}^A}\;. &(27)\end{aligned}$$

Commutation and anticommutation relations among these derivatives (or bases) are given by

$$\begin{array}{l} [D^{++}, D_i^+] = 0 \ , \quad [D^{++}, D_i^-] = -D_i^+ \ , \quad [D^{++}, D^i_{\dot{A}}] = 0 \\ [D^{--}, D_i^+] = D_i^- \ , \quad [D^{--}, D_i^-] = 0 \ , \quad [D^{--}, D^i_{\dot{A}}] = 0 \\ [D^{++}, D^0] = -2D^{++} \ , \quad [D^{--}, D^0] = 2D^{--} \\ [D^0, D_i^+] = D_i^+ \ , \quad [D^0, D_i^-] = -D_i^- \ , \quad [D^0, D^i_{\dot{A}}] = 0 \end{array} \tag{28}$$

$$\begin{array}{rcl} \{D_i^+, D_j^+\} & = & \{D_i^+, D_j^-\} = \{D_i^-, D_j^+\} = \{D_i^-, D_j^-\} = 0 \\ \{D^i_{\dot{A}}, D^j_{\dot{B}}\} & = & 0 \\ \{D_i^+, D^j_{\dot{A}}\} & = & -i2(-1)^{i(i-1)/2}\ \delta_i^j\ u^A D_{A\dot{A}} \\ \{D_i^-, D^j_{\dot{A}}\} & = & i2(-1)^{i(i-1)/2}\ \delta_i^j\ \bar{\omega}^A D_{A\dot{A}}\ . \end{array} \tag{29}$$

The covariantization of anticommutators (29) are identical to the constraints in (23)-(26). The extra gauge potentials A^{++}, A^{--} do not involve in the anticommutators. The commutators in (28) mean that it is necessary to make $A^{\pm\pm}$, A^0 vanish in order to satisfy the constraints (23)-(26) in the extended superspace. These constraints are hard to solve by themselves. Our strategy to solve these is to carry out 'gauge transformaitons' in the extended superspace such that there are non-zero $A^{\pm\pm}$.

Let us look at one of the constraints, $\{\mathcal{D}_i^+, \mathcal{D}_j^+\} = \{D_i^+ + A_i^+, D_j^+ + A_j^+\} = 0$ with $A_i^+ = (-1)^{i(i-1)/2}\, u^A A_{Ai}$ ($A_{Ai} = e^M_{Ai} D_M + \Omega^{mn}_{Ai}\Sigma^{mn}$). Notice that A_i^+ can be expressed as a pure gauge form, $A_i^+ = -D_i^+ g\, g^{-1}$ such that $\{\mathcal{D}_i^+, \mathcal{D}_j^+\} = g\{D_i^+, D_j^+\}g^{-1} = 0$, where $g(x^\mu, \theta^{Ai}, \bar{\theta}^i_{\dot{A}}; u^A, \bar{\omega}^A)$ is some matrix, realizing gauge transformations on the extended superspace. One can eliminate A_i^+, using such a matrix. In doing so, the additional potentials $A^{\pm\pm}$ are no longer zero, rather, in this new gauge we have

$$\begin{array}{rcl} A'^+_i & = & 0 \\ A'^-_i & = & g^{-1}A_i^- g + g^{-1}D_i^- g \\ A'^i_{\dot{A}} & = & g^{-1}A^i_{\dot{A}} g + g^{-1}D^i_{\dot{A}} g \\ A'^{++} & = & g^{-1}D^{++}g \\ A'^{--} & = & g^{-1}D^{--}g \\ A'^0 & = & g^{-1}D^0 g. \end{array} \tag{30}$$

Note that D^0 is a charge operator, assigning +1 charge to u^A and -1 charge to $\bar{\omega}^A$. Since this charge is to be preserved for any potentials under a 'gauge transformation,' we require

$$D^0 g = \left(u^A \frac{\partial}{\partial u^A} - \bar{\omega}^A \frac{\partial}{\partial \bar{\omega}^A} \right) g = 0. \tag{31}$$

This implies $g(x, \theta, \bar{\theta}; \langle u\bar{\omega}\rangle)$, where $\langle u\bar{\omega}\rangle$ is the inner product of spinors, $\langle u\bar{\omega}\rangle = \epsilon_{AB} u^A \bar{\omega}^B$ $= u^A \bar{\omega}_A = \bar{\omega}^A u_A$. The $\langle u\bar{\omega}\rangle$ dependence may lead to (31), however, as one can easily seen, this keep $A'^{\pm\pm}$ vanishing. We are looking for a solution of the constraints (23)-(26) by executing a 'gauge transformation' in the extended supersupace such that we have non-vanishing $A'^{\pm\pm}$. This leads us to introduce the quantities

$$u^A \theta^i_A = \xi^i \quad , \quad \bar{\omega}^A \theta^i_A = \bar{\xi}^i \qquad (i = 1, 2, \cdots, 8) \tag{32}$$

We can parametrize $g = g(x, \theta, \bar{\theta}; \xi^i, \bar{\xi}^i)$ such that it is linear in ξ^i as well as in $\bar{\xi}^i$. This parametrization naturally leads $D^0 g = 0$ and non-zero $A'^{\pm\pm}$ by use of $D^{++}\bar{\xi}^i = u^A \frac{\partial}{\partial \bar{\omega}^A}\bar{\xi}^i = \xi^i$ and $D^{--}\xi^i = -\bar{\omega}^A \frac{\partial}{\partial u^A}\xi^i = -\bar{\xi}^i$.

For simplicity, let us write the gauge potentials without primes. In the new gauge, $A_i^+ = 0$, the gauged versions of the anticommutation relations (29) become

$$
\begin{aligned}
D_i^+ A_j^- = D_j^+ A_i^- &= 0 && (33)\\
D_i^- A_j^- + D_j A_i^- + \{A_i^-, A_j^-\} &= 0 && (34)\\
D_i^+ A_{\dot{A}}^j &= -i2(-1)^{i(i-1)/2}\, \delta_i^j u^A A_{A\dot{A}} && (35)\\
D_i^- A_{\dot{A}}^j + D_{\dot{A}}^j A_i^- + \{A_i^-, A_{\dot{A}}^j\} &= i2(-1)^{i(i-1)/2}\, \delta_i^j \bar{\omega}^A A_{A\dot{A}} && (36)\\
D_{\dot{A}}^i A_{\dot{B}}^j + D_{\dot{B}}^j A_{\dot{A}}^i + \{A_{\dot{A}}^i, A_{\dot{B}}^j\} &= 0 && (37)
\end{aligned}
$$

where we omit the trivial relation $\{D_i^+, D_j^+\} = 0$. The gauged versions of the commutation relations (28) become

$$
\begin{aligned}
D^{++}A^{--} - D^{--}A^{++} + [A^{++}, A^{--}] &= 0 && (38)\\
D^0 A^{++} &= 2A^{++} && (39)\\
D^0 A^{--} &= -2A^{--} && (40)\\
D_i^+ A^{++} &= 0 && (41)\\
D^{\pm\pm} A_{\dot{A}}^i - D_{\dot{A}}^i A^{\pm\pm} + [A^{\pm\pm}, A_{\dot{A}}^i] &= 0 && (42)\\
D^{\pm\pm} A_i^- - D_i^- A^{\pm\pm} + [A^{\pm\pm}, A_i^-] &= 0 && (43)\\
D_i^+ A^{--} &= -A_i^- && (44)\\
D^0 A_i^- &= -A_i^- && (45)\\
D^0 A_{\dot{A}}^i &= 0 && (46)
\end{aligned}
$$

where we also omit the trivial relation $[D^0, D_i^+] = 0$. Equations (39) and (40) imply that the group element g further has a dependence on $\xi^i\bar{\xi}^i$, that is, it is parametrized by $g(x, \theta, \bar{\theta}; \xi^i\bar{\xi}^i)$ rather than $g(x, \theta, \bar{\theta}; \xi^i, \bar{\xi}^i)$. Parametrization of $A_{\dot{A}}^i = A_{\dot{A}}^i(x, \theta, \bar{\theta}; \xi^i\bar{\xi}^i)$ and $A_{Ai} = A_{Ai}(x, \theta, \bar{\theta}; \xi^i\bar{\xi}^i)$ $(A_i^- = -(-1)^{i(i-1)/2}\bar{\omega}^A A_{Ai})$ is also compatible with (45) and (46). The expression of $A'^i_{\dot{A}}$ in (30), however, includes the term $g^{-1}D_{\dot{A}}^i g$. Because of the spinorial derivative $D_{\dot{A}}^i$ acting on $g(x, \theta, \bar{\theta}; \xi^i\bar{\xi}^i)$, this term potentially change the degree of homogeneity of ξ's or $\bar{\xi}$'s in $A'^i_{\dot{A}}$ by the 'gauge transformations' in the extended superspace. We then further impose

$$
D_{\dot{A}}^i\, g = 0\ . \qquad (47)
$$

Notice that we need to have $D_{Ai}\, g \neq 0$, since otherwise, $A_i^- = -D_i^+ A^{--} = -g^{-1}D_i^+ D^{--} g = g^{-1}(D^{--}D_i^+ - D_i^-)g$ vanishes and we will have a trivial solution. The chirality condition (47) means that the x^μ-dependence of g always comes in the form of $y^\mu = x^\mu - i\theta^{Ai}\sigma^\mu_{A\dot{A}}\bar{\theta}_i^{\dot{A}}$. This allows us to parametrize g as $g(y^\mu, \theta; \xi^i\bar{\xi}^i)$.

Let us recapitulate the results we have obtained so far. In terms of $g(y^\mu, \theta; \xi^i\bar{\xi}^i)$ we parametrize the gauge potentials as

$$
\begin{aligned}
A^{++} &= g^{-1}D^{++}g\\
A^{--} &= g^{-1}D^{--}g\\
A_i^+ &= 0 && (48)
\end{aligned}
$$

$$
\begin{aligned}
A_i^- &= -D_i^+ A^{--} = -D_i^+ (g^{-1} D^{--} g) \\
A_{\dot{A}}^i &= g^{-1} A^{(-g)}{}^i_{\dot{A}} g
\end{aligned}
$$

where $A^{(-g)}{}^i_{\dot{A}}$ is defined in an ordinary superspace, *i.e.*, $A^{(-g)}{}^i_{\dot{A}} = A^{(-g)}{}^i_{\dot{A}}(x, \theta, \bar{\theta})$. The gauged version of this, $A^i_{\dot{A}}$, is then parametrized as $A^i_{\dot{A}}(x, \theta, \bar{\theta}; \xi^i \bar{\xi^i})$ in general. In addition to the above set of potentials, we also have $A_{A\dot{A}} = A_{A\dot{A}}(x, \theta, \bar{\theta}; u, \bar{\omega})$. With the expressions in (48), one can straightforwardly check the equations (33), (34), (38) and (42). As we have seen, the relations (39), (40), (45) and (46) are imbedded in the parametrization (48).

The rest of the constraint equations can be understood as follows. We consider the equation (41) as an analyticity condition on A^{++}. Following an idea of harmonic superspace, we regard A^{++} as an unconstrained analytic function with which every potential is to be expressed. With (38) and (41), it is easy to check one of the relations in (43) involving D^{++} and A^{++}. We may have $D_i^- A^{--} = 0$ as a consequence of (41) and, with this relation, the other equation in (43) involving D^{--}, A^{--} also holds. The equation (38) can alternatively be considered as a defining equation for A^{--} in terms of A^{++} (or an expansion of A^{++}'s) as is first shown in [13]. The equation (44) then shows A_i^- is given by A^{++}. The rest of the constraints can be considered similarly, namely, we regard the equation (37) as a defining equation for $A^i_{\dot{A}}$ and the equations (35), (36) as that of $A_{A\dot{A}}$ in terms of $A^i_{\dot{A}}$. Since $A^i_{\dot{A}}$ can be given by a function of A^{++}, all gauge potentials (in extended superspace) are then expressed by $A^{++} = g^{-1} D^{++} g$, the unconstrained analytic (chiral) function of $(y^\mu, \theta; \xi^i, \bar{\xi^i})$. The parametrization of $g^{-1}(y^\mu, \theta; \xi^i \bar{\xi^i})$ and $D^{++} g \equiv g^{++}(y^\mu, \theta; \xi^i \xi^i)$ indicates that A^{++} depends on the combinations of both $\xi^i \bar{\xi^i}$ and $\xi^i \xi^i$. This implies that A^{++} contains antiholomorphic factor in terms of the spinors (which is different from what happens in the Yang-Mills theory).

We end our discussion with the following few remarks. The equation (38) can be used to determine a proper group element g. It is possible to obtain this equation as an equation of motion for a gauged Wess-Zumino-Witten (WZW) action [11]. By use of the Polyakov-Wiegman identity, this gauged WZW action becomes a WZW action whose target space is our extended superspace $\mathbf{CP}^{3|8}$. As in the Yang-Mills case, we expect that current correlators of this WZW model describes multigraviton tree level amplitudes. It is also possible to interpret graviton amplitudes in the same manner as gluon amplitudes, with an introduction of appropriate Chan-Paton factors [14]. In this context, the graviton amplitudes can arise from the so-called super-ambitwistor space $\mathbf{CP}^{3|4} \times \mathbf{CP}^{3|4}$.

Acknowledgments

The author would also like to thank Professor V.P. Nair for helpful discussions.

References

[1] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. **B76** (1978) 409.

[2] E. Cremmer and S. Ferrara, Phys. Lett. **B91** (1980) 61.

[3] L. Brink and P. Howe, Phys. Lett. **B91** (1980) 384.

[4] P.S. Howe, Phys. Lett. **B415** (1997) 149, hep-th/9707184.

[5] R. Grimm, M. Sohnius and J. Wess, Nucl. Phys. **B133** (1978) 275; M. Sohnius, ibid., **B136** (1978) 461.

[6] B.E.W. Nilssoin, Nucl. Phys. **B188** (1981) 176.

[7] E. Witten, Nucl. Phys. **B266** (1986) 245.

[8] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **1** (1984) 469; ibid. **2** 155; A.A. Rosly and A.S. Schwarz, Commun. Math. Phys. **105** (1986) 645; B.M. Zupnik, Phys. Lett. **B183** (1987) 175; Nucl. Phys. Proc. Suppl. **102** (2001) 278.

[9] A. Galperin, Nguyen Ahn Ky and E. Sokatchev, Class. Quant. Grav. **4** (1987) 1235; A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, ibid. **4** (1987) 1225.

[10] E. Ivanov and A. Sutulin, Nucl. Phys. **B432** (1994) 246; B **483** (1997) 531(E); S. Bellucci and E. Ivanov, Nucl. Phys. **B587** (2000) 445, hep-th/0003154.

[11] Y. Abe, V.P. Nair and M.-I. Park, Phys.Rev. **D71** (2005) 025002, hep-th/0408191.

[12] Ch. Devchand and V. Ogievetsky, Nucl. Phys. **B444** (1995) 381, hep-th/9501061.

[13] B.M. Zupnik, Phys. Lett. **B 183** (1987) 175.

[14] V. P. Nair, Phys.Rev. **D71** (2005) 121701, hep-th/0501143; Y. Abe, Phys. Lett. **B623** (2005) 126, hep-th/0504174.