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2015 J. Phys.: Conf. Ser. 597 012017

(http://iopscience.iop.org/1742-6596/597/1/012017)

Painlevé IV Hamiltonian systems and coherent states

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Abstract. Schrödinger Hamiltonians with third-order differential ladder operators are linked to the Painlevé IV equation. Some of these appear from applying SUSY QM to the harmonic oscillator. Departing from them, we will build coherent states as eigenstates of the annihilation operator, then as displaced versions of the extremal states, both involving the third-order ladder operators, and finally as displaced extremal states using linearized ladder operators. To each Hamiltonian corresponds two families of coherent states for fixed ladder operators: one in the infinite dimension subspace associated with the oscillator spectrum and another in the finite dimension one generated by the eigenstates created by SUSY QM.

1. Introduction

For a long time, coherent states (CS) for physical systems of different nature have been successfully derived through the several available definitions [1–10]. For instance, if the Hilbert space $\mathcal{H}$ is of finite dimension they have been built up through the group theoretical construction of Perelomov [6]. On the other hand, for systems as the harmonic oscillator, such that $\mathcal{H}$ is generated by a discrete infinite set of orthogonal eigenstates of the Hamiltonian, they can be derived using Barut-Girardello's definition, as eigenstates of the annihilation operator [5]. It would be important to address the subject for systems of mixed type, for which $\mathcal{H}$ turns out to be the direct sum of a finite dimension subspace and an infinite dimension one. Let us note that a subset of Painlevé IV Hamiltonian systems, called in this way because they are determined by solutions of the Painlevé IV equation [11], has been recently identified [12–16]. In this work we will explore the coherent states construction for systems in such a subset.

In order to do that, this paper has been organized as follows. In Section 2, the second-order polynomial Heisenberg algebras (PHA) and the general one-dimensional Schrödinger Hamiltonians ruled by them will be studied. In Section 3 the family of kth order supersymmetric (SUSY) partners of the harmonic oscillator Hamiltonian will be constructed, while in Section 4 the subfamily of Painlevé IV Hamiltonian systems, ruled as well by second-order PHA, will be identified. In Section 5 the coherent states construction for the third-order differential ladder operators generating the second-order algebra will be addressed. In Section 6 the same will be done for a pair of linearized annihilation and creation operators built from the third-order ones. Finally, Section 7 contains our conclusions.

2. Second order PHA

Systems ruled by a second-order polynomial Heisenberg algebra possess three generators $\{H, \mathcal{L}^-, \mathcal{L}^+\}$ which satisfy [13–16]:

$$\begin{aligned} \left[H, \mathcal{L}^\pm\right] &= \pm\mathcal{L}^\pm, & (1)\\ \left[\mathcal{L}^-, \mathcal{L}^+\right] &= N_3(H+1) - N_3(H) = P_2(H), & (2)\\ N_3(H) &\equiv \mathcal{L}^+\mathcal{L}^- = (H-\varepsilon_1)(H-\varepsilon_2)(H-\varepsilon_3). & (3)\end{aligned}$$

The simplest realization arises for H taking Schrödinger's form

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \tag{4}$$

where the potential $V(x)$ is to be determined, while $\mathcal{L}^+$ is a third-order differential ladder operator factorized as

$$\begin{aligned} \mathcal{L}^+ &= L_a^+ L_b^+, & (5)\\ L_a^+ &= \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right], & (6)\\ L_b^+ &= \frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g(x)\frac{\mathrm{d}}{\mathrm{d}x} + h(x)\right]. & (7)\end{aligned}$$

By solving the resulting system of equations we arrive at

$$\begin{aligned} f &= g + x, & (8)\\ h &= \frac{g'}{2} - \frac{g^2}{2} - 2xg - x^2 + a, & (9)\\ V &= \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \varepsilon_1 - \frac{1}{2}, & (10)\end{aligned}$$

where $g(x)$ satisfies

$$\begin{aligned} g'' &= \frac{g'^2}{2g} + \frac{3}{2}g^3 + 4xg^2 + 2(x^2-a)g + \frac{b}{g}, & (11)\\ a &= \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 - 1, \qquad b = -2(\varepsilon_2-\varepsilon_3)^2, & (12)\end{aligned}$$

which is the Painlevé IV equation [11]. Thus, if we find a solution $g(x)$ to Eq. (11), for given extremal energies $\varepsilon_i, i = 1, 2, 3$, then the potential $V(x)$ and the associated third-order ladder operators $\mathcal{L}^\pm$ become determined. This is the reason to call Painlevé IV Hamiltonian system to those ruled by the second-order PHA of Eqs. (1-3), with H and $\mathcal{L}^\pm$ given by Eqs. (4-12).

3. Harmonic oscillator SUSY partners

As is well known, a generic k-th order SUSY transformation applied to the harmonic oscillator produces a new Hamiltonian [17–20] (see also [21])

$$H_k = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_k(x), \tag{13}$$

whose potential (indeed a family) is expressed in terms of the initial one $V_0(x) = x^2/2$ as

$$V_k(x) = \frac{x^2}{2} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{\ln W\left[u(x,\epsilon_0), \ldots, u(x,\epsilon_{k-1})\right]\right\}, \tag{14}$$

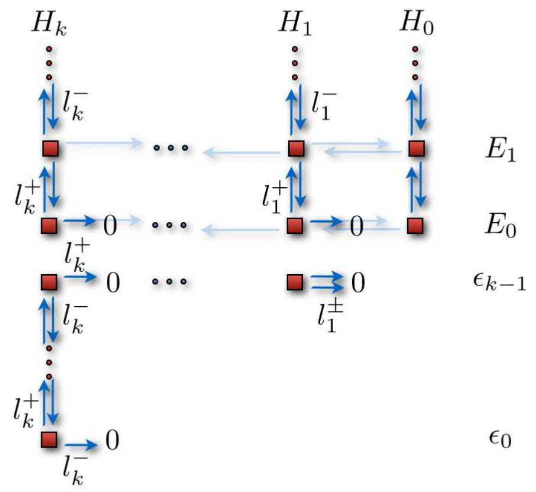

Figure 1. Action of the operators $l_j^{\pm}$ onto the eigenvectors of $H_j, j = 1, \ldots, k$.

where $u(x, \epsilon_j), j = 0, \ldots, k-1$ are k seed solutions of the stationary Schrödinger equation associated to $\epsilon_0 < \ldots < \epsilon_{k-1} < E_0 = \frac{1}{2}$, which are given by

$$u(x,\epsilon)= e^{-\frac{x^2}{2}} \left[{}_1F_1 \left(\tfrac{1-2\epsilon}{4}, \tfrac{1}{2}, x^2\right) + 2\nu x \frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})} \; {}_1F_1 \left(\tfrac{3-2\epsilon}{4}, \tfrac{3}{2}; x^2\right) \right]. \tag{15}$$

Note that $V_k(x)$ is non-singular if $|\nu_{k-j}| < 1$ for j odd and $|\nu_{k-j}| > 1$ for j even, $j = 1, 2, \ldots, k$. The spectrum of H_k becomes:

$$\mathrm{Sp}\,[H_k] = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}, E_n = n + 1/2, n = 0, 1, \ldots\}. \tag{16}$$

The family of potentials $V_k(x)$ of Eq. (14), in general, is generated by $2k$ free parameters: the k factorization energies $\epsilon_0, \ldots, \epsilon_{k-1}$ and the corresponding constants $\nu_0, \ldots, \nu_{k-1}$.

4. Non-trivial realizations of PIV Hamiltonian systems

Among the family of potentials $V_k(x)$ of Eq. (14), ruled in general by a $2k$th order PHA, there is a subfamily having associated also a second-order PHA which consists of Hamiltonians generated by connected seed solutions [12–16] (see also [20])

$$u(x, \epsilon_{k-j-1}) = (a^-)^j u(x, \epsilon_{k-1}), \qquad \epsilon_{k-j-1} = \epsilon_{k-1} - j, \quad j = 0, \ldots, k-1, \tag{17}$$

where the only free seed $u(x, \epsilon_{k-1})$ is such that $\epsilon_{k-1} < E_0 = 1/2$ and $|\nu_{k-1}| < 1$. As a consequence, the Hamiltonian H_k has also third-order differential ladder operators $l_k^{\pm}$ such that

$$[H_k, l_k^{\pm}] = \pm l_k^{\pm}, \tag{18}$$

$$[l_k^-, l_k^+] = N(H_k + 1) - N(H_k), \tag{19}$$

$$N(H_k) = l_k^+ l_k^- = \left(H_k - \tfrac{1}{2}\right)(H_k - \epsilon_0)(H_k - \epsilon_{k-1} - 1). \tag{20}$$

It turns out that now $\mathcal{H} = \mathcal{H}_{\rm iso} \oplus \mathcal{H}_{\rm new}$, where the action of $l_k^{\pm}$ on the basis of eigenvectors $|n^k\rangle$ of H_k in $\mathcal{H}_{\rm iso}$ (such that $H_k|n^k\rangle = E_n|n^k\rangle, \ n = 0, 1, \ldots$) becomes

$$l_k^-|n^k\rangle = \sqrt{(E_n - E_0)(E_n - \epsilon_0)(E_n - \epsilon_0 - k)}|n - 1^k\rangle, \tag{21}$$

$$l_k^+|n^k\rangle = \sqrt{(E_{n+1} - E_0)(E_{n+1} - \epsilon_0)(E_{n+1} - \epsilon_0 - k)}|n + 1^k\rangle. \tag{22}$$

On the other hand, on the eigenstates $|\epsilon_j^k\rangle$ of H_k in $\mathcal{H}_{\rm new}$ (such that $H_k|\epsilon_j^k\rangle = \epsilon_j|\epsilon_j^k\rangle,\ j = 0,\dots k-1$) this action turns out to be:

$$l_k^-|\epsilon_j^k\rangle = \sqrt{(\epsilon_j - E_0)(\epsilon_j - \epsilon_0)(\epsilon_j - \epsilon_0 - k)}|\epsilon_{j-1}^k\rangle, \tag{23}$$

$$l_k^+|\epsilon_j^k\rangle = \sqrt{(\epsilon_{j+1} - E_0)(\epsilon_{j+1} - \epsilon_0)(\epsilon_{j+1} - \epsilon_0 - k)}|\epsilon_{j+1}^k\rangle. \tag{24}$$

An scheme representing this action is given in Figure 1.

5. PIV coherent states

Once the systems under study, and their annihilation and creation operators $l_k^\pm$, have been identified, let us build the corresponding coherent states [22, 23].

5.1. Barut-Girardello coherent states

First, let us look for them as eigenstates of l_k^- on each of the two subspaces, namely:

$$l_k^-|z^k\rangle = z|z^k\rangle. \tag{25}$$

On the subspace $\mathcal{H}_{\rm iso}$ we have

$$|z_{\rm iso}^k\rangle = \sum_{n=0}^{\infty} c_n|n^k\rangle. \tag{26}$$

Then we get,

$$|z_{\rm iso}^k\rangle = c_0(z,z)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\sqrt{\frac{\Gamma(E_0-\epsilon_0+1)\Gamma(E_0-\epsilon_0-k+1)}{\Gamma(E_0-\epsilon_0+1+n)\Gamma(E_0-\epsilon_0-k+1+n)}}|n^k\rangle, \tag{27}$$

where the normalization factor $c_0(z,z)$ is obtained from the function

$$c_0(a,b) \equiv [{}_0F_2(E_0-\epsilon_0+1, E_0-\epsilon_0-k+1; a^*b)]^{-1/2}. \tag{28}$$

The properties of the CS $|z_{\rm iso}^k\rangle$ of Eq. (27) are the following:

– The so-called reproducing kernel turns out to be:

$$\langle z'^k_{\rm iso}|z^k_{\rm iso}\rangle = \frac{c_0(z',z')c_0(z,z)}{c_0^2(z',z)}. \tag{29}$$

A plot illustrating its modulus for $z' = -5 + i,\ \epsilon_0 = -4,\ k = 4$ is shown in Figure 2.
– There is continuity of the CS $|z_{\rm iso}^k\rangle$ in z, i.e., $\left\||z'^k_{\rm iso}\rangle - |z^k_{\rm iso}\rangle\right\| \to 0$ when $z \to z'$.
– The restriction of the identity operator $\mathbb{I}$ on $\mathcal{H}_{\rm iso}$ admits the following decomposition:

$$\int |z^k_{\rm iso}\rangle\langle z^k_{\rm iso}|\,\mu_1(z){\rm d}z = \mathbb{I}|_{\mathcal{H}_{\rm iso}}, \tag{30}$$

with the positive definite measure μ_1 given by [24] ($r = |z|$)

$$\mu_1(z) \equiv \frac{G^{3\ 0}_{0\ 3}\left(0, E_0-\epsilon_0, E_0-\epsilon_0-k\,|\,r^2\right)}{\pi c_0^2(z,z)\Gamma(E_0-\epsilon_0+1)\Gamma(E_0-\epsilon_0-k+1)}, \tag{31}$$

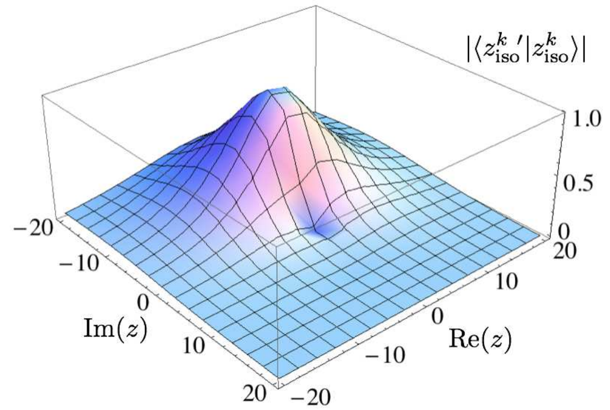

Figure 2. Modulus of the reproducing kernel of Eq. (29) for $z' = -5 + i,\ \epsilon_0 = -4,\ k = 4$.

where G is a Meijer G function.

– A CS $|z_{\rm iso}^k\rangle$ evolves into another CS of the same kind,

$$U(t)|z_{\rm iso}^k\rangle = e^{-iH_k t}|z_{\rm iso}^k\rangle = e^{-iE_0 t}|z(t)_{\rm iso}^k\rangle, \quad z(t) = ze^{-it}. \tag{32}$$

– The mean energy value $\langle H_k\rangle_{\rm iso} = \langle z_{\rm iso}^k|H_k|z_{\rm iso}^k\rangle$ acquires the form:

$$\langle H_k\rangle_{\rm iso} = \frac{1}{2} + \frac{r^2\,{}_0F_2(E_0-\epsilon_0+2, E_0-\epsilon_0-k+2; r^2)}{(E_0-\epsilon_0+1)(E_0-\epsilon_0-k+1)\,{}_0F_2(E_0-\epsilon_0+1, E_0-\epsilon_0-k+1; r^2)}. \tag{33}$$

– State probability. For a system in a CS $|z_{\rm iso}^k\rangle$, the probability $p_n^{\rm iso}(z) \equiv |\langle n^k|z_{\rm iso}^k\rangle|^2$ that an energy measurement gives the value E_n can be written as:

$$p_n^{\rm iso}(z) = c_0^2(z,z)\frac{|z|^{2n}}{n!}\frac{\Gamma(E_0-\epsilon_0+1)\Gamma(E_0-\epsilon_0-k+1)}{\Gamma(E_0-\epsilon_0+1+n)\Gamma(E_0-\epsilon_0-k+1+n)}. \tag{34}$$

On the other hand, on $\mathcal{H}_{\rm new}$ the only eigenstate of l_k^- is the extremal state $|\epsilon_0^k\rangle$ with eigenvalue $z = 0$, i.e., there is no family of CS in $\mathcal{H}_{\rm new}$ generated through this definition.

5.2. *Perelomov coherent states*

Now, let us look for the CS as the action of the "displacement" operator (which is not unitary)

$$D(z) = \exp\left(-\tfrac{1}{2}|z|^2\right)\exp\left(zl_k^+\right)\exp\left(-z^*l_k^-\right), \tag{35}$$

onto the extremal states of the system.

On $\mathcal{H}_{\rm new}$ we calculate $|z_{\rm new}^k\rangle = N_z' D(z)|\epsilon_0^k\rangle$, with N_z' being a normalization factor, leading to:

$$|z_{\rm new}^k\rangle \;=\; N(z,z)\left[\sum_{j=0}^{k-1}\sqrt{(E_0-\epsilon_0-j)_j(k-j)_j}\,\frac{z^j}{\sqrt{j!}}|\epsilon_j^k\rangle\right], \tag{36}$$

where $N(z,z)$ is obtained from the function

$$N(a,b) \;\equiv\; \left[\sum_{j=0}^{k-1}\frac{(a^*b)^j}{j!}(E_0-\epsilon_0-j)_j(k-j)_j\right]^{-1/2}. \tag{37}$$

The properties of the CS $|z^k_{\rm new}\rangle$ of Eq. (36) are the following:
– The reproducing kernel is now:

$$\langle z'^k_{\rm new}|z^k_{\rm new}\rangle = \frac{N(z',z')N(z,z)}{N^2(z',z)}. \tag{38}$$

– Once again there is continuity in the label z, since $\left\||z'^k_{\rm new}\rangle - |z^k_{\rm new}\rangle\right\| \to 0$ when $z \to z'$.
– The restriction of the identity operator on $\mathcal{H}_{\rm new}$ is expressed as

$$\int |z^k_{\rm new}\rangle\langle z^k_{\rm new}|\, \mu_2(z){\rm d}z = \mathbb{I}|_{\mathcal{H}_{\rm new}}, \tag{39}$$

where the positive definite measure μ_2 reads now:

$$\mu_2(z) \equiv \frac{G^{1\ 2}_{2\ 1}\left({}^{-k,\epsilon_0-E_0}_{\quad 0} \mid r^2\right)}{\pi N^2(z,z)\Gamma(E_0-\epsilon_0)\Gamma(k)}. \tag{40}$$

– A CS $|z^k_{\rm new}\rangle$ evolves into another CS of the same kind,

$$U(t)|z^k_{\rm new}\rangle = e^{-iH_kt}|z^k_{\rm new}\rangle = e^{-i\epsilon_0t}|z(t)^k_{\rm new}\rangle, \quad z(t) = ze^{-it}. \tag{41}$$

– The mean energy $\langle H_k\rangle_{\rm new} = \langle z^k_{\rm new}|H_k|z^k_{\rm new}\rangle$ now is given by

$$\langle H_k\rangle_{\rm new} = \epsilon_0 + N^2(z,z)\left[\sum_{j=0}^{k-1} j(E_0-\epsilon_0-j)_j(k-j)_j\frac{|z|^{2j}}{j!}\right]. \tag{42}$$

– State probability. For a system in a CS $|z^k_{\rm new}\rangle$, the probability $p^{\rm new}_j(z) \equiv |\langle\epsilon^k_j|z^k_{\rm new}\rangle|^2$ that an energy measurement gives the value ϵ_j is now:

$$p^{\rm new}_j(z) = N^2(z,z)(E_0-\epsilon_0-j)_j(k-j)_j\frac{|z|^{2j}}{j!}. \tag{43}$$

On the other hand, on $\mathcal{H}_{\rm iso}$ the only normalizable CS of kind $D(z)|0^k\rangle$ appears for $z=0$, i.e., we do not get a family of Perelomov coherent states on $\mathcal{H}_{\rm iso}$ but just the single CS $|0^k\rangle$.

6. Linearized PIV coherent states

We have derived families of CS on $\mathcal{H}_{\rm iso}$ and $\mathcal{H}_{\rm new}$ by using Barut-Girardello's definition for the former and Perelomov's one for the latter. It would be important to produce CS in both subspaces using just one of these two definitions. This can be done by introducing new annihilation and creation operators whose actions onto the eigenstates of H_k are simpler than the $l^\pm_k$ ones [23].

Thus, let us introduce the so-called linearized annihilation and creation operators as follows:

$$\ell^+_k \equiv \sigma(H_k)l^+_k, \qquad \ell^-_k \equiv \sigma(H_k+1)l^-_k, \tag{44}$$

$$\sigma(H_k) = [(H_k-\epsilon_0)(H_k-\epsilon_0-k)]^{-1/2}, \tag{45}$$

which act on the eigenvectors of H_k in a simplified way:

$$\ell^-_k|n^k\rangle = \sqrt{n}\,|n-1^k\rangle, \tag{46}$$

$$\ell^+_k|n^k\rangle = \sqrt{n+1}\,|n+1^k\rangle, \tag{47}$$

$$\ell^+_k|\epsilon^k_j\rangle = (1-\delta_{j,k-1})\sqrt{\epsilon_{j+1}-E_0}\,|\epsilon^k_{j+1}\rangle, \tag{48}$$

$$\ell^-_k|\epsilon^k_j\rangle = (1-\delta_{j,0})\sqrt{\epsilon_j-E_0}\,|\epsilon^k_{j-1}\rangle. \tag{49}$$

We construct the CS using Perelomov's definition which involves these ladder operators $\ell^\pm_k$.

Journal of Physics: Conference Series **597** (2015) 012017 doi:10.1088/1742-6596/597/1/012017

6.1. Linearized PIV coherent states on $\mathcal{H}_{\rm iso}$

On $\mathcal{H}_{\rm iso}$ the operators $\{H_k, \ell_k^-, \ell_k^+\}$ satisfy the Heisenberg-Weyl algebra. Let us use the displacement operator

$$\mathcal{D}(z) = \exp\left(-\frac{1}{2}|z|^2\right)\exp\left(z\ell_k^+\right)\exp\left(-z^*\ell_k^-\right). \tag{50}$$

Hence, the linearized CS $|z_{\rm isol}^k\rangle = \mathcal{D}(z)|0^k\rangle$ acquire the form

$$|z_{\rm isol}^k\rangle = \exp\left(-\frac{|z|^2}{2}\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n^k\rangle. \tag{51}$$

Several properties of the CS $|z_{\rm isol}^k\rangle$ are the same as for the standard CS, namely:
– The reproducing kernel becomes:

$$\langle z_{\rm isol}'^k|z_{\rm isol}^k\rangle = \exp\left(-\frac{|z|^2}{2} - \frac{|z'|^2}{2} + z'^*z\right). \tag{52}$$

– There is continuity in the label z, i.e., $\left\||z_{\rm isol}'^k\rangle - |z_{\rm isol}^k\rangle\right\| \to 0$ when $z \to z'$.
– The restriction of the identity operator on $\mathcal{H}_{\rm iso}$ decomposes into

$$\frac{1}{\pi}\int |z_{\rm isol}^k\rangle\langle z_{\rm isol}^k|\, {\rm d\,Re}(z)\, {\rm d\,Im}(z) = \mathbb{I}|_{\mathcal{H}_{\rm iso}}. \tag{53}$$

– There is temporal stability in the CS $|z_{\rm isol}^k\rangle$, namely,

$$U(t)|z_{\rm isol}^k\rangle = e^{-iE_0t}|z(t)_{\rm isol}^k\rangle, \qquad z(t) = ze^{-it}. \tag{54}$$

– The mean energy $\langle H_k\rangle_{\rm isol} = \langle z_{\rm isol}^k|H_k|z_{\rm isol}^k\rangle$ becomes the standard one,

$$\langle H_k\rangle_{\rm isol} = |z|^2 + E_0. \tag{55}$$

– State probability. For a system in a CS $|z_{\rm isol}^k\rangle$, the probability $p_n^{\rm isol}(z) = |\langle n^k|z_{\rm isol}^k\rangle|^2$ that an energy measurement gives the value E_n is now a Poisson distribution:

$$p_n^{\rm isol}(z) = e^{-|z|^2}\frac{|z|^{2n}}{n!}. \tag{56}$$

Although the previous properties of the CS $|z_{\rm isol}^k\rangle$ are the same as for the standard CS, this does not mean that the associated wavefunctions are Gaussian, which is illustrated in Figure 3, where the modulus of the CS wavefunction (blue curve) is plotted for $\nu_3 = 0$, $\epsilon_3 = -5/2$, $z = 1.5$ with $\langle H_k\rangle_{\rm isol} = 11/4$. The corresponding potential $V_4(x)$ (purple curve) becomes a rational extension of the harmonic oscillator, where the eigenfunctions $\langle x|n^k\rangle$ are expressed in terms of exceptional Hermite polynomials [25, 26].

The Barut-Girardello CS of Eq. (27) and the linearized ones of Eq. (51) coexist in the same subspace $\mathcal{H}_{\rm iso}$, thus they can be compared to each other. In Figure 4 the mean energy values $\langle H_k\rangle_{\rm iso}$ and $\langle H_k\rangle_{\rm isol}$ as functions of $r = |z|$ for both sets of CS are plotted. In purple we can observe the slow increasing behavior of $\langle H_k\rangle_{\rm iso}$ for the Barut-Girardello CS (see Eq. (33)); on the other hand, for the linearized PIV CS on $\mathcal{H}_{\rm iso}$ it is seen in blue the standard half parabola with a minimum $\langle H_k\rangle_{\rm isol} = 1/2$ at $r = 0$ (see Eq. (55)). The system under consideration corresponds to a SUSY partner of the harmonic oscillator generated for $k = 4$ with four connected energy levels at $\epsilon_0 = -11/2$, $\epsilon_1 = -9/2$, $\epsilon_2 = -7/2$ and $\epsilon_3 = -5/2$.

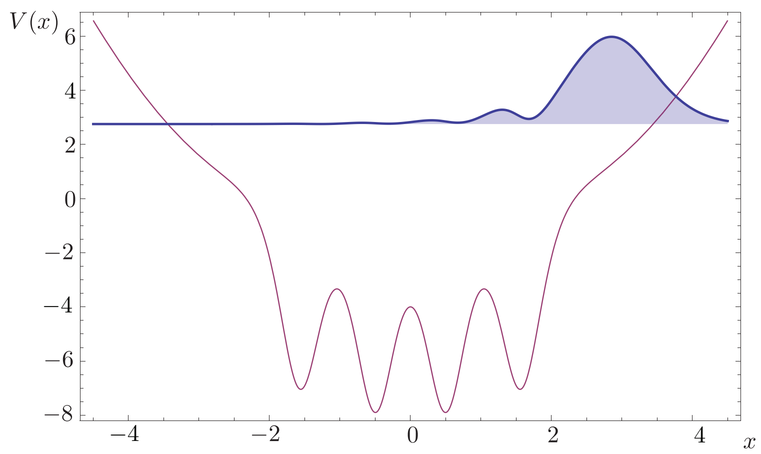

Figure 3. Modulus of the wavefunction $\langle x|z^k_{\rm isol}\rangle$ for $\nu_3 = 0$, $\epsilon_3 = -5/2$, $z = 1.5$, with $\langle H_k\rangle_{\rm isol} = 11/4$.

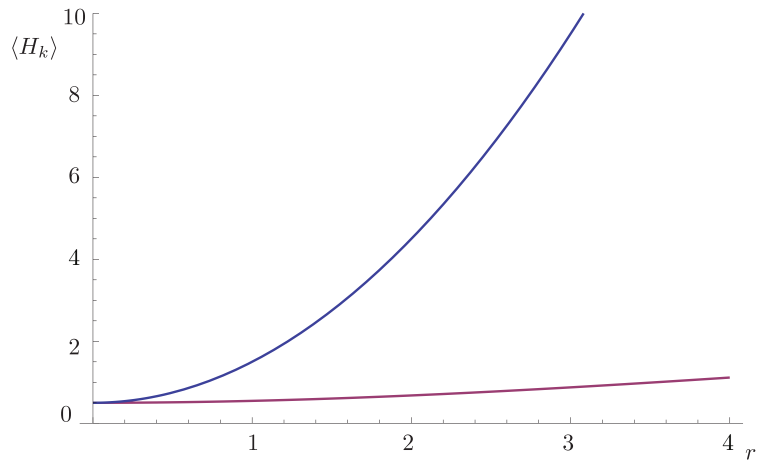

Figure 4. Mean energies $\langle H_k\rangle_{\rm iso}$ and $\langle H_k\rangle_{\rm isol}$ as functions of $r = |z|$ for the Barut-Girardello CS (purple curve) and the linearized PIV CS (blue curve) on $\mathcal{H}_{\rm iso}$, Eqs. (33) and (55) respectively. The system under consideration corresponds to a SUSY partner of the harmonic oscillator for $k = 4$ and $\epsilon_0 = -11/2$.

6.2. Linearized PIV coherent states on $\mathcal{H}_{\rm new}$

On $\mathcal{H}_{\rm new}$ the operators $\{H_k, \ell^-_k, \ell^+_k\}$ do not satisfy the Heisenberg-Weyl algebra. However, we will use the same displacement operator of Eq. (50) to act on the extremal state $|\epsilon^k_0\rangle \in \mathcal{H}_{\rm new}$. Thus, the linearized CS $|z^k_{\rm newl}\rangle = C'_z D(z)|\epsilon^k_0\rangle$ become now

$$|z^k_{\rm newl}\rangle = C(z,z)\sum_{j=0}^{k-1}\frac{(iz)^j}{j!}\sqrt{\frac{1}{\Gamma(E_0-\epsilon_0-j)}}|\epsilon^k_j\rangle, \tag{57}$$

where

$$C(a,b) = \left[\sum_{j=0}^{k-1}\frac{(a^*b)^j}{(j!)^2}\frac{1}{\Gamma(E_0-\epsilon_0-j)}\right]^{-1/2}. \tag{58}$$

The properties of $|z^k_{\text{newl}}\rangle$ are now non-standard:
– The reproducing kernel is given by:

$$\langle z'^k_{\text{newl}}|z^k_{\text{newl}}\rangle = \frac{C(z',z')C(z,z)}{C^2(z',z)}. \tag{59}$$

– There is once again continuity in the label z, namely, $\big\||z'^k_{\text{newl}}\rangle - |z^k_{\text{newl}}\rangle\big\| \to 0$ when $z \to z'$.
– The resolution of the identity on $\mathcal{H}_{\text{new}}$ turns out to be:

$$\int_{\mathbb{C}} |z^k_{\text{newl}}\rangle\langle z^k_{\text{newl}}|\, \mu_3(z)\mathrm{d}z = \mathbb{I}|_{\mathcal{H}_{\text{new}}}, \tag{60}$$

where the positive definite measure μ_3 becomes now:

$$\mu_3(z) = \frac{f_3(r)}{\pi C^2(z,z)}, \tag{61}$$

$$f_3(r) = G^{2\ 1}_{1\ 2}\left(\begin{smallmatrix}\epsilon_0 - E_0 \\ 0,0\end{smallmatrix}\,\middle|\, r^2\right) = \Gamma^2(E_0+1-\epsilon_0)U(E_0+1-\epsilon_0, 1; r^2). \tag{62}$$

– Temporal stability. A CS $|z^k_{\text{newl}}\rangle$ evolves once again in a CS,

$$U(t)|z^k_{\text{newl}}\rangle = e^{-i\epsilon_0 t}|z(t)^k_{\text{newl}}\rangle, \quad z(t) = ze^{-it}. \tag{63}$$

– The mean energy $\langle H_k\rangle_{\text{newl}} = \langle z^k_{\text{newl}}|H_k|z^k_{\text{newl}}\rangle$ becomes now:

$$\langle H_k\rangle_{\text{newl}} = \epsilon_0 + C^2(z,z)\sum_{j=0}^{k-1} \frac{j|z|^{2j}}{(j!)^2}\frac{1}{\Gamma(E_0-\epsilon_0-j)}. \tag{64}$$

– State probability. For a system in a CS $|z^k_{\text{newl}}\rangle$, the probability $p_j^{\text{newl}}(z) = |\langle\epsilon_j^k|z^k_{\text{newl}}\rangle|^2$ that an energy measurement gives the value ϵ_j is not longer a Poisson distribution:

$$p_j^{\text{newl}}(z) = \frac{|z|^{2j}}{(j!)^2}\frac{C^2(z,z)}{\Gamma(E_0-\epsilon_0-j)}. \tag{65}$$

Once again, the wavefunction $\langle x|z^k_{\text{newl}}\rangle$ is not Gaussian, as can be seen in the example of Figure 5 (blue curve) built for $\nu_3 = 0,\ \epsilon_3 = -5/2,\ z = 2e^{-1.59i}$, with $\langle H_k\rangle_{\text{newl}} = -3.0923$.

In Figure 6 the mean energy values $\langle H_k\rangle_{\text{new}}$ and $\langle H_k\rangle_{\text{newl}}$ as functions of $r = |z|$ are shown for the Perelomov CS (purple curve) and the linearized PIV CS (blue curve) on $\mathcal{H}_{\text{new}}$ (see Eqs. (42) and (64) respectively) with parameters $\epsilon_0 = -11/2$ and $k = 4$. The minimum for both curves corresponds to ϵ_0 when $r = 0$, while the maximum becomes ϵ_3 when $r \to \infty$. It can be seen that $\langle H_k\rangle_{\text{newl}}$ for the linearized PIV CS on $\mathcal{H}_{\text{new}}$ grows slower than for the Perelomov ones.

7. Conclusions

In this paper we have constructed several CS for a subfamily of Painlevé IV Hamiltonian systems which are connected with the PIV equation and have third-order differential ladder operators. The corresponding Hilbert space is of type $\mathcal{H} = \mathcal{H}_{\text{iso}} \oplus \mathcal{H}_{\text{new}}$, thus on each subspace we have derived two independent sets of CS.

As for a criterion for choosing a certain type of CS, this depends on the physical situation we are dealing with. For instance, if we are studying the semiclassical limit of the system the CS in $\mathcal{H}_{\text{iso}}$ would be appropriate, since their mean energy value can be made arbitrarily large. However, if we want to characterize a “more quantum” system’s behavior it would be better to use either the CS in $\mathcal{H}_{\text{new}}$ or those in $\mathcal{H}_{\text{iso}}$ for $|z|$ small.

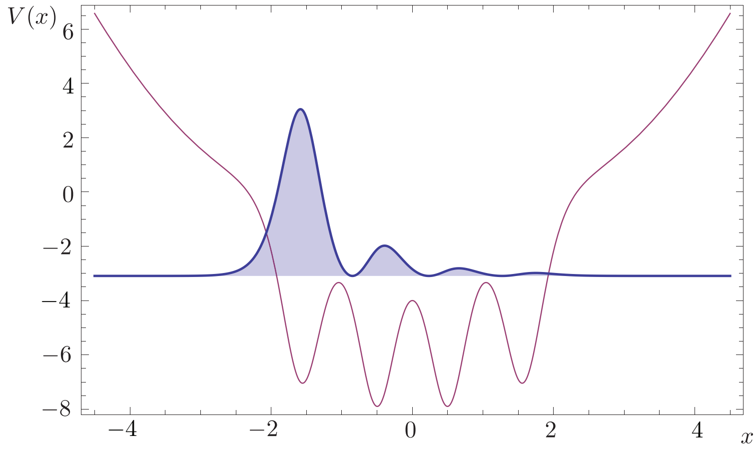

Figure 5. Modulus of the wavefunction $\langle x|z^k_{\rm newl}\rangle$ for $\nu_3 = 0$, $\epsilon_3 = -5/2$, $z = 2e^{-1.59i}$ with $\langle H_k\rangle_{\rm newl} = -3.0923$.

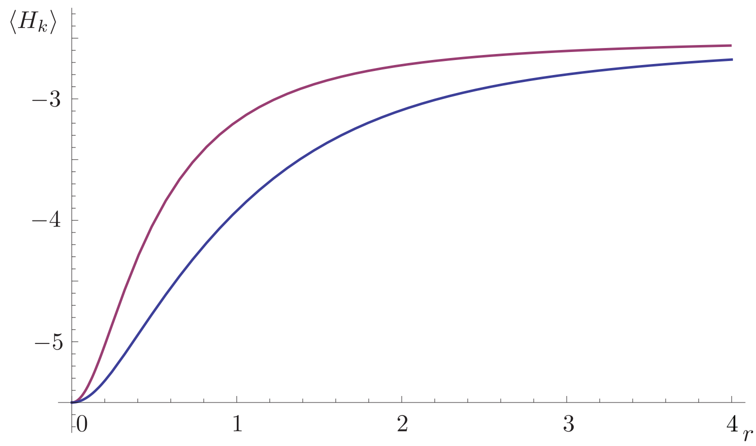

Figure 6. Mean energies $\langle H_k\rangle_{\rm new}$ and $\langle H_k\rangle_{\rm newl}$ as functions of r for the Perelomov CS (purple curve) and the linearized PIV CS (blue curve) on $\mathcal{H}_{\rm new}$, Eqs. (42) and (64) respectively. The system under consideration corresponds to a SUSY partner of the harmonic oscillator for $k = 4$ and $\epsilon_0 = -11/2$.

Acknowledgments

The authors acknowledge the support of Conacyt (Mexico), Project 152574. DB and ACA also acknowledge Conacyt fellowships 207672 and 207577.

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