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On motivic multiple t values, Saha's basis conjecture, and generators of alternating MZV's

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Abstract

We give an evaluation for the stuffle-regularised $t^{*,V}(\{2\}^a, 1, \{2\}^b)$ as a polynomial in single-zeta values, $\log(2)$ and V. We then apply this to establish some linear independence results of certain sets of motivic multiple t values. In particular, we prove the elements of Saha's conjectural basis are linearly independent, on the motivic level, and that the (suitably regularised) elements $t^{\mathfrak{m}}(\{1, 2\}^{\times})$ form a basis for both the (extended) motivic MtV's and the alternating MZV's.

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1 Introduction

The *multiple zeta value* (MZV) with indices $k_1, \ldots, k_d \in \mathbb{Z}_{\geq 1}$, and $k_d \geq 2$ for convergence reasons, is defined by

$$\zeta(k_1, \ldots, k_d) := \sum_{0<n_1<\cdots<n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}.$$

As is common, we call d the *depth* of the MZV and $k_1 + \cdots + k_d$ the *weight*. Although the cases of depth $d = 1, 2$ were already studied by Euler, the research into the case of general depth d only started in the early 1990s with work of Hoffman [14] and Zagier [26], with many theorems and identities being proven and many conjectures formulated since then. These values have also earned a prominent place in high energy physics, as part of the calculation (in special cases) of Feynman integrals and scattering amplitudes (see [3] as a starting point).

In [16], Hoffman studied the *multiple t values* (MtV's) defined by restricting to MZV-like sums with odd denominators,

$$t(k_1, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{(2n_1 - 1)^{k_1} \cdots (2n_d - 1)^{k_d}},$$

with again $k_i \in \mathbb{Z}_{\geq 1}$ and $k_d > 1$ for convergence, and the same notion of weight and depth. (Be aware that Hoffman uses the other convention on MZV's, with summation indices given by $n_1 > \cdots > n_d > 0$, which has the effect of reversing argument strings.) Therein Hoffman compared and contrasted the algebraic and combinatorial properties of MtV's and MZV's, establishing that MtV's have many similarities with MZV's, but some distinct differences of there own. (In particular, both MtV's and MZV's have a stuffle-product, and symmetric sum formulae [16, Section 3]. MZV's admit a duality relation such as $\zeta(1, 2) = \zeta(3)$, but MtV's appear to have no such identities: already [16, Appendix A] shows that $t(3)$ and $t(1, 2)$ are unrelated. However MtV's conjecturally admit a derivation [16, Conjecture 2.1] which is realised in Appendix A therein as a formal differentiation with respect to $\log(2)$. We refer to Remark 5.10 below for an interpretation and explanation of this derivation as the action of D_1 on the motivic level.)

By writing

$$t(k_1, \ldots, k_d) := \sum_{0<n_1<\cdots<n_d} \frac{(1 - (-1)^{n_1})}{2\, n_1^{k_1}} \cdots \frac{(1 - (-1)^{n_d})}{2\, n_d^{k_d}},$$

one obtains a formula (cf. [16, Corollary 4.1]) for the MtV's in terms of so-called alternating MZV's. Namely

$$t(k_1, \ldots, k_d) = \frac{1}{2^d} \sum_{\varepsilon_i \in \{\pm 1\}} \varepsilon_1 \cdots \varepsilon_d \zeta \binom{\varepsilon_1, \ldots, \varepsilon_d}{k_1, \ldots, k_d}, \tag{1}$$

where

$$\zeta \binom{\varepsilon_1, \ldots, \varepsilon_d}{k_1, \ldots, k_d} := \sum_{0<n_1<\cdots<n_d} \frac{\varepsilon_1^{n_1} \cdots \varepsilon_d^{n_d}}{n_1^{k_1} \cdots n_d^{k_d}}$$

is the *alternating MZV* with signs ε_i and arguments k_i (also called the coloured MZV of level $N = 2$; level here referring to the order of the roots of unity involved). Often, if all $\varepsilon_i \in \{\pm 1\}$, one denotes arguments k_i which have associated sign $\varepsilon_i = -1$ by $\overline{k_i}$. There are notions of *regularisation* which we cover more fully in Section 2, which allow the divergent MZV's and MtV's to be assigned consistent finite values.

Our first main result, in Section 3, gives an evaluation for the stuffle-regularised multiple t value $t^{*,V}(\{2\}^a, 1, \{2\}^b)$, analogous to the evaluation for $\zeta(\{2\}^a, 3, \{2\}^b)$ established by Zagier [27], and the evaluation for $t(\{2\}^a, 3, \{2\}^b)$ established by Murakami [22], both of which were used to establish the linear independence and basis properties of certain motivic MZV's and MtV's respectively ([4] in the zeta-value case, and [22] for the t-value case).

Theorem 1.1 (Theorem 3.3 below) *The following evaluation holds for any $a, b \in \mathbb{Z}_{\geq 0}$, for $t^{*,V}$ the stuffle-regularised MtV's with $t^{*,V}(1) = V$.*

$$\begin{aligned} & t^{*,V}(\{2\}^a, 1, \{2\}^b) = \\ & -\sum_{r=1}^{a+b} (-1)^r 2^{-2r} \left[\binom{2r}{2a} + \frac{2^{2r}}{2^{2r}-1} \binom{2r}{2b} \right] \zeta(\overline{2r+1}) t(\{2\}^{a+b-r}) \\ & + \delta_{a=0} \log(2) t(\{2\}^b) + \delta_{b=0} (V - \log(2)) t(\{2\}^a)\,, \end{aligned}$$

where we write $\{k\}^n = \overbrace{k, \ldots, k}^{n}$ for the argument k repeated n times, and $\delta_\bullet$ is the Kronecker delta symbol, equal to 1 if the condition $\bullet$ holds, and 0 otherwise.

Because a subset of the MtV's which appear in this evaluation series are divergent, we must understand the asymptotics of certain ${}_3F_2$ hypergeometric series (and thus the ${}_4F_3$ series from which they originate), via results from the Evans–Stanton/Ramanujan asymptotic [8], in order to extract the evaluation with a generating series approach. By extracting the special case $a = 0, b = n$ of Theorem 1.1 in Section 3.3, we also answer a question posed in [6].

We then utilise the arithmetic properties of the coefficients to establish some linear independence properties of certain sets of motivic MtV's. In Section 4 we state the main definitions, properties and theorems pertaining to the framework of motivic MZV's. One can often think of the motivic MZV's and MtV's as some algebraically defined 'formal' analogue to their analytic counterparts, which have more rigid structure and better properties. Conjecturally the motivic versions reflect all of the relations between the real-valued versions, but they certainly do not introduce new relations. In Section 5 we discuss the regularised distribution relations, and extend the formulae given by Murakami [22] for D_r, to all motivic MtV's. In Section 6 we use this to lift Theorem 1.1 to a motivic version.

Then in Section 7, we establish that the elements conjectured by Saha [25] to be a basis of (convergent) MtV's are, at least, linearly independent. This establishes a lower bound of F_N on the dimension of the space of (convergent) motivic MtV's of weight N, where $F_n = F_{n-1} + F_{n-2}$ is the n-th Fibonacci number, with $F_1 = 1$, $F_2 = 1$.

Theorem 1.2 (Corollary 7.15 below) *Let*

$$S = \{t^{\mathfrak{m}}(k_1, \ldots, k_{r-1}, k_r + 1) \mid k_i \in \{1, 2\}\}$$

be the set of elements in Saha's basis conjecture. Then the elements of S are linearly independent.

For example, this establishes that $t^{\mathfrak{m}}(1, 2)$ and $t^{\mathfrak{m}}(3)$ are linearly independent in weight 3, as are $t^{\mathfrak{m}}(1, 1, 2)$, $t^{\mathfrak{m}}(2, 2)$ and $t^{\mathfrak{m}}(1, 3)$ in weight 4.

Finally in Section 8 we introduce the Hoffman t one-two elements—those MtV's with arguments exactly 1 or 2—in analogy with the Hoffman elements $\zeta(k_1, \ldots, k_r)$, $k_i \in \{2, 3\}$ for classical MZV's. We show the one-two elements form a basis of both (extended) motivic MtV's and alternating motivic MZV's, under a certain shuffle regularisation and certain stuffle regularisation.

Theorem 1.3 (Corollary 8.19, and Corollary 8.26, below) *Let*

$$H^{\bullet} = \{t^{\mathfrak{m},\bullet}(k_1, \ldots, k_r) \mid k_i \in \{1, 2\}\}$$

be the Hoffman one-two elements, for $\bullet = *$ *or* $\bullet = ш$*, where the shuffle regularisation arises from* $\zeta^{\mathfrak{m},ш}(1) = 0$ *and the stuffle regularisation has* $t^{\mathfrak{m},*}(1) = \lambda \log^{\mathfrak{m}}(2)$*,* λ *of the form* $\frac{2a+1}{b} \in \mathbb{Q}$ *with* $a, b \in \mathbb{Z}$*. (In particular,* $\lambda = \frac{1}{2}, 1$ *are allowed.) Then the elements in H are linearly independent, and span the space of both (extended) motivic MtV's and motivic alternating MZV's.*

As a corollary, we see the space of alternating MZV's of weight N (under shuffle-regularisation with $\zeta^{ш,0}(1) = 0$ or stuffle-regularisation with $\zeta^{*,0}(1) = 0$ and extended MtV's (under shuffle-regularisation induced by $\zeta^{ш,0}(1) = 0$, or stuffle-regularisation with $t^{*,V}(1) = V$ for $V = \lambda \log(2)$, $\lambda = \frac{2a+1}{b} \in \mathbb{Q}$, $a, b \in \mathbb{Z}$) coincide. In particular they have dimension F_{N+1}. We also indicate how badly this can fail for certain 'singular' regularisation parameters in Section 8.2.

We also include some further examples of the motivic Galois descent (to motivic MZV's) for $t^{\mathfrak{m}}(k_1, \ldots, k_d)$ for particular families which include arguments $k_i = 1$. For example, in Proposition 5.12 we show

$$t^{\mathfrak{m}}(\{2\}^a, 1, \{2\}^b, 3, \{2\}^c),$$

is a linear combination of motivic MZV's whenever $a \geq 1$. This shows that Murakami's motivic Galois descent of $t^{\mathfrak{m}}(k_1, \ldots, k_d)$, all $k_i \geq 2$ [22, Theorem 8] is not exhaustive, and raises the question of more generally characterising when such a motivic MtV can be written as a linear combination of motivic MZV's.

2 Relating regularisations of multiple zeta values and multiple *t* values

In this section we recall, compare and contrast the different notions of regularisation which apply to MtV's. In particular, we need to understand how the stuffle regularisation of t values with $t^{*,V}(1) = V$ relates to the regularisation of t values induced by the stuffle regularisation of the underlying zeta values at $\zeta^{*,U}(1) = U$, or the shuffle regularisation of the zeta values with $\zeta^{\sqcup\mathchoice{}{}{}{}\!\sqcup,W}(1) = W$.

2.1 Stuffle and stuffle regularisation for MZV's and MtV's

As already noted in the introduction, MZV's and MtV's are only defined when the last argument $k_d \geq 2$, otherwise the series is divergent. However, one can use the stuffle product structure to give a consistent definition to any MZV or MtV with trailing 1's, in terms of a single parameter assigned to $\zeta(1) := \zeta^{*,U}(1) = U$ or $t(1) := t^{*,V}(1) = V$ respectively. Alternatively one can utilise the iterated integral representation to define a(nother) regularisation. We briefly recall the details here, see [18] for full details in the case of classical MZV's and the extension presented in [28, §13.3] for the case of cyclotomic MZV's. (See also §6 and the rest of [17] for the general background on quasi-shuffle algebras, and their applications to multiple zeta values.)

Stuffle-regularisation of MZV's and MtV's: Define the alphabet

$$Z = \{z_{k,\varepsilon} \mid k \geq 1, \varepsilon \in \{\pm 1\}\},$$

with the letter product $z_{k,\varepsilon} \diamond z_{\ell,\eta} = z_{k+\ell,\varepsilon\eta}$ on $\mathbb{Q}Z$. On $\mathfrak{A}^1_* := \mathbb{Q}\langle Z \rangle$, the $\mathbb{Q}$-vector space of words over the alphabet Z, the $\diamond$-product induces a stuffle-product $*$ given by

$$\begin{aligned}(z_{k,\varepsilon} w_1) * (z_{\ell,\eta} w_2) &= z_{k,\varepsilon}(w_1 * z_{\ell,\eta} w_2) + z_{\ell,\eta}(z_{k,\varepsilon} w_1 * w_2) + (z_{k,\varepsilon} \diamond z_{\ell,\eta})(w_1 * w_2) \\ &= z_{k,\varepsilon}(w_1 * z_{\ell,\eta} w_2) + z_{\ell,\eta}(z_{k,\varepsilon} w_1 * w_2) + z_{k+\ell,\varepsilon\eta}(w_1 * w_2)\,.\end{aligned}$$

On the subspace $\mathfrak{A}^0_*$ of *convergent words* (also called 'admissible'), namely those words which do not end in $z_{1,1}$, the map

$$\begin{aligned}\zeta \colon (\mathfrak{A}^0_*, *) &\to \mathbb{R} \\ z_{k_1,\varepsilon_1} \cdots z_{k_d,\varepsilon_d} &\mapsto \zeta\binom{\varepsilon_1, \ldots, \varepsilon_d}{k_1, \ldots, k_d}\end{aligned}$$

is an algebra homomorphism. This is because $*$ corresponds to the multiplication of (alternating) MZV's as series, given by interleaving the summation indices with equality allowed. For example (recall $\overline{k_i}$ means the sign $\varepsilon_i = -1$ is associated to k_i),

$$\begin{aligned}
\zeta(\overline{k_1}, k_2)\zeta(\overline{\ell_1}) &= \sum_{0<n_1<n_2} \sum_{0<m_1} \frac{(-1)^{k_1}}{n_1^{k_1} n_2^{k_2}} \cdot \frac{(-1)^{\ell_1}}{m_1^{\ell_1}} \\
&= \Bigg(\sum_{0<n_1<n_2<m_1} + \sum_{0<n_1<m_1<n_2} + \sum_{0<m_1<n_1<n_2} \\
&\qquad + \sum_{0<n_1<n_2=m_1} + \sum_{0<n_1=m_1<n_2} \Bigg) \frac{(-1)^{k_1}}{n_1^{k_1} n_2^{k_2}} \cdot \frac{(-1)^{\ell_1}}{m_1^{\ell_1}} \\
&= \zeta(\overline{k_1}, k_2, \overline{\ell_1}) + \zeta(\overline{k_1}, \overline{\ell_1}, k_2) + \zeta(\overline{\ell_1}, \overline{k_1}, k_2) \\
&\quad + \zeta(\overline{k_1}, \overline{k_2 + \ell_1}) + \zeta(k_1 + \ell_1, k_2)\,,
\end{aligned}$$

which corresponds to the computing $z_{k_1,-1} z_{k_2,1} * z_{\ell_1,-1}$.

It is then a standard result [15, 23] that $\mathbb{Q}\langle Z\rangle \cong \mathfrak{A}_*^0[z_{1,1}]$ (with the product given by $*$). This is proven by recursion, as

$$w z_{1,1}^n - \frac{1}{n}(w z_{1,1}^{n-1} * z_{1,1}) \tag{2}$$

is a sum of words ending in strictly fewer $z_{1,1}$ letters. Finally the map $\zeta \colon (\mathfrak{A}_*^0, *) \to \mathbb{R}$ can be extended uniquely to an algebra morphism

$$\zeta^{*,U} \colon \mathbb{Q}\langle Z\rangle \cong \mathfrak{A}_*^0[z_{1,1}] \to \mathbb{R}[U]$$

by the sending $z_{1,1} \mapsto U$. This enables us to define the stuffle-regularised MZV's as follows.

Definition 2.1 (*Stuffle-regularised MZV's*) The stuffle-regularisation of the alternating MZV is defined by

$$\zeta^{*,U}\binom{\varepsilon_1, \ldots, \varepsilon_d}{k_1, \ldots, k_d} := \zeta^{*,U}(z_{k_1,\varepsilon_1} \cdots z_{k_d,\varepsilon_d}).$$

With $a \neq 1$, we can for example compute—using the recursion from (2) as the key step—that,

$$\begin{aligned}
\zeta^{*,U}(a, 1) &= \zeta^{*,U}(a)\zeta^{*,U}(1) - \zeta^{*,U}(1, a) - \zeta^{*,U}(a+1) \\
&= \zeta(a)U - \zeta(1, a) - \zeta(a+1)
\end{aligned} \tag{3}$$

Then

$$\begin{aligned} &\zeta^{*,U}(a,1,1)\\ &= \tfrac{1}{2}\zeta^{*,U}(a,1)\zeta^{*,U}(1) - \zeta^{*,U}(1,a,1) - \zeta^{*,U}(a,2) - \zeta^{*,U}(a+1,1)\\ &= \tfrac{1}{2}U^2\zeta(a) - U\zeta(a+1) - U\zeta(1,a) + \tfrac{1}{2}\zeta(a+2)\\ &\quad + \zeta(1,a+1) + \tfrac{1}{2}\zeta(2,a) - \tfrac{1}{2}\zeta(a,2) + \zeta(1,1,a). \end{aligned} \tag{4}$$

Similarly the map

$$t\colon (\mathfrak{A}^0_*, *) \to \mathbb{R}$$
$$z_{k_1,\varepsilon_1}\cdots z_{k_d,\varepsilon_d} \mapsto t\begin{pmatrix}\varepsilon_1,\ldots,\varepsilon_d\\ k_1,\ldots,k_d\end{pmatrix}$$

is an algebra homomorphism, which can be extend to $t^{*,V}\colon (\mathfrak{A}^0_*, *) \to \mathbb{R}[V]$ by sending $z_{1,1} \mapsto V$. (Here we have implicitly introduced alternating MtV's,

$$t\begin{pmatrix}\varepsilon_1,\ldots,\varepsilon_d\\ k_1,\ldots,k_d\end{pmatrix} := \sum_{0<n_1<\cdots<n_d} \frac{\varepsilon_1^{n_1}\cdots\varepsilon_d^{n_d}}{(2n_1-1)^{k_1}\cdots(2n_d-1)^{k_d}},$$

although we are still only interested in the case where all $\varepsilon_i = 1$.) From this we define the stuffle-regularised MtV's as follows.

Definition 2.2 (*Stuffle-regularised MtV's*) The stuffle-regularised MtV is defined by

$$t^{*,V}(k_1,\ldots,k_d) := t^{*,V}(z_{k_1,1}\cdots z_{k_d,1}).$$

Then one immediately has that the same formulae as in (3) and (4) hold with ζ replaced by t and U by V.

Natural choices for $\zeta^{*,U}(1)$ and $t^{*,V}(1)$; extended MtV's: Although we are now free to choose any value we like for $\zeta^{*,U}(1) = U$ or $t^{*,V}(1) = V$, some choices are more natural or convenient than others.

As shown in the proof of Theorem 1 [18], the classical asymptotic

$$\zeta^{\mathrm{tr}}_M(1) := \sum_{0<n<M} \frac{1}{n} = \log(M) + \gamma + O(M^{-1}),$$

(here $\gamma = 0.577\ldots$ is the Euler–Mascheroni constant) implies the so-called *truncated* MZV has the following asymptotic

$$\begin{aligned} \zeta^{\mathrm{tr}}_M(k_1,\ldots,k_d) &:= \sum_{0<n_1<\cdots<n_d<M} \frac{1}{n_1^{k_1}\cdots n_d^{k_d}}\\ &= Z_{k_1,\ldots,k_d}(\log(M)+\gamma) + O(M^{-1}\log^J M), \end{aligned}$$

for some J, as $M \to \infty$, where $Z_{k_1,\ldots,k_d}(T) = \zeta^{*,T}(k_1, \ldots, k_d) \in \mathbb{R}[T]$ is some polynomial. Then one can naturally look at the constant term of this polynomial, to 'renormalise' and 'remove' the $\log(M)$ dependence. This corresponds to taking $T = 0$, so we can naturally assign $U = \zeta^{*,U}(1) = 0$ in the above stuffle-regularisation of MZV's.

Applying the same prescription to the truncated MtV's gives

$$
\begin{aligned}
t_M^{\mathrm{tr}}(1) &:= \sum_{0 \le n < M} \frac{1}{2n+1} = \sum_{1 \le n < 2M} \frac{1}{n} - \sum_{1 \le n < M} \frac{1}{2n} \\
&= \Big(\log(2M) + \gamma + O\big((2M)^{-1}\big)\Big) - \frac{1}{2}\Big(\log(M) + \gamma + O\big(M^{-1}\big)\Big) \\
&= \log(2) + \frac{1}{2}\big(\log(M) + \gamma \big) + O\big(M^{-1}\big) .
\end{aligned}
$$

This suggests that one should naturally define

$$
V = t^{*,V}(1) = \log(2), \tag{5}
$$

by taking the constant term of the above, when viewed as a polynomial in $\log(M) + \gamma$.

Since the space of weight 1 MtV's $t(k_1, \ldots, k_d)$ with $k_1 + \cdots + k_d = 1$ and $k_d > 1$ is 0 dimensional, we already see that defining $t^{*,V}(1) = \log(2)$ extends the space of MtV's. We make the following definition, for clarity, with regard to convergent and regularised MtV's.

Definition 2.3 (*Convergent and extended MtV's*) A multiple t value $t(k_1, \ldots, k_d)$ is called

(i) *convergent* if $k_d \ge 2$, and
(ii) *extended* if $k_d \ge 1$

The *space of convergent MtV's* refers the space generated by all MtV's $t(k_1, \ldots, k_d)$ which have $k_d \ge 2$, whereas the *space of extended MtV's* denotes that generated by all MtV's $t(k_1, \ldots, k_d)$ allowing $k_d > 1$ or $k_d = 1$, under some particular (specified) regularisation.

Shuffle regularisation of MZV's: Likewise, it is well known how to write MZV's (or more generally multiple polylogarithms) as iterated integrals. Namely

$$
\zeta \binom{\varepsilon_1, \ldots, \varepsilon_d}{k_1, \ldots, k_d} = (-1)^d I(0; \eta_1, \{0\}^{k_1 - 1}, \eta_2, \{0\}^{k_2 - 1}, \ldots, \eta_d, \{0\}^{k_d - 1}; 1) \tag{6}
$$

where the notation $\{k\}^n = \overbrace{k, \ldots, k}^{n}$ denotes k repeated n times, $\eta_i = \prod_{j=i}^d \varepsilon_j^{-1}$, and

$$
I(x_0; x_1, \ldots, x_N; x_{N+1}) := \int_{x_0 < s_1 < \cdots < s_N < x_{n+1}} \omega(x_1; s_1) \wedge \cdots \wedge \omega(x_N; s_N), \tag{7}
$$

is the iterated integral of the family of differential forms $\omega(x; s) = \frac{\mathrm{d}s}{s-x}$. Note that this integral only converges when $x_0 \neq x_1$ and $x_N \neq x_{N+1}$.

Integrals of this form multiply by the shuffle product, which corresponding to interleaving the integration indices, although here equality of indices gives sets of measure zero, and so no contribution to the result. For example

$$\begin{aligned}
\zeta(2)\zeta(2) &= I(0; 1, 0; 1) I(0; 1, 0; 1) \\
&= \int_{0<s_1<s_2<1} \frac{\mathrm{d}s_1}{s_1 - 1} \wedge \frac{\mathrm{d}s_2}{s_2} \int_{0<s_3<s_4<1} \frac{\mathrm{d}s_3}{s_3 - 1} \wedge \frac{\mathrm{d}s_4}{s_4} \\
&= \int_{0<s_1<s_2<r_1<r_2} \frac{\mathrm{d}s_1}{s_1 - 1} \wedge \frac{\mathrm{d}s_2}{s_2} \wedge \frac{\mathrm{d}r_1}{r_1 - 1} \wedge \frac{\mathrm{d}r_2}{r_2} \\
&\quad + \int_{0<s_1<r_1<s_2<r_2} \frac{\mathrm{d}s_1}{s_1 - 1} \wedge \frac{\mathrm{d}r_1}{r_1 - 1} \wedge \frac{\mathrm{d}s_2}{s_2} \wedge \frac{\mathrm{d}r_2}{r_2} + 4 \text{ more terms} \\
&= \zeta(2, 2) + \zeta(1, 3) + 4 \text{ more terms} \\
&= 2\zeta(2, 2) + 4\zeta(1, 3)
\end{aligned}$$

Introduce the alphabet $Y = \{e_0, e_1, e_{-1}\}$, with letter product $e_i \diamond e_j = 0$ on $\mathbb{Q}Y$. On $\mathfrak{B}^1_{\sqcup\!\sqcup} = \mathbb{Q}\langle Y \rangle$, the $\diamond$-product induces the shuffle product $\sqcup\!\sqcup$, given by

$$\begin{aligned}
(e_i w_1) \sqcup\!\sqcup (e_j w_2) &= e_i(w_1 \sqcup\!\sqcup e_j w_2) + e_2(e_i w_1 \sqcup\!\sqcup w_2) + (e_i \diamond e_j)(w_1 \sqcup\!\sqcup w_2) \\
&= e_i(w_1 \sqcup\!\sqcup e_j w_2) + e_j(e_i w_1 \sqcup\!\sqcup w_2) \,.
\end{aligned}$$

On the subspace $\mathfrak{B}^0_{\sqcup\!\sqcup}$ of *convergent words*, namely those which do not *start* in e_0, and do not *end* in e_1, the map

$$\begin{aligned}
\zeta \colon (\mathfrak{B}^0_{\sqcup\!\sqcup}, \sqcup\!\sqcup) &\to \mathbb{R} \\
e_{\eta_1} e_0^{n_1 - 1} \cdots e_{\eta_d} e_0^{n_d - 1} &\mapsto \zeta\begin{pmatrix} \eta_2/\eta_1, \eta_3/\eta_2, \ldots, 1/\eta_d \\ n_1 \quad, n_2 \quad, \ldots, n_d \end{pmatrix} \\
&\qquad = (-1)^d I(0; e_{\eta_1}, \{e_0\}^{n_1 - 1}, \ldots, e_{\eta_d}, \{e_0\}^{n_d - 1}; 1) \,,
\end{aligned}$$

is an algebra homomorphism, as $\sqcup\!\sqcup$ encodes the product of iterated integrals, as explained above. (Note: the sign $1/\eta_d$ appearing in the last ζ argument involves the 1 from the upper bound of integration.)

It is again a standard result [15, 23], provable by recursion, that $\mathfrak{B}^1_{\sqcup\!\sqcup} \cong \mathfrak{B}^0_{\sqcup\!\sqcup}[e_0, e_1]$ (with the product given by $\sqcup\!\sqcup$). One simply notices that

$$e_0^n w - \frac{1}{n} e_0 \sqcup\!\sqcup (e_0^{n-1} w), \quad \text{and} \quad w e_1^n - \frac{1}{n} (w e_1^{n-1}) \sqcup\!\sqcup e_1, \tag{8}$$

are a sum of words which start with strictly fewer e_0 letters, and a sum of words which end with strictly fewer e_1 letters, respectively. Finally, the map $\zeta \colon (\mathfrak{B}^0_{\sqcup\!\sqcup}, \sqcup\!\sqcup) \to \mathbb{R}$ can be extended uniquely to an algebra morphism

$$\zeta^{\sqcup\!\sqcup, W_1, W_2} \colon \mathbb{Q}\langle Y \rangle \cong \mathfrak{B}^0_{\sqcup\!\sqcup}[e_0, e_1] \mapsto \mathbb{R}[W_1, W_2],$$

by sending $e_1 \mapsto W_1$, $e_0 \mapsto W_2$. We can therefore define the shuffle regularised MZV's (with leading 0's) as follows.

Definition 2.4 (*Shuffle-regularised MZV's*) The shuffle-regularisation of the alternating MZV with ℓ leading 0's is defined by

$$\zeta_\ell^{⧢,W_1,W_2}\begin{pmatrix}\varepsilon_1,\ldots,\varepsilon_d\\k_1,\ldots,k_d\end{pmatrix} := \zeta^{⧢,W_1,W_2}(e_0^\ell e_{\eta_1} e_0^{k_1-1}\cdots e_{\eta_d} e_0^{k_d-1}),$$

with $\eta_i = \prod_{j=i}^d \varepsilon_j^{-1}$. As with the usual convention for alternating MZV's, one may write $\overline{k_i}$ to denote $\varepsilon_i = -1$, and suppress $\varepsilon_1,\ldots,\varepsilon_d$ from the notation of $\zeta_\ell^{⧢,W_1,W_2}$.

We will typically be interested in either the case where $\ell = 0$, in which case the dependence on W_2 drops out, or in the case where $\ell \geq 0$, and we will set $W_1 = W_2 = 0$, to agree with the period of the corresponding motivic MZV (see Section 4 below). More generally, from the duality of multiple zeta values (which arises from the functoriality of iterated integrals under $s_i \mapsto 1 - s_i$, along with path reversal) the one can argue informally that, since

$$\zeta_0(1) = -I(0;1;1) \overset{s_i\mapsto 1-s_i}{=} -I(1;0;0) \overset{\text{reverse}}{=} I(0;0;1) = \zeta_1(\emptyset),$$

it makes sense to impose that $W_1 = \zeta_0^{⧢,W_1,W_2}(1)$ and $W_2 = \zeta_1^{⧢,W_1,W_2}(\emptyset)$ are equal in the shuffle-regularisation. (This informal calculation is made precise by considering the asymptotic expansion of $\int_\varepsilon^{1-\varepsilon}\frac{\mathrm{d}t}{t-1} = -\int_\varepsilon^{1-\varepsilon}\frac{\mathrm{d}t}{t-0}$, as a polynomial in $\log\varepsilon$, as $\varepsilon \to 0$, cf. [12, §2.9] or [9, §3.6.5–3.6.6].)

We therefore refine the definition of the shuffle-regularised MZV's as follows.

Definition 2.5 (*Shuffle-regularised MZV's, refined*) The shuffle-regularisation of the alternating MZV with ℓ leading 0's is defined by

$$\zeta_\ell^{⧢,W}\begin{pmatrix}\varepsilon_1,\ldots,\varepsilon_d\\k_1,\ldots,k_d\end{pmatrix} := \zeta^{⧢,W,W}(e_0^\ell e_{\eta_1} e_0^{k_1-1}\cdots e_{\eta_d} e_0^{k_d-1}),$$

with $\eta_i = \prod_{j=i}^d \varepsilon_j^{-1}$. As with the usual convention for alternating MZV's, one may write $\overline{k_i}$ to denote $\varepsilon_i = -1$, and suppress $\varepsilon_1,\ldots,\varepsilon_d$ from the notation of $\zeta_\ell^{⧢,W}$.

For example, we compute—using the recursions in (8) as the key step—the following shuffle-regularised MZV's,

$$\begin{aligned}\zeta^{⧢,W}(2,1,1) &= \frac{1}{2}W^2\zeta(2) - 2W\zeta(1,2) + 3\zeta(1,1,2)\,,\\ \zeta_1^{⧢,W_1,W_2}(2,1,1) &= \frac{1}{2}\zeta(2)W_2W_1^2 - W_1^2\zeta(3) - 2W_1W_2\zeta(1,2)\\ &\quad + 4W_1\zeta(1,3) + W_1\zeta(2,2) + 3W_2\zeta(1,1,2)\\ &\quad - 6\zeta(1,1,3) - 2\zeta(1,2,2) - \zeta(2,1,2)\,,\end{aligned}$$

and by taking $W_1 = W_2 = W$,

$$\zeta_1^{ш,W}(2,1,1) = \frac{1}{2}\zeta(2)W^3 - W^2\zeta(3) - 2W^2\zeta(1,2) + 4W\zeta(1,3) + W\zeta(2,2) + 3W\zeta(1,1,2) - 6\zeta(1,1,3) - 2\zeta(1,2,2) - \zeta(2,1,2)\,.$$

We already remark here that $\zeta^{ш,W}(\underline{\mathbf{k}})$ and $\zeta^{*,U}(\underline{\mathbf{k}})$ are not the same even for $W = U$, but they are closely related, as we recall in Lemma 2.16 below.

In the case $W = 0$, one has the following explicit formula to unshuffle and regularise the initial 0's. (Compare property I2 in Section 2.4 of [4])

Lemma 2.6 (Unshuffling of initial 0's) *For any $\ell \geq 0$, any $\underline{\mathbf{k}} = (k_1, \ldots, k_d) \in \mathbb{Z}^d_{\geq 1}$ and any signs ε_i, we have*

$$\zeta_\ell^{ш,0}\begin{pmatrix}\boldsymbol{\varepsilon} \\ \underline{\mathbf{k}}\end{pmatrix} = (-1)^\ell \sum_{i_1+\cdots+i_d=\ell} \binom{k_1+i_1-1}{i_1} \cdots \binom{k_d+i_d-1}{i_d} \times \zeta^{ш,0}\begin{pmatrix}\boldsymbol{\varepsilon} \\ k_1+i_1, \ldots, k_d+i_d\end{pmatrix}. \tag{9}$$

Proof This is a straightforward proof by induction on ℓ, the case $\ell = 0$ is clearly true. So take $\ell > 0$, and assume the statement holds for $\ell - 1$. In $\mathfrak{B}^0_ш$, with $\eta_i = \prod_{j=i}^d \varepsilon_j^{-1}$, we compute the following product

$$\begin{aligned} &e_0^\ell e_{\eta_1} e_0^{k_1-1} \cdots e_{\eta_d} e_0^{k_d-1} - \tfrac{1}{\ell} e_0 \,ш\, e_0^{\ell-1} e_{\eta_1} e_0^{k_1-1} \cdots e_{\eta_d} e_0^{k_d-1} \\ &= -\sum\nolimits_{j=1}^d \frac{k_j}{\ell} \cdot e_0^{\ell-1} e_{\eta_1} e_0^{k_1-1} \cdots e_{\eta_j} e_0^{(k_j-1)+1} \cdots e_{\eta_d} e_0^{k_d-1} \end{aligned}$$

Applying $\zeta^{ш,0}$ gives

$$\zeta_\ell^{ш,0}\begin{pmatrix}\boldsymbol{\varepsilon} \\ k_1, \ldots, k_d\end{pmatrix} = -\sum_{j=1}^d \frac{k_j}{\ell} \zeta_{\ell-1}^{ш,0}\begin{pmatrix}\boldsymbol{\varepsilon} \\ k_1, \ldots, k_j+1, \ldots, k_d\end{pmatrix},$$

since $\zeta_1^{ш,0}(\emptyset) = \zeta^{ш,0}(e_0) = 0$. After substituting in (9) in the case $\ell - 1$, and simplifying, we obtain the result in the case ℓ. This completes the induction, and the lemma holds. □

Shuffle-regularisation of MtV's: In contrast to MZV's, the integral representation of MtV's does not endow them with a nice shuffle-product structure. For example, one sees that in depth 1, we have

$$t(2) = -\int_{0<s_1<s_2<1} \frac{\mathrm{d}s_1}{s_1^2-1} \wedge \frac{\mathrm{d}s_2}{s_2},$$

whereas in depth 2

$$t(2,2) = \int_{0<s_1<s_2<s_3<s_4<1} \frac{\mathrm{d}s_1}{s_1^2-1} \wedge \frac{\mathrm{d}s_2}{s_2} \wedge \frac{s_3\mathrm{d}s_3}{s_3^2-1} \wedge \frac{\mathrm{d}s_4}{s_4}.$$

In particular, the first differential form is of a unique type, and any further forms corresponding to higher depth and more arguments have an extra s_i in the numerator, i.e.

$$\frac{s_i\mathrm{d}s_i}{s_i^2-1} \quad \text{versus} \quad \frac{\mathrm{d}s_1}{s_1^2-1}.$$

Therefore, after taking the shuffle product (valid for iterated integrals of any families of differential forms) of two copies this representation of $t(2)$, one has to disentangle cases the forms when

$$\frac{\mathrm{d}s_j}{s_j^2-1}$$

appears after the first position, and when

$$\frac{s_1\mathrm{d}s_1}{s_1^2-1}$$

appears in the first position.

To cut to the point: we shall take the expression for t in terms of ζ from (1), and use this as the basis for defining the shuffle-regularised version of t by shuffle-regularising the MZV's therein.

Definition 2.7 (*Shuffle-regularised MtV's*) The shuffle-regularisation of the MtV is defined by

$$t^{ш,W}(k_1,\ldots,k_d) := \frac{1}{2^d} \sum_{\varepsilon_i\in\{\pm1\}} \varepsilon_1\cdots\varepsilon_d \zeta^{ш,W}\binom{\varepsilon_1,\ldots,\varepsilon_d}{k_1,\ldots,k_d}. \tag{10}$$

2.2 Compatibility of the *t* and zeta stuffle-regularisations

Motivated by the above definition of $t^{ш,W}(k_1,\ldots,k_d)$ in (10), we can likewise introduce a regularised version of the MtV's given by stuffle-regularising the underlying MZV's

Definition 2.8 (*Zeta-stuffle-regularised MtV's*) The stuffle-regularisation of the MtV given by stuffle-regularising the underlying MZV expression is defined by

$$t^{\zeta,W}(k_1,\ldots,k_d) := \frac{1}{2^d} \sum_{\varepsilon_i\in\{\pm1\}} \varepsilon_1\cdots\varepsilon_d \zeta^{*,W}\binom{\varepsilon_1,\ldots,\varepsilon_d}{k_1,\ldots,k_d}. \tag{11}$$

The natural question is how $t^{\zeta,W}$ and $t^{*,V}$ are related? Firstly, we note that the stuffle product of t values and of the underlying zeta values is compatible in a precise sense. To make this statement formal, we must first recall and introduce some additional algebraic notation.

Recall the algebraic setup from earlier: $\mathfrak{A}^1_* := \mathbb{Q}\langle z_{k,\varepsilon} \mid k \geq 1, \varepsilon \in \{\pm 1\}\rangle$, with the stuffle product given by

$$\begin{aligned}&(z_{k,\varepsilon} w_1) * (z_{\ell,\eta} w_2) = \\ &z_{k,\varepsilon}(w_1 * z_{\ell,\eta} w_2) + z_{\ell,\eta}(z_{k,\varepsilon} w_1 * w_2) + z_{k+\ell,\varepsilon\eta}(w_1 * w_2).\end{aligned}$$

The map $\zeta \colon \mathfrak{A}^0_* \to \mathbb{R}$ from convergent words (those not ending in $z_{1,1}$) is an algebra homomorphism. Introduce the alphabet $C = \{c_{k,p} \mid k \geq 1, p \in \{0, 1\}\}$, with letter product on $\mathbb{Q}C$ given by

$$c_{k,p} \diamond c_{\ell,q} = \delta_{p=q} c_{k+\ell,p}.$$

This induces the stuffle product

$$\begin{aligned}&(c_{k,p} w_1) *_t (c_{\ell,q} w_2) \\ &= c_{k,p}(w_1 *_t c_{\ell,q} w_2) + c_{\ell,q}(c_{k,p} w_1 *_t w_2) + \delta_{p=q} c_{k+\ell,p}(w_1 *_t w_2),\end{aligned}$$

on $\mathfrak{C}^1_* := \mathbb{Q}\langle C\rangle$. On the convergent words $\mathfrak{C}^0_*$, i.e. those not ending in $c_{1,0}$ or $c_{1,1}$, the map

$$\begin{aligned}t \colon (\mathfrak{C}^0_*, *_t) &\to \mathbb{R} \\ c_{k_1,p_1} \cdots c_{k_d,p_d} &\mapsto \sum_{\substack{0<n_1<\cdots<n_d \\ n_j \equiv p_j \pmod 2}} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}},\end{aligned}$$

is an algebra homomorphism; the stuffle product $*_t$ encodes the multiplication of such series, as interleaved indices can only be equal in the case they satisfy the same parity constraint (whence the term $\delta_{p=q}$ in $*_t$).

Note that with all $p_i = 1$, $t(c_{k_1,1} \cdots c_{k_d,1}) = t(k_1, \ldots, k_d)$ is the MtV we are interested in. Whereas, with $p_i = 1$, i odd, $p_i = 0$, i even, the map $t(c_{k_1,p_1} \cdots c_{k_d,p_d})$ produces the multiple (capital) T values (MTV's) in the sense of Kaneko-Tsumura [19, 20].

Define a $\mathbb{Q}$-linear map

$$\begin{aligned}\sigma \colon \mathfrak{C}^1_* &\to \mathfrak{A}^1_* \\ c_{k_1,p_1} \cdots c_{k_d,p_d} &\mapsto \frac{1}{2^d} \sum_{\varepsilon_j \in \{\pm 1\}} \varepsilon_1^{p_1} \cdots \varepsilon_d^{p_d} \cdot z_{k_1,\varepsilon_1} \cdots z_{k_d,\varepsilon_d}.\end{aligned}$$

Then immediately we have that $t = \zeta \circ \sigma$ on $\mathfrak{C}^0_*$. This is just the generalisation of (1) to odd or even summation indices, where this parity can be forced by inserting the factor

$$\frac{1-(-1)^{n_i}}{2} \quad \text{or} \quad \frac{1+(-1)^{n_i}}{2},$$

respectively.

Lemma 2.9 *The map* $\sigma : \mathfrak{C}^1_* \to \mathfrak{A}^1_*$ *is an isomorphism* (of vector spaces).

Proof Fix d, and $k_1, \ldots, k_d$. Then σ maps words of the form $c_{k_1,p_1} \cdots c_{k_d,p_d}$ to (sums of) words of the form $z_{k_1,\varepsilon_1} \cdots z_{k_d,\varepsilon_d}$. In this basis of the subspace and image, with $(p_1, \ldots, p_d)$ and $(\varepsilon_1, \ldots, \varepsilon_d)$ ordered lexicographically, σ is described by the $2^d \times 2^d$ matrix

$$M_d := \left(\varepsilon_1^{p_1} \cdots \varepsilon_d^{p_d}\right)_{(p_1,\ldots,p_d)\in\{0,1\}^d, (\varepsilon_1,\ldots,\varepsilon_d)\in\{\pm1\}^d}.$$

By the lexicographic ordering of $(p_1, \ldots, p_d)$ and $(\varepsilon_1, \ldots, \varepsilon_d)$, we can write

$$M_d = \begin{pmatrix} 1^0 \cdot M_{d-1} & 1^1 \cdot M_{d-1} \\ (-1)^0 \cdot M_{d-1} & (-1)^1 \cdot M_{d-1} \end{pmatrix} = \begin{pmatrix} M_{d-1} & M_{d-1} \\ M_{d-1} & -M_{d-1} \end{pmatrix}$$

as a recursively defined block Vandermonde matrix, with

$$M_0 = \begin{pmatrix} 1 \end{pmatrix}, M_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

By the properties of block determinants, we see by induction that $\det(M_d) = \det(-2M_{d-1}^2) = (-2)^{2^{d-1}} \det(M_{d-1})^2 = (-2)^{d2^{d-1}}$, which is in particular non-zero. Hence σ is invertible on this subspace, and by the block diagonal nature of σ in general, it is an isomorphism in general. □

Lemma 2.10 *The stuffle product of MtV's and the stuffle product of MZV's are compatible in the following sense. For any* $w_1, w_2 \in \mathfrak{C}^1_*$

$$\sigma(w_1 *_t w_2) = \sigma(w_1) * \sigma(w_2)$$

That is to say, σ *is an algebra (iso)morphism.*

Proof We prove thus by induction on the length of w_1 plus w_2. For $w_1 = \mathbb{1}$, the empty word, then both sides are just $\sigma(w_2 *_t \mathbb{1}) = \sigma(w_2) = \sigma(w_2) * \mathbb{1}$, likewise for $w_2 = \mathbb{1}$. Now assuming both $w_1 = c_{k,p}w_1'$, $w_2 = c_{\ell,q}w_2' \neq \mathbb{1}$, we compute (using linearity of σ),

$$\begin{aligned} &\sigma((c_{k,p}w_1') *_t (c_{\ell,q}w_2')) \\ &= \sigma\left(c_{k,p}(w_1' *_t c_{\ell,q}w_2') + c_{\ell,q}(c_{k,p}w_1' *_t w_2') + \delta_{p=q}c_{k+\ell,p}(w_1' *_t w_2')\right) \\ &= \sigma\left(c_{k,p}(w_1' *_t c_{\ell,q}w_2')\right) + \sigma\left(c_{\ell,q}(c_{k,p}w_1' *_t w_2')\right) + \delta_{p=q}\sigma\left(c_{k+\ell,p}(w_1' *_t w_2')\right). \end{aligned}$$

Now observe that directly from the definition

$$\sigma(c_{k,p}w_1') = \frac{1}{2}\sum_{\varepsilon\in\{\pm1\}} \varepsilon^p z_{k,\varepsilon}\sigma(w_1').$$

So using this (twice), and the induction assumption,

$$\begin{aligned}
&\sigma((c_{k,p}w_1') *_t (c_{\ell,q}w_2')) \\
&= \frac{1}{2}\sum_{\varepsilon\in\{\pm1\}} \Big\{\varepsilon^p z_{k,\varepsilon}\sigma\big(w_1' *_t c_{\ell,q}w_2'\big) + \varepsilon^q z_{\ell,\varepsilon}\sigma\big(c_{k,p}w_1' *_t w_2'\big) \\
&\qquad\qquad + \delta_{p=q}\varepsilon^p z_{k+\ell,\varepsilon}\sigma\big(w_1' *_t w_2'\big)\Big\}. \\
&= \frac{1}{2}\sum_{\varepsilon\in\{\pm1\}} \Big\{\varepsilon^p z_{k,\varepsilon}\big(\sigma(w_1') * \sigma(c_{\ell,q}w_2')\big) + \varepsilon^q z_{\ell,\varepsilon}\big(\sigma(c_{k,p}w_1') * \sigma(w_2')\big) \\
&\qquad\qquad + \delta_{p=q}\varepsilon^p z_{k+\ell,\varepsilon}\big(\sigma(w_1') * \sigma(w_2')\big)\Big\}. \\
&= \frac{1}{4}\sum_{\varepsilon,\eta\in\{\pm1\}} \Big\{\varepsilon^p z_{k,\varepsilon}\big(\sigma(w_1') * \big(\eta^q z_{\ell,\eta}\sigma(w_2')\big)\big) + \eta^q z_{\ell,\eta}\big(\big(\varepsilon^p z_{k,\varepsilon}\sigma(w_1')\big) * \sigma(w_2')\big)\Big\} \\
&\quad + \frac{1}{2}\sum_{\varepsilon\in\{\pm1\}} \delta_{p=q}\varepsilon^p z_{k+\ell,\varepsilon}\big(\sigma(w_1') * \sigma(w_2')\big).
\end{aligned}$$

By the recursion for $*$, with $z_{k,\varepsilon} \diamond z_{\ell,\eta} = z_{k+\ell,\varepsilon\eta}$, this can be rewritten as

$$\begin{aligned}
&= \frac{1}{4}\sum_{\varepsilon,\eta\in\{\pm1\}} \big(\varepsilon^p z_{k,\varepsilon}\sigma(w_1')\big) * \big(\eta^q z_{\ell,\eta}\sigma(w_2')\big) \\
&\quad -\frac{1}{4}\sum_{\varepsilon,\eta\in\{\pm1\}} \varepsilon^p\eta^q z_{k+\ell,\varepsilon\eta}\big(\sigma(w_1') * \sigma(w_2')\big) \\
&\quad +\frac{1}{2}\sum_{\varepsilon\in\{\pm1\}} \delta_{p=q}\varepsilon^p z_{k+\ell,\varepsilon}\big(\sigma(w_1') * \sigma(w_2')\big).
\end{aligned} \tag{12}$$

In the third summand, put $\varepsilon = \eta\hat{\varepsilon}$, with $\hat{\varepsilon} \in \{\pm1\}$. Then $\varepsilon\eta = \hat{\varepsilon}\eta^2 = \hat{\varepsilon}$, and $\sum_{\eta\in\{\pm1\}}\eta^{p+q} = 2\delta_{p=q}$, so

$$\begin{aligned}
&\frac{1}{4}\sum_{\varepsilon,\eta\in\{\pm1\}} \varepsilon^p\eta^q z_{k+\ell,\varepsilon\eta}\big(\sigma(w_1') * \sigma(w_2')\big) \\
&= \frac{1}{4}\sum_{\hat{\varepsilon},\eta\in\{\pm1\}} \hat{\varepsilon}^p\eta^{p+q} z_{k+\ell,\hat{\varepsilon}}\big(\sigma(w_1') * \sigma(w_2')\big) \\
&= \frac{1}{2}\sum_{\varepsilon\in\{\pm1\}} \delta_{p=q}\varepsilon^p z_{k+\ell,\varepsilon}\big(\sigma(w_1') * \sigma(w_2')\big)
\end{aligned}$$

This shows that the last two terms in (12) cancel, from which we obtain

$$\begin{aligned}\sigma((c_{k,p}w_1') *_t (c_{\ell,q}w_2')) &= \frac{1}{4}\sum_{\varepsilon,\eta\in\{\pm1\}} \big(\varepsilon^p z_{k,\varepsilon}\sigma(w_1')\big) * \big(\eta^q z_{\ell,\eta}\sigma(w_2')\big) \\ &= \sigma(c_{k,p}w_1') * \sigma(c_{\ell,q}w_2'),\end{aligned}$$

which completes the proof by induction. □

Corollary 2.11 *The stuffle product of MtV agrees with the stuffle product of the underlying MZV's of each factor.*

Proof Using $t = \zeta \circ \sigma$, and the algebra isomorphism property of σ above, this follows directly:

$$t(w_1 *_t w_2) = t\big(\sigma^{-1}(\sigma(w_1) * \sigma(w_2))\big) = \zeta(\sigma(w_1) * \sigma(w_2)). \qquad \square$$

Corollary 2.12 *It holds that*

$$t^{*,V} = t^{\zeta,2V-\log(2)}.$$

That is the stuffle regularisation of MtV's with parameter $t^{*,V}(1) = V$ corresponds to stuffle regularisation of the underlying MZV's with parameter $\zeta^{*,W}(1) = W$, for $W = 2V - \log(2)$.

Proof Since the stuffle products are compatible, both $t^{*,V}$ and $t^{\zeta,W}$ are lifts of the algebra homomorphism t from the algebra $\overline{\mathfrak{A}}^0_*$ of convergent (or 'admissible') words (those not ending in z_1) of $\mathbb{Q}\langle z_k \mid k \geq 1\rangle$ to the algebra $\overline{\mathfrak{A}}^1_* := \mathbb{Q}\langle z_k \mid k \geq 1\rangle \cong \overline{\mathfrak{A}}^0_*[z_1]$ of all words. (We do not want to invoke alternating MtV's here.) Their value on z_1 determine them completely, and so we obtain agreement when

$$V = t^{*,V}(1) = t^{\zeta,W}(1) = \frac{1}{2}(\zeta^{*,W}(1) - \zeta(\overline{1})) = \frac{1}{2}(W + \log(2)).$$

Hence the relation $W = 2V - \log(2)$ follows. □

In some sense, this means the most natural regularisation for multiple t values, when defined formally as a sum of alternating MZV's via (1), has $t^{*,V}(1) = \frac{1}{2}\log(2)$. We already saw in (5) above that $t^{*,V}(1) = \log(2)$ is another very natural regularisation for MtV's, and so these will be the two cases of most interest.

2.3 Relations between regularisations of alternating MZV's

We establish (or recall) some relationships between regularisations with different parameters, and between the shuffle and stuffle regularisations of *alternating* MZV's, which will be useful when applied to MtV's. In all of the lemmas that follow in this section, let $\underline{\mathbf{k}} = (k_1, \ldots, k_d)$ be an index with barred entries, such that $k_d \neq 1$.

Lemma 2.13 *The stuffle regularisations with parameter* $\zeta^{*,T}(1) = T$ *and* $\zeta^{*,S}(1) = S$ *are related as follows,*

$$\zeta^{*,T}(\underline{\mathbf{k}}, \{1\}^{\alpha}) = \sum_{i=0}^{\alpha} \zeta^{*,S}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \frac{(T-S)^i}{i!}.$$

Proof We actually prove a stronger statement, which claims that this regularisation formula holds for MZV's at arbitrary roots of unity. Consider the alphabet $\hat{Z} = \{z_{n,\theta} \mid n \geq 1 \in \mathbb{Z}, \theta \in \mathbb{C}, |\theta| = 1\}$, with letter product $z_{n,\theta} \diamond z_{m,\phi} = z_{n+m,\theta\phi}$ on $\mathbb{Q}\hat{Z}$, and the induced stuffle product on $\mathbb{Q}\langle\hat{Z}\rangle$ given by

$$\begin{aligned}(z_{n,\theta} w) * (z_{m,\phi} v) &= (z_{n,\theta} \diamond z_{m,\phi})(w * v) + z_{n,\theta}(w * z_{m,\phi} v) + z_{m,\phi}(z_{n,\theta} w * v) \\ &= z_{n+m,\theta\phi}(w * v) + z_{n,\theta}(w * z_{m,\phi} v) + z_{m,\phi}(z_{n,\theta} w * v)\,.\end{aligned}$$

Then $*$ describes the product of multiple zeta values (at arbitrary roots of unity) under the map $z_{n_1,\theta_1} \cdots z_{n_d,\theta_d} \mapsto \zeta\binom{\theta_1,\ldots,\theta_d}{n_1,\ldots,n_d}$. By expanding out with the stuffle product we see the following. For any convergent word $w_0 = w_0' z_{n,\theta}$ with $(n,\theta) \neq (1,1)$,

$$\sum_{i=0}^{\alpha} \frac{(-1)^i}{i!} z_{1,1}^{*i} * (w_0 z_{1,1}^{\alpha-i})$$

is a sum of purely convergent words; there is a pairwise cancellation of any words ending in $z_{1,1}$. On the other hand, this expression is a stuffle-polynomial in $z_{1,1}$, whose constant term is the word $w_0 z_{1,1}^{\alpha}$. In the regularisation where $z_{1,1} \mapsto 0$, only the constant term of this polynomial is left, and we see

$$\operatorname{reg}_0^*(w_0 z_{1,1}^{\alpha}) = \sum_{i=0}^{\alpha} \frac{(-1)^i}{i!} z_{1,1}^{*i} * w_0 z_{1,1}^{\alpha-i}.$$

(Here reg_T^* denotes the isomorphism $\mathbb{Q}\langle\hat{Z}\rangle \cong \mathbb{Q}\langle\hat{Z}\rangle^0[T]$ obtained from $\mathbb{Q}\langle\hat{Z}\rangle \cong \mathbb{Q}\langle\hat{Z}\rangle^0[z_{1,1}]$, with $\mathbb{Q}\langle\hat{Z}\rangle^0$ the convergent words (not ending in $z_{1,1}$), by sending $z_{1,1} \mapsto T$, so $\zeta^{*,T} = \zeta \circ \operatorname{reg}_T^*$.) By substituting the above expression for the case $\operatorname{reg}_0^*(w_0 z_{1,1}^{\alpha-i})$ into the following, and switching the order of summation, we see

$$\sum_{i=0}^{\alpha} \frac{1}{i!} \operatorname{reg}_0^*(w_0 z_{1,1}^{\alpha-i}) * z_{1,1}^{*i} = w_0 z_{1,1}^{\alpha}.$$

Since the left hand side is now a polynomial in $z_{1,1}$ with convergent coefficients (by virtue of being a regularised expression already), we can apply $\zeta_{T,\mathrm{reg}} = \zeta \circ \operatorname{reg}_T^*$, to obtain

$$\zeta^{*,T}\begin{pmatrix}\underline{\boldsymbol{\phi}}, \{1\}^{\alpha} \\ \underline{\mathbf{n}}, \{1\}^{\alpha}\end{pmatrix} = \sum_{i=0}^{\alpha} \zeta^{*,0}\begin{pmatrix}\underline{\boldsymbol{\phi}}, \{1\}^{\alpha-i} \\ \underline{\mathbf{n}}, \{1\}^{\alpha-i}\end{pmatrix} \frac{T^i}{i!},$$

where the letters in $w_0 = z_{n_1,\phi_1} \cdots z_{n_d,\phi_d}$ induce the arguments $\underline{\mathbf{n}}$ with signs $\underline{\boldsymbol{\phi}}$ in the MZV's. (Note that this formula is already established for classical MZV's in [18, Proposition 10, Equation (5.10), and Corollary 5].)

Now multiply both sides of the preceding equation by u^α and sum on α to obtain

$$\begin{aligned}\sum_{\alpha=0}^{\infty} \zeta^{*,T}\binom{\underline{\boldsymbol{\phi}}, \{1\}^\alpha}{\underline{\mathbf{n}}, \{1\}^\alpha} u^\alpha &= \sum_{\alpha=0}^{\infty}\sum_{i=0}^{\alpha} \zeta^{*,0}\binom{\underline{\boldsymbol{\phi}}, \{1\}^{\alpha-i}}{\underline{\mathbf{n}}, \{1\}^{\alpha-i}} \frac{u^\alpha T^i}{i!} \\ &= \sum_{\alpha=0}^{\infty} \zeta^{*,0}\binom{\underline{\boldsymbol{\phi}}, \{1\}^\alpha}{\underline{\mathbf{n}}, \{1\}^\alpha} u^\alpha \cdot \sum_{i=0}^{\infty} \frac{u^i T^i}{i!} \\ &= \sum_{\alpha=0}^{\infty} \zeta^{*,0}\binom{\underline{\boldsymbol{\phi}}, \{1\}^\alpha}{\underline{\mathbf{n}}, \{1\}^\alpha} u^\alpha \cdot \exp(Tu).\end{aligned}$$

From this we see

$$\sum_{\alpha=0}^{\infty} \zeta^{*,T}\binom{\underline{\boldsymbol{\phi}}, \{1\}^\alpha}{\underline{\mathbf{n}}, \{1\}^\alpha} u^\alpha \cdot \exp(-Tu)$$

is independent of T, so by equating the T and the S regularisation we obtain

$$\sum_{\alpha=0}^{\infty} \zeta^{*,T}\binom{\underline{\boldsymbol{\phi}}, \{1\}^\alpha}{\underline{\mathbf{n}}, \{1\}^\alpha} u^\alpha = \sum_{\alpha=0}^{\infty} \zeta^{*,S}\binom{\underline{\boldsymbol{\phi}}, \{1\}^\alpha}{\underline{\mathbf{n}}, \{1\}^\alpha} u^\alpha \cdot \exp((T-S)u).$$

Comparing the coefficient of u^α establishes the claim for MZV's at arbitrary roots of unit; when $\underline{\boldsymbol{\phi}} \in \{\pm 1\}$, one reduces to the case of alternating MZV's as stated in the lemma. □

Lemma 2.14 *The shuffle regularisations with parameter* $\zeta^{*,T}(1) = T$ *and* $\zeta^{*,S}(1) = S$ *are related as follows,*

$$\zeta^{ш,T}(\underline{\mathbf{k}}, \{1\}^\alpha) = \sum_{i=0}^{\alpha} \zeta^{ш,S}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \frac{(T-S)^i}{i!}.$$

Proof The proof of this is analogous to the above proof for the stuffle regularisation (and is also shown in the case of MZV's at arbitrary roots of unity). Namely consider the alphabet $\hat{Y} = \{e_0\} \cup \{e_\eta \mid \eta \in \mathbb{C}, |\eta| = 1\}$. Then under the induced shuffle product

$$(e_a w) ш (e_b v) = e_a(w ш e_b v) + e_a(e_b w ш v)$$

the algebra $\mathbb{Q}\langle\hat{Y}\rangle$ encodes the shuffle product of MZV's (at arbitrary roots of unity) under the map

$$e_0^k e_{\eta_1} e_0^{n_1-1} \cdots e_{\eta_d} e_0^{n_d-1} \mapsto \zeta_k \begin{pmatrix} \eta_2/\eta_1, \eta_3/\eta_2, \ldots, 1/\eta_d \\ n_1 \quad , n_2 \quad , \ldots, n_d \end{pmatrix}$$
$$= (-1)^d I(0; \{e_0\}^k, e_{\eta_1}, \{e_0\}^{n_1-1}, \ldots, e_{\eta_d}, \{e_0\}^{n_d-1}; 1),$$

(where the 1 in $1/\eta_d$ in the last sign comes from the upper bound of the integral).

For any convergent word $w_0 = e_a w_0' e_b$ with $e_a \neq e_0$ and $e_b \neq e_1$,

$$\sum_{i=0}^{\alpha} \frac{(-1)^i}{i!} e_1^{⧢ i} ⧢ (w_0 e_1^{\alpha-i}).$$

is a sum of purely convergent words; there is a pairwise cancellation of any words ending in e_1. On the other hand, this expression is a shuffle-polynomial in e_1, whose constant term is the word $w_0 e_1^\alpha$. In the regularisation where $e_1 \mapsto 0$, only the constant term of this polynomial is left, and we see

$$\mathrm{reg}_0^{⧢}(w_0 e_1^\alpha) = \sum_{i=0}^{\alpha} \frac{(-1)^i}{i!} e_1^{⧢ i} ⧢ w_0 e_1^{\alpha-i}.$$

(Here $\mathrm{reg}_T^{⧢}$ denotes the isomorphism $\mathbb{Q}\langle\hat{Y}\rangle \cong \mathbb{Q}\langle\hat{Y}\rangle^0[T]$, with $\mathbb{Q}\langle\hat{Y}\rangle^0$ the admissible words (those not ending in e_1), obtained from $\mathbb{Q}\langle\hat{Y}\rangle^0 \cong \mathbb{Q}\langle\hat{Y}\rangle^0[e_1]$ by sending $e_1 \mapsto T$, so $\zeta^{⧢,T} = \zeta \circ \mathrm{reg}_T^{⧢}$.) By substituting the above expression for the case $\mathrm{reg}_0^{⧢}\, w_0 e_1^{\alpha-i}$ into the following, and switching the order of summation, we see

$$\sum_{i=0}^{\alpha} \frac{1}{i!} \mathrm{reg}_0^{⧢}(w_0 e_1^{\alpha-i}) ⧢ e_1^{⧢ i} = w_0 e_1^\alpha.$$

Since the left hand side is now a polynomial in e_1 with convergent coefficients (by virtue of being a regularised expression already), we can apply the regularisation map with $e_1 \mapsto T$ to obtain (in zeta notation already)

$$\zeta^{⧢,T} \begin{pmatrix} \underline{\boldsymbol{\phi}}, \{1\}^\alpha \\ \underline{\mathbf{n}}, \{1\}^\alpha \end{pmatrix} = \sum_{i=0}^{\alpha} \zeta^{⧢,0} \begin{pmatrix} \underline{\boldsymbol{\phi}}, \{1\}^{\alpha-i} \\ \underline{\mathbf{n}}, \{1\}^{\alpha-i} \end{pmatrix} \frac{T^i}{i!},$$

where the letters in $w_0 = e_{\eta_1} e_0^{n_1-1} \cdots e_{\eta_d} e_0^{n_d-1}$ induce the arguments $\underline{\mathbf{n}}$ with signs $\underline{\boldsymbol{\phi}}$ in the MZV's. (Note that this formula is already established for classical MZV's in [18, Proposition 10, Equation (5.9), and Corollary 5].)

Now multiplying both sides of the preceding equation by u^α, and summing on α to form the generating series shows that

$$\sum_{\alpha=0}^{\infty} \zeta^{⧢,T} \begin{pmatrix} \underline{\boldsymbol{\phi}}, \{1\}^\alpha \\ \underline{\mathbf{n}}, \{1\}^\alpha \end{pmatrix} u^\alpha \cdot \exp(-Tu)$$

is independent of T, so by equating the T and the S regularisation we obtain the claim. □

The following is first proven in [18] for classical MZV's. It is convenient however to recall the details for application to later lemmas; moreover we need a version which holds also for alternating MZV's [28, Theorem 13.3.9].

Definition 2.15 (*Linear map ρ*) Define an $\mathbb{R}$ linear map $\rho\colon \mathbb{R}[T] \to \mathbb{R}[T]$ by

$$\rho(e^{Tu}) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right) e^{Tu},\ |u| < 1.$$

So $\rho(1) = 1$, $\rho(T) = T$, $\rho(T^2) = T^2 + \zeta(2)$ and $\rho(T^3) = T^3 + 3\zeta(2)T - 2\zeta(3)$ are the initial few values.

Then the map ρ gives us the translation between shuffle and stuffle regularisation, as follows.

Lemma 2.16 (Theorem 1, [18], generalised in Theorem 13.3.9, [28]) *For any index $\underline{\mathbf{m}} = (m_1, \ldots, m_d)$, where $m_d = 1$ is permitted, and the entries may be barred, the shuffle regularisation with parameter $\zeta^{⧢,T}(1) = T$ and the stuffle regularisation with the same parameter $\zeta^{*,T}(1) = T$ are related as follows.*

$$\zeta^{⧢,T}(\underline{\mathbf{m}}) = \rho\big(\zeta^{*,T}(\underline{\mathbf{m}})\big).$$

At this point it is instructive to notice that

$$\exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right) = \left(1 + \sum_{n=1}^{\infty} \zeta^{*,0}(\{1\}^n) u^n\right)^{-1}.$$

This can be seen via Corollary 2 in [18], or rather via Corollary 1 upon applying the regularisation-evaluation map $\zeta^{*,0} = \zeta \circ \operatorname{reg}_0^*$ (or $Z \circ \operatorname{reg}_0^*$ in the notation of [18]). Directly, one sees that the regularisation has parameter $\zeta^{*,0}(1) = 0$ since the $\zeta(1)$ term on the left hand side is not present, and so has been regularised to 0. More generally, applying $\zeta^{*,T} = \zeta \circ \operatorname{reg}_T^*$ one has:

$$\exp\left(-Tu + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right) = \left(1 + \sum_{n=1}^{\infty} \zeta^{*,T}(\{1\}^n) u^n\right)^{-1}.$$

Lemma 2.17 *The stuffle regularisation with parameter $\zeta^{*,T}(1) = T$ may be expressed via the shuffle regularisation with parameter $\zeta^{⧢,0}(1) = 0$ and the 'periodic' MZV $\zeta^{*,T}(\{1\}^i)$, as follows. For any index $\underline{\mathbf{k}} = (k_1, \ldots, k_d)$ with barred entries and $k_d \neq 1$,*

$$\zeta^{*,T}(\underline{\mathbf{k}}, \{1\}^\alpha) = \sum_{i=0}^{\alpha} \zeta^{⧢,0}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \zeta^{*,T}(\{1\}^i)$$

Proof We apply Lemma 2.13 in the case $S = 0$ to write

$$\zeta^{*,T}(\underline{\mathbf{k}}, \{1\}^{\alpha}) = \sum_{i=0}^{\alpha} \zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \frac{T^i}{i!}.$$

Note now $\zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^{\alpha-i})$ is a combination of convergent MZV's (after being regularised with parameter $\zeta^{*,0}(1) = 0$). Since ρ is $\mathbb{R}$-linear, application of ρ to convert to the shuffle product only applies to the $\frac{T^i}{i!}$ part of the summand. That is to say, we have

$$\zeta^{ш,T}(\underline{\mathbf{k}}, \{1\}^{\alpha}) = \rho\big(\zeta^{*,T}(\underline{\mathbf{k}}, \{1\}^{\alpha})\big) = \sum_{i=0}^{\alpha} \zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \rho\Big(\frac{T^i}{i!}\Big)$$

Multiply both sides by u^{α}, and sum on α to form the generating series

$$\sum_{\alpha=0}^{\infty} \zeta^{ш,T}(\underline{\mathbf{k}}, \{1\}^{\alpha}) u^{\alpha} = \sum_{\alpha=0}^{\infty} \sum_{i=0}^{\alpha} \zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \rho\Big(\frac{T^i}{i!}\Big) u^{\alpha}.$$

Interchange the summation order, then set $\alpha \to \alpha + i$

$$\begin{aligned} &= \sum_{i=0}^{\infty} \sum_{\alpha=i}^{\infty} \zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \rho\Big(\frac{T^i}{i!}\Big) u^{\alpha} = \sum_{i=0}^{\infty} \sum_{\alpha=0}^{\infty} \zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^{\alpha}) \rho\Big(\frac{T^i}{i!}\Big) u^{i+\alpha} \\ &= \sum_{\alpha=0}^{\infty} \zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^{\alpha}) u^{\alpha} \cdot \sum_{i=0}^{\infty} \rho\Big(\frac{T^i}{i!}\Big) u^i . \end{aligned}$$

One sees that the sum involving ρ is simply $\rho(e^{Tu})$, which may be replaced by the expression in Definition 2.15, to give

$$= \sum_{\alpha=0}^{\infty} \zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^{\alpha}) u^{\alpha} \cdot \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n \right) e^{Tu}$$

We therefore have

$$\begin{aligned} &\sum_{\alpha=0}^{\infty} \zeta^{ш,T}(\underline{\mathbf{k}}, \{1\}^{\alpha}) u^{\alpha} \cdot e^{-Tu} \cdot \exp\left(Tu - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n \right) \\ &= \sum_{\alpha=0}^{\infty} \zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^{\alpha}) u^{\alpha} \cdot e^{Tu}. \end{aligned}$$

By Lemma 2.14 and Lemma 2.13, respectively

$$\sum_{\alpha=0}^{\infty} \zeta^{\sqcup\!\sqcup,T}(\underline{\mathbf{k}}, \{1\}^\alpha) u^\alpha \cdot e^{-Tu} = \sum_{\alpha=0}^{\infty} \zeta^{\sqcup\!\sqcup,0}(\underline{\mathbf{k}}, \{1\}^\alpha) u^\alpha, \text{ and}$$
$$\sum_{\alpha=0}^{\infty} \zeta^{*,0}(\underline{\mathbf{k}}, \{1\}^\alpha) u^\alpha \cdot e^{Tu} = \sum_{\alpha=0}^{\infty} \zeta^{*,T}(\underline{\mathbf{k}}, \{1\}^\alpha) u^\alpha .$$

Finally, it follows from the observation above that

$$\exp\left(Tu - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right) = 1 + \sum_{n=1}^{\infty} \zeta^{*,T}(\{1\}^n) u^n.$$

Making these substitutions, and extracting the coefficient of u^α establishes the claim. □

2.4 Relating stuffle and shuffle regularised MtV's

Finally, we give a concrete and explicit relationship between the stuffle and shuffle regularised MtV's for an arbitrary choice of parameters.

Proposition 2.18 *Let $\underline{\mathbf{k}} = (k_1, \ldots, k_d)$ unbarred, such that $k_d \neq 1$. Then the stuffle regularisation of MtV's at parameter $t^{*,V}$ and the shuffle regularisation $t^{\sqcup\!\sqcup,0}$ induced by the representation of MtV's as alternating MZV's, with $\zeta^{\sqcup\!\sqcup,0}(1) = 0$, are related as follows.*

$$t^{*,V}(\underline{\mathbf{k}}, \{1\}^\alpha) = \sum_{i=0}^{\alpha} t^{\sqcup\!\sqcup,0}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \cdot \frac{1}{2^i} \zeta^{*,2V-\log(2)}(\{1\}^i). \tag{13}$$

Proof This is clearly true when $\alpha = 0$, and no regularisation is necessary, so we assume $\alpha > 0$. Now apply the expression for t in terms of alternating MZV's in (1), and Corollary 2.12 to write

$$t^{*,V}(\underline{\mathbf{k}}, \{1\}^\alpha) = \frac{1}{2^{d+\alpha}} \sum_{\substack{\underline{\boldsymbol{\varepsilon}}=(\varepsilon_1,\ldots,\varepsilon_d),\\ \varepsilon_k,\delta_1,\ldots,\delta_\alpha \in \{\pm 1\}}} \varepsilon_1 \cdots \varepsilon_d \cdot \delta_1 \cdots \delta_\alpha \zeta^{*,2V-\log(2)} \begin{pmatrix} \underline{\boldsymbol{\varepsilon}}, \delta_1, \ldots, \delta_\alpha \\ \underline{\mathbf{k}}, 1, \ldots, 1 \end{pmatrix}.$$

For notation simplicity, we shall always write $\underline{\boldsymbol{\varepsilon}} = (\varepsilon_1, \ldots, \varepsilon_d)$, and drop the explicit reference to $\in \{\pm 1\}$ from the summation; this should be taken as implied for whatever selection of signs we specify in the sum. Now gather the terms in this sum by the number of trailing $\delta_i = 1$ signs. One has

$$= \frac{1}{2^{d+\alpha}} \sum_{j=0}^{\alpha} \sum_{\substack{\underline{\boldsymbol{\varepsilon}}, \\ \delta_1,\ldots,\delta_{\alpha-1-j}, \\ \delta_{\alpha-j}=-1}} \varepsilon_1 \cdots \varepsilon_d \cdot \delta_1 \cdots \delta_{\alpha-j} \zeta^{*,2V-\log(2)} \begin{pmatrix} \underline{\boldsymbol{\varepsilon}}, \delta_1, \ldots, \delta_{\alpha-j}, \{1\}^j \\ \underline{\mathbf{k}}, 1, \ldots, 1, \{1\}^j \end{pmatrix}.$$

Application of Lemma 2.17 allows us to convert the $\zeta^{*,T}$ regularisation to $\zeta^{\sqcup\!\sqcup,0}$ corrected by $\zeta^{*,T}(\{1\}^n)$, which gives

$$= \frac{1}{2^{d+\alpha}} \sum_{j=0}^{\alpha} \sum_{\substack{\underline{\boldsymbol{\varepsilon}}, \\ \delta_1,\ldots,\delta_{\alpha-1-j}, \\ \delta_{\alpha-j}=-1}} \varepsilon_1 \cdots \varepsilon_d \cdot \delta_1 \cdots \delta_{\alpha-j} \sum_{i=0}^{j} \zeta^{\sqcup\!\sqcup,0} \begin{pmatrix} \underline{\boldsymbol{\varepsilon}}, \delta_1, \ldots, \delta_{\alpha-j}, \{1\}^{j-i} \\ \underline{\mathbf{k}}, 1, \ldots, 1, \{1\}^{j-i} \end{pmatrix} \times \zeta^{*,2V-\log(2)}(\{1\}^i)$$

Moving the sum over i outside the sum over signs (of which it is independent), and then interchanging the j and i summation order gives

$$= \frac{1}{2^{d+\alpha}} \sum_{i=0}^{\alpha} \sum_{j=i}^{\alpha} \sum_{\substack{\underline{\boldsymbol{\varepsilon}}, \\ \delta_1,\ldots,\delta_{\alpha-1-j}, \\ \delta_{\alpha-j}=-1}} \varepsilon_1 \cdots \varepsilon_d \cdot \delta_1 \cdots \delta_{\alpha-j} \cdot \zeta^{\sqcup\!\sqcup,0} \begin{pmatrix} \underline{\boldsymbol{\varepsilon}}, \delta_1, \ldots, \delta_{\alpha-j}, \{1\}^{j-i} \\ \underline{\mathbf{k}}, 1, \ldots, 1, \{1\}^{j-i} \end{pmatrix} \times \zeta^{*,2V-\log(2)}(\{1\}^i)$$

$$= \frac{1}{2^{d+\alpha}} \sum_{i=0}^{\alpha} \sum_{j=0}^{\alpha-i} \sum_{\substack{\underline{\boldsymbol{\varepsilon}}, \\ \delta_1,\ldots,\delta_{\alpha-i-1-j}, \\ \delta_{\alpha-i-j}=-1}} \varepsilon_1 \cdots \varepsilon_d \cdot \delta_1 \cdots \delta_{\alpha-i-j} \cdot \zeta^{\sqcup\!\sqcup,0} \begin{pmatrix} \underline{\boldsymbol{\varepsilon}}, \delta_1, \ldots, \delta_{\alpha-i-j}, \{1\}^{j} \\ \underline{\mathbf{k}}, 1, \ldots, 1, \{1\}^{j} \end{pmatrix} \times \zeta^{*,2V-\log(2)}(\{1\}^i)$$

One now recognises that the sum over j and the sum over signs with $\delta_{\alpha-i-j} = -1$ is just the expression for the sum over all signs, gathered by the number of trailing 1's. So we can rewrite this to be

$$= \frac{1}{2^{d+\alpha}} \sum_{i=0}^{\alpha} \sum_{\substack{\underline{\boldsymbol{\varepsilon}}, \\ \delta_1,\ldots,\delta_{\alpha-i}}} \varepsilon_1 \cdots \varepsilon_d \cdot \delta_1 \cdots \delta_{\alpha-i} \cdot \zeta^{\sqcup\!\sqcup,0} \begin{pmatrix} \underline{\boldsymbol{\varepsilon}}, \delta_1, \ldots, \delta_{\alpha-i} \\ \underline{\mathbf{k}}, 1, \ldots, 1 \end{pmatrix} \times \zeta^{*,2V-\log(2)}(\{1\}^i)$$

Lastly, we recognise the sum over signs to be $2^{d+\alpha-i} t^{\sqcup\!\sqcup,0}(\underline{\mathbf{k}}, \{1\}^{\alpha-i})$, so after making this replacement, and cancelling the powers of 2, we obtain the claim in the proposition. □

3 Evaluation of the stuffle-regualrised $t^{*,V}(\{2\}^a, 1, \{2\}^b)$

In this section we prove the following evaluation for the stuffle-regularised $t^{*,V}(\{2\}^a, 1, \{2\}^b)$, with $t^{*,V}(1) = V$. Namely

$$\begin{aligned} t^{*,V}(\{2\}^a, 1, \{2\}^b) = \\ -\sum_{r=1}^{a+b}(-1)^r 2^{-2r}\left[\binom{2r}{2a} + \frac{2^{2r}}{2^{2r}-1}\binom{2r}{2b}\right]\zeta(\overline{2r+1})t(\{2\}^{a+b-r}) \\ + \delta_{a=0}\log(2)t(\{2\}^b) + \delta_{b=0}(V - \log(2))t(\{2\}^a), \end{aligned} \tag{14}$$

where $\delta_\bullet$ is the Kronecker delta symbol, equal to 1 if the condition $\bullet$ holds, and 0 otherwise. One can write this very explicitly, if desired, as a polynomial in single zeta values, and powers of π, using the following evaluation from [16]:

$$t(\{2\}^a) = \frac{\pi^{2a}}{2^{2a}(2a)!}, \tag{15}$$

and the evaluation $\zeta(\overline{2r+1}) = -(1-2^{-2r})\zeta(2r+1)$, for $r > 0$. (Note $\zeta(\overline{1}) = -\log(2)$, while $\zeta(1)$ is divergent and must be regularised to make sense.)

In order to prove this identity, we first convert it to a generating series identity. For this purpose introduce the following functions.

Definition 3.1 (*Functions* $A(z)$, $B(z)$) For $|z| < 1$, define the $A(z)$ and $B(z)$ via the following power series

$$\begin{aligned} A(z) &:= \sum_{r=1}^{\infty}\zeta(2r+1)z^{2r}, \\ B(z) &:= \sum_{r=1}^{\infty}(1-2^{-2r})\zeta(2r+1)z^{2r} = \sum_{r=1}^{\infty}-\zeta(\overline{2r+1})z^{2r}. \end{aligned}$$

Remark 3.2 The functions $A(z)$ and $B(z)$ are the same as defined in Zagier's evaluation of $\zeta(\{2\}^a, 3, \{2\}^b)$ in [27], and Murakami's evaluation of $t(\{2\}^a, 3, \{2\}^b)$ in [22]. It is noted in the proof of Proposition 2 in [27] that they can be expressed via the digamma function $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, as follows:

$$A(z) = \psi(1) - \tfrac{1}{2}(\psi(1+z) + \psi(1-z)), \quad B(z) = A(z) - A(\tfrac{z}{2}).$$

In this form the functions $A(z)$ and $B(z)$ analytically continue to the whole complex plane, with simple poles at $z \in \mathbb{Z}\backslash\{0\}$.

It is a routine manner to sum (a tweaked version of) the generating series of the right-hand side to see the claim is equivalent to the following Theorem. For details of such summation techniques, we refer to the corresponding evaluations in both [27, proof of Proposition 2] and [22, proof of Proposition 13].

Theorem 3.3 *The following generating series evaluation holds for the stuffle-regularised* $t^{*,V}$, *with* $t^{*,V}(1) = V$,

$$\begin{aligned}\sum_{a,b\geq 0} (-1)^{a+b} t^{*,V}(\{2\}^a, 1, \{2\}^b) \cdot (2x)^{2a} (2y)^{2b} \\ = \frac{1}{2}\cos(\pi x)(A(x-y) + A(x+y) + 2(V - \log(2))) \\ + \frac{1}{2}\cos(\pi y)(B(x-y) + B(x+y) + 2\log(2)) \,,\end{aligned}$$

where

$$A(z) = \psi(1) - \tfrac{1}{2}(\psi(1+z) + \psi(1-z)) = \sum_{r=1}^{\infty} \zeta(2r+1) z^{2r} \,,$$

$$B(z) = A(z) - A(\tfrac{z}{2}) = \sum_{r=1}^{\infty} (1 - 2^{-2r}) \zeta(2r+1) z^{2r} = \sum_{r=1}^{\infty} -\zeta(\overline{2r+1}) z^{2r} \,.$$

3.1 Proof of Theorem 3.3

Firstly, recall that the ${}_pF_{p-1}$ hypergeometric function is defined as

$${}_pF_{p-1}\left[\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_{p-1}\end{matrix} ; x\right] := \sum_{m=0}^{\infty} \frac{\{a_1\}_m \cdots \{a_p\}_m}{\{b_1\}_m \cdots \{b_{p-1}\}_m} \frac{x^m}{m!} \,,$$

where $\{a\}_m = a(a+1)\cdots(a+m-1)$ is the ascending Pochhammer symbol. Asymptotic and transformation properties of the ${}_4F_3$ and ${}_3F_2$ will play a key role in the proof of our generating series evaluation.

In order to prove this theorem, we utilise a multiple t-polylogarithm type function, defined as follows.

Definition 3.4 (*Multiple t-polylogarithm*) For a choice of indices $s_1, \ldots, s_d \in \mathbb{Z}_{\geq 0}$, the Ti functions is defined by

$$\mathrm{Ti}_{s_1,\ldots,s_d}(x_1, \ldots, x_d) := \sum_{0<n_1<\cdots<n_d} \frac{x_1^{2n_1-1} \cdots x_d^{2n_d-1}}{(2n_1-1)^{s_1} \cdots (2n_d-1)^{s_d}} \,,$$

which converges when $|x_1 \cdots x_i| < 1$, for $i = 1, \ldots, d$.

Remark 3.5 Closely related functions, at least for depth $d = 1$ and weight 2, are already studied in Lewin's book [21] under the names 'the inverse tangent integral' (Chapter 2 of [21])

$$\widetilde{\mathrm{Ti}}_2(x) = \mathrm{Im}\,\mathrm{Li}_2(\mathrm{i}x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)^2}$$

and ‘Legendre’s chi-function’ (Section 1.8 of [21])

$$\chi_2(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)^2} = \frac{1}{2}\operatorname{Li}_2(x) - \frac{1}{2}\operatorname{Li}_2(-x).$$

Lewin actually uses the notation Ti for his function, but I write $\widetilde{\mathrm{Ti}}$ here to avoid confusion with the function introduced above. Moreover, the notation χ_2 is Lewin's choice, supplanting the too general notation ϕ originally used by Legendre.

Recently, Rudenko [24, Section 5.5] has also introduced essentially the same functions, and established [24, Lemma 5.18] a coproduct property of Ti which is consistent with the coaction formula (Proposition 5.7 below) of the MtV's. Rudenko's formulation occurs in the Lie coalgebra of multiple polylogarithms, wherein one is free to ignore product terms and certain weight 1 terms; for the special case where $x_1 = \cdots = x_d = 1$ one obtains MtV's, and the formula in Proposition 5.7 refines the coproduct of the corresponding MtV's.

The function Ti from Definition 3.4 is related to the classical multiple polylogarithm functions $\operatorname{Li}_{s'_1,\ldots,s'_d}$ in an analogous way to how the multiple t-value $t(s_1,\ldots,s_d)$ is related to the classical multiple zeta values $\zeta(s'_1,\ldots,s'_d)$ in (1). An explicit formula can be given, exactly as for t values, by inserting a factor $\frac{1}{2}(1-(-1)^{n_i})$ into the numerator for $i = 1,\ldots,d$, which allows one to extend the range of summation of the denominators and exponents from just odd integers, to all positive integers. Namely

$$\begin{aligned}
\operatorname{Ti}_{s_1,\ldots,s_d}(x_1,\ldots,x_d) &= \sum_{0<n_1<\cdots<n_d} \frac{x_1^{2n_1-1}\cdots x_d^{2n_d-1}}{(2n_1-1)^{s_1}\cdots(2n_d-1)^{s_d}} \\
&= \sum_{0<n_1<\cdots<n_d} \frac{(1-(-1)^{n_1})\cdots(1-(-1)^{n_d})}{2^d} \frac{x_1^{n_1}\cdots x_d^{n_d}}{n_1^{s_1}\cdots n_d^{s_d}} \\
&= \frac{1}{2^d} \sum_{\varepsilon_1,\ldots,\varepsilon_d\in\{\pm1\}} \varepsilon_1\cdots\varepsilon_d \operatorname{Li}_{s_1,\ldots,s_d}(\varepsilon_1 x_1,\ldots,\varepsilon_d x_d)\,.
\end{aligned}$$

We note also that when $s_d > 1$, the special value $\operatorname{Ti}_{s_1,\ldots,s_d}(1,\ldots,1) = t(s_1,\ldots,s_d)$ is exactly the multiple t value of the given indices, as in this case the MtV is convergent. We find, however, that

$$\operatorname{Ti}_1(z) = \sum_{n_1=1}^{\infty} \frac{z^{2n_1-1}}{2n_1-1} = \tanh^{-1}(z),$$

so in particular $\lim_{z\to1^-}\operatorname{Ti}_1(z) = \infty$.

Now, let us turn out attention to

$$\begin{aligned}&\mathrm{Ti}_{\{2\}^a,1,\{2\}^b}(\{1\}^a, z, \{1\}^b)\\&= \sum_{\substack{0<n_1<\cdots<n_a<r\\<m_1<\cdots<m_b}} \frac{1}{(2n_1-1)^2\cdots(2n_a-1)^2} \cdot \frac{z^{2r-1}}{2r-1} \cdot \frac{1}{(2m_1-1)^2\cdots(2m_b-1)^2}\,.\end{aligned}$$

We will establish that a certain limit involving a similar generating series of these $\mathrm{Ti}_{\{2\}^a,1,\{2\}^b}$-polylogs can be used to give the desired generating series of $t^{*,V=0}$ values. We find

$$\begin{aligned}&\sum_{a,b\geq 0} (-1)^{a+b}\, \mathrm{Ti}_{\{2\}^a,1,\{2\}^b}(\{1\}^a, z, \{1\}^b) \cdot (2x)^{2a}(2y)^{2b}\\&\quad= \sum_{r=1}^{\infty} \prod_{\ell<r} \Big(1 - \frac{4x^2}{(2\ell-1)^2}\Big) \cdot \frac{z^{2r-1}}{2r-1} \cdot \prod_{k>r} \Big(1 - \frac{4y^2}{(2k-1)^2}\Big)\\&\quad= \cos(\pi y) \sum_{r=1}^{\infty} \prod_{\ell<r} \Big(1 - \frac{4x^2}{(2\ell-1)^2}\Big) \cdot \frac{z^{2r-1}}{2r-1} \cdot \prod_{k\leq r} \Big(1 - \frac{4y^2}{(2k-1)^2}\Big)^{-1}\\&\quad= \frac{z\cos(\pi y)}{1-4y^2} \cdot {}_4F_3\left[\begin{matrix}1,\ \frac{3}{2},\ \frac{1}{2}-x,\ \frac{1}{2}+x\\ \frac{1}{2},\ \frac{3}{2}-y,\ \frac{3}{2}+y\end{matrix}\,;z^2\right].\end{aligned} \tag{16}$$

One checks directly that the summand above is expressible in the required form for the ${}_4F_3$ hypergeometric function.

Now the divergent part (as $z \to 1^-$) of this generating series arises from

$$\sum_{a\geq 0} (-1)^a\, \mathrm{Ti}_{\{2\}^a,1}(\{1\}^a, z) \cdot (2x)^{2a}\,.$$

We notice here that by stuffle-regularising,

$$\begin{aligned}\mathrm{Ti}_{\{2\}^a,1}(\{1\}^a, z) &= t(\{2\}^a)\,\mathrm{Ti}_1(z) - \sum_{i=0}^{a-1} \mathrm{Ti}_{\{2\}^i,1,\{2\}^{a-i}}(\{1\}^i, z, \{1\}^{a-i})\\&\quad - \sum_{i=0}^{a-1} \mathrm{Ti}_{\{2\}^i,3,\{2\}^{a-1-i}}(\{1\}^i, z, \{1\}^{a-1-i}).\end{aligned} \tag{17}$$

So one can write that

$$\sum_{a\geq 0} (-1)^a\, \mathrm{Ti}_{\{2\}^a,1}(\{1\}^a, z) \cdot (2x)^{2a} = \tanh^{-1}(z)\cos(\pi x) + f(x,z)$$

where

$$f(x,z) = -\sum_{a=0}^{\infty}(-1)^a\Bigg(\sum_{i=0}^{a-1}\mathrm{Ti}_{\{2\}^i,1,\{2\}^{a-i}}(\{1\}^i, z, \{1\}^{a-i}) \\ + \sum_{i=0}^{a-1}\mathrm{Ti}_{\{2\}^i,3,\{2\}^{a-1-i}}(\{1\}^i, z, \{1\}^{a-1-i})\Bigg)\cdot(2x)^{2a}.$$

(Note that $\cos(\pi x)$ arises as the generating series of $t(\{2\}^a)$, after incorporating the normalisation factors $(-1)^a$ and $(2x)^{2a}$ above. Namely

$$\sum_{a=0}^{\infty}(-1)^a t(\{2\}^a)\cdot(2x)^{2a} = \sum_{a=0}^{\infty}(-1)^a\frac{\pi^{2a}}{2^{2a}(2a)!}\cdot(2x)^{2a} = \cos(\pi x),$$

wherein we have substituted the evaluation of $t(\{2\}^a)$ from [16], given in (15) above.) We see that at $z=1$,

$$\begin{aligned}&\sum_{i=0}^{a-1}\mathrm{Ti}_{\{2\}^i,1,\{2\}^{a-i}}(\{1\}^{a+1}) + \sum_{i=0}^{a-1}\mathrm{Ti}_{\{2\}^i,3,\{2\}^{a-1-i}}(\{1\}^a)\\ &= \sum_{i=0}^{a-1}t(\{2\}^i,1,\{2\}^{a-i}) + \sum_{i=0}^{a-1}t(\{2\}^i,3,\{2\}^{a-1-i})\\ &= t^{*,V=0}(1)t(\{2\}^a) - t^{*,V=0}(\{2\}^a,1)\\ &= -t^{*,V=0}(\{2\}^a,1)\,.\end{aligned}$$

So that $f(x,1)$ (or at least the limit $\lim_{z\to1^-}$ thereof) satisfies

$$f(x,1) = \sum_{a=0}^{\infty}(-1)^a t^{*,V=0}(\{2\}^a,1)\cdot(2x)^{2a}.$$

Now subtract (17) from (16), and take the limit $\lim_{z\to1^-}$. From this we see that the generating series of stuffle-regularised (at $V=0$) MtV's is obtained by computation of the following limit

$$\begin{aligned}&\sum_{a,b\geq0}(-1)^{a+b}t^{*,V=0}(\{2\}^a,1,\{2\}^b)\cdot(2x)^{2a}(2y)^{2b}\\ &= \lim_{z\to1^-}\frac{z\cos(\pi y)}{1-4y^2}\cdot{}_4F_3\left[\begin{matrix}1,\ \frac{3}{2},\ \frac{1}{2}-x,\ \frac{1}{2}+x\\ \frac{1}{2},\ \frac{3}{2}-y,\ \frac{3}{2}+y\end{matrix};z^2\right] - \tanh^{-1}(z)\cos(\pi x)\,.\end{aligned} \tag{18}$$

We now apply some transformation properties of ${}_4F_3$ in order to reduce this to a combination of ${}_3F_2$ functions, whose asymptotic behaviour is established by [8]. First make use of the contiguous function relation

$$b \cdot {}_4F_3\left[\begin{matrix} a,\ b+1,\ c,\ d \\ p,\ q,\ r \end{matrix};z\right] - a \cdot {}_4F_3\left[\begin{matrix} a+1,\ b,\ c,\ d \\ p,\ q,\ r \end{matrix};z\right] + (a-b) \cdot {}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d \\ p,\ q,\ r \end{matrix};z\right] = 0$$

in the case $(a, b, c, d) = (1, \frac{1}{2}, \frac{1}{2}-x, \frac{1}{2}+x)$, $(p, q, r) = (\frac{1}{2}, \frac{3}{2}-y, \frac{3}{2}+y)$, to obtain the following reduction of our ${}_4F_3$ to a combination of ${}_3F_2$'s. We find

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} 1,\ \frac{3}{2},\ \frac{1}{2}-x,\ \frac{1}{2}+x \\ \frac{1}{2},\ \frac{3}{2}-y,\ \frac{3}{2}+y \end{matrix};z^2\right] \\ &= 2 \cdot {}_3F_2\left[\begin{matrix} 2,\ \frac{1}{2}-x,\ \frac{1}{2}+x \\ \frac{3}{2}-y,\ \frac{3}{2}+y \end{matrix};z^2\right] - {}_3F_2\left[\begin{matrix} 1,\ \frac{1}{2}-x,\ \frac{1}{2}+x \\ \frac{3}{2}-y,\ \frac{3}{2}+y \end{matrix};z^2\right] \end{aligned} \tag{19}$$

The second term is convergent at $z = 1$, and can be evaluated via Whipple's theorem (see Section 3.4 in [1]) to give (after some simplification with the reflection formula of the Γ-function) that

$$\begin{aligned} {}_3F_2\left[\begin{matrix} 1,\ \frac{1}{2}-x,\ \frac{1}{2}+x \\ \frac{3}{2}-y,\ \frac{3}{2}+y \end{matrix};1\right] &= \frac{(1-2y)(1+2y)}{2(x-y)(x+y)} \sec(\pi y) \sin\left(\tfrac{\pi}{2}(x-y)\right) \sin\left(\tfrac{\pi}{2}(x+y)\right) \\ &= -\frac{1-4y^2}{\cos(\pi y)} \frac{\cos(\pi x)-\cos(\pi y)}{4(x^2-y^2)} . \end{aligned}$$

To deal with the first term, we need to recall the Evans-Stanton/Ramanujan asymptotic for 0-balanced ${}_3F_2$ hypergeometric functions.

Theorem 3.6 (Evans–Stanton 1984 [8], Ramanujan) *If* $a+b+c = d+e$, *and* $\mathrm{Re}(c) > 0$*, then as* $u \to 1^-$*,*

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)} \cdot {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix};u\right] = -\log(1-u) + L + O((1-u)\log(1-u)),$$

where

$$L = -2\gamma - \frac{\Gamma'(a)}{\Gamma(a)} - \frac{\Gamma'(b)}{\Gamma(b)} + \sum_{k=1}^{\infty} \frac{\{d-c\}_k\,\{e-c\}_k}{\{a\}_k\,\{b\}_k\,k}.$$

Here $\gamma \approx 0.577\ldots$ *is the Euler–Mascheroni constant, and* $\{x\}_k = x(x+1)\cdots(x+k-1)$ *is the ascending Pochhammer symbol.*

If we apply this asymptotic (with $c = 2$, and $a, b = \frac{1}{2} \pm x$, via the symmetry of ${}_3F_2$ in its upper arguments) to the first term on the right hand side of (19), and recall

$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, we obtain the asymptotic formula

$$\begin{aligned}
&\frac{4}{1-4y^2} \frac{\cos(\pi y)}{\cos(\pi x)} {}_3F_2\left[\begin{matrix} 2, \ \frac{1}{2}-x, \ \frac{1}{2}+x \\ \frac{3}{2}-y, \ \frac{3}{2}+y \end{matrix}; z^2\right] \\
&= -\log(1-z^2) - 2\gamma - \psi(\tfrac{1}{2}-x) - \psi(\tfrac{1}{2}+x) \\
&\quad + \sum_{k=1}^{\infty} \frac{\left\{-\frac{1}{2}-y\right\}_k \left\{-\frac{1}{2}+y\right\}_k}{k\left\{\frac{1}{2}-x\right\}_k \left\{\frac{1}{2}+x\right\}_k} + O((1-z^2)\log(1-z^2)) \,.
\end{aligned}$$

We also note

$$4A(2x) - 2A(x) = -4\log(2) - 2\gamma - \psi(\tfrac{1}{2}-x) - \psi(\tfrac{1}{2}+x),$$

so that the digamma combination above can be rewritten via the function A defined earlier. Applying these results to (18), we find

$$\begin{aligned}
\text{RHS (18)} = \frac{\cos(\pi x) - \cos(\pi y)}{4(x^2-y^2)} + \frac{1}{2}\cos(\pi x) \\
\cdot \left(4A(2x) - 2A(x) + 2\log(2) + \sum_{k=1}^{\infty} \frac{\left\{-\frac{1}{2}-y\right\}_k \left\{-\frac{1}{2}+y\right\}_k}{k\left\{\frac{1}{2}-x\right\}_k \left\{\frac{1}{2}+x\right\}_k}\right).
\end{aligned}$$

We note next that

$$\sum_{k=1}^{\infty} \frac{\left\{-\frac{1}{2}-y\right\}_k \left\{-\frac{1}{2}+y\right\}_k}{k\left\{\frac{1}{2}-x\right\}_k \left\{\frac{1}{2}+x\right\}_k} = \frac{\mathrm{d}}{\mathrm{d}Z}\Big|_{Z=0} {}_3F_2\left[\begin{matrix} -\frac{1}{2}-y, \ -\frac{1}{2}+y, \ Z \\ \frac{1}{2}-x, \ \frac{1}{2}+x \end{matrix}; 1\right].$$

Compare Proposition 1 in [27] for a similar summation, which we will in fact reduce this to. Using the contiguous function relation

$$\begin{aligned}
(a-b)p \cdot {}_3F_2\left[\begin{matrix} a, \ b, \ c \\ p, \ q \end{matrix}; z\right] - b(a-p) \cdot {}_3F_2\left[\begin{matrix} a, \ 1+b, \ c \\ 1+p, \ q \end{matrix}; z\right] \\
+ a(b-p) \cdot {}_3F_2\left[\begin{matrix} 1+a, \ b, \ c \\ 1+p, \ q \end{matrix}; z\right] = 0
\end{aligned}$$

in the case $(a, b, c) = (-\frac{1}{2}-y, -\frac{1}{2}+y, Z)$, $(p, q) = (\frac{1}{2}-x, \frac{1}{2}+x)$, we find (note the sign of y is different in various places in the coefficient of each ${}_3F_2$ on the right hand side) that

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}-\frac{1}{2}-y,\ -\frac{1}{2}+y,\ Z\\ \frac{1}{2}-x,\ \frac{1}{2}+x\end{matrix};1\right]\\
&\quad=-\frac{(1-x+y)(1-2y)}{2y(1-2x)}{}_3F_2\left[\begin{matrix}-\left(\frac{1}{2}+y\right),\ \frac{1}{2}+y,\ Z\\ 1-\left(x-\frac{1}{2}\right),\ 1\ \ +\left(x-\frac{1}{2}\right)\end{matrix};1\right]\\
&\qquad+\frac{(1-x-y)(1+2y)}{2y(1-2x)}{}_3F_2\left[\begin{matrix}-\left(-\frac{1}{2}+y\right),\ -\frac{1}{2}+y,\ Z\\ 1-\left(x-\frac{1}{2}\right),\ 1+\left(x-\frac{1}{2}\right)\end{matrix};1\right],
\end{aligned}
\tag{20}
$$

and the same expression upon replacing ${}_3F_2$ with $\frac{\mathrm{d}}{\mathrm{d}Z}\big|_{Z=0}{}_3F_2$ on both sides. Both hypergeometric functions derivatives are now of the form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}Z}\Big|_{Z=0}{}_3F_2\left[\begin{matrix}-X,\ X,\ Z\\ 1-Y,\ 1+Y\end{matrix};1\right]&=\left[A(X+Y)+A(X-Y)-2A(Y)\right]\\
&\quad-\frac{\sin(\pi X)}{\sin(\pi Y)}\left[B(X+Y)-B(X-Y)\right],
\end{aligned}
$$

the evaluation of which here follows as essentially the punchline to Section 4 of Zagier's evaluation of $\zeta(\{2\}^a,3,\{2\}^b)$ in [27] after combining the results of Sections 2 and 3 therein. (Namely the equality of $F(x,y)=\widehat{F}(x,y)$ established in the proof Theorem 1 in [27], plus the expressions in Propositions 1 and 2 of [27], gives the above evaluation.)

Substituting this evaluation into $\frac{\mathrm{d}}{\mathrm{d}Z}\big|_{Z=0}$ of (20), and substituting the resulting Pochhammer sum evaluation into the ${}_4F_3$ limit produces an elementary expression for the generating series of $(-1)^{a+b}t^{*,V=0}(\{2\}^a,1,\{2\}^b)$ in terms of A, B, sine and cosine.

This elementary generating series expression can be simplified as follows. Firstly, apply the digamma duplication relation

$$
\psi\left(z+\tfrac{1}{2}\right)=-\psi(z)+2\psi(2z)-2\log(2),
$$

to obtain that

$$
A\left(-\tfrac{1}{2}+x\right)=\tfrac{1}{2x-1}-A(x)+2A(2x)+2\log(2);
$$

use this to eliminate $A(-\frac{1}{2}+x)$ from the resulting expression in favour of $2A(2x)-A(x)$. Then use the digamma functional equation

$$
\psi(z+1)=\psi(z)+\tfrac{1}{z},
$$

to obtain (along with duplication in the case of B), that

$$
A(x+1)=A(x)-\tfrac{1}{2(1-x)}-\tfrac{1}{2x},\quad B(x+1)=-\tfrac{1}{2x(x+1)}-2\log(2)-B(x).
$$

Use these to replace $B(1-x+y)$, $B(-1+x+y)$, $A(1-x+y)$, $A(-1+x+y)$ by $-B(-x+y)$, $-B(x+y)$, $A(-x+y)$, $A(x+y)$, respectively. Since both $A(x)$ and

$B(x)$ are even functions, the expression now simplifies directly, and one readily finds

$$\begin{aligned}&\sum_{a,b\geq 0}(-1)^{a+b}t^{*,V=0}(\{2\}^a,1,\{2\}^b)\cdot(2x)^{2a}(2y)^{2b}\\&=\frac{1}{2}\cos(\pi x)(A(x-y)+A(x+y)-2\log(2))\\&\quad+\frac{1}{2}\cos(\pi y)(B(x-y)+B(x+y)+2\log(2)).\end{aligned}\tag{21}$$

The generating series for the general regularisation is recovered upon noting that

$$t^{*,V}(\{2\}^a,1)=Vt(\{2\}^a)+t^{*,V=0}(\{2\}^a,1),$$

i.e. the constant term in the regularisation polynomial is the regularisation at parameter $V=0$. Since

$$\sum_{a\geq 0}(-1)^a Vt(\{2\}^a)\cdot(2x)^{2a}=V\cos(\pi x),$$

as already noted above without the V, this gives the necessary correction term to add to the right hand side of (21) to find the generating series for the general regularisation. Doing so gives the equality stated in Theorem 3.3, and so completes the proof. □

3.2 Evaluation of shuffle-regularised $t^{⧢,W}(\{2\}^a,1,\{2\}^b)$

Although the shuffle regularisation $t^{⧢,0}$, arising from $\zeta^{⧢,0}(1)=0$ is most important, we can in fact compute the regularisation for any $t^{⧢,W}$ arising from $\zeta^{⧢,W}(1)=W$ with equal ease. Clearly, if $b>0$

$$t^{⧢,W}(\{2\}^a,1,\{2\}^b)=t(\{2\}^a,1,\{2\}^b),$$

as no regularisation is necessary. However when $b=0$ we compute via (1)—with the convention that $\underline{\boldsymbol{\varepsilon}}=(\varepsilon_1,\ldots,\varepsilon_a)$ and the sum over $\underline{\boldsymbol{\varepsilon}},\delta$ implies all choice of signs in $\{\pm 1\}$—that

$$t^{⧢,W}(\{2\}^a,1)=\frac{1}{2^{a+1}}\sum_{\underline{\boldsymbol{\varepsilon}},\delta}\varepsilon_1\cdots\varepsilon_a\cdot\delta\cdot\zeta^{⧢,W}\begin{pmatrix}\underline{\boldsymbol{\varepsilon}},\delta\\\{2\}^a,1\end{pmatrix}$$

Since the alternating MZV ends with at most a single entry 1 (with sign 1), we know from Lemma 2.16 that the shuffle and the stuffle regularisation in this case are exactly equal. This is because the $\mathbb{R}$-linear map ρ from Definition 2.15 appearing in Lemma 2.16 has $\rho(1)=1$ and $\rho(T)=T$, so leaves a linear regularisation polynomial unchanged. Hence

$$
\begin{aligned}
&= \frac{1}{2^{a+1}} \sum_{\underline{\boldsymbol{\varepsilon}},\delta} \varepsilon_1 \cdots \varepsilon_a \cdot \delta \cdot \zeta^{*,W} \begin{pmatrix} \underline{\boldsymbol{\varepsilon}}, \delta \\ \{2\}^a, 1 \end{pmatrix} \\
&= t^{*, \frac{1}{2}(W+\log(2))}(\{2\}^a, 1).
\end{aligned}
$$

Recall from Corollary 2.12: the stuffle regularisation of $t^{*,V}(1) = V$ corresponds to the stuffle regularisation of $\zeta^{*,U}(1) = U$ where $U = 2V - \log(2)$, hence the change in regularisation parameter in the last line.

This amounts to saying the shuffle regularised version $t^{\sqcup\!\sqcup,W}$ of the generating series in Theorem 3.3 is obtained simply by changing the regularisation parameter on the RHS to $\frac{1}{2}(W + \log(2))$. Hence we have the following proposition.

Proposition 3.7 *The following generating series evaluation holds for the shuffle-regularised $t^{\sqcup\!\sqcup,W}$, induced by $\zeta^{\sqcup\!\sqcup,W}(1) = W$,*

$$
\begin{aligned}
&\sum_{a,b\geq 0} (-1)^{a+b} t^{\sqcup\!\sqcup,W}(\{2\}^a, 1, \{2\}^b) \cdot (2x)^{2a}(2y)^{2b} \\
&\quad = \frac{1}{2}\cos(\pi x)(A(x-y) + A(x+y) + (W - \log(2))) \\
&\quad\quad + \frac{1}{2}\cos(\pi y)(B(x-y) + B(x+y) + 2\log(2)) ,
\end{aligned}
$$

where

$$
\begin{aligned}
A(z) &= \psi(1) - \tfrac{1}{2}(\psi(1+z) + \psi(1-z)) = \sum_{r=1}^{\infty} \zeta(2r+1) z^{2r} , \\
B(z) &= A(z) - A(\tfrac{z}{2}) = \sum_{r=1}^{\infty} (1 - 2^{-2r})\zeta(2r+1) z^{2r} = \sum_{r=1}^{\infty} -\zeta(\overline{2r+1}) z^{2r} .
\end{aligned}
$$

From this follows an explicit evaluation, analogous to (14), by replacing V with $\frac{1}{2}(W + \log(2))$ therein:

$$
\begin{aligned}
&t^{\sqcup\!\sqcup,W}(\{2\}^a, 1, \{2\}^b) \\
&= -\sum_{r=1}^{a+b} (-1)^r 2^{-2r} \left[\binom{2r}{2a} + \frac{2^{2r}}{2^{2r}-1} \binom{2r}{2b} \right] \zeta(\overline{2r+1}) t(\{2\}^{a+b-r}) \\
&\quad + \delta_{a=0} \cdot \log(2) t(\{2\}^b) + \delta_{b=0} \cdot \frac{1}{2}(W - \log(2)) t(\{2\}^a),
\end{aligned} \tag{22}
$$

where $\delta_\bullet$ is the Kronecker delta symbol, equal to 1 if the condition $\bullet$ holds, and 0 otherwise.

3.3 Evaluation of $t(1, \{2\}^n)$

In order to answer a question posed in [6], we turn to the special case of $t(1, \{2\}^n)$, for $n \geq 1$. Here we extract from (14), the following evaluation for $t(1, \{2\}^n)$, where $n \geq 1$

$$t(1, \{2\}^n) = \\ -\sum_{r=1}^{n}(-1)^r 2^{-2r}\left[1 + \frac{2^{2r}}{2^{2r}-1}\binom{2r}{2n}\right]\zeta(\overline{2r+1})t(\{2\}^{n-r}) + \log(2)t(\{2\}^n).$$

Since $\binom{2r}{2n} = 0$ for $r < n$, and $\binom{2r}{2n} = 1$ for $r = n$, this can be written as

$$= \log(2)t(\{2\}^n) - \sum_{r=1}^{n-1}(-1)^r 2^{-2r}\zeta(\overline{2r+1})t(\{2\}^{n-r}) \\ - (-1)^n 2^{-2n}\left[1 + \frac{2^{2n}}{2^{2n}-1}\right]\zeta(\overline{2n+1}).$$

Now the first term can be incorporated as the $r = 0$ term of the sum, giving

$$= -\sum_{r=0}^{n-1}(-1)^r 2^{-2r}\zeta(\overline{2r+1})t(\{2\}^{n-r}) - (-1)^n 2^{-2n}\left[1 + \frac{2^{2n}}{2^{2n}-1}\right]\zeta(\overline{2n+1})$$

Now substitute in the evaluation of $t(\{2\}^a) = \frac{\pi^{2a}}{2^{2a}(2a)!}$ from (15), and convert the last term to a classical (non-alternating) MZV, to obtain

$$= \frac{1}{2^{2n}}\left(\sum_{r=0}^{n-1}(-1)^r(-\zeta(\overline{2r+1}))\frac{\pi^{2(n-r)}}{(2(n-r))!} + (-1)^n 2\big(1 - 2^{-2n-1}\big)\zeta(2n+1)\right)$$

This confirms Conjecture 4.5 on the evaluation of $t(1, \{2\}^n)$ stated in [6] (be aware, the opposite MZV/MtV convention is used therein). The authors of [6] also write the Dirichlet eta function $\eta(m) = (1-2^{1-m})\zeta(m)$, with $\eta(1) = \log(2)$, in place of $-\zeta(\overline{m})$ used herein.

4 Motivic framework

In this section we briefly recall the setup of motivic iterated integrals framework introduced by Brown [4, 5] (extending that of Goncharov [12, 13]). We define the motivic (alternating) MZV's and the motivic MtV's; we introduce the necessary combinatorial operations and fundamental properties of these objects which will play a key role from Section 6 onwards.

4.1 Goncharov's motivic iterated integrals

In [13], Goncharov upgraded the iterated integrals $I(x_0; x_1, \ldots, x_N; x_{N+1})$, $x_i \in \overline{\mathbb{Q}}$ (see (7) above for the definition), to framed mixed Tate motives, in order to define motivic iterated integrals

$$I^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1})$$

living in a graded (by the weight N) connected Hopf algebra $\mathcal{A} = \mathcal{A}_\bullet(\overline{\mathbb{Q}})$. The Hopf algebra $\mathcal{A}$ is the ring of regular functions on the unipotent part of the motivic Galois group. In [13], they are denoted by $I^{\mathscr{M}}$, but when incorporated into Brown's motivic framework, they are better denoted by $I^{\mathfrak{u}}$ for the unipotent part. (The component consisting of weight N integrals is denoted $\mathcal{A}_N$.)

The motivic iterated integrals satisfy relations of a 'geometric' origin, arising from change of variables in an iterated integral, the results of Stoke's theorem, or from the linearity of domain and integrand. The coproduct Δ on this Hopf algebra is computed via Theorem 1.2 in [13] as

$$\Delta I^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1}) = \sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=N+1}} I^{\mathfrak{u}}(x_0; x_{i_1}, \ldots, x_{i_k}; x_{N+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{u}}(x_{i_p}; x_{i_p+1}, \ldots, x_{i_{p+1}-1}; x_{i_{p+1}}).$$

In this Hopf algebra, the motivic version of $\zeta^{\mathfrak{u}}(2) = -I^{\mathfrak{u}}(0; 1, 0; 1) = 0$, or more fundamentally, the Lefschetz motive $\mathbb{L}^{\mathfrak{u}}$, a motivic version of $\mathrm{i}\pi$, vanishes so that $(\mathrm{i}\pi)^{\mathfrak{u}} = \mathbb{L}^{\mathfrak{u}} = 0$.

4.2 Brown's $\mathcal{A}$-comodule of motivic iterated integrals

The motivic iterated integrals $I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1})$ in the sense of Brown [4, 5] are elements of the weight-graded $\mathcal{A}$-comodule $\mathcal{H}$ of regular functions on the torsor of tensor isomorphisms between Betti and de Rham realisations. (The weight N graded component of $\mathcal{H}$ is denoted $\mathcal{H}_N$. These integrals do depend implicitly on a path γ from x_0 to x_{N+1}, but for our purposes typically $x_0, x_{N+1} \in \{0, 1\}$, and then the canonical straight line path $\mathrm{dch}\colon [0,1] \to [0,1]$, $t \mapsto t$ is sufficient.)

This comodule is endowed with a coaction $\Delta\colon \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ which, as noted in [5], is given by the same formula as Goncharov's coproduct, transposed to this setting, i.e.

$$\Delta I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1}) = \sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=N+1}} \prod_{p=0}^{k} I^{\mathfrak{u}}(x_{i_p}; x_{i_p+1}, \ldots, x_{i_{p+1}-1}; x_{i_{p+1}}) \otimes I^{\mathfrak{m}}(x_0; x_{i_1}, \ldots, x_{i_k}; x_{N+1}). \tag{23}$$

(We have switched the order of the factors for later convenience.) We will mainly use the derivation operations D_r defined as a linearised, weight-graded part of the coaction (see Section 4.4 below), but it will be useful to keep in mind from where these operations originate, particularly when considering how they act on primitive elements.

In Brown's setting $\zeta^{\mathfrak{m}}(2) = -I^{\mathfrak{m}}(0; 1, 0; 1) \neq 0$, and therefore much more information about motivic iterated integrals is retained. In particular, the coaction can fix identities up to primitive elements (namely motivic MZV's of depth 1, at some root of unity $\zeta^{\mathfrak{m}}\binom{\exp(2\pi i a/b)}{n}$). More concretely the coaction can be used to fix the coefficient of product terms involving $\zeta(2n)$, in contrast to the coproduct above. (One can think of Goncharov's motivic iterated integrals as $\mathcal{H}/\zeta^{\mathfrak{m}}(2)\mathcal{H}$, wherein $\zeta^{\mathfrak{m}}(2)$ is killed.)

One has a well-defined $\mathbb{Q}$-algebra homomorphism per, called the period map,

$$\begin{gathered}\mathrm{per}\colon \mathcal{H} \to \mathbb{C} \\ I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1}) \mapsto I^{Ш,0}(x_0; x_1, \ldots, x_N; x_{N+1}),\end{gathered}$$

which means that the classical iterated integrals satisfy all motivically true relations. Conjecturally, the space of motivic iterated integrals is isomorphic to the space of classical iterated integrals. This conjecture is a special case of the Grothendieck period conjecture, which posits that the period map per (in the most general setting) is injective, so that all relations are motivic ('geometric') in origin, i.e. there are no 'spurious' or 'coincidental' relations on the level of numbers.

We briefly recall some main relations satisfied by the motivic iterated integrals.

(i) Unit: $I^{\mathfrak{m}}(x_0; x_1) = 1$ in weight 0,
(ii) Trivial integration: $I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1}) = 0$ if $x_0 = x_{N+1}$ and $N \geq 1$,
(iii) Path composition: for any $y \in \overline{\mathbb{Q}}$,

$$I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1}) = \sum_{i=0}^{N} I^{\mathfrak{m}}(x_0; x_1, \ldots, x_i; y) I^{\mathfrak{m}}(y; x_{i+1}, \ldots, x_N; x_{N+1})$$

(iv) Path reversal: $I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1}) = (-1)^N I^{\mathfrak{m}}(x_{N+1}; x_N, \ldots, x_1; x_0)$
(v) Homothety: if $x_0 \neq x_1$, and $x_N \neq x_{N+1}$, then for any $\alpha \in \overline{\mathbb{Q}}$,

$$I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1}) = I^{\mathfrak{m}}(\alpha \cdot x_0; \alpha \cdot x_1, \ldots, \alpha \cdot x_N; \alpha \cdot x_{N+1})$$

Tangential base-points: In the cases where $x_0 = x_1$ or $x_N = x_{N+1}$ the motivic iterated integrals depends even on the tangential base-points of the path from x_0 to x_{N+1}.

More formally, for this process, we replace 0 and 1 with a tangential base-points $\vec{1}_0$ and $\overrightarrow{-1}_1$ which denote the tangent vector $\vec{1}$ at the point 0, and $\overrightarrow{-1}$ at the point 1, respectively, which are the tangent vectors for the straight line path dch: $[0, 1] \to [0, 1]$. The details of iterated integrals with tangential base points (and the motivic versions thereof) can be found in [9, Section 3.7, Section 4.5]. The notation with tangential base-points is helpful to identify when certain transformations and relations are invalid in the case of regularised integrals.

One can use the shuffle product to write any

$$I^{\mathfrak{m}}(x_0; \{x_0\}^a, x_1, \ldots, x_N, \{x_{N+1}\}^b; x_{N+1})$$

as a polynomial in $I^{\mathfrak{m}}(x_0; x_0; x_{N+1})$,$I^{\mathfrak{m}}(x_0; x_{N+1}; x_{N+1})$ and integrals of the form $I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1})$, $x_1 \neq x_0, x_N \neq x_{N+1}$ (in a similar manner to (8)). Then the calculation $I^{\mathfrak{m}}(a; b; c) = \log^{\mathfrak{m}}(b-c) - \log^{\mathfrak{m}}(b-a)$, with $\log^{\mathfrak{m}}(0) := 0$ (cf. [13, Equation 6] and thereafter) allows one to understand $I^{\mathfrak{m}}(x_0; \{x_0\}^a, x_1, \ldots, x_N, \{x_{N+1}\}^b; x_{N+1})$ in general.

For the case $x_0 = 0, x_1 = 1$, we find $I^{\mathfrak{m}}(0; 0; 1) = I^{\mathfrak{m}}(0; 1; 1) = 0$, so that per $I^{\mathfrak{m}}(0; x_1, \ldots, x_n; 1)$ is the shuffle-regularised version $I^{ш,0}(0; x_1, \ldots, x_n; 1)$, (with $I^{ш,0}$ defined analogously to $\zeta^{ш,0}$ in Section 2.1, extending I as a shuffle-homomorphism to all divergent words, and sending $e_0, e_1 \mapsto 0$, to give $I^{ш,0}(0; 0; 1) = I^{ш,0}(0; 1; 1) = 0$).

The homothety property fails for $I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N, x_{N+1})$ if $x_0 = x_1$ or if $x_N = x_{N+1}$, because in this case the integral depends on the vector of the tangential base-points at x_0 or x_{N+1}, which are changed when we scale by $x_i \mapsto x_i\alpha$. This point is glossed over in [10, Section 2.3], [11, Section 2.2] and in [22, Section 2]. However, whenever the homothety property is applied in [22], one only needs it to hold modulo products and $\zeta^{\mathfrak{u}}(2)$, i.e. in the Lie coalgebra $\mathcal{L} = \mathcal{A}_{>0}/\mathcal{A}_{>0} \cdot \mathcal{A}_{>0}$ which we introduce momentarily (in Section 4.4 below). This version of homothety does hold in general for $x_i \in \{0, \pm 1\}$, and so Murakami's conclusions are valid; see Remark 5.8 below for more explicit details.

4.3 Motivic multiple zeta values and motivic multiple *t* values

For $\ell \in \mathbb{Z}_{\geq 0}$, any index $\underline{\mathbf{k}} = (k_1, \ldots, k_d)$ with $k_i \in \mathbb{Z}_{\geq 1}$ and any choice of signs $\varepsilon_i \in \{\pm 1\}$, we define the *(alternating) motivic multiple zeta values* by

$$\zeta_\ell^{\mathfrak{m}}\begin{pmatrix}\varepsilon_1, \ldots, \varepsilon_d \\ k_1, \ldots, k_d\end{pmatrix} := (-1)^d I^{\mathfrak{m}}(0; \{0\}^\ell, \eta_1, \{0\}^{k_1-1}, \eta_2, \{0\}^{k_2-1}, \ldots, \eta_d, \{0\}^{k_d-1}; 1),$$

where $\{k\}^n = \overbrace{k, \ldots, k}^{n}$ denotes the argument k repeated n times, and $\eta_i = \prod_{j=i}^d \varepsilon_j^{-1}$. This arises by transposing (6) to the motivic world, as a definition, and extending to the case of leading 0's (which as already indicated in the tangential base-point discussion above) amounts to an analogue of shuffle-regularisation with $\zeta^{\mathfrak{m}}(1) = 0$, as explained in Section 2.1 for $\zeta^{ш,0}$.

When $\ell > 0$, this integral is computed in the same manner as described in Section 2.1, in particular via (9) in Lemma 2.6 to express $\zeta_\ell^{\mathfrak{m}}$ in terms of $\zeta_0^{\mathfrak{m}} = \zeta^{\mathfrak{m}}$. We have a further property

(vi) Unshuffling of 0's:

$$\zeta_\ell^{\mathfrak{m}}\binom{\varepsilon_1,\dots,\varepsilon_d}{k_1,\dots,k_d} = (-1)^\ell \sum_{i_1+\cdots+i_d=\ell} \binom{k_1+i_1-1}{i_1}\cdots\binom{k_d+i_d-1}{i_d} \cdot \zeta_0^{\mathfrak{m}}\binom{\varepsilon_1,\dots,\varepsilon_d}{k_1+i_1,\dots,k_d+i_d}.$$

When $\ell = 0$, we can write $\zeta^{\mathfrak{m}}$ instead of $\zeta_0^{\mathfrak{m}}$. When all signs $\varepsilon_i = 1$, we can write

$$\zeta_\ell^{\mathfrak{m}}\binom{1,\dots,1}{k_1,\dots,k_d} =: \zeta_\ell^{\mathfrak{m}}(k_1,\dots,k_d),$$

and will refer to this as a *(non-alternating) motivic MZV*. As shorthand notation, we also write $\overline{k_i}$ to denote the argument k_i which has associated sign $\varepsilon_i = -1$, and write $\zeta^{\mathfrak{m}}$ with only one row of arguments.

It is convenient to give notation to the space of all motivic MZV's and all motivic alternating MZV's, within the space of all motivic iterated integrals.

Definition 4.1 (*Space of (alternating) motivic MZV's*) Let $\mathcal{H}^{(1)}$ be the $\mathbb{Q}$-vector space generated by all (non-alternating) motivic MZV's. Likewise, let $\mathcal{H}^{(2)}$ be the $\mathbb{Q}$-vector space generated by all alternating motivic MZV's. Moreover, write $\mathcal{H}_N^{(1)}$, or $\mathcal{H}_N^{(2)}$ for the space of weight N (non-alternating) motivic MZV's, and weight N alternating motivic MZV's respectively.

We then define the motivic multiple t values, using (1) as follows.

Definition 4.2 (*Motivic multiple t value*) For any index $\underline{\mathbf{k}} = (k_1,\dots,k_d)$, $k_i \in \mathbb{Z}_{\geq 1}$, the *motivic multiple t value* $t^{\mathfrak{m}}(k_1,\dots,k_d)$ is defined by

$$t^{\mathfrak{m}}(k_1,\dots,k_d) := \frac{1}{2^d} \sum_{\varepsilon_i \in \{\pm 1\}} \varepsilon_1 \cdots \varepsilon_d \, \zeta^{\mathfrak{m}}\binom{\varepsilon_1,\dots,\varepsilon_d}{k_1,\dots,k_d}.$$

It will often be convenient to work with the following rescaled version

$$\widetilde{t}^{\mathfrak{m}}(k_1,\dots,k_d) := 2^{|\underline{\mathbf{k}}|-d} \sum_{\varepsilon_i \in \{\pm 1\}} \varepsilon_1 \cdots \varepsilon_d \, \zeta^{\mathfrak{m}}\binom{\varepsilon_1,\dots,\varepsilon_d}{k_1,\dots,k_d},$$

where $|\underline{\mathbf{k}}| = k_1 + \cdots + k_d$ denotes the weight of the index $\underline{\mathbf{k}}$.

We call a motivic MtV $t^{\mathfrak{m}}(k_1,\dots,k_d)$ or $\widetilde{t}^{\mathfrak{m}}(k_1,\dots,k_d)$

(i) a *convergent motivic MtV* if $k_d \geq 2$, and
(ii) an *extended motivic MtV* if $k_d \geq 1$.

By view of (1) and (10), we see that the image of $t^{\mathfrak{m}}(k_1,\dots,k_d)$ under the period map,

$$\mathrm{per}(t^{\mathfrak{m}}(k_1,\dots,k_d)) = t^{⧢,0}(k_1,\dots,k_d),$$

gives the shuffle-regularised multiple t value $t^{\sqcup\!\sqcup,0}(k_1,\ldots,k_d)$ arising from the shuffle regularisation with parameter $\zeta^{\sqcup\!\sqcup,0}(1)=0$. Under the period map, the convergent motivic MtV's give convergent MtV's in the sense of Definition 2.3, and in particular correspond to convergent series. Likewise the extended motivic MtV's correspond to extended MtV's in the sense of Definition 2.3, and require regularisation to be defined.

4.4 Derivations D_r, and the kernel of $D_{<N}$

Finally, we turn to one of the most useful features of the motivic MZV's, the combinatorial operations D_r arising from the coaction, which allow us to recursively find and verify identities.

Recall the coaction $\Delta\colon \mathcal{H}\to\mathcal{A}\otimes_{\mathbb{Q}}\mathcal{H}$ defined in (23). We wish to consider a linearised version of this, which is computationally less complex to calculate, but still very rich in information. By the earlier remark, we have that $\mathcal{A}=\mathcal{H}/\mathcal{H}\zeta^{\mathfrak{m}}(2)$. Moreover, introduce the linearised quotient of $\mathcal{A}$—which then has the structure of a Lie coalgebra—defined by

$$\mathcal{L}=\mathcal{A}_{>0}/\mathcal{A}_{>0}\cdot\mathcal{A}_{>0}.$$

Here $\mathcal{A}_{>0}$ denotes the elements of weight >0, and $\mathcal{A}_{>0}\cdot\mathcal{A}_{>0}$ is then the non-trivial products in $\mathcal{A}$. Likewise $\mathcal{L}_N$ denotes the weight N graded component of $\mathcal{L}$. Denote by $I^{\mathfrak{l}}$ and $\zeta^{\mathfrak{l}}$, the image of $I^{\mathfrak{m}}$ and $\zeta^{\mathfrak{m}}$ respectively, in $\mathcal{L}$.

Definition 4.3 (*Derivation D_r*) For any $r\geq 1$, define the derivation

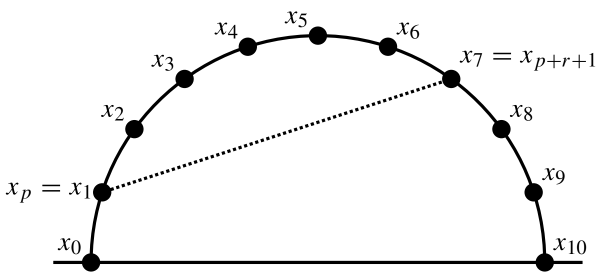
$$D_r\colon\mathcal{H}\to\mathcal{L}_r\otimes_{\mathbb{Q}}\mathcal{H}$$

as the composition of $\Delta-(1\otimes\mathrm{id})$ with $\pi_r\otimes\mathrm{id}$, where π_r is the projection $\mathcal{A}\to\mathcal{L}\to\mathcal{L}_r$, to the weight r graded component $\mathcal{L}_r\subset\mathcal{L}$.

Essentially D_r is given by the terms in Δ which have weight r in the left hand factor, and are irreducible. Therefore, one has the following explicit and combinatorial formula to compute D_r,

$$D_r\big(I^{\mathfrak{m}}(x_0;x_1,\ldots,x_N;x_{N+1})\big)=$$
$$\sum_{p=0}^{N-r} I^{\mathfrak{l}}(x_p;x_{p+1},\ldots,x_{p+r};x_{p+r+1})\otimes I^{\mathfrak{m}}(x_0;x_1,\ldots,x_p,x_{p+r+1},\ldots,x_N;x_{N+1})$$

Often, the following mnemonic picture is used to describe this formula.



The terms in D_r correspond to segments cut out of the semicircular polygon with vertices labelled by the integral parameters $x_0, x_1, \ldots, x_N, x_{N+1}$. (In this picture, $N = 9, r = 5, p = 1$.) Each term corresponds to a particular segment which cuts off a small polygon with r interior points. The small polygon $(x_1, x_2, \ldots, x_7)$ above gives the left hand factor $I^{\mathfrak{l}}(x_1; x_2, \ldots, x_6; x_7)$ in the formula, while the main polygon, containing the integration endpoints x_0 and $x_{N+1} = x_{10}$ gives rise to the right hand factor $I^{\mathfrak{m}}(x_0; x_1 , x_7, x_8, x_9; x_{10})$, by deleting the interior points from the segment.

The following theorems illustrate the power and information contained in these operations. For $N \geq 1$, write

$$D_{<N} = \bigoplus\nolimits_{1 \leq 2r+1 \leq N} D_{2r+1}$$

as the overall combination of all (relevant) derivations in weight $< N$. (Note that $D_1 \equiv 0$ on $\mathcal{H}_N^{(1)}$, so its inclusion will not change the statement of Brown's Theorem appreciably. However, D_1 is important for Glanois' Theorem below.)

Theorem 4.4 (Brown, Theorem 3.3 [4]) *The kernel of $D_{<N}$ on motivic MZV's is 1 dimensional in weight N, and spanned by $\zeta^{\mathfrak{m}}(N)$,*

$$\ker D_{<N} \cap \mathcal{H}_N^{(1)} = \zeta^{\mathfrak{m}}(N)\mathbb{Q}.$$

This (often) allows one to recursively lift identities of real MZV's to motivic MZV's, by recursively verifying $D_{<N}$ vanishes, and using the numerical identity to fix the final unknown coefficient of $\zeta^{\mathfrak{m}}(N)$ via the period map.

This was extended by Glanois [10, 11] to the case of alternating motivic MZV's (and motivic MZV's at higher roots of unity).

Theorem 4.5 (Glanois, Corollary 2.4.5 [10], Theorem 2.2 [11]) *The kernel of $D_{<N}$ on alternating motivic MZV's is 1 dimensional in weight N, and spanned by $\zeta^{\mathfrak{m}}(\overline{N})$,*

$$\ker D_{<N} \cap \mathcal{H}_N^{(2)} = \zeta^{\mathfrak{m}}(\overline{N})\mathbb{Q}.$$

This again (often) allows one to recursively lift identities of real alternating MZV's to alternating motivic MZV's, by recursively verifying $D_{<N}$ vanishes, and using the numerical identity to fix the final unknown coefficient of $\zeta^{\mathfrak{m}}(\overline{N})$ via the period map.

For $N > 1$, one can take $\zeta^{\mathfrak{m}}(N)$ instead as the generator, however, for $N = 1$, one must take $\zeta^{\mathfrak{m}}(\overline{1}) = -\log^{\mathfrak{m}}(2)$, as $\zeta^{\mathfrak{m}}(1) = 0$.

Remark 4.6 The analogous result for higher roots of unity is not true, as further primitive elements come into play. For example $\zeta^{\mathfrak{m}}(N)$ and $\zeta^{\mathfrak{m}}\binom{\exp(2\pi \mathrm{i}/3)}{N}$ are both primitive for Δ, and therefore vanish under all derivations D_r. However, (after application of the period map, to take real and imaginary parts), one sees they are linearly independent. Therefore the kernel of $D_{<N}$ on motivic MZV's at 3rd roots of unity is (at least) two dimensional. Glanois gives such characterisations in more cases in Corollary 2.4.5 [10].

Finally, Glanois also studied when alternating motivic MZV's Galois descend to be(come) linear combinations of (non-alternating) motivic MZV's. The following Theorem gives a criterion to check this recursively using D_r.

Theorem 4.7 (Glanois, [10, Corollary 5.1.3], [11, Corollary 2.4]) *Let $\mathfrak{z} \in \mathcal{H}^{(2)}$ be a motivic alternating MZV. Then $\mathfrak{z} \in \mathcal{H}^{(1)}$, i.e. $\mathfrak{z}$ is a linear combination of (non-alternating) motivic MZV's, if and only if*

(i) $D_1(\mathfrak{z}) = 0$, *and*
(ii) $D_{2r+1}\mathfrak{z} \in \mathcal{L}^{(1)}_{2r+1} \otimes \mathcal{H}^{(1)}$ *for all* $r \geq 1$,

where $\mathcal{L}^{(1)}_{2r+1}$ is the subspace of $\mathcal{L}$ generated by all (non-alternating) motivic MZV's of weigh $2r+1$.

5 Regularised distribution relations, and the derivations D_r

In [22], Murakami calculated the derivations D_r on motivic MtV's of the form $t^{\mathfrak{m}}(k_1, \ldots, k_d)$ with each $k_i > 1$. The case where some $k_i = 1$ was not treated, as the distribution relations used in the proof would not hold exactly. For the purposes of treating the more general case, we need to consider the case of regularised distribution relations, and verify them on the motivic level.

5.1 Classic and motivic regularised distribution relations

The (convergent) distribution relation of 'level $N = 2$' states that if $k_d > 1$, the following holds

$$2^{k_1+\cdots+k_d-d} \sum_{\varepsilon_1,\ldots,\varepsilon_d \in \{\pm 1\}} \zeta\binom{\varepsilon_1, \ldots, \varepsilon_d}{k_1, \ldots, k_d} = \zeta(k_1, \ldots, k_d), \tag{24}$$

This immediately follows from a corresponding distribution relation for multiple polylogarithms which holds on the power-series level

$$2^{k_1+\cdots+k_d-d} \sum_{\varepsilon_1,\ldots,\varepsilon_d \in \{\pm 1\}} \mathrm{Li}_{k_1,\ldots,k_d}(\varepsilon_1 x_1, \ldots, \varepsilon_d x_d) = \mathrm{Li}_{k_1,\ldots,k_d}(x_1^2, \ldots, x_d^2),$$

by setting $x_i = 1$. The distribution relations are known to be motivic, and Murakami indeed even verified this again to be the case in Proposition 10 of [22], at least when all $k_i > 1$. Geometrically, they follow by taking the pullback under $s \mapsto s^2$ (for level $N = 2$), and analogously in general. For example

$$\sum_{\varepsilon_1, \varepsilon_2 \in \{\pm 1\}} \zeta \binom{\varepsilon_1, \varepsilon_2}{a, b} = \\ I(0; 1, \{0\}^{a-1}, 1, \{0\}^{b-1}; 1) + I(0; 1, \{0\}^{a-1}, -1, \{0\}^{b-1}; 1) \\ + I(0; -1, \{0\}^{a-1}, 1, \{0\}^{b-1}; 1) + I(0; -1, \{0\}^{a-1}, -1, \{0\}^{b-1}; 1).$$

where as always a_i within the bounds of the integral represents the form $\frac{\mathrm{d}s}{s - a_i}$, as in (7) above. By linearity of integration, the forms can be combined as

$$\frac{\mathrm{d}s}{s-1} + \frac{\mathrm{d}s}{s+1} = \frac{2s\mathrm{d}s}{s^2-1},$$

and so we can write the combination of integrals as

$$= \int\limits_{0 < s_1 < \cdots < s_{a+b} < 1} \frac{2s_1 \mathrm{d}s_1}{s_1^2 - 1} \wedge \overbrace{\frac{\mathrm{d}s_2}{s_2} \wedge \cdots \wedge \frac{\mathrm{d}s_a}{s_a}}^{a-1 \text{ terms}} \wedge \frac{2s_{a+1}\mathrm{d}s_{a+1}}{s_{a+1}^2 - 1} \wedge \overbrace{\frac{\mathrm{d}s_{a+2}}{s_{a+2}} \wedge \cdots \wedge \frac{\mathrm{d}s_{a+b}}{s_{a+b}}}^{b-1 \text{ terms}},$$

Now set $y_i = s_i^2$, for which the bounds $0 < s_1 < \cdots < s_{a+b} < 1$ become $0 < y_1 < \cdots < y_{a+b} < 1$, and the forms become

$$\frac{\mathrm{d}y_i}{2y_i} = \frac{\mathrm{d}s_i}{s_i} \qquad \frac{\mathrm{d}y_i}{y_i - 1} = \frac{2s_i \mathrm{d}s_i}{s_i^2 - 1}.$$

This means the integral is equal to

$$= \int\limits_{0 < y_1 < \cdots < y_{a+b} < 1} \frac{\mathrm{d}y_1}{y_1 - 1} \wedge \overbrace{\frac{\mathrm{d}y_2}{2y_2} \wedge \cdots \wedge \frac{\mathrm{d}y_a}{2y_a}}^{a-1 \text{ terms}} \wedge \frac{\mathrm{d}y_{a+1}}{y_{a+1} - 1} \wedge \overbrace{\frac{\mathrm{d}y_{a+2}}{2y_{a+2}} \wedge \cdots \wedge \frac{\mathrm{d}y_{a+b}}{2y_{a+b}}}^{b-1 \text{ terms}} \\ = \frac{1}{2^{a+b-2}} \zeta(a, b)$$

Therefore (modulo some formalities to translate this carefully), they indeed have a geometric ('motivic') origin.

On the level of real numbers we claim the following regularised version of the distribution relations of level $N = 2$ holds. (A regularised version of the distribution relations is discussed in general in [28, Section 13.3.4], alternatively one can understand this via the asymptotic expansion discussed in [12, Proposition 2.19 and Lemma 2.21].)

Proposition 5.1 *For $\underline{\mathbf{k}} = (k_1, \ldots, k_d)$, with $k_d \neq 1$ an index, and any $\alpha \geq 0$, the following regularised version of the distribution relation holds*

$$2^{k_1+\cdots+k_d-d} \sum_{\substack{\underline{\boldsymbol{\epsilon}}=(\epsilon_1,\ldots,\epsilon_d) \\ \underline{\boldsymbol{\delta}}=(\delta_1,\ldots,\delta_\alpha) \\ \epsilon_i,\delta_j \in \{\pm 1\}}} \zeta^{\sqcup\!\sqcup, W} \begin{pmatrix} \underline{\boldsymbol{\epsilon}}, \underline{\boldsymbol{\delta}} \\ \underline{\mathbf{k}}, \{1\}^\alpha \end{pmatrix} - \zeta^{\sqcup\!\sqcup, W}(\underline{\mathbf{k}}, \{1\}^\alpha)$$
$$= \sum_{h=1}^{\alpha} \zeta^{\sqcup\!\sqcup, W}(\underline{\mathbf{k}}, \{1\}^{\alpha-h}) \frac{(-\log(2))^h}{h!} .$$

Remark 5.2 Firstly, note that the power of 2 is still given by weight minus depth; the additional α many 1's increase both weight and depth by α, which cancels. Also by the change of regularisation formula from Lemma 2.14, with $S = W$, $T = W - \log(2)$, we can in fact rewrite the identity as

$$2^{k_1+\cdots+k_d-d} \sum_{\substack{\underline{\boldsymbol{\epsilon}}=(\epsilon_1,\ldots,\epsilon_d) \\ \underline{\boldsymbol{\delta}}=(\delta_1,\ldots,\delta_\alpha) \\ \epsilon_i,\delta_j \in \{\pm 1\}}} \zeta^{\sqcup\!\sqcup, W} \begin{pmatrix} \underline{\boldsymbol{\epsilon}}, \underline{\boldsymbol{\delta}} \\ \underline{\mathbf{k}}, \{1\}^\alpha \end{pmatrix} = \zeta^{\sqcup\!\sqcup, W-\log(2)}(\underline{\mathbf{k}}, \{1\}^\alpha).$$

Proof of Proposition 5.1 Recall the algebraic setup as in Section 2.1, with the alphabet $Y = \{e_0, e_1, e_{-1}\}$, letter product $e_i \diamond e_j = 0$, and the induced shuffle product $e_i w_1 \sqcup\!\sqcup e_j w_2 = e_i(w_1 \sqcup\!\sqcup e_j w_2) + e_j(e_i w_1 \sqcup\!\sqcup w_2)$ on $\mathfrak{B}^1_{\sqcup\!\sqcup} = \mathbb{Q}\langle Y\rangle$. On the convergent words $\mathfrak{B}^0_{\sqcup\!\sqcup}$ (those not starting e_0 and not ending e_1), the map

$$\begin{aligned} \zeta \colon (\mathfrak{B}^0_{\sqcup\!\sqcup}, \sqcup\!\sqcup) &\to \mathbb{R} \\ e_{\eta_1} e_0^{n_1-1} \cdots e_{\eta_d} e_0^{n_d-1} &\mapsto \zeta \begin{pmatrix} \eta_2/\eta_1, \eta_3/\eta_2, \ldots, 1/\eta_d \\ n_1 \quad , n_2 \quad , \ldots, n_d \end{pmatrix} \\ &\qquad = (-1)^d I(0; e_{\eta_1}, \{e_0\}^{n_1-1}, \ldots, e_{\eta_d}, \{e_0\}^{n_d-1}; 1) , \end{aligned}$$

where $\eta_j = \prod_{i=j}^d \varepsilon_i^{-1}$, is an algebra homomorphism. It extends uniquely to a homomorphism $\zeta^{\sqcup\!\sqcup, W} \colon (\mathfrak{B}^0_{\sqcup\!\sqcup}[e_1], \sqcup\!\sqcup) \to \mathbb{R}[W]$, by requiring $e_1 \mapsto W$. Like before $\mathfrak{B}^0_{\sqcup\!\sqcup}[e_1]$ is isomorphic to the space of words not starting e_0. (This $\zeta^{\sqcup\!\sqcup, W}$ agrees with Definition 2.5, as no word starts in e_0.)

Consider now

$$w = c e_0^{k_1-1} \cdots c e_0^{k_d-1} c^\alpha ,$$

where $c = \lambda e_{-1} + \mu e_1$ is an arbitrary linear combination of letters e_{-1}, e_1. By the same argument as in Lemma 2.14, along with the linearity of $\sqcup\!\sqcup$, we see that

$$\sum_{h=0}^{\alpha} \left(c e_0^{k_1-1} \cdots c e_0^{k_d-1} c^{\alpha-h} \right) \sqcup\!\sqcup \frac{(-1)^h c^{\sqcup\!\sqcup h}}{h!} =: f(c) \tag{25}$$

is a sum of words which do not end in c, as there is a pairwise cancellation when applying the recursive definition of $\sqcup\!\sqcup$. Moreover $f(c)$ is a sum of words of weight $k_1 + \cdots + k_d + \alpha$ and with $d + \alpha$ many c's in each word. Applying the convergent distribution relation shows that

$$\zeta^{\sqcup\!\sqcup, W}\big(2^{k_1+\cdots+k_d+\alpha-(d+\alpha)} f(e_{-1} + e_1) - f(e_1)\big) = 0, \tag{26}$$

since when expanded out all words have depth $d + \alpha$, i.e. $d + \alpha$ many non-e_0 entries, and $f(e_{-1} + e_1)$ sums over all choices of signs $c = e_{\pm 1}$ independently (both in $\mathfrak{B}^1_{\sqcup\!\sqcup}$ and in the MZV's after applying $\zeta^{\sqcup\!\sqcup, W}$ as the correspondence $(\eta_1, \ldots, \eta_d) \leftrightarrow (\varepsilon_1, \ldots, \varepsilon_d) = (\eta_2/\eta_1, \eta_3/\eta_2, \ldots, 1/\eta_d)$ is a bijection and maps $\{\pm 1\}^d$ to $\{\pm 1\}^d$).

Hence expanding out each f in (26) via (25), we obtain

$$2^{k_1+\cdots+k_d+\alpha-(d+\alpha)} \sum_{h=0}^{\alpha} \sum_{\substack{\underline{\boldsymbol{\epsilon}}=(\epsilon_1,\ldots,\epsilon_d) \\ \underline{\boldsymbol{\delta}}=(\delta_1,\ldots,\delta_{\alpha-h}) \\ \epsilon_i,\delta_j \in \{\pm 1\}}} \zeta^{\sqcup\!\sqcup, W} \begin{pmatrix} \underline{\boldsymbol{\epsilon}}, \underline{\boldsymbol{\delta}} \\ \underline{\mathbf{k}}, \{1\}^{\alpha-h} \end{pmatrix} \frac{(-W + \log(2))^h}{h!}$$
$$= \sum_{h=0}^{\alpha} \zeta^{\sqcup\!\sqcup, W}(\underline{\mathbf{k}}, \{1\}^{\alpha-h}) \frac{(-W)^h}{h!}\,.$$

Taking the generating series $\sum_{\alpha=0}^{\infty} \bullet\, X^\alpha$ of both sides of this gives

$$2^{k_1+\cdots+k_d-d} \sum_{\alpha=0}^{\infty} \Bigg(\sum_{\substack{\underline{\boldsymbol{\epsilon}}=(\epsilon_1,\ldots,\epsilon_d) \\ \underline{\boldsymbol{\delta}}=(\delta_1,\ldots,\delta_{\alpha}) \\ \epsilon_i,\delta_j \in \{\pm 1\}}} \zeta^{\sqcup\!\sqcup, W} \begin{pmatrix} \underline{\boldsymbol{\epsilon}}, \underline{\boldsymbol{\delta}} \\ \underline{\mathbf{k}}, \{1\}^{\alpha} \end{pmatrix} \Bigg) X^\alpha \cdot \exp((-W + \log(2))h)$$
$$= \sum_{\alpha=0}^{\infty} \zeta^{\sqcup\!\sqcup, W}(\underline{\mathbf{k}}, \{1\}^{\alpha}) \cdot \exp(-Wh)\,.$$

Moving all of the exponentials to the right-hand side, and extracting the coefficient of X^α gives the claimed identity. ◻

In particular, the weight $w > 1$ distribution relations holds modulo products, whether or not regularisation is necessary. In the case of weight 1 however, we find

$$2^0 \left(\zeta^{\sqcup\!\sqcup, W} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \zeta^{\sqcup\!\sqcup, W} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) - \zeta^{\sqcup\!\sqcup, W}(1) = -\log(2),$$

which is non-zero modulo products (at least assuming the usual conjectures), as this is a logarithm (i.e. weight 1). More precisely, on the motivic level, the weight 1 distribution identity (with $W = 0$) is clearly satisfied, and then $\log^{\mathfrak{l}}(2)$ (the motivic logarithm $\log^{\mathfrak{m}}(2)$ modulo products) does not vanish, as it lives in a the weight 1 component.

We now specialise to the case $W = 0$, in line with the usual prescription via the tangential base-points of the straight line path $\gamma\colon [0, 1] \to [0, 1]$, $\gamma(t) = t$. In this prescription: $\zeta^{\sqcup\!\sqcup,0}(1) = \zeta_1^{\sqcup\!\sqcup,0}(\emptyset) = 0$. Using this, we can extend Proposition 5.1 to the case of $\zeta_\ell^{\sqcup\!\sqcup,0}(\underline{\mathbf{k}}, \{1\}^\alpha)$, wherein the integral representation starts with a string $\{0\}^\ell$ of ℓ many 0's.

Corollary 5.3 *For $\underline{\mathbf{k}} = (k_1, \ldots, k_d)$, with $k_d \neq 1$ an index, any $\alpha \geq 0$, and any $\ell \geq 0$, the following regularised version of the distribution relation holds*

$$2^{k_1+\cdots+k_d+\ell-d} \sum_{\substack{\underline{\boldsymbol{\epsilon}}=(\epsilon_1,\ldots,\epsilon_d) \\ \underline{\boldsymbol{\delta}}=(\delta_1,\ldots,\delta_\alpha) \\ \epsilon_i,\delta_j \in \{\pm 1\}}} \zeta_\ell^{\sqcup\!\sqcup,0}\begin{pmatrix} \underline{\boldsymbol{\epsilon}}, \underline{\boldsymbol{\delta}} \\ \underline{\mathbf{k}}, \{1\}^\alpha \end{pmatrix} - \zeta_\ell^{\sqcup\!\sqcup,0}(\underline{\mathbf{k}}, \{1\}^\alpha)$$
$$= \sum_{i=1}^{\alpha} \zeta_\ell^{\sqcup\!\sqcup,0}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \frac{(-\log(2))^i}{i!}$$

Proof In the case $\alpha = 0$, applying the unshuffling of starting 0's from (9) in Lemma 2.6 shows that

$$\zeta_\ell^{\sqcup\!\sqcup,0}\begin{pmatrix} \underline{\boldsymbol{\varepsilon}} \\ \underline{\mathbf{k}} \end{pmatrix} = (-1)^\ell \sum_{i_1+\cdots+i_d=\ell} \binom{k_1+i_1-1}{i_1} \cdots \binom{k_d+i_d-1}{i_d}$$
$$\times \zeta\begin{pmatrix} \underline{\boldsymbol{\varepsilon}} \\ k_1+i_1, \ldots, k_d+i_d \end{pmatrix}.$$

This reduces ζ_ℓ to convergent zetas on the right hands side, to which the distribution relation applies exactly. So after summing over all choices of signs, and applying the usual distribution relation, one finds

$$\sum_{\substack{\underline{\boldsymbol{\varepsilon}}=(\varepsilon_1,\ldots,\varepsilon_d) \\ \varepsilon_i \in \{\pm 1\}}} \zeta_\ell^{\sqcup\!\sqcup,0}\begin{pmatrix} \underline{\boldsymbol{\varepsilon}} \\ \underline{\mathbf{k}} \end{pmatrix}$$
$$= 2^{k_1+\cdots+k_d+\ell-d}(-1)^d \sum_{i_1+\cdots+i_d=\ell} \binom{k_1+i_1-1}{i_1} \cdots \binom{n_d+i_d-1}{i_d}$$
$$\times \zeta\begin{pmatrix} \{1\}^d \\ k_1+i_1, \ldots, k_d+i_d \end{pmatrix}$$
$$= 2^{k_1+\cdots+k_d+\ell-d} \zeta_\ell^{\sqcup\!\sqcup,0}(\underline{\mathbf{k}}),$$

where the last equality arises by applying the unshuffling process again. The case with trailing 1's follows by applying the proof of Proposition 5.1 again, mutatis mutandis, with the word $w = e_0^\ell c e_0^{k_1-1} \cdots c e_0^{k_d-1}$. □

Remark 5.4 In principle, one can also give a version of the unshuffling identity (9) in Lemma 2.6 which holds for different regularisation parameters, and so one can extend Corollary 5.3 to the general regularisation parameter (even taking different regularisations with $\zeta^{\sqcup\!\sqcup,W,W'}(1) = W$ and $\zeta_1^{\sqcup\!\sqcup,W,W'}(\emptyset) = W'$), by tracking and incorporating the product terms involving W and W'.

The proofs of Proposition 5.1 and Corollary 5.3 proceeded purely by using the shuffle product of iterated integrals and the non-regularised distribution relations, so the result holds true on the motivic level as well, as both ingredients are already known to be motivic. So as a result, we obtain the following corollaries.

Corollary 5.5 *For* $\underline{\mathbf{k}} = (k_1, \ldots, k_d)$, *with* $k_d \neq 1$ *an index, any* $\alpha \geq 0$, *and any* $\ell \geq 0$ *the following regularised version of the distribution relation holds for motivic multiple zeta values.*

$$2^{k_1+\cdots+k_d+\ell-d} \sum_{\substack{\underline{\boldsymbol{\epsilon}}=(\epsilon_1,\ldots,\epsilon_d) \\ \underline{\boldsymbol{\delta}}=(\delta_1,\ldots,\delta_\alpha) \\ \epsilon_i,\delta_j\in\{\pm1\}}} \zeta_\ell^{\mathfrak{m}}\left(\begin{matrix}\underline{\boldsymbol{\epsilon}}, \underline{\boldsymbol{\delta}} \\ \underline{\mathbf{k}}, \{1\}^\alpha\end{matrix}\right) - \zeta_\ell^{\mathfrak{m}}(\underline{\mathbf{k}}, \{1\}^\alpha)$$

$$= \sum_{i=1}^{\alpha} \zeta_\ell^{\mathfrak{m}}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \frac{(-\log^{\mathfrak{m}}(2))^i}{i!} .$$

Corollary 5.6 *For* $\underline{\mathbf{k}} = (k_1, \ldots, k_d)$, *with* $k_d \neq 1$ *an index, any* $\alpha \geq 0$, *and any* $\ell \geq 0$, *the following regularised version of the distribution relation holds for motivic multiple zeta values of weight* $w > 1$ *or* $\ell > 0$ *modulo products*

$$2^{k_1+\cdots+k_d+\ell-d} \sum_{\substack{\underline{\boldsymbol{\epsilon}}=(\epsilon_1,\ldots,\epsilon_d) \\ \underline{\boldsymbol{\delta}}=(\delta_1,\ldots,\delta_\alpha) \\ \epsilon_i,\delta_j\in\{\pm1\}}} \zeta_\ell^{\mathfrak{l}}\left(\begin{matrix}\underline{\boldsymbol{\epsilon}}, \underline{\boldsymbol{\delta}} \\ \underline{\mathbf{k}}, \{1\}^\alpha\end{matrix}\right) = \zeta_\ell^{\mathfrak{l}}(\underline{\mathbf{k}}, \{1\}^\alpha) .$$

In the case of weight 1 and $\ell = 0$, *the distribution relation modulo products has an extra* $-\log^{\mathfrak{l}}(2)$ *correction, namely*

$$\zeta^{\mathfrak{l}}(1) + \zeta^{\mathfrak{l}}(\overline{1}) = \zeta^{\mathfrak{l}}(1) - \log^{\mathfrak{l}}(2).$$

5.2 Derivations on $\widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d)$

Now that we have the motivic version of the distribution relations for arbitrary arguments, we may directly generalise Murakami's computation of D_r given in Proposition 11 of [22].

Proposition 5.7 (Generalisation of Proposition 11, [22]) *Let* $\underline{\mathbf{k}} = (k_1, \ldots, k_d) \in (\mathbb{Z}_{\geq 1})^d$ *be an index. Write* $\underline{\mathbf{k}}_{i,j} = (k_i, \ldots, k_j)$ *for a subindex of* $\underline{\mathbf{k}}$ *and* $|(a_1, \ldots, a_r)| = a_1 + \cdots + a_r$ *for the total (weight) of an index. Then the derivation* D_r, r *odd, is computed as follows*

$$D_r\big(\widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d)\big) = \sum_{1\leq j\leq d} \delta_{|\underline{\mathbf{k}}_{1,j}|=r} \widetilde{t}^{\mathfrak{l}}(k_1, \ldots, k_j) \otimes \widetilde{t}^{\mathfrak{m}}(k_{j+1}, \ldots, k_d) \tag{27}$$

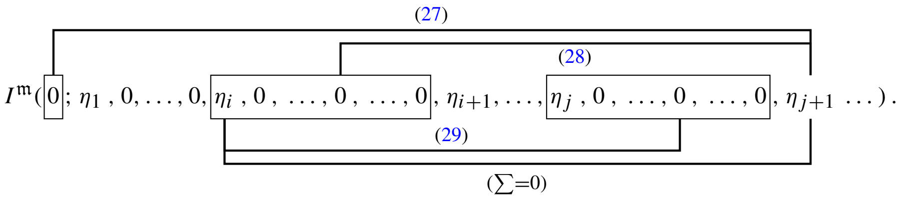
$$+ \sum_{1 \leq i < j \leq d} \delta_{|\underline{\mathbf{k}}_{i+1,j}| \leq r < |\underline{\mathbf{k}}_{i,j}|-1} \big(\zeta^{\mathfrak{l}}_{r - |\underline{\mathbf{k}}_{i+1,j}|}(k_{i+1}, \ldots, k_j) - \delta_{r=1} \log^{\mathfrak{l}}(2) \big) \otimes \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{i-1}, |\underline{\mathbf{k}}_{i,j}| - r, k_{j+1}, \ldots, k_d) \tag{28}$$

$$- \sum_{1 \leq i < j \leq d} \delta_{|\underline{\mathbf{k}}_{i,j-1}| \leq r < |\underline{\mathbf{k}}_{i,j}|-1} \big(\zeta^{\mathfrak{l}}_{r - |\underline{\mathbf{k}}_{i,j-1}|}(k_{j-1}, \ldots, k_i) - \delta_{r=1} \log^{\mathfrak{l}}(2) \big) \otimes \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{i-1}, |\underline{\mathbf{k}}_{i,j}| - r, k_{j+1}, \ldots, k_d) \tag{29}$$

Proof The proof of this proposition works in precisely the same way as Murakami's proof of the special case, where each $k_i \geq 2$, given in [22, Proposition 11]. In particular, the terms arise from the following cuts on the integral defining $\widetilde{t}^{\mathfrak{m}}$

$$\begin{aligned} &(-1)^d 2^{d-k_1-\cdots-k_d} \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d) \\ &= \sum_{\eta_i \in \{\pm 1\}} \eta_1 I^{\mathfrak{m}}(0; \eta_1, \{0\}^{k_1-1}, \eta_2, \{0\}^{k_2-1}, \ldots, \eta_d, \{0\}^{k_d-1}; 1). \end{aligned}$$

The correspondence in particular is as follows



The only difference from Murakami's proof [22, Proposition 11, proof] comes when computing D_1, wherein terms (28) and (29) are simplified using the regularised distribution relation, and so pick up an extra $-\log^{\mathfrak{l}}(2)$ in weight 1. □

Remark 5.8 We should note here, again, that the motivic iterated integrals $I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1})$ satisfy the homothety

$$I^{\mathfrak{m}}(\lambda a_0; \lambda a_1, \ldots, \lambda a_n; \lambda a_{n+1}) = I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}),$$

if $a_0 \neq a_1$ and $a_n \neq a_{n+1}$. However, if one of these is actually an equality, the integral depend on the tangential base-points of the path, and the homothety can actually change these, so the equality does not in general hold. Viz:

$$\begin{aligned} I^{\mathfrak{m}}(\overrightarrow{1}_0; 0, 1; -1) &= I^{\mathfrak{m}}(\overrightarrow{1}_0; 0; -1) I^{\mathfrak{m}}(\overrightarrow{1}_0; 1; -1) - I^{\mathfrak{m}}(\overrightarrow{1}_0; 1, 0; -1) \\ &= (\mathrm{i}\pi)^{\mathfrak{m}} \log^{\mathfrak{m}}(2) + \zeta^{\mathfrak{m}}(\overline{2}) \\ I^{\mathfrak{m}}(\overrightarrow{1}_0; 0, -1; 1) &= I^{\mathfrak{m}}(\overrightarrow{1}_0; 0; 1) I^{\mathfrak{m}}(\overrightarrow{1}_0; -1; 1) - I^{\mathfrak{m}}(\overrightarrow{1}_0; -1, 0; 1) \\ &= 0 \cdot \log^{\mathfrak{m}}(2) + \zeta^{\mathfrak{m}}(\overline{2}), \end{aligned}$$

where $\overrightarrow{1}_0$ denotes the tangential base-point at 0 with tangent vector in the direction $\overrightarrow{1}$. So there is a difference of $(\mathrm{i}\pi)^{\mathfrak{m}} \log^{\mathfrak{m}}(2)$ between the homotheties. However, one can say that in general (for $a_i \in \{0, \pm 1\}$), with weight $w > 1$, that

$$I^{\mathfrak{l}}(\lambda a_0; \lambda a_1, \ldots, \lambda a_n; \lambda a_{n+1}) = I^{\mathfrak{l}}(a_0; a_1, \ldots, a_n; a_{n+1}).$$

In weight 1, homotheties by λ (with $|\lambda| = 1$) will rotate the tangential base-point, and so contribute some rational times $(i\pi)^{\mathfrak{l}} = 0$ (already $(i\pi)^{\mathfrak{u}} = 0$, even before killing products) by considering the decomposition of paths $I^{\mathfrak{l}}(\overrightarrow{\lambda}_0; 0; a) = I^{\mathfrak{l}}(\overrightarrow{\lambda}_0; 0; \overrightarrow{1}_0) + I^{\mathfrak{l}}(\overrightarrow{1}_0; 0; a)$. So the homothety property still holds.

This property is applied in various places in Murakami's proof of Proposition 11 (and some earlier results upon which it is dependent). But in every case, it is applied to the $I^{\mathfrak{l}}$ part of the coaction, and so is valid.

We now make some observations which simplify the calculation of D_r when $r = 1$. Note that D_r, for $r > 1$ is computed with exactly the same formula as Murakami [22, Proposition 11], we have merely extended the range of validity to all (shuffle regularised) multiple t values. So let us focus on the case D_1.

Proposition 5.9 (Calculation of D_1) *Let* $\underline{\mathbf{k}} = (k_1, \ldots, k_d) \in (\mathbb{Z}_{\geq 1})^d$ *be an index. Then*

$$D_1 \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d) = \delta_{k_1=1} \cdot 2 \log^{\mathfrak{l}}(2) \otimes \widetilde{t}^{\mathfrak{m}}(k_2, \ldots, k_d) \\ -\delta_{k_d=1} \log^{\mathfrak{l}}(2) \otimes \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{d-1}).$$

That is D_1 acts by deconcatenation of trailing 1's and of leading 1's (with coefficient 2).

Proof Assuming $r = 1$, we consider how terms (28) and (29) can contribute. For (28), the delta condition requires $|\underline{\mathbf{k}}_{i+1,j}| \leq 1 < |\underline{\mathbf{k}}_{i,j}| - 1$. The first condition forces $i + 1 = j$, and $\underline{\mathbf{k}}_{i+1} = 1$, so $\underline{\mathbf{k}}_{i,j} = (\alpha, 1)$, for some $\alpha > 1$. One has that $|\underline{\mathbf{k}}_{i,j}| - r = \alpha + 1 - 1 = \alpha$. In this case, we contribute

$$(\zeta_0(1) - \log^{\mathfrak{l}}(2)) \otimes \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, \alpha, \underline{\mathbf{k}}_{j+1,d}),$$

which can be seen as deleting the 1 following $\alpha = k_i$ in $\underline{\mathbf{k}}_{i,j} = (\alpha, 1)$.

Likewise, for (29), the delta condition requires $|\underline{\mathbf{k}}_{i,j-1}| \leq 1 < |\underline{\mathbf{k}}_{i,j}| - 1$. The first condition forces $i = j - 1$, and $\underline{\mathbf{k}}_i = 1$, so $\underline{\mathbf{k}}_{i,j} = (1, \alpha)$, for some $\alpha > 1$. One has that $|\underline{\mathbf{k}}_{i,j}| - r = \alpha + 1 - 1 = \alpha$. In this case, we contribute

$$-(\zeta_0(1) - \log^{\mathfrak{l}}(2)) \otimes \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, \alpha, \underline{\mathbf{k}}_{j+1,d}),$$

which can be seen as deleting the 1 proceeding $\alpha = k_{i+1}$ in $\underline{\mathbf{k}}_{i,j} = (1, \alpha)$.

This means that for any subindex $(\alpha, \{1\}^n, \beta)$, $\alpha, \beta > 1$ appearing in $\underline{\mathbf{k}}$, the term from deleting the 1 after α cancels with the term from deleting the 1 before β. The only terms which can survive this process are of the form $(\{1\}^n, \beta)$ at the start of $\underline{\mathbf{k}}$, and $(\alpha, \{1\})$ at the end of $\underline{\mathbf{k}}$. Combined with the pre-existing deconcatenation term (27), we obtain the claimed expression. □

Remark 5.10 (Hoffman's derivation with respect to log(2)) In [16, Conjecture 2.1], Hoffman conjectures that the algebra of MtV's admits a derivation d which acts on $t(k_1, \dots, k_d)$ by

$$\mathrm{d}t(k_1, \dots, k_d) = \begin{cases} t(k_2, \dots, k_d) & \text{if } k_1 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\underline{\mathbf{k}} = (k_1, \dots, k_d)$, with $k_d \neq 1$, so that $\widetilde{t}^{\,\mathfrak{m}}(\underline{\mathbf{k}})$ is a convergent motivic MtV, one obtains the formula

$$D_1 \widetilde{t}^{\,\mathfrak{m}}(\underline{\mathbf{k}}) = \delta_{k_1=1} \log^{\mathfrak{l}}(2) \otimes \widetilde{t}^{\,\mathfrak{m}}(k_2, \dots, k_d).$$

Since D_1 acts as derivation in the sense

$$D_1(XY) = (1 \otimes X) D_1 Y + (1 \otimes Y) D_1 X,$$

after projecting $\log^{\mathfrak{l}}(2) \mapsto 1$ we see that Hoffman's conjectural derivation is nothing but the action of D_1 on the motivic MtV's, in the convergent case.

Moreover, one also notes that for $\ell_1, \dots \ell_f, n \in \mathbb{Z}$, $n \geq 2$, we have

$$D_1 \zeta^{\mathfrak{m}}(\ell_1, \dots, \ell_f, \overline{n}) = 0.$$

This is because the strings $\{0, 1, -1\}$, $\{0, -1, 1\}$, $\{-1, 1, 0\}$, $\{1, -1, 0\}$ do not occur in the integral representation of the MZV. These strings lead to $\log^{\mathfrak{l}}(2)$ factors, whereas $I^{\mathfrak{l}}(1; 0; -1) = I^{\mathfrak{l}}(-1; 0; 1) = 0$. The algebra basis of the MZV Data Mine [2] exclusively invokes alternating MZV's of the form

$$\zeta(\ell_1, \dots, \ell_f, \overline{n}),$$

with ℓ_i, n odd. So one sees that Hoffman's claim, with regard to the action of d as differentiation wrt log(2) on the formulae in Appendix A of [16] is generally valid, for this specific choice of basis.

For example, the following identity is verified by the Data Mine, and so actually holds on the motivic level

$$\begin{aligned} t^{\mathfrak{m}}(1, 3, 2) = & -\frac{2}{21} t^{\mathfrak{m}}(6) - \frac{3}{196} t^{\mathfrak{m}}(3)^3 - \frac{1}{2} t^{\mathfrak{m}}(2) \zeta^{\mathfrak{m}}(1, \overline{3}) + \frac{1}{4} \zeta^{\mathfrak{m}}(1, \overline{5}) \\ & - \frac{1}{2} t^{\mathfrak{m}}(5) \log^{\mathfrak{m}}(2) + \frac{4}{7} t^{\mathfrak{m}}(2) t^{\mathfrak{m}}(3) \log^{\mathfrak{m}}(2) \,. \end{aligned}$$

Application of D_1 (after scaling to write it via $\widetilde{t}$, so Proposition 5.9 can be applied, and rescaling afterwards) leads to

$$\log^{\mathfrak{l}}(2) \otimes t^{\mathfrak{m}}(3, 2) = \log^{\mathfrak{l}}(2) \otimes \Big(-\frac{1}{2} t^{\mathfrak{m}}(5) + \frac{4}{7} t^{\mathfrak{m}}(2) t^{\mathfrak{m}}(3) \Big),$$

or equivalently

$$t^{\mathfrak{m}}(3,2) = -\frac{1}{2}t^{\mathfrak{m}}(5) + \frac{4}{7}t^{\mathfrak{m}}(2)t^{\mathfrak{m}}(3),$$

as expected from Hoffman's claim.

Remark 5.11 The formula for $D_1 \widetilde{t}^{\mathfrak{m}}$ in Proposition 5.9 shows immediately that the convergent $\widetilde{t}^{\mathfrak{m}}(1, \underline{\mathbf{k}})$ cannot be a motivic MZV, as $D_1 \widetilde{t}^{\mathfrak{m}}(1, \underline{\mathbf{k}}) = \log^{\mathfrak{l}}(2) \otimes \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}) \neq 0$. On the other hand, this gives us a place and means to search for other Galois descent candidates.

Proposition 5.12 *Let $a, b, c, n \in \mathbb{Z}_{\geq 0}$, such that $a \geq 1$ and $n \geq 1$. Then the motivic multiple t value*

$$\tau = \widetilde{t}^{\mathfrak{m}}(\{2\}^a, 1, \{2\}^b, 2n+1, \{2\}^c)$$

is always a (linear combination of) motivic MZV's.

Proof From the above remark, we know $D_1 \tau = 0$. We must only check the second part of Glanois's motivic Galois descent criterion from Theorem 4.7, namely that $D_{2r+1}\tau \in \mathcal{L}^{(1)}_{2r+1} \otimes \mathcal{H}^{(1)}$, i.e. the parts of the coaction are already motivic MZV's.

We first note that the deconcatenation term (27) takes the form

$$\widetilde{t}^{\mathfrak{l}}(\{2\}^a, 1, \{2\}^{r-a}) \otimes \widetilde{t}^{\mathfrak{m}}(\{2\}^{b-(r-a)}, 2n+1, \{2\}^c)$$

The left hand factor is (modulo products!) a motivic MZV by Theorem 6.1 below. (The terms $\log^{\mathfrak{m}}(2)$ only appear as products in weight > 1, so vanish when we project to $\mathcal{L}$.) The right hand factor is a motivic MZV by Theorem 8 in [22]; therein Murakami showed that whenever $\underline{\mathbf{k}} \in (\mathbb{Z}_{\geq 2})^d$ is an index with all entries ≥ 2, then $\widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}) \in \mathcal{H}^{(1)}$ is a motivic MZV.

Then for the terms (28) and (29), one only needs to consider the right hand factor, as the left hand one is already an MZV. One can also assume $\underline{\mathbf{k}}_{i,j}$ does not contain 1, for if it does contain 1, then the condition $r < |\underline{\mathbf{k}}_{i,j}| - 1$ means that $|\underline{\mathbf{k}}_{i,j}| - r > 1$, so that the subindex we removed is replaced with ≥ 2. Hence by Theorem 8 [22] is already a motivic MZV. More generally, we note that in (28) by subtracting the delta condition from $|\underline{\mathbf{k}}_{i,j}|$, one has

$$k_i \geq |\underline{\mathbf{k}}_{i,j}| - r > 1$$

So the replacement value $|\underline{\mathbf{k}}_{i,j}| - r$ for the entire subindex $\underline{\mathbf{k}}_{i,j}$ is between 2 and k_i, the left endpoint. Likewise in (29), the replacement is between 2 and k_j, the right endpoint.

We have the following subindices which exhaust all remaining possible cases. The subindex D here may start or end at $2n+1$,.

$$\widetilde{t}^{\mathfrak{m}}(\overbrace{2\,,\;\ldots,2}^{A}\,,1\,,\overbrace{2\,,\;\ldots,2}^{B}\,,2n+1\,,\overbrace{2\,,\;\ldots,2}^{C}\,)\,.$$

$$\underbrace{\ldots,2\,,2n+1\,,2\,,}_{D}$$

We already note though that A, B and C cannot in fact contribute. The replacement value $k_i = k_j = 2 = |\underline{\mathbf{k}}_{i,j}| - r > 1$ must be 2. But this implies $\underline{\mathbf{k}}_{i,j} = r + 2$ is odd. So we are left with the case D, and for the same reason the replacement must be even in this case, namely:

Subindex	$\underline{\mathbf{k}}_{i,j}$	$\lvert\underline{\mathbf{k}}_{i,j}\rvert - r$	$\widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, \lvert\underline{\mathbf{k}}\rvert_{i,j} - r, \underline{\mathbf{k}}_{j+1,d})$
D	$(\{2\}^{\alpha}, 2n+1, \{2\}^{\beta})$	2	$\widetilde{t}^{\mathfrak{m}}(\{2\}^{a}, 1, \{2\}^{\gamma}, 2, \{2\}^{\delta})$

Since this MtV is of the form $\widetilde{t}^{\mathfrak{m}}(\{2\}^{a}, 1, \{2\}^{b})$ with $a, b > 0$, it is a motivic multiple zeta value via Theorem 6.1 below. □

It would be interesting to see how far this proof can be generalised, and whether one can give some complete combinatorial criterion for when $\widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}})$ descends to a motivic MZV. Certainly other families of motivic MtV's which descend seem to exist. A promising candidate is as follows: let $a, n \in \mathbb{Z}_{\geq 1}$, and $\underline{\mathbf{k}}$, $\underline{\boldsymbol{\ell}}$ be indices containing only even entries. Then it appears that the following MtV, a generalisation of the above, is also a motivic multiple zeta value.

$$\widetilde{t}^{\mathfrak{m}}(\{2\}^{a}, 1, \underline{\mathbf{k}}, 2n+1, \underline{\boldsymbol{\ell}}) \overset{?}{\in} \mathcal{H}^{(1)}.$$

There are also indices with multiple 1's that Galois descend, such as

$$\widetilde{t}^{\mathfrak{m}}(\{2, 1, 3\}^{2}) \in \mathcal{H}^{(1)},$$

although this pattern does not seem to continue. Once can check (via the MZV Data Mine [2]) that $D_7 \widetilde{t}^{\mathfrak{m}}(\{2, 1, 3\}^{3}) \notin \mathcal{L}_7^{(1)} \otimes \mathcal{H}_{11}^{(1)}$.

6 Lift to a motivic $\widetilde{t}^{\mathfrak{m}}(\{2\}^{a}, 1, \{2\}^{b})$ evaluation

The aim of this section is to first lift the evaluation for $t^{\sqcup\!\sqcup, W=0}(\{2\}^{a}, 1, \{2\}^{b})$ given in Proposition 3.7 (more precisely, the explicit version given in (22) thereafter) to an identity amongst motivic multiple t values.

The shuffle version, with regularisation parameter $W = 0$ is the key identity for the rest of this work, since the motivic MZV's are naturally and almost-always regularised in this manner. Moreover, via Proposition 2.18, we can express the stuffle regularisation at arbitrary parameters $t^{*,V}(1) = V$ via the shuffle regularised version at $t^{\sqcup\!\sqcup, W=0}$. This will be used to sidestep later the issue of how to take the motivic stuffle regularisation.

Theorem 6.1 *The following motivic identity holds for all* $a, b \geq 0$

$$\begin{aligned}&\widetilde{t}^{\mathfrak{m}}(\{2\}^a, 1, \{2\}^b)\\&= \sum_{r=1}^{a+b}(-1)^{r+1}\cdot 2\left[\binom{2r}{2a} + \frac{2^{2r}}{2^{2r}-1}\binom{2r}{2b}\right]\zeta^{\mathfrak{m}}(\overline{2r+1})\widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b-r})\\&\quad + \delta_{a=0}2\cdot\log^{\mathfrak{m}}(2)\widetilde{t}^{\mathfrak{m}}(\{2\}^b) - \delta_{b=0}\log^{\mathfrak{m}}(2)\widetilde{t}^{\mathfrak{m}}(\{2\}^a),\end{aligned} \tag{30}$$

Before we begin, it will be useful for later purposes to recall the motivic identity proven in [22] for $\widetilde{t}^{\mathfrak{m}}(\{2\}^a, 3, \{2\}^b)$. This also gives us an opportunity to compare and contrast the two evaluations, which in the MZV case would be equal by duality.

Theorem 6.2 (Murakami, [22, Theorem 22]) *The following motivic identity holds for all* $a, b \geq 0$

$$\begin{aligned}&\widetilde{t}^{\mathfrak{m}}(\{2\}^a, 3, \{2\}^b) =\\&\sum_{r=1}^{a+b+1}(-1)^{r+1}\cdot 2\left[\binom{2r}{2a+1} + (1-2^{-2r})\binom{2r}{2b+1}\right]\zeta^{\mathfrak{m}}(2r+1)\cdot\widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b+1-r})\end{aligned} \tag{31}$$

6.1 Proof of Theorem 6.1

After application of the period map, the identity in the theorem reduces to ($2^{2a+2b+1}$ times) the identity in (22), with $W = 0$. Therefore we only need to verify that

$$D_{2r+1}\big(\text{LHS (30)}\big) - D_{2r+1}\big(\text{RHS (30)}\big) = 0$$

This will show that the purported identity lies in the kernel of $D_{<N}$, $N = 2a+2b+1$. Hence by Glanois's Theorem (Theorem 4.5) it holds up to an additive constant $c\zeta^{\mathfrak{m}}(\overline{N})$, and application of the period map shows that $c = 0$. This will verify that the identity holds on the motivic level, as claimed.

Write

$$\begin{aligned}L^{a,b} &= \widetilde{t}^{\mathfrak{m}}(\{2\}^a, 1, \{2\}^b)\\R^{a,b} &= -\sum_{r=1}^{a+b}(-1)^{r}\cdot 2\left[\binom{2r}{2a} + \frac{2^{2r}}{2^{2r}-1}\binom{2r}{2b}\right]\zeta^{\mathfrak{m}}(\overline{2r+1})\widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b-r})\\&\quad + \delta_{a=0}2\cdot\log^{\mathfrak{m}}(2)\widetilde{t}^{\mathfrak{m}}(\{2\}^b) - \delta_{b=0}\log^{\mathfrak{m}}(2)\widetilde{t}^{\mathfrak{m}}(\{2\}^a)\,,\end{aligned}$$

for the left and right hand side of the purported identity. In order to check

$$D_{2r+1}\big(L^{a,b}\big) - D_{2r+1}\big(R^{a,b}\big) = 0,$$

we proceed inductively on $a + b$.

In the case $a = b = 0$, we immediately find

$$L^{0,0} = \widetilde{t}^{\mathfrak{m}}(1) = \zeta^{\mathfrak{m}}(1) - \zeta^{\mathfrak{m}}(\overline{1}) = \log^{\mathfrak{m}}(2) = R^{0,0}$$

So we may take $a + b > 0$.

Lemma 6.3 *The following expression for $D_{2r+1}L^{a,b}$ holds for any $a, b \geq 0$, and $0 \leq r \leq a + b$,*

$$D_{2r+1}L^{a,b} = D_{2r+1}\widetilde{t}^{\mathfrak{m}}(\{2\}^a, 1, \{2\}^b) = \pi(\widehat{\xi}^r_{a,b}) \otimes \widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b-r}),$$

where $\pi : \mathcal{H}^{(2)} \to \mathcal{L}^{(2)}$ denotes the projection, and $\widehat{\xi}^r_{a,b}$ is given by (the sums running over all indices $\alpha, \beta \geq 0$ satisfying $\alpha + \beta = r$)

$$\begin{aligned}\widehat{\xi}^r_{a,b} = {} & \delta_{r=0}\delta_{a=0}\log^{\mathfrak{m}}(2) - \delta_{r=0}\delta_{b=0}\log^{\mathfrak{m}}(2) + \delta_{a\leq r}\widetilde{t}^{\mathfrak{m}}(\{2\}^a, 1, \{2\}^{r-a}) \\ & + \sum_{\substack{\alpha\leq a-1\\ \beta\leq b}} \zeta^{\mathfrak{m}}_0(\{2\}^\alpha, 1, \{2\}^\beta) - \sum_{\substack{\alpha\leq a\\ \beta\leq b-1}} \zeta^{\mathfrak{m}}_0(\{2\}^\beta, 1, \{2\}^\alpha),\end{aligned}$$

Proof This is a direct, if somewhat tedious, calculation which follows from Proposition 5.7. □

Now introduce the following notation from [4]

$$A^r_{a,b} = \binom{2r}{2a+2},\ B^r_{a,b} = (1 - 2^{-2r})\binom{2r}{2b+1},$$

and recall one of the main results proved therein.

Theorem 6.4 (Brown [4, Theorem 4.3]) *For all $a, b \geq$, the following identity amongst motivic MZV's holds*

$$\begin{aligned}&\zeta^{\mathfrak{m}}(\{2\}^a, 3, \{2\}^b) \\ &= \sum_{r=1}^{a+b+1} 2\cdot(-1)^{r-1}\cdot\left(-A^r_{a,b} + B^r_{a,b}\right)\zeta^{\mathfrak{m}}(2r+1)\zeta^{\mathfrak{m}}(\{2\}^{a+b+1-r}).\end{aligned}$$

We therefore have for $\alpha \geq 0, \beta > 0$ that

$$\begin{aligned}\zeta^{\mathfrak{l}}_0(\{2\}^\alpha, 1, \{2\}^\beta) &= \zeta^{\mathfrak{l}}_0(\{2\}^{\beta-1}, 3, \{2\}^\alpha) \\ &= 2(-1)^{\alpha+\beta}\left(A^{\alpha+\beta}_{\beta-1,\alpha} - B^{\alpha+\beta}_{\beta-1,\alpha}\right)\zeta^{\mathfrak{l}}(2\alpha+2\beta+1), \\ \zeta^{\mathfrak{l}}_0(\{2\}^\alpha, 1) &= \zeta^{\mathfrak{l}}_1(\{2\}^\alpha) = 2(-1)^\alpha\zeta^{\mathfrak{l}}(2\alpha+1). \end{aligned} \tag{32}$$

The first follows by duality and extracting the coefficient of $\zeta^{\mathfrak{m}}(2a + 2b + 3)$ in Theorem 6.4. The second follows by shuffle regularising, or from the stuffle product, as shown in Lemma 3.8 [4].

By the induction assumption, we are also granted $\widetilde{t}^{\mathfrak{l}}(1) = \log^{\mathfrak{l}}(2)$, and that for $0 < a' + b' < a + b$ we have

$$\widetilde{t}^{\mathfrak{l}}(\{2\}^{a'}, 1, \{2\}^{b'}) =$$
$$2(-1)^{1+a'+b'}\left[\binom{2a'+2b'}{2a'} + \frac{2^{2(a'+b')}}{2^{2(a'+b')}-1}\binom{2a'+2b'}{2b'}\right]\zeta^{\mathfrak{l}}(\overline{2a'+2b'+1}). \quad (33)$$

Case D_1: We check explicitly and directly the case $r = 0$, because it can have a distinctly different form, on account of the $\delta_{r=0}\log^{\mathfrak{l}}(2)$ terms. Explicitly we find (also directly from Proposition 5.9)

$$\begin{aligned}
\widehat{\xi}^{0}_{a,b} &= \delta_{a=0}\log^{\mathfrak{m}}(2) - \delta_{b=0}\log^{\mathfrak{m}}(2) + \delta_{a\leq 0}\widetilde{t}^{\mathfrak{m}}(\{2\}^{a}, 1, \{2\}^{-a}) \\
&\quad + \sum_{\substack{\alpha\leq a-1\\ \beta\leq b\\ \alpha+\beta=0}} \zeta^{\mathfrak{m}}_{0}(\{2\}^{\alpha}, 1, \{2\}^{\beta}) - \sum_{\substack{\alpha\leq a\\ \beta\leq b-1\\ \alpha+\beta=0}} \zeta^{\mathfrak{m}}_{0}(\{2\}^{\beta}, 1, \{2\}^{\alpha}) \\
&= \delta_{a=0}\log^{\mathfrak{m}}(2) - \delta_{b=0}\log^{\mathfrak{m}}(2) + \delta_{a\leq 0}\widetilde{t}^{\mathfrak{m}}(\{2\}^{0}, 1, \{2\}^{0}) \\
&\quad + \zeta^{\mathfrak{m}}_{0}(\{2\}^{0}, 1, \{2\}^{0}) - \zeta^{\mathfrak{m}}_{0}(\{2\}^{0}, 1, \{2\}^{0}) \\
&= \delta_{a=0}2\cdot\log^{\mathfrak{m}}(2) - \delta_{b=0}\log^{\mathfrak{m}}(2)
\end{aligned}$$

So that since $D_1 = D_{2\cdot 0+1}$ with $r = 0$, we have

$$D_1 L^{a,b} = \big(\delta_{a=0}2\cdot\log^{\mathfrak{m}}(2) - \delta_{b=0}\log^{\mathfrak{m}}(2)\big)\otimes\widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b}).$$

Whereas, directly from $R^{a,b}$, we can compute the following. We make use of some simple properties of D_{2r+1}, such as the derivation and that $\zeta^{\mathfrak{m}}(N)$ and $\zeta^{\mathfrak{m}}(\overline{N})$ are primitive for the coaction, viz.: $\Delta\zeta^{\mathfrak{m}}(N) = 1\otimes\zeta^{\mathfrak{m}}(N) + \zeta^{\mathfrak{l}}(N)\otimes 1$. Overall this means

$$\begin{aligned}
D_{2r+1}XY &= (1\otimes Y)D_{2r+1}X + (1\otimes X)D_{2r+1}Y\,, \\
D_{2r+1}\zeta^{\mathfrak{m}}(N) &= \begin{cases} 0\,, & \text{if } 2r+1\neq N \\ \zeta^{\mathfrak{l}}(N)\,, & \text{if } 2r+1 = N\,, \end{cases}
\end{aligned}$$

the latter also for N replaced by $\overline{N}$, in particular also for $\zeta^{\mathfrak{m}}(\overline{1}) = -\log^{\mathfrak{m}}(2)$. Applying these to the computation of $D_1 R^{a,b}$ gives the following

$$\begin{aligned}
&D_1 R^{a,b} = \\
&-\sum_{r'=1}^{a+b}(-1)^{r'}2\left[\binom{2r'}{2a} + \frac{2^{2r'}}{2^{2r'}-1}\binom{2r'}{2b}\right]\Big((1\otimes\zeta^{\mathfrak{m}}(\overline{2r'+1}))\overbrace{D_1\widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b-r'})}^{=0} \\
&\qquad\qquad + \underbrace{D_1\zeta^{\mathfrak{m}}(\overline{2r'+1})}_{=\delta_{2r'+1=1}}(1\otimes\widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b-r'}))\Big) \\
&\quad + \delta_{a=0}2\cdot\Big(D_1\log^{\mathfrak{m}}(2)(1\otimes\widetilde{t}^{\mathfrak{m}}(\{2\}^{b}) + (1\otimes\log^{\mathfrak{m}}(2))\underbrace{D_1\widetilde{t}^{\mathfrak{m}}(\{2\}^{b})}_{=0}\Big) \\
&\quad - \delta_{b=0}\Big(D_1\log^{\mathfrak{m}}(2)(1\otimes\widetilde{t}^{\mathfrak{m}}(\{2\}^{a})) + (1\otimes\log^{\mathfrak{m}}(2))\underbrace{(D_1\widetilde{t}^{\mathfrak{m}}(\{2\}^{a}))}_{=0}\Big)\,,
\end{aligned}$$

So all terms vanish apart from the two terms involving $D_1 \log^{\mathfrak{m}}(2)$, which leads to

$$\begin{aligned} D_1 R^{a,b} &= \delta_{a=0} 2 \cdot \log^{\mathfrak{l}}(2) \otimes \widetilde{t}^{\,\mathfrak{m}}(\{2\}^b) - \delta_{b=0} \log^{\mathfrak{l}}(2) \otimes \widetilde{t}^{\,\mathfrak{m}}(\{2\}^a) \\ &= \big(\delta_{a=0} 2 \cdot \log^{\mathfrak{l}}(2) - \delta_{b=0} \log^{\mathfrak{l}}(2)\big) \otimes \widetilde{t}^{\,\mathfrak{m}}(\{2\}^{a+b}) \,. \end{aligned}$$

The last simplification holds because $R^{a,b}$ has total weight $2a + 2b + 1$, so the right hand tensor factor of D_1 must have weight $2a+2b$, irrespective of checking the various cases of the Kronecker delta conditions.

In particular, we have that $D_1 L^{a,b} = D_1 R^{a,b}$ in this case.

Case $r > 0$**:** Now we turn to the case $r > 0$, which will have no extra $\log^{\mathfrak{l}}(2)$ contribution. We find it helpful to separate out the terms where $\beta = 0$ or $\beta > 0$ in the sum involving $\zeta_0^{\mathfrak{l}}(\{2\}^\alpha, 1, \{2\}^\beta)$, and similarly for the one involving $\zeta_0^{\mathfrak{l}}(\{2\}^\beta, 1, \{2\}^\alpha)$. This is on account of the different form of the coefficient of $\zeta^{\mathfrak{l}}(2r + 1)$ therein might take. We have that

$$\begin{aligned} &\widehat{\xi}^r_{a,b} = \\ &\delta_{a \leq r} \widetilde{t}^{\,\mathfrak{m}}(\{2\}^a, 1, \{2\}^{r-a}) + \sum_{\substack{\alpha \leq a-1 \\ 1 \leq \beta \leq b}} \zeta_0^{\mathfrak{m}}(\{2\}^\alpha, 1, \{2\}^\beta) - \sum_{\substack{1 \leq \alpha \leq a \\ \beta \leq b-1}} \zeta_0^{\mathfrak{m}}(\{2\}^\beta, 1, \{2\}^\alpha) \\ &+ \sum_{\substack{\alpha \leq a-1 \\ \beta = 0}} \zeta_0^{\mathfrak{m}}(\{2\}^\alpha, 1, \{2\}^\beta) - \sum_{\substack{\alpha = 0 \\ \beta \leq b-1}} \zeta_0^{\mathfrak{m}}(\{2\}^\beta, 1, \{2\}^\alpha) \end{aligned}$$

Since we sum over $\alpha + \beta = r$, the last two summations resolve to a Kronecker delta condition, namely $\delta_{r \leq a-1}$ and $\delta_{r \leq b-1}$ respectively. Making the substitutions for the various $\zeta_0^{\mathfrak{l}}$ using (32) and for $\widetilde{t}^{\,\mathfrak{l}}$ from (33) by induction, we find

$$\begin{aligned} &\pi(\widehat{\xi}^r_{a,b}) = \\ &\delta_{a \leq r} 2 \cdot (-1)^{r+1} \left(\binom{2r}{2a} + \frac{2^{2r}}{2^{2r} - 1} \binom{2r}{2r - 2a} \right) \zeta^{\mathfrak{l}}(\overline{2r+1}) \\ &+ \Bigg\{ \sum_{\substack{\alpha \leq a-1 \\ 1 \leq \beta \leq b}} 2(-1)^r \big(A^r_{\beta-1,\alpha} - B^r_{\beta-1,\alpha}\big) - \sum_{\substack{1 \leq \alpha \leq a \\ \beta \leq b-1}} 2(-1)^r \big(A^r_{\alpha-1,\beta} - B^r_{\alpha-1,\beta}\big) \\ &\quad + \delta_{r \leq a-1} 2(-1)^r - \delta_{r \leq b-1} 2(-1)^r \Bigg\} \zeta^{\mathfrak{l}}(2r+1) \end{aligned}$$

If we make the change of variables $\beta = \beta' + 1$, in the first sum, and $\alpha = \alpha' + 1$ in the second sum, then the (implicit) summation range $\alpha + \beta = r$ is converted to $\alpha + \beta' + 1 = r$ and $\alpha' + 1 + \beta = r$ respectively. (We shall write this explicitly from now on.) Doing so, and simplifying the expression coming from $\widetilde{t}^{\,\mathfrak{l}}$ with $\zeta^{\mathfrak{m}}(\overline{2r+1}) = -(1 - 2^{-2r})\zeta^{\mathfrak{m}}(2r + 1)$, gives

$$\pi(\widehat{\xi}^r_{a,b}) = 2(-1)^r \Bigg\{ \delta_{a\le r}\big(2-2^{-2r}\big)\binom{2r}{2a} + \delta_{r\le a-1} - \delta_{r\le b-1} + \sum_{\substack{\alpha\le a-1\,,\beta\le b-1\\ \alpha+\beta=r-1}} \big(A^r_{\beta,\alpha} - B^r_{\beta,\alpha}\big) - \sum_{\substack{\alpha\le a-1\,,\beta\le b-1\\ \alpha+\beta=r-1}} \big(A^r_{\alpha,\beta} - B^r_{\alpha,\beta}\big) \Bigg\} \zeta^{\mathfrak{l}}(2r+1)$$

Now we may apply the following Lemma

Lemma 6.5 (Brown, [4, Lemma 4.2]) *For any* $a, b \ge 0$, *and* $1 \le r \le a+b+1$ *we have*

$$\sum_{\substack{\alpha<a,\beta\le b\\ \alpha+\beta+1=r}} A^r_{\alpha,\beta} - \sum_{\substack{\alpha\le a\,,\beta<b\\ \alpha+\beta+1=r}} A_{\beta,\alpha} + \delta_{b\ge r} - \delta_{a\ge r} = 0, \text{ and}$$

$$\sum_{\substack{\alpha\le a\,,\beta\le b\\ \alpha+\beta+1=r}} B^r_{\alpha,\beta} - \sum_{\substack{\alpha\le a\,,\beta<b\\ \alpha+\beta+1=r}} B_{\beta,\alpha} = B^r_{a,b}.$$

In the case $a-1, b-1$, we may apply the lemma when $r \le (a-1)+(b-1)+1 = a+b-1$. But we are computing D_{2r+1} with $3 \le 2r+1 < 2a+2b+1$, i.e. $3 \le 2r+1 \le 2a+2b-1$ or equivalently $1 \le r \le a+b-1$. Application of this lemma (taking care with the range of summation indices, some are $<$ while others are $\le$) gives

$$\begin{aligned}
&\pi(\widehat{\xi}^r_{a,b}) \\
&= 2(-1)^r \Bigg\{ \delta_{a\le r}\cdot\big(2-2^{-2r}\big)\binom{2r}{2a} + \delta_{r\le a-1} - \delta_{r\le b-1} - \delta_{a\le r} A^r_{a-1,r-a} \\
&\qquad + \delta_{b\le r} A^r_{b-1,r-b} - \sum_{\substack{\alpha<a-1\\ \beta\le b-1\\ \alpha+\beta=r-1}} A^r_{\alpha,\beta} + \sum_{\substack{\alpha\le a-1\\ \beta<b-1\\ \alpha+\beta=r-1}} A^r_{\beta,\alpha} \\
&\qquad - \delta_{b\le r} B^r_{b-1,r-b} + \sum_{\substack{\alpha\le a-1\\ \beta\le b-1\\ \alpha+\beta=r-1}} B^r_{\alpha,\beta} - \sum_{\substack{\alpha\le a-1\\ \beta<b-1\\ \alpha+\beta=r-1}} B^r_{\beta,\alpha} \Bigg\} \zeta^{\mathfrak{l}}(2r+1) \\
&= 2(-1)^r \Bigg\{ \delta_{a\le r}\cdot\big(2-2^{-2r}\big)\binom{2r}{2a} + \delta_{r\le a-1} - \delta_{r\le b-1} \\
&\qquad - \delta_{a\le r} A^r_{a-1,r-a} + \delta_{b\le r} A^r_{b-1,r-b} + \delta_{b-1\ge r} - \delta_{a-1\ge r} \\
&\qquad - \delta_{b\le r} B^r_{b-1,r-b} + B^r_{a-1,b-1} \Bigg\} \zeta^{\mathfrak{l}}(2r+1) \\
&= 2(-1)^r \Bigg\{ \delta_{a\le r}\cdot\big(2-2^{-2r}\big)\binom{2r}{2a} - \delta_{a\le r} A^r_{a-1,r-a} + \delta_{b\le r} A^r_{b-1,r-b} \\
&\qquad - \delta_{b\le r} B^r_{b-1,r-b} + B^r_{a-1,b-1} \Bigg\} \zeta^{\mathfrak{l}}(2r+1)
\end{aligned}$$

We now make a number of straight forward simplifications. Namely, $A^r_{a,b}$ only depends on a, r, and $B^r_{a,b}$ only depends on b, r. Moreover since $r \geq 1$, if $a \geq r$ or $a < 0$, then already $A^r_{a,b} = 0$, likewise if $b \geq r$ or $b < 0$ then $B^r_{a,b} = 0$. So we find

$$\pi(\widehat{\xi}^r_{a,b}) = 2(-1)^r \bigg\{ (2 - 2^{-2r})\binom{2r}{2a} - A^r_{a-1,b-1} + A^r_{b-1,a-1} - B^r_{b-1,r-b} + B^r_{a-1,b-1} \bigg\} \zeta^{\mathfrak{l}}(2r+1)$$

Note that $-B^r_{b-1,r-b} + B^r_{a-1,b-1} = 0$ just by their definitions, so overall we obtain

$$\pi(\widehat{\xi}^r_{a,b}) = 2(-1)^r \bigg\{ (1 - 2^{-2r})\binom{2r}{2a} + \binom{2r}{2b} \bigg\} \zeta^{\mathfrak{l}}(2r+1)$$

Therefore

$$\begin{aligned} D_{2r+1} L^{a,b} &= \pi(\widehat{\xi}^r_{a,b}) \otimes \widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b-r}) \\ &= 2(-1)^r \bigg\{ (1 - 2^{-2r})\binom{2r}{2a} + \binom{2r}{2b} \bigg\} \zeta^{\mathfrak{l}}(2r+1) \otimes \widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b-1}) \end{aligned}$$

gives us the derivation of the left hand side of (30), for $r > 0$.

On the other hand, a direct computation of $D_{2r+1} R^{a,b}$ gives us that

$$\begin{aligned} D_{2a+1} R^{a,b} &= (-1)^{r+1} \cdot 2 \bigg[\binom{2r}{2a} + \frac{2^{2r}}{2^{2r}-1} \binom{2r}{2b} \bigg] \zeta^{\mathfrak{l}}(\overline{2r+1}) \otimes \widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b-r}) \\ &= 2(-1)^r \bigg[(1 - 2^{-2r})\binom{2r}{2a} + \binom{2r}{2b} \bigg] \zeta^{\mathfrak{l}}(2r+1) \otimes \widetilde{t}^{\mathfrak{m}}(\{2\}^{a+b-r}) \end{aligned}$$

is the derivation of the right hand side of (30), for $r > 0$.

Conclusion: We have shown that $D_{2r+1} L^{a,b} - R^{a,b} = 0$ for $0 \leq r \leq a + b - 1$, hence $L^{a,b} - R^{a,b} \in \ker D_{<N}$. Therefore by Glanois's theorem, we know that

$$L^{a,b} - R^{a,b} = c\zeta^{\mathfrak{m}}(2a + 2b + 1),$$

for some $c \in \mathbb{Q}$. Then by applying the period map, we reduce to the numerically valid identity in Proposition 3.7 (with $W = 0$), and hence see that $c = 0$. Therefore the identity $L^{a,b} = R^{a,b}$ is true on the motivic level, and this complete the proof. ◻

7 Independence of Saha's elements

We now turn to the first application of this motivic identity. We show that the elements that Saha conjectured [25] to be a basis for convergent MtV's are, at least, linearly independent.

We recall briefly Saha's conjecture.

Conjecture 7.1 (Saha, [25]) *Let*

$$\mathcal{B}^S := \{t(k_1, \ldots, k_{m-1}, k_m + 1) \mid k_i \in \{1, 2\}\}.$$

Then $\mathcal{B}^S$ is a basis for convergent MtV's. Moreover, the weight w component of $\mathcal{B}^S$ is

$$\mathcal{B}^S_w = \{t(k_1, \ldots, k_{m-1}, k_m + 1) \mid k_i \in \{1, 2\}, k_1 + \cdots + k_m = w - 1\},$$

which has cardinality $\#\mathcal{B}^S_N = F_N$, for $N > 1$. Here $F_n = F_{n-1} + F_{n-2}$ is the n-th Fibonacci number, with $F_1 = F_2 = 1$.

We note that the arguments of such MtV's can be written as an arbitrary word in 1's and 2's, followed by either a 2 or a 3. We can therefore schematically describe the set of arguments as follows

$$w \in (\{1, 2\}^\times \oplus 2) \cup (\{1, 2\}^\times \oplus 3),$$

where $\oplus$ denotes concatenation of words.

Definition 7.2 (*Saha filtration*) For, $w \in (\{1, 2\}^\times \oplus 2) \cup (\{1, 2\}^\times \oplus 3)$, we define the level of w to be $\deg_1 w + \deg_3 w$, i.e. the total number of 1's and 3's in the word. We define $\mathbb{Q}$-subspace of $\mathcal{H}^{(2)}$, and the level $\leq \ell$ piece of the level filtration by

$$\begin{aligned} \mathcal{H}^S &:= \langle t^{\mathfrak{m}}(w) \mid w \in (\{1, 2\}^\times \oplus 2) \cup (\{1, 2\}^\times \oplus 3) \rangle_{\mathbb{Q}}\,, \\ S_\ell \mathcal{H}^S &:= \langle t^{\mathfrak{m}}(w) \mid w \in (\{1, 2\}^\times \oplus 2) \cup (\{1, 2\}^\times \oplus 3), \text{ s.t. } \deg_1 w + \deg_3 w \leq \ell \rangle_{\mathbb{Q}} \end{aligned}$$

The associated graded to this filtration is then given by

$$\operatorname{gr}^S_\ell \mathcal{H}^S := S_\ell \mathcal{H}^S / S_{\ell-1} \mathcal{H}^S.$$

Example 7.3 The level ≤ 1 part of this filtration is generated by the following elements

$$S_1 \mathcal{H}^S = \langle t^{\mathfrak{m}}(\{2\}^a, 1, \{2\}^b), t^{\mathfrak{m}}(\{2\}^c, 3), t^{\mathfrak{m}}(\{2\}^d) \mid a, c, d \geq 0, b \geq 1 \rangle_{\mathbb{Q}},$$

whereas the level ≤ 0 part of this filtration is generated by

$$S_0 \mathcal{H}^S = \langle t^{\mathfrak{m}}(\{2\}^d) \mid d \geq 0 \rangle_{\mathbb{Q}}.$$

Lemma 7.4 *The Saha-level is motivic. More precisely, the following holds for all $r' \geq 0$*

$$D_{2r'+1} S_\ell \mathcal{H}^S \subseteq \mathcal{L}^{(2)}_{2r+1} \otimes_{\mathbb{Q}} S_{\ell-1} \mathcal{H}^S.$$

Proof Let $r \geq 0$ be odd, and $\underline{\mathbf{k}} = (k_1, \ldots, k_d) \in (\{1, 2\}^\times \oplus 2) \cup (\{1, 2\}^\times \oplus 3)$ with level ℓ. We consider how to compute $D_r \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d)$ via Proposition 5.7.

Firstly, if the deconcatenation term (27) $\widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_j) \otimes \widetilde{t}^{\mathfrak{m}}(k_{j+1}, \ldots, k_d)$ contributes, then the string $(k_1, \ldots, k_j)$ of odd weight must contain a 1 or a 3. Hence the level of $(k_{j+1}, \ldots, k_d)$ is reduced.

Now, if the term (28) contributes, we must satisfy the conditions $|\underline{\mathbf{k}}_{i+1,j}| \leq r < |\underline{\mathbf{k}}_{i,j}| - 1$. This means that $k_i \geq |\underline{\mathbf{k}}_{i,j}| - r > 1$, so that $\underline{\mathbf{k}}_{i,j} - r = 2, 3$. The case $|\underline{\mathbf{k}}_{i,j}| = 3$ could occurs if $k_i = 3$, and this can only occur if $k_i = k_d$ with $k_d = 3$, so that $i = j$. But this is excluded from the sum, so $|\underline{\mathbf{k}}_{i,j}| - r = 2$. Since r is odd, this implies $|\underline{\mathbf{k}}_{i,j}|$ is also odd, and so the subindex must contain (at least) one 1 or 3. This is replaced by a 2, and so the level is reduced.

Likewise, if (29) contributes, we must have $|\underline{\mathbf{k}}_{i,j-1}| \leq r < |\underline{\mathbf{k}}_{i,j}| - 1$. This means $k_j \geq |\underline{\mathbf{k}}_{i,j}| - r > 1$, so that $|\underline{\mathbf{k}}_{i,j}| - r = 2, 3$. The case $|\underline{\mathbf{k}}_{i,j}| - r = 2$ is analogous to the previous: $|\underline{\mathbf{k}}_{i,j}|$ is odd, so contains at least one 1 or 3. This is replaced by a 2 and so the level is reduced. Now, though, $|\underline{\mathbf{k}}_{i,j}| - r = 3$ occurs if $j = d$ and $k_d = 3$. But we see that $|\underline{\mathbf{k}}_{i,j}|$ must be even, and already contains a three (from $k_j = k_d = 3$). Therefore it must also contain at least one 1. Since a 3 and a 1 are replaced with a single 3, at the end of the string as $j = d$, the element again is a Saha element, and of lower level. □

From this lemma, we obtain a level-graded derivation

$$\operatorname{gr}_\ell^S D_{2r+1} \colon \operatorname{gr}_\ell^S \mathcal{H}^S \to \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} \operatorname{gr}_{\ell-1}^S \mathcal{H}^S$$

Moreover we claim, this map lands in the subspace of $\mathcal{L}_{2r+1}$ generated by the single zeta element $\zeta^{\mathfrak{l}}(\overline{2r+1})$.

Lemma 7.5 *For $\ell \geq 1$, $r' \geq 0$, the level-graded derivation $\operatorname{gr}_\ell^S D_{2r'+1}$ satisfies*

$$\operatorname{gr}_\ell^S D_{2r'+1}\big(\operatorname{gr}_\ell^S \mathcal{H}^S\big) \subseteq \zeta^{\mathfrak{l}}(\overline{2r'+1})\mathbb{Q} \otimes_{\mathbb{Q}} \operatorname{gr}_{\ell-1}^S \mathcal{H}^S.$$

Proof Let $r \geq 1$ be odd, and $\underline{\mathbf{k}} = (k_1, \ldots, k_d) \in (\{1,2\}^\times \oplus 2) \cup (\{1,2\}^\times \oplus 3)$ with level ℓ. We consider how to compute $D_r \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d)$ via Proposition 5.7, and more carefully track the contributions when we take elements of level $\ell - 1$ in the right hand tensor factor. For $r = 1$, this is clear: $\mathcal{H}_1^{(2)} = \langle \log^{\mathfrak{m}}(2) \rangle_{\mathbb{Q}} = \langle \zeta^{\mathfrak{m}}(\overline{1}) \rangle_{\mathbb{Q}}$, as weight 1 motivic alternating MZV's are spanned by $\log^{\mathfrak{m}}(2)$. So the $\mathcal{L}$-factor of D_1 can only be a multiple of $\zeta^{\mathfrak{l}}(\overline{1})$, for dimensional reasons. So we can assume $r > 1$

With (27), the deconcatenation term $\widetilde{t}^{\mathfrak{l}}(k_1, \ldots, k_j) \otimes \widetilde{t}^{\mathfrak{m}}(k_{j+1}, \ldots, k_d)$, we see that for $\underline{\mathbf{k}}_{j+1,d}$ to have level $\ell - 1$, a single 1 or 3 must have been removed. Therefore if $\underline{\mathbf{k}}_{1,j} = (\{2\}^a, 1, \{2\}^b)$, we know via Theorem 6.1, that $\widetilde{t}^{\mathfrak{l}}(\underline{\mathbf{k}}_{1,j}) \in \zeta^{\mathfrak{l}}(\overline{2r+1})\mathbb{Q}$. Likewise, if $\underline{\mathbf{k}}_{1,j} = (\{2\}^a, 3)$ (for if it contains a 3, it must be that $j = d$, and $k_d = 3$.), one has from Murakami's evaluation in Theorem 6.2, that $\widetilde{t}^{\mathfrak{l}}(\underline{\mathbf{k}}_{1,j}) \in \zeta^{\mathfrak{l}}(2r+1)\mathbb{Q} = \zeta^{\mathfrak{l}}(\overline{2r+1})\mathbb{Q}$.

Then we turn to the contribution from (28), and recall the considerations in the proof of Lemma 7.4. Namely, $|\underline{\mathbf{k}}_{i,j}| - r = 2$, $k_i = 2$, which forces certain behaviour onto $\underline{\mathbf{k}}_{i,j}$. If $|\underline{\mathbf{k}}_{i,j}| - r = 2$, then $\underline{\mathbf{k}}_{i,j}$ must contain an odd number of 1's and 3's. But for level-grading reasons, it actually must contain exactly one such, which if it were a 3, must appear in the last position. We have the following cases.

$\lvert\underline{\mathbf{k}}_{i,j}\rvert - r$	$\underline{\mathbf{k}}_{i,j}$	Contribution to D_r
2	$(2, \{2\}^a, 1, \{2\}^b)$	$\zeta^{\mathfrak{l}}_{2-2}(\{2\}^a, 1, \{2\}^b) \otimes \widetilde{t}^{\,\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$
2	$(2, \{2\}^a, 3)$	$\zeta^{\mathfrak{l}}_{2-2}(\{2\}^a, 3) \otimes \widetilde{t}^{\,\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$

In either case, we see via Theorem 6.4, or rather (32) thereafter, that each $\zeta^{\mathfrak{l}}_{\alpha}(\underline{\mathbf{k}}_{i+1,j}) \in \zeta^{\mathfrak{l}}(\overline{2r+1})\mathbb{Q}$.

Likewise, from (29), we have $\lvert\underline{\mathbf{k}}_{i,j}\rvert - r = 2, 3$. If $\lvert\underline{\mathbf{k}}_{i,j}\rvert - r = 2$, then $k_j = 2, 3$ and $\lvert\underline{\mathbf{k}}_{i,j}\rvert$ contains an odd number of 1's and 3's. In the level-graded, it therefore must contain exactly one 1 or one three (where a 3 would appear at the end). Otherwise $\lvert\underline{\mathbf{k}}_{i,j}\rvert - r = 3$, so $k_j = 3$, and $\lvert\underline{\mathbf{k}}_{i,j}\rvert$ contains an even number of 1's and 3's. As it already must contain a 3 at the end (since $k_j = 3$ and so $j = d$), it must also contain 1 somewhere else.

Be aware that we must reverse $\underline{\mathbf{k}}_{i,j}$ when inserting it into $\zeta^{\mathfrak{l}}$ in term (29). We have the following cases.

$\lvert\underline{\mathbf{k}}_{i,j}\rvert - r$	$\underline{\mathbf{k}}_{i,j}$	Contribution to D_r
2	$(\{2\}^a, 1, \{2\}^b, 2)$	$-\,\zeta^{\mathfrak{l}}_{2-2}(\{2\}^b, 1, \{2\}^a) \otimes \widetilde{t}^{\,\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$
2	$(\{2\}^a, 3)$	$-\zeta^{\mathfrak{l}}_{3-2}(\{2\}^a) \otimes \widetilde{t}^{\,\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$
3	$(\{2\}^a, 1, \{2\}^b, 3)$	$-\zeta^{\mathfrak{l}}_{3-3}(\{2\}^b, 1, \{2\}^a) \otimes \widetilde{t}^{\,\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 3)$

Once again, we see via Theorem 6.4, or rather (32) thereafter, that in every case the term $\zeta^{\mathfrak{l}}_{\alpha}(\underline{\mathbf{k}}_{j-1,i}) \in \zeta^{\mathfrak{l}}(\overline{2r+1})\mathbb{Q}$. □

We now look at the action of these derivations on elements of a given level, and package them together into the following linear map.

Definition 7.6 For all $N, \ell \geq 1$, let $\partial^S_{N,\ell}$ be the linear map

$$\partial^S_{N,\ell} \colon \operatorname{gr}^S_\ell \mathcal{H}^S_N \to \bigoplus_{1 \leq 2r+1 \leq N} \operatorname{gr}^S_{\ell-1} \mathcal{H}^S_{N-2r-1},$$

defined by first applying $\bigoplus_{1\leq 2r+1 \leq N} \operatorname{gr}^S_\ell D_{2r+1}\big|_{\operatorname{gr}^S_\ell \mathcal{H}^S_N}$, and then sending all $\log^{\mathfrak{m}}(2) \mapsto \frac{1}{2}$, $\zeta^{\mathfrak{l}}(2r+1) \mapsto 2^{2r-1}$, $r > 0$ to by the projection

$$\widetilde{\pi}_{2r+1} \colon \mathbb{Q}\zeta^{\mathfrak{l}}(\overline{2r+1}) \to \mathbb{Q}$$
$$\begin{cases} \log^{\mathfrak{l}}(2) \mapsto \frac{1}{2}\,, & \text{if } r = 0\,, \\ \zeta^{\mathfrak{l}}(2r+1) \mapsto 2^{2r-1}\,, & \text{if } r > 0 \end{cases}$$

The goal is to show that the maps $\partial^S_{N,\ell}$ are injective for $\ell \geq 1$. Then by recursion, we will establish the elements of level ℓ are linearly independent (otherwise $\partial^S_{N,\ell}$ would construct a non-trivial relation of strictly smaller level).

Definition 7.7 (*Matrix basis*) Let $\ell, N \geq 1$, with $N \equiv \ell \pmod 2$. Define the following sets

$$B_{S,N,\ell} := \{w \in (\{1,2\}^{\times} \oplus 2) \cup (\{1,2\}^{\times} \oplus 3) \mid \deg_1 w + \deg_3 w = \ell, |w| = N\}$$
$$B'_{S,N,\ell} := \{w \in (\{1,2\}^{\times} \oplus 2) \cup (\{1,2\}^{\times} \oplus 3) \mid \deg_1 w + \deg_3 w = \ell{-}1, |w| < N\}$$

In the case $\ell = 1$, the set $B'_{S,N,\ell}$ also includes the empty word (of weight 0 and level 0). Sort both sides in reverse colexicographic order (i.e. reading right to left, and the largest first), with $3 < 1 < 2$. In this ordering, all terms ending with a 3 will appear last.

A counting argument shows that $\#B_{S,N,\ell} = \#B'_{S,N,\ell}$ for all such choices of ℓ, N. These basis elements will be used to define the matrix form of the linear map $\partial^S_{N,\ell}$, and the claim of injectivity corresponds to non-zero determinant.

Example 7.8 For $N = 8$, $\ell = 2$, we have

$$B_{S,N,\ell} = \{11222\,, 12122\,, 21122\,, 12212\,, 21212\,, 22112\,, 1223\,, 2123\,, 2213\},$$
$$B'_{S,N,\ell} = \{1222\,, 2122\,, 122\,, 2212\,, 212\,, 12\,, 223\,, 23\,, 3\}.$$

Definition 7.9 (*Matrix of* $\partial^S_{N,\ell}$) For $\ell \geq 1$, $N \geq 1$, with $N \equiv \ell \pmod 2$, let

$$M_{S,N,\ell} := \big(f^{w}_{w'}\big)_{w \in B_{S,N,\ell}, w' \in B'_{S,N,\ell}}$$

be the matrix of $\partial^S_{N,\ell}$ with respect to the bases $B_{S,N,\ell}$ and $B'_{S,N,\ell}$. Here f'^{w}_{w} denotes the coefficient of $\widetilde{t}^{\mathfrak{m}}(w')$ in $\partial^S_{N,\ell}\widetilde{t}^{\mathfrak{m}}(w)$, and in the matrix w corresponds to rows, and w' to columns.

It will be helpful to introduce some notation to talk more directly about there coefficients of $\zeta^{\mathfrak{l}}(2r+1)$ in various identities.

Definition 7.10 Write $c_{2^a 3 2^b}$, $c_{2^a 1}$, $d_{2^a 1 2^b}$, $d_{2^a 3 2^b}$ to be the coefficient such that

$$\begin{aligned}
\zeta^{\mathfrak{l}}(\{2\}^a, 3, \{2\}^b) = \zeta^{\mathfrak{l}}(\{2\}^b, 1, \{2\}^{a+1}) &= c_{2^a 3 2^b}\zeta^{\mathfrak{l}}(2a+2b+3) \\
\zeta^{\mathfrak{l}}(\{2\}^a, 1) = \zeta^{\mathfrak{l}}_1(\{2\}^a) &= c_{2^a 1}\zeta^{\mathfrak{l}}(2a+1) \\
\widetilde{t}^{\mathfrak{l}}(\{2\}^a, 1, \{2\}^b) &= d_{2^a 1 2^b}\zeta^{\mathfrak{l}}(2a+2b+1) \\
\widetilde{t}^{\mathfrak{l}}(\{2\}^a, 3, \{2\}^b) &= d_{2^a 3 2^b}\zeta^{\mathfrak{l}}(2a+2b+3)\,.
\end{aligned}$$

Moreover note that $d_1 = 2$ so that $\widetilde{t}^{\mathfrak{l}}(1) = d_1 \cdot \frac{1}{2}\log^{\mathfrak{l}}(2)$. From the computations in Theorem 6.4 and (32) thereafter, and from Theorem 6.1 and Theorem 6.2, we have the following explicit formulae.

$$\begin{aligned}
c_{2^a 3 2^b} &= 2(-1)^{a+b}\left(-\binom{2a+2b+2}{2a+2} + (1-2^{-2a-2b-2})\binom{2a+2b+2}{2b+1}\right) \\
c_1 &= 0 \\
c_{2^a 1} &= 2(-1)^a
\end{aligned}$$

$$
\begin{aligned}
d_{2^a 1 2^b} &= 2(-1)^{a+b}\left((1-2^{-2a-2b})\binom{2a+2b}{2a} + \binom{2a+2b}{2b}\right) \\
&= 4(-1)^{a+b}(1-2^{-2a-2b-1})\binom{2a+2b}{2a} \\
d_{2^a 3 2^b} &= 2(-1)^{a+b}\left(\binom{2a+2b+2}{2a+1} + (1-2^{-2a-2b-2})\binom{2a+2b+2}{2b+1}\right) \\
&= 4(-1)^{a+b}(1-2^{-2a-2b-3})\binom{2a+2b+2}{2a+1}
\end{aligned}
$$

Example 7.11 For $N = 8, \ell = 2$, the matrix $M_{S,8,2}$ is as follows; the first row and column label the elements of $B'_{S,8,2}$ and $B_{S,8,2}$ respectively.

	1222	2122	122	2212	212	12	223	23	3
11222	1	0	$-2c_{21}$	0	0	$-8c_{221}$	0	0	0
12122	0	1	$2d_{12}-2c_{21}$	0	0	$8c_{23}-8c_{32}$	0	0	0
21122	0	0	$2d_{21}$	0	$-2c_{21}$	0	0	0	0
12212	0	0	$2c_{21}$	1	$2d_{12}-2c_{21}$	$-8c_{23}+8c_{32}+8d_{122}$	0	0	0
21212	0	0	0	0	$2d_{21}$	$8d_{212}$	0	0	0
22112	0	0	0	0	$2c_{21}$	$8d_{221}$	0	0	0
1223	0	0	$2c_3-2c_{21}$	0	0	$8c_{23}-8c_{221}$	1	$2d_{12}-2c_{21}$	$8d_{122}-8c_{221}$
2123	0	0	0	0	$2c_3-2c_{21}$	0	0	$2d_{21}-2c_{21}$	$8d_{212}-8c_{32}$
2213	0	0	0	0	0	0	0	$2c_{21}-2c_3$	$8d_{221}-8c_{23}$

The entries 1 in the matrix arise from both the deconcatenation term $2\widetilde{\tau}^{\mathfrak{l}}(1)$ which appears in D_1 as per Proposition 5.9. With the projection $\log^{\mathfrak{l}}(2) \to \frac{1}{2}$, this combination gives 1 above.

After substituting the values for $c_\bullet$ and $d_\bullet$ using the formulae above, we obtain the matrix

	1222	2122	122	2212	212	12	223	23	3
11222	1	0	4	0	0	−16	0	0	0
12122	0	1	−3	0	0	−80	0	0	0
21122	0	0	−7	0	4	0	0	0	0
12212	0	0	−4	1	−3	111	0	0	0
21212	0	0	0	0	−7	186	0	0	0
22112	0	0	0	0	−4	31	0	0	0
1223	0	0	6	0	0	−60	1	−3	15
2123	0	0	0	0	6	0	0	−3	150
2213	0	0	0	0	0	0	0	−6	75

We notice already that the matrix has odd entries on the diagonal, and all entries below the diagonal are even. Therefore the matrix is upper triangle modulo 2 with 1's on the whole diagonal. So it has determinant $\equiv 1 \pmod 2$ and is invertible. We aim to show this is a general phenomenon for level $\ell > 1$. In fact, we shall show that modulo 2, $\partial^S_{N,\ell}$ acts by deconcatenation, so that the only entries $d_\bullet$ occur above (or rather right) of the main diagonal inclusive.

Remark 7.12 In the case of level $\ell = 1$, the matrix actually has even determinant, and so the above considerations would fail. However, we will show that the evenness of the determinant arises exactly from the single even entry in the last row. This single entry, is the deconcatenation term from $\partial^S_{N,1}\widetilde{t}^{\mathfrak{m}}(\{2\}^a,3) = d_{2^a3}\emptyset$, and so by expanding about the last row, we would reduce to d_{2^a3} times a determinant which is invertible modulo 2, at least once we prove this previous claim.

Lemma 7.13 *Let $\ell > 1$ and $w \in B_{S,N,\ell}$. Then every coefficient of $\widetilde{t}^{\mathfrak{m}}(u)$, $u \in B'_{S,N,\ell}$ in*

$$\partial^S_{N,\ell}\widetilde{t}^{\mathfrak{m}}(w) - \sum_{\substack{w=uv \\ \deg_3 u+\deg_1 u=1}} 2^{|u|-2} d_u \widetilde{t}^{\mathfrak{m}}(v),$$

is an even integer.

Proof Let $\underline{\mathbf{k}} \in (\{1,2\}^\times \oplus 2) \cup (\{1,2\}^\times \oplus 3)$, with $\ell = \deg_1 \underline{\mathbf{k}} + \deg_3 \underline{\mathbf{k}}$. We consider how to compute $\mathrm{gr}_\ell\, D_r \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}})$ via Proposition 5.7 and the simplification in Proposition 5.9 for D_1. For $r = 1$, we immediately find

$$\begin{aligned}\mathrm{gr}_\ell\, D_1 \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}) &= \widetilde{\pi}_1(2\log^{\mathfrak{l}}(2))\delta_{k_1=1}\widetilde{t}^{\mathfrak{m}}(k_2,\ldots,k_d) \\ &= \delta_{k_1=1}2^{-1}d_1\widetilde{t}^{\mathfrak{m}}(k_2,\ldots,k_d)\end{aligned}$$

Now if we assume $r > 1$, we have

$$\begin{aligned}&\mathrm{gr}_\ell\, D_r\big(\widetilde{t}^{\mathfrak{m}}(k_1,\ldots,k_d)\big) = \\ &\sum_{1\le j\le d} \delta_{|\underline{\mathbf{k}}_{1,j}|=r}\widetilde{\pi}_r\big(\widetilde{t}^{\mathfrak{l}}(k_1,\ldots,k_j)\big)\cdot\widetilde{t}^{\mathfrak{m}}(k_{j+1},\ldots,k_d) \qquad (34)\\ &+\sum_{1\le i<j\le d} \delta_{|\underline{\mathbf{k}}_{i+1,j}|\le r<|\underline{\mathbf{k}}_{i,j}|-1}\widetilde{\pi}_r\big(\zeta^{\mathfrak{l}}_{r-|\underline{\mathbf{k}}_{i+1,j}|}(k_{i+1},\ldots,k_j)\big) \\ &\qquad\cdot\widetilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},|\underline{\mathbf{k}}_{i,j}|-r,k_{j+1},\ldots,k_d) \qquad (35)\\ &-\sum_{1\le i<j\le d} \delta_{|\underline{\mathbf{k}}_{i,j-1}|\le r<|\underline{\mathbf{k}}_{i,j}|-1}\widetilde{\pi}_r\big(\zeta^{\mathfrak{l}}_{r-|\underline{\mathbf{k}}_{i,j-1}|}(k_{j-1},\ldots,k_i)\big) \\ &\qquad\cdot\widetilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},|\underline{\mathbf{k}}_{i,j}|-r,k_{j+1},\ldots,k_d) \qquad (36)\end{aligned}$$

We note that since $\ell > 1$, $\mathrm{gr}_\ell\, D_{|\underline{\mathbf{k}}|}\widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}) = \widetilde{\pi}_{|\underline{\mathbf{k}}|}\big(\widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}})\big)\widetilde{t}^{\mathfrak{m}}(\emptyset) = 0$, since $\emptyset$—the empty word—has level $0 < \ell - 1$. (For this, consider how to compute D_{2r+1} via the graded parts of the coaction $\Delta(x) = 1\otimes x + x\otimes 1 + \Delta'(x)$; the part with full weight in the left hand factor is then clearly $x\otimes 1$.) This means the deconcatenation part in (34) can never involve $\widetilde{t}^{\mathfrak{l}}(\ldots,3)$, so must be of the form $\widetilde{\pi}_{2r+1}\big(\widetilde{t}^{\mathfrak{l}}(\{2\}^a,1,\{2\}^b)\big) = 2^{2a+2b-1}d_{2^a12^b}$.

Now consider (35). According to the table of cases in Lemma 7.5, we have the following contributions.

$\lvert\underline{\mathbf{k}}_{i,j}\rvert - r$	$\underline{\mathbf{k}}_{i,j}$	Contribution to $\mathrm{gr}_\ell\, D_r$
$2, b=0$	$(2,\{2\}^a,1,\{2\}^b)$	$2^{2a-1}c_{2^a1}\widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1},2,\underline{\mathbf{k}}_{j+1,d})$
$2, b>0$	$(2,\{2\}^a,1,\{2\}^b)$	$2^{2a+2b-1}c_{2^{b-1}32^a}\widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1},2,\underline{\mathbf{k}}_{j+1,d})$
2	$(2,\{2\}^a,3)$	$2^{2a+1}c_{2^a3}\widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1},2,\underline{\mathbf{k}}_{j+1,d})$

Likewise for (36), we have the following contributions.

$\lvert\underline{\mathbf{k}}_{i,j}\rvert - r$	$\underline{\mathbf{k}}_{i,j}$	Contribution to $\mathrm{gr}_\ell\, D_r$
$2, a = 0$	$(\{2\}^a, 1, \{2\}^b, 2)$	$-\,2^{2b-1} c_{2^b 1}\, \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$
$2, a > 0$	$(\{2\}^a, 1, \{2\}^b, 2)$	$-2^{2a+2b-1} c_{2^{a-1} 3 2^b}\, \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$
2	$(\{2\}^a, 3)$	$-2^{2a-1} c_{2^a 1}\, \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$
$3, a = 0$	$(\{2\}^a, 1, \{2\}^b, 3)$	$-2^{2b-1} c_{2^b 1}\, \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 3)$
$3, a > 0$	$(\{2\}^a, 1, \{2\}^b, 3)$	$-2^{2a+2b-1} c_{2^{a-1} 3 2^b}\, \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 3)$

The key points to observe now are that

$$2^{2a-1} c_{2^a 1} = \begin{cases} 0 & a = 0 \\ 2^{2a}(-1)^a & a > 0 \end{cases} \tag{37}$$

$$2^{2a+2b+1} c_{2^a 3 2^b} = \\ (-1)^{a+b}\left(-2^{2a+2b+2}\binom{2a+2b+2}{2a+2} + (2^{2a+2b+2}-1)\binom{2a+2b+2}{2b+1}\right), \tag{38}$$

hence both coefficients are always even integers, and so $\equiv 0 \pmod 2$. For $c_{2^a 3 2^b}$ it follows by writing $\binom{2a+2b+2}{2b+1} = \frac{2(a+b+1)}{2a+1}\binom{2a+2b+1}{2b+1}$. Compare Corollary 4.4 [4] for a more precise statement about $\nu_2(c_{2^a 3 2^b})$, the 2-adic valuation thereof, which was one of the key lemmas in Brown's proof of the linear independence of $\zeta^{\mathfrak{m}}(\{2,3\}^\times)$.

Since all terms arising from $\zeta^{\mathfrak{l}}$ have even coefficient in $\mathrm{gr}_\ell\, D_r$, we see that the only remaining terms arise from the deconcatenation part, and so the lemma follows. □

Theorem 7.14 *For $N, \ell \geq 1$, the matrix $M_{S,N,\ell}$ is invertible.*

Proof We proceed in a similar way as to the proofs by Murakami [22, Theorem 36], and Brown [4, Corollary 6.2].

We assume initially that $\ell > 1$, so that $d_{2^a 3}$ does not appear as an entry. For $\ell > 1$, consider the map

$$\begin{aligned} B'_{S,N,l} &\to B_{S,N,\ell} \\ u &\mapsto 2^r 1 u\,, \end{aligned}$$

where r is the unique integer such that $|2^r 1 u| = N$. This map is a bijection, and preserves the ordering of both $B'_{S,N,\ell}$ and $B_{S,N,\ell}$. That is to say, $u < v$ if and only if $2^r 1 u < 2^{r'} 1 v$, which holds as we are in the reverse colexicographic (reading right to left, largest first) order with $3 < 1 < 2$. The diagonal entries of $M_{S,N,\ell}$ are of the form $f_u^{2^r 1 u} = 2^{2a-1} d_{2^a 1} + 2n$, $n\mathbb{Z}$. Moreover, the only other non-even entries in the column indexed by $u \in B'_{S,N,\ell}$ occur for rows indexed by $w = 2^a 1 2^b u$, however since $2^r 1 u < 2^a 1 2^b u$, these occur above the diagonal. These entries are also integral since

$$\begin{aligned} 2^{2a+2b-1} d_{2^a 1 2^b} &= (-1)^{a+b} 2^{2a+2b+1}(1 - 2^{-2a-2b-1})\binom{2a+2b}{2a} \\ &= (-1)^{a+b}(2^{2a+2b+1} - 1)\binom{2a+2b}{2a}. \end{aligned}$$

We finally note that

$$2^{2a-1} d_{2^a 1} = (-1)^a (2^{2a+1} - 1) \equiv 1 \pmod 2,$$

so that $f_u^{2^r 1u} \equiv 1 \pmod 2$. This means the matrix is integral, and modulo 2 it reduces to an upper triangle matrix with leading diagonal equal to 1. Hence $M_{S,N,\ell}$ has determinant $\equiv 1 \pmod 2$, and so is invertible.

When $\ell = 1$, we note that all of the previous steps apply for all words $w = 2^a 1 2^b$, $b \geq 0$ indexing the rows. However, we obtain as the last row corresponding to the word $w = 2^a 3$, the row vector

$$(0, \ldots, 0, 2^{2a+1} d_{2^a 3}).$$

since one must deconcatenate the entire word to reduce the level by 1. Expand the determinant out about the last row, and we reduce to the submatrix involving words $w = 2^a 1 2^b$, $b > 0$ indexing the rows, and 2^a, $a > 0$ indexing the columns. This submatrix is integral, and modulo 2 it is upper triangle with 1's on the diagonal. Hence has non-zero determinant. Since

$$\begin{aligned} 2^{2a+1} d_{2^a 3} &= 2^{2a+3} (-1)^a (1 - 2^{-2a-3}) \binom{2a+2}{2a+1} \\ &= (-1)^a (2^{2a+3} - 1)(2a+2) \\ &\neq 0 \end{aligned}$$

the determinant of $M_{S,N,\ell}$ is still non-zero, and so $M_{S,N,\ell}$ is also invertible when $\ell = 1$. □

Corollary 7.15 *The Saha elements*

$$\{\widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{d-1}, k_d + 1) \mid k_i \in \{1, 2\}\}$$

are linearly independent.

Proof We proceed by induction on the level, as in [4, Theorem 7.4], [22, Corollary 38]. The elements of level $\ell = 0$ are of the form $\widetilde{t}^{\mathfrak{m}}(\{2\}^n)$, which are linearly independent because weight is a grading on $\mathcal{H}^{(2)}$. Now suppose the elements

$$\{\widetilde{t}^{\mathfrak{m}}(w) \mid w \in (\{1, 2\}^\times \oplus 2) \cup (\{1, 2\}^\times \oplus 3), \deg_1 w + \deg_w 3 \leq \ell - 1\},$$

of level $\leq \ell - 1$ are linearly independent. Since weight is a grading on $\mathcal{H}^{(2)}$, any non-trivial linear relation between elements of level ℓ can be assumed as homogeneous of some weight N. By Theorem 7.14, the map $\partial_{N,\ell}^S$ is injective as the matrix of the map is invertible. Application of $\partial_S^{N,\ell}$ to a non-trivial linear relation between level ℓ elements produces a non-trivial linear relation of strictly smaller level, which does not exist by the induction assumption. So the elements of level ℓ are also linearly independent, which completes the proof by induction. □

Corollary 7.16 *The space* $\mathcal{T}_N^{\text{conv}}$ *of convergent motivic MtV's has dimension* $\geq F_N$ *in weight* $N > 1$; *here* $F_n = F_{n-1} + F_{n-2}$ *are the Fibonacci numbers, with* $F_1 = F_2 = 1$.

Remark 7.17 For strictly convergent motivic MtV's, we do not appear yet to have the correct upper bound to show the Saha elements are a basis. The motivic MtV's fit into the following inclusions

$$\mathcal{H}_N^{(1)} \subseteq \mathcal{T}_N^{\text{conv}} \subseteq \mathcal{T}_N^{\text{ext}} \subseteq \mathcal{H}_N^{(2)},$$

where $\mathcal{T}_N^{\text{conv}}$ denotes the space of convergent motivic MtV's (with last argument ≥ 2) of weight N, and $\mathcal{T}_N^{\text{ext}}$ denotes the space of all shuffle regularised motivic MtV's of weight N. (The first inclusion follows from Murakami's motivic Galois [22, Theorem 8] descent showing $\widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d) \in \mathcal{H}^{(1)}$, whenever all $k_i \geq 2$. The upper bound of $\dim_{\mathbb{Q}} \mathcal{H}_N^{(2)} \leq F_{N+1}$ (established in [7]) only gives us the bound that $F_N \leq \dim_{\mathbb{Q}} \mathcal{T}_N^{\text{conv}} \leq F_{N+1}$. Below in Corollary 8.20, we will show however that $\mathcal{T}^{\text{ext}} = \mathcal{H}^{(2)}$ using the independence of the Hoffman one-two elements.

8 The Hoffman one-two elements as a basis

We now turn to the second application of the motivic identity. We show that the elements whose arguments consist of only 1's and 2's are linearly independent as motivic MtV's (analogous to Hoffman's conjectured (motivically true) basis of MZV's as those with arguments 2's and 3's). Dimension counting then shows that the elements—F_{N+1} many in weight N—must be a basis for motivic MtV's and alternating motivic MZV's, as these spaces have known dimensions $\leq F_{N+1}$.

Definition 8.1 (*Hoffman t filtration*) For, $w \in \{1, 2\}^\times$, we define the level of w to be $\deg_1 w$, i.e. the total number of 1 in the word. We define $\mathbb{Q}$-subspace of $\mathcal{H}^{(2)}$, and the level $\leq \ell$ piece of the level filtration by

$$\begin{aligned} \mathcal{H}^H &:= \langle t^{\mathfrak{m}}(w) \mid w \in \{1, 2\}^\times \rangle_{\mathbb{Q}}, \\ H_\ell \mathcal{H}^H &:= \langle t^{\mathfrak{m}}(w) \mid w \in \{1, 2\}^\times, \text{ s.t. } \deg_1 w \leq \ell \rangle_{\mathbb{Q}}. \end{aligned}$$

The associated graded to this filtration is then given by

$$\operatorname{gr}_\ell^H \mathcal{H}^H := H_\ell \mathcal{H}^H / H_{\ell-1} \mathcal{H}^H.$$

Example 8.2 The level ≤ 1 part of this filtration is generated by the following elements

$$H_1 \mathcal{H}^H = \langle t^{\mathfrak{m}}(\{2\}^a, 1, \{2\}^b), t^{\mathfrak{m}}(\{2\}^c) \mid a, b, c \geq 0 \rangle_{\mathbb{Q}},$$

whereas the level ≤ 0 part of this filtration is generated by

$$H_0 \mathcal{H}^H = \langle t^{\mathfrak{m}}(\{2\}^c) \mid c \geq 0 \rangle_{\mathbb{Q}}.$$

Lemma 8.3 *The Hoffman-level is motivic. More precisely, the following holds for all* $r' \geq 0$,

$$D_{2r'+1} H_\ell \mathcal{H}^H \subseteq \mathcal{L}^{(2)}_{2r+1} \otimes_{\mathbb{Q}} H_{\ell-1} \mathcal{H}^H.$$

Proof The proof is essentially the same as for Lemma 7.4, except the cases $\underline{\mathbf{k}}_{i,j} - r = 3$ cannot occur, since every argument $k_i \leq 2$. □

From this lemma, we obtain a level-graded derivation

$$\mathrm{gr}^H_\ell D_{2r+1} \colon \mathrm{gr}^H_\ell \mathcal{H}^H \to \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} \mathrm{gr}^H_{\ell-1} \mathcal{H}^H.$$

Moreover we claim, this map lands in the subspace of $\mathcal{L}_{2r+1}$ generated by the single zeta element $\zeta^{\mathfrak{l}}(\overline{2r+1})$.

Lemma 8.4 *For* $\ell \geq 1$, $r' \geq 0$, *the level-graded derivation* $\mathrm{gr}^H_\ell D_{2r'+1}$ *satisfies*

$$\mathrm{gr}^H_\ell D_{2r'+1}\big(\mathrm{gr}^H_\ell \mathcal{H}^H\big) \subseteq \zeta^{\mathfrak{l}}(\overline{2r'+1})\mathbb{Q} \otimes_{\mathbb{Q}} \mathrm{gr}^H_{\ell-1} \mathcal{H}^H.$$

Proof With $r = 0$, the claim is clear as $\mathcal{L}^{(2)}_1$ is generated by $\log^{\mathfrak{l}}(2)$. So let $r \geq 0$ be odd, and $\underline{\mathbf{k}} = (k_1, \ldots, k_d) \in \{1, 2\}^\times$ with level ℓ. We consider how to compute $D_r \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d)$ via Proposition 5.7, and more carefully track the contributions when we take elements of level $\ell - 1$ in the right hand tensor factor.

With (27), the deconcatenation term $\widetilde{t}^{\mathfrak{l}}(k_1, \ldots, k_j) \otimes \widetilde{t}^{\mathfrak{m}}(k_{j+1}, \ldots, k_d)$, we see that for $\underline{\mathbf{k}}_{j+1,d}$ to have level $\ell - 1$, a single 1 must have been removed. Therefore if $\underline{\mathbf{k}}_{1,j} = (\{2\}^a, 1, \{2\}^b)$, we know via Theorem 6.1, that $\widetilde{t}^{\mathfrak{l}}(\underline{\mathbf{k}}_{1,j}) \in \zeta^{\mathfrak{l}}(\overline{2r+1})\mathbb{Q}$.

Then we turn to the contribution from (28), and recall the considerations in the proof of Lemma 8.3. Namely, $|\underline{\mathbf{k}}_{i,j}| - r = 2$, $k_i = 2$, which forces certain behaviour onto $\underline{\mathbf{k}}_{i,j}$. If $|\underline{\mathbf{k}}_{i,j}| - r = 2$, then $\underline{\mathbf{k}}_{i,j}$ must contain an odd number of 1's. But for level-grading reasons, it actually must contain exactly one such. We have the following case.

$\lvert\underline{\mathbf{k}}_{i,j}\rvert - r$	$\underline{\mathbf{k}}_{i,j}$	Contribution to D_r
2	$(2, \{2\}^a, 1, \{2\}^b)$	$\zeta^{\mathfrak{l}}_{2-2}(\{2\}^a, 1, \{2\}^b) \otimes \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$

So via (32), we have $\zeta^{\mathfrak{l}}_\alpha(\underline{\mathbf{k}}_{i+1,j}) \in \zeta^{\mathfrak{l}}(\overline{2r+1})\mathbb{Q}$.

Likewise, from (29), we have $|\underline{\mathbf{k}}_{i,j}| - r = 2$, with $k_j = 2$ so $|\underline{\mathbf{k}}_{i,j}|$ contains an odd number of 1's. In the level-graded, it therefore must contain exactly one 1. Be aware that we must reverse $\underline{\mathbf{k}}_{i,j}$ when inserting it into $\zeta^{\mathfrak{l}}$ in term (29). We have the following case.

$\lvert\underline{\mathbf{k}}_{i,j}\rvert - r$	$\underline{\mathbf{k}}_{i,j}$	Contribution to D_r
2	$(\{2\}^a, 1, \{2\}^b, 2)$	$-\zeta^{\mathfrak{l}}_{2-2}(\{2\}^b, 1, \{2\}^a) \otimes \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$

Once again, the term $\zeta^{\mathfrak{l}}_\alpha(\underline{\mathbf{k}}_{j-1,i}) \in \zeta^{\mathfrak{l}}(\overline{2r+1})\mathbb{Q}$ via (32). □

We now look at the action of these derivations on elements of a given level, and package them together into the following linear map.

Definition 8.5 For all $N, \ell \geq 1$, let $\partial^H_{N,\ell}$ be the linear map

$$\partial^H_{N,\ell} \colon \operatorname{gr}^H_\ell \mathcal{H}^H_N \to \bigoplus_{1 \leq 2r+1 \leq N} \operatorname{gr}^H_{\ell-1} \mathcal{H}^H_{N-2r-1},$$

defined by first applying $\bigoplus_{1\leq 2r+1 \leq N} \operatorname{gr}^H_\ell D_{2r+1}\big|_{\operatorname{gr}^H_\ell \mathcal{H}^H_N}$, and then sending all $\log^{\mathfrak{l}}(2) \mapsto \frac{1}{2}$, $\zeta^{\mathfrak{l}}(2r+1) \mapsto 2^{2r-1}$, $r > 0$ to by the projection

$$\widetilde{\pi}_{2r+1} \colon \mathbb{Q}\zeta^{\mathfrak{l}}(\overline{2r+1}) \to \mathbb{Q}$$
$$\begin{cases} \log^{\mathfrak{m}}(2) \mapsto \frac{1}{2}\,, & \text{if } r = 0\,, \\ \zeta^{\mathfrak{l}}(2r+1) \mapsto 2^{2r-1}\,, & \text{if } r > 0. \end{cases}$$

The goal is to show that the maps $\partial^H_{N,\ell}$ are injective for $\ell \geq 1$. Then by recursion, we will establish the elements of level ℓ are linearly independent (otherwise $\partial^H_{N,\ell}$ would construct a non-trivial relation of strictly smaller level).

Definition 8.6 (*Matrix basis*) Let $\ell, N \geq 1$, with $N \equiv \ell \pmod 2$. Define the following sets

$$B_{H,N,\ell} := \{w \in \{1,2\}^\times \mid \deg_1 w = \ell, |w| = N\},$$
$$B'_{H,N,\ell} := \{w \in \{1,2\}^\times \mid \deg_1 w = \ell - 1, |w| < N\}.$$

In the case $\ell = 1$, the set $B'_{H,N,\ell}$ also includes the empty word (of weight 0 and level 0). Sort both sides in reverse colexicographic order (i.e. reading right to left, largest first), with $1 < 2$.

A counting argument shows that $\#B_{H,N,\ell} = \#B'_{H,N,\ell}$ for all such choices of ℓ, N. These basis elements will be used to define the matrix form of the linear map $\partial^H_{N,\ell}$, and the claim of injectivity corresponds to non-zero determinant.

Example 8.7 For $N = 8, \ell = 2$, we have

$$\begin{aligned} B_{H,N,\ell} = \{&11222\,, 12122\,, 21122\,, 12212\,, 21212\,, \\ &22112\,, 12221\,, 21221\,, 22121\,, 22211\}, \\ B'_{H,N,\ell} = \{&1222\,, 2122\,, 122\,, 2212\,, 212\,, 12\,, 2221\,, 221\,, 21, 1\}. \end{aligned}$$

Definition 8.8 (*Matrix of* $\partial^H_{N,\ell}$) For $\ell \geq 1$, $N \geq 1$, with $N \equiv \ell \pmod 2$, let

$$M_{H,N,\ell} := \left(f^w_{w'}\right)_{w \in B_{H,N,\ell}, w' \in B'_{H,N,\ell}}$$

be the matrix of $\partial^H_{N,\ell}$ with respect to the bases $B_{H,N,\ell}$ and $B'_{H,N,\ell}$. Here f'^w_w denotes the coefficient of $\widetilde{t}^{\,\mathfrak{m}}(w')$ in $\partial^H_{N,\ell} \widetilde{t}^{\,\mathfrak{m}}(w)$, and in the matrix w corresponds to rows, and w' to columns.

Example 8.9 For $N = 8$, $\ell = 2$, the matrix $M_{H,8,2}$ is as follows; the first row and column label the elements of $B'_{H,8,2}$ and $B_{H,8,2}$ respectively.

	1222	2122	122	2212	212	12	2221	221	21	1
11222	1	0	$-2c_{21}$	0	0	$-8c_{221}$	0	0	0	0
12122	0	1	$2d_{12}-2c_{21}$	0	0	$8c_{23}-8c_{32}$	0	0	0	0
21122	0	0	$2d_{21}$	0	$-2c_{21}$	0	0	0	0	0
12212	0	0	$2c_{21}$	1	$2d_{12}-2c_{21}$	$-8c_{23}+8c_{32}+8d_{122}$	0	0	0	0
21212	0	0	0	0	$2d_{21}$	$8d_{212}$	0	0	0	0
22112	0	0	0	0	$2c_{21}$	$8d_{221}$	0	0	0	0
12221	$-\frac{1}{2}$	0	$2c_{21}$	0	0	$8c_{221}$	1	$2d_{12}-2c_{21}$	$8d_{122}-8c_{221}$	$32d_{1222}$
21221	0	$-\frac{1}{2}$	0	0	$2c_{21}$	0	0	$2d_{21}-2c_{21}$	$8c_{23}-8c_{32}+8d_{212}$	$32d_{2122}$
22121	0	0	0	$-\frac{1}{2}$	0	0	0	$2c_{21}$	$-8c_{23}+8c_{32}+8d_{221}$	$32d_{2212}$
22211	0	0	0	0	0	0	$-\frac{1}{2}$	$2c_{21}$	$8c_{221}$	$32d_{2221}$

The entries 1 in the matrix arise from both the deconcatenation term $2\widetilde{t}^{\,\mathfrak{l}}(1)$ removing a leading 1 which appears in D_1 as per Proposition 5.9, the entries $-\frac{1}{2}$ correspond to the deconcatenation term $-\widetilde{t}^{\,\mathfrak{l}}(1)$ removing a trailing 1 which appear in D_1. With the projection $\log^{\mathfrak{l}}(2) \to \frac{1}{2}$, these combinations give 1 and $-\frac{1}{2}$ respectively.

After substituting the values for $c_\bullet$ and $d_\bullet$ using the formulae above, we obtain the matrix

	1222	2122	122	2212	212	12	2221	221	21	1
11222	1	0	4	0	0	−16	0	0	0	0
12122	0	1	−3	0	0	−80	0	0	0	0
21122	0	0	−7	0	4	0	0	0	0	0
12212	0	0	−4	1	−3	111	0	0	0	0
21212	0	0	0	0	−7	186	0	0	0	0
22112	0	0	0	0	−4	31	0	0	0	0
12221	$-\frac{1}{2}$	0	−4	0	0	16	1	−3	15	−127
21221	0	$-\frac{1}{2}$	0	0	−4	0	0	−3	106	−1905
22121	0	0	0	$-\frac{1}{2}$	0	0	0	−4	111	−1905
22211	0	0	0	0	0	0	$-\frac{1}{2}$	−4	16	−127

We notice already that the matrix is block *lower* triangular; the blocks correspond to the number of trailing ones in the quotients, which is also the number of trailing 1's when deconcatenating the maximal string $2^a 1$ from the start of the basis words. Each diagonal block except the last (i.e. here only the first, but in general all further intermediate ones too) is *upper* triangular modulo 2, so has determinant $\neq 0$. Expanding the determinant of the last block about its first column produces two matrices with integer entries, which are equivalent to triangular matrices mod 2, so the first block has determinant $\frac{\text{odd}}{2}$. Overall the full matrix has the same property: the determinant is in $\frac{1}{2} + \mathbb{Z}$.

We aim to show this is a general phenomenon for level $\ell \geq 1$. In fact, we shall show that $\partial^H_{N,\ell}$ never increases the number of trailing 1's, explaining the block triangular appearance; moreover, for a fixed number of trailing 1's we show that $\partial^H_{N,\ell}$ acts by deconcatenation, modulo 2, explaining the upper triangular appearance of each block. (Special care must be given for the first row, where an extra $-\frac{1}{2}$ is produced.)

We introduce a partition of the basis set $B'_{H,N,\ell}$ by the number of trailing 1's.

Definition 8.10 (*Trailing 1's*) Write

$$T'_{\alpha,N,\ell} := \{w' \in B'_{H,N,\ell} \mid w' \text{ has exactly } \alpha \text{ trailing 1's}\}.$$

More precisely, one can define "exactly α trailing 1's" as a word of the form 1^α or $w21^\alpha$, where $w \in \{1,2\}^\times$, to obtain

$$T'_{\alpha,N,\ell} = B'_{S,N,\ell} \cap \Big(\{1^\alpha\} \cup \{w21^\alpha \mid w \in \{1\,2\}^\times\} \Big).$$

We note that $T'_{\alpha,N,\ell} = \emptyset$ if $\alpha \geq \ell$ as a word of level $< \ell$ cannot contain ℓ trailing 1's. Whereas $T'_{\ell-1,N,\ell} = \{2^{\frac{1}{2}(N-\ell)} 1^{\ell-1}, \ldots, 21^{\ell-1}, 1^{\ell-1}\}$. So we certainly have as a disjoint union that

$$B'_{S,N,\ell} = \bigcup_{0\leq\alpha,N,\ell} T'_{\alpha,N,\ell}.$$

Now consider the bijection

$$\begin{aligned} \phi \colon B'_{S,N,\ell} &\to B_{H,N,\ell} \\ u &\mapsto 2^a 1u\,, \end{aligned}$$

where a is the unique value such that $2^a 1u$ has weight N. We pull back the partition $T'_{\alpha,N,\ell}$ to define

$$T_{\alpha,N,\ell} = \{w \in B'_{H,N,\ell} \mid \phi^{-1}(w) \in T'_{\alpha,N,\ell}\}.$$

Note that the inverse $\phi^{-1}(w)$ is obtained by taking the suffice when deconcatenating w after the first 1.

Lemma 8.11 *For $w \in B_{H,N,\ell}$, with $w \neq 2^{\frac{1}{2}(N-\ell)}1^\ell$ then the following holds. The word w has α trailing 1's if and only if*

$$w \in T_{\alpha,N,\ell}.$$

However, the word $w = 2^{\frac{1}{2}(N-\ell)}1^\ell$ lies in $T_{\ell-1,N,\ell}$.

Proof Firstly, we check the case $w = 2^{\frac{1}{2}(N-\ell)}1^\ell$. Deconcatenating at the first 1 tells us that $\phi(w) = 1^{\ell-1}$ which has exactly $\ell - 1$ trailing 1's. So $w \in T_{\ell-1,N,\ell}$ as claimed.

Now take any other word v of level ℓ. It cannot have 1^ℓ trailing 1's, so is of necessarily of the form $v'21^\alpha$, with $\alpha < \ell$, for some $v' \in \{1,2\}^\times$, where $\deg_1 v' \geq \ell - \alpha \geq 1$. This means the first 1 in v occurs somewhere in v'. Deconcatenating after this, gives a suffice of the form $v''21^\alpha$, so that $\phi(v) \in T'_{\alpha,N,\ell}$, meaning $v \in T_{\alpha,N,\ell}$ as claimed.

Conversely, given $v \in T_{\alpha,N,\ell}$, we know that $\phi^{-1}(v) \in T'_{\alpha,N,\ell}$, so that $\phi^{-1}(v) = v'21^\alpha$ or $\phi^{-1}(v) = 1^\alpha$, with $\alpha = \ell - 1$. The former case leads to $v = 2^a 1 v' 21^\alpha$

which ends in exactly α trailing 1's. The latter case leads to $v = 2^{\frac{1}{2}(N-\ell)}1^\ell$, which we already excluded. □

Now we claim that the map $\partial^H_{N,\ell}$ never increases the number of trailing 1's.

Lemma 8.12 *For all words $w \in T_{\alpha,N,\ell}$, the image under $\partial^H_{N,\ell}$ satisfies*

$$\partial^H_{N,\ell} w = \sum_{\substack{w' \in T_{\beta,N,\ell} \\ \beta \leq \alpha}} f^w_{w'} \widetilde{t}^{\mathfrak{m}}(w').$$

That is to say, $\partial^H_{N,\ell} w$ only involves words with $\leq \alpha$ trailing 1's.

Proof By Lemma 8.11, we know that the set $T_{\alpha,N,\ell}$ is characterised as the words $w \in B_{N,\ell}$ ending with α many trailing 1's, except that when $\alpha = \ell - 1$, where also include the word $2^{\frac{1}{2}(N-\ell)}1^\ell \in T_{\ell-1,N,\ell}$. (Note that since $T'_{\ell,N,\ell} = \emptyset$, the set $T_{\ell,N,\ell}$ would also be empty.)

Purely for level-filtration reasons (see Lemma 8.3), the word $w = 2^{\frac{1}{2}(N-\ell)}1^\ell$ must map to a sum of words with $< \ell$ trailing 1's. We may therefore assume $\alpha \leq \ell - 1$, and $w \in T_{\alpha,N,\ell}$ genuinely ends in 1^α.

We consider the terms which arise in $\partial^H_{N,\ell} \widetilde{t}^{\mathfrak{m}}(w)$ via the cases in Lemma 8.4 for $\mathrm{gr}_\ell D_{2r'+1} \widetilde{t}^{\mathfrak{m}}(w)$, with w viewed as a tuple. Take the deconcatenation term

$$\widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,j}) \otimes \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{j+1,d}).$$

Since $\underline{\mathbf{k}}_{j+1,d}$ is a suffix of $\underline{\mathbf{k}}$ it clearly has $\leq \alpha$ trailing 1's. (Either the cut $\underline{\mathbf{k}}_{1,j}$ ends before the first trailing 1, in which case we have exactly as many trailing 1's. Or the cut ends after this point, and we have even reduced the number of trailing 1's as $\underline{\mathbf{k}}_{j+1,d} = (\{1\}^\beta)$ with $\beta < \alpha$).

On the other hand, if $r = 1$, and we take the deconcatenation term

$$\delta_{k_d=1} \widetilde{t}^{\mathfrak{l}}(1) \otimes \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{d-1})$$

which occurs in D_1, then we have certainly removed a trailing 1 if this term is non-zero.

Now consider the cases as is the tables in Lemma 8.4, which come from replacing a subindex $\underline{\mathbf{k}}_{i,j}$ by $|\underline{\mathbf{k}}_{i,j}| - r$. In both cases we see the replacement is by a 2. So if $\underline{\mathbf{k}}_{i,j}$ ends before the trailing 1's, we do not increase their number. Otherwise $\underline{\mathbf{k}}_{i,j}$ ends within the string of trailing 1's, and some set 1^β, $\beta > 0$ of them are replaced by a 2, leaving strictly fewer trailing 1's, namely $1^{\alpha-\beta}$.

The block (lower) triangularity corresponds to the ordering, wherein we have the reverse colexicographic order (reading right to left, largest first), with $1 < 2$. So $1^\gamma > w_0 21^\alpha > w'_0 21^\beta$, for any $\beta < \alpha \leq \gamma$. □

This lemma has established that the matrix $M_{H,N,\ell}$ of $\partial^H_{N,\ell}$ is block (lower) triangular, and that the diagonal blocks are square. (We constructed $T_{\alpha,N,\ell}$ as the preimage of $T'_{\alpha,N,\ell}$ under a bijection of the bases.) We therefore reduction the question of injectivity of $\partial^H_{N,\ell}$, equivalently the determinant of $M_{H,N,\ell}$ being non-zero, to a question of understanding the determinants of these diagonal blocks.

Definition 8.13 For $\ell \geq 1$, $N \geq 1$, $\alpha \leq \ell - 1$, with $N \equiv \ell \pmod 2$, let

$$M_{\alpha,H,N,\ell} := \left(f^w_{w'}\right)_{w \in T_{\alpha,N,\ell}, w' \in T'_{\alpha,N,\ell}}$$

be the diagonal block of $M_{H,N,\ell}$ corresponding to α trailing 1's (after deconcatenating $2^a 1$ for $T_{\alpha,N,\ell}$, or immediately for $T'_{\alpha,N,\ell}$).

Lemma 8.14 *For $\alpha < \ell - 1$, the restriction of $\partial_{N,\ell}$ to $T_{\alpha,N,\ell}$, and projected to the $T'_{\alpha,N,\ell}$ satisfies the following. For $w \in T_{\alpha,N,\ell}$, the coefficient of every word $\widetilde{t}^{\mathfrak{m}}(u)$ with $u \in T'_{\alpha,N,\ell}$, in*

$$\partial^H_{N,\ell} \widetilde{t}^{\mathfrak{m}}(w) - \sum_{\substack{w=uv \\ \deg_1 u = 1}} 2^{|u|-2} d_u \widetilde{t}^{\mathfrak{m}}(v)$$

is an even integer.

Proof Since $\alpha < \ell - 1$ we know $w \neq 2^{\frac{1}{2}(N-\ell)} 1^\ell$, so we do not have to worry about the deconcatenation term $\widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{d-1})$ contributing: it has strictly fewer than $\ell - 1$ trailing 1's.

The terms in

$$\sum_{\substack{w=uv \\ \deg_1 u = 1}} 2^{|u|-2} d_u \widetilde{t}^{\mathfrak{m}}(v)$$

are the deconcatenation terms from Lemma 8.4, after using $\widetilde{\pi}_{|u|} = \widetilde{\pi}_{2a+2b+1}$ to project the factor $\widetilde{t}^{\mathfrak{l}}(u) = \widetilde{t}^{\mathfrak{l}}(\{2\}^a, 1, \{2\}^b) = d_u \zeta^{\mathfrak{l}}(2a + 2b + 1)$.

The remaining terms (whether or not they have fewer trailing 1's), arise from the the (28) and (29) terms (or rather their images in $\mathrm{gr}^H_\ell D_{2r+1}$). They are categorised by the cases listed in Lemma 8.4, namely

$\lvert\underline{\mathbf{k}}_{i,j}\rvert - r$	$\underline{\mathbf{k}}_{i,j}$	Contribution to D_r
2	$(2, \{2\}^a, 1, \{2\}^b)$	$\zeta^{\mathfrak{l}}_{2-2}(\{2\}^a, 1, \{2\}^b) \otimes \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$
2	$(\{2\}^a, 1, \{2\}^b, 2)$	$-\zeta^{\mathfrak{l}}_{2-2}(\{2\}^b, 1, \{2\}^a) \otimes \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}_{1,i-1}, 2, \underline{\mathbf{k}}_{j+1,d})$

In each case, the coefficient of $\widetilde{t}^{\mathfrak{m}}(v)$ after projecting via $\widetilde{\pi}_{2r+1}$ has one of the following forms (depending on whether $b = 0$ or $b > 0$ in case 1, likewise case 2)

$$2^{2a-1}c_{2^a 1} = \begin{cases} 0 & a = 0 \\ 2^{2a}(-1)^a & a > 0 \end{cases}$$

$$2^{2a+2b+1}c_{2^a 3 2^b} =$$
$$(-1)^{a+b}\left(-2^{2a+2b+2}\binom{2a+2b+2}{2a+2} + (2^{2a+2b+2}-1)\binom{2a+2b+2}{2b+1}\right).$$

This is exactly the same claim as in (37) and (38), so as before these coefficients are indeed even integers. □

In the same manner as Theorem 7.14, it follows that the diagonal block $M_{\alpha,H,N,\ell}$ corresponding to $T_{\alpha,N,\ell}$ and $T'_{\alpha,N,\ell}$ is invertible, for $\alpha < \ell - 1$.

Proposition 8.15 *The diagonal block $M_{\alpha,H,N,\ell}$ is invertible for $\alpha < \ell - 1$, in particular it has non-zero determinant.*

Proof Recall the bijection ϕ between the bases, sending $u \in B'_{H,N,\ell}$ to $2^a 1u$, for the unique a such that $2^a 1u$ has weight N. This map defined the set $T_{\alpha,N,\ell}$ as the preimage of $T'_{\alpha,N,\ell}$.

By the previous result, Lemma 8.14, we know that $M_{\alpha,H,N,\ell}$ is upper triangular modulo 2, the terms above the diagonal arising from the deconcatenation terms. That is to say: the diagonal entries of $M_{\alpha,H,N,\ell}$ are of the form $f_u^{2^r 1u} = 2^{2a-1}d_{2^a 1} + 2n$, $n \in \mathbb{Z}$. The only other non-even entries in the column indexed by $u \in B'_{\alpha,H,N,\ell}$ occur for rows indexed by $w = 2^a 12^b u$, however since $2^r 1u < 2^a 12^b u$, these occur above the diagonal.

As the diagonal terms are given by

$$2^{2a-1}d_{2^a 1} = (-1)^a(2^{2a+1} - 1) \equiv 1 \pmod 2,$$

we see that $f_u^{2^r 1u} \equiv 1 \pmod 2$. Therefore $M_{\alpha,H,N,\ell}$ is upper triangular, modulo 2, and has 1's on the diagonal. It therefore has determinant $\equiv 1 \pmod 2$, and so $M_{\alpha,H,N,\ell}$ has non-zero determinant. □

We now turn to the case $\alpha = \ell - 1$. In this case the following modification of Lemma 8.14 holds.

Lemma 8.16 *For $\alpha = \ell - 1$, the restriction of $\partial_{N,\ell}$ to $T_{\alpha,N,\ell}$, and projected to the $T'_{\alpha,N,\ell}$ satisfies the following. For $w \in T_{\alpha,N,\ell}$, the coefficient of every word $\widetilde{t}^{\mathfrak{m}}(u)$ with $u \in T'_{\alpha,N,\ell}$, in*

$$\partial^H_{N,\ell}\widetilde{t}^{\mathfrak{m}}(w) - \sum_{\substack{w=uv \\ \deg_1 u = 1}} 2^{|u|-2}d_u \widetilde{t}^{\mathfrak{m}}(v) + \frac{1}{2}\widetilde{t}^{\mathfrak{m}}(2^{\frac{1}{2}(N-\ell)}1^{\ell-1})$$

is an even integer.

Proof The proof of Lemma 8.14 goes through unchanged, except that we must also consider the case $w \neq 2^{\frac{1}{2}(N-\ell)}1^{\ell}$. Even for this, the argument about the deconcatenation (27) and other terms in Lemma,8.4 goes through unchanged.

The only additional term we must consider is the term arising from deconcatenating a trailing 1, namely $-\log^{\mathfrak{l}}(1) \otimes \widetilde{\mathcal{I}}^{\mathfrak{m}}(k_1, \ldots, k_{d-1})$ which appears in Proposition 5.9. This is the additional term above, and so the proof is complete. □

We note now that this additional term occurs in the first column, last row of the matrix $M_{\alpha,H,N,\ell}$ because the word indexing the column is $2^{\frac{1}{2}(N-\ell)}1^{\ell-1} > \cdots 21^{\ell-1} > 1^{\ell-1}$ while the word indexing the row is $2^{\frac{1}{2}(N-\ell)}1^{\ell} < w21^{\ell-1}$, for any w.

Finally we can show that the diagonal block $M_{\ell-1,H,N,\ell}$ is also invertible, or equivalently has non-zero determinant.

Proposition 8.17 *The diagonal block $M_{\alpha,H,N,\ell}$ is invertible for $\alpha = \ell-1$, in particular it has non-zero determinant.*

Proof The above observation tells us that the first column of the matrix $M_{\alpha,H,N,\ell}$, $\alpha = \ell - 1$, consists of a single entry $\frac{1}{2}$ at the bottom, a single entry $\frac{1}{2}d_1 = 1$ at the top, and (potentially) a number of even entries. However, since this column is indexed by $2^{\frac{1}{2}(N-\ell)}1^{\ell-1}$, this column corresponds to the computation of $\operatorname{gr}_\ell D_1$. Therefore there are no other entries in this column since $\zeta_0^{\mathfrak{l}}(1) = 0$. Now expand out the determinant about this column.

The minor $A_{1,1}$ corresponding to the $(1, 1)$ entry of $M_{\alpha,H,N,\ell}$ is again an upper triangular matrix modulo 2, as it arises from deleting the first row and column of $M_{\alpha,H,N,\ell}$. That matrix is itself integral and upper triangular modulo 2, after removing the entry $-\frac{1}{2}$ in the first column (compare the argument in Lemma 8.14 and Lemma 8.16). So as entry $-\frac{1}{2}$ plays no role in the $(1, 1)$ cofactor, the integrality and upper triangularity modulo 2 holds. Likewise the diagonal entries are equal to 1, and so we find $C_{1,1} = \det A_{1,1} \equiv 1 \pmod 2$. This means $C_{1,1} = 2x + 1 \in \mathbb{Z}$ is an odd integer.

The minor $A_{1,\frac{1}{2}(N-\ell)+1}$ corresponding to the bottom entry of the first column, is given by the an explicit formula modulo 2. We note the rows are indexed by $2^{a-1}12^{\frac{1}{2}(N-\ell)+1-a}1^{\ell-1}$, for $1 \leq a \leq \frac{1}{2}(N-\ell)$, so that $2^{\frac{1}{2}(N-\ell)}1^{\ell}$ is avoided. Likewise the columns are indexed by $2^{\frac{1}{2}(N-\ell)-c}1^{\ell-1}$, for $1 \leq c \leq \frac{1}{2}(N-\ell)$, so that $2^{\frac{1}{2}(N-\ell)}1^{\ell-1}$ is avoided. This means that modulo 2, the minor is given by

$$
\begin{aligned}
\left(A_{1,\frac{1}{2}(N-\ell)+1}\right)_{a,c=1}^{\frac{1}{2}(N-\ell)} &= 2^{2c-1} d_{2^{a-1}12^{c+1-a}} \\
&= (-1)^c (2^{2c+1} - 1) \binom{2c}{2a-2} \\
&\equiv \binom{2c}{2a-2} \pmod 2.
\end{aligned}
$$

We notice the following: when $a = c + 1$,

$$
\binom{2c}{2a-2} = 1
$$

so the minor has 1's on the subdiagonal. When $a = 1$

$$\binom{2c}{2a-2} = 1,$$

so the minor has 1's in the first row. Summing down each column we note that

$$\sum_{a=2}^{\frac{1}{2}(N-\ell)} \binom{2c}{2a-2} = \begin{cases} 2^{2c-1} - 1 & 1 \le c < \frac{1}{2}(N-\ell) \\ 2^{2c-1} - 2 & c = \frac{1}{2}(N-\ell), \end{cases}$$

since in the latter case the term $\binom{2c}{2c}$ on the subdiagonal is not part of the matrix. This means that if we subtract the sum of the remaining rows of the minor from the first row, we obtain a single 1 in the final column. This establishes that the modulo 2 the minor is equivalent a permutation of an upper triangular matrix with 1's on the diagonal (move the last column to the start). Hence $C_{1,\frac{1}{2}(N-\ell)+1} = \det(A_{1,\frac{1}{2}(N-\ell)+1}) \equiv 1 \pmod 2$, and since $A_{1,\frac{1}{2}(N-\ell)+1}$ actually has integer entries, we have $C_{1,\frac{1}{2}(N-\ell)+1} = 2y+1 \in \mathbb{Z}$ is an odd integer.

Finally, we assemble the determinant of $M_{\alpha,H,N,\ell}$ to be

$$\begin{aligned} &(-1)^{1+1} \cdot C_{1,1} + (-1)^{(\frac{1}{2}(N-\ell)+1)+1} C_{1,\frac{1}{2}(N-\ell)+1} \\ &= (2x+1) + \frac{1}{2}(2y+1) \\ &= (2x+y+1) + \frac{1}{2}\,. \end{aligned}$$

In particular it is in $\frac{1}{2} + \mathbb{Z}$, and so cannot be 0. □

From these two propositions follows immediately the invertibility of the whole matrix $M_{H,N,\ell}$.

Corollary 8.18 *The matrix $M_{H,N,\ell}$ is invertible.*

Proof The matrix $M_{H,N,\ell}$ is block upper triangular by Lemma 8.12 and the discussion thereafter. The diagonal blocks are invertible square matrices by Proposition 8.15 and Proposition 8.17, hence $M_{H,N,\ell}$ itself is invertible. □

Corollary 8.19 *The Hoffman one-two elements*

$$\{\widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d) \mid k_i \in \{1, 2\}\}$$

are linearly independent.

Proof We proceed by induction on the level, as in [4, Theorem 7.4], [22, Corollary 38]. The elements of level $\ell = 0$ are of the form $\widetilde{t}^{\mathfrak{m}}(\{2\}^n)$, which are linearly independent because weight is a grading on $\mathcal{H}^{(2)}$. Now suppose the elements

$$\{\widetilde{t}^{\mathfrak{m}}(w) \mid w \in \{1, 2\}^{\times}, \deg_1 w \le \ell - 1\},$$

of level $\leq \ell - 1$ are linearly independent. Since weight is a grading on $\mathcal{H}^{(2)}$, any non-trivial linear relation between elements of level ℓ can be assumed as homogeneous of some weight N. By Corollary 8.18, the map $\partial^{H}_{N,\ell}$ is injective as the matrix of the map is invertible. Application of $\partial^{N,\ell}_{H}$ to a non-trivial linear relation between level ℓ elements produces a non-trivial linear relation of strictly smaller level, which does not exist by the induction assumption. So the elements of level ℓ are also linearly independent, which completes the proof by induction. □

Corollary 8.20 *The elements*

$$\{\widetilde{t}^{\mathfrak{m}}(w) \mid w \in \{1,2\}^{\times}\},$$

form a basis for the space of:

(i) *motivic extended shuffle-regularised multiple t values,*
(ii) *alternating (shuffle-regularised) motivic multiple zeta values*

In particular these spaces agree, and extended shuffle-regularised motivic multiple t values have dimension F_{N+1} in weight N, where $F_k = F_{k-1} + F_{k-2}$ with $F_1 = F_2 = 1$ is the sequence of Fibonacci numbers.

Proof From their definition as sums of alternating motivic MZV's, we know the following inclusion holds

$$\mathcal{T}^{\text{ext}}_{N} \subset \mathcal{H}^{(2)}_{N},$$

where $\mathcal{T}^{\text{ext}}_{N}$ denote the space of all shuffle-regularised motivic MtV's of weight N. However the upper bound $\dim_{\mathbb{Q}} \mathcal{H}^{(2)} \leq F_{N+1}$ is established in [7] (in fact already $=$), and the lower bound $F_{N+1} \leq \mathcal{T}^{\text{ext}}_{N}$ from the explicit collection of independent elements shows that all of these inclusions are equalities and the dimensions is exactly F_{N+1} in weight N. □

8.1 Stuffle regularised Hoffman one-two elements

We now wish to extend the independence result on the Hoffman one-two elements from the case of shuffle regularised MtV's to the more natural case of stuffle regularised MtV's. On the motivic level, we shall do this by viewing Proposition 2.18 as a definition.

Definition 8.21 (*Motivic $\widetilde{t}^{\mathfrak{m},*,V}$*) Let $\underline{\mathbf{k}} = (k_1, \ldots, k_d)$, such that $k_d \neq 1$. Then the stuffle regularised motivic MtV with $\widetilde{t}^{\mathfrak{m},*,V}(1) = 2V \in \mathcal{H}^{(2)}$ is defined by

$$\widetilde{t}^{\mathfrak{m},*,V}(\underline{\mathbf{k}}, \{1\}^{\alpha}) := \sum_{i=0}^{\alpha} \widetilde{t}^{\mathfrak{m}}(\underline{\mathbf{k}}, \{1\}^{\alpha-i}) \cdot \zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^{i}), \tag{39}$$

where $\zeta^{\mathfrak{m},*,U}(\{1\}^i)$ is given by the coefficient of u^i in

$$\exp\Big(Uu - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta^{\mathfrak{m}}(n) u^n\Big).$$

Regardless then of the technicalities of defining a stuffle-regularisation on the motivic level, one knows that

$$\operatorname{per} \widetilde{t}^{\,\mathfrak{m},*,V}(\underline{\mathbf{k}}, \{1\}^\alpha) = \widetilde{t}^{\,*,V}(\underline{\mathbf{k}}, \{1\}^\alpha).$$

In particular

$$\begin{aligned}\operatorname{per} \widetilde{t}^{\,\mathfrak{m},*,V}(1) &= \widetilde{t}^{\,*,V}(1) + \zeta^{*,2V-\log(2)}(1) = \log(2) + (2V - \log(2)) \\ &= 2V = 2t^{V,*}(1) = \widetilde{t}^{\,V,*}(1)\,,\end{aligned}$$

as per the definition of $\widetilde{t}$. So $\widetilde{t}^{\,\mathfrak{m},*,V}$ corresponds to the regularisation of $t^{*,V}(1) = V$. Therefore any linear independence and basis results will successfully translate over to the classical real valued versions as spanning set results, along with whatever identities we establish motivically.

Naturally the question of how to compute $D_{2r+1} \widetilde{t}^{\,\mathfrak{m},*,V}(\underline{\mathbf{k}}, \{1\}^\alpha)$ now arises, but for this we appeal again to the derivation property of D_{2r+1}, namely

$$D_{2r+1}(XY) = (1 \otimes Y) D_{2r+1} X + (1 \otimes X) D_{2r+1} Y.$$

We first give a lemma about the action of D_{2r+1} on $\zeta^{\mathfrak{m},*,U}(\{1\}^i)$.

Lemma 8.22 *The action of D_{2r+1} on $\zeta^{\mathfrak{m},*,U}(\{1\}^i)$, with $U \in \mathcal{H}^{(2)}$, is given by*

$$D_{2r+1}\zeta^{\mathfrak{m},*,U}(\{1\}^i) = \begin{cases} \zeta^{\mathfrak{l},*,U}(\{1\}^{2r+1}) \otimes \zeta^{\mathfrak{m},*,U}(\{1\}^{i-(2r+1)}) & \text{if } i \geq 2r+1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof The case $2r + 1 > i$ is clear, as in this case there can be no (non-zero) weight $i - (2r + 1)$ factor in the right hand factor of D_{2r+1}. So we assume $2r + 1 \leq i$.

We know that $D_{2r+1}\zeta^{\mathfrak{m}}(s) = \delta_{s=2r+1}\zeta^{\mathfrak{l}}(2r + 1) \otimes 1$ since $\zeta^{\mathfrak{m}}(s)$ is primitive for the coaction Δ. More generally, if $k_i \neq 2r + 1$, for any $1 \leq i \leq n$, then

$$\begin{aligned}&D_{2r+1}\zeta^{\mathfrak{m}}(2r + 1)^\ell \zeta^{\mathfrak{m}}(k_1) \cdots \zeta^{\mathfrak{m}}(k_n) \\ &\quad = \zeta^{\mathfrak{l}}(2r + 1) \otimes \ell \zeta^{\mathfrak{m}}(2r + 1)^{\ell-1} \zeta^{\mathfrak{m}}(k_1) \cdots \zeta^{\mathfrak{m}}(k_n)\,.\end{aligned}$$

So, when acting on a polynomial $p(\zeta^{\mathfrak{m}}(2r+1), \zeta^{\mathfrak{m}}(k_1), \ldots, \zeta^{\mathfrak{m}}(k_n))$ in single motivic zeta values, the right hand tensor factor is (formally) the derivative of $p(\zeta^{\mathfrak{m}}(2r + 1), \zeta^{\mathfrak{m}}(k_1), \ldots,$
$\zeta^{\mathfrak{m}}(k_n))$ with respect $\zeta^{\mathfrak{m}}(2r + 1)$, and the left hand tensor factor is simply $\zeta^{\mathfrak{l}}(2r + 1)$.

More rigorously, the right hand factor of action of D_{2r+1} mimics the action of $\frac{\mathrm{d}}{\mathrm{d}z_{2r+1}}$ on the polynomial $p(z_{2r+1}, z_{k_1}, \ldots, z_{k_n})$, under the correspondence $\zeta^{\mathfrak{m}}(m) \leftrightarrow z_m$. So we are justified now in proceeding via this formal derivative with respect to $\zeta^{\mathfrak{m}}(2r+1)$.

If $2r+1 > 1$, applying this formal differentiation operation to

$$\sum_{i=0}^{\infty} \zeta^{\mathfrak{m},*,U}(\{1\}^i)u^i = \exp\Big(Uu - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta^{\mathfrak{m}}(n)u^n\Big), \tag{40}$$

viewed as a generating series, leads to the following (extending D_{2r+1} by linearity to the coefficients of a power series):

$$\begin{aligned} &D_{2r+1} \sum_{i=0}^{\infty} \zeta^{\mathfrak{m},*,U}(\{1\}^i)u^i \\ &= \zeta^{\mathfrak{l}}(2r+1) \otimes \frac{\mathrm{d}}{\mathrm{d}\zeta^{\mathfrak{m}}(2r+1)} \exp\Big(Uu - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta^{\mathfrak{m}}(n)u^n\Big) \\ &= \zeta^{\mathfrak{l}}(2r+1) \otimes \frac{(-1)^{2r}}{2r+1} u^{2r+1} \exp\Big(Uu - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta^{\mathfrak{m}}(n)u^n\Big) \end{aligned}$$

So by comparing the coefficient of u^i on both sides, we obtain

$$D_{2r+1}\zeta^{\mathfrak{m},*,U}(\{1\}^i) = \frac{1}{2r+1} \zeta^{\mathfrak{l}}(2r+1) \otimes \zeta^{\mathfrak{m},*,U}(\{1\}^{i-(2r+1)}).$$

It remains to note that

$$\zeta^{\mathfrak{l},*,U}(\{1\}^{2r+1}) = \frac{1}{2r+1} \zeta^{\mathfrak{l}}(2r+1),$$

by extracting the irreducible contribution in (40).

The corresponding result holds for $2r+1 = 1$, mutatis mutandis, by the view that $\zeta^{\mathfrak{m},*,U}(1) = U \in \mathcal{H}^{(2)}$. So in particular $\zeta^{\mathfrak{m},*,U}(1)$ is some rational multiple $\lambda\zeta^{\mathfrak{m}}(\overline{1})$ of $\zeta^{\mathfrak{m}}(\overline{1}) = -\log^{\mathfrak{m}}(2)$, and so primitive for the coaction. Namely $D_1 U = D_1 \lambda \log^{\mathfrak{m}}(2) = \lambda(\log^{\mathfrak{l}}(2) \otimes 1) = (U)^{\mathfrak{l}}$. Likewise, $\zeta^{\mathfrak{l},*,U}(1) = (U)^{\mathfrak{l}}$, so the left hand tensor factor is also just $\zeta^{\mathfrak{l},*,U}(1)$ in this case. □

Then we compute the derivation D_{2r+1} on the stuffle-regularised motivic MtV's as follows. We claim it is given by the essentially same formula as in Proposition 5.7, with $\widetilde{t}^{\bullet}$ replaced by $\widetilde{t}^{\bullet,*,V}$, and a (potential) additional term deconcatenating 1's from the end.

Proposition 8.23 (Derivation D_r on $\widetilde{t}^{\mathfrak{m},*,V}$) *Let* $\underline{\mathbf{k}} = (k_1, \ldots, k_d) \in (\mathbb{Z}_{\geq 1})^d$ *be an index. Write* $\underline{\mathbf{k}}_{i,j} = (k_i, \ldots, k_j)$ *for a subindex of* $\underline{\mathbf{k}}$ *and* $|(a_1, \ldots, a_r)| = a_1 + \cdots + a_r$ *for the total (weight) of an index. Then the derivation* D_r*, r odd, is computed on the stuffle regularised* $\widetilde{t}^{\mathfrak{m},*,V}$ *as follows*

$$D_r\big(\widetilde{t}^{\,\mathfrak{m},*,V}(k_1,\ldots,k_d)\big) = \sum_{1\le j\le d} \delta_{|\underline{\mathbf{k}}_{1,j}|=r}\, \widetilde{t}^{\,\mathfrak{l}}(k_1,\ldots,k_j) \otimes \widetilde{t}^{\,\mathfrak{m},*,V}(k_{j+1},\ldots,k_d) \tag{41}$$

$$+ \sum_{1\le i<j\le d} \delta_{|\underline{\mathbf{k}}_{i+1,j}|\le r<|\underline{\mathbf{k}}_{i,j}|-1}\big(\zeta^{\mathfrak{l}}_{r-|\underline{\mathbf{k}}_{i+1,j}|}(k_{i+1},\ldots,k_j) - \delta_{r=1}\log^{\mathfrak{l}}(2)\big) \otimes \widetilde{t}^{\,\mathfrak{m},*,V}(k_1,\ldots,k_{i-1},|\underline{\mathbf{k}}_{i,j}|-r,k_{j+1},\ldots,k_d) \tag{42}$$

$$- \sum_{1\le i<j\le d} \delta_{|\underline{\mathbf{k}}_{i,j-1}|\le r<|\underline{\mathbf{k}}_{i,j}|-1}\big(\zeta^{\mathfrak{l}}_{r-|\underline{\mathbf{k}}_{i,j-1}|}(k_{j-1},\ldots,k_i) - \delta_{r=1}\log^{\mathfrak{l}}(2)\big) \otimes \widetilde{t}^{\,\mathfrak{m},*,V}(k_1,\ldots,k_{i-1},|\underline{\mathbf{k}}_{i,j}|-r,k_{j+1},\ldots,k_d) \tag{43}$$

$$+ \delta_{(k_{d-r},\ldots,k_d)=(1,\ldots,1)} \cdot \zeta^{\mathfrak{l},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^r) \otimes \widetilde{t}^{\,\mathfrak{m},*,V}(k_1,\ldots,k_{d-r}) \tag{44}$$

Proof We treat this based on the number of trailing 1's in the $\underline{\mathbf{k}}$. Write $\underline{\mathbf{k}} = (k_1,\ldots,k_{d-\alpha},\{1\}^\alpha)$, with $k_{d-\alpha}\ne 1$, and apply the derivation property to

$$\begin{aligned} &D_r\widetilde{t}^{\,\mathfrak{m},*,V}(k_1,\ldots,k_{d-\alpha},\{1\}^\alpha) \\ &= \sum_{\ell=0}^{\alpha} D_r\widetilde{t}^{\,\mathfrak{m}}(k_1,\ldots,k_{d-\alpha},\{1\}^{\alpha-\ell}) \cdot \zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell) \\ &= \sum_{\ell=0}^{\alpha} \Big\{\big(1\otimes\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell)\big)\cdot D_r\widetilde{t}^{\,\mathfrak{m}}(k_1,\ldots,k_{d-\alpha},\{1\}^{\alpha-\ell}) \\ &\quad + \big(1\otimes\widetilde{t}^{\,\mathfrak{m}}(k_1,\ldots,k_{d-\alpha},\{1\}^{\alpha-\ell})\big)\cdot D_r\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell)\Big\}. \end{aligned}$$

We compute the second term of the sum to be

$$\begin{aligned} &(1\otimes\widetilde{t}^{\,\mathfrak{m}}(k_1,\ldots,k_{d-\alpha},\{1\}^{\alpha-\ell}))\cdot D_r\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell) \\ &= \big(1\otimes\widetilde{t}^{\,\mathfrak{m}}(k_1,\ldots,k_{d-\alpha},\{1\}^{\alpha-\ell})\big) \\ &\quad \cdot\big(\delta_{r\le\ell}\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^r)\otimes\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^{\ell-r})\big) \\ &= \zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^r) \\ &\quad \otimes\big(\delta_{r\le\ell}\widetilde{t}^{\,\mathfrak{m}}(k_1,\ldots,k_{d-\alpha},\{1\}^{\alpha-\ell})\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^{\ell-r})\big). \end{aligned}$$

The sum $\sum_{\ell=0}^{\alpha}$ then restricts to $\sum_{\ell=r}^{\alpha}$ because of the Kronecker delta, so we find

$$\begin{aligned} &\sum_{\ell=0}^{\alpha}(1\otimes\widetilde{t}^{\,\mathfrak{m}}(k_1,\ldots,k_{d-\alpha},\{1\}^{\alpha-\ell}))\cdot D_r\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell)\big) \\ &= \zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^r)\otimes\sum_{\ell=r}^{\alpha}\widetilde{t}^{\,\mathfrak{m}}(k_1,\ldots,k_{d-\alpha},\{1\}^{\alpha-\ell})\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^{\ell-r})\big) \\ &= \delta_{r\le\alpha}\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^r)\otimes\widetilde{t}^{\,\mathfrak{m},V,*}(k_1,\ldots,k_{d-\alpha},\{1\}^{\alpha-r}) \end{aligned}$$

This gives the last term (44).

Now consider the first term of the sum. We need to apply the previous formula from Proposition 5.7 for $D_r \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{d-\alpha}, \{1\}^\alpha)$. We obtain

$$\sum_{\ell=0}^{\alpha} \big(1 \otimes \zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell)\big) \cdot D_r \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{d-\alpha}, \{1\}^{\alpha-\ell}) = \tag{45}$$

$$\sum_{\ell=0}^{\alpha} \Bigg\{ \sum_{1 \leq j \leq d-\ell} \delta_{|\underline{\mathbf{k}}_{1,j}|=r} \widetilde{t}^{\mathfrak{l}}(k_1, \ldots, k_j) \otimes \widetilde{t}^{\mathfrak{m}}(k_{j+1}, \ldots, k_{d-\ell}) \zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell)$$

$$+ \sum_{1 \leq i < j \leq d-\ell} \delta_{|\underline{\mathbf{k}}_{i+1,j}| \leq r < |\underline{\mathbf{k}}_{i,j}|-1} \big(\zeta^{\mathfrak{l}}_{r-|\underline{\mathbf{k}}_{i+1,j}|}(k_{i+1}, \ldots, k_j) - \delta_{r=1} \log^{\mathfrak{l}}(2)\big)$$

$$\otimes \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{i-1}, |\underline{\mathbf{k}}_{i,j}| - r, k_{j+1}, \ldots, k_{d-\ell}) \zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell)$$

$$- \sum_{1 \leq i < j \leq d-\ell} \delta_{|\underline{\mathbf{k}}_{i,j-1}| \leq r < |\underline{\mathbf{k}}_{i,j}|-1} \big(\zeta^{\mathfrak{l}}_{r-|\underline{\mathbf{k}}_{i,j-1}|}(k_{j-1}, \ldots, k_i) - \delta_{r=1} \log^{\mathfrak{l}}(2)\big)$$

$$\otimes \widetilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{i-1}, |\underline{\mathbf{k}}_{i,j}| - r, k_{j+1}, \ldots, k_{d-\ell}) \zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell) \Bigg\}$$

The sum over ℓ and over $i < j$ (respectively j) interchange as follows

$$\sum_{\ell=0}^{\alpha} \sum_{1 \leq i < j \leq d-\ell} = \sum_{1 \leq i < j \leq d} \sum_{\ell=0}^{\min(\alpha, d-j)},$$

$$\sum_{\ell=0}^{\alpha} \sum_{1 \leq j \leq d-\ell} = \sum_{1 \leq j \leq d} \sum_{\ell=0}^{\min(\alpha, d-j)}.$$

We than note that the upper bound of the ℓ-summation is given exactly by the total number of trailing 1's contained across the $\widetilde{t}^{\mathfrak{m}}$ and $\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}$ arguments. No further 1's can be introduced, as $|\underline{\mathbf{k}}_{i,j}| - r > 1$ which follows immediately from the inequality $r < |\underline{\mathbf{k}}_{i,j}| - 1$ in the Kronecker deltas. Then if, for example, $j = d - \alpha$, then $k_{j+1} = 1$ while $k_j = k_{d-\alpha} \neq 1$, so the subindex $(k_{j+1}, \ldots, k_{d-\ell}) = (k_{d-\alpha+1}, \ldots, k_{d-\ell})$ consists of $\alpha - \ell$ many 1's. And indeed $\alpha - \ell$ from $\widetilde{t}^{\mathfrak{m}}$ and ℓ from $\zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}$ give α overall, equal to $\min(\alpha, d-j) = \alpha$. Whereas if $j = d - \alpha + 1$, we already remove the first $1 = k_j$ from the subindex, leaving $\alpha - 1 - \ell$ many 1's in $\widetilde{t}^{\mathfrak{m}}$ and $\alpha - 1$ overall, agreeing with $\min(\alpha, d-j) = \alpha - 1$.

This means that after summing over ℓ we obtain the corresponding $\widetilde{t}^{\mathfrak{m},*,V}$ value in each case, namely

$$\sum_{\ell=0}^{\min(\alpha, d-j)} \widetilde{t}^{\mathfrak{m}}(k_{j+1}, \ldots, k_{d-\ell}) \zeta^{\mathfrak{m},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^\ell) = \widetilde{t}^{\mathfrak{m},*,V}(k_{j+1}, \ldots, k_{d-\ell}),$$

and likewise for the other terms, as per Definition 8.21. The sum over ℓ only affects the right hand tensor factors, as the left hand ones are independent of ℓ, so we readily obtain the remaining terms (41), (42) and (43) from the three summands in (45). This completes the proof. □

Now fix $V = \lambda \log^{\mathfrak{m}}(2)$, $\lambda \in \mathbb{Q}$. One can then proceed in the same way as Lemma 8.3 and Lemma 8.4 to conclude that the Hoffman-stuffle filtration

$$\mathcal{H}^{H,*} := \langle t^{\mathfrak{m},*,\lambda \log^{\mathfrak{m}}(2)}(w) \mid w \in \{1,2\}^\times \rangle_{\mathbb{Q}}$$
$$H_{\ell,*}\mathcal{H}^{H,*} := \langle t^{\mathfrak{m},*,\lambda \log^{\mathfrak{m}}(2)}(w) \mid w \in \{1,2\}^\times \,,\ \text{s.t. } \deg_1 w \leq \ell \rangle_{\mathbb{Q}} \,.$$

is motivic, of a particular form. Namely

$$\mathrm{gr}_\ell^{H,*} D_{2r'+1}(\mathrm{gr}_\ell^{H,*} \mathcal{H}^{H,*}) \subseteq \zeta^{\mathfrak{l}}(\overline{2r'+1})\mathbb{Q} \otimes_{\mathbb{Q}} \mathrm{gr}_{\ell-1}^{H,*} \mathcal{H}^{H,*}.$$

In particular: terms (41),(42),(43) give exactly the same contributions as previously, except for replacing $\widetilde{t}^{\,\mathfrak{m}}$ with $\widetilde{t}^{\,\mathfrak{m},*,V}$ in the *right* hand tensor factor. Finally

$$\zeta^{\mathfrak{l},*,2V-\log^{\mathfrak{m}}(2)}(\{1\}^{2r'+1}) = \begin{cases} (2\lambda-1)\log^{\mathfrak{l}}(2) & \text{if } 2r'+1 = 1 \\ \frac{1}{2r'+1}\zeta^{\mathfrak{l}}(2r'+1) & \text{if } 2r'+1 > 1, \end{cases}$$

and the right hand factor $\delta_{(k_{d-r},\ldots,k_d)=(1,\ldots,1)} \widetilde{t}^{\,\mathfrak{m},*,V}(k_1,\ldots,k_{d-r})$ is obtained by removing $2r'+1$ ones from the end of the t value. So the contribution from (44) lands in the space $\zeta^{\mathfrak{l}}(\overline{2r'+1})\mathbb{Q} \otimes_{\mathbb{Q}} \mathrm{gr}_{\ell-1}^{H,*} \mathcal{H}^{H,*}$. In fact, if $2r'+1 > 1$, we remove at least 3 ones, and so reduce the level by 3, which vanishes in $\mathrm{gr}_{\ell-1}^{H,*} \mathcal{H}^{H,*}$. Whereas, if $2r'+1 = 1$, this term contributes

$$\delta_{k_d=1}(2\lambda-1)\log^{\mathfrak{l}}(2) \otimes \widetilde{t}^{\,\mathfrak{m},*,V}(k_1,\ldots,k_{d-1}).$$

Note also, this is the only place the regularisation parameter λ enters the calculation.

In particular, the additional $\delta_{k_d=1}(2\lambda-1)\widetilde{\pi}^1\big(\log^{\mathfrak{l}}(2)\big) \cdot \widetilde{t}^{\,\mathfrak{m},*,V}(k_1,\ldots,k_{d-1})$ term combines with the original term

$$-\delta_{k_d=1}\widetilde{\pi}_1\big(\log^{\mathfrak{l}}(2)\big)\widetilde{t}^{\,\mathfrak{m},*,V}(k_1,\ldots,k_{d-1})$$

coming from deconcatenating at the end. (This arises in (43); by the same argument as in Proposition 5.9, one knows that only the two extremal terms, removing initial or terminal 1's, actually contribute to D_1.) One then introduces the linear map $\partial_{N,\ell}^{H,*}$ as

$$\partial_{N,\ell}^{H,*} \colon \mathrm{gr}_\ell^{H,*} \mathcal{H}_N^{H,*} \to \bigoplus_{1 \leq 2r+1 \leq N} \mathrm{gr}_{\ell-1}^{H,*} \mathcal{H}_{N-2r-1}^{H,*},$$

by first applying $\bigoplus_{1\leq 2r+1 \leq N} \mathrm{gr}_\ell^{H,*} D_{2r+1}\big|_{\mathrm{gr}_\ell^{H,*}\mathcal{H}_N^{H,*}}$, then projecting $\zeta^{\mathfrak{l}}(2r'+1)$, $\log^{\mathfrak{l}}(2)$ to $\mathbb{Q}$ via $\widetilde{\pi}_{2r'+1}$ as in Definition 8.5.

Then define the matrix $M_{H,*,N,\ell}$ as the matrix of $\partial^{H,*}_{N,\ell}$ as in Definition 8.8, with respect to the bases $B_{H,N,\ell}$, $B'_{H,N,\ell}$ given in Definition 8.6, after replacing $\widetilde{\tau}^{\mathfrak{m}}$ with $\widetilde{\tau}^{\mathfrak{m},*,V}$.

Now, one notes that the to compute the matrix $M_{H,*,N,\ell}$ one replaces each term $-\frac{1}{2}$ in the matrix $M_{H,N,\ell}$ arising from $\widetilde{\pi}_1\big(\log^{\mathfrak{l}}(2)\big) = \frac{1}{2}$ by the coefficient $\frac{1}{2}(\lambda-1) - \frac{1}{2} = \lambda - 1$.

Example 8.24 For $N = 8$, $\ell = 2$, the matrix $M_{H,*,8,2}$ is as follows; the first row and column label the elements of $B'_{H,8,2}$ and $B_{H,8,2}$ respectively.

	1222	2122	122	2212	212	12	2221	221	21	1
11222	1	0	$-2c_{21}$	0	0	$-8c_{221}$	0	0	0	0
12122	0	1	$2d_{12}-2c_{21}$	0	0	$8c_{23}-8c_{32}$	0	0	0	0
21122	0	0	$2d_{21}$	0	$-2c_{21}$	0	0	0	0	0
12212	0	0	$2c_{21}$	1	$2d_{12}-2c_{21}$	$-8c_{23}+8c_{32}+8d_{122}$	0	0	0	0
21212	0	0	0	0	$2d_{21}$	$8d_{212}$	0	0	0	0
22112	0	0	0	0	$2c_{21}$	$8d_{221}$	0	0	0	0
12221	$\lambda-1$	0	$2c_{21}$	0	0	$8c_{221}$	1	$2d_{12}-2c_{21}$	$8d_{122}-8c_{221}$	$32d_{1222}$
21221	0	$\lambda-1$	0	0	$2c_{21}$	0	0	$2d_{21}-2c_{21}$	$8c_{23}-8c_{32}+8d_{212}$	$32d_{2122}$
22121	0	0	0	$\lambda-1$	0	0	0	$2c_{21}$	$-8c_{23}+8c_{32}+8d_{221}$	$32d_{2212}$
22211	0	0	0	0	0	0	$\lambda-1$	$2c_{21}$	$8c_{221}$	$32d_{2221}$

We note now that when $\lambda = \frac{1}{2}$, the matrices $M_{H,*,N,\ell}$ and $M_{H,N,\ell}$ are identical, and therefore the stuffle-regularised matrix is also invertible. Moreover, when $\lambda = 1$ the last diagonal block (corresponding to the original block $M_{\ell-1,H,N,\ell}$ ending in $\ell - 1$ trailing 1's) is now upper triangular modulo 2 (compare Lemma 8.16), and so also again establishes that $M_{H,*,N,\ell}$ is an invertible matrix. More generally we have the following.

Proposition 8.25 *Suppose λ has the form $\frac{2a+1}{b} \in \mathbb{Q}$, with $a, b \in \mathbb{Z}$. Then the matrix $M_{H,*,N,\ell}$ is invertible.*

Proof The previous result Lemma 8.12 carries through to show the matrix is block triangular. The result Lemma 8.14 also carries over to show the diagonal blocks corresponding to $< \ell - 1$ trailing 1's are upper triangular modulo 2, and are therefore invertible. The proof of Lemma 8.16 is adapted to show that the determinant of the last block has the form

$$(2x+1) + (\lambda-1)(2y+1),$$

with $x, y \in \mathbb{Z}$. For λ of the above form, this is

$$2(x-y) + \frac{(2a+1)(2y+1)}{b},$$

which cannot be 0, as the numerator of the fraction is odd. □

The proofs of Corollary 8.19 and Corollary 8.20 now directly generalise to this case, giving

Corollary 8.26 *Let* $V = \lambda \log^{\mathfrak{m}}(2)$, *with* $\lambda = \frac{2a+1}{b} \in \mathbb{Q}$ *and* $a, b \in \mathbb{Z}$. *Then the elements*

$$\{\widetilde{t}^{\mathfrak{m},*,V}(w) \mid w \in \{1,2\}^{\times}\},$$

are linearly independent. Moreover, they form a basis for the space of:

(i) *motivic extended stuffle-regularised multiple t values with* $\widetilde{t}^{\mathfrak{m},*,V}(1) = 2V$,
(i′) *motivic extended shuffle-regularised multiple t values,*
(ii) *alternating (shuffle-regularised) motivic multiple zeta values*

In particular all of these spaces agree, and extended stuffle-regularised motivic multiple t values with $\widetilde{t}^{\mathfrak{m},*,V}(1) = 2V$ *have dimension* F_{N+1} *in weight* N, *where* $F_k = F_{k-1} + F_{k-2}$ *with* $F_0 = F_1 = 1$ *is the sequence of Fibonacci numbers.*

8.2 Singular regularisation parameters

The proof of Proposition 8.25 breaks down irrevocably in certain cases, in a way that is unavoidable. For example, for $N = 8$, $\ell = 2$ as above, one can see that $\lambda = \frac{242}{91}$ leads to determinant 0 in the last diagonal block. This corresponds to the a linear dependence between regularised elements of level $\ell \leq 2$ in weight 8.

For $V = \frac{242}{91}\log(2)$, one has the following identity between stuffle-regularised MtV's (and a corresponding identity of stuffle-regularised motivic MtV's), as verified via the Data Mine [2]

$$\begin{aligned}
&t^{*,V}(2,2,2,1,1) = \\
&\tfrac{345998}{24843} t^{*,V}(2,2,2,2) \\
&\quad - \tfrac{22801}{8281} t^{*,V}(1,1,2,2,2) - \tfrac{11023}{8281} t^{*,V}(1,2,1,2,2) + \tfrac{1661}{1183} t^{*,V}(1,2,2,1,2) \\
&\quad - \tfrac{22801}{8281} t^{*,V}(2,1,1,2,2) - \tfrac{919}{637} t^{*,V}(2,1,2,1,2) - \tfrac{17257}{8281} t^{*,V}(2,2,1,1,2) \\
&\quad + \tfrac{151}{91} t^{*,V}(1,2,2,2,1) + \tfrac{73}{91} t^{*,V}(2,1,2,2,1) - \tfrac{11}{13} t^{*,V}(2,2,1,2,1)\,.
\end{aligned}$$

Such a regularisation parameter should be termed *singular*, as the matrix of $\partial^{H,*}_{N,\ell}$ is singular. The following $V = \lambda \log^{\mathfrak{m}}(2)$ are singular regularisation parameters, first appearing at the indicated weight.

N	1	3	5	7	9	11	13	15	17
λ	0	2	$\frac{28}{11}$	$\frac{242}{91}$	$\frac{64472}{23479}$	$\frac{712586}{252913}$	$\frac{8156772916}{2873825507}$	$\frac{1002618956134}{348754372637}$	$\frac{6597362406922672}{2270331930729959}$

A (potential) new parameter λ appears in level 1, odd weight, corresponding to last diagonal block with 0 trailing 1's, and a reduction of

$$\widetilde{t}^{\mathfrak{m},*,V}(\{2\}^n, 1) = \sum_{i=0}^{n+1} c_i \, \widetilde{t}^{\mathfrak{m}}(\{2\}^i, 1, \{2\}^{n-i}),$$

for some $c_i \in \mathbb{Q}$. In weight $2N+1$ this reduction can also be obtained more directly from the identity in Theorem 6.1, when written in matrix form with rows indexed by

c_i and columns by $\zeta^{\mathfrak{m}}(\overline{2r'+1})\,\widetilde{t}^{\mathfrak{m}}(\{2\}^{N-r})$, which essentially encodes (the last block of) such $M_{H,*,2N+1,\ell=1}$.

Once such a parameter appears, it renders nonsensical the matrices $M_{H,*,N,\ell}$ of higher weight in that level, as the basis of lower level elements $B'_{H,N,\ell}$ is no longer linearly independent. One can strip trailing 1's from the last diagonal block, without changing the combinatorial of the matrix entries, to see that every singular regularisation parameter arises from the level 1 relation.

The sequence $(\lambda_i)_{i=1}^{\infty}$ of singular regularisation parameters appears to satisfy a number of properties. We end with the following conjecture.

Conjecture 8.27 *The sequence* $(\lambda_i)_{i=1}^{\infty} = (0, 2, \frac{28}{11}, \frac{242}{91}, \frac{64472}{23479}, \ldots)$ *of singular regularisation parameters satisfies the following:*

(i) *the sequence is increasing* $\lambda_{i+1} > \lambda_i$,
(ii) *the sequence is bounded* $\lambda_i < 3$, *for all* i,
(iii) *the sequence has limit* $\lim_{i\to\infty} \lambda_i = 3$.

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Conflict of interest The corresponding author states that there is no conflict of interest.

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