Analysis of gravitational waves from inflation model with minimal, non-minimal, and non-minimal derivative coupling of scalar field from Horndeski theory

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Abstract. Inflation theory provides solutions for problems in cosmology, such as horizon problem and flatness problem. The study of Gravitational waves production in cosmic inflation era not prove the general inflation theory, but also to differentiate in detail among specific models. In this research, we use inflation model with minimal, non-minimal, and non-minimal derivative coupling of scalar field without potential from Horndeski Theory. From this model, we calculate scalar and tensor perturbation equations and then obtain its equation of gravitational waves, spectral index for each perturbation mode and tensor-to-scalar ratio. Spectral index and tensor-to-scalar ratio nearly scale-invariant and agree with observational data for some H_0, ζ, and ξ. Gravitational waves remain constant during inflation and start oscillates when its modes enter the horizon. Energy of this gravitational waves is scale-invariant for modes that re-enter horizon during radiation dominated era and rises toward lower frequencies.

1. Introduction

Standard cosmology gives us good explanations of creation and evolution of our Universe. However, standard cosmology still cannot explain some problems in cosmology, such as horizon problem and flatness problem. Inflation theory states that the Universe went through an exponential expansion of space in the beginning of time, and thus provides solutions for these problems [1, 2]. Various inflation model have been developed [3, 4, 5], but there is still no observation that can ensure what is the most appropriate model for our Universe. Therefore, scientists are trying to find observational signatures of inflation theory, one of them is production of primordial gravitational waves. Primordial gravitational waves are not predicted in the framework of non-inflationary models, making them a conclusive evidence of Inflation [6].

Inflation scenario can produce gravitational waves because of cosmological perturbation in that era, specifically the tensor perturbation. Quantum fluctuations in the inflationary scenario can explain the presence of this perturbation. The expansion in the early universe stretch the wavelength of these fluctuations to scales greater than horizon, so they become classical [2]. The amplitude of the gravitational wave can be described with the tensor-to-scalar ratio r, which is the ratio between the tensor and scalar power spectrum amplitudes for a given scale k. The current prediction on r comes from the Planck Collaboration, which yields $r < 0.064$ for

$k = 0.002\text{Mpc}^{-1}$ [7] and $r < 0.081$ for $k = 0.02\text{Mpc}^{-1}$ [7]. The important thing is that, different inflation model predict different characteristic of gravitational waves. Even in the simplest single-field model, different cosmological parameter or scenario result in different value of r. So if we can observe these gravitational waves, we can prove inflation theory and also determine the most appropriate model for inflation.

In this paper, our model is based on scalar-tensor theory which involve extra scalar-field coupled with tensor metric like [8, 9, 10]. Horndeski theory provide general form of Lagrangian of a scalar-tensor theory which free from *ghost* [11]. Therefore with this theory, we construct an inflation model consists with minimal, non-minimal, and non-minimal derivative coupling of scalar field without potential to avoid fine-tuning problem when we decide the form of its potential [12]. Previous research has been studied this model with equal non-minimal coupling constant ζ and non-minimal derivative coupling constant ξ [10]. In this paper we use more general form of these constants, so ζ and ξ are not necessary equal. Thereafter, from perturbation analysis on this model, we can formulate characteristics of its gravitational waves.

In the next Section, we write our model of Inflation from Horndeski theory with ADM formalism. In Section 3, we derive its background solution and see combination of coupling constants that fulfill de Sitter expansion and decaying scalar-field. In Section 4, we calculate scalar and tensor perturbation equations and then obtain its equation of gravitational waves, spectral index for each perturbation mode and tensor-to-scalar ratio. In Section 5, we describe characteristic of gravitational waves from this model such as wave solution after inflation and their energy. Finally in Section 6, we discuss our conclusion.

2. Setup Model

Lagrangian from Horndeski theory contains coupled scalar field with gravitation [11]

$$L = \sum_{i=2}^{5} L_i, \tag{1}$$

with

$$L_2 = G_2(\phi, X), \tag{2}$$
$$L_3 = -G_3(\phi, X)\Box\phi, \tag{3}$$
$$L_4 = G_4(\phi, X)R - 2G_{4X}(\phi, X)\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right], \tag{4}$$
$$L_5 = G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)\left[(\Box\phi)^3 - 3(\Box\phi)(\phi_{;\mu\nu}\phi^{;\mu\nu}) + 2(\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma})\right], \tag{5}$$

with $G_\mu\nu$ is Einstein tensor, R is Ricci scalar, $G_{iX} = \partial G_i/\partial X$, $\Box$ is d'Alembert operator ($\Box\phi \equiv g^{\mu\nu}\phi_{;\mu\nu}$) and G_i is coefficient function depended on scalar field ϕ and its kinetic energy $X \equiv \phi_{;\mu}\phi^{;\mu}$.

Inflation model on this research consists with kinetic term of scalar field (G_2), minimal coupling (first term of G_4), non-minimal coupling (second term of G_4), non-minimal derivative coupling (G_5) of scalar field

$$G_2 = -\frac{X}{2}, \qquad G_3 = 0, \qquad G_4 = \frac{1}{2} - \frac{1}{2}\zeta\phi^2, \qquad G_5 = \xi\phi. \tag{6}$$

Thus, the Lagrangian for this model reads,

$$L = -\frac{X}{2} + \left(\frac{1}{2} - \frac{1}{2}\zeta\phi^2\right)R + \xi\phi G_{\mu\nu}\phi^{;\mu\nu}. \tag{7}$$

3. Background Analysis

We use Friedmann-Robertson-Walker (FRW) metric in Arnowitt-Deser-Misner (ADM) formalism [13] to obtain background field equations and perturbation analysis later

$$ds^2 = -N^2 dt^2 + a^2(t)\delta_{ij}dx^i dx^j \tag{8}$$

with $N(t)$ is lapse function and $a(t)$ is scale factor. Lapse function for FRW metric is $N(t) = 1$, but here we will substitute the value of $N(t)$ in the end of calculation after varying the Lagrangian with this variable. Now the Lagrangian depended on several variables a, N, and ϕ.

$$L_2 = -\frac{X}{2} = \frac{\dot{\phi}^2}{2N^2}, \tag{9}$$

$$L_3 = 0, \tag{10}$$

$$L_4 = \frac{6}{N^2}\left(\frac{1}{2} - \frac{1}{2}\zeta\phi^2\right)\left(\dot{H} + 2H^2 - H\frac{\dot{N}}{N}\right) \tag{11}$$

$$L_5 = \xi\phi\frac{3H}{N^4}\left[H\ddot{\phi} + \left(3H^2 + 2\dot{H} - 3H\frac{\dot{N}}{N}\right)\dot{\phi}\right]. \tag{12}$$

Its field equations is obtained from varying the Lagrangian with respect to these variables.

$$\begin{aligned} 3H^2 + 2\dot{H} = &-\frac{1}{2}\dot{\phi}^2 + \zeta((3H^2 + 2\dot{H})\phi^2 + 4H\phi\dot{\phi} + 2(\phi\ddot{\phi} + \dot{\phi}^2)) \\ &- \xi(4H\dot{\phi}\ddot{\phi} + (3H^2 + 2\dot{H})\phi^2) \end{aligned} \tag{13}$$

$$(\ddot{\phi} + 3H\dot{\phi})(1 - 6\xi H^2) + 6\zeta(\dot{H} + 2H^2)\phi - 12\xi H\dot{H}\dot{\phi} = 0, \tag{14}$$

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + 3\zeta(H^2\phi^2 + 2H\phi\dot{\phi}) - 9\xi H^2\dot{\phi}^2. \tag{15}$$

These three equations can be simplified into two equation with combining equation (13) and equation (14)

$$\dot{H} + \frac{1}{2}\dot{\phi}^2 + \zeta(H\phi\dot{\phi} - \dot{H}\phi^2 - \phi\dot{\phi} - \dot{\phi}^2) + \xi(2H\dot{\phi}\ddot{\phi} + (\dot{H} - 3H^2)\phi^2) = 0, \tag{16}$$

One of the most general solution of inflation theory is inflation with de Sitter expansion. In this condition, Hubble parameter is constant so it produces perfect exponential expansion [1].

$$H = H_0 \Rightarrow a(t) \sim e^{H_0 t}, \dot{H} = 0 \tag{17}$$

With this condition, the background field equations become

$$\ddot{\phi} + 3H_0\dot{\phi} + \frac{12\zeta H_0^2}{(1 - 6\xi H_0^2)}\phi = 0. \tag{18}$$

$$\left(\frac{1}{2} - \zeta - 3\xi H_0^2\right)\dot{\phi}^2 + \zeta H_0\phi\dot{\phi} - \zeta\phi\ddot{\phi} + 2\xi H_0\dot{\phi}\ddot{\phi} = 0. \tag{19}$$

These equations can be combined with eliminate $\phi\ddot{\phi}$ and $\dot{\phi}\ddot{\phi}$ terms,

$$\left(\frac{1}{2} - \zeta - 9\xi H_0^2\right)\dot{\phi}^2 + \frac{12\zeta^2 H_0^2}{(1 - 6\xi H_0^2)}\phi^2 + \left(\frac{4\zeta H_0 - 48\zeta\xi H_0^3}{1 - 6\xi H_0^2}\right)\phi\dot{\phi} = 0. \tag{20}$$

This is a quadratic equation respect to $\dot{\phi}$. Solution from this equation is

$$\dot{\phi} = \frac{2H_0\zeta\phi}{\left(\frac{1}{2} - \zeta - 9\xi H_0^2\right)} \left(\frac{1 - 12\xi H_0^2}{1 - 6\xi H_0^2}\right) \left(-1 \pm \sqrt{1 - 3\left(\frac{1}{2} - \zeta - 9\xi H_0^2\right) \frac{1 - 6\xi H_0^2}{(1 - 12\xi H_0^2)^2}}\right) \tag{21}$$

Then we integrate the equation until we get solutions for ϕ,

$$\phi = e^{\lambda t}, \tag{22}$$

with

$$\lambda = \frac{2H_0\zeta}{\left(\frac{1}{2} - \zeta - 9\xi H_0^2\right)} \left(\frac{1 - 12\xi H_0^2}{1 - 6\xi H_0^2}\right) \left(-1 \pm \sqrt{1 - 3\left(\frac{1}{2} - \zeta - 9\xi H_0^2\right) \frac{1 - 6\xi H_0^2}{(1 - 12\xi H_0^2)^2}}\right) \tag{23}$$

After the Inflation, we expect the effect of scalar field to disappear. This condition require ϕ to decay throughout time, specifically in this case λ has to be negative. For example in the figure 1, it shows combination of ζ and ξ that fulfill this condition for Hubble parameter $H_0 = 60, 80, 100, 120$.

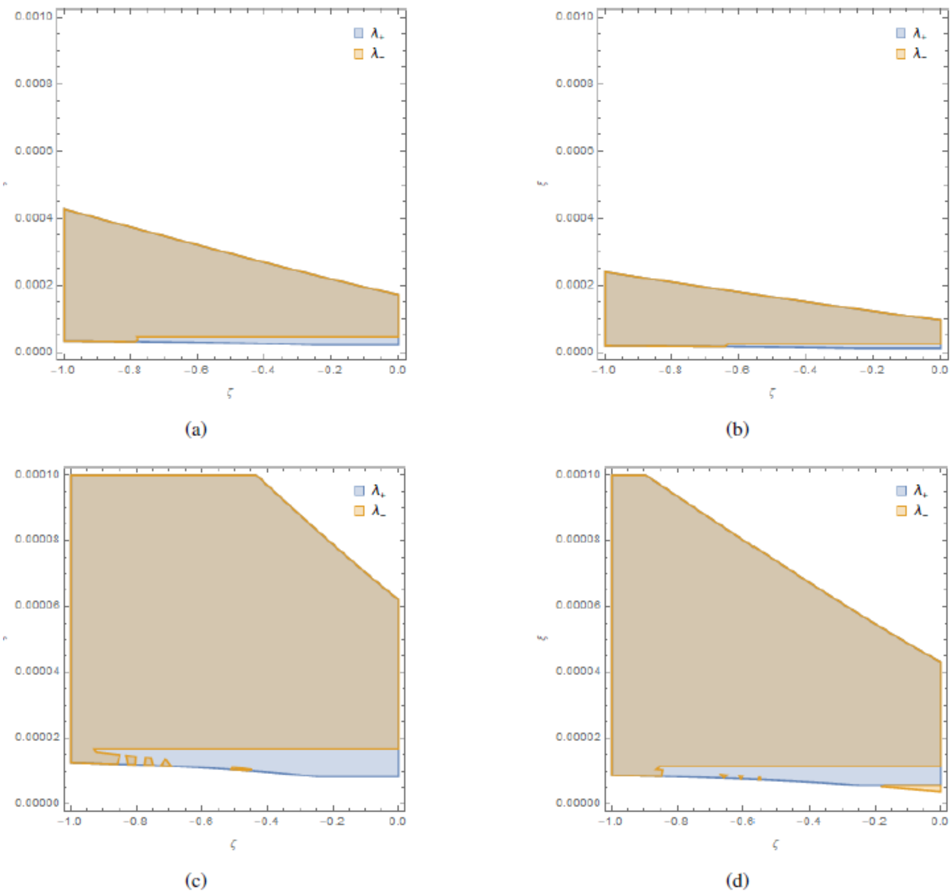


Figure 1. Combination of coupling constants ζ and ξ that cause decaying of scalar field for (a) $H_0 = 60$, (b) $H_0 = 80$, (c) $H_0 = 100$, and (d) $H_0 = 120$. The orange area represent λ_{-} and the blue area represent λ_{+}.

4. Perturbation Analysis

We consider Maldacena gauge for metric in this perturbation analysis to fix time and space reparameterization [14],

$$ds^2 = -dt^2 + h_{ij}dx^i dx^j, \tag{24}$$

with h_{ij} is three dimensional metric

$$h_{ij} = a^2(t)e^{2\Theta}(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma_{lj}), \tag{25}$$

Θ is scalar perturbation and γ is tensor perturbation. Tensor perturbation is tracelesss and divergenceless, so $\gamma_{ii} = 0$ and $\partial_i\gamma_{ij} = 0$. Tensor perturbation γ_{ij} can be expressed into two modes of polarization

$$\gamma_{ij} = h_+\hat{\gamma_{ij}}^+ + h_\times\hat{\gamma_{ij}}^\times \tag{26}$$

with $\hat{\gamma_{ij}}^+$ and $\hat{\gamma_{ij}}^\times$ in Fourier space satisfy normalization condition $\hat{\gamma_{ij}}(\mathbf{k})\hat{\gamma_{ij}}(\textbf{-k})^* = 2$ and $\hat{\gamma_{ij}}^+(\mathbf{k})\hat{\gamma_{ij}}^\times(\textbf{-k})^* = 0$.

4.1. Scalar Perturbation

We need to expand Lagrangian (7) until second order to analize the perturbation [15]. Expanding the background Lagrangian until second order leads into following action for scalar perturbation,

$$S_2^{(s)} = \int d^4x a^3(t)Q_s\left[\dot{\Theta}^2 - \frac{c_s^2}{a^2}(\partial\Theta)^2\right], \tag{27}$$

with

$$Q_s \equiv \frac{2L_\mathcal{S}[3\mathcal{B}^2 + 4L_\mathcal{S}(2L_N + L_{NN})]}{\mathcal{W}^2}, \tag{28}$$

$$c_s^2 \equiv \frac{2}{Q_s}(\dot{\mathcal{M}} + H\mathcal{M} - \mathcal{E}), \tag{29}$$

and,

$$L_\mathcal{S} = \frac{1}{2}\left(1 - \zeta\phi^2 + \xi\dot{\phi}^2\right), \tag{30}$$

$$L_N = -\dot{\phi}^2 - 6H\zeta\phi\dot{\phi} + 6H^2\dot{\phi}^2\xi, \tag{31}$$

$$L_{NN} = 3\dot{\phi}^2 + 12H\zeta\phi\dot{\phi} - 18H^2\dot{\phi}^2\xi \tag{32}$$

$$\mathcal{B} = -2\zeta\phi\dot{\phi} + 4H\xi\dot{\phi}^2, \tag{33}$$

$$\mathcal{W} = 2(H - \zeta(H\phi^2 + \phi\dot{\phi}) - 3H\xi\dot{\phi}^2), \tag{34}$$

$$\mathcal{M} = \frac{\left(1 - \zeta\phi^2 + \xi\dot{\phi}^2\right)^2}{2(H - \zeta(H\phi^2 + \phi\dot{\phi}) - 3H\xi\dot{\phi}^2)}, \tag{35}$$

$$\mathcal{E} = \frac{1}{2}\left(1 - \zeta\phi^2 - \xi\dot{\phi}^2\right). \tag{36}$$

Varying the action with respect to Θ, we get the equation of motion in Fourier space

$$\ddot{\Theta} + \left(3H + \frac{\dot{Q}_s}{Q_s}\right)\dot{\Theta} + c_s^2\frac{k^2}{a^2}\Theta = 0. \tag{37}$$

The solution is obtained with applying Bunch-Davies vacuum function,

$$\Theta(k,\tau) = \frac{iHe^{-ic_s k\tau}}{2(c_s k)^{3/2}\sqrt{Q_s}}(1 + ic_s k\tau). \tag{38}$$

Equation (38) at super-horizon condition with $\tau \approx 0$ has the following form

$$\lim_{\tau\to 0}\Theta(k,\tau) = \frac{iH}{2(c_s k)^{3/2}\sqrt{Q_s}}. \tag{39}$$

Power spectrum of scalar perturbation at super-horizon condition is

$$P_\theta = \frac{k^3}{2\pi^2}|\Theta(k,\tau)|^2 = \frac{H^2}{8\pi^2 c_s^3 Q_s}. \tag{40}$$

Thus, the spectral index for scalar perturbation when leaves the horizon ($c_s k = aH$) reads

$$n_s - 1 = \left.\frac{d\ln P_\Theta}{d\ln k}\right|_{c_s k = aH} = 2\frac{\dot{H}}{H^2} - \frac{\dot{Q}_s}{HQ_s} - 3\frac{\dot{c}_s}{Hc_s}. \tag{41}$$

Consider de Sitter expansion condition and substitute ϕ solution that we already got at equation (22) into scalar spectral index (41), we now get n_s as function of ζ, ξ, H_0, and t. Moreover, for inflation to solve cosmological problems we need $H_0 t > 60$ [16]. Therefore, we create contour plot of n_s as a function of ζ and ξ for some $H_0 t > 60$ cases. From figure 2, we can see that different value of H_0 does not give a significant difference value of n_s. These n_s agree with observational data from Planck Collaboration $n_s = 0.9665 \pm 0.0038$ [7] for some combination of H_0, ζ, and ξ with some deviation.

4.2. Tensor Perturbation

Second order action for tensor perturbation is

$$S_2^{(t)} = \sum_{\lambda=+,\times}\int d^4x a^3 Q_t\left[\dot{h}_\lambda^2 - \frac{c_t^2}{a^2}(\partial h_\lambda)^2\right], \tag{42}$$

with

$$Q_t = \frac{1}{4}\left(1 - \zeta\phi^2 + \xi\dot{\phi}^2\right), \tag{43}$$

$$c_t^2 = \frac{\left(1 - \zeta\phi^2 - \xi\dot{\phi}^2\right)}{\left(1 - \zeta\phi^2 + \xi\dot{\phi}^2\right)}. \tag{44}$$

Varying the action with respect to h_λ, we get the equation of motion for each polarization in Fourier space

$$\ddot{h}_\lambda + \left(3H + \frac{\dot{Q}_t}{Q_t}\right)\dot{h}_\lambda + c_t^2\frac{k^2}{a^2}h_\lambda = 0. \tag{45}$$

After that we change the derivative with respect to time t into derivative with respect to conformal time τ,

$$h_\lambda'' + \left(2\frac{a'}{a} + \frac{Q_t'}{Q_t}\right)h_\lambda' + c_t^2 k^2 h_\lambda = 0. \tag{46}$$

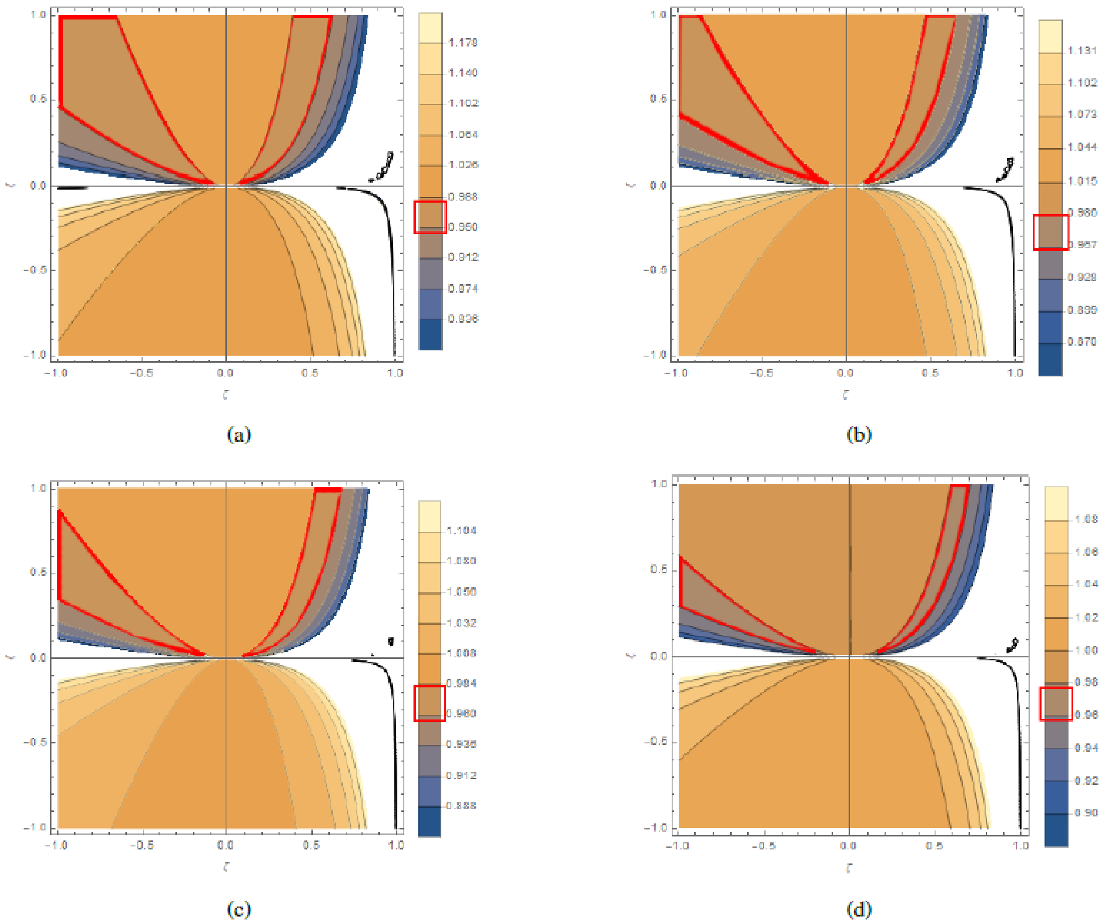


Figure 2. Contour plot of n_s for (a) $H_0 = 60$, (b) $H_0 = 80$, (c) $H_0 = 100$, and (d) $H_0 = 120$. The red line shows area that agree with observational data.

Equation (46) is equation of gravitational waves from tensor perturbation at inflationay epoch. Solution of this equation with Bunch-Davies vacuum function is

$$h_\lambda(k,\tau) = \frac{iHe^{-ic_t k\tau}}{2(c_t k)^{3/2}\sqrt{Q_t}}(1 + ic_t k\tau). \tag{47}$$

Equation (47) at super-horizon condition $\tau \approx 0$ reads

$$\lim_{\tau\to 0} h_\lambda(k,\tau) = \frac{iH}{2(c_t k)^{3/2}\sqrt{Q_t}}. \tag{48}$$

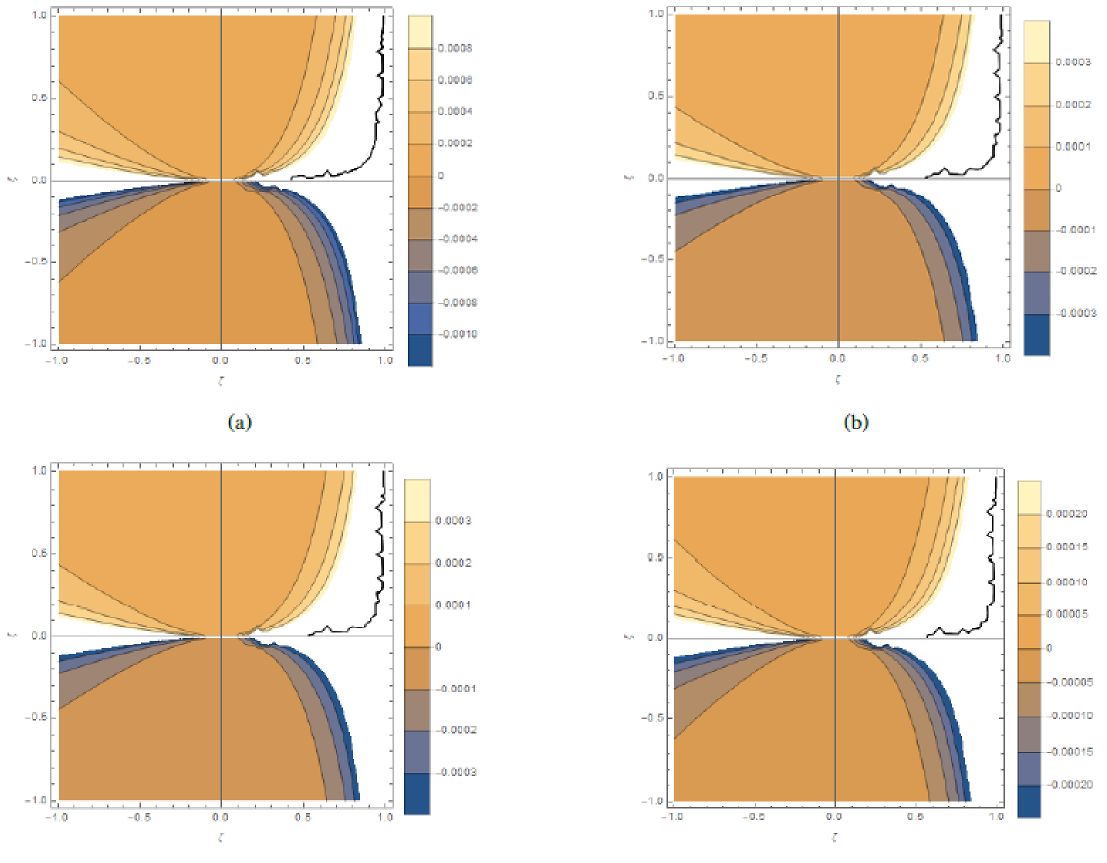
Power spectrum of tensor perturbation at super-horizon condition is

$$P_h = \frac{k^3}{2\pi^2}\sum_{\lambda=+,\times} 2|h_\lambda|^2 = \frac{H^2}{2\pi^2 c_t^3 Q_t}. \tag{49}$$

Thus, spectral index for this perturbation when leaves the horizon ($c_t k = aH$) reads,

$$n_t = \left.\frac{d\ln P_h}{d\ln k}\right|_{c_t k = aH} = 2\frac{\dot{H}}{H^2} - \frac{\dot{Q}_t}{HQ_t} - 3\frac{\dot{c}_t}{Hc_t}. \tag{50}$$

Consider the same condition with scalar analysis, we have n_t as function of ζ, ξ, H_0, and t. Therefore, we create contour plot of n_t as a function of ζ and ξ for some $H_0 t > 60$ cases. From figure 3, we can see that different value of H_0 does not give a significant difference value of n_t. These n_t nearly scale invariant $n_t \simeq 0$ with some deviation.



(a) (b) (c) (d)

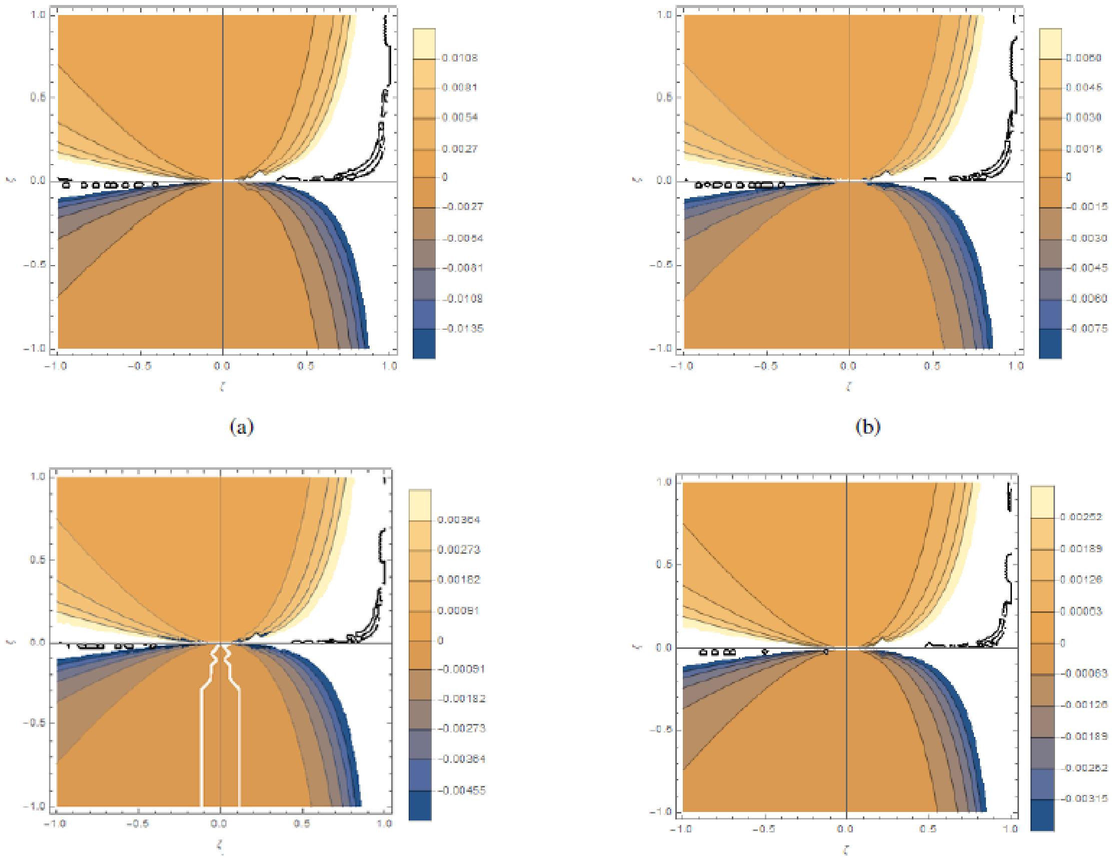
Figure 3. Contour plot of n_t for (a) $H_0 = 60$, (b) $H_0 = 80$, (c) $H_0 = 100$, and (d) $H_0 = 120$.

4.3. Tensor-to-Scalar Ratio

Tensor-to-scalar ratio r describe the amplitude of gravitational waves in form of ratio between tensor and scalar power spectrum amplitude. At inflation era, these power spectrum nearly scale-invariant so tensor-to-scalar at this case reads

$$r = \frac{P_h}{P_\Theta} = 4\frac{c_s^3 Q_s}{c_t^3 Q_t}. \tag{51}$$

Consider the same condition with previous analysis, we now have r as function of ζ, ξ, H_0, and t. Therefore, we create contour plot of r as a function of ζ and ξ for some $H_0 t > 60$ cases.From figure 4, we can see that different value of H_0 does not give a significant difference value of r. These r nearly scale-invariant and agree with observational data from Planck Collaboration $r < 0.064$ for $k = 0.002$ and $r < 0.081$ for $k = 0.02$ [7]. Furthermore, we will use certain combination of H_0, ζ, and ξ that also fulfill the condition of negative λ or decaying of scalar field for analysis of gravitational waves characteristic.



(a) (b) (c) (d)

Figure 4. Contour plot of r for (a) $H_0 = 60$, (b) $H_0 = 80$, (c) $H_0 = 100$, and (d) $H_0 = 120$.

5. Gravitational Waves Characteristics

When inflation ends, primordial gravitational waves will enter horizon again, start with mode with the shortest wavelength. Inside the horizon, these modes begin to oscillate with damped amplitude because of expansion of the Universe [6, 17]. Afterwards, These gravitational waves will curve the background curvature of the Universe because of energy they carry.

5.1. Post Inflation Solution

When radiation dominated the Universe, scale factor evolves as $a \sim \tau$ so the equation of gravitational waves reads

$$\tau^2 h''_{rad} + 2\tau h'_{rad} + c_t^2 k^2 \tau^2 h_{rad} = 0. \tag{52}$$

In this equation, Q'_t/Q_t term is already neglected due to its constant value after inflation ends. This is now a spherical Bessel equation with solution

$$h_{rad}(\tau) = C_1 j_0(c_t k\tau) + C_2 y_0(c_t k\tau), \tag{53}$$

with j_n is spherical Bessel function of the first kind and y_n is spherical Bessel function of the second kind. Gravitational waves is constant outside the horizon with amplitude (48) until inflation ends. To obtain its particular solution, we use initial condition when inflation ends, so the solution is

$$h_{rad}(\tau) = h_{k,prim} j_0(c_t k \tau). \tag{54}$$

with $h_{k,prim}$ is gravitational waves amplitude when they enter the horizon,

$$h_{k,prim} = \frac{H}{2(c_t k)^{3/2}\sqrt{Q_t}}. \tag{55}$$

For modes that enter the horizon during matter dominated universe era, scale factor evolves as $a \sim \tau^2$ so equation (46) reads,

$$\tau^2 h''_{mat} + 4\tau h'_{mat} + c_t^2 k^2 \tau^2 h_{mat} = 0. \tag{56}$$

This is also spherical Bessel equation with solution

$$h_{mat}(\tau) = \frac{1}{\tau}(A_1 j_1(c_t k\tau) + A_2 y_1(c_t k\tau)). \tag{57}$$

To obtain its particular solution, we use initial condition when inflation ends, so the solution is

$$h_{mat}(\tau) = h_{k,prim}\frac{3 j_1(c_t k\tau)}{c_t k\tau}. \tag{58}$$

Although we can explain the evolution of gravitational waves that enter the horizon during radiation dominated era with equation (54), but these waves eventually went through transition from radiation dominated into matter dominated era. This transition could affect the evolution of the gravitational waves. Moreover if the transition occurs instantaneously, we can see the equation (57) at radiation-matter equality era

$$\frac{1}{\tau_{eq}}(A_1(k) j_1(c_t k\tau_{eq}) + A_2(k) y_1(c_t k\tau_{eq})) = h_{k,prim} j_0(c_t k\tau_{eq}), \tag{59}$$

with τ_{eq} is conformal time at radiation-matter equality. See also its first derivative,

$$\frac{1}{\tau_{eq}}(A_1(k) j_2(c_t k\tau_{eq}) + A_2(k) y_2(c_t k\tau_{eq})) = h_{k,prim} j_1(c_t k\tau_{eq}). \tag{60}$$

From equations (59) and (60), we get following A_1 and A_2

$$A_1(k) = h_{k,prim}\tau_{eq}\left(\frac{3}{2c_t k\tau_{eq}} - \frac{\cos(2c_t k\tau_{eq})}{(2c_t k\tau_{eq})} + \frac{\sin(2c_t k\tau_{eq})}{(c_t k\tau_{eq})^2}\right), \tag{61}$$

$$A_2(k) = h_{k,prim}\tau_{eq}\left(-1 + \frac{1}{(c_t k\tau_{eq})^2} - \frac{\cos(2c_t k\tau_{eq})}{(c_t k\tau_{eq})^2} - \frac{\sin(2c_t k\tau_{eq})}{(2c_t k\tau_{eq})}\right). \tag{62}$$

Define $B_1(k) \equiv A_1(k)/h_{k,prim}$ and $B_2(k) \equiv A_2(k)/h_{k,prim}$, solution for gravitational waves that enter the horizon during radiation dominated era and consider the radiation-matter equality era is

$$h_{inst}(\tau) = \begin{cases} h_{k,prim} j_0(c_t k\tau), & \tau < \tau_{eq} \\ h_{k,prim}\frac{1}{\tau}(B_1(k) j_1(c_t k\tau) + B_2(k) y_1(c_t k\tau)), & \tau > \tau_{eq} \end{cases}. \tag{63}$$

Generally, post inflation solution of gravitational waves can be expressed in transfer functions $\mathcal{T}(\tau, k)$. Transfer function describes evolution of gravitational waves when enter the horizon after inflation ends

$$h(\tau) = h_{k,prim}\mathcal{T}(\tau, k). \tag{64}$$

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Transfer functions for radiation dominated era, radiation-matter equality era and matter dominated era are

$$\mathcal{T}(\tau < \tau_{eq}, k > k_{eq}) = j_0(c_t k\tau), \tag{65}$$
$$\mathcal{T}(\tau > \tau_{eq}, k > k_{eq}) = \tau(B_1(k)j_1(c_t k\tau) + B_2(k)y_1(c_t k\tau)), \tag{66}$$
$$\mathcal{T}(\tau > \tau_{eq}, k < k_{eq}) = \frac{3j_1(c_t k\tau)}{c_t k\tau}, \tag{67}$$

with k_{eq} is wave number for wave that enter the horizon at τ_{eq}.

In this analysis we assume for each H_0 gravitational waves exist with every wave number or frequency. For example here, the wave number of h_{rad} that we use is $k = 0.01/\text{Mpc}$ while for h_{mat} the wave number is $k = 0.0001/\text{Mpc}$. This is considering gravitation modes with $k \geq k_{eq} \sim 0.01/\text{Mpc}$ enter the horizon earlier at radiation dominated universe, while $k < k_{eq}$ enter the horizon afterwards [17].

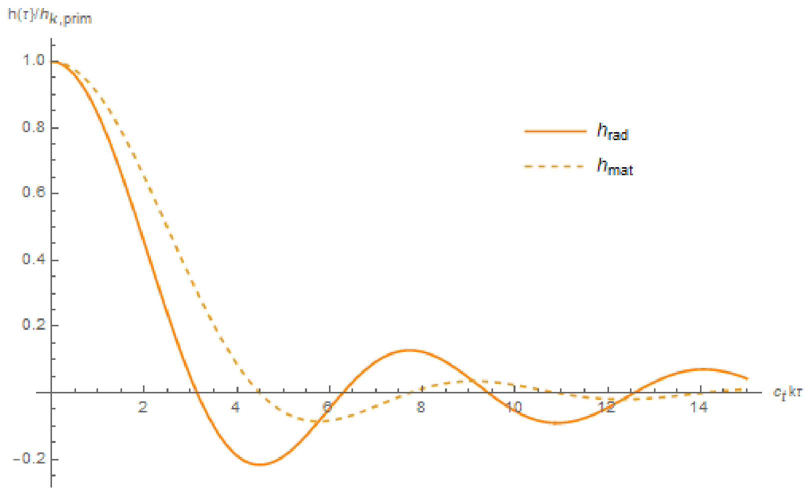

Figure 5. Evolution of gravitational waves that enter the horizon during radiation dominated era (54) and matter dominated era (58).

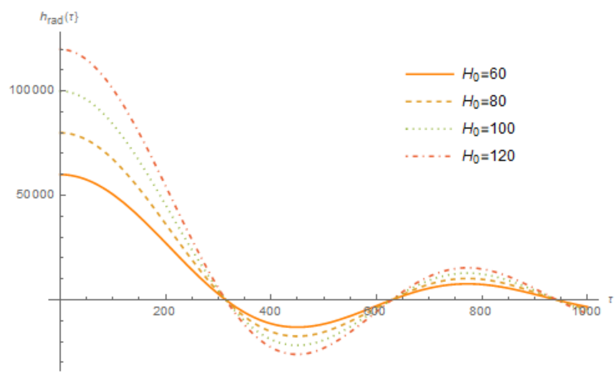

Figure 6. Plots of h_{rad} for $k = 0.01/\text{Mpc}$, $\zeta = -10^{-2}$ and $\xi = 4 \times 10^{-5}$ with variation of H_0.

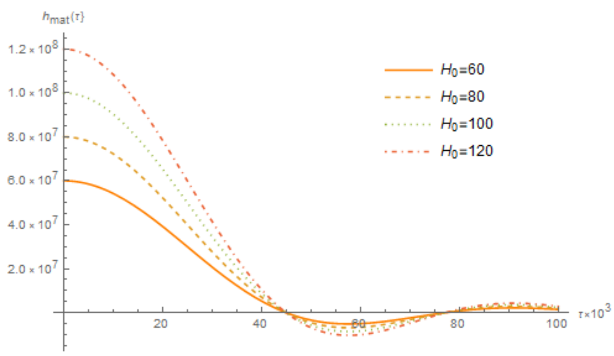

Figure 7. Plots of h_{mat} for $k = 0.0001//\text{Mpc}$, $\zeta = -10^{-2}$ and $\xi = 4 \times 10^{-5}$ with variation of H_0.

From figure 5, gravitational waves that enter during radiation dominated era and matter dominated era have different evolution due to different rate of expansion. Variation of H_0 for each wave number k (or frequency) affect their amplitude because H_0 correspond to the energy scale of inflation. The energy scale increases as H_0 increases. Higher the energy scale means inflation stretch the space larger, which also stretch the gravitational waves. Thus we have gravitational waves that increase as H_0 increases.

5.2. Energy of Gravitational Waves

Energy density of gravitational waves can be expressed in the following equation

$$\rho_h = \frac{1}{16\pi G a^2}\langle h_+^{\prime 2} + h_\times^{\prime 2}\rangle \tag{68}$$

Substitute equation (64) into this equation so we get

$$\rho_h = \frac{1}{8\pi G a^2}\langle h_{prim}^2\rangle \mathcal{T}'(\tau,k)^2. \tag{69}$$

From this equation, because of $h_{k,prim} \propto a^{-1}$ so $\rho_h \propto a^{-4}$, it is the same with energy density of radiation. From the definition of power spectrum, energy density can be expressed in this variable

$$\rho_h = \frac{1}{32\pi G a^2}\int \frac{dk}{k} P_{h,prim}\mathcal{T}'(\tau,k)^2, \tag{70}$$

with $P_{h,prim}$ is power spectrum of gravitational waves when enter the horizon (49).

Define relative spectral density as energy density per logarithmic scale, normalized to the critical denity,

$$\Omega_h(\tau,k) \equiv \frac{\bar{\rho}_h}{\rho_c}, \tag{71}$$

with

$$\bar{\rho}_h = \frac{d\rho_h}{d\ln k}. \tag{72}$$

Substitute (70) into relative spectral density so we get

$$\Omega_h(\tau,k) = \frac{1}{32\rho_c \pi G a^2} P_{h,prim}\mathcal{T}'(\tau,k)^2. \tag{73}$$

From Friedmann equation, critical density reads

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{74}$$

so relative spectral density from gravitational waves reads,

$$\Omega_h(\tau,k) = \frac{1}{12H^2a^2} P_{h,prim}\mathcal{T}'(\tau,k)^2. \tag{75}$$

To see the relative spectral density at present time $\tau = \tau_0$, we use transfer functions (66) and (67). Each of this transfer function describes modes that enter horizon before and after radiation-matter equality. They represent evolution of gravitational waves at radiation dominated era. Derivative respect to τ for each transfer function reads,

$$\mathcal{T}'(\tau > \tau_{eq}, k > k_{eq}) = -\frac{c_t k}{\tau}(B_1(k)j_2(c_t k\tau) + B_2(k)y_2(c_t k\tau)) \tag{76}$$

$$\mathcal{T}'(\tau > \tau_{eq}, k < k_{eq}) = -\frac{3j_2(c_t\tau)}{\tau}. \tag{77}$$

In this analysis, we use cosmological parameter $\tau_0 = 13515$ Mpc and $k_{eq} = 0.010339/\text{Mpc}$ which corresponding to $\Omega_m = 1 - \Omega_r$, $\Omega_r h^2 = 4.15 \times 10^{-5}$, and $h = 0.7$ [7]. Plot of $\Omega_h(\tau_0, k)$ to $c_t k$ for $H_0 = 60$, $\zeta = -0.085$, and $\xi = 5 \times 10^{-5}$ at present time is represented in figure 8.

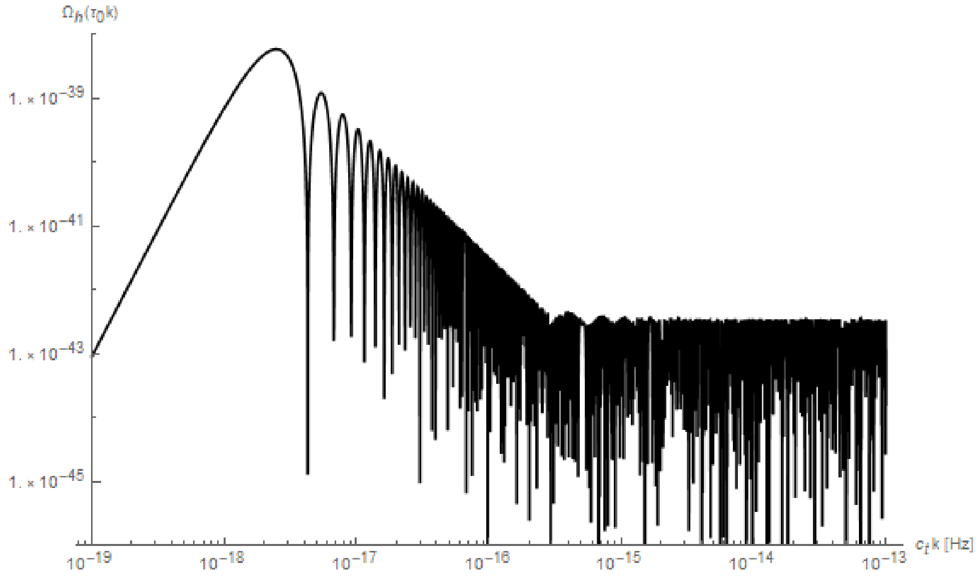

Figure 8. Plot of $\Omega_h(\tau_0, k)$ to $c_t k$ for $H_0 = 60$, $\zeta = -0.085$, and $\xi = 5 \times 10^{-5}$.

Although the spectrum that we got from figure 8 show us rapid oscillations, we do not expect to observe these as the observations are only sensitive to average power over some time in frequency. During radiation dominated era (high frequencies $k \geq k_{eq}$) we see that $\Omega_h(\tau_0, k)$ is independent of k when averaged out, which leads to a scale-invariant spectrum. On the other hand $\Omega_h(\tau_0, k) \sim k^{-2}$ at lower frequencies.

6. Conclusion

Generally, gravitational waves from our model have similar behavior with gravitational waves from slow-roll inflation model. The differences are the value of their characteristics, like the amplitude. Inflation stretches gravitational waves wavelengths to scale larger than horizon, make them do not behave like waves but have an almost frozen amplitude (48). When gravitational waves modes re-enter the horizon sequentially, they are start oscillating inside the horizon with damped amplitude due to expansion of the universe. Mode that enter the horizon during radiation dominated era has different evolution with mode that enter during matter dominated era like shown in figure 5. Variation of H_0 for the same k affect amplitude of gravitational waves (see figure 6 and figure 7) because H_0 correspond to the energy during Inflation. Gravitational waves evolve like radiation $\bar{\rho}_h \propto a^{-4}$, so relative spectral density Ω_h during radiation dominated era is constant whereas during matter dominated era evolves with $\Omega_h \propto a^{-1}$, so mode that enter the horizon later is less damped. Thus, their energy is scale-invariant at $k \geq k_{eq}$ and rises at $k < k_{eq}$ (see figure 8).

This research can be developed furthermore. For instance, we can analyze effect of anisotropic part of energy-momentum tensor to gravitational waves equation. We also can see effect of primordial gravitational waves to temperature fluctuation and polarization of CMB.

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