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Fermions in a plane

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Abstract. It is a well known feature of 2+1-dimensions d that there exist two inequivalent fundamental representations of the Dirac gamma matrices. As a consequence, a parity invariant Lagrangian can only be constructed by incorporating both the representations. We obtain explicit solutions of the Dirac equation in both the representations to confirm these features. In addition to the parity invariance, we argue that to obtain the complete particle spectrum, both the representations need to be considered simultaneously.

1. Introduction

There has been a growing interest in the last few years in theories with extra dimensions which have the capability of reducing the string mass scale M_s several orders of magnitudes lower than the Planck mass scale of 1.9×10^{16} TeV, see e.g., [1]. In extreme scenarios, [2], it can be as low as of the order of TeV. In such cases, we can nip the gauge hierarchy problem in the bud. In view of this exciting possibility, it is timely to revise various properties of field theories in higher dimensions. Theories in even space-time dimensions are rather similar to the ones in 4-dimensions. However, odd dimensions can have striking differences, [3, 4]. In the present brief report, we study fermions in a plane. Through constructing the explicit solutions of the Dirac equation, we see that the particle spectrum is incomplete. Thus to complete the spectrum, we must take into account both the inequivalent representations of the gamma matrices.

2. Particle Spectrum and Symmetries

Starting from the Lagrangian $\mathcal{L} = \bar{\psi}(i\hbar\, \partial\!\!\!/ - mc)\psi$, we can choose the following representation of the gamma matrices in a plane : $\gamma^0 = \sigma_3, \gamma^1 = i\sigma_1$, and $\gamma^2 = i\sigma_2$ (the A representation hereafter), where σ_i are the Pauli matrices. We then readily obtain the solutions of the free Dirac equation :

$$\psi_A^P(x) = \psi_1(x) = \begin{pmatrix} 1 \\ \frac{c(p_y - ip_x)}{E+mc^2} \end{pmatrix} e^{-\frac{i}{\hbar}x\cdot p} \equiv u_A^P(p) e^{-\frac{i}{\hbar}x\cdot p} ,$$

$$\psi_A^N(x) = \psi_2(x) = \begin{pmatrix} \frac{c(p_y + ip_x)}{E+mc^2} \\ 1 \end{pmatrix} e^{\frac{i}{\hbar}x\cdot p} \equiv v_A^P(p) e^{\frac{i}{\hbar}x\cdot p} . \qquad (1)$$

Choosing the normalization of the spinors to be such that there are $2E$ number of particles per unit volume, we have $u_1\bar{u}_1 = \not{p}c + mc^2$ and $v_1\bar{v}_1 = \not{p}c - mc^2$. This is to say that the completeness relations are not hampered by the fact there is just one particle spinor and one anti-particle spinor. The above relation permits us to define the projection operators $\Lambda_\pm = (\pm \not{p} + mc)/2mc$ which project out the particle and ant-particle spinors respectively. Therefore, everything is apparently in order. However, there are several reasons to believe that the above Lagrangian fails to incorporate the complete description of physical reality :

- **Particle Spectrum:** There are two independent solutions, one corresponding to a particle (P) and the other to an anti-particle (N). In a plane, there is just one orbital angular momentum which we can define as $L = r_x p_y - r_y p_x$. It does not commute with the Hamiltonian $H = \gamma^0(\vec{\gamma} \cdot \vec{p} + mc^2)$. However, if we define the spin operator as $\Sigma = (\hbar/2)\gamma^0$, the total angular momentum $J = L + \Sigma$ is a conserved quantity. It is easy to see that for the particle at rest, i.e., for $p_x = p_y = 0$, u_1 and v_1 are eigenfunctions of Σ with eigenvalues $\hbar/2$ and -$\hbar/2$ respectively. It implies a natural interpretation of the solution u_1 as that of a particle with spin say clockwise and of v_1 as that of an anti-particle with spin anti-clockwise. Obviously, the particle spectrum is incomplete because the particle with spin anti-clockwise and the anti-particle with spin clockwise are absent.
- **Parity (and Charge Conjugation) Invariance:** The incompleteness of the spectrum is also implied by a different argument. In a plane, just like any other odd dimensions, parity operation is defined by reversing the signs of all but one coordinate. Let us suppose that under parity transformation, $r_x \to -r_x$ and $r_y \to r_y$. Consequently, spin, being an angular momentum, changes sign. Therefore, particle with clockwise spin and anti-clockwise spin are related through the parity transformation. But one of these particles is not a solution of the Dirac equation. Similarly, charge conjugation operation relates the particle of a given spin to the anti-particle of the same spin. Again, one of these particles does not correspond to the solutions obtained.

These apparent paradoxes get resolved thanks to the well-known fact that for odd d, there exist two inequivalent representations, [4]. In the planar case, we can choose R_2 to be $\gamma^0 = \sigma_3, \gamma^1 = i\sigma_1, \gamma^{2\prime} = -\gamma^2 = -i\sigma_2$ We transform the corresponding solutions ϕ_1 and ϕ_2 of the Dirac equation to ψ_B^P and ψ_B^N for obvious particle identification as follows :

$$\psi_B^P(x) = i\gamma^2\phi_1 = \begin{pmatrix} \frac{c(p_y+ip_x)}{E+mc^2} \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar}p\cdot x} = u_B^P(p)e^{-\frac{i}{\hbar}p\cdot x} \; ,$$

$$\psi_B^N(x) = i\gamma^2\phi_2 = \begin{pmatrix} 1 \\ \frac{c(p_y-ip_x)}{E+mc^2} \end{pmatrix} e^{\frac{i}{\hbar}p\cdot x} = u_B^N(p)e^{\frac{i}{\hbar}p\cdot x} \; . \tag{2}$$

Looking at the stationary case, $p_x = p_y = 0$, by applying the spin operator, we can see that ψ_A^P and ψ_B^P correspond to particles with opposite spins. Similarly, ψ_A^N and ψ_B^N correspond to anti-particles with opposite spins. Moreover, one can check that the parity transformation $\mathcal{P}$ implies : $(u_A^P)^\mathcal{P} = -\gamma^1 u_B^P$ and $(v_A^P)^\mathcal{P} = \gamma^1 v_B^P$. Thus the parity transformation indeed swaps the spinors in the two inequivalent representations. It converts particle of one spin to the particle of opposite spin, and the same is true for the anti-particle. The Lagrangian which takes into account both the representations is, [4] :

$$\mathcal{L} \;=\; \bar{\psi_A}(i\hbar \not{\partial} - mc)\psi_A + \bar{\psi_B}(i\hbar \not{\partial} + mc)\psi_B \; . \tag{3}$$

It completes the particle spectrum and is also parity invariant.

3. Conclusions

We demonstrate that by taking into account both the inequivalent fundamental representations of the gamma matrices for odd number of space-time dimensions, the resulting Lagrangian not only is parity and charge-conjugation invariant but also completes the particle spectrum.

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