# Romanovski Polynomials in Selected Physics Problems

A. P. Raposo[1], H. J. Weber[2], D. Alvarez-Castillo[3], M. Kirchbach[3],

[1]*Facultad de Ciencias, Av. Salvador Nava s/n, San Luis Potosí,*

*S.L.P. 78290, Universidad Autónoma de San Luis Potosí, México,*

[2]*Department of Physics, University of Virginia,*

*Charlottesville, VA 22904, USA and*

[3] *Instituto de Física, Av. Manuel Nava 6, San Luis Potosí,*

*S.L.P. 78290, Universidad Autónoma de San Luis Potosí, México*

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## Abstract

We briefly review the five possible real polynomial solutions of hypergeometric differential equations. Three of them are the well known classical orthogonal polynomials, but the other two are different with respect to their orthogonality properties. We then focus on the family of polynomials which exhibits a finite orthogonality. This family, to be referred to as the Romanovski polynomials, is required in exact solutions of several physics problems ranging from quantum mechanics and quark physics to random matrix theory. It appears timely to draw attention to it by the present study. Our survey also includes several new observations on the orthogonality properties of the Romanovski polynomials and new developments from their Rodrigues formula.

## I. INTRODUCTION

Several physics problems ranging from ordinary–and supersymmetric quantum mechanics to applications of random matrix theory in nuclear and condensed matter physics are ordinarily resolved in terms of Jacobi polynomials of purely imaginary arguments and parameters that are complex conjugate to each other. Depending on whether the degree $n$ of these polynomials is even or odd, they appear either genuinely real or purely imaginary. The fact is that all the above problems are naturally resolved in terms of manifestly real orthogonal polynomials. These real polynomials happen to be related to the above Jacobi polynomials by the purely imaginary phase factor, $i^n$, much like the phase relationship between the hyperbolic and the trigonometric functions, i.e. $\sin ix = i \sinh x$. These polynomials have first been reported by Sir Edward John Routh [1] in 1884, and then were rediscovered within the context of probability distributions by Vsevolod Romanovski [2] in 1929. They are known in the mathematics literature under the name of "Romanovski" polynomials.

Romanovski polynomials may be derived as the polynomial solutions of the ODE

$$(1+x^2)\frac{d^2R(x)}{dx^2} + t(x)\frac{dR(x)}{dx} + \lambda R(x) = 0, \tag{1}$$

with $t(x)$ a polynomial, at most a linear, which is a particular subclass of the hypergeometric differential equations [3], [4]. Other subclasses give rise to the well known classical orthogonal polynomials of Hermite, Laguerre and Jacobi [4], [5]. Romanovski polynomials are not so widespread as the others in applications. But in recent years several problems have been solved in terms of this family of polynomials (Schrödinger equation with the hyperbolic Scarf and the trigonometric Rosen-Morse potentials [6, 7], Klein-Gordon equation with equal vector and scalar potentials [8], certain classes of non-central potential problems as well [9]) and so they deserve a closer look and be placed on equal footing with the classical orthogonal polynomials.

In this context, our goal is threefold. First of all, it is to establish the orthogonality properties of these polynomials. This is achieved by the same methods as for any other hypergeometric differential equation. Our second goal is to explain their use as orthogonal eigenfunctions of some Hamiltonian operators. Third, Eq. (1) has been described in [10] as a complexification of the Jacobi ODE, a general expression that can be written as

$$(1-x^2)\frac{d^2P(x)}{dx^2} + t(x)\frac{dP(x)}{dx} + \lambda P(x) = 0, \tag{2}$$

where $t(x)$ is again an arbitrary polynomial of at most first degree, but not necessarily the same as in Eq. (1). If that were the case, solutions to Eq. (1) would be the complexification of the solutions to Eq. (2), that is, the complexification of the Jacobi polynomials. Hence, our final goal is to clarify this relationship.

We deal with all these issues in the following way: In Section II we give a classification of hypergeometric equations placing Eq. (1) among them. Next, in Section III we show some expected properties of the $R_n^{(\alpha,\beta)}(x)$ functions as solutions of a hypergeometric ODE such as: being indeed polynomials, recurrence relations; and the absence of another, namely general orthogonality. In Section IV we compare the polynomials $R_n^{(\alpha,\beta)}(x)$ with the complexified Jacobi polynomials. In Section V we show some examples of physical problems whose solutions lead to Romanovski polynomials. Section VI sheds light on some peculiarities of orthogonal polynomials as part of quantum mechanics wave functions. In the final Section VII we summarize our conclusions.

## II. CLASSIFICATION OF HYPERGEOMETRIC DIFFERENTIAL EQUATIONS

A hypergeometric equation [3] is an ODE of the form

$$s(x)F''(x) + t(x)F'(x) + \lambda F(x) = 0, \tag{3}$$

where the unknown is a real function of real variable $F : \mathcal{U} \to \mathbf{R}$, where $\mathcal{U} \subset \mathbf{R}$ is some open subset of the real line, and $\lambda \in \mathbf{R}$ a corresponding eigenvalue, and where the functions $s$ and $t$ are real polynomials of at most second order and first order, respectively. Here the prime stands for differentiation with respect to the variable $x$. This class of ODEs is very well known both from the mathematical and the physical points of view. From the mathematical one, many properties that its solutions exhibit make them interesting in their own right. For instance, the classical orthogonal polynomials [3], [11], [12] (Hermite, Laguerre and Jacobi polynomials, the latter including as particular cases Legendre, Chebyshev and Gegenbauer polynomials) are solutions of particular subfamilies of hypergeometric ODEs. From the physical point of view, many of the exact solutions to the eigenvalue equation of a quantum mechanical Hamilton operator lead to an equation of the hypergeometric kind: harmonic oscillator, Coulomb potential, the trigonometric Rosen-Morse and Scarf potentials, hyperbolic Rosen-Morse and hyperbolic Scarf potentials.

As to our goal, the mathematical properties we are interested in are the following (refer to [3] for a detailed study and proofs of these statements). The leading property, which gives the differential equation its name “hypergeometric,” is that if $F(x)$ is a solution to Eq. (3), then the derivative $F'(x)$ is a solution to another hypergeometric equation that is closely related to the former:

$$s(x)(F'(x))'' + t^{(1)}(x)(F'(x))' + \lambda^{(1)}F'(x) = 0, \tag{4}$$

where $t^{(1)}(x) \equiv t(x) + s'(x)$ and $\lambda^{(1)} \equiv \lambda + t'(x)$. Iteratively, it is easy to show that the $m$th derivative, $F^{(m)}(x)$ is a solution of

$$s(x)(F^{(m)}(x))'' + t^{(m)}(x)(F^{(m)}(x))' + \lambda^{(m)}F^{(m)}(x) = 0, \tag{5}$$

where now $t^{(m)}(x) \equiv t(x)+ms'(x)$ and $\lambda^{(m)} \equiv \lambda+mt'(x)+\frac{1}{2}m(m-1)s''(x)$. The next result is that, for any $n \in \{0, 1, 2, \dots\}$, there exists a polynomial $F_n(x)$ of degree $n$, together with a constant $\lambda_n$ which satisfy Eq. (3). The constant is given by

$$\lambda_n = -n\left(t'(x) + \frac{1}{2}(n-1)s''(x)\right). \tag{6}$$

The last result, together with the former, tells us that $F_n(x)$ and its derivatives, $F_n^{(m)}(x)$, are solutions to similar equations. By means of a weight function, it is possible to write down a formula which gives all these polynomials at once. A weight function $w(x)$ associated with Eq. (3) is a solution of Pearson’s differential equation

$$[s(x)w(x)]' = t(x)w(x), \tag{7}$$

that assures the self-adjointness of the differential operator of the hypergeometric ODE. Then, the generalized Rodrigues formula gives the $m$th derivative of the polynomial $F_n(x)$ as

$$F_n^{(m)}(x) = N_{nm}\frac{1}{w(x)s(x)^m}\frac{d^{n-m}}{dx^{n-m}}\left[w(x)s(x)^n\right], \quad 0 \le m \le n, \tag{8}$$

where $N_{nm}$ is a normalization constant. This constant is related to the coefficient $a_n$ of the term of degree $n$ in the polynomial $F_n(x)$ by the expression

$$N_{nm} = \frac{(-1)^{n-m}n!\,a_n}{\prod_{k=m}^{n-1}\lambda_n^{(k)}}, \tag{9}$$

which is valid for $0 \leq m \leq n-1$ and $n \geq 1$. Equation (8), with $m = 0$, gives the classical Rodrigues formula

$$F_n(x) = N_n \frac{1}{w(x)} \frac{d^n}{dx^n} \left[ w(x) s(x)^n \right], \tag{10}$$

where we have identified $F_n^{(0)}(x) = F_n(x)$ and $N_{n0} = N_n$.

If the functions $s(x)$ and $w(x)$ satisfy yet another condition, namely both being positive within an interval $(a, b)$ and

$$\lim_{x \to a} s(x) w(x) x^l - \lim_{x \to b} s(x) w(x) x^l = 0\,, \tag{11}$$

for any nonnegative integer $l$, then the family of polynomials is orthogonal with respect to the weight function $w$, i.e.

$$\begin{gathered} \int_a^b w(x) F_m(x) F_n(x) \, \mathrm{d}x = (f_n)^2 \delta_{mn}, \\ \forall m, n \in \{0, 1, 2, \dots\}, \end{gathered} \tag{12}$$

where $f_n$ is the norm of the polynomials. Hence, all hypergeometric ODEs admit a family of polynomial solutions. But this family is not orthogonal for all hypergeometric ODEs.

The fact that a solution $F(x)$ and its derivatives $F^{(m)}(x)$ obey hypergeometric ODEs with the same coefficient $s(x)$, Eqs. (3) and (5), suggests a classification in terms of the polynomial $s(x)$. Moreover, a classification according to the roots of $s(x)$ has proved useful and provides a characterization of the solutions [3],[4],[5]. There are five classes in this scheme, as $s(x)$ may be a constant, a first degree polynomial or a second order one with two distinct real roots, one real root or, finally, two complex conjugate, not real, roots. In addition, it is useful to note that an affine change of variable (i.e., $x \to a\,x + b, \quad a \neq 0$) does preserve the hypergeometric character of Eq. (3) and the kind of roots of polynomial $s(x)$. Then, in each class, we may consider only a canonical form of the equation, to which any other can be reduced by an affine change of the independent variable.

1. *Polynomial $s(x)$ is a constant:*

We take as canonical form

$$H''(x) - 2\alpha x H'(x) + \lambda H(x) = 0, \tag{13}$$

where $\alpha \in \mathbf{R}$ is an arbitrary constant, i.e., we have here a one-parameter family of ODEs. We call it generalized Hermite equation (the equation with $\alpha = 1$ is called Hermite equation). The polynomials are a generalization of Hermite polynomials, denoted $\{H_n^{(\alpha)}\}$,

$n \in \{0, 1, 2, \dots\}$. The weight function is

$$w(x) = e^{-\alpha x^2}. \tag{14}$$

For $\alpha > 0$ the additional conditions for orthogonality, Eq. (11), are fulfilled in the interval $(-\infty, \infty)$, hence we get an orthogonality relation:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} H_m^{(\alpha)}(x) H_n^{(\alpha)}(x)\, \mathrm{d}x = (h_n)^2 \delta_{mn}, \qquad \forall m, n \in \{0, 1, 2, \dots\},\ \alpha > 0. \tag{15}$$

2. *Polynomial $s(x)$ is of the first degree:*

The canonical form of the ODE is

$$xL''(x) + t(x)L'(x) + \lambda L(x) = 0, \tag{16}$$

which we call generalized Laguerre equation. The first-order polynomial $t$ is still arbitrary, so this is a two-parameter family of ODEs. If $t(x)$ is written as $t(x) = -\alpha x + \beta + 1$, with $\alpha, \beta \in \mathbf{R}$, the parameters (actually, Eq. (16) is called associated Laguerre equation in the case $\alpha = 1$, and Laguerre equation if $\alpha = 1$ and $\beta = 0$), then the weight function is

$$w(x) = x^{\beta} e^{-\alpha x}, \tag{17}$$

and the polynomials are written $\{L_n^{(\alpha,\beta)}\}$, $n \in \{0, 1, 2, \dots\}$. If $\alpha, \beta > 0$, the condition of Eq. (11) is fulfilled and one gets orthogonality in the interval $[0, \infty)$ as

$$\int_{0}^{\infty} x^{\beta} e^{-\alpha x} L_m^{(\alpha,\beta)}(x) L_n^{(\alpha,\beta)}(x)\, \mathrm{d}x = (l_n)^2 \delta_{mn}, \qquad \forall m, n \in \{0, 1, 2, \dots\},\ \alpha, \beta > 0. \tag{18}$$

3. *Polynomial $s(x)$ is of the second degree, with two different real roots:*

The canonical form of the ODE is

$$(1 - x^2)P''(x) + t(x)P'(x) + \lambda P(x) = 0, \tag{19}$$

which is known as Jacobi equation. It is customary to write the arbitrary polynomial $t(x)$ in the form $t(x) = \beta - \alpha - (\alpha + \beta + 2)x$, where $\alpha, \beta \in \mathbf{R}$ are the parameters. Then, for each pair $(\alpha, \beta)$, the Rodrigues formula defines a family of polynomials, the Jacobi polynomials, denoted $\{P_n^{(\alpha,\beta)}\}$, $n \in \{0, 1, 2, \dots\}$ with the weight function given by

$$w(x) = (1 - x)^{\alpha} (1 + x)^{\beta}. \tag{20}$$

If parameters $\alpha$ and $\beta$ satisfy $\alpha, \beta > -1$, the additional condition of Eq. (11) is fulfilled in the interval $(-1, 1)$, so there is an orthonormalization relation:

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x = (p_n)^2 \delta_{mn}, \qquad \forall m, n \in \{0, 1, 2, \dots\}, \alpha, \beta > -1. \tag{21}$$

Some particular cases received special names: Gegenbauer polynomials if $\alpha = \beta$, Chebyshev I and II if $\alpha = \beta = \pm 1/2$, Legendre polynomials if $\alpha = \beta = 0$.

4. *Polynomial $s(x)$ is of the second degree, with one double real root:*

We choose as canonical form of the ODE

$$x^2 B''(x) + t(x) B'(x) + \lambda B(x) = 0, \tag{22}$$

If the arbitrary first order polynomial is written as $t(x) = (\alpha + 2)x + \beta$, with $\alpha, \beta \in \mathbf{R}$ the parameters, then the weight function is

$$w^{(\alpha,\beta)}(x) = x^{\alpha} e^{-\frac{\beta}{x}}. \tag{23}$$

We write the polynomials as $\{B_n^{(\alpha,\beta)}\}$, $n \in \{0, 1, 2, \dots\}$, which are called Bessel polynomials [13] (they were also given under type V in Ref. [2] and classified in Ref. [4]). There is no combination of any particular values of the parameters and any interval which satisfies Eq. (11), so neither of these families is orthogonal with respect to the weight function (23).

5. *Polynomial $s(x)$ is of the second degree, with two complex roots:*

The canonical form of the ODE for this case is chosen as

$$(1 + x^2) R''(x) + t(x) R'(x) + \lambda R(x) = 0, \tag{24}$$

which is the one studied in [4]–[6], and [14]–[15]. A note of caution: the solutions introduced in [6] and [7] seem to be different, but this is due just to a different form given to the arbitrary polynomial $t(x)$. A careful review shows that both papers are dealing with the same ODE, namely Eq. (24), so the solutions must be the same up to a constant factor. Writing the polynomial $t(x)$ as $t(x) = 2\beta x + \alpha$, with $\alpha, \beta \in \mathbf{R}$, we have again a two-parameter family of ODEs with their respective families of polynomials which we denote $\{R_n^{(\alpha,\beta)}\}$, $n \in \{0, 1, 2, \dots\}$. With this notation (which differs slightly from [6],[7],[16]) the weight function is

$$w^{(\alpha,\beta)}(x) = (1 + x^2)^{\beta - 1} \mathrm{e}^{-\alpha \cot^{-1} x}. \tag{25}$$

Upon comparison with Romanovski's original work [2], we conclude $\{R_n^{(\alpha,\beta)}\}$ are the Romanovski polynomials. In Section III we study the properties of these polynomials.

In Ref. [4] polynomial solutions of linear homogeneous 2nd-order ODEs

$$\begin{aligned} s(x)y_n''(x) + t(x)y_n'(x) &= n[(n-1)e + 2\varepsilon]y_n(x), \\ s(x) = ex^2 + 2fx + g, \quad t(x) &= 2\varepsilon + \gamma, \\ n \in \{0, 1, \ldots\}, \ e, f, g, \varepsilon, \gamma &\in \mathbf{R}\,, \end{aligned} \tag{26}$$

are classified upon substituting the finite power series

$$\begin{aligned} y_n(x) &= \sum_{k=0}^{n} \frac{a_{n,k}}{k!}(x+c)^k, \\ a_{n,n} &\neq 0, \quad c \in \mathbf{C}\,, \end{aligned} \tag{27}$$

and analyzing the resulting recursions among the coefficients. No other solutions other than the polynomials given above are found. Their orthogonality properties are derived by means of the spectral theorem (of Favard [12]).

To sum up, all hypergeometric equations fall into one of these five classes. Three of them give rise to the very well studied classical orthogonal polynomials (Jacobi, Laguerre and Hermite) or a slight generalization of them. A fourth one has not attracted much attention due to, we guess, the lack of general orthogonality. Finally, a fifth class is the family of ODEs we are dealing with here.

## III. DEFINITION AND PROPERTIES OF ROMANOVSKI POLYNOMIALS

We focus now on the Romanovski polynomials $R_n^{(\alpha,\beta)}$ and study some well known, and some new properties they have. We start by writing down the explicit Rodrigues formula, Eq. (10), for this case, that we take as their definition. For each $\alpha, \beta \in \mathbf{R}$ and each $n \in \mathbf{N} = \{0, 1, 2, \ldots\}$ we define the function $R_n^{(\alpha,\beta)}$, by the Rodrigues formula

$$R_n^{(\alpha,\beta)}(x) \equiv \frac{1}{w^{(\alpha,\beta)}(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[w^{(\alpha,\beta)}(x) s(x)^n\right], \tag{28}$$

where

$$w^{(\alpha,\beta)}(x) \equiv (1+x^2)^{\beta-1} \mathrm{e}^{-\alpha \cot^{-1} x}, \tag{29}$$

is the weight function, same as in equation (25), and

$$s(x) \equiv 1 + x^2, \tag{30}$$

is the coefficient of the second derivative of the hypergeometric differential equation (24). Notice that we have chosen the normalization constants $N_n = 1$, which is equivalent to make a choice of the coefficient of highest degree in the polynomial, as given by equation (9), which takes the form

$$a_n = \frac{1}{n!}\prod_{k=0}^{n-1}\left[2\beta(n-k)+n(n-1)-k(k-1)\right], \qquad n \geq 1. \tag{31}$$

Notice that the coefficient $a_n$ does not depend on the parameter $\alpha$, but only on $\beta$ and, for particular values of $\beta$, $a_n$ is zero (i.e., for all the values $\beta = \frac{k(k-1)-n(n-1)}{2(n-k)}$ where $k = 0, \ldots, n-1$). This observation poses a problem that we will address somewhere else. For later reference, we write explicitly the polynomials of degree 0, 1 and 2

$$R_0^{(\alpha,\beta)}(x) = 1, \tag{32}$$

$$R_1^{(\alpha,\beta)}(x) = \frac{1}{w^{(\alpha,\beta)}(x)}\left(w'^{(\alpha,\beta)}(x)s(x) + s'(x)w^{(\alpha,\beta)}(x)\right) = t^{(\alpha,\beta)}(x) = 2\beta x + \alpha, \tag{33}$$

$$\begin{aligned} R_2^{(\alpha,\beta)}(x) &= \frac{1}{w^{(\alpha,\beta)}(x)}\frac{d}{dx}[s^2(x)w'^{(\alpha,\beta)}(x) + 2s(x)s'(x)w^{(\alpha,\beta)}(x)] \\ &= \frac{1}{w^{(\alpha,\beta)}(x)}\frac{d}{dx}\left(s(x)w^{(\alpha,\beta)}(x)(t^{(\alpha,\beta)}(x) + s'(x))\right) \\ &= (2x + t^{(\alpha,\beta)}(x))t^{(\alpha,\beta)}(x) + (2 + t'^{(\alpha,\beta)}(x))s(x) \\ &= (2\beta+1)(2\beta+2)x^2 + 2(2\beta+1)\alpha x + (2\beta + \alpha^2 + 2) \end{aligned} \tag{34}$$

that derive from the Rodrigues formula (28) in conjunction with Pearson's ODE (7).

The whole set of Romanovski polynomials is spanned by the three parameters $\alpha$, $\beta$ and $n$

$$\{R_n^{(\alpha,\beta)} \,:\, \alpha, \beta \in \mathbf{R},\, n \in \mathbf{N}\}. \tag{35}$$

In order to study their properties we have found it useful to classify them in families. The properties are stated for each family. We have found two different classifications in families of different kinds which share different properties, so we distinguish them in the following two subsections.

### A. The $\mathcal{R}^{(\alpha,\beta)}$ families

The family $\mathcal{R}^{(\alpha,\beta)}$ contains the polynomials with fixed parameters $\alpha$ and $\beta$.

$$\mathcal{R}^{(\alpha,\beta)} \equiv \{R_n^{(\alpha,\beta)} \,:\, n \in \mathbf{N}\}. \tag{36}$$

This family has one polynomial, and only one, of each degree; two different families do not share any polynomial in common and the union of all of them gives the whole set of Romanovski polynomials (i.e., they form a partition of this set).

The first property of one of these families is that which led to their construction: the family $\mathcal{R}^{(\alpha,\beta)}(x)$ comprises all the polynomial solutions of the hypergeometric differential equation

$$(1+x^2)R''(x) + (2\beta x + \alpha)R'(x) + \lambda R(x) = 0, \tag{37}$$

where $\lambda$ is a constant which, for the solution $R_n^{(\alpha,\beta)}(x)$, is given by $\lambda_n = -n(2\beta + n - 1)$.

Other characteristic properties of classical polynomials are present in the $\mathcal{R}^{(\alpha,\beta)}$ family too, such as a differential recursion relation and an expression for a generating function in closed form.

The differential recursion relation is obtained from the Rodrigues formula, Eq. (28), for the polynomial $R_{n+1}(x)$ (superscripts $(\alpha,\beta)$ omitted for clarity).

$$R_{n+1}(x) = \frac{1}{w(x)}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[w(x)s(x)^{n+1}\right]. \tag{38}$$

Then, because of the very definition of the weight function, Eq. (7), it is easy to see that

$$\left[w(x)s(x)^{n+1}\right]' = w(x)s(x)^n[2(\beta+n)x + \alpha]. \tag{39}$$

Upon substitution in Eq. (38) and a straightforward derivation one gets

$$R_{n+1}(x) = \frac{1}{w(x)}\left([2(\beta+n)x+\alpha]\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[w(x)s(x)^n\right] + 2n(\beta+n)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[w(x)s(x)^n\right]\right). \tag{40}$$

In the first term, the formula for $R_n(x)$ is recognized, while in the second term its derivative, $R_n'(x)$, appears (by means of Eq. (8) applied to the present case). The result, which makes use of Eqs. (9) and (31), is the following differential recursion relation

$$2(\beta+n)(1+x^2)\frac{\mathrm{d}R_n^{(\alpha,\beta)}(x)}{\mathrm{d}x} = (2\beta+n-1)\left(R_{n+1}^{(\alpha,\beta)}(x) - [2(\beta+n)x+\alpha]R_n^{(\alpha,\beta)}(x)\right). \tag{41}$$

An integral representation of the Romanovski polynomials is obtained by means of the Cauchy's integral formula. As the weight function can be extended to the complex plane, where it is analytic except at points $\pm i$, we can use Cauchy's integral formula to get

$$w^{(\alpha,\beta)}(x)s(x)^n = \frac{1}{2\pi i}\int_\gamma \frac{w^{(\alpha,\beta)}(z)s(z)^n}{z-x}\,\mathrm{d}z, \tag{42}$$

where $x$ is real but $z$ is a complex variable, and $\gamma$ is a closed curve in the complex plane, enclosing point $x$, but not $\pm i$. Substituting this equation into the definition of Romanovski polynomials, Eq. (28), we get

$$R_n^{(\alpha,\beta)}(x) = \frac{1}{2\pi i w^{(\alpha,\beta)}(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_\gamma \frac{w^{(\alpha,\beta)}(z)s(z)^n}{z-x}\,\mathrm{d}z. \tag{43}$$

The $n$ derivatives with respect to $x$ can be easily performed under the integral sign, giving rise to the following integral representation:

$$R_n^{(\alpha,\beta)}(x) = \frac{n!}{2\pi i w^{(\alpha,\beta)}(x)}\int_\gamma \frac{w^{(\alpha,\beta)}(z)s(z)^n}{(z-x)^{n+1}}\,\mathrm{d}z. \tag{44}$$

This representation is useful in calculating the generating function, as explained in [3]. A generating function, $R^{(\alpha,\beta)}(x,y)$, of the family $\mathcal{R}^{(\alpha,\beta)}$ is a function that is analytic in the variable $y$ and whose Taylor expansion in the variable $y$ has the form

$$R^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{\infty}\frac{y^k}{k!}R_k^{(\alpha,\beta)}(x). \tag{45}$$

Upon substitution of Eq. (44) in previous equation, and interchanging the order of integral and sum signs (which is allowed since the function is analytic), we get

$$R^{(\alpha,\beta)}(x,y) = \frac{1}{2\pi i w^{(\alpha,\beta)}(x)}\int_\gamma \frac{w^{(\alpha,\beta)}(z)}{z-x}\sum_{k=0}^{\infty}\frac{y^k s(z)^k}{(z-x)^k}\,\mathrm{d}z. \tag{46}$$

The sum is a geometric series, which can be summed as

$$\sum_{k=0}^{\infty}\left(\frac{ys(z)}{z-x}\right)^k = \frac{1}{1-\frac{ys(z)}{z-x}} = \frac{z-x}{z-x-ys(z)}, \tag{47}$$

provided $|\frac{ys(z)}{z-x}| < 1$. In this case, the expression for the generating function results in an integral which can be easily evaluated by the method of residues.

$$R^{(\alpha,\beta)}(x,y) = \frac{1}{2\pi i w^{(\alpha,\beta)}(x)}\int_\gamma \frac{w^{(\alpha,\beta)}(z)}{z-x-ys(z)}\,\mathrm{d}z = \frac{1}{w^{(\alpha,\beta)}(x)}\mathrm{Res}\left(\frac{w^{(\alpha,\beta)}(z)}{z-x-ys(z)}, z_1\right), \tag{48}$$

where $z_1$ is one of the roots (the one closer to $x$ so it is the only one enclosed by $\gamma$) of the second order polynomial in $z$ in the denominator: $z - x - ys(z) = -yz^2 + z - (x + y)$. The residue is

$$\mathrm{Res}\left(\frac{w^{(\alpha,\beta)}(z)}{z - x - ys(z)}, z_1\right) = \frac{-yw^{(\alpha,\beta)}(z_1)}{\sqrt{1 - 4y(x + y)}}, \tag{49}$$

where $z_1 = \frac{1}{2y}(1 - \sqrt{1 - 4y(x + y)})$. Direct substitution gives the final form for the generating function:

$$R^{(\alpha,\beta)}(x, y) = \frac{y(1 + A^2)^{\beta-1}\mathrm{e}^{-\alpha\cot^{-1}A}}{(2yA - 1)\,(1 + x^2)^{\beta-1}\mathrm{e}^{-\alpha\cot^{-1}x}}, \tag{50}$$

where

$$A = \frac{1 - \sqrt{1 - 4y(x + y)}}{2y}. \tag{51}$$

The next issue to study is the orthogonality properties inside one family. In contrast with the classical orthogonal polynomials, these families are not orthogonal with respect to the weight function $w^{(\alpha,\beta)}(x)$ in the natural interval $(-\infty, \infty)$ as the following counterexample shows.

Let us consider the family of polynomials with $\alpha = 0$ and $\beta = 0$ and, within it, the polynomials of degree 0 and 2 which, upon substitution in Eqs. (32) and (34), result

$$R_0^{(0,0)}(x) = 1, \tag{52}$$

$$R_2^{(0,0)}(x) = 2(x^2 + 1). \tag{53}$$

Then, an integral of the form of that in Eq. (12), in the interval $(-\infty, \infty)$, takes the form

$$\int_{-\infty}^{\infty} w^{(0,0)}(x)R_0^{(0,0)}(x)R_2^{(0,0)}(x)\,\mathrm{d}x = 2\int_{-\infty}^{\infty}\mathrm{d}x, \tag{54}$$

which does not converge. Notice that this poses also a problem with the normalization: polynomial $R_2^{(0,0)}(x)$, for instance, is not normalizable. The most that can be said is the following theorem [2],[4] that we state and prove, which establishes (the so-called finite) orthogonality among a few of the polynomials in the family.

> If $R_m^{(\alpha,\beta)}(x)$ and $R_n^{(\alpha,\beta)}(x)$, $m \neq n$, are Romanovski polynomials of degree $m$ and $n$ respectively, then
>
> $$\int_{-\infty}^{\infty} w^{(\alpha,\beta)}(x)R_m^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(x)\,\mathrm{d}x = 0 \tag{55}$$
>
> if, and only if, $m + n < 1 - 2\beta$.

The proof follows the steps of the usual proof of orthogonality of the polynomial solutions of hypergeometric equations. Express Eq. (37) in the adjoint form with the aid of the weight function (it is possible thanks to the Pearson's relation in Eq. (7)) for $R_m(x)$ (in the proof we drop the superscripts $(\alpha, \beta)$ as they are fixed):

$$[w(x)s(x)R'_m(x)]' + \lambda_m w(x)R_m(x) = 0. \tag{56}$$

Multiply by $R_n(x)$, and do the same as above but interchange $m$ and $n$. Then subtract and integrate, which leads to

$$\int_{-\infty}^{\infty} \{R_n(x)[w(x)s(x)R'_m(x)]' - R_m(x)[w(x)s(x)R'_n(x)]'\} \, \mathrm{d}x \\ + (\lambda_m - \lambda_n) \int_{-\infty}^{\infty} w(x)R_m(x)R_n(x) \, \mathrm{d}x = 0. \tag{57}$$

The second integral is the orthogonality integral of Eq. (12). The first integral, upon integration by parts, vanishes except for the boundary terms, so we are left with (because $\lambda_m \neq \lambda_n$)

$$\int_{-\infty}^{\infty} w(x)R_m(x)R_n(x) \, \mathrm{d}x = \frac{1}{\lambda_m - \lambda_n} \left\{w(x)s(x)[R_m(x)R'_n(x) - R'_m(x)R_n(x)]\right\}_{-\infty}^{\infty}. \tag{58}$$

The boundary term must be evaluated as

$$\lim_{x\to\infty} M(x) - \lim_{x\to-\infty} M(x), \tag{59}$$

where

$$M(x) = \Big((1+x^2)^{\beta} \mathrm{e}^{-\alpha \cot^{-1} x}[R_m(x)R'_n(x) - R'_m(x)R_n(x)]\Big). \tag{60}$$

Both limits must exist separately. Notice that the limit of the exponential factor is a positive constant (1 or $\mathrm{e}^{-\alpha\pi}$). The other factors behave as a power of $x$ with exponent $(2\beta+m+n-1)$. Then, both limits exist if, and only if, this exponent is negative, in which case both are zero, as claimed.

With a similar argument the following is proved, but we omit the explicit proof:

For the family of polynomials $\mathcal{R}^{(\alpha,\beta)}$ only the polynomials $R_n^{(\alpha,\beta)}(x)$ with $n < -\beta$ are normalizable, i.e., the integral

$$\int_{-\infty}^{\infty} w^{(\alpha,\beta)}(x) \left(R_n^{(\alpha,\beta)}(x)\right)^2 \, \mathrm{d}x, \tag{61}$$

converges.

So, for the family $\mathcal{R}^{(\alpha,\beta)}$ only the polynomials in a finite subset are normalizable, and only a finite number of couples are orthogonal. This type of orthogonality is sometimes called finite orthogonality.

### B. The $\mathcal{Q}^{(\alpha,\beta)}$ families

In order to calculate explicitly a polynomial $R_n^{(\alpha,\beta)}(x)$ one has to evaluate the $n$th derivative $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(w^{(\alpha,\beta)}(x)s(x)^n\right)$, step by step: first $\frac{\mathrm{d}}{\mathrm{d}x}\left(w^{(\alpha,\beta)}(x)s(x)^n\right)$, then $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(w^{(\alpha,\beta)}(x)s(x)^n\right)$, and so on. It is useful to notice the following relation holds

$$w^{(\alpha,\beta)}(x)s(x)^n = w^{(\alpha,\beta+n)}(x), \tag{62}$$

as can be seen directly in the definitions of $w^{(\alpha,\beta)}(x)$ and $s(x)$, Eq. (25). Thus, it is enough to have an expression for the $\nu$th derivative of $w^{(\alpha,\beta)}(x)$. In looking for it one soon discovers the structure of these derivatives: the $\nu$th derivative can be factorized as $w(x)s(x)^{-\nu}$ and a polynomial of degree $\nu$. For instance, it is easy to calculate directly the first derivative (compare with Pearson's ODE (7))

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[w^{(\alpha,\beta)}(x)\right] = w^{(\alpha,\beta)}(x)s(x)^{-1}[2(\beta-1)x+\alpha]. \tag{63}$$

By induction, it is easy to prove the statement in general: for fixed $\alpha$ and $\beta$, the $\nu$th derivative, $\nu \in \mathbf{N}$, of $w^{(\alpha,\beta)}(x)$ has the form

$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\left[w^{(\alpha,\beta)}(x)\right] = Q_\nu(x)w^{(\alpha,\beta)}(x)s(x)^{-\nu}, \tag{64}$$

where $Q_\nu(x)$ is a polynomial of degree $\nu$. The case $\nu = 0$ is trivial, and the case $\nu = 1$ is shown in Eq. (63). The step from case $\nu$ to $\nu + 1$ is a straightforward derivation. So, for given $\alpha$, $\beta$ and $\nu$ we define the polynomial $Q_\nu^{(\alpha,\beta)}$ as

$$Q_\nu^{(\alpha,\beta)}(x) \equiv \frac{s(x)^\nu}{w^{(\alpha,\beta)}(x)}\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\left(w^{(\alpha,\beta)}(x)\right). \tag{65}$$

Using Pearson's ODE (7) as above the first three polynomials for fixed $\alpha$ and $\beta$ are

$$Q_0^{(\alpha,\beta)}(x) = 1, \tag{66}$$

$$Q_1^{(\alpha,\beta)}(x) = 2(\beta-1)x+\alpha, \tag{67}$$

$$Q_2^{(\alpha,\beta)}(x) = 2(\beta-1)(2\beta-3)x^2 + 2\alpha(2\beta-3)x + 2(\beta-1) + \alpha^2. \tag{68}$$

By looking at these expressions, and upon comparison with the first three $R_n^{(\alpha,\beta)}(x)$ polynomials, one readily suspects some relation between the polynomials $Q_\nu(x)$ and $R_\nu(x)$. This relation indeed exists and is the following.

The polynomial $Q_\nu^{(\alpha,\beta)}(x)$ is a Romanovski polynomial, specifically

$$Q_\nu^{(\alpha,\beta)} = R_\nu^{(\alpha,\beta-\nu)}. \tag{69}$$

The proof is a straightforward manipulation of Eq. (28), applied to $R_\nu^{(\alpha,\beta-\nu)}$ with the aid of Eq. (62):

$$\begin{aligned} R_\nu^{(\alpha,\beta-\nu)}(x) &= \frac{1}{w^{(\alpha,\beta-\nu)}(x)} \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} \left[ w^{(\alpha,\beta-\nu)}(x) s(x)^\nu \right] \\ &= \frac{1}{w^{(\alpha,\beta)}(x) s(x)^{-\nu}} \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} \left[ w^{(\alpha,\beta)}(x) \right]. \end{aligned} \tag{70}$$

But the last term is precisely the definition of $Q_\nu^{(\alpha,\beta)}(x)$, Eq. (65).

We now define the family $\mathcal{Q}^{(\alpha,\beta)}$ as

$$\mathcal{Q}^{(\alpha,\beta)}(x) \equiv \{ Q_\nu^{(\alpha,\beta)} \ : \ \nu \in \mathbf{N} \} = \{ R_\nu^{(\alpha,\beta-\nu)} \ : \ \nu \in \mathbf{N} \}. \tag{71}$$

This family contains Romanovski polynomials with one fixed superscript and the other running with the degree. As in the case of the $\mathcal{R}$ families, the $\mathcal{Q}^{(\alpha,\beta)}$ family contains one, and only one, polynomial of each degree; two different families do not share any polynomial in common and the union of all the families gives the whole set of Romanovski polynomials (i.e. they constitute another partition of this set).

We now study the properties of the polynomials in the family $\mathcal{Q}^{(\alpha,\beta)}$.

In the first place we give some Rodrigues-looking expressions for the $Q_\nu^{(\alpha,\beta)}$ polynomials. The polynomials in the family $\mathcal{Q}^{(\alpha,\beta)}$ obey the following formulas:

$$Q_\nu^{(\alpha,\beta)}(x) = \frac{1}{w^{(\alpha,\beta-\nu)}(x)} \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} \left[ w^{(\alpha,\beta-\nu)}(x) s(x)^\nu \right], \tag{72}$$

$$Q_\nu^{(\alpha,\beta)}(x) = \frac{1}{w^{(\alpha,\beta)}(x) s(x)^{-\nu}} \frac{\mathrm{d}^{\nu-\mu}}{\mathrm{d}x^{\nu-\mu}} \left[ w^{(\alpha,\beta)}(x) s(x)^{-\mu} Q_\mu^{(\alpha,\beta)}(x) \right]. \tag{73}$$

The first one arises right from the definition in Eq. (65) and Eq. (62). For the second formula, take $\mu$ derivatives of $w^{(\alpha,\beta)}(x)$ in Eq. (65) and apply the definition of the $Q_\mu^{(\alpha,\beta)}(x)$ polynomial.

A second property is the one which makes worthy the definition of the families $\mathcal{Q}^{(\alpha,\beta)}$, for it captures the exact relation between a Romanovski polynomial and its derivative, and expresses it in a simple fashion. We already know that if $R_n^{(\alpha,\beta)}(x)$ is a Romanovski polynomial, so is its derivative, because it is a solution of the same hypergeometric equation for the parameters $(\alpha, \beta+1)$. So $\frac{\mathrm{d}R_n^{(\alpha,\beta)}(x)}{\mathrm{d}x}$ and $R_{n-1}^{(\alpha,\beta+1)}(x)$ must be proportional. Working through Eqs. (8), (9) and (31), the exact relation results:

$$\frac{\mathrm{d}R_n^{(\alpha,\beta)}(x)}{\mathrm{d}x} = n(2\beta+n-1)\, R_{n-1}^{(\alpha,\beta+1)}(x). \tag{74}$$

This equation, when expressed in terms of the $Q_\nu(x)$ polynomials, takes the form

$$\frac{\mathrm{d}Q_\nu^{(\alpha,\beta)}(x)}{\mathrm{d}x} = \nu(2\beta+\nu-1)\, Q_{\nu-1}^{(\alpha,\beta)}(x), \tag{75}$$

which tells us that in the family $\mathcal{Q}^{(\alpha,\beta)}$, each polynomial is the derivative (up to a constant factor) of the following polynomial in the same family.

For a third property, let us note first that the family $\mathcal{Q}^{(\alpha,\beta)}$ obeys the following differential recurrence relation:

$$Q_{\nu+1}^{(\alpha,\beta)}(x) = s(x)\frac{\mathrm{d}Q_\nu^{(\alpha,\beta)}(x)}{\mathrm{d}x} + [2(\beta+\nu-1)x+\alpha]Q_\nu^{(\alpha,\beta)}(x). \tag{76}$$

For a proof, derive the definition of $Q_\nu^{(\alpha,\beta)}(x)$, Eq. (65), and get

$$\frac{\mathrm{d}Q_\nu^{(\alpha,\beta)}(x)}{\mathrm{d}x} = -\frac{1}{(w^{(\alpha,\beta-\nu)}(x))^2}\frac{\mathrm{d}w^{(\alpha,\beta-\nu)}(x)}{\mathrm{d}x}\frac{\mathrm{d}^\nu w^{(\alpha,\beta)}(x)}{\mathrm{d}x^\nu} + \frac{1}{w^{(\alpha,\beta-\nu)}(x)}\frac{\mathrm{d}^{\nu+1}w^{(\alpha,\beta)}(x)}{\mathrm{d}x^{\nu+1}}.$$

The second term on the right is $s^{-1}(x)Q_{\nu+1}^{(\alpha,\beta)}(x)$. The first term in the right, after the derivation of $w^{(\alpha,\beta-\nu)}(x)$, gives $s^{-1}(x)[2(\beta-\nu-1)x+\alpha]Q_\nu^{(\alpha,\beta)}(x)$. The result arises after reordering. Then, the substitution of the derivative formula, Eq. (75), gives a three term recurrence relation.

$$Q_{\nu+1}^{(\alpha,\beta)}(x) - [2(\beta+\nu-1)x+\alpha]Q_\nu^{(\alpha,\beta)}(x) - \nu(2\beta+\nu-1)(1+x^2)Q_{\nu-1}^{(\alpha,\beta)}(x) = 0, \tag{77}$$

from which the polynomials can be efficiently generated, in contrast to the Rodrigues formulas. A fourth property states that $Q_\nu^{(\alpha,\beta)}(x)$ is the polynomial solution to the hypergeometric differential equation

$$(1+x^2)Q''(x) + [2(\beta-\nu)x+\alpha]Q'(x) + \lambda_\nu Q(x) = 0, \tag{78}$$

where $\lambda_\nu = -\nu(2\beta - \nu - 1)$. This equation is just the hypergeometric differential equation (37) for the polynomial $R_\nu^{(\alpha,\beta-\nu)}(x)$, which is $Q_\nu^{(\alpha,\beta)}(x)$, so it is proved.

The fifth property is that there exists a generating function $Q^{(\alpha,\beta)}(x,y)$ in closed form for the family $\mathcal{Q}^{(\alpha,\beta)}$, which is an analytic function whose Taylor expansion in the variable $y$ is given by

$$Q^{(\alpha,\beta)}(x,y) = \sum_{\nu=0}^{\infty} \frac{y^\nu}{\nu!} Q_\nu^{(\alpha,\beta)}(x). \tag{79}$$

By substituting the definition of $Q_\nu^{(\alpha,\beta)}(x)$, Eq. (65), in previous equation and grouping factors and setting $z = x + ys(x)$ we get

$$\begin{aligned} Q^{(\alpha,\beta)}(x,y) &= \frac{1}{w^{(\alpha,\beta)}(x)} \sum_{\nu=0}^{\infty} \frac{[ys(x)]^\nu}{\nu!} \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} \left[w^{(\alpha,\beta)}(x)\right] \\ &= \frac{1}{w^{(\alpha,\beta)}(x)} \sum_{\nu=0}^{\infty} \frac{(z-x)^\nu}{\nu!} \frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu} \left[w^{(\alpha,\beta)}(z)\right] |_{z=x}, \end{aligned} \tag{80}$$

which is a Taylor expansion of the function inside the derivative at the point $(x + ys(x))$ with base point $x$. Thus, the summation of the series is given by $w^{(\alpha,\beta)}$ at the point $B = x + ys(x) = x + y(1 + x^2)$. The result is

$$Q^{(\alpha,\beta)}(x,y) = \frac{(1+B^2)^{\beta-1} \mathrm{e}^{-\alpha \cot^{-1} B}}{(1+x^2)^{\beta-1} \mathrm{e}^{-\alpha \cot^{-1} x}}, \tag{81}$$

from which numerous recursion relations, such as Eq. (77), may be derived as usual [17], [18], [19].

We now address an orthogonality property of the $Q_\nu^{(\alpha,\beta)}(x)$ polynomials. Polynomials in the family $\mathcal{Q}^{(\alpha,\beta)}$, with $\beta < \varepsilon - \frac{1}{2}$, satisfy the following relation:

$$\int_{-\infty}^{\infty} \frac{w^{(\alpha,\beta)}(x)}{s(x)^{\frac{\varepsilon}{2}}} \frac{Q_m^{(\alpha,\beta)}(x)}{s(x)^{\frac{m}{2}}} \frac{Q_n^{(\alpha,\beta)}(x)}{s(x)^{\frac{n}{2}}} \, \mathrm{d}x = 0, \tag{82}$$

where $m \neq n$ and $\varepsilon = 1$ if $m + n$ is odd, and $\varepsilon = 2$ if $m + n$ is even. This is an orthogonality integral between the functions $Q_m^{(\alpha,\beta)}/s(x)^{m/2}$ and $Q_n^{(\alpha,\beta)}/s(x)^{n/2}$ built on top of the polynomials in $\mathcal{Q}^{(\alpha,\beta)}$. In contrast to the orthogonality relations in the $\mathcal{R}^{(\alpha,\beta)}$ families, which are valid only for a finite subfamily of polynomials, Eq. (82) applies to the whole family $\mathcal{Q}^{(\alpha,\beta)}$. In terms of the Romanovski polynomials in Eq. (69) the integral in Eq. (82) takes the following form

$$\int_{-\infty}^{\infty} \sqrt{w^{(\alpha,\beta-m)}(x)} \, R_m^{(\alpha,\beta-m)}(x) \sqrt{w^{(\alpha,\beta-n)}(x)} \, R_n^{(\alpha,\beta-n)}(x) \frac{1}{s(x)^{\frac{\varepsilon}{2}}} \, \mathrm{d}x = 0, \tag{83}$$

which can be read as orthogonality within the infinite sequence of polynomials $R_k^{(\alpha,\beta-k)}$ with a running parameter attached to the polynomial degree.

In fact, Eq. (82) stands for two different results which require separate proofs. In any case, since $m \neq n$, we can take $m > n$. Let us consider first the case of even $m+n$. Then, the integral of Eq. (82) is

$$O_{m,n} = \int_{-\infty}^{\infty} \frac{w^{(\alpha,\beta)}(x)}{s(x)} \frac{Q_m^{(\alpha,\beta)}(x)}{s(x)^{\frac{m}{2}}} \frac{Q_n^{(\alpha,\beta)}(x)}{s(x)^{\frac{n}{2}}} \, \mathrm{d}x. \tag{84}$$

Upon substitution of $Q_m^{(\alpha,\beta)}(x)$ by its definition, Eq. (65), we get

$$O_{m,n} = \int_{-\infty}^{\infty} s(x)^{\frac{1}{2}(m-n)-1} Q_n^{(\alpha,\beta)}(x) \frac{\mathrm{d}^m}{\mathrm{d}x^m} w^{(\alpha,\beta)}(x) \, \mathrm{d}x. \tag{85}$$

Because $m+n$ is even and $m>n$, then $m-n-2$ is an even, nonnegative, integer. Thus, $s(x)^{\frac{1}{2}(m-n)-1}$ is a polynomial of degree $m-n-2$ and $s(x)^{\frac{1}{2}(m-n)-1} Q_n^{(\alpha,\beta)}(x)$ is a polynomial of degree $m-2$, which we call $P_{m-2}$. Then, after $m-1$ integrations by parts, $m-1$ derivatives are applied to $P_{m-2}$ so it vanishes and we are left only with the boundary terms.

$$O_{m,n} = \sum_{k=1}^{m-1} (-1)^{k-1} \left[ \frac{\mathrm{d}^{k-1} P_{m-2}(x)}{\mathrm{d}x^{k-1}} \frac{\mathrm{d}^{m-k} w^{(\alpha,\beta)}(x)}{\mathrm{d}x^{m-k}} \right]_{-\infty}^{\infty}. \tag{86}$$

For each $k$, the derivative of $P_{m-2}$ is a polynomial of degree $m-k-1$ whereas the $m-k$ derivative of $w^{(\alpha,\beta)}$ is given in terms of the polynomial $Q_{m-k}^{(\alpha,\beta)}$, again by its definition in Eq. (65). Then, the $k$ boundary term results

$$\frac{\mathrm{d}^{k-1} P_{m-2}(x)}{\mathrm{d}x^{k-1}} \frac{\mathrm{d}^{m-k} w^{(\alpha,\beta)}(x)}{\mathrm{d}x^{m-k}} = \mathrm{e}^{-\alpha \cot^{-1} x} (1+x^2)^{\beta-m+k-1} \tilde{P}_{2m-2k-1}(x), \tag{87}$$

where $\tilde{P}_{2m-2k-1}$ is a polynomial of degree $2m-2k-1$. The asymptotic behavior of this term at $\pm\infty$ is the same as $x^{2\beta-3}$ and, thus, it goes to zero if, and only if, $\beta < \frac{3}{2}$.

The proof of the case with $(m+n)$ odd is similar.

## IV. RELATIONSHIP BETWEEN ROMANOVSKI POLYNOMIALS AND JACOBI POLYNOMIALS

Romanovski and Jacobi polynomials are closely related. In fact, it is common that Romanovski polynomials are referred to as complexified Jacobi polynomials [1], [10]. In this section we are showing which is the precise relationship between them and which is not:

Romanovski polynomials can indeed be obtained from a generalization of Jacobi polynomials to the complex plane, but not through the complexification of Jacobi polynomials, which is a different issue. Let us distinguish both concepts.

For ease of reference, we recall here equations (1) and (2), which are the equations Romanovski polynomials and Jacobi polynomials solve, respectively.

$$(1+x^2)R'' + t(x)R' + \lambda R = 0, \tag{88a}$$

$$(1-x^2)P'' + t(x)P' + \lambda P = 0, \tag{88b}$$

where $t(x)$ is a polynomial of, at most, first degree.

The argument of the complexification is based on the fact that the change from $x$ to $ix$ transforms the coefficient $(1-x^2)$ in Eq. (88b) into $(1+x^2)$, the coefficient in Eq. (88a). But caution is needed with this idea. Complexification is a transformation which takes real valued functions of a real variable into complex functions of a real variable. If $g : \mathcal{U} \subset \mathbf{R} \to \mathbf{R}$ is such a real valued function, defined in an open subset of the real line, we define the function $\widetilde{g} : \mathcal{U}' \subset \mathcal{U} \to \mathbf{C}$ by the recipe (wherever it makes sense)

$$\widetilde{g}(x) \equiv g(ix). \tag{89}$$

The new function $\widetilde{g}$ may, or may not, be well defined in all the points of $\mathcal{U}$ and may, or may not, inherit the continuity and differentiability properties of $g$ in all points of $\mathcal{U}$ (think, for an instance, of the function $g(x) = (1+x^2)^{-1}$). In the case $g$ happens to be a polynomial it is easy to see that both continuity and differentiability are indeed respected. Then, the derivatives of $g$ and $\widetilde{g}$ with respect to $x$ satisfy the following identity:

$$\begin{gathered} \widetilde{g}^{(n)} = i^n \widetilde{g^{(n)}}, \\ n \in \{0, 1, \ldots\}. \end{gathered} \tag{90}$$

Complexification respects the sum and product operations, i.e., $\widetilde{f+g} = \widetilde{f} + \widetilde{g}$ and $\widetilde{f\,g} = \widetilde{f}\widetilde{g}$ (it is a ring homomorphism from a ring of real valued functions of a real variable to the ring of complex valued functions of a real variable). Thus, if the function $g$ is a solution to a linear differential equation, then $\widetilde{g}$ is a solution of the complexification of that linear differential equation. The application of this argument to the Jacobi polynomials gives the result that $\widetilde{P_n^{(\alpha,\beta)}}(x) \equiv P_n^{(\alpha,\beta)}(ix)$ verifies the complexification of Eq. (88b), namely

$$(1+x^2)(\widetilde{P})'' + i\,t(ix)(\widetilde{P})' - \lambda \widetilde{P} = 0, \tag{91}$$

where the prime still stands for derivative with respect to the real variable $x$. But equation (91) is not the same as equation (88a) unless $i\,t(ix)$ is real. If we write $t(x) = \beta - \alpha - (\alpha + \beta + 2)x$, as is customary in the Jacobi equation (see Eq. (19)), we need $(\alpha + \beta + 2)$ to be real and $(\beta - \alpha)$ to be imaginary, which is achieved only if $\alpha$ and $\beta$ are complex and $\beta = \alpha^*$. Hence we have to consider the functions $P_n^{(\alpha,\beta)}(ix)$ with complex parameters $\alpha$ and $\beta$ which are no longer the complexification of the classical Jacobi polynomials as described above. So, the complexification of Jacobi polynomials does not result in the Romanovski polynomials. But, even in case it did, not all the properties of Jacobi polynomials would be translated to properties of Romanovski polynomials: only those which made use of theorems like equation (90), which relates the derivatives. For instance, there is no theorem relating integrals of a complexified function and the original function; thus, all the properties depending on integrations, such as the orthogonality, would have no translation to the complexified version.

An alternative scenario is to extend the definition of Jacobi polynomials to the complex plane: complex variable $z$, complex parameters $\alpha$ and $\beta$ and, obviously, complex values. In [20] this definition has been successfully given as

$$P_n^{(\alpha,\beta)}(z) \equiv \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (1-z)^k (1+z)^{n-k} \tag{92}$$

or, equivalently, by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2^n n!} (1-z)^{-\alpha} (1+z)^{-\beta} \frac{d^n}{dz^n} \left[ (1-z)^{n+\alpha} (1+z)^{n+\beta} \right], \tag{93}$$

which are formally the same as the classical ones except now $z, \alpha, \beta \in \mathbf{C}$ while $n \in \{0, 1, 2, \ldots\}$. These polynomials solve the complex Jacobi ODE

$$(1-z^2)P''(z) + [\beta - \alpha - (\alpha + \beta + 2)z]P'(z) + (\alpha + \beta + 1 + n)nP(z) = 0, \tag{94}$$

where the prime stands now for the derivative with respect to the complex variable $z$.

The specialization of the variable to the imaginary axis, $z = ix$, and the parameters to $\beta = \alpha^*$ leaves us with Eq. (88a) (notice the change from $\mathrm{d}/\mathrm{d}z$ to $\mathrm{d}/\mathrm{d}x$ gives an extra $i$), so these complex Jacobi polynomials solve the differential equation that define the Romanovski polynomials. One has to prove now that functions $P_n^{(\alpha,\alpha^*)}(z)$ restricted to $z = ix$ are real valued or, at least, proportional to a real valued one. This is easily achieved by computing the complex conjugate of the function $P_n^{(\alpha,\alpha^*)}(ix)$ in terms of the definition in Eq. (92)

$$P_n^{(\alpha,\alpha^*)}(ix)^* = \frac{(-1)^n}{2^n} \sum_{k=0}^{n} \binom{n+\alpha^*}{n-k} \binom{n+\alpha}{k} (1+ix)^k (1-ix)^{n-k}. \tag{95}$$

With a change in the summation index from $k$ to $l = n - k$, we get

$$P_n^{(\alpha,\alpha^*)}(ix)^* = (-1)^n P_n^{(\alpha,\alpha^*)}(ix), \tag{96}$$

i.e., for even $n$, $P_n^{(\alpha,\alpha^*)}(ix)$ is real, while for odd $n$, it is imaginary. Hence, the combination $i^n P_n^{(\alpha,\alpha^*)}(ix)$ is a real function for all $n$. Finally, because the polynomial solutions of a hypergeometric differential equation are unique for each degree, up to a constant factor, we conclude that $i^n P_n^{(\alpha,\alpha^*)}(ix)$ is the Romanovski polynomial of degree $n$ with parameters $-2\Im(\alpha)$ and $-(\Re(\alpha)+1)$. In other words, complex Jacobi polynomials do provide another characterization of the Romanovski polynomials via

$$R_n^{(\alpha,\beta)}(x) = i^n P_n^{(1-\beta-\frac{i}{2}\alpha, 1-\beta+\frac{i}{2}\alpha)}(ix), \tag{97}$$

(with suitably chosen normalization constants for the Jacobi polynomials). However, this alternative characterization is of no help when it comes to study the orthogonality properties, because the orthogonality properties of the complex Jacobi polynomials are not well known. In [20] the authors state some new results on the orthogonality along some particular paths in the complex plane. For instance, in their Eq. (4.3) an orthogonality relation is given along the imaginary axis, which is our case, but only for a special case demanding real, not integer, parameters, which is not our case. To our knowledge, at the present time there are no results concerning the orthogonality of these complex polynomials which would provide an alternative approach to the results on orthogonality stated in Subsection III A.

The argument presented here states that Romanovski polynomials are just a subset of complex Jacobi polynomials. Therefore it may seem that Romanovski polynomials are, somehow, subordinated to the Jacobi polynomials. But the whole argument could be reversed if we had a definition of complex Romanovski polynomials as the one given in Ref. [20] (here reproduced in Eq. (92)). If that would be the case, it would not be surprising to get a relation of the form

$$P_n^{(\alpha,\beta)}(x) = -(i^n) R_n^{(i(\alpha-\beta), \frac{1}{2}(\alpha+\beta)+1)}(ix),$$

where, now, $P_n^{(\alpha,\beta)}(x)$ is a real Jacobi polynomial and $R_n^{(i(\alpha-\beta),\frac{1}{2}(\alpha+\beta)+1)}(ix)$ would be a complex Romanovski polynomial. But, as we do not have such a definition, this last formula is nothing but a conjecture.

## V. ROMANOVSKI POLYNOMIALS IN SELECTED QUANTUM MECHANICS PROBLEMS

The Romanovski polynomials are part of the exact solutions of several problems in ordinary and supersymmetric quantum mechanics. In this section we review a few prominent cases. The selection of the examples certainly reflects personal preferences and does not pretend to be complete.

In general, the exactly soluble Schrödinger equations enjoy a special status because most of them describe phenomena that play a key role in physics. Suffice it to mention in that regard such textbook examples as the description of the spectrum of the hydrogen atom in terms of the Coulomb potential [21], or the description of vibrational modes in molecules and nuclei in terms of the Hulthen and Morse potentials [22], [23]. More recently, exactly soluble potentials acquired importance within the context of supersymmetric quantum mechanics (SUSYQM) which considers the special class of Schrödinger equations $(H(z) - E)\Psi(z) = 0$, with $H(z)$ standing for the Hamiltonian (of the one-dimensional, real variable $z$), and $E$ for the energy, which allow [24] a factorization of $H(z)$ according to $H(z) = \mathcal{A}^+(z)\mathcal{A}^-(z) + E_{\mathrm{gst}}$, and $\mathcal{A}^-(z)\Psi_{\mathrm{gst}}(z) = 0$. SUSYQM provides a powerful technique for finding the exact solutions of Schrödinger equations. To be specific, any excited state can be obtained by the successive action on the ground state, $\Psi_{\mathrm{gst}}(z)$, of an appropriate number of creation operators, $\mathcal{A}^+(z)$, defined in terms of the so-called superpotential, $\mathcal{U}(z)$, as

$$\mathcal{A}^{\pm}(z) \equiv \left( \pm \frac{\hbar}{\sqrt{2\mu}} \frac{\mathrm{d}}{\mathrm{d}z} + \mathcal{U}(z) \right).$$

Supersymmetric quantum mechanics governs a family of exactly soluble potentials (see Refs. [25]– [28] for details) two of which are the so-called hyperbolic Scarf and trigonometric Rosen-Morse potentials, that have been solved recently in [6], [7], [16] in terms of the Romanovski polynomials as discussed in the next two subsections. The third subsection is devoted to applications of the Romanovski polynomials in random matrix theory.

### A. Romanovski polynomials in problems with non-central electric potentials

The (one-dimensional) Schrödinger equation with the hyperbolic Scarf potential is

$$\left(-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}z^2}+V_h(z)-E\right)\Psi(z)=0\,,$$
$$V_h(z)\equiv[B^2-A(A+1)]\frac{1}{\cosh^2 z}-B(2A+1)\tanh z\frac{1}{\cosh z}\,. \tag{98}$$

This equation appears, among others, in the problem of a particle within a non-central scalar potential, a result due to Ref. [9]. In denoting such a potential by $V(r,\theta)$, one can make for it the specific choice of

$$V(r,\theta)=V_1(r)+\frac{V_2(\theta)}{r^2}\,,$$
$$V_2(\theta)=-b\cot\theta. \tag{99}$$

An interesting phenomenon is the electrostatic non-central potential in which case $V_1(r)$ is the Coulomb potential. The corresponding Schrödinger equation

$$\left[-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]+V(r,\theta)\right]\Psi(r,\theta,\varphi)$$
$$=E\Psi(r,\theta,\varphi)\,, \tag{100}$$

is solved in the standard way by separating variables. As long as the potential does not depend on the azimuthal angle, one assumes

$$\Psi(r,\theta,\phi)=\mathcal{R}(r)\Theta(\theta)e^{im\phi}\,. \tag{101}$$

The radial and angular differential equations for $\mathcal{R}(r)$ and $\Theta(\theta)$ are then found as

$$\frac{d^2\mathcal{R}(r)}{dr^2}+\frac{2}{r}\frac{d\mathcal{R}(r)}{dr}+\left[\frac{2\mu}{\hbar^2}[V_1(r)+E]-\frac{l(l+1)}{r^2})\right]\mathcal{R}(r)=0, \tag{102}$$

and

$$\frac{d^2\Theta(\theta)}{d\theta^2}+\cot(\theta)\frac{d\Theta(\theta)}{d\theta}+\left[l(l+1)-\frac{2\mu V_2(\theta)}{\hbar^2}-\frac{m^2}{\sin^2\theta}\right]\Theta(\theta)=0\,, \tag{103}$$

with $l(l+1)$ being the separation constant. From now on we will focus on the second equation. Notice that for $V_2(\theta)=0$, and upon changing variables from $\theta$ to $\cos\theta$, the last equation transforms into the associated Legendre equation and correspondingly

$$\Theta(\theta)\xrightarrow{V_2(\theta)\to 0}P_l^m(\cos\theta)\,, \tag{104}$$

an observation that will become important below.

Following Ref. [9] one begins with substituting the polar angle variable by a new variable, $z$, introduced via $\theta = f(z)$, with $f$ to be determined. This leads to the new equation

$$\left[\frac{d^2}{dz^2} + \left[-\frac{f''(z)}{f'(z)} + f'(z)\cot f(z)\right]\frac{d}{dz} + \left[-\frac{2\mu}{\hbar^2}V_2(f(z)) + l(l+1) - \frac{m^2}{\sin^2 f(z)}\right] f'^{\,2}(z)\right]\psi(z) = 0 \quad (105)$$

with $f'(z) \equiv \frac{df(z)}{dz}$, and $\psi(z)$ defined as $\psi(z) \equiv \Theta(f(z))$. Next one can require that $f'(z)$ approaches zero at $z=0$ like $\sin z$, meaning, $\lim_{z\to 0} f'(z)/\sin z = 1$, and define $f(z)$ via

$$\frac{f''(z)}{f'(z)} = f'(z)\cot f(z)\,. \quad (106)$$

The latter equation is solved by $f(z) = 2\tan^{-1} e^z$. With this relation one finds that

$$\sin\theta = \frac{1}{\cosh z}, \quad \cos\theta = -\tanh z\ , \quad (107)$$

and consequently, $f'(z) = \sin f(z) = \operatorname{sech} z$. Upon substituting the last relations into Eqs. (99), and (105), one arrives at

$$\frac{d^2\psi(z)}{dz^2} + \left[l(l+1)\frac{1}{\cosh^2 z} - \frac{2\mu}{\hbar^2} b\tanh z\frac{1}{\cosh z} - m^2\right]\psi(z) = 0\,. \quad (108)$$

In taking in consideration Eqs. (98), (107) one realizes that the letter equation is precisely the one-dimensional Schrödinger equation with the hyperbolic Scarf potential and with

- $l(l+1)$ playing the role of $-(B^2 - A(A+1))/\left(\hbar^2/(2\mu)\right)$,
- $m^2$ playing the role of $-E_n/\left(\hbar^2/(2\mu)\right)$,
- $b$ playing the role of $-B(2A+1)$.

This equation has been solved in terms of the Romanovski polynomials in Ref. [6] upon substituting $\sinh z = x$. Notice that there the weight function was defined as $(1+x^2)^{-p}e^{q\tan^{-1}x}$, and the polynomials have been labeled correspondingly as $R_n^{(p,q)}(x)$, following [29]. A comparison with Eq. (25) allows identifying $p \longrightarrow -\beta+1$, $q \longrightarrow \alpha$. In terms of the notations of the present work, the result of Ref. [6] can be cast into the following form:

$$\begin{aligned}\psi_n(z) &= C_n[1+(\sinh z)^2]^{\frac{\beta}{2}-\frac{1}{4}}\ \mathrm{e}^{\frac{\alpha}{2}\tan^{-1}\sinh z} R_n^{(\alpha,\beta)}(\sinh z),\\ \alpha &= -2B, \quad \beta = -A + \frac{1}{2}, \quad E_n = -(A-n)^2\,,\end{aligned} \quad (109)$$

with $C_n$ being a normalization constant. Back to the $\theta$ variable and in making use of the equality $x \stackrel{\text{def}}{:=} \sinh z = -\cot\theta$, we find

$$\Theta(\theta) = \psi_n(\sinh^1(-\cot\theta)) = C_n[1+(\cot\theta)^2]^{\frac{\beta}{2}-\frac{1}{4}}\ \mathrm{e}^{\frac{\alpha}{2}\tan^{-1}(-\cot\theta)} R_n^{(\alpha,\beta)}(-\cot\theta), \tag{110}$$

showing that the angular part of the exact solution to the non-central potential under consideration is defined by the Romanovski polynomials. In turning off the non-central piece of the potential, the angular part of the solutions will become the standard spherical harmonics, $Y_l^m(\theta,\phi) = P_l^m(\theta)\ \mathrm{e}^{im\phi}$, which will produce a relationship between the Romanovski polynomials and the associated Legendre functions, an issue to be considered in more detail at the end of this section.

Now, in accord with the theorem on the finite orthogonality of the Romanovski polynomials in Eq. (55), also only a finite number of eigen-wave functions to the hyperbolic Scarf potential appears orthogonal,

$$\int_{-\infty}^{+\infty} \psi_n(z)\psi_m(z)dz = C_m C_n \int_{\infty}^{+\infty} w^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(x) = \delta_{mn}, \qquad m+n \leq 2A\,. \tag{111}$$

This finite orthogonality reflects the finite number of bound states within the potential under consideration.

Next, it is quite instructive to consider the case of a vanishing $V_2(\theta)$, i.e. $b=0$, and compare Eq. (108) to Eq. (98) for $B=0$. From now on we will give all quantities in units of $\hbar = 1 = 2\mu$. In this case

- Eq. (108) reduces to the equation for the associated Legendre polynomials, $P_l^m(\cos\theta)$,
- $l$ becomes $A$,
- $m^2$ becomes $(l-n)^2$,
- Eq. (98) produces $R_n^{(\alpha=0,\beta=-l+\frac{1}{2})}(x)$ as part of its solutions,

which allows one to relate $n$ to $l$ and $m$ as $m = l - n$. In taking into account Eq. (104) together with $\cot\theta = -\sinh z$ provides the following relationship between the associated Legendre functions and the Romanovski polynomials

$$P_l^m(\cos\theta) = \text{const}[1+(\cot\theta)^2]^{-\frac{l}{2}} R_{m+l}^{(0,\frac{1}{2}-l)}(-\cot\theta)\,, \qquad m+l = n \in \{0,1,\ldots,l\}. \tag{112}$$

In substituting the latter equation into the orthogonality integral between the associated Legendre functions,

$$\int_{-1}^{1} P_l^m(\cos\theta) P_{l'}^m(\cos\theta)\, \mathrm{d}\cos\theta = 0, \qquad l \neq l', \tag{113}$$

we find

$$\int_{-1}^{1} (1+(\cot\theta)^2)^{-\frac{l+l'}{2}} R_{l+m}^{(0,\frac{1}{2}-l)}(-\cot\theta) R_{l'+m}^{(0,\frac{1}{2}-l')}(-\cot\theta) \mathrm{d}\cos\theta = 0, \qquad l \neq l'. \tag{114}$$

The latter relationship amounts to the following orthogonality integral

$$\int_{-\infty}^{\infty} (1+(\sinh z)^2)^{-\frac{l+l'}{2}} R_n^{(0,\frac{1}{2}-l)}(\sinh z) R_{n'}^{(0,\frac{1}{2}-l')}(\sinh z)(\mathrm{sech}\ z)^2 \mathrm{d}z = \int_{-\infty}^{\infty} \sqrt{w^{(0,\frac{1}{2}-l)}(x)} R_n^{(0,\frac{1}{2}-l)}(x) \sqrt{w^{(0,\frac{1}{2}-l')}(x)} R_{n'}^{(0,\frac{1}{2}-l')}(x) \frac{1}{s(x)}\, \mathrm{d}x = 0,$$
$$x = \sinh z, \quad l - n = l' - n' = m \geq 0, \quad l \neq l'. \tag{115}$$

Careful inspection shows that this equation is nothing but a particular case of the orthogonality relation in the family of polynomials $\mathcal{Q}^{(\alpha,\beta)}$, established in Eq. (82) and translated to the $R_n^{(\alpha,\beta)}$ notation in Eq. (83).

A further example is given by the Klein-Gordon equation with equal scalar and vector potentials. It has been shown in Ref. [8] that the former can be reduced to the corresponding Schrödinger equation. Therefore, in case one uses the hyperbolic Scarf potential in the above Klein-Gordon equation, one will face again the Romanovski polynomials as part of its exact solutions.

### B. Romanovski polynomials in quark physics.

The interaction of quarks, the fundamental constituents of the baryons, are governed by Quantum Chromodynamics (QCD) which is a non-Abelian gauge theory with gauge bosons being the so called gluons. QCD predicts that the quark interactions run from one- to many gluon exchanges over gluon self-interactions, the latter being responsible for the so-called quark confinement, where highly energetic quarks remain trapped but behave as (asymptotically) free particles at high energies and momenta. The QCD equations are nonlinear and

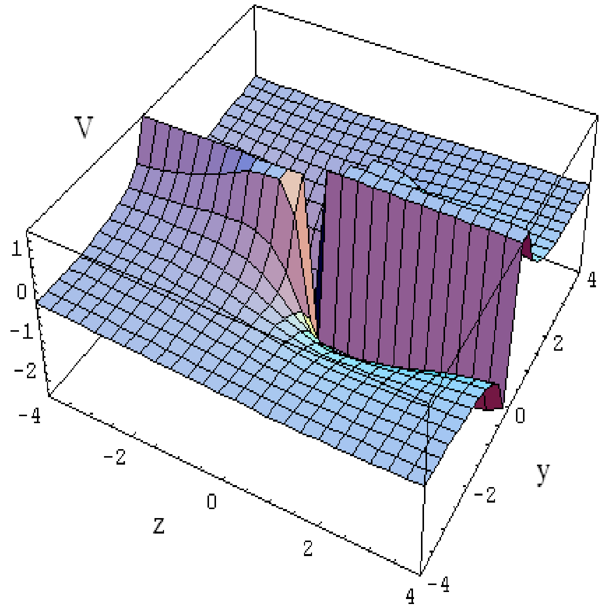


FIG. 1: The non-central potential $V(r,\theta)$, here displayed in its intersection with the $x=0$ plane, i.e. for $r=\sqrt{y^2+z^2}$, and $\theta=\tan^{-1}\frac{y}{z}$. The polar angle part of its exact solutions is expressed in terms of the Romanovski polynomials.

complicated due to the gluonic self-interaction processes and their solution requires employment of highly sophisticated techniques such as discretization of space time, so-called lattice QCD. Lattice QCD calculations of the properties of hadrons, which are all strongly interacting composite particles, predict a linear confinement potential with increasing energy. The one-gluon exchange potential, which is Coulomb-like, $\sim 1/r$, adds to the linear confinement potential, $\sim r$, a combination that is believed to provide the basic properties of two-body (mainly quark-antiquark) systems as concluded from quark model calculations. In contrast to this, three quark systems (baryons) have been believed for a long time to involve more complicated interactions depending on the internal quark degrees of freedom such as their spins, isospins, flavors, and combinations of them, while the potential in coordinate space has been considered of lesser relevance and modeled preferably by means of the harmonic oscillator. However, in so doing, one encounters the problem of a serious excess of predicted baryon excitations in comparison with data [30] (so called "missing resonances").

More recently, the structure of the baryon spectra has been re-analyzed in Refs. [31] with the emphasis on the light quark resonances. The result was the observation of a striking grouping of resonances with different spins and parities in narrow mass bands separated by significant spacings. More specifically, it was found that to a very good accuracy, the nucleon excitation levels carry the same degeneracies as the levels of the electron with spin in the hydrogen atom, though the splittings of the former are quite different from those of the latter. Namely, compared to the hydrogen atom, the baryon level splittings contain, in addition to the Balmer term, also its inverse but of opposite sign. The same was found to be valid for the excitation spectrum of the so called $\Delta(1232)$ particle, the most important baryon excitation after the nucleon. The appeal of these results lies in the fact that no state drops out of the systematics, on the one side, and that the number of "missing" states predicted by it is significantly less than within all preceding schemes. The observed degeneracies in the spectra of the light quark baryons have been attributed in Ref. [32] to the dominance of a quark–antiquark configuration in baryon structure. Within the light of these findings, the form of the potential in configuration space acquires importance anew. In Refs. [7],[16] the case was made that the trigonometric Rosen-Morse potential provides precisely degeneracies and level splittings as required by the light quark baryon spectra. The trigonometric Rosen-Morse potential in the parametrization of Ref. [16] reads:

$$v_{tRM}(z) = -2b\cot z + l(l+1)\frac{1}{\sin^2 z}\,, \tag{116}$$

with $l$ standing for the relative angular momentum between the quark and the di-quark in units, as usual, of $\hbar = 1 = 2\mu$, and $z = \frac{r}{d}$ is a dimensionless variable built with a suited length scale $d$.

The reason for the success of this potential in quark physics is that it captures the essential traits of the QCD quark-gluon dynamics in interpolating between the Coulomb potential (associated with the one-gluon exchange) and the infinite wall potential (associated with the trapped but asymptotically free quarks) while passing through a linear confinement region (as predicted by lattice QCD) (see Fig. 2). It is quite instructive to perform the Taylor expansion of the potential of interest,

$$v(z)_{tRM} \approx -\frac{2b}{z} + \frac{2b}{3}z + \frac{l(l+1)}{z^2} + \frac{l(l+1)}{15}z^2 + ... \tag{117}$$

This expansion clearly reveals the proximity of the cot term to the Coulomb-plus-linear confinement potential, and the proximity of the $\csc^2$ term to the standard centrifugal barrier. The great advantage of the trigonometric Rosen-Morse potential over the linear-plus-Coulomb potential is that while the latter is neither especially symmetric, nor exactly soluble,

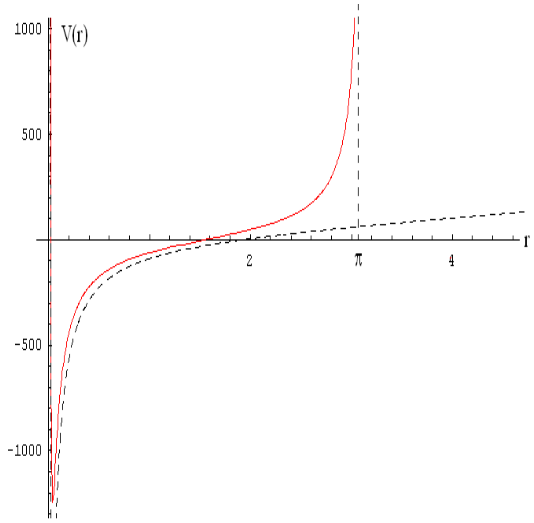


FIG. 2: The trigonometric Rosen-Morse potential (solid line) and its proximity to the Coulomb-plus-linear potential as predicted by lattice QCD (dashed line) for the toy values $l = 1, b = 50$ of the parameters.

the former is both, it has the dynamical $O(4)$ symmetry (as the hydrogen atom) and is exactly soluble. The exact solutions of the, now three dimensional, Schrödinger equation with $v_{tRM}(z)$ from Eq. (116) have been constructed in [16] on the basis of the one-dimensional solutions found in [7] and read:

$$\psi_n(\cot^{-1} x) = (1+x^2)^{-\frac{n+l}{2}} e^{-\frac{b}{n+l+1}\cot^{-1} x} C_n^{\left(-(n+l)+1, \frac{2b}{n+l+1}\right)}(x)\,, \tag{118}$$

with $x = \cot z$. The $C$ polynomials from [7] are Romanovski polynomials but with parameters that depend on the degree of the polynomial. The following identification is valid:

$$\begin{aligned} C_n^{(-(n+l)+1, \frac{2b}{n+l+1})}(x) &\equiv R_n^{(\alpha_n,\beta_n)}(x), \\ \alpha_n = \frac{2b}{n+l+1}, \quad \beta_n &= -(n+l)+1, \quad n \in \{0,1,2,\dots\} \end{aligned} \tag{119}$$

The Schrödinger wave functions are orthogonal, as they are eigenfunctions of a Hamilton operator. Their orthogonality integral (here in $z$ space) reads

$$\int_0^{\pi} dz\ \psi_n(z)\psi_{n'}(z) = \delta_{nn'}\,. \tag{120}$$

The orthogonality of the wave functions $\psi_n(z)$ implies in $x$ space orthogonality of the $R_n^{(\alpha_n,\beta_n)}(x)$ polynomials with respect to $w^{(\alpha_n,\beta_n)}(x)\frac{dz}{dx}$ due to the variable change. As long as

$\frac{d\cot^{-1}x}{dx} = -1/(1+x^2) = -1/s(x)$ then the orthogonality integral takes the form

$$\int_{-\infty}^{\infty} \frac{dx}{s(x)} \sqrt{w^{(\alpha_n,\beta_n)}(x)} R_n^{(\alpha_n,\beta_n)}(x) \sqrt{w^{(\alpha_{n'},\beta_{n'})}(x)} R_{n'}^{(\alpha_{n'},\beta_{n'})}(x) = \delta_{nn'} \ . \tag{121}$$

To recapitulate, the Romanovski polynomials have been shown to be important ingredients of the wave functions of quarks in accord with QCD quark-gluon dynamics.

### C. Romanovski polynomials in random matrix theory

Random matrix theory was pioneered by Wigner [33] for the sake of modeling spectra of heavy nuclei which are characterized by complicated interactions between large numbers of protons and neutrons. Wigner's idea was to limit the infinite dimensional Hamiltonian matrix in configuration space to a finite, real quadratic ($N \times N$), and symmetric, matrix with elements being chosen at random from a suitable probability density distribution, say, the Gaussian one. Along this line one can then model the densities of the nuclear states as averages over the weighted sets of matrices. The advantage of this method is that as $N \longrightarrow \infty$, the (normalized) eigenvalues of any randomly chosen matrix approach the limits of the corresponding system averages, much like the general limit theorem. The probability density distribution (p.d.f.) of the eigenvalues of the Gaussian ensemble of random matrices is given by (the presentation in this section closely follows Ref. [34]):

$$\frac{1}{C_N} e^{-\frac{1}{2}\sum_{j=1}^{N}\lambda_j^2} \prod_{1\le j<k\le N} |\lambda_k - \lambda_j|, \tag{122}$$

where $C_N$ is the normalization constant and $\lambda_i$ are the eigenvalues. Besides the Gaussian random ensemble there are other random matrix ensembles under consideration in quantum physics such as the circular Jacobi ensemble and the Cauchy ensemble, and precisely these are of interest in the present section. However, their definitions require us to go beyond the ensembles of random real matrices and consider matrices with complex entries. To be more specific, one considers random ensembles composed by symmetric, unitary matrices, in which case the theory is not developed from an explicit distribution density function for their elements but rather from the requirement of the existence of a certain appropriate uniform measure. The random unitary matrix ensemble is special not only because it forms a group but mainly because this group is compact and allows for the definition of the so called *Haar* volume, which then provides the uniform measure on the space as required above.

**Definition:** The circular unitary ensemble is the group of unitary matrices $\mathbf{U}$ endowed with the volume form $(d_H\mathbf{U}) = \frac{1}{C}\left(\mathbf{U}^\dagger\mathbf{U}\right) = id\mathbf{M}_2$ with $\mathbf{M}_2$ hermitian.

The eigenvalues of the circular ensembles are confined to the unit circle, i.e. to $\lambda_j = e^{i\theta_j}$ with $-\pi < \theta_j < \pi$. The associated probability density distribution of the eigenvalues is then given by

$$\begin{gathered}\frac{1}{C}\prod_{1\le j<k\le N}|e^{i\theta_k} - e^{i\theta_j}|\;,\\ -\pi < \theta_l < \pi\,.\end{gathered}\tag{123}$$

More generally, an ensemble of unitary and symmetric matrices has an eigenvalues p.d.f of the form (in the notations of Ref. [34])

$$\begin{aligned}&\prod_{l=1}^{N} w_2(z_l)\prod_{1\le j\le k\le N}|z_k - z_j|,\\ &z = e^{i\theta} = e^{\frac{2\pi i x}{L}},\quad \theta\in[0,2\pi),\quad x\in[0,L)\,,\end{aligned}\tag{124}$$

where $w_2(z_l)$ is a specific weight function. The *circular Jacobi ensemble* is specified by

$$w_2(z) = |1-z|^{2a}\,.\tag{125}$$

A relevant research goal in quantum physics is finding the spacings in the spectra of the circular Jacobi ensemble. Compared to the state densities, the calculation of gap probabilities in the spectra deserves special efforts. In this section we review briefly the concept for the calculation of gap probabilities in the circular Jacobi ensemble by means of the so called *Cauchy random matrix ensemble*, a venue that will conduct us one more time to the Romanovski polynomials .

To begin with, one considers the mapping

$$e^{i\theta} = \frac{1+i\lambda}{1-i\lambda}\,,\tag{126}$$

which maps each point $\lambda$ on the real line to a point $\theta$ on the unit circle (measured anticlockwise from the origin) via a stereographic projection. Changing correspondingly variables in Eqs. (124)–(125) amounts to the following eigenvalue p.d.f.:

$$\begin{gathered}\prod_{l=1}^{N}(1+\lambda_j^2)^{-N-a}\prod_{1\le j\le k\le N}|\lambda_k - \lambda_j|\,,\\ \lambda_j\in(-\infty,+\infty)\,.\end{gathered}\tag{127}$$

As long as one recognizes in the weight function the Cauchy weight, the random unitary matrix ensemble generated this way is termed the Cauchy ensemble. On the other hand, a comparison with the weight function of the Romanovski polynomials reveals the Cauchy weight as $w^{(0,-N-a+1)}(x)$, an observation that will acquire a profound importance in the following.

Back to the main goal, the gap probability, or better, the probability for no eigenvalues in a region $I$, denoted by $E(0,I)$, and for the case of any ensemble is now calculated as [34]

$$E(0,I) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_I dx_1 ... \int_I dx_n \det \sum_{l=0}^{N-1} \sqrt{w_2(x_i)} p_l(x_i) \sqrt{w_2(x_j)} p_l(x_j) \ , \qquad (128)$$

where $p_l(x)$ with $l = 0, 1, 2, ...$ stand for the orthogonal polynomials associated with the weight function $w_2(x)$. In other words, knowing the orthogonal polynomials is crucial for the calculation of gap probabilities in any random matrix ensemble. In the specific case under consideration, one seems to have two options in the choice for those polynomials, Romanovski versus Jacobi polynomials in accord with their relationship established in Eq. (97). The choice is clearly in favor of the Romanovski polynomials because the formalism developed for calculating $E(0,I)$ (see Ref. [35] for details) is based on *real coupled differential equations*. In choosing the Romanovski polynomials one, to speak with the authors of Ref. [34], avoids the clumsy and unnecessary work of recasting the formalism on the circle, i.e. in terms of the complexified Jacobi polynomials. In summary, the Romanovski polynomials (termed Cauchy weight polynomials in Ref. [34]) provide a natural and comfortable tool for finding all the results for the circular Jacobi ensemble from those of the Cauchy ensemble.

## VI. ON THE ORTHOGONALITY RELATIONS

We have shown in the previous sections not one but several different orthogonality relations among the Romanovski polynomials. In this section we comment on this issue.

First we have shown, in Eq. (55), a finite orthogonality in the family $\mathcal{R}^{(\alpha,\beta)}$ (that of polynomials with fixed parameters $\alpha$ and $\beta$). It is the equivalent relation to the well known orthogonality of the Hermite, Laguerre and Jacobi polynomials, except that in these cases, there is complete orthogonality. The finite orthogonality, however, is required as such in the solution of the wave eigenfunctions of the hyperbolic Scarf potential, see Eq. (111) in subsection V A or Ref. [6]: in this case only a finite number of states are bounded; precisely those which are normalizable. The orthogonality relation in the $\mathcal{Q}^{(\alpha,\beta)}$ family, Eq. (82), is a complete, not finite, orthogonality: it is valid for all the polynomials in the family. The difference with the previous is that the $\mathcal{Q}^{(\alpha,\beta)}$ family is made up with Romanovski polynomials with one fixed parameter but the other running attached to the degree. This different orthogonality also find its application in, for instance, Eq. (115) in subsection V A.

Finally another physics problem, the eigenfunctions of the trigonometric Rosen-Morse potential studied in [7] and [10] and revised in Subsection V B, has given rise to yet another orthogonality relation: the one in Eq. (121), which is very similar to the orthogonality in the $\mathcal{Q}^{(\alpha,\beta)}$ family, but not equivalent. The polynomials involved in Eq. (121) have both parameters, $\alpha$ and $\beta$ running with the degree, as shown in Eq. (119). This last orthogonality is proved not directly as the others, but by means of the Schrödinger equation where it comes from: as the functions involved are the eigenfunctions of a self-adjoint operator, they are orthogonal. Here, thus, it seems as if the Schrödinger equation carefully chooses, from the set of all Romanovski polynomials, another family with a special combination between parameters and degrees such that another orthogonality relation surfaces. This kind of fine tuned combination of parameters is not completely new. Here is a well known instance: the radial part of the well-known solution of the hydrogen atom, which is given by

$$\mathcal{R}_{nl}(x_n) = N_{nl}\frac{x_n^{\frac{\beta_l}{2}}}{\sqrt{x_n}}\mathrm{e}^{-\frac{x_n}{2}}L_{n-l-1}^{(1,\beta_l)}(x_n), \quad \beta_l = 2l+1, \quad x_n = a_n r, \tag{129}$$

where $x_n$ is the dimensionless but $n$ dependent variable (see also Problem 13.2.11 in Ref. [19]), while $r$ is the radial one. Here $L_m^{(\alpha,\beta)}(x)$ is the generalized Laguerre polynomial of degree $m$ as introduced after Eq. (17). Notice that Laguerre polynomials of different degrees in Eq. (129) emerge within different potential strengths, $Ze^2/a_n$, and, henceforth, the orthogonality relation given by the Schrödinger equation

$$\begin{gathered}\int_0^\infty \frac{x_n^{\frac{\beta_l}{2}}}{\sqrt{x_n}}\mathrm{e}^{-\frac{x_n}{2}}L_{m_{(n,l)}}^{(1,\beta_l)}(x_n)\frac{x_{n'}^{\frac{\beta_l}{2}}}{\sqrt{x_{n'}}}\mathrm{e}^{-\frac{x_{n'}}{2}}L_{m_{(n',l)}}^{(1,\beta_l)}(x_{n'})\, x_n^2 dx_n = 0,\\ \beta_k = 2k+1, \quad m_{(n,k)} = n-k-1, \quad n \neq n', \quad x_{n'} = \frac{a_{n'}}{a_n}x_n,\end{gathered} \tag{130}$$

is not equivalent to the orthogonality given by the weight function, Eq. (18), here restated for $\alpha = 1$

$$\begin{gathered}\int_0^\infty x^{\frac{\beta}{2}}e^{-\frac{x}{2}}L_m^{(1,\beta)}(x)x^{\frac{\beta}{2}}e^{-\frac{x}{2}}L_{m'}^{(1,\beta)}(x)dx = 0,\\ m \neq m',\ \beta > 0.\end{gathered} \tag{131}$$

In particular, notice that Eq. (131), when $\beta \in \mathbf{N}$, is recovered from Eq. (130) in the case $l' = l$, but for $\beta \notin \mathbf{N}$ both formulas are completely different.

In the Introduction we said that, perhaps, the lack of general orthogonality of Romanovski polynomials has been seen as a weakness and because of it they have not attracted as much attention as the classical orthogonal polynomials. Now we have shown that, far from being a weakness, the various orthogonality relations of Romanovski polynomials give them new appealing properties which widens their possible applications.

## VII. CONCLUSIONS

We have presented a fairly complete description of the Romanovski polynomials as solutions of the hypergeometric differential equation (1), properties derived from it and some applications, with the following prominent items:

1. We have described a complete classification of the hypergeometric differential equations in order to place Eq. (1) in its proper context.

2. We have described completely, in Eq. (28) the polynomial solutions to Eq. (1), which are the Romanovski polynomials. We have also stated some known and some new properties of these polynomials. We have proposed different partitions of the set of all Romanovski polynomials into families which allows one to express the plethora of properties in a simpler and more ordered form. This approach can be applied as well to the other four classes of polynomial solutions of hypergeometric equations: Hermite, Laguerre, Jacobi and Bessel.

3. In particular we have stated exact results about several orthogonality relations among the Romanovski polynomials. We have shown that a family of Romanovski polynomials, solutions to the same hypergeometric equation, is not completely orthogonal, but exhibits a finite orthogonality, Eq. (55). However, we have found two other orthogonality relations, in families with running parameters (attached to the degree of the polynomial) which provide infinite orthogonality, Eqs. (82) and (121).

4. The relationship between Romanovski polynomials and Jacobi polynomials has been precisely stated: Romanovski polynomials cannot be obtained as just a complexification of Jacobi polynomials (i.e., change $x$ by $ix$), but they can be realized as a

particularization of complex Jacobi polynomials (an extension to the complex plane with complex parameters). Yet, despite this relation, these complex Jacobi polynomials are not completely understood so, for instance, the orthogonality properties of Romanovski polynomials cannot be derived, at the present time, from properties of the Jacobi polynomials.

5. We have presented three instances of the use of Romanovski polynomials in actual physics problems. In particular the polynomials introduced in [7] and [6] are recognized as Romanovski polynomials. The orthonormality relations shown in these references are explained in this context.

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