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Geometry of 1-lightlike submanifolds in anti-de Sitter n-space

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In this paper we investigate the differential geometry of 1-lightlike submanifolds in anti-de Sitter n-space as an application of the theory of Legendrian singularities. Based on some theory of lightlike submanifolds, we also introduce the notion of 1-lightlike horospherical Gauss curvature, which is important for us to study the singularities of 1-lightlike horospherical hypersurfaces. Moreover, we discuss the related geometric property of 1-lightlike horospherical hypersurfaces in anti-de Sitter n-space.

1. Introduction

Anti-de Sitter n-space, denoted by AdS^n, is a maximally symmetric semi-Riemannian manifold with constant negative scalar curvature. It is best known for its role in the anti-de Sitter/conformal field theory correspondence. In the theory of relativity, anti-de Sitter space is a vacuum solution of Einstein's field equation with a negative cosmological constant. It is well known that there exist spacelike submanifolds, timelike submanifolds and lightlike submanifolds in semi-Riemannian space. Lightlike submanifolds appear in many physics papers. For example, the lightlike submanifolds are of interest because they provide models of different horizon types such as event horizons of Kerr black holes, isolated horizons, Cauchy horizons, Kruskal horizons and Killing horizons [2, 6, 14, 15, 22, 28–30, 32]. Lightlike submanifolds are also studied in the theory of electromagnetism (see, for example, [7, 31]).

During the last four decades, singularity theory has enjoyed rapid development. French mathematician R. Thom (Fields medallist) first put forward the philosophical idea of applying singularity theory to the study of differential geometry. The natural connection between geometry and singularities relies on the basic fact that the contacts of a submanifold with the models (invariant under the action of a suitable transformation group) of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions or, equivalently, of their associated Lagrangian and/or Legendrian maps [1, 26]. Porteous applied the thoughts of Thom to the study of Euclidean geometry [27]. On this basis, Bruce and Giblin have systematically discussed the classification of singularities, singularities' stability and the relationship between the singularities and the geometry invariants of submanifolds in Euclidean space and obtained a number of good results [4]. The first attempts to apply the singularity theory to non-Euclidean geometry were undertaken in 1996 by the second author and Izumiya. The singularities of spacelike and timelike submanifolds in Minkowski space have been studied extensively by, among others, the second author and by Izumiya *et al*. [16–21]. However, to the best of the authors' knowledge, there are not many results on submanifolds immersed in anti-de Sitter space, in particular from the view point of singularity theory, and there is even less literature regarding the singularities of lightlike submanifolds. The only paper regarding the singularities of lightlike submanifolds in anti-de Sitter space, by the second and the third author *et al*., considered lightlike surfaces generated by spacelike curves [5]. Because the research methods regarding the lightlike submanifolds are limited by the degeneracy of the lightlike submanifolds, the studies of many more general lightlike submanifolds, regarding the singularities, have not been considered in the literature. In fact, for non-degenerate submanifolds (spacelike and timelike submanifolds) in semi-Riemannian space, many of the classical results from Riemannian geometry have semi-Riemannian counterparts. Non-degenerate submanifolds can be studied using approaches similar to those taken in positive definite Riemannian geometry. However, lightlike submanifolds have many properties that are very different from spacelike and timelike submanifolds. In other words, lightlike submanifold theory has many results with no Riemannian analogues. In the geometry of lightlike submanifolds, difficulties arise because the arc length vanishes, making it impossible to normalize the tangent vector and to define the induced geometric objects (such as linear connection, second fundamental form, Gauss and Weingarten equations) on the lightlike submanifolds as done in non-degenerate submanifolds. This is why the singularities of lightlike submanifolds cannot be widely studied.

In the study of the differential geometry of lightlike submanifolds, several authors have devoted their work to the research of properties of lightlike submanifolds and have successfully solved some difficult problems [3, 8–12]. Thus, we use these fundamental results of differential geometry as our basic tools in researching the geometry of lightlike submanifolds. We first considered null Cartan curves, one-dimensional isotropic submanifolds, being one of the four types of lightlike submanifolds [34, 35]. In [34], we investigated the singularities of ruled null surfaces of the principal normal indicatrix to a null Cartan curve in de Sitter 3-space; we classified the singularities of ruled null surfaces by using Bruce's singularity theory. As an extension of our previous work [34, 35], the current study focuses on the 1-lightlike submanifolds of the higher dimension and codimension in AdS^n. Compared with the lightlike surfaces mentioned in [5], lightlike submanifolds dealt with in this paper are more general. In fact, because the lightlike surface is generated by a spacelike curve presented in [5], and the spacelike curve is a 1-dimensional non-degenerate submanifold, they still used approaches similar to those taken in positive definite Riemannian geometry to deal with the lightlike surface. But this approach is ineffective in dealing with general lightlike submanifolds. With the help of differential geometry theory of lightlike submanifolds, we have also found an approach that can be applied to the study of more general lightlike submanifolds. We apply the theory of Legendrian singularities to investigate the differential geometry of m-dimensional 1-lightlike submanifolds in AdS^n. We introduce the notion of the *1-lightlike horospherical hypersurfaces* of 1-lightlike submanifolds by using spacelike unit normal vector fields. In particular, we define the *1-lightlike horospherical Gauss curvature*, through which singularities of the 1-lightlike horospherical hypersurface correspond to the points at which the 1-lightlike horospherical Gauss curvature vanishes. It is quite different from the definition of the Gauss–Kronecker curvature adapted for non-degenerate submanifolds in [19, 20]. The definition of the 1-lightlike horospherical Gauss curvature also induces a new definition of umbilical points for a 1-lightlike submanifold. We call the singular point of the 1-lightlike horospherical hypersurface a *1-lightlike horospherical parabolic point*, and the 1-lightlike submanifold is tangent to a *hyperhorosphere* at the 1-lightlike horospherical parabolic point. If we assume a hypothesis of theorem 5.6, then a contact type of a hyperhorosphere and a 1-lightlike submanifold corresponds to a singular type of 1-lightlike horospherical hypersurface, that is, the singularity of the 1-lightlike horospherical hypersurface of a 1-lightlike submanifold can describe the contact of the submanifold with a hyperhorosphere.

This paper has the following structure. We begin in § 2 with the differential geometry of anti-de Sitter spaces and semi-Euclidean spaces. In § 3, we consider general 1-lightlike submanifolds in AdS^n and study their basic properties. We define the 1-lightlike horospherical height function (family) on a 1-lightlike submanifold and show that the discriminant set is a 1-lightlike horospherical hypersurface (see proposition 3.1). In § 4, we further show that the 1-lightlike horospherical height function of a 1-lightlike submanifold is a Morse family (see proposition 4.1). Therefore, the 1-lightlike horospherical hypersurface of a 1-lightlike submanifold is the wavefront set of a Legendrian submanifold. In § 5, we study the contact of submanifolds with hyperhorospheres as an application of the theory of Legendrian singularities and discuss geometric properties of singularities of horospherical hypersurfaces. We consider generic properties of lightlike submanifolds in § 6. Throughout the paper, all maps and manifolds are C^∞ unless otherwise stated; similarly, submanifolds of semi-Euclidean spaces are always assumed to be semi-Riemannian.

2. Preliminaries

Let $\mathbb{R}^{n+1}_2$ denote the $(n+1)$-dimensional semi-Euclidean space with index 2, that is to say, the manifold $\mathbb{R}^{n+1}$ with a flat semi-Euclidean metric $\langle\cdot,\cdot\rangle$ such that, for any two vectors $\boldsymbol{x} = (x_1, x_2, \dots, x_{n+1})$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_{n+1})$ in $\mathbb{R}^{n+1}$,

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -x_1y_1 - x_2y_2 + x_3y_3 + \cdots + x_{n+1}y_{n+1}.$$

We say that a vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_2 \setminus \{\boldsymbol{0}\}$ is *spacelike, null* (*lightlike*) or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle$ is positive, zero or negative, respectively. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_2$ and a real number c, we define the hyperplane with pseudo-normal vector $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_2 \colon \langle \boldsymbol{x}, \boldsymbol{v}\rangle = c\}.$$

We call $HP(\boldsymbol{v}, c)$ a *Lorentz hyperplane*, a *semi-Euclidean hyperplane with index 2* or a *null hyperplane* if $\boldsymbol{v}$ is *timelike, spacelike* or *null*, respectively.

We now introduce a typical semi-Riemannian manifold. We set

$$\mathrm{AdS}^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_2 \colon \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}.$$

It is well known that AdS^n is a complete semi-Riemannian manifold with constant sectional curvature -1. We call AdS^n the *anti-de Sitter n-space.*

We also define a *unit pseudo n-sphere with index 2* by

$$S^n_2 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_2 \colon \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},$$

an *n-dimensional* (*open*) *nullcone* with vertex $\boldsymbol{a}$ by

$$\Lambda^n_{\boldsymbol{a}} = \{\boldsymbol{x} = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_2 \colon \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\} \setminus \{\boldsymbol{a}\};$$

when $\boldsymbol{a} = \boldsymbol{0}$, we simply denote $\Lambda^n_{\boldsymbol{0}}$ by Λ^n. Let $\boldsymbol{x} \colon U \to \mathrm{AdS}^n$ be an m-dimensional regular submanifold of AdS^n (i.e. an embedding), where $U \subset \mathbb{R}^m$ is an open subset and $m \geqslant 2$, $n \geqslant m + 2$. We identify $M = \boldsymbol{x}(U)$ with U through the embedding $\boldsymbol{x}$. If $\langle \cdot, \cdot \rangle$ is degenerate on the tangent bundle TM of M, we say that M is a *lightlike submanifold* of AdS^n. Next, we introduce some basic notions about lightlike submanifolds (see [3, 8–12]). Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$ module of smooth sections of a vector bundle E (same notation for any other vector bundle) over M. For a degenerate tensor field $\langle \cdot, \cdot \rangle$ on M, there exists locally a vector field $\xi \in \Gamma(TM)$ such that $\langle \xi, X \rangle = 0$ for any $X \in \Gamma(TM)$. Then, for each tangent space T_pM, we have that $T_pM^{\perp} = \{\boldsymbol{u} \in T_p\mathrm{AdS}^n \colon \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0 \ \forall \boldsymbol{v} \in T_pM\}$, which is a degenerate n-dimensional subspace of $T_p\mathrm{AdS}^n$. The *radical subspace* of T_pM (denoted as $\mathrm{Rad}\, T_pM$) is defined by $\mathrm{Rad}\, T_pM = \{\xi_p \in T_pM \colon \langle \xi_p, X \rangle = 0 \ \forall X \in T_pM\}$. The dimension of $\mathrm{Rad}\, T_PM = T_pM \cap T_PM^{\perp}$ depends on $p \in M$. The submanifold M of AdS^n is said to be an *r-lightlike submanifold* if the mapping

$$\begin{aligned} \mathrm{Rad}\, TM \colon M &\to TM \\ p &\mapsto \mathrm{Rad}\, T_pM \end{aligned}$$

defines a smooth distribution of rank r on M. $\mathrm{Rad}\, TM$ is called the *radical distribution.*

In this paper, we study the 1-lightlike submanifold M of AdS^n. Consider a complementary distribution $S(TM)$ of $\mathrm{Rad}\, TM$ in TM. Clearly, $S(TM)$ is orthogonal to $\mathrm{Rad}\, TM$ and non-degenerate with respect to $\langle \cdot, \cdot \rangle$. Let a complementary vector subbundle to $\mathrm{Rad}\, TM$ in $TM^{\perp}$ be denoted by $S(TM^{\perp})$. We call $S(TM)$ and $S(TM^{\perp})$ a screen distribution and a screen transversal vector bundle of M, respectively. We suppose that $S(TM^{\perp})$ is of constant index 1 on M. Similarly, let $\mathrm{tr}\, TM$ and $\mathrm{ltr}\, TM$ be complementary (but not orthogonal) vector bundles to TM in $T\mathbb{R}^{n+1}_2|_M$ and to $\mathrm{Rad}\, TM$ in $S(TM^{\perp})^{\perp}$, respectively. We call $\mathrm{ltr}\, TM$ and $\mathrm{tr}\, TM$ a lightlike transversal vector bundle and a transversal vector bundle of M, respectively. For the 1-lightlike submanifold M of AdS^n, we have the fact that there exists a unique vector subbundle $\mathrm{ltr}\, TM$ of $S(TM^{\perp})^{\perp}$ of rank 1 such that, for any $\boldsymbol{\xi} \in \Gamma(\mathrm{Rad}\, TM)$, $\boldsymbol{\xi} \neq 0$ on M, there exists a unique $\boldsymbol{\eta} \in (\mathrm{ltr}\, TM)$ of $S(TM^{\perp})^{\perp}$

satisfying (see [11, p. 144, theorem 1.2])

$$\langle \xi, \boldsymbol{\eta} \rangle = 1, \qquad \langle \boldsymbol{\eta}, \boldsymbol{\eta} \rangle = 0.$$

We obtain that

$$\begin{aligned} \operatorname{tr} TM &= \operatorname{ltr} TM \perp S(TM^{\perp}), \\ T\mathbb{R}^{n+1}_2|_M &= TM \oplus \operatorname{tr} TM \\ &= S(TM) \perp S(TM^{\perp}) \perp (\operatorname{Rad} TM \oplus \operatorname{ltr} TM). \end{aligned} \tag{2.1}$$

Consider the following local field of frames of $\mathbb{R}^{n+1}_2$ along M:

$$\{\boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_m}, \boldsymbol{\eta}, \boldsymbol{w}_1, \dots, \boldsymbol{w}_{n-m}\}, \tag{2.2}$$

where $\boldsymbol{x}_{u_i} = \partial \boldsymbol{x}/\partial u_i$, $\{\boldsymbol{x}_{u_1} = \boldsymbol{\xi}\}$ is a lightlike basis of $\Gamma(\operatorname{Rad} TM)$, $\{\boldsymbol{x}_{u_2}, \dots, \boldsymbol{x}_{u_m}\}$ is a basis of $\Gamma(S(TM))$, $\{\boldsymbol{\eta}\}$ is a lightlike basis of $\Gamma(\operatorname{ltr} TM)$ and $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_{n-m}\}$ is a basis of $\Gamma(S(TM^{\perp}))$. We can also choose $\boldsymbol{w}_{n-m} = \boldsymbol{x}$ as a normal vector; then (2.2) becomes

$$\{\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{x}_{u_2}, \dots, \boldsymbol{x}_{u_m}, \boldsymbol{\eta}, \boldsymbol{w}_1, \dots, \boldsymbol{w}_{n-m-1}\}.$$

The following ranges of the indices are used in this paper (unless otherwise stated): $i, j \in \{2, \dots, m\}$ and $r, r' \in \{1, \dots, n-m-1\}$.

The local field of frames satisfies

$$\langle \boldsymbol{\eta}, \boldsymbol{\eta} \rangle = \langle \boldsymbol{\eta}, \boldsymbol{x} \rangle = \langle \boldsymbol{\eta}, \boldsymbol{w}_r \rangle = \langle \boldsymbol{\eta}, \boldsymbol{x}_{u_i} \rangle = \langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle = \langle \boldsymbol{x}_{u_i}, \boldsymbol{w}_r \rangle = \langle \boldsymbol{x}, \boldsymbol{w}_r \rangle = 0, \tag{2.3}$$

$$\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle = \langle \boldsymbol{\xi}, \boldsymbol{w}_r \rangle = \langle \boldsymbol{\xi}, \boldsymbol{x}_{u_i} \rangle = 0, \qquad \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 1, \tag{2.4}$$

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, \qquad \langle \boldsymbol{w}_r, \boldsymbol{w}_{r'} \rangle = \delta_{rr'}, \tag{2.5}$$

$$\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_i} \rangle > 0,$$

where $\delta_{rr'}$ is the Kronecker function. According to (2.1) we have the Gauss formulae and the Weingarten formulae for the 1-lightlike submanifold M of AdS^n:

$$\bar{\nabla}_X Y = \nabla_X Y + h^{\ell}(X, Y) + h^s(X, Y), \tag{2.6}$$

$$\bar{\nabla}_X V = -A(V, X) + D^{\ell}_X V + D^s_X V \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(\operatorname{tr}(TM))$, where $\nabla_X Y$, $A(V, X)$ belongs to $\Gamma(TM)$, $\{h^{\ell}, D^{\ell}_X\}$ is the $\Gamma(\operatorname{ltr}(TM))$-value and $\{h^s, D^s_X\}$ is the $\Gamma(S(TM^{\perp}))$-value.

We now introduce the pseudo-Riemannian metric

$$\mathrm{d}s^2 = \sum_{i,j=1}^{m} g_{ij}\, \mathrm{d}u_i\, \mathrm{d}u_j$$

on $M = X(U)$, where $g_{ij}(u) = \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}_{u_j}(u) \rangle$ for any $u \in U$. We denote the local lightlike second fundamental forms and the local screen second fundamental forms of M on U by $\{h^\ell_{ik}\}$ and $\{h^s_\alpha\}$, respectively. From (2.6), (2.7), we derive

$$\bar{\nabla}_{\boldsymbol{x}_{u_i}}\boldsymbol{x}_{u_k} = \nabla_{\boldsymbol{x}_{u_i}}\boldsymbol{x}_{u_k} + h^\ell_{ik}(\boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_k})\boldsymbol{\eta} + \sum_{\alpha=1}^{n-m-1} h^s_\alpha(\boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_k})\boldsymbol{w}_\alpha, \tag{2.8}$$

$$\bar{\nabla}_{\boldsymbol{x}_{u_i}}\boldsymbol{\eta} = -\sum_{j=1}^{m}\tau^j_i(\boldsymbol{x}_{u_i}, \boldsymbol{w})\boldsymbol{x}_{u_j} - \theta^\ell_i(\boldsymbol{x}_{u_i}, \boldsymbol{\eta})\boldsymbol{\eta} + \sum_{\alpha=1}^{n-m-1}\rho^\alpha_i(\boldsymbol{x}_{u_i}, \boldsymbol{\eta})\boldsymbol{w}_\alpha, \tag{2.9}$$

$$\bar{\nabla}_{\boldsymbol{x}_{u_i}}\boldsymbol{w} = -\sum_{j=1}^{m}\sigma^j_i(\boldsymbol{x}_{u_i}, \boldsymbol{w})\boldsymbol{x}_{u_j} - \nu^\ell_i(\boldsymbol{x}_{u_i}, \boldsymbol{w})\boldsymbol{\eta} + \sum_{\alpha=1}^{n-m-1}\mu^\alpha_i(\boldsymbol{x}_{u_i}, \boldsymbol{w})\boldsymbol{w}_\alpha \tag{2.10}$$

for any $\boldsymbol{w} \in T_pM^\perp$, where $h^\ell_{k1}(\boldsymbol{x}_{u_k}, \boldsymbol{x}_{u_1}) = 0$ (see [11, p. 157, proposition 2.2]).

Let $T_pM^\perp = \mathrm{Rad}\,T_pM \perp S(T_pM^\perp)$ be the normal space of M at $p = \boldsymbol{x}(u)$ in $\mathbb{R}^{n+1}_2$; we define $N_p(M) = T_pM^\perp \cap T_p\mathrm{AdS}^n$ and $\overline{T_pM} = S(T_pM) \perp \mathrm{ltr}\,T_pM$. We call $\overline{T_pM}$ the *corrected tangent space* of M at $p = \boldsymbol{x}(u)$. We arbitrarily choose a normal section $\boldsymbol{w}(u) \in N_p(M)$. By (2.1), we have $\boldsymbol{w}_{u_i}(u) \in \overline{T_pM} \oplus T_pM^\perp$.

Consider the projections

$$\pi^{s\ell} \colon \overline{T_pM} \oplus T_pM^\perp \to \overline{T_pM}$$

and

$$\pi^N \colon \overline{T_pM} \oplus T_pM^\perp \to T_pM^\perp.$$

Let $\mathrm{d}\boldsymbol{w}_u \colon T_uU \to \overline{T_pM} \oplus T_pM^\perp$ be the derivative of $\boldsymbol{w}$. We define that $\mathrm{d}\boldsymbol{w}^{s\ell}_u = \pi^{s\ell} \circ \mathrm{d}\boldsymbol{w}_u$ and $\mathrm{d}\boldsymbol{w}^N_u = \pi^N \circ \mathrm{d}\boldsymbol{w}_u$. For any $\boldsymbol{w} \in N_{p_0}(M)$, we call the linear transformation

$$S^{\boldsymbol{w}}_{p_0} = \mathrm{d}\boldsymbol{w}^{s\ell}_{u_0} \colon T_{p_0}M \to \overline{T_{p_0}M}$$

the *1-lightlike $\boldsymbol{w}$-shape operator* of $M = \boldsymbol{x}(U)$ at $p_0 = \boldsymbol{x}(u_0)$. For a spacelike unit vector $\boldsymbol{w} \in N_{p_0}(M)$, we define

$$S^{\boldsymbol{x}+\boldsymbol{w}}_{p_0} = \mathrm{d}\boldsymbol{x}_{u_0} + \mathrm{d}\boldsymbol{w}^{s\ell}_{u_0};$$

we call $S^{\boldsymbol{x}+\boldsymbol{w}}_{p_0}$ the *1-lightlike horospherical $\boldsymbol{w}$-shape operator* of $M = \boldsymbol{x}(U)$ at $p_0 = \boldsymbol{x}(u_0)$. For a given basis $\{\boldsymbol{\xi}\}$ of $\mathrm{Rad}\,T_{p_0}M$ and $\{\boldsymbol{\eta}\}$ of $\mathrm{ltr}\,T_{p_0}M$ satisfying $\langle\boldsymbol{\xi}, \boldsymbol{\eta}\rangle = 1$, we define an isomorphic mapping

$$A_{p_0} \colon \overline{T_{p_0}M} \to T_{p_0}M$$

such that, for any $\sum_{i=2}^m \lambda_i\boldsymbol{x}_{u_i} + \tau\boldsymbol{\eta} \in \overline{T_{p_0}M}$,

$$A_{p_0}\left(\sum_{i=2}^m \lambda_i\boldsymbol{x}_{u_i} + \tau\boldsymbol{\eta}\right) = \sum_{i=2}^m \lambda_i\boldsymbol{x}_{u_i} + \tau\boldsymbol{\xi}.$$

For any unit spacelike normal vector $\boldsymbol{w} \in N_{p_0}(M)$, we define the linear operators

$$LT^{\boldsymbol{w}}_{p_0} = A_{p_0} \circ S^{\boldsymbol{w}}_{p_0} \colon T_{p_0}M \to T_{p_0}M$$

and

$$LT^{\boldsymbol{x}+\boldsymbol{w}}_{p_0} = A_{p_0} \circ S^{\boldsymbol{x}+\boldsymbol{w}}_{p_0} \colon T_{p_0}M \to T_{p_0}M.$$

The *1-lightlike horospherical Gauss curvature* with respect to $\boldsymbol{w}$ at $p_0 = \boldsymbol{x}(u_0)$ is defined to be

$$K_\ell(\boldsymbol{w})(p_0) = \det(S^{\boldsymbol{x}+\boldsymbol{w}}_{p_0}).$$

It is clear that

$$\det(S^{\boldsymbol{x}+\boldsymbol{w}}_{p_0}) = \det(A^{-1}_{p_0} \circ LT^{\boldsymbol{x}+\boldsymbol{w}}_{p_0}) = \det(A^{-1}_{p_0}) \det(LT^{\boldsymbol{x}+\boldsymbol{w}}_{p_0}) = \det(LT^{\boldsymbol{x}+\boldsymbol{w}}_{p_0}).$$

We denote the eigenvalues of $LT^{\boldsymbol{w}}_{p_0}$ and $LT^{\boldsymbol{x}+\boldsymbol{w}}_{p_0}$ by $k^{\boldsymbol{w}}_{\ell,i}(p_0)$ and $\bar{k}^{\boldsymbol{w}}_{\ell,i}(p_0)$, which we respectively call a *1-lightlike $\boldsymbol{w}$-principal curvature* and a *1-lightlike horospherical $\boldsymbol{w}$-principal curvature* at p_0 with respect to $\boldsymbol{w}$. We say that a point $p_0 = \boldsymbol{x}(u_0)$ is a *1-lightlike horospherical $\boldsymbol{w}$-umbilic point* if $LT^{\boldsymbol{x}+\boldsymbol{w}}_{p_0} = \bar{k}^{\boldsymbol{w}}_\ell(p_0) id_{T_{p_0}} M$, where $\bar{k}^{\boldsymbol{w}}_\ell(p_0) = \bar{k}^{\boldsymbol{w}}_{\ell,i}(p_0)$ for all $i \in \{1, \dots, m\}$. We say that $M = \boldsymbol{x}(U)$ is *totally 1-lightlike horospherical $\boldsymbol{w}$-umbilic* if all points on M are 1-lightlike horospherical $\boldsymbol{w}$-umbilic. We say that the unit normal vector field $\boldsymbol{w}$ is *1-lightlike parallel at p_0* if $-\boldsymbol{w}_{u_1}(u_0) = \boldsymbol{\xi}(u_0)$ and $-\boldsymbol{w}_{u_i}(u_0) \in S(T_{p_0}M)$ for any $i \in \{2, \dots, m\}$. We simply say that $\boldsymbol{w}$ is *1-lightlike parallel* if it is 1-lightlike parallel at all points of M.

Considering the hypersurface defined by $HP(\boldsymbol{v}, c) \cap \mathrm{AdS}^n$, we say that $HP(\boldsymbol{v}, c) \cap \mathrm{AdS}^n$ is an *elliptic hyperquadric* or a *hyperbolic hyperquadric* if $HP(\boldsymbol{v}, c)$ is a Lorentz hyperplane or a semi-Euclidean hyperplane with index 2, respectively. We say that $HP(\boldsymbol{v}, c) \cap \mathrm{AdS}^n$ is a *hyperhorosphere* if $HP(\boldsymbol{v}, c)$ is null hyperplane.

PROPOSITION 2.1. *Let $M = \boldsymbol{x}(U)$ be a 1-lightlike submanifold of AdS^n of dimension m. Then the following assertions hold.*

(1) *Suppose that M is 1-lightlike horospherical $\boldsymbol{w}$-umbilic at $p_0 = \boldsymbol{x}(u_0)$ and that $\boldsymbol{\xi}$ and $\boldsymbol{x}_{u_i}$ are eigenvectors for $LT^{\boldsymbol{w}}_{p_0}$ with respect to the eigenvalues $k^{\boldsymbol{w}}_{\ell,1}(u_0)$ and $k^{\boldsymbol{w}}_{\ell,i}(u_0)$, respectively. Then, $\bar{k}^{\boldsymbol{w}}_\ell(u_0) = k^{\boldsymbol{w}}_{\ell,1}(u_0) = k^{\boldsymbol{w}}_{\ell,i}(u_0) - 1$ for every $i \in \{2, \dots, m\}$.*

(2) *Suppose that $M = \boldsymbol{x}(U)$ is 1-lightlike horospherical $\boldsymbol{w}$-umbilic and that the spacelike unit normal vector field $\boldsymbol{w} \in N_p(M)$ satisfies $-\boldsymbol{w}_{u_1}(u_0) = \boldsymbol{\xi}(u_0)$ and $-\boldsymbol{w}_{u_i}(u_0) \in S(T_{p_0}M)$ for any $i \in \{2, \dots, m\}$. Then, $\bar{k}^{\boldsymbol{w}}_\ell(u)$ is constant, $k = 0$ and there exists a vector $\boldsymbol{v} \in \mathrm{AdS}^n$ such that M is a part of the hyperhorosphere $HP(\boldsymbol{v}, -1) \cap \mathrm{AdS}^n$.*

Proof.

(1) As M is 1-lightlike horospherical $\boldsymbol{w}$-umbilic at $p_0 = \boldsymbol{x}(u_0)$, we derive that

$$\begin{aligned} -A_p \circ \pi^{s\ell} \circ (\boldsymbol{x} + \boldsymbol{w})_{u_1} &= -A_p \circ \pi^{s\ell} \circ \boldsymbol{x}_{u_1} - A_p \circ \pi^{s\ell} \circ \boldsymbol{w}_{u_1} \\ &= -A_p \circ \pi^{s\ell} \circ \boldsymbol{w}_{u_1} \\ &= k^{\boldsymbol{w}}_{\ell,1}(u_0)\boldsymbol{\xi} \end{aligned}$$

and

$$\begin{aligned} -A_p \circ \pi^{s\ell} \circ (\boldsymbol{x} + \boldsymbol{w})_{u_i} &= -A_p \circ \pi^{s\ell} \circ \boldsymbol{x}_{u_i} - A_p \circ \pi^{s\ell} \circ \boldsymbol{w}_{u_i} \\ &= -\boldsymbol{x}_{u_i} + k^{\boldsymbol{w}}_{\ell,i}(u_0)\boldsymbol{x}_{u_i} \\ &= (k^{\boldsymbol{w}}_{\ell,i}(u_0) - 1)\boldsymbol{x}_{u_i}. \end{aligned}$$

It is easy to check that $\bar{k}^{\boldsymbol{w}}_\ell(u_0) = k^{\boldsymbol{w}}_{\ell,1}(u_0) = k^{\boldsymbol{w}}_{\ell,i}(u_0) - 1$ for every $i \in \{2, \dots, m\}$.

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(2) Since $\boldsymbol{w}$ is a 1-lightlike parallel spacelike unit normal vector field along M, we have that

$$-A_p \circ \pi^{s\ell} \circ (\boldsymbol{x}+\boldsymbol{w})_{u_1} = 0 = -\boldsymbol{\xi} - \boldsymbol{w}_{u_1}$$

and

$$-A_p \circ \pi^{s\ell} \circ (\boldsymbol{x}+\boldsymbol{w})_{u_i} = -(\boldsymbol{x}+\boldsymbol{w})_{u_i} = -\boldsymbol{x}_{u_i} - \boldsymbol{w}_{u_i}.$$

By the assumption, we have that $-\boldsymbol{\xi} - \boldsymbol{w}_{u_1} = \bar{k}^{\boldsymbol{w}}_{\ell}(u)\boldsymbol{\xi} = 0$, $-\boldsymbol{x}_{u_i} - \boldsymbol{w}_{u_i} = \bar{k}^{\boldsymbol{w}}_{\ell}(u)\boldsymbol{x}_{u_i} = 0$, so we have that $\boldsymbol{x}+\boldsymbol{w}$ is a constant vector $\boldsymbol{v}$ and $\bar{k}^{\boldsymbol{w}}_{\ell}(u) = 0$. Moreover, we have that $\langle \boldsymbol{x}, \boldsymbol{v}\rangle = -1$. Since $\boldsymbol{v}$ is a lightlike vector, this means that M is contained in a hyperhorosphere. □

PROPOSITION 2.2. *Under the above notation, the 1-lightlike horospherical Gauss curvature with respect to any spacelike unit normal vector*

$$\boldsymbol{w} = \mu\boldsymbol{\xi} + \sum_{i=2}^{m} \lambda_i \boldsymbol{x}_{u_i} + \sum_{r=1}^{n-m-1} \omega_r \boldsymbol{w}_r \in N_p(M)$$

is given by

$$K_\ell(\boldsymbol{w})(p) = \frac{\det(\bar{h}_{ij}(\boldsymbol{w}))}{\det(g_{\alpha\beta})}, \tag{2.11}$$

where λ_i, ω_r are real numbers and

$$g_{\alpha\beta} = \begin{cases} 1, & \alpha=\beta=1, \\ \langle \boldsymbol{x}_{u_\alpha}, \boldsymbol{x}_{u_\beta}\rangle, & \alpha\neq 1 \text{ or } \beta \neq 1, \end{cases}$$

$$\bar{h}_{ij}(\boldsymbol{w}) = \langle -\bar{\nabla}_{\boldsymbol{x}_{u_i}} \boldsymbol{x}+\boldsymbol{w}, \boldsymbol{x}_{u_j}\rangle.$$

Proof. Using (2.3), (2.4) and (2.10), we obtain that

$$\begin{aligned} \langle -\bar{\nabla}_{\boldsymbol{x}_{u_i}} \boldsymbol{x}+\boldsymbol{w}, \boldsymbol{x}_{u_1}\rangle &= \langle \nu^{\ell}_i(\boldsymbol{x}_{u_i}, \boldsymbol{x}+\boldsymbol{w})\boldsymbol{\eta}, \boldsymbol{x}_{u_1}\rangle + \left\langle \sum_{\alpha=2}^{m} \sigma^{\alpha}_i(\boldsymbol{x}_{u_i}, \boldsymbol{x}+\boldsymbol{w})\boldsymbol{x}_{u_\alpha}, \boldsymbol{x}_{u_1} \right\rangle \\ &= \nu^{\ell}_i(\boldsymbol{x}_{u_i}, \boldsymbol{x}+\boldsymbol{w}) g_{11} + \sum_{\alpha=2}^{m} \sigma^{\alpha}_i(\boldsymbol{x}_{u_i}, \boldsymbol{x}+\boldsymbol{w}) g_{\alpha 1}, \end{aligned}$$

where $g_{11} = \langle \boldsymbol{\eta}, \boldsymbol{x}_{u_1}\rangle = 1$, $g_{\alpha 1} = \langle \boldsymbol{x}_\alpha, \boldsymbol{x}_{u_1}\rangle$. Similarly, we have that

$$\begin{aligned} \langle -\bar{\nabla}_{\boldsymbol{x}_{u_i}} \boldsymbol{x}+\boldsymbol{w}, \boldsymbol{x}_{u_\beta}\rangle &= \left\langle \sum_{\alpha=2}^{m} \sigma^{\alpha}_i(\boldsymbol{x}_{u_i}, \boldsymbol{x}+\boldsymbol{w})\boldsymbol{x}_{u_\alpha}, \boldsymbol{x}_{u_\beta} \right\rangle \\ &= \sum_{\alpha=2}^{m} \sigma^{\alpha}_i(\boldsymbol{x}_{u_i}, \boldsymbol{w}) g_{\alpha\beta}, \end{aligned}$$

where $g_{\alpha\beta} = \langle \boldsymbol{x}_\alpha, \boldsymbol{x}_{u_\beta}\rangle$. Moreover, we have that

$$\begin{aligned} \nu^{\ell}_i(\boldsymbol{x}_{u_i}, \boldsymbol{x}+\boldsymbol{w}) &= \langle -\bar{\nabla}_{\boldsymbol{x}_{u_i}} \boldsymbol{x}+\boldsymbol{w}, \boldsymbol{x}_{u_1}\rangle g^{11}, \\ \sigma^{j}_i(\boldsymbol{x}_{u_i}, \boldsymbol{x}+\boldsymbol{w}) &= \sum_{\beta=2}^{m} \langle -\bar{\nabla}_{\boldsymbol{x}_{u_i}} \boldsymbol{x}+\boldsymbol{w}, \boldsymbol{x}_{u_\beta}\rangle g^{\beta j}, \end{aligned}$$

where $(g^{\beta j}) = (g_{\beta j})^{-1}$. We define that $\varsigma_i^1 = \nu_i^\ell(\boldsymbol{x}_{u_i}, \boldsymbol{x}+\boldsymbol{w})$ and $\varsigma_i^j = \sigma_i^j(\boldsymbol{x}_{u_i}, \boldsymbol{x}+\boldsymbol{w})$ for $i = 1, \dots, m$, $j = 2, \dots, m$. We can then check that

$$K_\ell(\boldsymbol{w})(p) = \det(S_p^{\boldsymbol{x}+\boldsymbol{w}}) = \det(\varsigma_i^j) = \det((\bar{h}_{i\beta}(\boldsymbol{w}))(g^{\beta j})) = \frac{\det(\bar{h}_{ij}(\boldsymbol{w}))}{\det(g_{\alpha\beta})},$$

which clearly proves our assertion. □

We let $K_\ell(\boldsymbol{w}_0)(u_0)$ denote the 1-lightlike horospherical Gauss curvature at $\boldsymbol{p}_0 = \boldsymbol{x}(u_0)$ with respect to $\boldsymbol{w}_0 = \boldsymbol{w}(u_0)$. We say that $\boldsymbol{p} = \boldsymbol{x}(u_0)$ is a *1-lightlike horospherical parabolic point* (or, more briefly, a 1-$LH(\boldsymbol{w}_0)$-*parabolic point*) of $M = \boldsymbol{x}(U)$ if $K_\ell(\boldsymbol{w}_0)(u_0) = 0$. We also say that $\boldsymbol{p} = \boldsymbol{x}(u_0)$ is a *1-lightlike horospherical flat point* (or, more briefly, a 1-$LH(\boldsymbol{w}_0)$-*flat point*) of $M = \boldsymbol{x}(U)$ if $p = \boldsymbol{x}(u_0)$ is a 1-lightlike horospherical umbilic point and $K_\ell(\boldsymbol{w}_0)(u_0) = 0$.

3. A 1-lightlike horospherical hypersurface of 1-lightlike submanifolds in AdS^n

In this section we define a 1-lightlike horospherical hypersurface from the 1-lightlike submanifold in AdS^n, and we introduce the 1-lightlike height function in order to study the singularities of 1-lightlike horospherical hypersurfaces.

Let $\boldsymbol{x}\colon U \to \mathrm{AdS}^n$ be a 1-lightlike submanifold of codimension $n-m$ in anti-de Sitter space and let $p = \boldsymbol{x}(u)$; we choose orthonormal sections $\{\boldsymbol{\xi}(u), \boldsymbol{w}_1(u), \dots, \boldsymbol{w}_{n-m-1}(u)\}$ of $N_p(M)$, where $\boldsymbol{\xi}(u)$ is a lightlike vector and $\boldsymbol{w}_r(u)$ are spacelike unit vectors for $r \in \{1, \dots, n-m-1\}$. We define a map $\boldsymbol{\rho}\colon U \times \mathbb{R} \times S^{n-m-2} \to S_2^n$ by

$$\boldsymbol{\rho}(u, \boldsymbol{\omega}) = \mu\boldsymbol{\xi}(u) + \sum_{r=1}^{n-m-1} \omega_r \boldsymbol{w}_r(u),$$

where $\boldsymbol{\omega} = (\mu, \omega_1, \dots, \omega_{n-m-1}) \in \mathbb{R} \times S^{n-m-2}$ with $\sum_{r=1}^{n-m-1} \omega_r^2 = 1$.

Let $\boldsymbol{x}\colon U \to \mathrm{AdS}^n$ be a 1-lightlike submanifold of codimension $n-m$. We define a family of functions $H\colon U \times \Lambda^n \to \mathbb{R}$ by $H(u, \boldsymbol{v}) = \langle \boldsymbol{x}(u), \boldsymbol{v}\rangle + 1$, where $\boldsymbol{v} = (v_0, v_1, \dots, v_n) \in \Lambda^n$. We call H the *1-lightlike horospherical height function* on $M = \boldsymbol{x}(U)$. Using the notation $h_{v_0} = H(u, \boldsymbol{v}_0)$ for any $\boldsymbol{v}_0 \in \Lambda^n$, we have the following proposition.

Proposition 3.1.

(1) $H(u, \boldsymbol{v}) = 0$ *if and only if there exist real numbers* $\mu, \tau, \lambda_i \in \mathbb{R}$ *and* $\omega_r \in \mathbb{R}$ *such that* $\boldsymbol{v} = \boldsymbol{x}(u) + \mu\boldsymbol{\xi}(u) + \lambda_2\boldsymbol{x}_{u_2}(u) + \dots + \lambda_m\boldsymbol{x}_{u_m}(u) + \tau\boldsymbol{\eta}(u) + \omega_1\boldsymbol{w}_1(u) + \dots + \omega_{n-m-1}\boldsymbol{w}_{n-m-1}(u)$.

(2) $H(u, \boldsymbol{v}) = \partial H/\partial u_i(u, \boldsymbol{v}) = 0$ *if and only if*

$$\boldsymbol{v} = \boldsymbol{x}(u) + \mu\boldsymbol{\xi}(u) + \sum_{r=1}^{n-m-1} \omega_r \boldsymbol{w}_r(u).$$

Proof. (1) Consider the following local field of frames of $T_p\mathbb{R}_2^{n+1}$ along M:

$$\{\boldsymbol{x}(u), \boldsymbol{x}_{u_1}(u) = \boldsymbol{\xi}(u), \boldsymbol{x}_{u_2}(u), \dots, \boldsymbol{x}_{u_m}(u), \boldsymbol{\eta}(u), \boldsymbol{w}_1(u), \dots, \boldsymbol{w}_{n-m-1}(u)\},$$

where $\boldsymbol{p} = \boldsymbol{x}(u)$ and there exist real numbers $\lambda_2, \dots, \lambda_m$, μ, τ, ω_0, $\omega_1, \dots, \omega_{n-m-1}$ such that

$$\boldsymbol{v} = \omega_0 \boldsymbol{x}(u) + \mu \boldsymbol{\xi}(u) + \sum_{i=2}^{m} \lambda_i \boldsymbol{x}_{u_2}(u) + \tau \boldsymbol{\eta}(u) + \sum_{r=1}^{n-m-1} \omega_r \boldsymbol{w}_r(u).$$

Therefore, $H(u, \boldsymbol{v}) = 0$ if and only if $1 = -\langle \boldsymbol{x}(u), \boldsymbol{v} \rangle = -\omega_0 \langle \boldsymbol{x}(u), \boldsymbol{x}(u) \rangle = \omega_0$.

(2) Because $\partial H/\partial u_i(u, \boldsymbol{v}) = \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{v} \rangle$, we obtain that

$$\tau \langle \boldsymbol{\xi}(u), \boldsymbol{\eta}(u) \rangle = \tau = 0$$

and

$$\sum_{i=2}^{m} \lambda_i \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}_{u_j}(u) \rangle = \sum_{i=2}^{m} \lambda_i g_{ij} = 0$$

for $j \in \{2, \dots, m\}$. Since (g_{ij}) is non-degenerate for $i, j \in \{2, \dots, m\}$, we have that $\lambda_i = 0$ for $i \in \{2, \dots, m\}$. Because $\boldsymbol{v} \in \Lambda^n$, the condition that $H(u, \boldsymbol{v}) = \partial H/\partial u_i(u, \boldsymbol{v}) = 0$ holds if and only if

$$\boldsymbol{v} = \boldsymbol{x}(u) + \mu \boldsymbol{\xi}(u) + \sum_{r=1}^{n-m-1} \omega_r \boldsymbol{w}_r(u) \quad \text{with} \quad \sum_{r=1}^{n-m-1} \omega_r^2 = 1.$$

This completes the proof. □

It follows that the discriminant set of H is

$$\mathfrak{D}_H = \{\boldsymbol{x}(u) + \boldsymbol{\rho}(u, \boldsymbol{\omega}) \mid (u, \boldsymbol{\omega}) \in U \times \mathbb{R} \times S^{n-m-2}\}.$$

For the 1-lightlike horospherical height function H, we have the set

$$\Sigma_*(H) = \{(u, \boldsymbol{v})\colon H(u, \boldsymbol{v}) = \partial H(u, \boldsymbol{v})/\partial u_i = 0\}.$$

We define a mapping

$$LH_x^1\colon U \times \mathbb{R} \times S^{n-m-2} \to \Lambda^n$$

by $LH_x^1(u, \boldsymbol{\omega}) = \boldsymbol{x}(u) + \boldsymbol{\rho}(u, \boldsymbol{\omega})$. Calling LH_x^1 the *1-lightlike horospherical hypersurface* of M, we remark that LH_x^1 depends on the choice of the pseudo-orthonormal frames of $T_pM^\perp = \operatorname{Rad} T_pM \perp S(T_pM^\perp)$.

Let

$$\{\boldsymbol{x}(u), \boldsymbol{\xi}(u), \boldsymbol{x}_{u_2}(u), \dots, \boldsymbol{x}_{u_m}(u), \boldsymbol{\eta}, \boldsymbol{w}_1(u), \dots, \boldsymbol{w}_{n-m-1}(u)\}$$

and

$$\{\boldsymbol{x}(u), \boldsymbol{\xi}'(u), \boldsymbol{x}'_{u_2}(u), \dots, \boldsymbol{x}'_{u_m}(u), \boldsymbol{\eta}', \boldsymbol{w}'_1(u), \dots, \boldsymbol{w}'_{n-m-1}(u)\}$$

be two frames of $T_p\mathbb{R}^n_2$. Consider the two pseudo-orthonormal frames of $T_pM^\perp$,

$$\{\boldsymbol{x}(u), \boldsymbol{\xi}(u), \boldsymbol{w}_1(u), \dots, \boldsymbol{w}_{n-m-1}(u)\}$$

and

$$\{\boldsymbol{x}(u), \boldsymbol{\xi}'(u), \boldsymbol{w}'_1(u), \dots, \boldsymbol{w}'_{n-m-1}(u)\}.$$

Because $\dim \operatorname{Rad} T_pM = 1$ and $\{\boldsymbol{w}_i(u)\}$ and $\{\boldsymbol{w}'_i(u)\}$ are the two bases of $S(T_pM^\perp)$, we have that $\boldsymbol{\xi}(u) = \zeta\boldsymbol{\xi}'(u)$ and

$$\boldsymbol{w}_i(u) = \sum_{j=1}^{n-m-1} \vartheta_i^j \boldsymbol{w}'_j(u) \quad (i = 1, \dots, n-m-1),$$

where

$$\zeta = \langle \boldsymbol{\xi}(u), \boldsymbol{\eta}'(u)\rangle, \qquad \vartheta_i^j = \langle \boldsymbol{w}_i(u), \boldsymbol{w}'_j(u)\rangle.$$

Define a diffeomorphism $\Phi\colon U \times \mathbb{R} \times S^{n-m-2} \to U \times \mathbb{R} \times S^{n-m-2}$ by

$$\Phi(u, \boldsymbol{\omega}) = \left(u, \left(\zeta\mu, \sum_{j=1}^{n-m-1} \vartheta_j^1 \omega_j, \dots, \sum_{j=1}^{n-m-1} \vartheta_j^{n-m-1}\omega_j\right)\right),$$

where $\boldsymbol{\omega} = (\mu, \omega_1, \dots, \omega_{n-m-1})$; we also define that

$$\boldsymbol{\rho}'(u, \boldsymbol{\omega}) = \mu\boldsymbol{\xi}'(u) + \sum_{r=1}^{n-m-1} \omega_r \boldsymbol{w}'_r(u).$$

It is easy to check that $\boldsymbol{\rho}(u, \boldsymbol{\omega}) = \boldsymbol{\rho}' \circ \Phi(u, \boldsymbol{\omega})$. Therefore, we have that

$$LH_x^1(u, \boldsymbol{\omega}) = \widehat{LH_x^1} \circ \Phi(u, \boldsymbol{\omega}),$$

where $\widehat{LH_x^1} = \boldsymbol{x}(u) + \boldsymbol{\rho}'(u, \boldsymbol{\omega})$. This means that $\widehat{LH_x^1}$ defines the same hypersurface as $LH_x^1(U \times \mathbb{R} \times S^{n-m-2})$, with a different parametrization.

Since we are interested in the singularity of $LH_x^1(U \times \mathbb{R} \times S^{n-m-2})$, we arbitrarily fix a pseudo-orthonormal frame

$$\{\boldsymbol{x}(u), \boldsymbol{\xi}(u), \boldsymbol{w}_1(u), \dots, \boldsymbol{w}_{n-m-1}(u)\}$$

of $T_pM^\perp = \operatorname{Rad} T_pM \perp S(T_pM^\perp)$ throughout the remainder of this paper. We can prove the following assertion.

Proposition 3.2. *Let $\boldsymbol{x}\colon U \to \mathrm{AdS}^n$ be an m-dimensional 1-lightlike submanifold. Then, $LH_x^1(u, \boldsymbol{\omega})$ is a constant map for some map $\psi\colon U \to \mathbb{R} \times S^{n-m-2}$ if and only if $M = \boldsymbol{x}(U)$ is a part of the hyperhorosphere $HP(\boldsymbol{v}, -1) \cap \mathrm{AdS}^n$.*

Proof. Suppose that $\boldsymbol{v}_0 = \boldsymbol{x}(u) + \boldsymbol{\rho}(u, \boldsymbol{\omega})$ is a constant lightlike vector. Because $\boldsymbol{\rho}(u, \boldsymbol{\omega})$ is a normal vector of M for any $u \in U$, we have that $\langle \boldsymbol{v}_0, \boldsymbol{x}(u)\rangle = -1$ for any $u \in U$. This means that $M \subset HP(\boldsymbol{v}_0, -1)$. On the other hand, if $M \subset HP(\boldsymbol{v}_0, -1) \cap \mathrm{AdS}^n$ for some lightlike vector $\boldsymbol{v}_0$, then $\langle \boldsymbol{v}_0, \boldsymbol{x}\rangle = -1$. It follows that we have that $\langle \boldsymbol{v}_0 - \boldsymbol{x}(u), \boldsymbol{x}(u)\rangle = 0$ for any $u \in U$. Moreover, we have that $\langle \boldsymbol{v}_0, \boldsymbol{x}_{u_i}(u)\rangle = 0$. Therefore, $\boldsymbol{v}_0 - \boldsymbol{x}(u)$ is a normal vector of M. We define a smooth mapping $\boldsymbol{\omega}\colon U \to \mathbb{R} \times S^{n-m-2}$ by

$$\boldsymbol{\omega}(u) = \langle \boldsymbol{v}_0 - \boldsymbol{x}(u), \boldsymbol{\eta}(u)\rangle \boldsymbol{\xi}(u) + \sum_{r=1}^{n-m-1} \langle \boldsymbol{v}_0 - \boldsymbol{x}(u), \boldsymbol{w}_r(u)\rangle \boldsymbol{w}_r(u).$$

We then have that $\boldsymbol{v}_0 - \boldsymbol{x}(u) = \boldsymbol{\rho}(u, \boldsymbol{\omega})$. This completes the proof. □

We have the following proposition.

Remark 3.3. By proposition 2.1, the condition of proposition 3.2 is equivalent to the condition that M is totally 1-lightlike horospherical $\boldsymbol{\rho}(u,\boldsymbol{\omega})$-umbilic, the normal vector field $\boldsymbol{\rho}(u,\boldsymbol{\omega})$ is 1-lightlike parallel and $K_\ell(\boldsymbol{\rho}(u,\boldsymbol{\omega}))(u)=0$.

Proposition 3.4. *The singular set of LH^1_x is given by*

$$\Sigma(LH^1_x)=\{(u,\boldsymbol{\omega})\in U\times\mathbb{R}\times S^{n-m-2}\colon K_\ell(\boldsymbol{\rho}(u,\boldsymbol{\omega}))(u)=0\}.$$

Proof. Let $h_v(u)$ be a 1-lightlike horospherical height function, with $\boldsymbol{v}\in\Lambda^n$. By straightforward calculation, the Hessian matrix of the 1-lightlike horospherical height function h_v at $p=\boldsymbol{x}(u)$ is given by $(\langle\bar{\nabla}_{\boldsymbol{x}_{u_i}}\boldsymbol{x}_{u_j}(u),\boldsymbol{v}\rangle)$, where $\boldsymbol{v}$ is a normal vector of M at p. Because $(u,\boldsymbol{v})\in\Sigma_*(H)$, we have that $\boldsymbol{v}=\boldsymbol{x}(u)+\boldsymbol{\rho}(u,\boldsymbol{\omega})$ for some $\boldsymbol{\omega}\in\mathbb{R}\times S^{n-m-2}$. By proposition 2.2, we know that

$$\begin{aligned}K_\ell(\boldsymbol{\rho}(u,\boldsymbol{\omega}))(u)&=\frac{\det(\bar{h}_{ij}(\boldsymbol{w}))}{\det(g_{\alpha\beta})}\\&=\frac{\det(-\langle\bar{\nabla}_{\boldsymbol{x}_{u_i}}\boldsymbol{x}(u)+\boldsymbol{\rho}(u,\boldsymbol{\omega}),\boldsymbol{x}_{u_j}(u)\rangle)}{\det(g_{\alpha\beta})}.\end{aligned}$$

As $\langle\boldsymbol{x}(u)+\boldsymbol{\rho}(u,\boldsymbol{\omega}),\boldsymbol{x}(u)+\boldsymbol{\rho}(u,\boldsymbol{\omega})\rangle=0$, it follows that

$$\begin{aligned}\det(-\langle\bar{\nabla}_{\boldsymbol{x}_{u_i}}\boldsymbol{x}(u)+\boldsymbol{\rho}(u,\boldsymbol{\omega}),\boldsymbol{x}_{u_j}(u)\rangle)&=\det(\langle\bar{\nabla}_{\boldsymbol{x}_{u_i}}\boldsymbol{x}_{u_j}(u),\boldsymbol{x}(u)+\boldsymbol{\rho}(u,\boldsymbol{\omega})\rangle)\\&=\det\operatorname{Hess}h_v(u).\end{aligned}$$

Therefore, $\det\operatorname{Hess}h_v(u)=0$ if and only if $K_\ell(\boldsymbol{\rho}(u,\boldsymbol{\omega}))(u)=0$. This completes the proof. □

As a consequence of proposition 3.4, we state the following.

Proposition 3.5. *For any 1-lightlike submanifold $\boldsymbol{x}\colon U\to\mathrm{AdS}^n$ and $\boldsymbol{v}_0=\boldsymbol{x}(u_0)+\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0)$, we have the following.*

(1) *$p=\boldsymbol{x}(u_0)$ is a 1-LH$(\boldsymbol{\rho}(u,\boldsymbol{\omega}))$-parabolic point if and only if*

$$\det\operatorname{Hess}h_{v_0}(u_0)=0.$$

(2) *$p=\boldsymbol{x}(u_0)$ is a 1-LH$(\boldsymbol{\rho}(u,\boldsymbol{\omega}))$-flat point if and only if the rank of* $\operatorname{Hess}h_{v_0}(u_0)$ *is* 0.

Here $\operatorname{Hess}h_{v_0}(u_0)$ *is the Hessian matrix of $h_{v_0}(u)=H(u,\boldsymbol{v}_0)$ at u_0.*

Example 3.6. Suppose that M is a regular 1-lightlike surface of AdS^5 given by $\boldsymbol{x}\colon U\to\mathrm{AdS}^5$, $u=(x_1,x_2)\mapsto(x_1,x_2,x_3,\dots,x_6)$, i.e. an embedding, where

$$x_3=\sin\Bigl(\ln\Bigl|x_1+\sqrt{x_1^2+1}\Bigr|\Bigr),\qquad x_4=\cos\Bigl(\ln\Bigl|x_1+\sqrt{x_1^2+1}\Bigr|\Bigr),$$
$$x_5=\frac{1}{\sqrt{2}}\Bigl(\sqrt{x_1^2+1}+\sqrt{x_2^2-3}\Bigr),\qquad x_6=\frac{1}{\sqrt{2}}\Bigl(\sqrt{x_1^2+1}-\sqrt{x_2^2-3}\Bigr).$$

where $x_2 > \sqrt{3}$. Let $\theta = \ln|x_1 + \sqrt{x_1^2+1}|$ and derive that

$$TM = \operatorname{span}\left\{ \boldsymbol{x}_{u_1} = \boldsymbol{\xi} = \frac{\partial}{\partial x_1} + \frac{1}{\sqrt{x_1^2+1}}\cos\theta\frac{\partial}{\partial x_3} - \frac{1}{\sqrt{x_1^2+1}}\sin\theta\frac{\partial}{\partial x_4} + \frac{x_1}{\sqrt{2(x_1^2+1)}}\frac{\partial}{\partial x_5} + \frac{x_1}{\sqrt{2(x_1^2+1)}}\frac{\partial}{\partial x_6}, \quad \boldsymbol{x}_{u_2} = \frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{x_2}{\sqrt{x_2^2-3}}\frac{\partial}{\partial x_5} - \frac{1}{\sqrt{2}}\frac{x_2}{\sqrt{x_2^2-3}}\frac{\partial}{\partial x_6} \right\}$$

and

$$\begin{aligned} TM^{\perp} &= \operatorname{span}\Bigg\{ \boldsymbol{\xi}, \boldsymbol{x}, \boldsymbol{w}_1 = \frac{1}{\sqrt{2(\sqrt{(x_1^2+1)(x_2^2-3)}+\frac{3}{2})^2+\frac{3}{2}}} \\ &\quad\times\Bigg[-x_2\frac{\partial}{\partial x_2} + ((-3-\sqrt{(x_1^2+1)(x_2^2-3)})\sin\theta - x_1\sqrt{x_2^2-3}\cos\theta)\frac{\partial}{\partial x_3} \\ &\quad + ((-3-\sqrt{(x_1^2+1)(x_2^2-3)})\cos\theta + x_1\sqrt{x_2^2-3}\sin\theta)\frac{\partial}{\partial x_4} + \sqrt{2(x_2^2-3)}\frac{\partial}{\partial x_6}\Bigg], \\ \boldsymbol{w}_2 &= \sqrt{\frac{(\sqrt{(x_1^2+1)(x_2^2-3)}+\frac{3}{2})^2+\frac{3}{4}}{6(x_1^2+1)(x_2^2-3)}} \\ &\quad\times\Bigg[x_2\left(1+\frac{(x_1^2+1)(x_2^2-3)-3}{(\sqrt{(x_1^2+1)(x_2^2-3)}+\frac{3}{2})^2+\frac{3}{4}}\right)\frac{\partial}{\partial x_2} \\ &\quad + \frac{3(((x_1^2+1)(x_2^2-3)+\sqrt{(x_1^2+1)(x_2^2-3)})\sin\theta - x_1\sqrt{x_2^2-3}(2+\sqrt{(x_1^2+1)(x_2^2-3)})\cos\theta)}{(\sqrt{(x_1^2+1)(x_2^2-3)}+\frac{3}{2})^2+\frac{3}{4}}\frac{\partial}{\partial x_3} \\ &\quad + \frac{3(((x_1^2+1)(x_2^2-3)+\sqrt{(x_1^2+1)(x_2^2-3)})\cos\theta + x_1\sqrt{x_2^2-3}(2+\sqrt{(x_1^2+1)(x_2^2-3)})\sin\theta)}{(\sqrt{(x_1^2+1)(x_2^2-3)}+\frac{3}{2})^2+\frac{3}{4}}\frac{\partial}{\partial x_4} \\ &\quad + \sqrt{2(x_2^2-3)}\frac{\partial}{\partial x_5} - \frac{\sqrt{2(x_2^2-3)}((x_1^2+1)(x_2^2-3)-3)}{(\sqrt{(x_1^2+1)(x_2^2-3)}+\frac{3}{2})^2+\frac{3}{4}}\frac{\partial}{\partial x_6}\Bigg]\Bigg\}, \end{aligned}$$

where $\boldsymbol{\xi}(u)$ and $\boldsymbol{w}_i(u)$ $(i=1,2)$ are lightlike and unit spacelike vectors, respectively, for each $u=(x_1,x_2)\in U$. It follows that $\operatorname{Rad} TM = \operatorname{span}\{\boldsymbol{\xi}\} = TM \cap TM^{\perp}$, that is, M is a 1-lightlike surface of AdS^5. We obtain the lightlike transversal vector

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bundle

$$\operatorname{ltr}(TM) = \operatorname{span}\bigg\{\boldsymbol{\eta} = \frac{1}{2}\bigg(-\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{x_1^2+1}}\cos\theta\frac{\partial}{\partial x_3} - \frac{1}{\sqrt{x_1^2+1}}\sin\theta\frac{\partial}{\partial x_4} + \frac{x_1}{\sqrt{2(x_1^2+1)}}\frac{\partial}{\partial x_5} + \frac{x_1}{\sqrt{2(x_1^2+1)}}\frac{\partial}{\partial x_6}\bigg)\bigg\}.$$

We can consider the 1-lightlike horospherical Gauss curvature of the 1-lightlike surface with respect to any normal vector field $\boldsymbol{\rho}$. Note that any unit spacelike normal vector $\boldsymbol{\rho} \in TM^\perp = \operatorname{Rad} TM \perp S(TM^\perp)$ is expressed as

$$\boldsymbol{\rho} = \mu\boldsymbol{\xi} + \omega_1\boldsymbol{w}_1 + \omega_2\boldsymbol{w}_2,$$

where μ, ω_1, ω_2 are real numbers and $\omega_1^2+\omega_2^2 = 1$. We give the 1-lightlike horospherical Gauss curvature of the 1-lightlike surface with respect to $\boldsymbol{\rho}$ at $p = \boldsymbol{x}(u)$ by

$$\begin{aligned}
K_\ell(\boldsymbol{\rho})(p) &= \det S_p^{\boldsymbol{x}+\boldsymbol{\rho}} \\
&= \frac{\det(-\langle\bar{\nabla}_{\boldsymbol{x}_{u_i}}(\boldsymbol{x}+\boldsymbol{\rho}), \boldsymbol{x}_{u_j}\rangle)}{\det(g_{\alpha\beta})} \\
&= \begin{vmatrix} -\langle\bar{\nabla}_{\boldsymbol{x}_{u_1}}(\boldsymbol{x}+\boldsymbol{\rho}), \boldsymbol{x}_{u_1}\rangle & -\langle\bar{\nabla}_{\boldsymbol{x}_{u_1}}(\boldsymbol{x}+\boldsymbol{\rho}), \boldsymbol{x}_{u_2}\rangle \\ -\langle\bar{\nabla}_{\boldsymbol{x}_{u_2}}(\boldsymbol{x}+\boldsymbol{\rho}), \boldsymbol{x}_{u_1}\rangle & -\langle\bar{\nabla}_{\boldsymbol{x}_{u_2}}(\boldsymbol{x}+\boldsymbol{\rho}), \boldsymbol{x}_{u_2}\rangle \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 3/(x_2^2-3) \end{vmatrix}^{-1} \\
&= \tfrac{1}{3}(x_2^2-3)\begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix} \\
&= \tfrac{1}{3}(x_2^2-3)AB,
\end{aligned}$$

where

$$\begin{aligned}
A &= -\langle\bar{\nabla}_{\boldsymbol{x}_{u_1}}(\boldsymbol{x}+\boldsymbol{\rho}), \boldsymbol{x}_{u_1}\rangle \\
&= \frac{3\sqrt{x_1^2+1}+2\sqrt{x_2^2-3}}{(x_1^2+1)^{3/2}\sqrt{(2\sqrt{(x_1^2+1)(x_2^2-3)}+\frac{3}{2})^2+\frac{3}{2}}}\omega_1 \\
&\quad - \frac{3}{\sqrt{6(x_1^2+1)(x_2^2-3)}} \\
&\qquad \times \bigg(\frac{(2x_1^2+1)(x_2^2-3)\sqrt{x_1^2+1}+(3x_1^2+1)\sqrt{x_2^2-3}}{(x_1^2+1)^{3/2}\sqrt{(\sqrt{(x_1^2+1)(x_2^2-3)}+\frac{3}{2})^2+\frac{3}{4}}}\bigg)\omega_2, \\
B &= -\langle\bar{\nabla}_{\boldsymbol{x}_{u_2}}(\boldsymbol{x}+\boldsymbol{\rho}), \boldsymbol{x}_{u_2}\rangle \\
&= \frac{3}{x_2^2-3}\bigg(-1+\omega_1-\bigg(1+\frac{\sqrt{(x_1^2+1)(x_2^2-3)}-3}{(\sqrt{(x_1^2+1)(x_2^2-3)}+\frac{3}{2})^2+\frac{3}{4}}\bigg)\omega_2\bigg).
\end{aligned}$$

The singular set of the 1-lightlike horospherical hypersurface $LH_x^1 = \boldsymbol{x}+\boldsymbol{\rho}$ is given by

$$\Sigma(LH_x^1) = \{(u,\boldsymbol{\omega}) \in U\times\mathbb{R}\times S^1 \colon AB = 0\}.$$

4. 1-lightlike horospherical hypersurfaces as wavefronts

In this section we naturally interpret the 1-lightlike horospherical hypersurfaces of M in Λ^n as a wavefront set in the framework of contact geometry. This is an analogous way to the differential geometry of hypersurfaces in hyperbolic space [20].

Let $\pi\colon PT^*(\Lambda^n) \to \Lambda^n$ be the projective cotangent bundles with the canonical contact structures. Consider the tangent bundle $\tau\colon TPT^*(\Lambda^n) \to PT^*(\Lambda^n)$ and the differential map $\mathrm{d}\pi\colon TPT^*(\Lambda^n) \to T(\Lambda^n)$ of π. For any $X \in TPT^*(\Lambda^n)$, there exists an element $\alpha \in T^*(\Lambda^n)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(\Lambda^n)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus, we can define the canonical contact structure on $PT^*(\Lambda^n)$ by

$$K = \{X \in TPT^*(\Lambda^n) \mid \tau(X)(\mathrm{d}\pi(X)) = 0\}.$$

On the other hand, we consider a point $\boldsymbol{v} = (v_0, v_1, \dots, v_n) \in \Lambda^n$; we have the relation

$$v_0 = \pm\sqrt{-v_1^2 + \sum_{j=2}^{n} v_j^2}.$$

So we adopt the coordinate system $(v_1, \dots, v_n)$ of Λ^n. We then have the trivialization $PT^*(\Lambda^n) \equiv \Lambda^n \times P\mathbb{R}^{n-1}$, and call $((v_1, \dots, v_n), [\xi_1 : \cdots : \xi_n])$ homogeneous coordinates of $PT^*(\Lambda^n)$, where $[\xi_1 : \cdots : \xi_n]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^{n-1})^*$. It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^{n} \mu_i \xi_i = 0$, where $\mathrm{d}\pi(X) = \sum_{i=1}^{n} \mu_i (\partial/\partial v_i)$. An immersion $i\colon L \to PT^*(\Lambda^n)$ is said to be a *Legendrian immersion* if $\dim L = n - 1$ and $\mathrm{d}i_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called the *Legendrian map* and the image $W(i) = \mathrm{image}(\pi \circ i)$ is called the *wavefront* of i. Moreover, i (or the image of i) is called the *Legendrian lift* of $W(i)$.

Let $F\colon (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k}\right)\colon (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R} \times \mathbb{R}^k, 0)$$

is non-singular, where $(\boldsymbol{q}, \boldsymbol{x}) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case we have a smooth $(n-1)$-dimensional submanifold $\Sigma_*(F) = \Delta^* F^{-1}(0)$, and a map germ $\Phi_F\colon (\Sigma_*(F), 0) \to PT^*\mathbb{R}^n$ defined by

$$\Phi_F(\boldsymbol{q}, \boldsymbol{x}) = \left(\boldsymbol{x}, \left[\frac{\partial F}{\partial x_1}(\boldsymbol{q}, \boldsymbol{x}), \dots, \frac{\partial F}{\partial x_n}(\boldsymbol{q}, \boldsymbol{x})\right]\right)$$

is a Legendrian immersion. We then have the following fundamental theorem of the theory of Legendrian singularities.

Proposition 4.1. *The 1-lightlike horospherical height function $H\colon U \times \Lambda^n \to \mathbb{R}$ is a Morse family.*

Proof. For any $\boldsymbol{v} = (v_0, v_1, \dots, v_n) \in \Lambda^n$, we have that

$$v_0 = \pm\sqrt{-v_1^2 + \sum_{j=2}^{n} v_j^2};$$

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we know that at least one of the $v_i (i = 0, 1)$ is not equal to zero. Without loss of generality, we assume that $v_0 > 0$, so

$$H(u, \boldsymbol{v}) = \langle \boldsymbol{x}(u), \boldsymbol{v} \rangle + 1 = -x_0 \sqrt{-v_1^2 + \sum_{j=2}^{n} v_j^2} - x_1(u)v_1 + \sum_{j=2}^{n} x_j(u)v_j + 1,$$

where $\boldsymbol{x}(u) = (x_0(u), x_1(u), \dots, x_n(u))$. We now prove that the mapping

$$\Delta^* H = (H, \partial H/\partial u_1, \dots, \partial H/\partial u_m)$$

is non-singular at any point $(u, \boldsymbol{v}) \in \Sigma_*(H)$. The Jacobian matrix of $\Delta^* H$ is given as

$$J\Delta^* H(u, v) = \left(\begin{array}{c|c} \dfrac{\partial H}{\partial u_j}(u, v)_{j=1,2,\dots,m} & \dfrac{\partial H}{\partial v_{j'}}(u, v)_{j'=1,2,\dots,n} \\ \hline \left(\dfrac{\partial^2 H}{\partial u_i \partial u_j}(u, v) \right)_{\substack{i=1,\dots,m \\ j=1,\dots,m}} & \left(\dfrac{\partial^2 H}{\partial u_{i'} \partial v_{j'}}(u, v) \right)_{\substack{i'=1,\dots,m \\ j'=1,\dots,n}} \end{array} \right).$$

We denote an $(m+1) \times n$ matrix B by

$$J\Delta^* H(u, \boldsymbol{v}) = \left(\begin{array}{c|c} \dfrac{\partial H}{\partial u_j}(u, \boldsymbol{v})_{j=1,2,\dots,m} & \\ \hline \left(\dfrac{\partial^2 H}{\partial u_i \partial u_j}(u, \boldsymbol{v}) \right)_{\substack{i=1,\dots,m \\ j=1,\dots,m}} & B \end{array} \right).$$

It is sufficient to show that $\operatorname{rank} B = m + 1$ at $(u, \boldsymbol{v}) \in \Sigma_*(H)$. We also denote an $(m+2) \times (n+1)$ matrix C by

$$C = \begin{pmatrix} -v_0 & -v_1 & v_2 & \dots & v_n \\ -x_0 & -x_1 & x_2 & \dots & x_n \\ -x_{0,u_1} & -x_{1,u_1} & x_{2,u_1} & \dots & x_{n,u_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{0,u_m} & -x_{1,u_m} & x_{2,u_m} & \dots & x_{n,u_m} \end{pmatrix}.$$

First, we prove that the rank of the matrix C is equal to $m + 2$. Let

$$\hat{\boldsymbol{v}} = (-v_0, -v_1, v_2, \dots, v_n),$$
$$\hat{\boldsymbol{x}}(u) = (-x_0, -x_1, x_2, \dots, x_n),$$
$$\hat{\boldsymbol{x}_{u_i}}(u) = (-x_{0,u_i}, -x_{1,u_i}, x_{2,u_i}, \dots, x_{n,u_i}).$$

By using the elementary column transformations, the matrix

$$(\boldsymbol{v}, \boldsymbol{x}(u), \boldsymbol{x}_{u_1}(u), \dots, \boldsymbol{x}_{u_m}(u))^{\mathrm{T}}$$

becomes the matrix

$$(\hat{\boldsymbol{v}}, \hat{\boldsymbol{x}}(u), \hat{\boldsymbol{x}}_{u_1}(u), \dots, \hat{\boldsymbol{x}}_{u_m}(u))^{\mathrm{T}}.$$

It follows that the rank of the matrix $(\boldsymbol{v}, \boldsymbol{x}(u), \boldsymbol{x}_{u_1}(u), \dots, \boldsymbol{x}_{u_m}(u))^{\mathrm{T}}$ is equal to the rank of the matrix $(\hat{\boldsymbol{v}}, \hat{\boldsymbol{x}}(u), \hat{\boldsymbol{x}}_{u_1}(u), \dots, \hat{\boldsymbol{x}}_{u_m}(u))^{\mathrm{T}}$. Since $\boldsymbol{v}$, $\boldsymbol{x}(u)$ and $\boldsymbol{x}_{u_i}(u)$ are linearly independent for all $(u, \boldsymbol{v}) \in \Sigma_*(H)$, $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{x}}(u)$ and $\hat{\boldsymbol{x}}_{u_i}(u)$ are also linearly independent; thus, we have $\operatorname{rank} C = m+2$.

Next, we prove that $\operatorname{rank} B = \operatorname{rank} C - 1$. We subtract the first row multiplied by x_0/v_0 from the second row of the matrix C, and subtract the first row multiplied by $x_{0,u_k}/v_0$ from the $(2+k)$th row for $k = 1, \dots, m$. We then have a matrix

$$C' = \left(\begin{array}{c|cccc} -v_0 & -v_1 & v_2 & \cdots & v_n \\ \hline 0 & & & & \\ \vdots & & & B & \\ 0 & & & & \end{array}\right).$$

Therefore, we have $\operatorname{rank} B = \operatorname{rank} C - 1 = \operatorname{rank} C' - 1 = m+1$. This completes the proof. □

By the above proposition, we can define the Legendrian lift of the 1-lightlike horospherical hypersurface as follows. We define $\boldsymbol{x}(u) = (x_0(u), \dots, x_n(u))$ and $LH_x(u, \boldsymbol{\omega}) = (\bar{\ell}_0(u, \boldsymbol{\omega}), \dots, \bar{\ell}_n(u, \boldsymbol{\omega}))$ as coordinate representations. Define a map

$$\mathcal{L}_x \colon U \times \mathbb{R} \times S^{n-m-2} \to PT^*(\Lambda^n)$$

by

$$\mathcal{L}_x(u, \boldsymbol{\omega}) = (LH^1_x(u, \boldsymbol{\omega}), [\bar{\ell}(u, \boldsymbol{\omega})]),$$

where

$$[\bar{\ell}(u,\boldsymbol{\omega})] = [-\bar{\ell}_1(u,\boldsymbol{\omega})x_0(u) + \bar{\ell}_0(u,\boldsymbol{\omega})x_1(u) : \cdots : -\bar{\ell}_n(u,\boldsymbol{\omega})x_0(u) + \bar{\ell}_0(u,\boldsymbol{\omega})x_n(u)].$$

We call F a *generating family* of Φ_F. Therefore, the wavefront is

$$W(\Phi_F) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{there exists } \boldsymbol{q} \in \mathbb{R}^k \\ \text{such that } F(\boldsymbol{q}, \boldsymbol{x}) = \dfrac{\partial F}{\partial q_1}(\boldsymbol{q}, \boldsymbol{x}) = \cdots = \dfrac{\partial F}{\partial q_k}(\boldsymbol{q}, \boldsymbol{x}) = 0 \end{array} \right\}.$$

We define $D_F = W(\Phi_F)$ and call it the *discriminant set* of F. Moreover, we have the following corollary.

Corollary 4.2. *For an m-dimensional embedding $\boldsymbol{x} \colon U \to \mathrm{AdS}^n$, $\mathcal{L}_x \colon U \times \mathbb{R} \times S^{n-m-2} \to PT^*(\Lambda^n)$ is a Legendrian immersion such that the 1-lightlike horospherical height function $H \colon U \times \Lambda^n \to \mathbb{R}$ of $\boldsymbol{x}(U) = M$ is a generating family of $\mathcal{L}_x$.*

5. Contact with hyperhorospheres

In this section we describe the contacts between the 1-lightlike submanifolds and the hyperhorospheres by applying Montaldi's theory [26].

Let X_i and Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, $\dim Y_1 = \dim Y_2$ and $\boldsymbol{y}_i \in X_i \cap Y_i$ for $i = 1, 2$. We say that the contact of X_1 and Y_1 at $\boldsymbol{y}_1$ is of the same type as the contact of X_2 and Y_2 at $\boldsymbol{y}_2$ if there exists a diffeomorphism germ $\Phi\colon (\mathbb{R}^n, \boldsymbol{y}_1) \to (\mathbb{R}^n, \boldsymbol{y}_2)$ such that $\Phi\colon ((X_1, \boldsymbol{y}_1)) = (X_2, \boldsymbol{y}_2)$ and $\Phi\colon ((Y_1, \boldsymbol{y}_1)) = (Y_2, \boldsymbol{y}_2)$. In this case we write that $K(X_1, Y_1; \boldsymbol{y}_1) = K(X_2, Y_2; \boldsymbol{y}_2)$. Two function germs $g_1, g_2\colon (\mathbb{R}^n, a_i) \to (\mathbb{R}, 0)$ $(i = 1, 2)$ are $\mathcal{K}$-equivalent if there exist a diffeomorphism germ $\Phi\colon (\mathbb{R}^n, a_1) \to (\mathbb{R}^n, a_2)$ and a function germ $\lambda\colon (\mathbb{R}^n, a_1) \to \mathbb{R}$ with $\lambda(a_1) \neq 0$ such that $f_1 = \lambda \cdot (g_2 \circ \Phi)$. In [26] Montaldi showed the following theorem.

THEOREM 5.1 (see [26]). *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i\colon (X_i, \boldsymbol{x}_i) \to (\mathbb{R}^n, \boldsymbol{y}_i)$ be immersion germs and let $f_i\colon (\mathbb{R}^n, \boldsymbol{y}_i) \to (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, \boldsymbol{y}_i) = (f_i^{-1}(0), \boldsymbol{y}_i)$. Then, $K(X_1, Y_1; \boldsymbol{y}_1) = K(X_2, Y_2; \boldsymbol{y}_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

Now consider the function $\mathcal{H}\colon \mathrm{AdS}^n \times \Lambda^n \to \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle + 1$. Given $\boldsymbol{v}_0 \in \Lambda^n$, we define $\mathfrak{h}_{v_0}(u) = \mathcal{H}(\boldsymbol{x}, \boldsymbol{v}_0)$, so we have that $\mathfrak{h}_{\boldsymbol{v}_0}^{-1}(0) = HP(\boldsymbol{v}_0, 1) \cap \mathrm{AdS}^n = \mathcal{HS}^{n-1}(\boldsymbol{v}_0, 1)$. Let $\boldsymbol{x}\colon U \to \mathrm{AdS}^n$ be an embedding of codimension $n - m$. For any $u_0 \in U$ and $\boldsymbol{\omega}_0 \in S^{n-m-2}$ we consider a lightlike vector $\boldsymbol{v}_0 = \boldsymbol{x}(u_0) + (u_0, \boldsymbol{\omega}_0) \in \Lambda^n$; it then follows from proposition 3.1 that:

(1)
$$\mathfrak{h}_{v_0} \circ \boldsymbol{x}(u_0) = \mathcal{H} \circ (\boldsymbol{x} \times id_{\Lambda^n})(u_0, \boldsymbol{v}_0) = H(u_0, \boldsymbol{v}_0) = 0,$$

(2)
$$\frac{\partial \mathfrak{h}_{v_0} \circ \boldsymbol{x}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, \boldsymbol{x}(u_0) + \boldsymbol{\rho}(u_0, \boldsymbol{\omega}_0)) = 0$$

for $i = 1, \dots, m$.

Hence, the hyperhorosphere $\mathfrak{h}_{\boldsymbol{v}_0}^{-1}(0) = \mathcal{HS}^{n-1}(\boldsymbol{v}_0, 1)$ is tangent to $M = \boldsymbol{x}(U)$ at $\boldsymbol{p} = \boldsymbol{x}(u_0)$. In this case, we call $\mathcal{HS}^{n-1}(\boldsymbol{v}_0, 1)$ the *tangent hyperhorosphere* of $M = \boldsymbol{x}(U)$ at $\boldsymbol{p}_0 = \boldsymbol{x}(u_0)$ with respect to $\boldsymbol{x}(u_0) + \boldsymbol{\rho}(u_0, \boldsymbol{\omega}_0)$, which we write as $\mathcal{HS}(\boldsymbol{x}, (u_0, \boldsymbol{\lambda}_0, \boldsymbol{\omega}_0))$.

We briefly review some results on the generating family of Legendrian map germs [19, 36, 37].

Let $i\colon (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i'\colon (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. We then say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H\colon (PT^*\mathbb{R}^n, p) \to (PT^*\mathbb{R}^n, p')$ such that H preserves fibres of π and $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^n$ at a point is said to be *Legendrian stable* if for every map with the given germ there exist a neighbourhood in the space of Legendre immersions (in the Whitney C^∞-topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has, in the second neighbourhood, a point at which its germ is Legendrian equivalent to the original germ.

Because the Legendrian lift $i\colon (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wavefront $W(i)$, we have the following simple but significant property of Legendrian immersion germs.

PROPOSITION 5.2 (see [37]). *Let*

$$i\colon (L, p) \subset (PT^*\mathbb{R}^n, p) \quad \textit{and} \quad i'\colon (L', p') \subset (PT^*\mathbb{R}^n, p')$$

be Legendrian immersion germs such that representatives of both map germs $\pi \circ i$ and $\pi \circ i'$ are proper and both regular sets are dense. Then, i, i' are Legendrian equivalent if and only if their wavefront sets $W(i)$, $W(i')$ are diffeomorphic as set germs.

The assumption in the above proposition is a generic condition for i, i'. In particular, if i, i' are Legendrian stable, then they satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, 0) \to \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n : h(0) = 0\}$. Let $F, G\colon (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *$\mathcal{P}-\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi\colon (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+n}}) = \langle G\rangle_{\mathcal{E}_{k+n}}$. Here, $\Psi^*\colon \mathcal{E}_{k+n} \to \mathcal{E}_{k+n}$ is the pullback $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F\colon (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is an *infinitesimal $\mathcal{K}$-versal deformation* of $f = F|_{\mathbb{R}^k \times 0}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}\bigg|_{\mathbb{R}_k \times \{0\}}, \dots, \frac{\partial F}{\partial x_n}\bigg|_{\mathbb{R}^k \times \{0\}} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k} \right\rangle_{\mathcal{E}_k}.$$

The main result in the theory of Legendrian singularities is the following.

THEOREM 5.3 (see [36]). *Let $F_i\colon (\mathbb{R}^{k_i} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ $(i = 1, 2)$ be Morse families. Then the following results hold.*

(1) *Φ_{F_1} and Φ_{F_2} are Legendrian equivalent if and only if F_1 and F_2 are stably $\mathcal{P}-\mathcal{K}$-equivalent.*

(2) *Φ_F is Legendrian stable if and only if F is an infinitesimal $\mathcal{K}$-versal deformation of $F|_{\mathbb{R}^k \times \{0\}}$.*

We now consider the contact of M with tangent hyperhorospheres at $p_0 \in M$ as an application of Legendrian singularity theory. Let $f_i\colon (N_i, \boldsymbol{x}_i) \to (P_i, \boldsymbol{y}_i)$ $(i = 1, 2)$ be C^∞ map germs. We say that f_1, f_2 are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi\colon (N_1, \boldsymbol{x}_1) \to (N_2, \boldsymbol{x}_2)$ and $\psi\colon (P_1, \boldsymbol{y}_1) \to (P_2, \boldsymbol{y}_2)$ such that $\psi \circ f_1 = f_2 \circ \phi$.

Let

$$LH^1_{\boldsymbol{x}_i}\colon (U \times \mathbb{R} \times S^{n-m-2}, (u_i, \boldsymbol{\omega}_i)) \to (\Lambda^n, \boldsymbol{v}_i) \quad (i = 1, 2)$$

be 1-lightlike horospherical hypersurface germs of the 1-lightlike submanifold germs

$$\boldsymbol{x}_i\colon (U, u_i) \to (\mathrm{AdS}^n, \boldsymbol{x}_i(u_i))$$

of codimension $n - m$. If the representatives of both map germs $LH^1_{x_i}$ are proper and the corresponding regular sets are dense in $(U \times \mathbb{R} \times S^{n-m-2}, (u_i, \boldsymbol{\omega}_i))$, it follows from proposition 5.2 that $LH^1_{x_1}$ and $LH^1_{x_2}$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian immersion germs

$$LH^1_{x_1}\colon (U \times \mathbb{R} \times S^{n-m-2}, (u_1, \boldsymbol{\omega}_1)) \to (\Lambda^n, \boldsymbol{v}_1)$$

and

$$LH^1_{x_2}\colon (U\times\mathbb{R}\times S^{n-m-2},(u_2,\boldsymbol{\omega}_2))\to(\Lambda^n,\boldsymbol{v}_2)$$

are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are $\mathcal{P}-\mathcal{K}$-equivalent by theorem 5.3. Here, $H_i\colon (U\times\Lambda^n,(u_i,\boldsymbol{v}_i))\to\mathbb{R}$ is the 1-lightlike horospherical height function germ of $\boldsymbol{x}_i$.

On the other hand, if we define that $h_{i,v_i}(u)=H_i(u,\boldsymbol{v}_i)$, then we have that $h_{i,v_i}(u)=\mathfrak{h}_{v_i}\circ\boldsymbol{x}_i(u)$. By theorem 5.1,

$$K(\boldsymbol{x}_1(U),\mathcal{HS}(\boldsymbol{x}_1,(u_1,\boldsymbol{\omega}_1)),\boldsymbol{v}_1)=K(\boldsymbol{x}_2(U),\mathcal{HS}(\boldsymbol{x}_2,(u_2,\boldsymbol{\omega}_2)),\boldsymbol{v}_2)$$

if and only if h_{1,v_1} and h_{1,v_2} are $\mathcal{K}$-equivalent. Therefore, we can apply the above arguments to our situation.

THEOREM 5.4. *Let $\boldsymbol{x}_i\colon (U,u_i)\to(\mathrm{AdS}^n,\boldsymbol{x}_i(u_i))$ $(i=1,2)$ be 1-lightlike submanifold germs of codimension $n-m$ such that the representatives of both map germs $LH^1_{x_i}$ are proper and the corresponding regular sets are dense in $(U\times\mathbb{R}\times S^{n-m-2},(u_i,\boldsymbol{\omega}_i))$. We then have the following assertions.*

(A) *The following conditions are equivalent:*

(1) *$LH^1_{x_1}$ and $LH^1_{x_2}$ are $\mathcal{A}$-equivalent,*

(2) *$LH^1_{x_1}$ and $LH^1_{x_2}$ are Legendrian equivalent,*

(3) *H_1 and H_2 are $\mathcal{P}-\mathcal{K}$-equivalent.*

(B) *If one of the above conditions holds for $\boldsymbol{x}_i$ $(i=1,2)$, then*

$$K(\boldsymbol{x}_1(U),\mathcal{HS}(\boldsymbol{x}_1,(u_1,\boldsymbol{\omega}_1)),\boldsymbol{v}_1)=K(\boldsymbol{x}_2(U),\mathcal{HS}(\boldsymbol{x}_2,(u_2,\boldsymbol{\omega}_2)),\boldsymbol{v}_2).$$

In this case, $(\boldsymbol{x}_1^{-1}(\mathcal{HS}(\boldsymbol{x}_1,(u_1,\boldsymbol{\omega}_1))),u_1)$ and $(\boldsymbol{x}_2^{-1}(\mathcal{HS}(\boldsymbol{x}_2,(u_2,\boldsymbol{\omega}_2))),u_2)$ are diffeomorphic as set germs.

Proof.

(A) By the assumption, the corresponding Legendrian lifts $\mathcal{L}_{x_i}$ satisfy the hypothesis of proposition 5.2. It follows from proposition 5.2 and theorem 5.3 that the conditions (1), (2) and (3) are equivalent.

(B) Suppose that H_1 and H_2 are $\mathcal{P}-\mathcal{K}$-equivalent. Then, h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent. By theorem 5.1, we have that

$$K(\boldsymbol{x}_1(U),\mathcal{HS}(\boldsymbol{x}_1,(u_1,\boldsymbol{\omega_1})),\boldsymbol{v}_1)=K(\boldsymbol{x}_2(U),\mathcal{HS}(\boldsymbol{x}_2,(u_2,\boldsymbol{\omega_2})),\boldsymbol{v}_2).$$

On the other hand, we have that $(\boldsymbol{x}_i^{-1}(\mathcal{HS}(\boldsymbol{x}_i,(u_i,\boldsymbol{\omega_i}))),u_i)=h_{i,v_i}(0)$. It follows that $(\boldsymbol{x}_1^{-1}(\mathcal{HS}(\boldsymbol{x}_1,(u_1,\boldsymbol{\omega_1}))),u_1)$ and $(\boldsymbol{x}_2^{-1}(\mathcal{HS}(\boldsymbol{x}_2,(u_2,\boldsymbol{\omega_2}))),u_2)$ are diffeomorphic as set germs because the $\mathcal{K}$-equivalence preserves the zero level sets. □

By the uniqueness result of the infinitesimal $\mathcal{K}$-versal deformation of a function germ, proposition 5.2 and theorem 5.3, we have the following classification result of Legendrian stable germs. For any map germ $f\colon(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, we define the local ring of f by $Q(f)=\mathcal{E}_n/f^*(\mathfrak{M}_p)\mathcal{E}_n$.

PROPOSITION 5.5 (see [19]). *Let* $F_i\colon (\mathbb{R}^k\times\mathbb{R}^n,0)\to(\mathbb{R},0)$ $(i=1,2)$ *be Morse families such that the corresponding* Φ_{F_i} *are Legendrian stable. The following conditions are then equivalent.*

(1) $W(\Phi_{F_1},0)$ *and* $W(\Phi_{F_2},0)$ *are diffeomorphic as germs.*

(2) Φ_{F_1} *and* Φ_{F_2} *are Legendrian equivalent.*

(3) $Q(f_1)$ *and* $Q(f_2)$ *are isomorphic as* $\mathbb{R}$*-algebras, where* $f_i = F_i|_{\mathbb{R}^k\times\{0\}}$.

For a 1-lightlike submanifold germ

$$\boldsymbol{x}\colon (U,u_0)\to(\mathrm{AdS}^n,\boldsymbol{x}(u_0)),$$

we call $(\boldsymbol{x}^{-1}(\mathcal{HS}(\boldsymbol{x},(u_0,\boldsymbol{\omega}_0))),u_0)$ the *tangent hyperhorospherical indicatrix germ* of $\boldsymbol{x}$ with respect to $\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0)$. By theorem 5.4, the diffeomorphism type of the tangent hyperhorospherical indicatrix germ is an invariant of the $\mathcal{A}$-classification among the 1-lightlike horospherical hypersurface germs for generic submanifolds.

In the case when the corresponding Legendrian immersion $\mathcal{L}_x$ is Legendrian stable, we have more detailed assertions. We now denote by $Q(\boldsymbol{x},(u_0,\boldsymbol{\omega}_0))$ the local ring of the function germ $h_{v_0}\colon (U,u_0)\to\mathbb{R}$, where $\boldsymbol{v}_0=\boldsymbol{x}(u_0)+\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0)$. We remark that we can explicitly write the local ring as

$$Q(\boldsymbol{x},u_0,\boldsymbol{\omega}_0)=C^\infty_{u_0}(U)/\langle\langle\boldsymbol{x}(u),\boldsymbol{x}(u_0)+\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0)\rangle+1\rangle_{C^\infty_{u_0}(U)},$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathfrak{M}_{u_0}(U)$.

THEOREM 5.6. *Let* $\boldsymbol{x}_i\colon (U,u_i)\to(\mathrm{AdS}^n,\boldsymbol{x}_i(u_i))$ $(i=1,2)$ *be 1-lightlike submanifold germs of codimension* $n-m$ *such that the corresponding Legendrian immersion germs* $LH^1_{x_i}\colon (U\times\mathbb{R}\times S^{n-m-2},(u_i,\boldsymbol{\omega}_i))\to(PT^*(\Lambda^n),\boldsymbol{z}_i)$ *are Legendrian stable. The following conditions are then equivalent:*

(1) *1-lightlike horospherical hypersurface germs* $LH^1_{x_1}$ *and* $LH^1_{x_2}$ *are* $\mathcal{A}$*-equivalent,*

(2) $\mathcal{L}_{x_1}$ *and* $\mathcal{L}_{x_2}$ *are Legendrian equivalent,*

(3) H_1 *and* H_2 *are* $\mathcal{P}-\mathcal{K}$*-equivalent,*

(4) h_{1,v_1} *and* h_{2,v_2} *are* $\mathcal{K}$*-equivalent,*

(5) $K(\boldsymbol{x}_1(U),\mathcal{HS}(\boldsymbol{x}_1,u_1),\boldsymbol{v}_1)=K(\boldsymbol{x}_2(U),\mathcal{HS}(\boldsymbol{x}_2,u_2),\boldsymbol{v}_2)$,

(6) $Q(\boldsymbol{x}_1,u_1;\boldsymbol{\omega}_1)$ *and* $Q(\boldsymbol{x}_2,u_2;\boldsymbol{\omega}_2)$ *are isomorphic as* $\mathbb{R}$*-algebras.*

Proof. We remark that if $\mathcal{L}_{x_i}$ is Legendrian stable, then the singular set $\Sigma(LH^1_{x_i})$ of the corresponding 1-lightlike horospherical hypersurface has no interior points as a subspace of $U\times\mathbb{R}\times S^{n-m-2}$. By theorem 5.4, the conditions (1), (2), (3) are equivalent. It follows from theorem 5.3 and proposition 5.5 that the conditions (2), (4) and (6) are equivalent. By theorem 5.1, the conditions (4) and (5) are equivalent. □

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We now consider the stratification of the ℓ-jet space $J^\ell(\mathbb{R}^k,\mathbb{R})$ such that the discriminant set of $\mathcal{K}$-versal deformations has the corresponding canonical stratification. By theorem 5.6, such a stratification should be $\mathcal{K}$-invariant, where we have the $\mathcal{K}$-action on $J^\ell(k,1)$ [23,24]. For this reason, we use Mather's canonical stratification here [13,25]. Let $A^\ell(k,1)$ be the canonical stratification of $J^\ell(k,1)\setminus W^\ell(k,1)$, where

$$W^\ell(k,1)=\{j^\ell f(0)\mid \dim_{\mathbb{R}}\mathcal{E}_k/((T_e\mathcal{K})(f)+\mathfrak{M}_k^\ell)\geqslant \ell\}.$$

We now define the stratification $A_0^\ell(\mathbb{R}^k,\mathbb{R})$ of $J^\ell(k,1)\setminus W^\ell(k,1)$ by

$$\mathbb{R}_k\times(\mathbb{R}\setminus 0)\times(J^\ell(k,1)\setminus W^\ell(k,1)),\qquad \mathbb{R}^k\times\{0\}\times A^\ell(k,1),$$

where $W^\ell(\mathbb{R}^k,\mathbb{R})\equiv\mathbb{R}^k\times\{0\}\times W^\ell(k,1)$. Wan showed that if $j_1^\ell F(0)\notin W^\ell(k,1)$ and $j_1^\ell F$ is transversal to $A_0^\ell(\mathbb{R}^k,\mathbb{R})$ [33], then $\pi_F\colon(F^{-1}(0),\mathbf{0})\to(\mathbb{R}^n,\mathbf{0})$ is an MT-stable map germ. Here, we call a map germ MT-stable if it is transversal to the canonical stratification of a jet space, which is introduced in [23].

In the next section, we prove that the assumption of the theorem is generic in the case when $n\leqslant 6$. For general dimensions, we need the topological theory.

PROPOSITION 5.7 (see [20]). *Let $F,G\colon(\mathbb{R}^k\times\mathbb{R}^n,0)\to(\mathbb{R},0)$ be Morse families such that π_F and π_G are MT-stable map germs. If $Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, then π_F and π_G are topological equivalent. Moreover, in this case, D_F and D_G are stratified equivalent.*

As a direct result of proposition 5.7, we have the following theorem.

THEOREM 5.8. *Let $\boldsymbol{x}_i\colon(U,u_i)\to(\mathrm{AdS}^n,\boldsymbol{x}_i(u_i))$ $(i=1,2)$ be 1-lightlike submanifold germs of codimension $n-m$ such that the map germ given by $\pi_{H_i}\colon(H_i^{-1}(\boldsymbol{v}_i),(u_i,\boldsymbol{v}_i))\to(\Lambda^n,\boldsymbol{v}_i)$ at any point $u_i\in U$ is an MT-stable map germ, where H_i is the 1-lightlike horospherical height function of x_i and $\boldsymbol{v}_i=\boldsymbol{x}_i(u_i)+\boldsymbol{\rho}(u_i,\boldsymbol{\omega}_i)$. If $Q(\boldsymbol{x}_1,(u_1,\boldsymbol{\omega}_1))$ and $Q(\boldsymbol{x}_2,(u_2,\boldsymbol{\omega}_2))$ are isomorphic as $\mathbb{R}$-algebras, then $LH^1_{x_1}$ and $LH^1_{x_2}$ are stratified equivalent as set germs.*

By the above results, we can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for function germs. The local ring of a function germ is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension (or Tyurina number) of a function germ is a numerical $\mathcal{K}$-invariant of function germs. We define that

$$H\text{-ord}(\boldsymbol{x},(u_0,\boldsymbol{\omega}_0))=\dim\frac{C^\infty_{u_0}(U)}{\langle\langle\boldsymbol{x}(u),\boldsymbol{v}_0\rangle+1,\langle\boldsymbol{x}_{u_i}(u),\boldsymbol{v}_0\rangle\rangle_{C^\infty_{u_0}(U)}},$$

where $C^\infty_{u_0}(U)=\{g\colon(U,u_0)\to R,\ \text{smooth}\}$ and $\boldsymbol{v}_0=\boldsymbol{x}(u_0)+\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0)$.

Usually H-ord$(\boldsymbol{x},(u_0,\boldsymbol{\omega}_0))$ is called the $\mathcal{K}$-codimension of h_{v_0}. However, we call it the *order of contact with the tangent hyperhorosphere* at $\boldsymbol{x}(u_0)$ with respect to $\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0)$. We also have the notion of corank of function germs:

$$H\text{-corank}(\boldsymbol{x},(u_0,\boldsymbol{\omega}_0))=m-\operatorname{rank}\operatorname{Hess}(h_{v_0}(u_0)).$$

By proposition 3.5, $\boldsymbol{x}(u_0)$ is a 1-$LH(\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0))$-parabolic point if and only if

$$H\text{-corank}(\boldsymbol{x},(u_0,\boldsymbol{\omega}_0))\geqslant 1.$$

Moreover, $\boldsymbol{x}(u_0)$ is a 1-$LH(\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0))$-flat point if and only if

$$H\text{-corank}(\boldsymbol{x},(u_0,\boldsymbol{\omega}_0)) = m.$$

On the other hand, a function germ $f\colon(\mathbb{R}^{n-1},\mathbf{a})\to\mathbb{R}$ has the A_k-type singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2\pm\cdots\pm u_{n-2}^2\pm u_{n-1}^{k+1}$. If H-corank$(\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0)) = m-1$, the 1-lightlike horospherical height function h_{v_0} has the A_k-type singularity at u_0 generically. In this case we have that H-ord$(\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0)) = k$. This number is equal to the order of contact. This is the reason why we call H-ord$(\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0))$ the order of contact with the tangent hyperhorosphere with the polar vector $\boldsymbol{v}_0 = \boldsymbol{x}(u_0)+\boldsymbol{\rho}(u_0,\boldsymbol{\omega}_0)$ at $\boldsymbol{x}(u_0)$.

6. Generic properties

In this section, we consider generic properties of 1-lightlike submanifolds in AdS^n. The main tool is the transversality theorem. Let U be an open subset of $\mathbb{R}^m$ and $\mathrm{Emb}(U,\mathrm{AdS}^n)$ be the space of embeddings $\boldsymbol{x}\colon U\to\mathrm{AdS}^n$ equipped with Whitney C^∞-topology. We define a function $\mathcal{H}\colon\mathrm{AdS}^n\times\Lambda^n\to\mathbb{R}$ by $\mathcal{H}(\boldsymbol{x},\boldsymbol{v}) = \langle\boldsymbol{x},\boldsymbol{v}\rangle+1$, and define $\mathfrak{h}_v(x) = \mathcal{H}(\boldsymbol{x},\boldsymbol{v})$. Then, $\mathfrak{h}_v$ is a submersion for any $\boldsymbol{v}\in\Lambda^*$. For any $\boldsymbol{x}\in\mathrm{Emb}(U,\mathrm{AdS}^n)$, we have that $H = \mathcal{H}\circ(\boldsymbol{x}\times\Lambda^n)$. We also have the ℓ-jet extension

$$j_1^\ell H\colon U\times\Lambda^n\to J^\ell(U,\mathbb{R})$$

defined by $j_1^\ell H(u,\boldsymbol{v}) = j^\ell\mathfrak{h}_v(u)$. We consider the trivialization $J^\ell(U,\mathbb{R})\equiv U\times\mathbb{R}\times J(m,1)$. For any submanifold $Q\subset J(m,1)$, we define that $\tilde{Q} = U\times\{0\}\times Q$. We then have the following proposition [34].

Proposition 6.1. *Let Q be a submanifold of $J(m,1)$. Then, the set*

$$T_Q = \{\boldsymbol{x}\in\mathrm{Emb}(U,\mathrm{AdS}^n)\mid j_1^\ell H \textit{ is transversal to } \tilde{Q}\}$$

is a residual subset of $\mathrm{Emb}(U,\mathrm{AdS}^n)$. *If Q is a closed subset, then T_Q is open.*

On the other hand, we already have the canonical stratification $A_0^\ell(U,\mathbb{R})$ of $J^\ell(\mathbb{R}^m,\mathbb{R})\setminus W^\ell(\mathbb{R}^m,\mathbb{R})$. By the above proposition and arguments in § 5, we have the following theorem.

Theorem 6.2. *There exists an open dense subset $\mathcal{O}\in\mathrm{Emb}(U,\mathrm{AdS}^n)$ such that for any $x\in\mathcal{O}$, the germ of the corresponding 1-lightlike horospherical hypersurface LH_x at each point is the critical part of an MT-stable map germ.*

In the case when $n\leqslant 6$, for any $\boldsymbol{x}\in\mathcal{O}$, the germ of the Legendre lift $\mathcal{L}_x$ of the 1-lightlike horospherical hypersurface LH_x^1 at each point is Legendrian stable.

We remark that we can also prove the multi-jet version of proposition 6.1. As an application of such a multi-jet transversality theorem, we can show that the 1-lightlike horospherical hypersurface LH_x^1 is the critical part of a (global) MT-stable map for a generic 1-lightlike submanifold $\boldsymbol{x}\colon U\to\mathrm{AdS}^n$. However, the arguments are rather tedious and we only consider local phenomenon in this paper, so we omit the proof.

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References

1 V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko. *Singularities of differentiable maps*, vol. I (Birkhäuser, 1986).

2 J. M. Bardeen, B. Carter and S. W. Hawking. The four laws of black hole mechanics. *Commun. Math. Phys.* **31** (1973), 161–170.

3 A. Bejancu. Lightlike curves in Lorentz manifolds. *Publ. Math. Debrecen* **44** (1994), 145–155.

4 J. W. Bruce and P. J. Giblin. *Curves and singularities*, 2nd edn (Cambridge University Press, 1992).

5 L. Chen, Q. Han, D. Pei and W. Sun. The singularities of null surfaces in anti-de Sitter 3-space. *J. Math. Analysis Applic.* **366** (2010), 256–265.

6 G. Clement. Black holes with a null Killing vector in three-dimensional massive gravity. *Class. Quant. Grav.* **26** (2009), 165002.

7 M. M. de Souza. The Lorentz–Dirac equation and the structures of spacetime. *Brazil J. Phys.* **28** (1998), 250–256.

8 K. L. Duggal. Constant curvature and warped product globally null manifolds. *J. Geom. Phys.* **43** (2002), 327–340.

9 K. L. Duggal. A report on canonical null curves and screen distributions for lightlike geometry. *Acta Appl. Math.* **95** (2007), 135–149.

10 K. L. Duggal. On scalar curvature in lightlike geometry. *J. Geom. Phys.* **57** (2007), 473–481.

11 K. L. Duggal and A. Bejancu. *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Mathematics and Its Applications, vol. 364 (Dordrecht: Kluwer Academic, 1996).

12 K. L. Duggal and D. H. Jin. Totally umbilical lightlike submanifolds. *Kōdai Math. J.* **26** (2003), 49–68.

13 C. G. Gibson, K. Wirthmüller, A. A. du Plessis and E. J. Looijenga. *Topological stability of smooth mappings*, Lecture Notes in Mathematics, vol. 552 (Springer, 1976).

14 E. Gourgoulhon and J. L. Jaramillo. A $3+1$ perspective on null hypersurfaces and isolated horizons. *Phys. Rep.* **423** (2006), 159–294.

15 S. W. Hawking. Black holes in general relativity. *Commun. Math. Phys.* **25** (1972), 152–166.

16 S. Izumiya and F. Tari. Projections of surfaces in the hyperbolic space along horocycles. *Proc. R. Soc. Edinb.* A **140** (2010), 399–418.

17 S. Izumiya, D. Pei and M. del Carmen Romero Fuster. Umbilicity of spacelike surfaces in Minkowski space. *Proc. R. Soc. Edinb.* A **134** (2004), 375–387.

18 S. Izumiya, D. Pei and M. del Carmen Romero Fuster. The horospherical geometry of surfaces in hyperbolic 4-space. *Israel J. Math.* **154** (2006), 361–379.

19 S. Izumiya, D. Pei, M. del Carmen Romero Fuster and M. Takahashi. The horospherical geometry of submanifolds in hyperbolic space. *J. Lond. Math. Soc.* **71** (2005), 779–800.

20 S. Izumiya, D. Pei and T. Sano. Singularities of hyperbolic Gauss maps. *Proc. Lond. Math. Soc.* **86** (2003), 485–512.

21 S. Izumiya, D. Pei and M. Takahashi. Evolutes of hypersurfaces in hyperbolic space. *Proc. Edinb. Math. Soc.* **47** (2004), 131–153.

22 M. Korzynski, J. Lewandowski and T. Pawlowski. Mechanics of multidimensional isolated horizons. *Class. Quant. Grav.* **22** (2005), 2001–2016.

23 J. Martinet. *Singularities of smooth functions and maps*, London Mathematical Society Lecture Note Series, vol. 58 (Cambridge University Press, 1982).

24 J. N. Mather. Stability of C^∞-mappings, IV, classification of stable germs by $\mathbb{R}$ algebras. *Publ. Math. IHES* **37** (1969), 223–248.

25 J. N. Mather. *How to stratify mappings and jet spaces*, Lecture Notes in Mathematics, vol. 553 (Springer, 1976).

26 J. A. Montaldi. On contact between submanifolds. *Michigan Math. J.* **33** (1986), 191–195.

27 I. R. Porteous. The normal singularities of a submanifold. *J. Diff. Geom.* **5** (1971), 543–564.

28 W. Rudnicki. Black hole interiors cannot be totally vicious. *Phys. Lett.* A **208** (1995), 53–58.

29 W. Rudnicki, R. J. Budzynski and W. Kondracki. Generalized strong curvature singularities and weak cosmic censorship in cosmological space-times. *Mod. Phys. Lett.* A **21** (2006), 1501–1509.

30 J. Sultana and C. C. Dyer. Cosmological black holes: a black hole in the Einstein–de Sitter universe. *Gen. Rel. Grav.* **37** (2005), 1347–1370.

31 S. I. Tertychniy. The black hole formed by electromagnetic radiation. *Phys. Lett.* A **96** (1983), 73–75.

32 M. Vincent and I. James. Symmetries of cosmological Cauchy horizons. *Commun. Math. Phys.* **89** (1983), 387–413.

33 Y. H. Wan. Generic deformations of varieties. *Trans. Am. Math. Soc.* **259** (1980), 107–119.

34 Z. Wang and D. Pei. Singularities of ruled null surfaces of the principal normal indicatrix to a null Cartan curve in de Sitter 3-space. *Phys. Lett.* B **689** (2010), 101–106.

35 Z. Wang and D. Pei. Null Darboux developable and pseudo-spherical Darboux image of null Cartan curve in Minkowski 3-space. *Hokkaido Math. J.* **40** (2011), 219–240.

36 V. M. Zakalyukin. Lagrangian and Legendrian singularities. *Funct. Analysis Applic.* **10** (1976), 23–31.

37 V. M. Zakalyukin. Reconstructions of fronts and caustics depending on a parameter and versality of mappings. *J. Sov. Math.* **27** (1984), 2713–2735.

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