Unified picture of Q-balls and boson stars via catastrophe theory

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Abstract

We make an analysis of Q-balls and boson stars via catastrophe theory.

1 Introduction

Among non-topological solitons which appear in $U(1)$-symmetric scalar fields, objects existing even in flat spacetime are called Q-balls [1], while objects supported by strong gravity are called boson stars [2]. Although the difference in theory between Q-balls and boson stars is solely the model parameters, the investigations of their properties have been carried out separately. This is because Q-balls and boson stars have been discussed in different contexts of particle physics or astrophysics. The purpose of the present paper is to obtain a unified picture of equilibrium solutions and their stability of Q-balls and boson stars. Gravitating Q-balls, or Q-stars, which are intermediate objects between Q-balls in flat spacetime and boson stars, have also been discussed [3–5]. Therefore, a unified analysis of Q-balls and boson stars is also important for the study of astrophysical models [6].

2 Analysis method of equilibrium Q-balls and boson stars

We begin with the action

$$\mathcal{S} \;=\; \int d^4x\sqrt{-g}\Big(\frac{\mathcal{R}}{16\pi G}-\frac{1}{2}g^{\mu\nu}\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi}-V(\phi)\Big), \tag{1}$$

where $\boldsymbol{\phi}=(\phi_1,\ \phi_2)$ is a SO(2)-symmetric scalar field and $\phi\equiv\sqrt{\boldsymbol{\phi}\cdot\boldsymbol{\phi}}=\sqrt{\phi_1^2+\phi_2^2}$. We assume a spherically symmetric and static spacetime, $ds^2=-\alpha^2(r)dt^2+A^2(r)dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)$. For the scalar field, we assume $(\phi_1,\phi_2)=\phi(r)(\cos\omega t,\sin\omega t)$.

Because of the symmetry, there is a conserved charge called Q-ball charge,

$$Q \;\equiv\; \int d^3x\sqrt{-g}g^{\mu\nu}(\phi_1\partial_\nu\phi_2-\phi_2\partial_\nu\phi_1)=\omega I, \quad \text{where} \quad I\equiv 4\pi\int\frac{Ar^2\phi^2}{\alpha}dr. \tag{2}$$

As for Q-balls, which are present even in flat spacetime, we suppose the potential,

$$V_3(\phi):=\frac{m^2}{2}\phi^2-\mu\phi^3+\lambda\phi^4 \quad \text{with} \quad m^2,\ \mu,\ \lambda>0\ , \tag{3}$$

which we call V_3 Model. Rescaling the quantities as

$$\tilde{t}\equiv mt,\quad \tilde{r}\equiv mr,\quad \tilde{\omega}\equiv\frac{\omega}{m},\quad \tilde{\mu}\equiv\frac{\mu}{\sqrt{\lambda}m},\quad \kappa\equiv\frac{m^2G}{\lambda},\quad \tilde{\phi}\equiv\frac{\sqrt{\lambda}}{m}\phi,\quad \tilde{V}\equiv\frac{\lambda}{m^4}V_3,\quad \tilde{Q}\equiv\lambda Q, \tag{4}$$

the field equations are rewritten as

$$A'+\frac{A}{2\tilde{r}}(A^2-1)=4\pi\kappa\tilde{r}A^3\left(\frac{\tilde{\phi}'^2}{2A^2}+\frac{\tilde{\omega}^2\tilde{\phi}^2}{2\alpha^2}+\tilde{V}\right), \tag{5}$$

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$$\alpha' + \frac{\alpha}{2\tilde{r}}(1-A^2) = 4\pi\kappa\tilde{r}\alpha A^2\left(\frac{\tilde{\phi}'^2}{2A^2}+\frac{\tilde{\omega}^2\tilde{\phi}^2}{2\alpha^2}-\tilde{V}\right), \tag{6}$$

$$\tilde{\phi}'' + \left(\frac{2}{\tilde{r}}+\frac{\alpha'}{\alpha}-\frac{A'}{A}\right)\tilde{\phi}' + \left(\frac{\tilde{\omega}A}{\alpha}\right)^2\tilde{\phi} = A^2\frac{d\tilde{V}}{d\tilde{\phi}}. \tag{7}$$

Let us discuss how we apply catastrophe theory to the present Q-ball or boson star system. An essential point is to choose *behavior variable(s)*, *control parameter(s)* and a *potential* in the Q-ball or boson star system appropriately. In [7] we argued that the total energy of the scalar field,

$$E_\phi \equiv \int d^3x\left\{\frac{\omega^2\phi^2}{2}+\frac{(\phi')^2}{2}+V\right\}, \tag{8}$$

is appropriate for a *potential* because the variation of E_ϕ under fixed Q, $\delta E_\phi/\delta\phi|_Q = 0$, reproduces the equilibrium field equation. A nontrivial issue in curved spacetime is the choice of the corresponding total energy since there are many definitions for total energy. However, we can conclude that the Hamiltonian energy E is appropriate which reduces to $E = \frac{M}{2}$, where M is the gravitational mass. We also use the normalized quantity $\tilde{E} \equiv \frac{\lambda}{m}E$.

Because the charge Q and the model parameter(s) of $V(\phi)$ can be given by hand, they should be regarded as *control parameters*. In flat spacetime, V_3 Model essentially has only one parameter, $\tilde{\mu}^2$. In curved spacetime, on the other hand, the normalized gravitational constant κ is another *control parameter*, which represents the strength of gravity.

To discuss a *behavior variable* we consider an one-parameter family of perturbed field configurations $\phi_x(r)$ near the equilibrium solution $\phi(r)$. Because $dE[\phi_x]/dx = (\delta E/\delta\phi_x)d\phi_x/dx = 0$ when ϕ_x is an equilibrium solution, x is a *behavior variable*. Although an explicit choice for x is not unique, we choose $\tilde{\omega}^2$ and $\tilde{\phi}(0)$ as *behavior variables*.

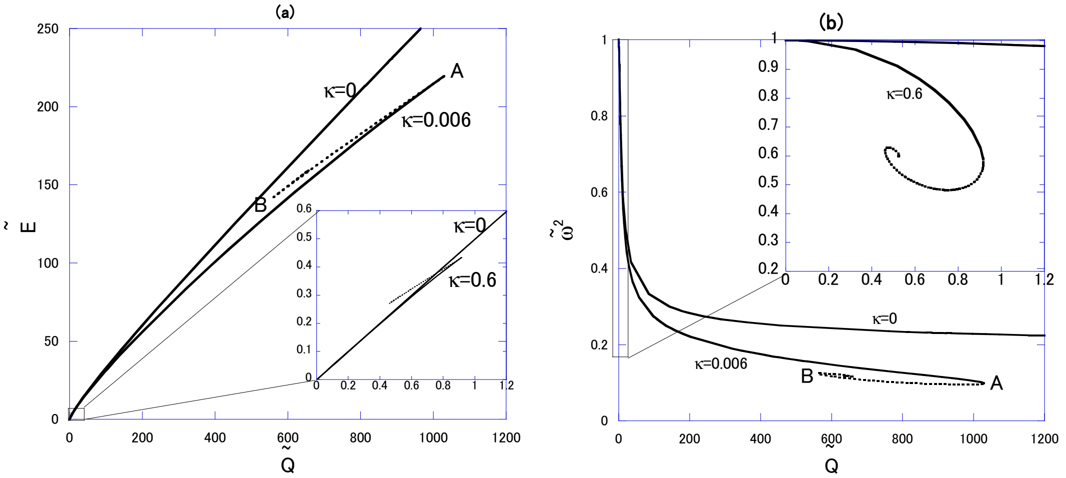

Figure 1: (a) $\tilde{Q}$-$\tilde{E}$ relation and (b) $\tilde{Q}$-$\tilde{\omega}^2$ relation for $\tilde{\mu}^2 = \frac{5}{3}$ in V_3 Model.

3 Stability of gravitating Q-balls and boson stars

3.1 Q-balls

We fix $\tilde{\mu}^2 = \frac{5}{3}$ as an example. Figure 1 shows a plot of $\tilde{Q}$ versus $\tilde{E}$ and that of $\tilde{Q}$ versus $\tilde{\omega}^2$ for equilibrium Q-ball solutions. In the case of $\kappa = 0$ there is one-to-one correspondence between $\tilde{Q}$ and $\tilde{E}$ while cusp structures appear in the case of $\kappa \neq 0$, as shown in (a). The maximum of $\tilde{Q}$ (labeled as A for $\kappa = 0.006$)

and the local minimum (labeled as B for $\kappa = 0.006$) appear in the case of gravitating Q-balls. At the point B, another cusp strcture appears and is far smaller than that at the point A. This sequences of cusp structure continue and we stopped calculation where the 4th cusp structure appears. The $\tilde{Q}$-maximum for $\kappa = 0.6$ is far smaller than that for $\kappa = 0.006$.

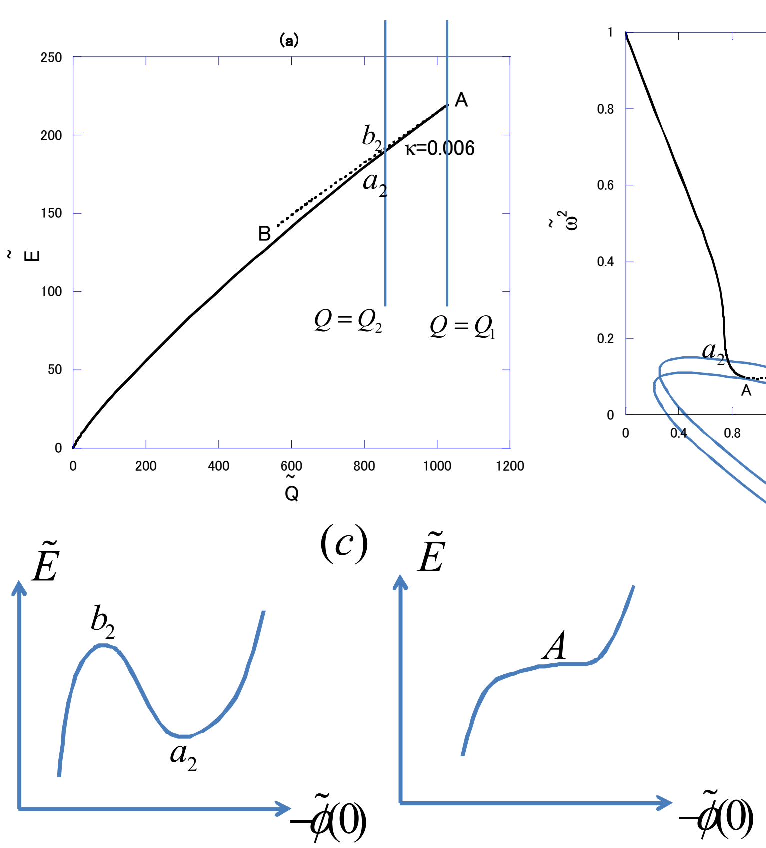

Figure 2: Stability interpretation via catastrophe theory for $\tilde{\mu}^2 = \frac{5}{3}$ for the gravitating case with $\kappa = 0.006$ (The qualitative properties are same for other κ).

We explain our interpretation via catastrophe theory by exemplifying the case with $\tilde{\mu}^2 = \frac{5}{3}$ and $\kappa = 0.006$ (Qualitative properties are not changed for other κ.). We identify $\tilde{Q}$ =const. lines in Fig. 2 (a) with the quadratic curves in Fig. 2 (b). If we adopt the view point that stability changes at the point A as we mentioned above and observe Fig. 2 (b), we notice that $(-\tilde{\phi}(0))$ is more appropriate for a *behavior variable* than $\tilde{\omega}^2$. Then, as we show in (c), a_2, b_2 and A can be interpreted as the potential minimum, maximum and the inflection point, respectively.

This case can be understood using the fold catastrophe $f(u) = u^3 + tu$ where u and t are the *bahavior variable* identified with $(-\tilde{\phi}(0))$ and the *control parameter* identified with $\tilde{Q}$, respectively.

We should reveal what causes the difference from the flat case. Naively speaking, solutions having larger $|A-1|$ at its peak have larger $(-\tilde{\phi}(0))$. We have confirmed that, independent of κ, solutions having $|A-1| \sim 1$ correspond to solutions expressed by dotted lines in Fig. 1. Therefore, we can suppose that the intrinsic difference from the flat case can be characterized by $|A-1|$.

3.2 Boson stars

Now we discuss boson stars with the potential V_3 with $\mu = 0$. Figure 3 shows plots of (a) $\tilde{Q}$-$\tilde{E}$, (b) $\tilde{Q}$-$\tilde{\omega}^2$ for equilibrium solutions of boson stars. Degenerate cusp and spiral structures are seen as in the case of gravitating Q-balls for $\tilde{\mu}^2 = \frac{5}{3}$. We have confirmed $|A-1| \sim 1$ at its peak in the solutions corresponding to the spiral curves or near the stability change points. We therefore conclude that, if gravity is so strong

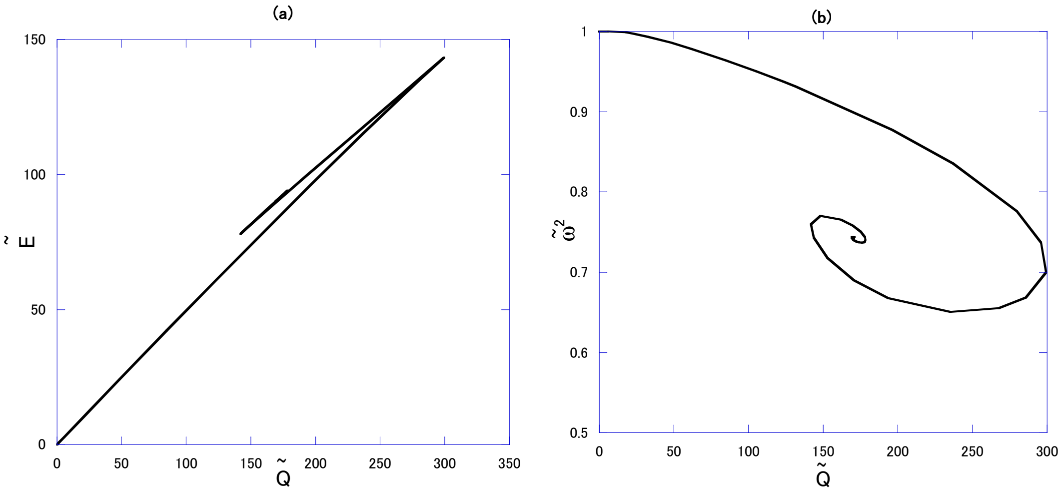

Figure 3: (a)$\tilde{Q}$-$\tilde{E}$, (b)$\tilde{Q}$-$\tilde{\omega}^2$ relations in the model of boson stars for $\kappa = 0.006$.

as $|A-1| \sim 1$ at its peak, catastrophic structures of Q-balls approach those of boson stars, regardless of the potential shape.

4 Conclusion and discussion

We have reanalyzed stability of gravitating Q-balls for a V_3 model and boson stars for a V_3 model with $\mu = 0$. For solutions with $|g^{rr} - 1| \sim 1$ at its peak, stability of Q-balls has been lost regardless of the potential parameters. As a result, phase relations, such as $\tilde{Q}$-$\tilde{E}$, approach those of boson stars, which tell us a unified picture of Q-balls and boson stars. Therefore, if we discuss the possibility of Q-balls or boson stars as dark matter candidates, our work would be useful.

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