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RESEARCH ARTICLE

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Geometrothermodynamics of black holes

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Abstract The thermodynamics of black holes is reformulated within the context of the recently developed formalism of geometrothermodynamics. This reformulation is shown to be invariant with respect to Legendre transformations, and to allow several equivalent representations. Legendre invariance allows us to explain a series of contradictory results known in the literature from the use of Weinhold's and Ruppeiner's thermodynamic metrics for black holes. For the Reissner–Nordström black hole the geometry of the space of equilibrium states is curved, showing a non trivial thermodynamic interaction, and the curvature contains information about critical points and phase transitions. On the contrary, for the Kerr black hole the geometry is flat and does not explain its phase transition structure.

Keywords Black hole thermodynamics, Phase transitions, Geometrothermodynamics, Thermodynamic metric

1 Introduction

The geometry of thermodynamics has been the subject of moderate research since the original works by Gibbs [1] and Caratheodory [2]. Results have been achieved in two different approaches. The first one consists in introducing metric structures on the space of thermodynamic equilibrium states $\mathscr{E}$, whereas the second group uses the contact structure of the so-called thermodynamic phase space $\mathscr{T}$. Weinhold [3; 4] introduced ad hoc on $\mathscr{E}$ a metric defined as the Hessian of the internal thermodynamic energy, where the derivatives are taken with respect to the extensive thermodynamic variables. Ruppeiner [5] introduced a metric which is conformally equivalent to Weinhold's metric, with the inverse of the temperature as the conformal factor. Results of the application of Ruppeiner's geometry have

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been reviewed in [6; 7; 8; 9; 10]. This approach has found applications also in the context of thermodynamics of black holes [11; 12; 13; 14; 15; 16].

The second approach, developed specially by Hermann [17] and Mrugala [18; 19], uses the natural contact structure of the phase space $\mathcal{T}$. Extensive and intensive thermodynamic variables are taken together with the thermodynamic potential to constitute well-defined coordinates on $\mathcal{T}$. A subspace of $\mathcal{T}$ is the space of thermodynamic equilibrium states $\mathcal{E}$, defined by means of a smooth embedding mapping $\varphi : \mathcal{E} \longrightarrow \mathcal{T}$. This implies that each system possesses its own space $\mathcal{E}$. On the other hand, on $\mathcal{T}$ it is always possible to introduce the fundamental Gibbs 1-form which, when projected on $\mathcal{E}$ with the pullback of φ, generates the first law of thermodynamics and the conditions for thermodynamic equilibrium. Furthermore, on $\mathcal{T}$ it is also possible to consider Riemannian structures [20; 21].

Geometrothermodynamics (GTD) [22; 23] was recently developed as a formalism that unifies the contact structure on $\mathcal{T}$ with the metric structure on $\mathcal{E}$ in a consistent manner, by considering only Legendre invariant metric structures on both $\mathcal{T}$ and $\mathcal{E}$. This last property is important in order to guarantee that the thermodynamic characteristics of a system do not depend on the thermodynamic potential used for its description. One simple metric has been used in GTD in order to reproduce geometrically the non critical and critical behavior of the ideal and van der Waals gas, respectively. In the present work we present a further application of GTD in general relativity, namely, we reformulate black hole thermodynamics and try to reproduce the phase transition structure of black holes by using one of the simplest metric structures that are included in GTD.

In general relativity, the gravitational field of the most general black hole is described by the Kerr–Newman [24] solution that corresponds to a rotating, charged black hole. The discovery by Bekenstein [25] that the behavior of the horizon area of a black hole resembles the behavior of the entropy of a classical thermodynamic system initiated an intensive and still ongoing investigation of what is now called thermodynamics of black holes [26; 27; 28]. Several attempts have been made in order to describe the thermodynamic behavior of black holes in terms of metrics defined on $\mathcal{E}$ [12; 13; 14; 15; 16]. In particular, Weinhold's and Ruppeiner's metrics were used to find a direct relationship between curvature singularities and divergencies of the heat capacity. Unfortunately, the results lead to completely contradictory statements. For instance, for the Kerr black hole Weinhold's metric predicts no phase transitions at all [13], whereas Ruppeiner's metric, with a very specific thermodynamic potential, predicts phase transitions which are compatible with the results of standard black hole thermodynamics [12]. It is one of the goals of this work to explain this contradiction by using an invariant approach. We will conclude that the origin of this inconsistency is due to the fact that
Weinhold's and Ruppeiner's metrics are not Legendre invariant, a property that makes them inappropriate for describing the geometry of thermodynamic systems. From the vast number of Legendre invariant metrics which are allowed in the context of GTD we choose probably the simplest one. This choice allows us to find Legendre invariant generalizations of Weinhold's and Ruppeiner's metrics. In the case of two-dimensional GTD, we apply these Legendre invariant metrics, and obtain consistent results.

This paper is organized as follows. In Sect. 2 we briefly review the fundamentals of GTD, and present a simple Legendre invariant metric. In Sect. 3 we apply GTD to thermodynamics of black holes in general, and find the simplest Legendre invariant generalizations of Weinhold's and Ruppeiner's metrics. In Sects. 4 and 5 we analyze the geometry of the Reissner–Nordström and Kerr black hole thermodynamics, by using Legendre invariant thermodynamic metrics. We show that in both cases the results are geometrically consistent. We find an agreement with the results of standard black hole thermodynamics in the case of the Reissner–Nordström solution. However, for Kerr black holes we show that the simplest Legendre invariant metrics do not reproduce the corresponding phase transition structure. Section 6 contains a brief analysis of the Fisher–Rao metric. Finally, Sect. 7 is devoted to discussions of our results and suggestions for further research. Throughout this paper we use units in which $G = c = k_{_B} = \hbar = 1$.

2 Review of geometrothermodynamics

Consider the $(2n+1)$-dimensional thermodynamic phase space $\mathscr{T}$ coordinatized by the thermodynamic potential Φ, extensive variables E^a, and intensive variables I^a $(a = 1,\dots,n)$. Consider on $\mathscr{T}$ a non-degenerate metric $G = G(Z^A)$, with $Z^A = \{\Phi, E^a, I^a\}$, and the Gibbs 1-form $\Theta = d\Phi - \delta_{ab} I^a dE^b$, with $\delta_{ab} = \mathrm{diag}(1,1,\dots,1)$. The set $(\mathscr{T}, \Theta, G)$ defines a contact Riemannian manifold [17; 21] if the condition $\Theta \wedge (d\Theta)^n \neq 0$ is satisfied. Moreover, the metric G is Legendre invariant if its functional dependence on Z^A does not change under a Legendre transformation [29]. The Gibbs 1-form Θ is also invariant with respect to Legendre transformations. Legendre invariance guarantees that the geometric properties of G do not depend on the thermodynamic potential used in its construction.

The n-dimensional subspace $\mathscr{E} \subset \mathscr{T}$ determined by the smooth mapping $\varphi : \mathscr{E} \longrightarrow \mathscr{T}$, that in terms of coordinates reads $\varphi : (E^a) \longmapsto (\Phi, E^a, I^a)$ with $\Phi = \Phi(E^a)$, is called the space of equilibrium thermodynamic states if the condition $\varphi^*(\Theta) = 0$ is satisfied, i.e.,

$$d\Phi = \delta_{ab} I^a dE^b, \quad \frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b. \tag{1}$$

The first of these equations corresponds to the first law of thermodynamics, whereas the second one is usually known as the condition for thermodynamic equilibrium [30]. In the GTD formalism, the last equation also means that the intensive thermodynamic variables are dual to the extensive ones. Notice that the mapping φ as defined above implies that the equation $\Phi = \Phi(E^a)$ must be explicitly given. In standard thermodynamics this is known as the fundamental equation from which all the equations of state can be derived [17; 30]. In this representation, the second law of thermodynamics is equivalent to the convexity condition on the thermodynamic potential $\partial^2\Phi/\partial E^a \partial E^b \geq 0$ [30; 31].

The thermodynamic potential satisfies the homogeneity condition $\Phi(\lambda E^a) = \lambda^\beta \Phi(E^a)$ for constant parameters λ and β. Using the first law of thermodynamics, it can easily be shown that this condition leads to the relations

$$\beta\Phi(E^a) = \delta_{ab} I^b E^a, \qquad (1-\beta)\delta_{ab} I^a dE^b + \delta_{ab} E^a dI^b = 0, \tag{2}$$

which are known as Euler's identity and Gibbs–Duhem relation.

The final ingredient of GTD is a non-degenerate metric structure g on $\mathscr{E}$ from which we demand to be compatible with the metric G on $\mathscr{T}$. This can be reached by using the pullback φ^* in such a way that g becomes naturally induced by G as $g = \varphi^*(G)$. As shown in [23], a Legendre invariant metric G induces a Legendre invariant metric g. Vice versa, a metric g on $\mathscr{E}$ is Legendre invariant only if it is induced by a Legendre invariant metric G on $\mathscr{T}$. It is in this sense that one can show that Weinhold's and Ruppeiner's metrics, which are defined on $\mathscr{E}$, are not Legendre invariant. Nevertheless, there is a vast number of metrics on $\mathscr{T}$ that satisfy the Legendre invariance condition. For instance, the metric structure

$$G = \Theta^2 + (\delta_{ab} E^a I^b)(\delta_{cd} dE^c dI^d), \tag{3}$$

where Θ is the Gibbs 1-form, is Legendre invariant and induces on $\mathscr{E}$ the metric

$$g = \Phi \frac{\partial^2 \Phi}{\partial E^a \partial E^b} dE^a dE^b. \tag{4}$$

An important feature of this metric is that it is flat for an ideal gas and non-flat for the van der Waals gas, with curvature singularities at the critical thermodynamic points [23]. This is an indication that it can be used as a Legendre invariant measure of thermodynamic interaction. Although this property is shared by other metrics on $\mathscr{E}$ in the following analysis we will use the specific choice (4) because of its simplicity.

Finally, we mention that the geometry of the metric $g = \varphi^*(G)$ is invariant with respect to arbitrary diffeomorphisms performed on $\mathscr{E}$. This can be shown by considering explicitly the components of g in terms of the components of G, and applying arbitrary Legendre transformations on G. This important property allows us to consider variational principles in GTD that impose additional conditions on the metric structures [38].

3 Black hole thermodynamics

Vacuum black holes in Einstein's theory are completely characterized by the mass M, angular momentum J, and electric charge Q. Although the statistical origin is still obscure, black holes possess thermodynamic properties specified through Hawking's temperature T, proportional to the surface gravity on the horizon, and entropy S proportional to the horizon area [25; 27]. All these parameters are related by means of the first law of black hole thermodynamics $dM = TdS + \Omega_H dJ + \phi dQ$ (see, for instance [26]), where Ω_H is the angular velocity on the horizon, and ϕ is the electric potential. For a given fundamental equation $M = M(S,J,Q)$ we have the conditions for thermodynamic equilibrium

$$T = \frac{\partial M}{\partial S}, \quad \Omega_H = \frac{\partial M}{\partial J}, \quad \phi = \frac{\partial M}{\partial Q}. \tag{5}$$

Thus, the phase space $\mathscr{T}$ for black hole thermodynamics is 7-dimensional with coordinates $Z^A = \{M, S, J, Q, T, \Omega_H, \phi\}$. The fundamental Gibbs 1-form is given by $\Theta = dM - TdS - \Omega_H dJ - \phi dQ$. The space of thermodynamic equilibrium

states $\mathscr{E}$ is 3-dimensional with coordinates $E^a = \{S,J,Q\}$, and is defined by means of the mapping

$$\varphi : \{S,J,Q\} \longmapsto \left\{ M(S,J,Q), S, J, Q, \frac{\partial M}{\partial S}, \frac{\partial M}{\partial J}, \frac{\partial M}{\partial Q} \right\}. \tag{6}$$

The mass M plays the role of thermodynamic potential that depends on the extensive variables S, J and Q. However, Legendre transformations allow us to introduce a set of seven additional thermodynamic potentials which depend on different combinations of extensive and intensive variables. The complete set of thermodynamic potentials can be written as

$$\begin{aligned}
M &= M(S,J,Q),\\
M_1 &= M_1(T,J,Q) = M - TS,\\
M_2 &= M_2(S,\Omega_H,Q) = M - \Omega_H J,\\
M_3 &= M_3(S,J,\phi) = M - \phi Q,\\
M_4 &= M_4(T,\Omega_H,Q) = M - TS - \Omega_H J,\\
M_5 &= M_5(T,J,\phi) = M - TS - \phi Q,\\
M_6 &= M_6(S,\Omega_H,\phi) = M - \Omega_H J - \phi Q,\\
M_7 &= M_7(T,\Omega_H,\phi) = M - TS - \Omega_H J - \phi Q.
\end{aligned} \tag{7}$$

Notice that the mapping φ can be defined in each case, independently of the chosen thermodynamic potential. On the other hand, since we are considering only Legendre invariant structures on $\mathscr{T}$ and $\mathscr{E}$, the characteristics of the underlying geometry for a given thermodynamic system will be independent of the thermodynamic potential. This is in agreement with standard thermodynamics. Consequently, in the mass representation of black hole thermodynamics described above, we have the freedom of choosing anyone of the potentials $M, M_1, \ldots, M_7$, without affecting the thermodynamic properties of black holes.

In the context of GTD, it is also possible to consider the entropy representation. In this case, the Gibbs 1-form of the phase space can be chosen as

$$\Theta_S = dS - \frac{1}{T} dM + \frac{\Omega_H}{T} dJ + \frac{\phi}{T} dQ. \tag{8}$$

The space of equilibrium states is then defined by the smooth mapping

$$\varphi_S : \{M,J,Q\} \longmapsto \{M, S(M,J,Q), J, Q, T(M,J,Q), \Omega_H(M,J,Q), \phi(M,J,Q)\}, \tag{9}$$

with

$$\frac{1}{T} = \frac{\partial S}{\partial M}, \quad \frac{\Omega_H}{T} = -\frac{\partial S}{\partial J}, \quad \frac{\phi}{T} = -\frac{\partial S}{\partial Q}, \tag{10}$$

such that $\varphi_S^*(\Theta_S) = 0$ leads to the first law. In the entropy representation the fundamental equation is now given by $S = S(M,J,Q)$, and the second law of thermodynamics corresponds to the concavity condition of the entropy function. Additional

representations can easily be analyzed within GTD, and the only object that is needed in each case is the smooth mapping φ which guarantees the existence of a well-defined space of equilibrium states. Clearly, the thermodynamic properties of black holes must be independent of the representation.

Now we consider metric structures on $\mathcal{E}$. For black holes, Weinhold's metric g^W is defined as the Hessian in the mass representation [3; 4], whereas Ruppeiner's metric g^R is given as minus the Hessian in the entropy representation [5]. From the analysis given above, it is clear that these metrics must be related by $g^W = Tg^R$. As we showed in [23], the main problem with Weinhold's and Ruppeiner's metrics is that they are not Legendre invariant. In GTD it is possible to derive, in principle, an infinite number of metrics which preserve Legendre invariance; nevertheless, according to Eq. (4), the simplest way to reach the Legendre invariance for g^W is to apply a conformal transformation, with the thermodynamic potential as the conformal factor. Consequently, the simplest Legendre invariant generalization of Weinhold's metric can be written in components as

$$g = Mg^W = M\frac{\partial^2 M}{\partial E^a \partial E^b} dE^a dE^b, \tag{11}$$

where $E^a = \{S, J, Q\}$. This Legendre invariant metric can also be written in terms of the components of Ruppeiner's metric as

$$g = MTg^R = -M\left(\frac{\partial S}{\partial M}\right)^{-1} \frac{\partial^2 S}{\partial F^a \partial F^b} dF^a dF^b, \tag{12}$$

with $F^a = \{M, J, Q\}$. Using the mass representation, in the phase space $\mathcal{T}$ the corresponding generating metric structure can be written as [cf. Eq. (3)]

$$\begin{aligned} G = {} & (dM - TdS - \Omega_H dJ - \phi dQ)^2 \\ & + (TS + \Omega_H J + \phi Q)(dTdS + d\Omega_H dJ + d\phi dQ). \end{aligned} \tag{13}$$

Notice that to obtain (11) we need to use Euler's identity for the conformal factor in front of the second term in (13), i.e., $\beta M = TS + \Omega_H J + \phi Q$. Thus, g as given in (11) is determined only up to the multiplicative constant β that, obviously, does not affect its geometry.

In the following sections we will analyze metrics (11) and (12) in the case of 2-dimensional GTD with $a, b = 1, 2$, $E^1 = S$, and E^2 will be chosen either as Q or as J, which corresponds to the Reissner–Nordström and Kerr black holes, respectively.

4 The Reissner–Nordström black hole

The Reissner–Nordström solution describes a static black hole with mass M and electric charge Q. The inner and outer event horizons are situated at r_- and r_+ so that the outer horizon area is $A = 4\pi r_+^2$, where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The extremal black hole corresponds to the value $r_+ = r_-$ and we suppose that $M^2 \geq Q^2$ in order

to avoid naked singularities. From the horizon area law it follows that the entropy of the black hole is given by

$$S = \frac{1}{4}A = \pi\left(M + \sqrt{M^2 - Q^2}\right)^2, \tag{14}$$

an expression that can be rewritten as

$$M = \frac{1}{2\sqrt{\pi S}}\left(\pi Q^2 + S\right). \tag{15}$$

This is the fundamental equation from which, according to Eq. (11), one easily calculates the Legendre invariant metric on the space of equilibrium states. Then

$$g = \frac{1}{S^2}\left(\pi Q^2 + S\right)\left[\frac{1}{16\pi S}\left(3\pi Q^2 - S\right)dS^2 - \frac{1}{2}QdSdQ + \frac{1}{2}SdQ^2\right]. \tag{16}$$

The convexity condition is not satisfied in general. Only the term g_{QQ} is always positive definite, whereas g_{QS} is positive only for negative values of the total charge. The component g_{SS} violates the convexity condition if $S > 3\pi Q^2$. The limiting value $S = 3\pi Q^2$ determines the turning point where the second law of thermodynamics becomes invalid. In terms of the horizons' radii and for a given radius r_+ of the outer horizon, this is equivalent to the statement that the convexity condition is valid only in the interval $r_- \in [r_+/3, r_+)$.

The scalar curvature corresponding to the metric (16) reads

$$R = -\frac{8\pi^2 Q^2 S^2(\pi Q^2 - 3S)}{(\pi Q^2 + S)^3(\pi Q^2 - S)^2}. \tag{17}$$

We see that the only curvature singularity occurs when $S = \pi Q^2$. This corresponds to the value $M = Q$, i.e., the extremal black hole. We interpret this result as an indication of the limit of applicability of GTD as a geometric model for equilibrium thermodynamics. This is also in accordance with the intuitive expectation that naked singularities show the limit of applicability of black hole thermodynamics.

Another interesting point is $S = \pi Q^2/3$, where the scalar curvature vanishes identically, leading to a flat geometry. At this point the scalar curvature changes its sign, and it is the only point where this happens. Notice that the value $S = \pi Q^2/3$ corresponds to $M = 2Q/\sqrt{3}$ or, equivalently, $r_+ = 3r_-$ which according to Davies [28] is exactly the point where the system is undergoing a phase transition.

It is interesting to note that the phase transition point has been analyzed in other works, using Weinhold's and Ruppeiner's metrics, with partially contradictory results. For instance, in [13] at the phase transition point $S = \pi Q^2/3$ nothing happens because Ruppeiner's metric is flat everywhere. On the other hand, in [12] this point corresponds to a true curvature singularity of Ruppeiner's geometry with a different thermodynamic potential. Moreover, in the same work the extremal black hole is described by a well-behaved metric with zero scalar curvature that, in principle, can be analytically extended to include the case of a naked singularity. We interpret these contradictory results as due to the use of metrics that

are not Legendre invariant. In fact, let us consider the simplest Legendre invariant generalization of Ruppeiner's metric (12) with the fundamental equation (14). Then

$$g = \frac{\pi M}{(M^2-Q^2)S}\left\{\left[2M^3-3MQ^2+2(M^2-Q^2)^{3/2}\right]dM^2 + 2Q^3 dMdQ - \left[M^3+(M^2-Q^2)^{3/2}\right]dQ^2\right\}. \tag{18}$$

From the corresponding scalar curvature one can see that it diverges at $M = Q$, and changes its sign at $M = 2Q/\sqrt{3}$, which coincides with the result obtained by using the Legendre invariant Weinhold metric given in Eq. (16). This solves the incompatibility problem between the results obtained by using geometries which do not preserve Legendre invariance.

What we learn in this case from the use of Legendre invariant metrics is that in GTD a phase transition can also be described by a change of sign of the scalar curvature, passing through a state of flat geometry. Although there is no singular behavior associated with this phase transition, we believe that the change of topology that occurs when going from a negative to a positive curvature could have drastic consequences for the underlying thermodynamics. A more detailed analysis will be necessary in order to clarify this issue.

5 The Kerr black hole

The Kerr solution describes the gravitational field of rotating black hole with mass M and angular momentum J. The inner and outer horizons are situated at r_- and r_+, where $r_\pm = M \pm \sqrt{M^2-a^2}$. The entropy is calculated as usual in terms of the area of the horizon

$$S = \frac{1}{4}A = 2\pi\left(M^2+\sqrt{M^4-J^2}\right) \tag{19}$$

which for the mass representation can be rewritten as

$$M = \sqrt{\frac{S}{4\pi}+\frac{\pi J^2}{S}}. \tag{20}$$

From Eq. (11) we get the corresponding Legendre invariant metric of the space of thermodynamic equilibrium states

$$g = \frac{\pi S}{S^2+4\pi^2J^2}\left[\left(\frac{3\pi^2J^4}{S^4}+\frac{3J^2}{2S^2}-\frac{1}{16\pi^2}\right)dS^2 - \frac{J}{S^3}\left(3S^2+4\pi^2J^2\right)dJdS+dJ^2\right]. \tag{21}$$

Unexpectedly, the curvature of this metric vanishes. The same result was obtained in [14] by using Weinhold's metric. This is a surprising result because it would mean that Kerr black holes do not show any statistical thermodynamic interaction. On the other hand, the standard thermodynamics of Kerr black holes is by no

means trivial and shows a very rich phase transition structure [28]. Moreover, the analysis performed in [12] using Ruppeiner's metric closely reproduces this structure. This contradictory result is again due to the use of non invariant metrics. In fact, if we consider in GTD the simplest Legendre invariant generalization of Ruppeiner's metric as given in Eq. (12), we obtain from the fundamental equation (19) the following metric

$$g = \frac{2\pi}{S(M^4 - J^2)} \left\{ \left[(M^4 - J^2)^{3/2} + M^2(M^4 - 3J^2) \right] dM^2 \right.$$
$$\left. + 2M^3 J dM dJ - \frac{M^4}{2} dJ^2 \right\}, \tag{22}$$

whose curvature also vanishes. This coincides with the result obtained by using the Legendre invariant generalization (21) of Weinhold's metric, but it also drastically differs from the result of [12] where a non zero curvature was obtained for the pure Ruppeiner metric with a special choice of the thermodynamic potential; instead of using M, the authors of [12] define an internal energy which in the notation used here corresponds to the entropy representation of the thermodynamic potential M_2 given in Eq. (13). One can easily show that the use of the original potential M leads to a different metric with non vanishing curvature, and the only curvature singularity appears at $M^2 = J$, i.e., in the extremal black hole limit, a result that does not reproduce the phase transition structure of the Kerr black hole. This shows that Ruppeiner's metric leads to completely different results, depending on the thermodynamic potential.

The main result of this Section is that Weinhold's and Ruppeiner's metrics in their Legendre invariant generalizations lead to the same result for the Kerr black hole. Unfortunately, the corresponding geometry in the space of equilibrium states is flat and does not reproduce the phase transition structure of the black hole. This is a negative result which calls for a reconsideration of the use of geometric structures in black hole thermodynamics.

6 The Fisher–Rao metric

In classical statistical mechanics, an alternative approach has been used to analyze the geometry of thermodynamic systems. The starting point is the probability density distribution which is given by the Gibbs measure

$$p(x|\theta) = \exp\left[-\theta_i H_i(x) - \ln Z(\theta)\right], \tag{23}$$

where $Z(\theta)$ is the partition function, $H_i(x)$ are Hamiltonian functions and θ^i, $i = 1, \ldots, n$ represent the n parameters characterizing the statistical model under consideration. It can be shown [32] that for each value of the parameters the square root of this density can be associated to a vector in the Hilbert space $\mathscr{H}$. Consequently, $\mathscr{H}$ contains the state space of the system, and the properties of the statistical system can be described by means of the embedding of $p(x|\theta)$ in $\mathscr{H}$. Once the Hilbert space is considered in the language of projective geometry, it is possible to generalize this embedding construction to include the cases of quantum mechanical dynamics of equilibrium states and pure quantum mechanics [33; 34].

The geometry that arises from the embedding turns out to possess a natural Riemannian metric, the Fisher-Rao metric in the classical case or the Fubini-Study metric in the quantum case.

For the classical Gibbs distribution (23) the Fisher-Rao metric takes the simple form [35]

$$g^{FR} = \frac{\partial^2 \ln Z(\theta)}{\partial\theta^i \partial\theta^j} d\theta^i d\theta^j. \tag{24}$$

The geometric properties of the manifold described by this metric has been analyzed for different statistical models. In the case of the van der Waals gas the parameters can be chosen as $\theta^1 = 1/T$, and $\theta^2 = P/T$, and the corresponding Hamiltonian functions are the internal energy U and volume V, respectively, so that $Z(\theta)$ is a function of temperature and pressure. The scalar curvature of this two-dimensional manifold turns out to diverge at the critical points, and the scaling exponent of the curvature near the transition points coincides with that of the correlation volume [6; 7]. Furthermore, in the limiting case of an ideal gas, the curvature vanishes and the manifold is flat. This is exactly the behavior shown by Ruppeiner's and Weinhold's geometry in these particular cases. In fact, it can be shown [36] that in general both metrics are related to the Fisher–Rao metric by means of Legendre transformations of the corresponding variables. This explains why their geometric properties are similar, and indicates that the Fisher–Rao metric is also not Legendre invariant.

To be more specific in the case under consideration in this work, we first mention that, due to its statistical origin, the components of the Fisher–Rao metric $g^{FR}_{ij}(\theta)$ are usually given in terms of the "inverse" of the thermodynamic variables: $\theta^1 = 1/T$, $\theta^2 = P/T$, etc. Since the relationships $\theta^i = \theta^i(E^a)$ must allow the inverse transformation, it is easy to show that in the coordinates used here the Fisher–Rao metric can be written as $g^{FR}_{ab} = \partial^2 \ln Z(E)/\partial E^a \partial E^b$. The partition function for black holes is given by (see, for instance [37])

$$Z = \exp\left[-\frac{1}{T}(M - TS - \Omega_H J - \phi Q)\right]. \tag{25}$$

It then follows that for the Reissner–Nordström black hole $Z = \exp(-M_5/T)$ and for the Kerr black hole $Z = \exp(-M_4/T)$, where the thermodynamic potentials M_4 and M_5 are related to the mass representation we are using here by the Legendre transformations given in Sect. 3. Consequently, the components of the Fisher–Rao metric for black holes are essentially given by $g^{FR}_{ab} = -\partial^2(M/T)/\partial E^a \partial E^b$. We now briefly explain how to show that this metric is not Legendre invariant. According to GTD, there must exist in the thermodynamic phase space $\mathcal{T}$ a metric G^{FR} which generates g^{FR} by means of the pullback $g^{FR} = \varphi^*(G^{FR})$. On $\mathcal{T}$ we perform an arbitrary Legendre transformation $Z^A \to \tilde{Z}^A$ which when acting on G^{FR} produces the Legendre transformed metric $\tilde{G}^{FR}$. Then the Legendre transformed Fisher–Rao metric $\tilde{g}^{FR}$ in the space of equilibrium states $\mathcal{E}$ is computed by $\tilde{g}^{FR} = \tilde{\varphi}^*(\tilde{G}^{FR})$, where $\tilde{\varphi}$ is the embedding mapping in the new coordinates (for more details see [23]). As a result we obtain that the functional dependence of $\tilde{g}^{FR}$ is completely different from that of g^{FR}; i.e., the Fisher–Rao metric is not Legendre invariant. Similar results can be obtained by using the original potentials M_4 and M_5.

The result of this section is that the Fisher–Rao metric for black holes is not Legendre invariant, and therefore cannot be used to solve the problem of contradictory results following from the application of Weinhold's and Ruppeiner's approaches.

7 Conclusions

In this work we formulated the thermodynamics of general relativistic black holes in the language of geometrothermodynamics. The general thermodynamic phase space $\mathcal{T}$ turns out to be 7-dimensional and it is always possible to introduce a smooth mapping from the 3-dimensional space of thermodynamic equilibrium states $\mathcal{E}$ to the phase space $\mathcal{T}$. Different formalisms based on different representations, such as the energy or the entropy representation, can easily be handled within GTD. The equivalence between all possible representations is an obvious consequence of the properties of the structures used in GTD. We present all the thermodynamic potentials that can be derived by means of arbitrary Legendre transformations, starting from the mass representation, and explain why the geometric properties of a thermodynamic system cannot depend on the chosen thermodynamic potential. From all Legendre invariant metric structures that can be introduced on $\mathcal{T}$ and, consequently, on $\mathcal{E}$, we choose a simple example which allows us to generalize other metrics used in the literature for investigating the geometric properties of thermodynamic systems.

We studied two-dimensional GTD in the case of the Reissner–Nordström and Kerr black holes, by using simple Legendre invariant generalizations of Weinhold's and Ruppeiner's metrics. Our results show that for the thermodynamics of the Reissner–Nordström black hole there exists a Legendre invariant geometry with non vanishing curvature. There is a true curvature singularity when the black hole becomes extremal. We interpret this result as indicating the limits of applicability of GTD in the sense that the thermodynamic processes associated with the black hole becoming extremal must be highly non trivial and related to non equilibrium thermodynamics, an issue that has not yet been considered within GTD. A second critical point occurs when $M = 2Q/\sqrt{3}$ (i.e., $r_+ = 3r_-$). At this point the scalar curvature changes its sign, passing through a state of flat geometry. It also coincides with a thermodynamic critical point where, according to Davies [28], the system is undergoing a phase transition. We propose that in GTD the change of topology, that happens when the scalar curvature changes its sign, can be associated to a drastic change of the thermodynamic properties of the system, like a phase transition. This question needs to be further analyzed in order to get a more concrete answer. Our results also solve an incompatibility existing in the literature. Aman et al. [13] used Ruppeiner's metric for the thermodynamics of the Reissner–Nordström black hole to show that its geometry has no critical points which could be related to phase transitions, because it is a flat geometry. On the other hand, Shen et al. [12] studied Ruppeiner's geometry and found a true curvature singularity at $r_+ = 3r_-$, corresponding to a second order phase transition. We proved that this contradiction is due to the use of metrics which do not preserve Legendre invariance. Our invariant generalizations of Weinhold's and Ruppeiner's metrics lead to compatible results and reinforce the prediction of a phase transition.

Our study of GTD in the case of the Kerr black hole also solves the contradiction in the results of Aman et al. [13] and Shen et al. [12]. We proved that the Legendre invariant generalizations of Weinhold's and Ruppeiner's metrics lead to the result that the underlying geometry is flat. This is a surprising result which does not coincide with the analysis of standard black hole thermodynamics that predicts the existence of phase transitions. A flat geometry implies that there is no thermodynamic interaction and, consequently, no phase transitions at all. This is a negative result which, in our opinion, implies that we should critically reconsider the application of geometry structures in black hole thermodynamics. However, this negative result can also be interpreted as implying that Weinhold's and Ruppeiner's metrics, even in their Legendre invariant version, are not suitable for describing the thermodynamics of black holes. The statistical origin of the Fisher–Rao metric could be thought to be an advantage, when compared with metrics introduced ad hoc. However, we have seen that Legendre invariance is not a property of this statistical metric.

An intriguing result was obtained by Shen et al. [12] for the Kerr black hole. They work in the entropy representation of Ruppeiner's metric with a different thermodynamic potential. Instead of using the mass M as an extensive variable, they consider the potential $M_2 = M - \Omega_H J$, and reproduce exactly the phase transition structure of the Kerr black hole [28]. Although we have seen that this result can drastically be changed by using M as thermodynamic potential, it would be interesting to consider the metric used by Shen et al. as a guide to find a generalization that would preserve Legendre invariance.

In GTD there exists, in principle, an infinite number of Legendre invariant metrics, and there is no reason to believe that all of them should be applicable to any thermodynamic system. We think that it is necessary to find additional criteria which would serve to select Legendre invariant metrics with certain specific properties. In this context, the application of variational principles in GTD could be useful. In fact, the metric induced on $\mathcal{E}$ by means of $g = \varphi^*(G)$ can be written in components as $(a, b = 1, \ldots, n,\ A, B = 0, \ldots, 2n)$

$$g_{ab} = \frac{\partial Z^A}{\partial E^a} \frac{\partial Z^B}{\partial E^b} G_{AB}. \tag{26}$$

Then, we can limit ourselves, for example, to only those metrics g_{ab} which define an extremal n-dimensional hypersurface on $\mathcal{E}$, i.e., metrics satisfying the "motion equations" following from the variation $\delta \int \sqrt{\det(g_{ab})} d^n x = 0$. This is equivalent to demanding that the mapping $\varphi : \mathcal{E} \longrightarrow \mathcal{T}$ determines a non linear sigma model. The resulting geodesic-like equations can be solved for a given fundamental equation and we obtain as a result the set of Legendre invariant metrics that can be used to describe the corresponding thermodynamic system. Also, for a given Legendre invariant metric one can find the set of fundamental equations that satisfy the corresponding equations. This task is currently under investigation [38].

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