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Bound state solution of the Schrödinger equation at finite temperature

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Abstract. In this article, the bound state solution of the modified radial Schrödinger equation is obtained for the sum of Cornell and inverse quadratic potential. Here in, the developed scheme is used to overcome the centrifugal part at the finite temperature and the energy eigenvalues and corresponding radial wave functions are defined for any angular momentum case via the Nikiforov-Uvarov methods. The present result are applied on the charmonium and bottomonium masses at finite and zero temperature. Our result are in good agreement with other theoretical and experimental results. The zero temperature limit of the energy spectrum and eigenfunctions is also founded. It is shown that the present approach can successfully be apply to the quarkonium systems at the finite temperature as well.

1. Introduction

It is well known that, one of the primary goals in the quantum mechanics is to find exact solutions of the wave equation because they contain all the necessary information on quantum system consideration. Since the wave function contains all essential information for full description of a quantum system, therefore, an analytical solution of the Schrödinger, Klein-Fock-Gordon and Dirac equations is of high importance in quantum mechanics [1, 2]. There are few potentials for which the Schrödinger equation are able solved exactly for all n_r radial and l orbital quantum states [1, 2, 3, 4]. In general, many quantum systems can only be treated by approximation methods or numerical solutions.

There are several potentials such as exponential-type potentials are still attracted the attention of many researchers. From this potentials include the Hulthén potential, the Manning Rosen potential, Woods-Saxon potential and the Eckart-type potential. It should be mentioned that essential contributions are concerned with the $l \neq 0$ wave case. Also the Cornell potential and mixed between the Cornell potential and the harmonic oscillator potential, also Morse potential as in [5, 6, 7, 8, 9] are suggested for solving the Schrödinger equation. The investigation of the heavy-meson systems such as bottomonium and charmonium are present essential interest because of its relies on entirely on the theory of quantum chromodynamics as in [10] and references therein. Heavy quarkonia have been suggested as hard probes of the quark-gluon plasma [11] since the modification of static interactions at finite temperature eventually implies a dissolution of heavy quarkonia bound states into the continuum of scattering states. Results of this effect is studied in a suppression of heavy quarkonia production in heavy-



ion collisions as an observable signal [12]. In work [13] solved D-dimensional Schrödinger equation for the Cornell potential at finite temperature.

Also, there are several works for solving the Schrödinger equation at finite temperature by using different methods, some of them can be seen in [14, 15, 16]. In work [17], the authors numerically solved the Schrödinger equation at finite temperature.

It would be interesting and an significant to solve the Schrödinger equation for the sum of Cornell [18, 19, 20] and inverse quadratic potential for $l \neq 0$, since it has been extensively used to describe the bound and continuum states of the interacting systems at finite temperature.

The combined potential considering in this study is obtained by sum of Cornell and inverse quadratic potential, since it has been extensively used to describe the bound and continuum states of the interacting systems.

$$V(r) = A \cdot r - \frac{B}{r} + \frac{C}{r^2}. \quad (1)$$

where A is a model parameter for the Coulomb strength, B is the string tension, C strength of the external field, and r is the interquark distance, respectively.

The Cornell potential consists of two terms, namely the Coulomb and linear terms. The Coulomb term is responsible for the interaction at small distances, which corresponds to the potential induced by one-gluon exchange between the quark and its anti-quark that dominated at short distances and the linear term leads to the confinement. The Cornell potential is used for mathematical modeling of the parton vibrations inside hadronic system and it constitutes an appropriate model for other physical situations.

Should be noted that the problem of obtaining the interquark potential is still open, its solution is necessary to find the mass spectra for coupled states and to describe the strong interaction characteristics of mesons.

The studying of the properties of mesons composed of a heavy quark and antiquark gives very important information of the heavy quark dynamics in sight hadrons. Should be notice heavy quarkonia have a rich spectroscopy with many narrow states lying under the threshold of open flavor production.

Therefore, it would be important and interesting to solve the Schrödinger equation for the sum of Cornell and inverse quadratic potential for arbitrary orbital $l \neq 0$ quantum number at finite temperature $T \neq 0$ using Nikiforov-Uvarov (NU) [21] method.

The rest of the present work is organized as follows. Bound-state Solution of the radial Schrödinger equation for the sum of Cornell and inverse quadratic potential by NU method is provided in Section 2. In Section 3 we present the numerical results for energy and mass spectrum and the corresponding normalized eigenfunctions. Finally, some concluding remarks are stated in Section 4.

2. Bound state Solution of the Radial Schrödinger equation for the sum of Cornell and inverse quadratic potential.

The Schrödinger equation in spherical coordinates is given as

$$\nabla^2 \psi + \frac{2\mu}{\hbar^2} [E - V(r)] \psi(r, \theta, \phi) = 0. \quad (2)$$

Considering this equation, the total wave function is written as

$$\psi(r, \theta, \phi) = R(r) Y_{l,m}(\theta, \phi), \quad (3)$$

Thus, the radial Schrödinger equation for the sum of Cornell and inverse quadratic potential defined in this form:

$$R''(r) + \frac{2}{r} R'(r) + \frac{2\mu}{\hbar^2} \left[E - \frac{l(l+1)\hbar^2}{2\mu r^2} + A \cdot r - \frac{B}{r} + \frac{C}{r^2} \right] R(r) = 0. \quad (4)$$

We assume $R(r) = \frac{1}{r}\chi(r)$ in (4) then radial Schrödinger equation becomes

$$\chi''(r) - \frac{2\mu}{\hbar^2} \left[E - \frac{\hbar^2 l(l+1)}{2\mu r^2} - A \cdot r + \frac{B}{r} - \frac{C}{r^2} \right] \chi(r) = 0. \quad (5)$$

Here μ the reduced mass for the quarkonium particle, for charmonium $\mu = \frac{m_c}{2}$, for bottomonium $\mu = \frac{m_b}{2}$.

In a thermal medium of temperature $T > 0$ the potential is modified by colour screening which can be parameterized in the form:

$$V_T(r) = \frac{A}{\mu_D(T)}(1 - \exp(-\mu_D(T)r)) - \frac{B}{r}\exp(-\mu_D(T)r) + \frac{C}{r^2}\exp(-\mu_D(T)r) \quad (6)$$

Here

$$A(T, r) = \frac{A}{\mu_D(T) \cdot r}(1 - \exp(-\mu_D(T)r)) \quad (7)$$

$$B(T, r) = B\exp(-\mu_D(T)r), \quad (8)$$

$$C(T, r) = C\exp(-\mu_D(T)r). \quad (9)$$

Here $\mu_D(T)$ is the Debye screening mass, which vanishes at $T \rightarrow 0$.

First we expand these (7-9) functions in a Taylor series around $r = 0$, then we obtain:

$$\begin{aligned} A(T, r) &= \frac{A}{\mu_D(T) \cdot r}(1 - \exp(-\mu_D(T)r)) = \frac{A}{\mu_D(T)r}(1 - 1 + \mu_D(T)r - \frac{1}{2}\mu_D^2(T) \cdot r^2) \\ &= A - \frac{A}{2}\mu_D(T) \cdot r \end{aligned} \quad (10)$$

$$B(T, r) = B(1 - \mu_D(T)r + \frac{1}{2}\mu_D^2(T) \cdot r^2) \quad (11)$$

$$C(T, r) = C(1 - \mu_D(T)r + \frac{1}{2}\mu_D^2(T) \cdot r^2) \quad (12)$$

Now substituting expression (10-12) into Eq.(6), then we obtain:

$$\begin{aligned} V(T, r) &= B\mu_D(T) + \frac{1}{2}C\mu_D^2(T) + (A - \frac{1}{2}B\mu_D^2(T))r - (B + C\mu_D(T))\frac{1}{r} - \\ &\quad \frac{1}{2}A\mu_D(T)r^2 + \frac{C}{r^2}. \end{aligned} \quad (13)$$

In Eq.(13) we use the following ansatz in order to make the potential more compact,

$$D = B\mu_D(T) + \frac{1}{2}C\mu_D^2(T), \quad F = A - \frac{1}{2}B\mu_D^2(T), \quad (14)$$

$$G = B + C\mu_D(T)), \quad L = \frac{1}{2}A\mu_D(T). \quad (15)$$

Then $V(T, r)$ have the form:

$$V(T, r) = D + Fr - \frac{G}{r} - Lr^2 + \frac{C}{r^2}. \quad (16)$$

Then Eq.(5) rewrite in this form:

$$\chi''(r) - \frac{2\mu}{\hbar^2} \left[E - \frac{\hbar^2 l(l+1)}{2\mu r^2} - D - Fr + \frac{G}{r} + Lr^2 - \frac{C}{r^2} \right] \chi(r) = 0. \quad (17)$$

Main aim to transform Eq.(17), the equation of the generalized hypergeometric-type which is in the form [21]:

$$\chi''(s) + \frac{\tilde{\tau}}{\sigma} \chi'(s) + \frac{\tilde{\sigma}}{\sigma^2} \chi(s) = 0, \quad (18)$$

In Eq.(17) changing $r = \frac{1}{x}$, then we obtain:

$$\chi''(x) - \frac{2x}{x^2} \chi'(x) + \frac{2\mu}{\hbar^2} \frac{1}{x^4} \left[E - \frac{\hbar^2}{2\mu} l(l+1)x^2 - D - \frac{F}{x} + Gx + \frac{L}{x^2} - Cx^2 \right] \chi(x) = 0. \quad (19)$$

For the solution Eq.(19) we will use approach, which in this approach is based on the expansion of $\frac{F}{x}$ and $\frac{L}{x^2}$ in a power series around the characteristic radius r_0 of meson up to the second order. In order to solve Eq.(19) for $l \neq 0$, we should make an approximation for the centrifugal term. For this we apply as Pekeris approximation [22], which it is helps to transform the centrifugal potential such that the modified equation can be solved by NU method. For this we introduce new variable $y = x - \delta$ where $\delta = 1/r_0$. After this we expand $\frac{F}{x}$ and $\frac{L}{x^2}$ into a series of powers around $y = 0$, then we obtain:

$$\begin{aligned} \frac{F}{x} &= \frac{F}{y + \delta} = \frac{F}{\delta} \left(1 - \frac{y}{\delta} + \frac{y^2}{\delta^2} \right) = \frac{F}{\delta} \left(1 - \frac{x - \delta}{\delta} + \frac{(x - \delta)^2}{\delta^2} \right) = \\ &= F \left(\frac{3}{\delta} - \frac{3x}{\delta^2} + \frac{x^2}{\delta^3} \right). \end{aligned} \quad (20)$$

Also similarly

$$\frac{L}{x^2} = \frac{L}{(y + \delta)^2} = \frac{L}{\delta^2} \left(1 + \frac{y}{\delta} \right)^{-2} = L \left(\frac{6}{\delta^2} - \frac{8x}{\delta^3} + \frac{3x^2}{\delta^4} \right). \quad (21)$$

Substituting Eqs.(20-21) into Eq.(19) then we obtain:

$$\begin{aligned} \chi''(x) - \frac{2x}{x^2} \chi'(x) + \frac{2\mu}{\hbar^2} \frac{1}{x^4} \left[\left(E - D - \frac{3F}{\delta} + \frac{6L}{\delta^2} \right) + \left(\frac{3F}{\delta^2} + G - \frac{8L}{\delta^3} \right) x + \right. \\ \left. + \left(-\frac{\hbar^2}{2\mu} l(l+1) - \frac{F}{\delta^3} - C + \frac{3L}{\delta^4} \right) x^2 \right] \chi(x) = 0. \end{aligned} \quad (22)$$

In the Eq.(22) we introduce new variable in order to make the differential equation more compact:

$$H = -\frac{2\mu}{\hbar^2}(E - D - \frac{3F}{\delta} + \frac{6L}{\delta^2}), \quad N = \frac{2\mu}{\hbar^2}(\frac{3F}{\delta^2} - \frac{8L}{\delta^3} + G), \quad (23)$$

$$Q = \frac{2\mu}{\hbar^2}(-\frac{\hbar^2}{2\mu}l(l+1) - \frac{F}{\delta^3} + \frac{3L}{\delta^4} - C). \quad (24)$$

Then Eq.(22) have the form:

$$\chi''(x) + \frac{2x}{x^2}\chi'(x) + \frac{1}{x^4}[-H + Nx + Qx^2]\chi(x) = 0. \quad (25)$$

Now, we can successfully apply NU method for defining eigenvalues of energy. By comparing Eq.(25) with Eq.(18) we can define the following:

$$\tilde{\tau}(x) = 2x, \quad \sigma(x) = x^2, \quad \tilde{\sigma}(x) = (-H + Nx + Qx^2). \quad (26)$$

If we take the following factorization

$$\chi(x) = \phi(x)y(x) \quad (27)$$

for the appropriate function $\phi(x)$, Eq.(25) takes the form of the well-known hypergeometric-type equation. The appropriate $\phi(x)$ function has to satisfy the following condition:

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)}, \quad (28)$$

where function $\pi(x)$ the polynomial of degree at most one is defined as

$$\begin{aligned} \pi(x) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma} &= \frac{2x - 2x}{2} \pm \sqrt{H - Nx - Qx^2 + kx^2} = \\ &\pm \sqrt{(k - Q)x^2 - Nx + H}. \end{aligned} \quad (29)$$

Finally, the equation, where $y(x)$ is one of its solutions, takes the form known as hypergeometric-type,

$$\sigma(x)y''(x) + \tau(x)y'(x) + \bar{\lambda}y(x) = 0, \quad (30)$$

where

$$\bar{\lambda} = k + \pi'(x). \quad (31)$$

and

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x). \quad (32)$$

The constant parameter k can be found by utilizing the condition that the expression under the square root has a double zero, i.e., its discriminant is equal to zero. Hence, we obtain

$$k = \frac{1}{4H}(N^2 + 4HQ). \quad (33)$$

Now substituting Eq.(33) into Eq.(29), then for $\pi(x)$ we obtain:

$$\pi(x) = \pm \frac{1}{2\sqrt{H}}(Nx - 2H). \quad (34)$$

According to NU method, from the two possible forms of the polynomial $\pi(x)$, we select the one for which the function $\tau(x)$ has the negative derivative. Another form is not suitable physically. Hence, the appropriate functions and $\pi(x)$ and $\tau(x)$ have the following forms:

$$\pi(x) = -\frac{1}{2\sqrt{H}}(Nx - 2H). \quad (35)$$

$$\pi'(x) = -\frac{N}{2\sqrt{H}}, \quad (36)$$

$$\tau(x) = 2x - \frac{Nx}{\sqrt{H}} + 2\sqrt{H}, \quad (37)$$

$$\tau'(x) = 2 - \frac{N}{\sqrt{H}}. \quad (38)$$

Also by Eq.(31), we can define the constant $\bar{\lambda}$ as

$$\bar{\lambda} = k + \pi'x = \frac{N^2}{4H} + Q - \frac{N}{2\sqrt{H}}. \quad (39)$$

Given a nonnegative integer n_r , the hypergeometric-type equation has a unique polynomial solution of degree n if and only if

$$\bar{\lambda} = \bar{\lambda}_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \quad (n = 0, 1, 2, \dots) \quad (40)$$

and $\bar{\lambda}_m \neq \bar{\lambda}_n$ for $m = 0, 1, 2, \dots, n-1$, then it follows that

$$\bar{\lambda}_{n_r} = -n_r(2 - \frac{N}{\sqrt{H}}) - n_r(n_r - 1) = -2n_r + \frac{N}{\sqrt{H}}n_r - n_r^2 + n_r = \frac{\sqrt{N}}{\sqrt{H}}n_r - n_r(n_r + 1) \quad (41)$$

$$\frac{N^2}{4H} + Q - \frac{N}{2\sqrt{H}} = \frac{N}{\sqrt{H}}n_r - n_r(n_r + 1). \quad (42)$$

We can solve Eq.(42) explicitly for H then we obtain:

$$\sqrt{H} = \frac{N}{(1+2n) \pm \sqrt{1-4Q}} \quad (43)$$

Substitute Eq.(43) into Eq.(23), then we obtain:

$$\sqrt{-\frac{2\mu}{\hbar^2}(E-D-\frac{3F}{\delta}+\frac{6L}{\delta^2})} = \frac{N}{(1+2n) \pm \sqrt{1-4Q}} \quad (44)$$

From Eq.(44) very easy we found energy spectrum in this form:

$$E = D + \frac{3F}{\delta} - \frac{6L}{\delta^2} - \frac{\hbar^2}{2\mu} \left[\frac{\frac{2\mu}{\hbar^2}(\frac{3F}{\delta^2} - \frac{8L}{\delta^3} + G)}{(1+2n_r) \pm \sqrt{1+4l(l+1)+8\mu\frac{F}{\delta^3}-24\mu\frac{L}{\delta^4}+8\mu C}} \right]^2. \quad (45)$$

If in Eq.(45) we take $T = 0$ then we obtain:

$$E = \frac{3A}{\delta} - \left[\frac{2\mu(\frac{3A}{\delta^2} + B)^2}{(1+2n) \pm \sqrt{1+4l(l+1)+8\mu\frac{A}{\delta^3}+8\mu C}} \right]. \quad (46)$$

If we take $T = 0$ and $C = 0$ in Eq.(45), then we obtain result [5]:

$$E = \frac{3A}{\delta} - \left[\frac{2\mu(\frac{3A}{\delta^2} + B)^2}{(1+2n) \pm \sqrt{1+4l(l+1)+8\mu\frac{A}{\delta^3}}} \right]. \quad (47)$$

Now, applying the Nikiforov-Uvarov method we can obtain the radial eigenfunctions. The appropriate $\pi(s)$ function must satisfy the following condition

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)} = \frac{\pm(\frac{Nx}{2\sqrt{H}} - \sqrt{H})}{x^2} = \pm(\frac{N}{2\sqrt{H}x} - \frac{\sqrt{H}}{x^2}), \quad (48)$$

Having substituted $\pi(x)$ and $\sigma(x)$ into Eq.(48) and solving first-order differential equation, it is easy to obtain

$$\phi(x) = x^{\frac{N}{2\sqrt{H}}} e^{-\frac{\sqrt{H}}{x}}, \quad (49)$$

Furthermore, the other part of the wave function $y_n(x)$ is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(x) = \frac{C_n}{\rho(x)} \frac{d^n}{dx^n} [\sigma^n(x)\rho(x)], \quad (50)$$

where C_n is a normalizing constant and $\rho(x)$ is the weight function which is the solution of the Pearson differential equation. The Pearson differential equation and $\rho(x)$ for our problem is given as

$$(\sigma\rho)' = \tau\rho. \quad (51)$$

It is easy to find the second part of the wave function from the definition of weight function

$$\rho(x) = x^{-N/\sqrt{H}} e^{-2\sqrt{H}/x}. \quad (52)$$

Substituting $\phi(x)$ and $y_{n_r}(x)$ into Eq.(27), we obtain

$$\chi_{n_r}(x) = C_{n_r l} x^{\frac{N}{2\sqrt{H}}} e^{\frac{\sqrt{H}}{x}} \frac{d^n}{dx^n} \left[x^{2n - \frac{N}{\sqrt{H}}} e^{-2\sqrt{H}/x} \right]. \quad (53)$$

In Eq.(53) changing $x = 1/r$ and using that $\chi(r) = rR(r)$, then we obtain:

$$\chi_{n_r}(r) = C_{n_r l} r^{-\frac{N}{2\sqrt{H}}} e^{\sqrt{H} \cdot r} \left(-r^2 \frac{d}{dr} \right)^n \left[r^{-2n + \frac{N}{\sqrt{H}}} e^{-2\sqrt{H} \cdot r} \right]. \quad (54)$$

Finally for radial wave function $R(r)$ we find:

$$R(r) = C_{n_r l} r^{-1 - \frac{N}{2\sqrt{H}}} e^{\sqrt{H} \cdot r} \left(-r^2 \frac{d}{dr} \right)^n \left[r^{-2n + \frac{N}{\sqrt{H}}} e^{-2\sqrt{H} \cdot r} \right]. \quad (55)$$

We calculate mass spectra of the heavy quarkonium system, for example charmonium and bottomonium mesons at finite temperature that quark and antiquark have flavor. For this we apply the following relation:

$$M = 2m + E_{nl}. \quad (56)$$

here m is bare mass of quarkonium.

So, at finite temperature $T \neq 0$ for mass of quarkonium system M we obtain:

$$M = 2m + D + \frac{3F}{\delta} - \frac{6L}{\delta^2} - \frac{\hbar^2}{2\mu} \left[\frac{\frac{2\mu}{\hbar^2} \left(\frac{3F}{\delta^2} - \frac{8L}{\delta^3} + G \right)}{(1+2n) \pm \sqrt{1 + 4l(l+1) + 8\mu \frac{F}{\delta^3} - 24\mu \frac{L}{\delta^4} + 8\mu C}} \right]^2. \quad (57)$$

The Eq.(57) represents the quarkonium mass at finite temperature. Here m is quarkonium bare mass for the charmonium or bottomonium mesons. By replacing $T = 0$ then we obtain quarkonium mass at zero temperature:

$$M = 2m + \frac{3A}{\delta} - \left[\frac{2\mu \left(\frac{3A}{\delta^2} + B \right)^2}{(1+2n) \pm \sqrt{1 + 4l(l+1) + 8\mu \frac{A}{\delta^3} + 8\mu C}} \right]. \quad (58)$$

By replacing $T = 0$ and $C = 0$ then we obtain quarkonium mass at zero temperature [5].

$$M = 2m + \frac{3A}{\delta} - \left[\frac{2\mu \left(\frac{3A}{\delta^2} + B \right)^2}{(1+2n) \pm \sqrt{1 + 4l(l+1) + 8\mu \frac{A}{\delta^3}}} \right]. \quad (59)$$

Should be noted that result at $T = 0$ and $C = 0$ full coincides with [5], in which the authors obtained the quarkonium energy and mass at zero temperature.

At the numerical calculations free parameters A , B , C and δ fitted with experimental data with using the Eq.(58).

The present results are in good agreement with available experimental data for all states of charmonium and bottomonium mesons. For the calculation the mass spectra of charmonium and bottomonium at finite temperature, we use the explicit form of $\mu_D(T)$ according to [23]:

$$\mu_D(T) = \gamma\alpha_s(T)T. \quad (60)$$

where $\gamma = 4\pi \cdot \eta c_\sigma$. At the numerical calculations we use $c_\sigma = 0.566 \pm 0.013$, $\eta = 2.06$ and $\gamma = 14.652 \pm 0.337$. In the numerical calculations for running coupling constant $\alpha_s(T)$ we will adopt following form at finite temperature:

$$\alpha_s(T) = \frac{2\pi}{(11 - \frac{2}{3}n_f)\ln(\frac{T}{\Lambda})}. \quad (61)$$

Here from lattice QCD we take for $\Lambda = \beta T_c$, and $\beta = 0.104 \pm 0.09$ [24].

In the numerical calculations we will apply as $N_f = 3$ with two light quarks of the same mass u and d and one heavier s and critical temperature $T_c = (169 \pm 16)MeV$, then for $\Lambda = \beta T_c = (17.6 \pm 3.2)MeV$ [25].

In Table 1 and 2 present mass spectra of charmonium and bottomonium for the sum of Cornell and inverse quadratic potential are given in comparison with experimental data and other theoretical calculations at $T = 0$, respectively.

3. Numerical Results and Discussion

Solutions of the modified radial Schrödinger equation for the sum of Cornell and inverse quadratic potential for $l \neq 0$ are obtained respectively within quantum mechanics by applying the Nikiforov-Uvarov method.

In Table 1 and 2 present mass spectra of charmonium and bottomonium for the sum of Cornell and inverse quadratic potential are given in comparison with experimental data and other theoretical calculations at the finite temperature $T \neq 0$, respectively.

In Fig.1 the dependence of the mass spectrum $1s$, $2s$, $3s$, $4s$ and $5s$ state of charmonium on the temperature T are plotted. These plots indicate that in the region $0 \text{ GeV}/c < T < 0.046 \text{ GeV}/c$, mass spectrum of the charmonium increases systemically with an increasing the temperature and has a maximum at point $T = 0.046 \text{ GeV}$, which it is correspond to $T/T_c = 0.272$, but in the region $0.046 \text{ GeV}/c < p_T < 0.156 \text{ GeV}/c$ mass spectrum of charmonium decrease with increasing the temperature.

In Fig.2 the dependence of the mass spectrum $1s$, $2s$, $3s$, $4s$ and $5s$ state of bottomonium on the temperature T are plotted. These plots indicate that in the region $0 \text{ GeV}/c < T < 0.046 \text{ GeV}/c$, mass spectrum of the charmonium increases systemically with increasing the temperature has a maximum in point at $T = 0.046 \text{ GeV}$, but in the region $0.046 \text{ GeV}/c < T < 0.156 \text{ GeV}/c$ mass spectrum of charmonium monotonically decrease with an increasing the temperature. The analysis of our calculations shows that the main reason for this depends on the phenomenological factors

These plots indicate that mass spectrum of charmonium and bottomonium is very sensitive to the choice of the quantum numbers and potential parameters.

Our detailed analysis show that this result opens a new possibilities for determining of the properties of the interactions in hadronic system. As a conclusion of the results presented in these tables and figures the numerical analyses obtained of the analytically solution is very

sensitive to the n_r radial and l orbital quantum numbers. The results are sufficiently accurate for practical purpose.

4. Conclusion

In this research we present analytical solution of the modified radial Schrödinger equation for the sum of Cornell and inverse quadratic potential are obtained within quantum mechanics by applying the Nikiforov-Uvarov method. The energy eigenvalues, mass spectrum and corresponding eigenfunctions are obtained for arbitrary l angular momentum quantum numbers.

The spectral problem of the Schrödinger equation with spherically symmetric potentials is important in the spectroscopy of complex chemical compounds and molecules. It is also important in describing the spectra of hadron resonances, specially mesons-quarkoniums systems. It is known, that despite the absence of a rigorous theoretical approach, potential models give a satisfactory description of the mass spectra for such systems as quarkonium. The interactions in such systems are usually represented by confining-type potentials. As an example, one can take the sum of Cornell and inverse quadratic potential consisting of three terms. One of the terms is responsible for the Coulomb interaction of quarks, second to the string interaction, which provides confinement and third terms corresponds inverse quadratic potential.

It is worth to mention that the extended Cornell potential is one of the important potential, and it is a subject of interest in many fields of physics and chemistry. The main results of this paper are the explicit and closed form expressions for the energy eigenvalues, the normalized wave functions and mass spectrum of the bottomonium and charmonium. The method presented in this paper is a systematic one and in many cases it is one of the most concrete works in this area.

The energy eigenvalues and eigenfunctions were obtained analytically for any l value at finite temperature $T \neq 0$ using Nikiforov-Uvarov method. Using the available experimental data, the obtained energy formula were used fit charmonium and bottomonium mass spectra from which then the potential parameters were determined. These parameters were then used to reproduce the mass spectra, which then were compared with the experimental results. The predictions from our model are found to be in good agreement with the experimental results. As a side result, the Hydrogen atom known spectrum is recovered.

It should be noticed that the analysis of our results and figures shows the bound state of the system more stable in the case $T \neq 0$. Hence, the temperature effects give more information about quantum systems instead of $T = 0$.

The numerical results obtained by using MATHEMATICA package programm.

Consequently, studying of analytical solution of the modified radial Schrödinger equation for the sum of Cornell and inverse quadratic potential take into account temperature effects could provide valuable information on the quantum mechanical dynamics at hadronic physics, atomic and molecule physics and also opens new window for further investigation.

We can conclude that our analytical results of this study are expected to enable new possibilities for pure theoretical and experimental physicist, because the results are exact and more general.

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<i>state</i>	Present work	[5]	[26]	[27]	[28]	Experiment [29]
1s	3.0969	3.096	3.068	3.078	3.096	3.096
2s	3.68697	3.686	3.697	3.581	3.686	3.686
3s	4.04143	4.040	4.144	4.085	3.984	4.040
4s	4.27086	4.269		4.589	4.150	4.263
5s	4.42783	4.425				4.421
1p	3.25581	3.255	3.526	3.415	3.433	
2p	3.77951	3.779	3.993	3.917	3.910	3.773
3p	4.09997					
4p	4.31021					
5p	4.45555					
1d	3.50471	3.504	3.829	3.749	3.767	

Table 1. Mass spectra of charmonium in GeV, $m_c = 1.209$, $A = 0.2$, $B = 1.244$, $C = 0.0029$, $\delta = 0.231$, $T = 0$.

<i>state</i>	Present work	[5]	[26]	[27]	[28]	Experiment [29]
1s	9.45851	9.460	9.447	9.510	9.460	9.460
2s	10.0218	10.023	10.012	10.038	10.023	10.023
3s	10.3539	10.355	10.353	10.566	10.280	10.355
4s	10.5661	10.567	10.629	11.094	10.420	10.580
5s	10.7098				6578	
1p	9.61781	9.619	9.900	9.862	9.840	
2p	10.1127	10.114	10.260	10.390	10.160	
3p	10.4106					
4p	10.6038					
5p	10.7362					
1d	10.2567	9.864	10.155	10.214	10.140	

Table 2. Mass spectra of bottomonium in GeV, $m_b = 4.823\text{GeV}$, $A = 0.2\text{GeV}^2$, $B = 1.569$, $\delta = 0.378\text{GeV}$, $C = 0.002$, $T = 0$.

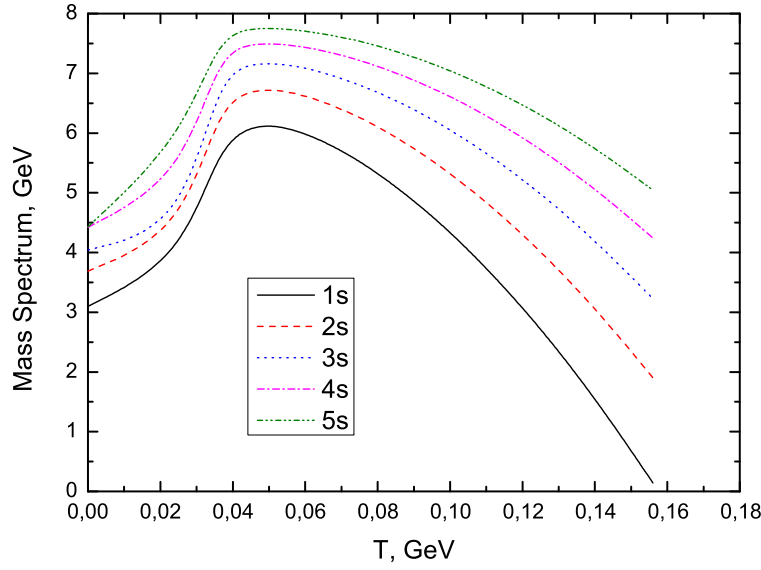


Figure 1. The mass spectrum of charmonium of $1s$, $2s$, $3s$, $4s$ and $5s$ states as a function of the temperature T at the $m_b = 1.209$, $A = 0.2\text{GeV}^2$, $B = 1.569\text{GeV}$, $C = 0.002$, $T_c = 0.169\text{GeV}$ and $\delta = 0.378$.

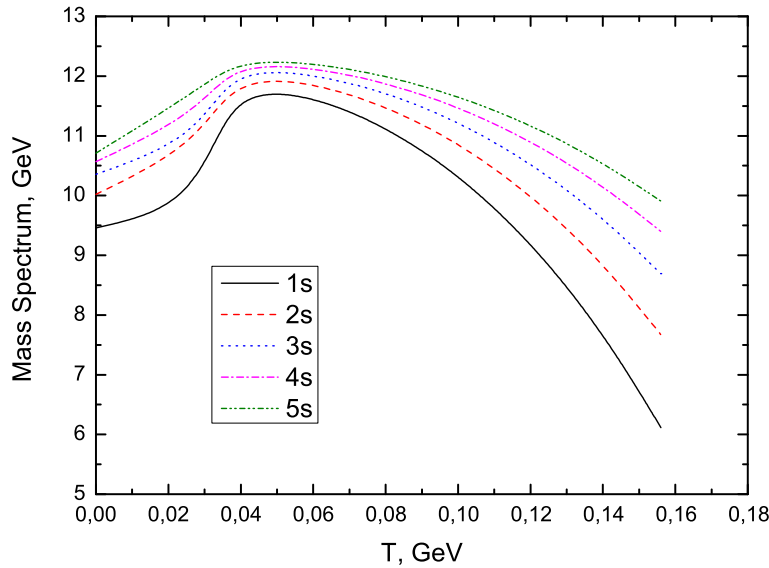


Figure 2. The mass spectrum of bottomonium of $1s$, $2s$, $3s$, $4s$ and $5s$ states as a function of the temperature T at the $m_b = 4.823$, $A = 0.2\text{GeV}^2$, $B = 1.244\text{GeV}$, $C = 0.0029$, $T_c = 0.169\text{GeV}$ and $\delta = 0.231$.