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Six Dimensional Spacetime with extra time dimensions: Casimir Effect, 6D Cosmology, and Experiments

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Abstract: I describe methods of treating six dimensional spacetimes with extra time dimensions, specifically the 6D space $\mathcal{M}^{3,3} \approx \mathbb{R}^{3,3}$ with three time dimensions. Compactification of such space along timelike dimensions will be discussed, and criticisms of extra timelike dimensions will be given. Much of mathematical fields, and effective string theories can result in multiple time dimensions with great applications. Problems dealing with ghosts and tachyons will be addressed.
I calculate Casimir forces in this spacetime will be revealed, and noting that infrared cutoffs are done for timelike compact direction as familiar ultraviolet cutoffs are done in regularizing the quantum spatial vacuum.This represents a T-duality for each torus cycle of UV becomes IR, as spacelike cycles become timelike, and vice-versa. Furthermore, applications to 6D cosmological and the cosmological constant problem will be shown to arrive at solutions for unresolved physics problems. Other applications of such split-time space, its algebraic representations, and applications in other fields of physics and beyond.

Contents

1 Introduction and outline

1.1 Introduction

String theory and M-theory effective theories have been studied down to simpler, and quite interesting and hopeful results to understanding the universal landscape of physics and the universe, in a simpler, lower dimensional effective field theory []. Furthermore, many dualities arise in quantum field theory, particle theory, and topological string theory [], [], as well as facilitating complex calculations in any of the above fields, knowing a simpler or even solved problem is dual to a very complicated and not understood model. Mathematicians also are adamant that the universe is spinorial, 6D with three time dimensions []. Duality in physics is more than symmetry 'a friend' - it is much more intimate and powerful and even incomprehensible! We will use use some dualities here, especially when it comes to T-duality of compactified field theories from 6D:

$$\pi : \mathcal{M}^{3,3} \to \mathcal{M}^{3,1},$$

where the 2-torus degrees of freedom are either projected out, integrated out, and heavy Kaluza-Klein modes removed for low-energy effective theory, and restricting gauge fields to the Cartan subalgebra of the gauge group, etc. Further simplifications can be made at low T as well, and perturbative expansions done, and compared to effective realistic QED particle theory.

The research we present here involves a few different cases, which we find are all dual to six dimensional compactifications to Minkowski space $\mathcal{M}^{3,1}$. This not only yields effective QED, as we will show, but has several applications in finance, sociophysics, biology, astrophysics and cosmological applications. Several unsolved problems can become more tractable as well, with help resolving the cosmological constant problem, for one [], and has a landscape that is more easily navigated, and with a plethora of dualities.

1.2 Motivation for 6D and extra time dimensions.

-Effective string theory interest [5].

-Mathematician's presentation and insistence on a signature $(3, 3)$ as a three time universe [3].

-Usefulness and current studies of 6D QFTs and topological field or string theories in understanding 3 and 4 dimensional field theories in general, and not to mention helpful dualities! [7].

-Dimensional reduction and compactification to 4D, yielding effective QED and QCD.

-Studies from 6D allow for methods of detecting extra dimensions, and evidence of time-like dimensions.

-Applications beyond physics, possibilities of resolving unsolved physics problems, plus to economics, sociodynamics, biology, and even universal psychology!

-Further evidence from current experiments on time crystals [], and the existence of 3D quasicrystals, being projections from regular 6D lattices (and generation of fractals, and algorithmic botany, it goes on....!

The usefulness of having a compact dimension to analytically study gauge theories in a theoretically-controlled manner at weak coupling for any gauge group has successfully been done recently for super Yang-Mills on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ with a mass for the gaugino. This is called deformed SYM (SYM*) studied in [4] and the deconfinement phase transition has been found at a critical gaugino mass (depending on the theta angle of the theory) to be first order for all gauge groups other than $SU(2)$ where it is second order [1]. It is conjectured that this zero temperature quantum phase transition is continuously related, as a function of gaugino mass m, to the thermal deconfinement transition of pure Yang-Mills as $m \to \infty$ at some critical deconfinement temperature T_c. Much supports this conjecture, including lattice studies where qualitative agreement to the zero temperature phase transition has been shown such as the order of the transition, its universality class of centre-symmetry breaking, and dependence on theta-angle [17].

This work considers a finite temperature study of $\mathcal{N} = 1$ super Yang-Mills (SYM) by having an additional compact direction along the time direction of size the inverse temperature of the theory $\beta = 1/T$. Having finite temperature breaks the supersymmetry of the theory if we take the (adjoint) fermions in the theory, the gluinos, to have anti-periodic boundary conditions along the thermal circle, and periodic ones for the gluons. This gives us then a full playground in which to study semi-classically a theory in lesser dimensions at finite temperature and with supersymmetry breaking, and eases the study of the deconfinement phase transition.

1.3 A motivational picture of 6D reality

One way to picture 6D dynamics from our our 4D perspective can be viewed as follows.

1.4 Outline and summary

The purpose of this paper is to motivate the pragmatism of 6D theories, especially with extra time dimensions, and find their effectiveness in providing effective 4D field theories from compactification of the extra dimensions. This includes modifications that can resolve several unresolved problems in field theory, particle theory and cosmology. Motivation for a 6D universe has been given as better more effective theories from string theory, with several applications to topological field theory as well. Mathematicians, such as Sparling, also insist on three time dimensions in a spinorial universe []. Experimental approaches to the detection of such extra dimensions will be mentioned briefly.

The paper proceeds as follows: in Section 2 I review the methods of evaluating the Casimir effect in spacetimes with extra dimensions, first at zero temperature, then at finite temperature. Dealing with extra timelike dimensions will be discussed in Section 4, as well as how resolving the cosmological constant problem can result. Casimir force calculations with extra time dimensions will be continued there.

Section 3 is devoted to a review of cases where the Casimir effect has moving boundary conditions in the compact directions, This is the so-called dynamical Casimir effect (DCE) and results in a low energy effective QED with photon modes coupling to oscillation modes of the boundaries, Experimental verification of the model in extra dimensions, spacelike or timelike, can result from these effective compactified theories. Resolutions also of criticisms of extra time dimensions (tachyons, unitarity, causality, and ghosts) will be explained.

Section 4 contains details of how to achieve effective field theories from compactifications with extra time dimensions. Casimir effect calculations in this case will be performed, and low energy effective 4D theories obtained will be compared to reality and methods of detecting differences from extra spatial dimensions will be noted, as well as how to detect them. Motivation will also be given with a discussion on how extra timelike dimensions lead to a positive semi-definite action required for the effective theory's validity.

Section 5 presents an electroweak unification that results from compactification from 6D to 4D, with extra spacelike or timelike dimensions. Further, an effective realistic QED is shown next to result from a vortex geometry in 6D, with gravity, and leading to realistic fermionic and bosonic states (W bosons etc.) and a resulting gauge-gravity unification, with charged fermions interactiong with photons and gravitons.

Section 6 shows how chirality issues can be resolved through 6D compectifications, as well as a supersymmetric enhancement from $\mathcal{N} = 1$ SUSY in 6D producing $\mathcal{N} = 2$ in the effective 4D theory. Later, I discuss how non-Abelian anomalies in 4D can be fixed to Abelian ones from the 6D framework.

Section 7 gives 6D cosmological models with extra space or time dimensions, and how the dynamics of the 6D metric can resolve questions about the past, present, and future of our Universe, including dark energy and dark matter, and their alternatives.

Section 8 reviews the role of non-perturbative objects, such as vortices or monopoles in 6D to 4D dynamics, particle theory, and field theories. Further investigations into higher rank gauge groups will be investigated, so as to achieve results of effective QCD-like theories, and their modifications in 4D.

Section 9 discusses possible past, present and future experimental methods for discovering extra dimensions, with emphasis on extra timelike dimensions, through measuring the Casimir effect and the dynamical Casimir effect. Proposals for future experiments will be detailed.

Section 10 summarizes the paper with important conclusions, and future work, still unresolved details, will be listed with suggestions.

The Appendix begins with part A reviewing Lie groups and Lie algebras in general, sets up the notation and definitions and explains the necessary concepts. Part B contains a complete derivation of the finite-temperature perturbative effective potential on $\mathcal{M}^{3,3}$ with more specific details on their zeta function regularizations at zero temperature and finite temperatures. Part C reviews monopole solutions (BPS and KK) for all gauge groups, including vortices that arise in some 6D compactifications.

2 $\mathcal{M}^{3,3}$, 6D Lorentz transformations, and mathematical properties to the Casimir effect.

In this section I examine

2.1 6D Lorentz transformations in $\mathcal{M}^{3,3}$

Before going further, let us review the basics of transformations in $\mathbb{R}^{3,3}$ preserving the 6-spacetime metric, that is Lorentz transformations of $SO(3,3)$, after a review of the basics.

2.2 $SO(3)$

$SO(3)$ is the group of rotations in $3d$ space and is a subgroup of the Lorentz group $SO(1,3)$. It is defined as the group of transformations which preserves the norm of vectors, or analogously which preserves the metric δ_{ij} (in $\mathbb{R}^3$ the metric is the identity matrix whose components are given by the Kronecker delta)

$$\begin{aligned} &R \in SO(3)\,, \forall v \in \mathbb{R}^3 \implies |Rv|^2 = |v|^2 \implies R_i^m R_j^n v^i v^j \delta_{mn} = v^i v^j \delta_{ij} \\ &\implies R_i^m R_j^n \delta_{mn} = \delta_{ij} \implies R^T \mathbb{I} R = \mathbb{I}\,, \end{aligned} \tag{2.1}$$

where we used Einstein notation (repeated indices imply a sum, e.g. $\delta_{ij}v^j = \sum_j \delta_{ij}v^j$).

In order to obtain the generators of the group we consider an infinitesimal transformation

$$R^T R = \mathbb{I} \implies (\mathbb{I} + \delta R^T)(\mathbb{I} + \delta R) = \mathbb{I} \implies \delta R + \delta R^T = 0\,, \tag{2.2}$$

which gives

$$\delta R = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}. \tag{2.3}$$

Consequently the generators of rotations turn out to be

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{2.4}$$

and the Lie algebra is given by

$$[X_i, X_j] = -\epsilon_{ijk} X_k\,. \tag{2.5}$$

To obtain the expression for a finite transformation from the infinitesimal generators we use the exponential map

$$R(\theta^1,\theta^2,\theta^3) = e^{\theta^i X_i} = \sum_n \frac{1}{n!}(\theta^i X_i)^n \,, \tag{2.6}$$

where $(\theta^1,\theta^2,\theta^3)$ are the angles of rotation around the three axes. If for simplicity we set $\theta^2=0$, $\theta^3=0$ and only $\theta^1\neq 0$, we obtain

$$R(\theta^1) = \sum_n \frac{1}{n!}(\theta^1 X_1)^n = \nVdash + \theta^1 X_1 + \frac{1}{2}(\theta^1 X_1)^2 + \frac{1}{6}(\theta^1 X_1)^3 + \frac{1}{24}(\theta^1 X_1)^4 + \frac{1}{120}(\theta^1 X_1)^5 + \dots \,. \tag{2.7}$$

Observe that $(X_1)^2 = -\mathbb{I}_1$, where

$$\mathbb{I}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \tag{2.8}$$

which allows us to rewrite Eq. (2.7) as

$$R(\theta^1) = \mathbb{I} + \mathbb{I}_1\left(-\frac{1}{2}(\theta^1)^2 + \frac{1}{24}(\theta^1)^4 + \dots\,\right) + X_1\left(\theta^1 - \frac{1}{6}(\theta^1)^3 + \frac{1}{120}(\theta^1)^5 + \dots\right) \,. \tag{2.9}$$

Observing moreover that

$$\begin{aligned} \cos(\theta^1) &= 1 - \frac{1}{2}(\theta^1)^2 + \frac{1}{24}(\theta^1)^4 + \dots\,, \\ \sin(\theta^1) &= \theta^1 - \frac{1}{6}(\theta^1)^3 + \frac{1}{120}(\theta^1)^5 + \dots\,, \end{aligned} \tag{2.10}$$

we obtain finally

$$R(\theta^1) = \begin{pmatrix} \cos(\theta^1) & \sin(\theta^1) & 0 \\ -\sin(\theta^1) & \cos(\theta^1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{2.11}$$

At the same way, in the general case $\theta^1\neq 0$, $\theta^2\neq 0$, $\theta^3\neq 0$, we have

$$R(\theta^1,\theta^2,\theta^3) = e^{\theta^i X_i} = \mathbb{I} + \sum_n \frac{1}{(2n+2)!}(\theta^i X_i)^{2n+2} + \sum_n \frac{1}{(2n+1)!}(\theta^i X_i)^{2n+1} \,. \tag{2.12}$$

Observing that

$$(\theta^i X_i)^3 = -\theta^2(\theta^i X_i) \,, \tag{2.13}$$

we can simplify the above summations as

$$\begin{aligned} \sum_n \frac{1}{(2n+1)!}(\theta^i X_i)^{2n+1} &= (\hat{\theta}^i X_i)\sum_n \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} \,, \\ \sum_n \frac{1}{(2n+2)!}(\theta^i X_i)^{2n+2} &= -(\hat{\theta}^i X_i)^2\sum_n \frac{(-1)^n}{(2n+2)!}\theta^{2n+2} \,, \end{aligned} \tag{2.14}$$

where $\hat{\theta}$ is the versor of the vector $\vec{\theta}$ and θ is its norm ($\vec{\theta} = \{\theta^1, \theta^2, \theta^3\}$, $\hat{\theta} = \vec{\theta}/|\vec{\theta}|$, $\theta = |\vec{\theta}|$) .

The general transformation $R(\vec{\theta})$ represents a rotation of an angle θ around the axis $\hat{\theta}$ and can be rewritten as

$$R(\vec{\theta}) = \mathbb{I} + (\hat{\theta}^i X_i)^2(1-\cos\theta) + (\hat{\theta}^i X_i)\sin\theta\,, \tag{2.15}$$

or explicitly in matrix notation

$$R = \begin{pmatrix} \cos\theta + \hat{\theta}_1^2(1-\cos\theta) & \hat{\theta}_3\sin\theta + \hat{\theta}_1\hat{\theta}_2(1-\cos\theta) & -\hat{\theta}_2\sin\theta + \hat{\theta}_1\hat{\theta}_3(1-\cos\theta) \\ -\hat{\theta}_3\sin\theta + \hat{\theta}_2\hat{\theta}_1(1-\cos\theta) & \cos\theta + \hat{\theta}_2^2(1-\cos\theta) & \hat{\theta}_1\sin\theta + \hat{\theta}_2\hat{\theta}_3(1-\cos\theta) \\ \hat{\theta}_2\sin\theta + \hat{\theta}_3\hat{\theta}_1(1-\cos\theta) & -\hat{\theta}_1\sin\theta + \hat{\theta}_3\hat{\theta}_2(1-\cos\theta) & \cos\theta + \hat{\theta}_3^2(1-\cos\theta) \end{pmatrix}. \tag{2.16}$$

2.3 $SO(3,3)$

$SO(3,3)$ is the special orthogonal group in $6d$ space $\mathbb{R}^6$ with metric η

$$\eta = \mathrm{diag}(-1,-1,-1,1,1,1)\,. \tag{2.17}$$

It is defined as the group of transformations which preserves the metric

$$\Lambda \in SO(3,3) \implies \Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma} = \eta_{\mu\nu} \implies \Lambda^T\eta\Lambda = \eta\,. \tag{2.18}$$

In order to obtain the generators of the group, we consider an infinitesimal transformation

$$\Lambda^\rho_\mu = \delta^\rho_\mu + \delta\Lambda^\rho_\mu\,, \tag{2.19}$$

where δ^ρ_μ is the Kronecker delta and $\delta\Lambda^\rho_\mu$ represents an infinitesimal perturbation. Substituting this expression into Eq. (2.18) gives

$$(\delta^\rho_\mu + \delta\Lambda^\rho_\mu)(\delta^\sigma_\nu + \delta\Lambda^\sigma_\nu)\eta_{\rho\sigma} = \eta_{\mu\nu} \implies \delta\Lambda^\rho_\mu\eta_{\rho\nu} + \delta\Lambda^\sigma_\nu\eta_{\sigma\mu} = 0\,. \tag{2.20}$$

The solution of this equation, in matrix form, is

$$\delta\Lambda = \begin{pmatrix} 0 & r_t^1 & r_t^2 & b_1^1 & b_1^2 & b_1^3 \\ -r_t^1 & 0 & r_t^3 & b_2^1 & b_2^2 & b_2^3 \\ -r_t^2 & -r_t^3 & 0 & b_3^1 & b_3^2 & b_3^3 \\ b_1^1 & b_2^1 & b_3^1 & 0 & r_s^1 & r_s^2 \\ b_1^2 & b_2^2 & b_3^2 & -r_s^1 & 0 & r_s^3 \\ b_1^3 & b_2^3 & b_3^3 & -r_s^2 & -r_s^3 & 0 \end{pmatrix}, \tag{2.21}$$

where we used the notation r_t^i to indicate the infinitesimal rotations in the time subspace, r_s^j to indicate the rotations in the spatial subspace, and b_m^n to indicate a boost concerning the temporal axis m and the spatial axis n. Following this classification, the generators of $SO(3,3)$ are

- Generators of rotations in time subspace

$$J_{RT}^{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_{RT}^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_{RT}^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.22}$$

- Generators of rotations in spatial subspace

$$J_{RS}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_{RS}^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, J_{RS}^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.23}$$

- Generators of boosts along the first time coordinate

$$J_{B}^{11} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_{B}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_{B}^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.24}$$

- Generators of boosts along the second time coordinate

$$J_{B}^{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_{B}^{22} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_{B}^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.25}$$

- Generators of boosts along the third time coordinate

$$J_{B}^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_{B}^{32} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J_{B}^{33} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.26}$$

The finite transformations along each generator are easily obtained using the exponential map. For example, the finite rotation along the first generator J_{RT}^{12} of the temporal rotations, which represents a rotation around the third temporal axis, is given by

$$R_T^{12}(\theta) = \sum_n \frac{1}{n!}(\theta J_{RT}^{12})^n = \mathbb{1} + \theta J_{RT}^{12} + \frac{1}{2}(\theta J_{RT}^{12})^2 + \frac{1}{6}(\theta J_{RT}^{12})^3 + \frac{1}{24}(\theta J_{RT}^{12})^4 + \frac{1}{120}(\theta J_{RT}^{12})^5 + \dots . \tag{2.27}$$

Observing that $(J_{RT}^{12})^2 = -\mathbb{1}_T^{12}$, where $\mathbb{1}_T^{12} = \mathrm{diag}(1,1,0,0,0,0)$, we can rewrite Eq. (2.27) as

$$R_T^{12}(\theta) = \mathbb{1} + \mathbb{1}_T^{12}\left(-\frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \dots\right) + J_{RT}^{12}\left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \dots\right). \tag{2.28}$$

Observing moreover that

$$\begin{aligned} \cos(\theta) &= 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \dots, \\ \sin(\theta) &= \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \dots, \end{aligned} \tag{2.29}$$

we obtain finally

$$R_T^{12}(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.30}$$

Similarly the finite boost along the generator J_B^{11} (concerning the first time coordinate and the first spatial coordinate) is given by

$$B^{11}(\xi) = \sum_n \frac{1}{n!}(\xi J_B^{11})^n = \mathbb{1} + \xi J_B^{11} + \frac{1}{2}(\xi J_B^{11})^2 + \frac{1}{6}(\xi J_B^{11})^3 + \frac{1}{24}(\xi J_B^{11})^4 + \frac{1}{120}(\xi J_B^{11})^5 + \dots, \tag{2.31}$$

where ξ is the rapidity (hyperbolic angle). Observing that $(J_B^{11})^2 = \mathbb{1}_B^{11}$, where $\mathbb{1}_B^{11} = \mathrm{diag}(1,0,0,1,0,0)$, we can rewrite Eq. (2.31) as

$$B^{11}(\xi) = \mathbb{1} + \mathbb{1}_B^{11}\left(+\frac{1}{2}\xi^2 + \frac{1}{24}\xi^4 + \dots\right) + J_B^{11}\left(\xi + \frac{1}{6}\xi^3 + \frac{1}{120}\xi^5 + \dots\right). \tag{2.32}$$

Observing moreover that

$$\begin{aligned} \cosh(\xi) &= 1 + \frac{1}{2}\xi^2 + \frac{1}{24}\xi^4 + \dots, \\ \sinh(\xi) &= \xi + \frac{1}{6}\xi^3 + \frac{1}{120}\xi^5 + \dots, \end{aligned} \tag{2.33}$$

we obtain finally

$$B^{11}(\xi) = \begin{pmatrix} \cosh(\xi) & 0 & 0 & \sinh(\xi) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sinh(\xi) & 0 & 0 & \cosh(\xi) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.34}$$

In order to show the relation between rapidity and velocity, assume to have two reference frames. We can consider the first to be fixed while the second is moving along the first time coordinate t_1 (but not the other temporal coordinates) and along the first spatial coordinate x_1 respect to the first frame. Then, we can restrict the 6-dimensional spacetime to the 2-dimensional subspace (t_1, x_1) and the relation between the two frames is given by (in Planck units $c = G = \hbar = 1$)

$$\begin{pmatrix} t'_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} \cosh(\xi) & \sinh(\xi) \\ \sinh(\xi) & \cosh(\xi) \end{pmatrix} \begin{pmatrix} t_1 \\ x_1 \end{pmatrix} . \tag{2.35}$$

The velocity v' in the second frame is obtained differentiating the coordinates

$$v' = \frac{dx'}{dt'} = \frac{\partial_t x' dt + \partial_x x' dx}{\partial_t t' dt + \partial_x t' dx} = \frac{(\sinh(\xi) + \cosh(\xi)v)dt + (\cosh(\xi) + \sinh(\xi)v^{-1})dx}{(\cosh(\xi) + \sinh(\xi)v)dt + (\sinh(\xi) + \cosh(\xi)v^{-1})dx} , \tag{2.36}$$

where $v = dx/dt$ is the velocity measured in the first frame. By setting $v = 0$ (which implies $dx = 0$), Eq. (2.36) reduces to

$$v' = \frac{\sinh(\xi)}{\cosh(\xi)} = \tanh(\xi) . \tag{2.37}$$

Observe that $v = 0$ measures the velocity of the first reference frame respect to itself. Consequently, v' is the velocity of the first frame measured from the second frame, i.e. it represents the relative velocity between the two frame. Hence, Eq. (2.37) gives the relation between the relative velocity v' between the reference frames and the rapidity ξ.

For completeness, we list the finite transformations along all generators of $SO(3,3)$

- Rotations along the generators in time subspace

$$R_T^{12} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , R_T^{13} = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , R_T^{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 & 0 & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.38}$$

- Rotations along the generators in spatial subspace

$$R_S^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , R_S^{13} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & 0 & \sin(\theta) \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} , R_S^{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} . \tag{2.39}$$

- Boosts along the first time coordinate

$$B^{11} = \begin{pmatrix} \cosh(\xi) & 0 & 0 & \sinh(\xi) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sinh(\xi) & 0 & 0 & \cosh(\xi) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , B^{12} = \begin{pmatrix} \cosh(\xi) & 0 & 0 & 0 & \sinh(\xi) & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sinh(\xi) & 0 & 0 & 0 & \cosh(\xi) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , B^{13} = \begin{pmatrix} \cosh(\xi) & 0 & 0 & 0 & 0 & \sinh(\xi) \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \sinh(\xi) & 0 & 0 & 0 & 0 & \cosh(\xi) \end{pmatrix} . \tag{2.40}$$

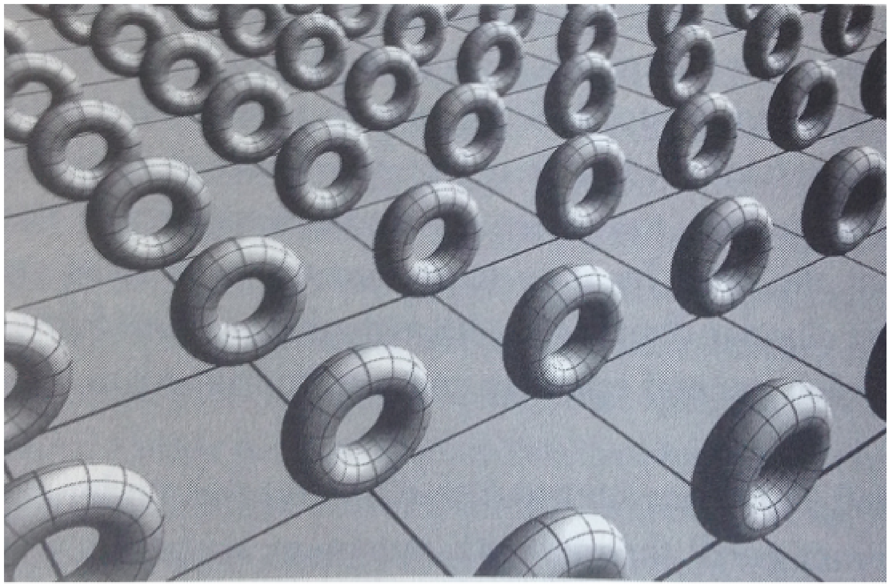

Figure 1. Two extra compact dimensions as a torus bundle on spacetime. Donuts everywhere!

- Boosts along the second time coordinate

$$B^{21} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cosh(\xi) & 0 & \sinh(\xi) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sinh(\xi) & 0 & \cosh(\xi) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B^{22} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cosh(\xi) & 0 & 0 & \sinh(\xi) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sinh(\xi) & 0 & 0 & \cosh(\xi) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B^{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cosh(\xi) & 0 & 0 & 0 & \sinh(\xi) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \sinh(\xi) & 0 & 0 & 0 & \cosh(\xi) \end{pmatrix}. \tag{2.41}$$

- Boosts along the third time coordinate

$$B^{31} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\xi) & \sinh(\xi) & 0 & 0 \\ 0 & 0 & \sinh(\xi) & \cosh(\xi) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B^{32} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\xi) & 0 & \sinh(\xi) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \sinh(\xi) & 0 & \cosh(\xi) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B^{33} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\xi) & 0 & 0 & \sinh(\xi) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sinh(\xi) & 0 & 0 & \cosh(\xi) \end{pmatrix}. \tag{2.42}$$

2.4 $T = 0$ Casimir dynamics and forces in different boundary conditions and geometries.

Let us look at the Casimir effect in the case of spacelike extra dimensions d, beginning with our case of 6D $\rightarrow$ 4D with a toric configuration $\mathcal{M} \approx \mathcal{M}(3,1) \times T2$. A general case will also be presented, beyond []. The case of timelike extra dimensions will be described later, and how the results are modified. Figure 1 shows what a torus compactified space looks, if we lived in 'flatland' with 2 dimensions. Donuts everywhere!

The contribution from the extra dimensional Casimir effect is simply the sum over KK modes, with some regularization techniques employed:

$$V = \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}\frac{d^4k}{(2\pi)^4}\ln(k^2+\mathcal{M}^2_{mn}) = -\frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}\frac{\partial}{\partial s}|_{s=0}\frac{d^4k}{(2\pi)^4}\ln(k^2+\mathcal{M}^2_{mn})^{-s},$$

$$= -\frac{1}{32\pi^2}\frac{\partial}{\partial s}|_{s=-2}\frac{1}{s(s+1)}\sum_{m,n}(\mathcal{M}^2_{mn})^{-s},$$

where $\mathcal{M}^2_{mn} = n^2/R_1^2 + m^2/R_2^2$ give the KK modes, and the T^2 has dimensions $R_1\times R_2$. The sum excludes the case where n and m are not both zero.

By performing the summation on the right, we get, using general zeta function regularization techniques (more details in general cases are given later, in [], [], and in an Appendix):

$$\sum_{(m,n)\neq\vec{0}}(\mathcal{M}^2_{mn})^{-s} = R_1^{2s}[2\zeta(2s) + 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}(\frac{R_1}{R_2})^{1-2s}\zeta(2s-1)+$$

$$+\frac{8\pi^4}{\Gamma(s)}(\frac{R_1}{R_2})^{\frac{1}{2}-s}\sum_{(m,n)\in\mathbb{N}^2_+}(\frac{n}{m})^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\frac{R_1}{R_2}mn)].$$

The Γ-function in the denominator is our friend: upon taking derivatives with respect to s, we get a finite result, which for the case of periodic (bosonic) boundary conditions, is calculated to be:

$$V^{++} = -\frac{1}{64\pi^2R_1^4}[\frac{3}{\pi^4}\zeta(5) + \frac{8\pi}{945}(\frac{R_1}{R_2})^5 + \frac{16}{\pi^2}(\frac{R_1}{R_2})^{5/2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\frac{m}{n})^{5/2}K_{-5/2}(2\pi\frac{R_1}{R_2}mn)].$$

Here, we have used, in taking the s-derivatives:

$$\zeta'(-4) = \frac{3}{4\pi^4}\zeta(5), \qquad \frac{\Gamma'(-2)}{\Gamma(-2)^2} = -2, \qquad \Gamma(-\frac{5}{2}) = -\frac{8}{15}\sqrt{\pi} \qquad \zeta'(-5) = -\frac{15}{4\pi^6}\zeta(6) = -1/252.$$

We can further use

$$K_{-5/2}(z) = \sqrt{\frac{\pi}{2}}(z^{-1/2} + 3z^{-3/2} + 3z^{-5/2})e^{-z},$$

we can obtain:

$$V^{++} = -\frac{1}{64\pi^2R_1^4}[\frac{3}{\pi^4}\zeta(5) + \frac{8\pi}{945}(\frac{R_1}{R_2})^5 + \frac{8L_3(\tau)}{\pi^2}(\frac{R_1}{R_2})^2 + \frac{12L_4(\tau)}{\pi^3}\frac{R_1}{R_2} + \frac{6L_5(\tau)}{\pi^4}],$$

where τ is the aspect ratio $\tau = R_1/R_2$ and[1] the L_n-functions can be written in terms of series of polylogarithms (with $q = e^{-2\pi\tau}$:

$$L_3(\tau) = \sum_{m=1}^{\infty} m^2 Li_3(q^m) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{\coth(\pi\tau n)}{n^2\sinh^2(\pi\tau n)},$$

[1] The area, $A = 4\pi^2R_1R_2$, is called the K ahler modulus, while in general (for a flat torus as a parallelogram of acute angle θ) $\tau = e^{i\theta}R_1/R_2$ is the modular parameter.

$$L_4(\tau) = \sum_{m=1}^{\infty} mLi_4(q^m) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^4\sinh^2(\pi\tau n)},$$

$$L_5(\tau) = \sum_{m=1}^{\infty} Li_5(q^m) = \sum_{n=1}^{\infty} 1/n^5(e^{2\pi\tau n}-1).$$

Upon direct computation, one finds a value for the Casimir energy (for $\tau = 1$):

$$V^{++} = -\frac{1}{4A^2}[\frac{8\pi^3}{945}\tau^3 + 8L_4(\tau) + \frac{12\tilde{L}_4}{\pi\tau} + \frac{3\zeta(5)+6L_5(\tau)}{\pi^2\tau^2}]$$

$$\approx -0.1502385/A^2,$$

and we note that it is attractive. It also has extreme values at the self-dual points $\theta \in \{\pi/2, 2\pi/3\}$, $\tau = 1$.

Let's now one of the directions be fermionic (anti-periodic boundary conditions), as a kind of thermal variation or $\mathcal{N} = 1$ supersymmetric case (the massive KK-modes before we can call W-bosons, and so in the SUSY case the KK modes we can call 'winos'). The result for the Casimir energy can be found similarly, with half-integers along the aperiodic direction ($\mathcal{M}^{+-}_{mn} \equiv n^2/R_1^2 + (m+\frac{1}{2})^2/R_2^2$):

$$\sum_{(m,n)\neq\vec{0}} (\mathcal{M}^{+-}_{mn})^{-2s} == R_1^{2s}[2\zeta(2s) + 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}(\frac{R_1}{R_2})^{1-2s}\zeta(2s-1,\frac{1}{2})+$$

$$+\frac{8\pi^4}{\Gamma(s)}(\frac{R_1}{R_2})^{\frac{1}{2}-s}\sum_{(m,n)\in\mathbb{N}^{2*}}(\frac{n}{m+\frac{1}{2}})^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\frac{R_1}{R_2}(m+\frac{1}{2})n)]$$

where the Hurwitz zeta function is used, and note $\zeta(2n-1,\frac{1}{2}) = (2^s-1)\zeta(s)$.

from which V^{+-} can be evaluated, using the $\tilde{L}$-functions:

$$\tilde{\sum}_{m=1}^{\infty}(m+\frac{1}{2})^2 Li_3(q^{\frac{1}{2}+m}) = \frac{1}{8}\sum_{n=1}^{\infty}\frac{\coth(\pi\tau n)+1}{n^3\sinh^3(\pi\tau n)},$$

$$\tilde{L}_4(\tau) = \sum_{m=1}^{\infty}(m+\frac{1}{2})Li_4(q^{m+1/2}) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{\coth(\pi\tau n)}{n^4\sinh^2(\pi\tau n)},$$

$$\tilde{L}_5(\tau) = \sum_{m=1}^{\infty} Li_5(q^{m+\frac{1}{2}}) = \frac{1}{2}\sum_{n=1}^{\infty} 1/n^5\sinh\pi\tau n.$$

After evaluating exactly ($\tau = 1$ again), we find a different result:

$$V^{+-} = -\frac{1}{4A^2}[\frac{-31\pi^3}{3780}\tau^3 + 8\tilde{L}_3(\tau) + \frac{12\tilde{L}_4}{\pi\tau} + \frac{6\tilde{L}_5(\tau)}{\pi^2\tau^2}]$$

$$\approx +0.0139727/A^2,$$

and we find the force to be attractive now![2]

We also note that interchanging the compact directions, in the 'SUSY' case, we find a T-duality $V^{+-} = V^{-+}(R_1 \iff R_2)$, so $\tau = 1/\tau$, and the results are the same in the cases above at $\tau = 1$.

For completion we give the result for V^{--}:

$$V^{--} = -\frac{1}{4A^2}[\frac{-31\pi^3}{3780}\tau^3 + 8\hat{L}_3(\tau) + \frac{12\hat{L}_4}{\pi\tau} + \frac{6\hat{L}_5(\tau)}{\pi^2\tau^2}],$$

where, as in [],

$$\hat{L}_3(\tau) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{n^3}(m+\frac{1}{2})^2 e^{-2\pi\tau n(m+1/2)} = -\tilde{L}_3(\tau) + \frac{1}{4}\tilde{L}_3(2\tau),$$

$$\hat{L}_4(\tau) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{n^4}(m+\frac{1}{2}) e^{-2\pi\tau n(m+1/2)} = -\tilde{L}_4(\tau) + \frac{1}{8}\tilde{L}_4(2\tau),$$

$$\hat{L}_5(\tau) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{n^3} e^{-2\pi\tau n(m+1/2)} = -\tilde{L}_5(\tau) + \frac{1}{16}\tilde{L}_5(2\tau).$$

The method of zeta function regularization is given in Appendix A.

We, combining these different boundary conditions, find a minimum in (A, τ)-space, and adding the vacuum energy:

$$\mathcal{E}_{vac} = \int d^6x\sqrt{G}\Lambda = A\int d^4x\Lambda,$$

a SUSY vacuum can produce the correct (nearly) vanishing cosmological constant! This is not so likely, but, as we will find, extra time dimensions help, especially the odd numbered ones. We can consider three time dimensions mostly. This will be discussed in later Sections as well.

We can do further approximations, say for the case $R_1 \ll R_2$, where we can identify $R_2 = \beta$ as an inverse temperature for a thermal direction, and achieve a simple effective 4D potential after several approximations, that yields a QED, and later on, a QCD-like theory, as if from nothing in the more plain 6D. This was done in [], and my thesis, for the case of $4D \to 2D$ instead. We attempt to do this now.

[2]Though it becomes repulsive for small $\tau \ll 1$, which is a UV/higher temperature regime, we can find a phase transition as in confinement [], though this leads to signature change of the metric as well.

2.5 Low energy vs. high energy approximate actions from the 6D Casimir effect.

As done in [], and my thesis [], we can look at high and low temperature expansions of the Casimir energy formulae obtained in the previous Section. This is the case for $R_1 \ll R_2$, where $R_1 = \beta$ is a thermal direction, with $\beta = 1/T$. In the next subsection we will see directly the finite temperature corrections to the Casimir effect, and show how these results compare, as well as to the case extra time dimensions.

We begin by

In the next subsection we look at finite temperature corrections to the Casimir effect, looking at the thermal photon modes as well. Later, we will look at oscillating boundaries (dynamical Casimir effect) as well, and find how 6D, can lead to an effective QED we observe today, plus measurable corrections and richer fields that may soon be detectable.

2.6 Finite temperature corrections to Casimir dynamics and forces in different dimensions.

We now investigate finite temperature considerations and corrections to Casimir forces calculated above. We begin with the basics from [] and [], and then apply to extra dimensions, geometries, and (later on) metrics. We recall the zero temperature result of the Casimir force:

$$F_c = -\frac{\pi^2 \hbar c}{240 a^4}.$$

and is an attractive force. The Casimir force has been difficult to measure, as the contribution is only $\approx 1.3\ mPa \approx 10^{-8}\ atm$, but certainly easier to measure than other quantum field phenomena, that have been measured to dozens of decimal places of accuracy.

Lifschitz in '55 [] calculated the result for the temperature dependent Casimir force, between two non-magnetic plates of dielectrix constant ϵ. It gets slightly more complicated in general,

$$F_c = -\frac{k_b T}{\pi}\sum_{m=0}^{\infty}\int_{\zeta_m}^{\infty} q^2 dq[\frac{A_m e^{-2qa}}{1 - A_m e^{-2qa}} + \frac{B_m e^{-2qa}}{1 - B_m e^{-2qa}}],$$

where $\zeta_m = \frac{2\pi m k_b T}{\hbar}$ are the Matsubara frequencies, as seen above, have aperiodic/thermal/timelike boundary conditions (so we take $\omega_m = i\zeta_m$), and hence the summation above having an IR lower bound. In the above formula, $A_m = (\frac{\epsilon p - s}{\epsilon p + s})^2$ and $B_m = (\frac{s-p}{p+s})^2$, with $s^2 = \epsilon - 1 + p^2$, and $p = qc/\zeta_m$. The A_m and B_m are the reflection coefficients TM (transverse magnetic) and TE (transverse electric) cases, where in either case either the electric field or the magnetic field are parallel to the plates.

We can look at the pure ideal metal plate case.

3 Moving boundary conditions: the dynamical Casimir effect and effective QED from six dimensions.

We now first consider the simple Casimir-parallel-plate example we started with, and now allow for moving boundaries (either forced/driven, or as a reaction to quantum field theory effects and finite temperature oscillations, on top of the Casimir force).

This is called the dynamical Casimir effect (DCE), and we propose it's more observable connection to effective QED from toroidal compactification from six dimensions. Second quantization of both QED vacuum excitations (photon and fermion modes), couple to the oscillation modes of the boundary walls[3]. We then describe this as an effective QED from 6D, and later mention the benefits of the DCE in experimental detection of Casimir effects and extra dimensional effects,

A difference from previous works, where you can peruse [], [], [], for the current status of the dynamical Casimir effect, is that we posit the possibility of extra timelike dimensions, as is motivated for various reasons in the Introduction section.

We summarize relevant methods in simple cases first, then move to extra dimensions, spacelike and timelike. More on the latter will be after the next section when methods for dealing with extra compact time dimensions is treated. Along the way, we will see many similarities and applications to brane cosmology, and with timelike bulk dimensions as well, with its own advantages. But let's begin with a simple case.

3.1 Wave equation and canonical quantization on a time dependent interval

Let us begin with a simple case of a scalar field $\Psi(t,\vec{x})$ and a time dependent interval $I(t) = [0,\ell(t)]$ with Dirichlet boundary conditions $\Phi(t,\vec{0}) = 0 = \Phi(t,\ell(t))$ for all t. The wave equation is $(\partial_t^2 - \partial_x^2)\Phi(t,x) = 0$ for one dimension. We can take a set (complete and orthonormal) of eigenfunctions $\{\phi_n\}$ of the spatial Laplacian $-\partial_x^2$ for a static interval $[0,\ell_0]$, with eigenvalues $k_n^2 = n^2\pi^2/\ell_0^2$, as well as 'instantaneous' eigenfunctions $\phi_n(t,x) = \sqrt{2/\ell(t)}\sin k_n(t)x$ with $k_n(t) = n\pi/\ell(t)$. By completeness and orthonormality, we can decompose Φ in terms of the eigenmodes: $\Phi(t,x) = \sum_{n=1}^{\infty} q - n(t)\phi_n(t,x)$, with the q_n the canonical variables. If we substitute this into the wave equation, multiply by $\phi_m(t,x)$ and integrate over $I(t)$, we get their equations of motion:

$$\ddot{q}_n + k_n^2(t)q_n - 2\sum_m M_{nm}(t)\dot{q}_n + \sum_m[\dot{M}_{mn}(t) - N_{nm}(t)]q_n = 0,$$

for coupling matrices

$$M_{nm}(t) = \int_0^{\ell(t)} dx\dot{\phi}_n\phi_m = \frac{\dot{\ell}}{\ell}(-1)^{n+m}\frac{2nm}{m^2-n^2},\ m\neq n,\quad 0,\ m=n,$$

[3]What exactly we mean by 'wall', is more abstract than you think in general cases, as you'll see later.

$$N_{nm}(t) = \int_0^{\ell(t)} dx \dot{\phi}_n \dot{\phi}_m = \sum_k M_{nk} M_{mk},$$

noting that M is anti-symmetric and N is symmetric.

For even simple $\ell(t)$, the differential cannot be solved exactly, but using conformal invariance of DCE, we can gain some useful knowledge of the solutions. Note also that the $k_n(t)$ and the $M_{nm}(t)$ give rise to the creation of particles (through the so called 'squeezing effect' and the 'acceleration effect', respectively. I'll mention later how this particle creation by the DCE differs from that of the creation of particles in an isotropic, homogeneous, expanding Universe, or heating/reheating in inflation models, or from time dependent electromagnetic fields, although many similarities will be mentioned (cosmological similarities can be seen from the cosmological-like parameter $\dot{\ell}/\ell$ in $M_{mn}(t)$ already:), and even similarities and differences to the Hawking radiation and the Unruh effect. By using the aforementioned conformal invariance, Moore gives a solution

$$\phi_n(t, x) = e^{-in\pi R(t+x)} - e^{-in\pi R(t-x)},$$

with

$$R(t + \ell(t)) - R(t - \ell(t) = 2.$$

We take $c = \hbar = 1$ here. We can note that the ebergy released from radiation/particles is $\mathcal{E} = \int \ddot{\ell}(t)^2 dt$, so no radiation is produced for a mirror in uniform motion.

Let's now turn to the canonical formalism and quantization for a massive real scalar field case.

$$\int d^2x \frac{1}{2}[(\partial_t \Phi)^2 - (\partial_c \Phi)^2 - m^2 \Phi^2,$$

and we get the same results as above with k_n^2 replaced by $\Omega_n^2(t) = k_n^2(t) + m^2]$. The Euler-Lagrange equations give the same equation for the canonical variables q_n previously in the massless case, with possibly different boundary conditions. Now for the Hamiltonian formulation, the canonical momentum variables are

$$p_n = \frac{\partial L}{\partial \dot{q}_n} = \dot{q}_n + \sum_m q_m M_{mn}(t),$$

giving the time dependent Hamiltonian

$$H(t) = \sum_n \dot{q}_n(t) p_n(t) - L(t) = H_{osc} + H_{int} = \frac{1}{2} \sum_n [p_n^2 + \Omega^2(t) q_n^2] = \sum_m \sum_n M_{nm}(t) q_n p_m.$$

Hamilton's equations then are

$$\dot{q}_n = \frac{\partial H}{\partial p_n} = p_n - \sum_m M_{mn} q_m.$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} = -[\Omega_n^3 q_n - \sum_m M_{nm} p_m].$$

Varying the action, gives

$$\delta S = -\int_{-}^{T} dt \int_0^{\ell(t)} dx(\Box_{(2)} + m^2\Phi)\delta\Phi - \int_0^T dt[(v\partial_t + \partial_x)\Phi)\delta\Phi|_{\ell(t)} - (\partial_x\Phi)\delta\Phi|_0,$$

where the boundary terms must banish for the wave equation to be satisfied. This restricts the boundary conditions to Dirichlet ones,or Neumann ones with $(v\partial_t + \partial_x)\Phi = 0$ at the boundary $\ell(t)$, and note that follows from a Lorentz transformation, as we note that boosts in the x-direction are preserved as above. Dirichlet boundary conditions at the moving boundary with Neumann ones at 0 are also allowed generally, but not vice-versa always.

We can find energy and momentum from the stress-energy tensor, whose non zero components are

$$T_{00} = \frac{1}{2}[\dot{\Phi}^2 + (\partial_x\Phi)^2 + m^2\Phi^2],$$

and with

$$\Pi(t,x) = \frac{\partial L}{\partial \Phi} = \dot{\Phi} = \sum_n p_n(t)\phi_n(t,x),$$

showing momentum modes can be expanded in the same eigenfunction basis. The energy is then

$$E = \int_{I(t)} T_{00} dx = \frac{1}{2}\sum_n [p_n^2 + \Omega_n^2(t) q_n^2] = H_{osc} \neq H,$$

and so the energy does not equal the Hamiltonian, which has interaction terms. We can can also find the rate of change of energy

$$\dot{E} = \int_{I(t)} dx \partial_t\Phi[\Box_{(2)} + m^2]\Phi + (\partial_x\Phi)\dot{\Phi}|_0^{\ell(t)} + vT_{00}[t,\ell(t)] = (T_{01} + vT_{00})[t,\ell(t)],$$

for Dirichlet or Neumann BCs at $x = 0$.

Quantization is simple, and done in Heisenberg picture, as we have time dependent operators replacing the canonical variables. The usual commutation relations are

$$[\hat{\Phi}(t,x), \hat{\Phi}(t,x')] = [\hat{\Pi}(t,x), \hat{\Pi}(t,x)] = 0, \quad [\hat{\Phi}(t,x), \hat{\Pi}(t,x')] = i\delta(x - x'),$$

where it follows

$$[\hat{q}_n, \hat{q}_m] = [\hat{p}_n, \hat{p}_m] = 0, \quad [\hat{q}_n, \hat{p}_n] = i\delta_{mn} \ \ \forall t.$$

The Hamiltonian $\hat{H}$ is the same as above, replacing the canonical variables with their operators. Since they are time dependent, the Heisenberg picture is well suited, with

$$\dot{\hat{\mathcal{O}}} = i[\hat{H}, \hat{\mathcal{O}}] + \partial\hat{\mathcal{O}}/\partial t,$$

for some explicit time dependence, in general in 3D time coming up. The states are what do not change in time in this picture, so $|\Psi(t)\rangle = |\Psi(t_0)\rangle,\ \forall t$. The Hamiltonian, as in classical mechanics, can be diagonalized into infinitely many independent harmonic oscillators, in an appropriate eigenbasis, so long as they are countable and so can have discrete eigenenergies, under quantization.

The vacuum will be defined by two times t_{in}, t_{out} where the moving mirror is at rest, as $\Omega_n^{in/out}(t < t_{in}, t > t_{out})] = const \neq 0$, not necessarily equal[4], with vanishing $M_{mn} = 0$ for t outside of $[t_{in}, t_{out}]$. The hamiltonian is diagonalized by

$$\hat{q}_n(t) =$$

$$\hat{p}_n(t) =$$

and annihilation and creation operators (with usual commutation relations

$$[\hat{a}_n, \hat{a}_n] = [\hat{a}_n^\dagger, \hat{a}_n^\dagger] = 0, \quad [\hat{a}_n, \hat{a}_m^\dagger] = \delta_{mn},$$

give rise to multiple particle states, spanning two Hilbert spaces $\mathcal{H}_{in}$, $\mathcal{H}_{out}$ by repeated application of creation operators to their respective vacua $|0, in/out\rangle$, both normalized. The particle number per state (degeneracy) is

$$\hat{N}_n^{in/out} = \hat{a}_n^{\dagger, in/out} \hat{a}_n^{in/out}, \qquad \hat{H}^{in/out} = \sum_n \Omega_n^{in/out} [\hat{N}_n^{in/out} + \frac{1}{2}],$$

where the vacuum energy is observed. Note that these vacuum states are only well defined if they have countably many states. As we will see later, extra time dimensions gives the vacuum a finite number of states, depending on the mass scale.

We can write the $\hat{a}_n^{in}$ in terms of the $\hat{a}_n^{out}$ by Bogoliubov transformations

$$\hat{a}_n^{out} = \sum_m [\mathcal{A}_{mn}(t_{out}) \hat{a}_m^{in} + \mathcal{B}_{mn}^*(t_{out}) \hat{a}_n^{\dagger,\ in}],$$

and so we find the number of produced particles as

$$\mathcal{N}_n^{out} = \langle 0, in | \hat{N}_n^{out} | 0, in \rangle = \sum_m |\mathcal{B}_{mn}(t_{out})|^2, \quad \mathcal{N}^{out} = \sum_n \mathcal{N}_n^{out}$$

if finite, like the case of extra time dimensions below. The energy output is then

$$\mathcal{E}^{out} = \sum_n \Omega_n^{out} \mathcal{N}_n^{out}.$$

Note $\hbar = 1$ again, and also that discontinuous wall motion can create large numbers of harmonics, and so this may not converge at the time endpoints, but with extra time dimensions,

[4] For both or either $\Omega_n^{\in/out} = 0$, the same quantization is as in bosonic string theory.

there is no worry. We define from now on the 'instantaneous vacuum' $|0, t\rangle$, such as is annihilated by all $\hat{a}_n(t)$ at its time t. The position and momentum modes are expressed in terms of the annihilation/creation operators as usual,

$$\hat{q}_n(t) = \frac{1}{\sqrt{2\Omega_n(t)}}[\hat{a}_n(t) + \hat{a}_n^\dagger(t)], \qquad \hat{p}_n(t) = i\sqrt{\frac{\Omega_n(t)}{2}}[\hat{a}_n^\dagger(t) - \hat{a}_n(t)].$$

Altogether, we get the Hamiltonian operator,

$$\hat{H}(t) = \sum_k \Omega_k(t)[\hat{a}_k^\dagger(t)\hat{a}_k(t)+\frac{1}{2}] = \frac{1}{2}i\sum_{k.m}\sqrt{\frac{\Omega_m(t)}{\Omega_k(t)}}M_{km}(t)[\hat{a}_m^\dagger(t)\hat{a}_k(t)-\hat{a}_k^\dagger(t)\hat{a}_m(t)-\hat{a}_m(t)\hat{a}_k^\dagger(t)+\hat{a}_k^\dagger(t)\hat{a}_m^\dagger(t)],$$

where the first term is the oscillator part and the second is the interaction term. We also get

$$\hat{\dot{a}}_n(t) = -i\Omega_n(t)\hat{a}_n(t) - \frac{1}{2}\sum_m[A_{mn}^-\hat{a}_m(t) + A_{mn}^+(t)\hat{a_m^\dagger}(t)],$$

where the $A_{mn}^\pm$ can be written in terms of the Bogoliubov coefficients [].

Let us now go to higher (brane) dimensions, and then to extra bulk dimensions. Begin with a $D = d+1$ dimensional Minkowski spacetime, with a massive real scalar field, satisfying the Klein-Gordon equation

$$[\Box_{(D)} + m^2]\Phi = 0,$$

with a solution set of plane waves:

$$u_{\vec{k}}(t, \vec{x}) = \frac{1}{\sqrt{2\Omega_{\vec{k}}(2\pi)^d}}e^{i\vec{k}\cdot\vec{x} - i\Omega_{\vec{k}}t},$$

which are eigenfunctions of ∂_t with eigenvalues $-i\Omega_{\vec{k}}$,
We now define an important inner product on the Hilbert space $\mathcal{H}$, mentioned above as those states obtained from the vacuum by repeated applications of creation operators $\hat{a^\dagger_h}$, as

$$(u_{\vec{k}}, u_{\vec{k}'}) = -i\int dx^d[u(\partial_t u^{*\prime}) - (\partial_t u)u^{*\prime}],$$

which satisfies

$$(u_{\vec{k}}, u_{\vec{k}'}) = \delta_{\vec{k}\vec{k}'}, \quad (u_{\vec{k}}^*, u_{\vec{k}'}^*) = -\delta_{\vec{k}\vec{k}'}, \quad (u_{\vec{k}}, u_{\vec{k}'}^*) = 0,$$

and the solution Φ can then be written as a sum over these orthonormal, complete set of modes

$$\Phi = \int_{\vec{k}} \hat{a}_{\vec{k}}u_{\vec{k}} + h.c.$$

with $\hat{a}_{\vec{k}}|0\rangle = 0, \quad \forall\vec{k}$. The Hamiltonian and momentum can be found from the non-zero components of the stress-energy tensor,

$$\hat{H} = \int \hat{T}_{00}d^dx = \frac{1}{2}\sum_{\vec{k}}\Omega_{\vec{k}}(\hat{a}_{\vec{k}}^\dagger\hat{a}_{\vec{k}} + \frac{1}{2}),$$

$$\hat{P}_i = \int \hat{T}_{0i} d^d x = \sum_{\vec{k}} k_i \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}}.$$

We can now pass to curved spacetimes, with a metric (easily generalized for extra bulk time dimensions, as we'll see later),

$$ds^2 = g_{00} dt^2 - g^\Sigma_{ij} dx^i dx^j,$$

on a Cauchy surface Σ with spacetime manifold $\mathcal{M} = \mathbb{R}_t \times \Sigma$. If $g_{00} = 1$, the metric is called ultrastatic. The action is

$$S = \int_{\mathcal{M}} dx^D \frac{1}{2} \sqrt{g} [(\partial_\mu \Phi)(\partial^\mu \Phi) - m^2 \Phi].$$

Varying the action with respect to the metric gives the Klein-Gordon equation with D'Alambertian

$$\Box_{(D)} = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu) = g^{00} \partial_t^2 + \frac{1}{\sqrt{g^\Sigma}} \partial_i (\sqrt{g^\Sigma} g^{ij} \partial_j),$$

where the metric on Σ is that induced from that on $\mathcal{M}$. The inner product introduced earlier also becomes

$$(u_1, u_2) = -i \int_\Sigma [u_1 (\partial_\mu u_2^*) - (\partial_\mu u_1) u_2^*] \sqrt{g^\Sigma} d\Sigma^\mu,$$

with 'surface' element normal $d\Sigma^\mu = n^\mu d\Sigma$, for a n^μ pointing in the time direction, normal to the spacelike manifold, such as $n^\mu = (1/\sqrt{g_{00}}, \vec{0})$. By Gauss' theorem, the integration measure is Σ-invariant.

The energy-momentum tensor can be found from varying the action:

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \Phi \partial^\mu \Phi + \frac{1}{2} m^2 g_{\mu\nu} \Phi^2,$$

giving a local conservation law

$$T^\mu_{\nu;\mu} = 0 = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} T^\mu_\nu) - \frac{1}{2} T^{\rho\kappa} \partial_\nu g_{\rho\kappa},$$

where the second term shows energy exchange between the scalar field and the gravity tensor field. If the manifold has Killing vectors ξ^μ (like $\partial_0 g_{\mu\nu} = 0$), the energy can be written

$$E = \int_\Sigma d\Sigma_\mu T^{\mu\nu} \xi_\nu,$$

which allows us to extend the DCE to curved spacetimes.

I mentioned previously how there are similarities and differences between the DCE and the Unruh effect, from the viewpoint of an accelerating detector (not the same as moving mirrors). This has a Killing field

$$\partial_\tau = \kappa (x \partial_t + t \partial_x),$$

in Rindler coordinates

$$t = \rho \sinh \kappa\tau, \quad x = \rho \cosh \kappa\tau, \quad \rightarrow \quad ds^2 = \kappa^2 \rho^2 d\tau^2 - d\rho^2,$$

for $(\rho, \tau) \in \mathbb{R}^+ \times \mathbb{R}$, and κ is called the 'surface gravity'. The solutions to the equations of motion in these coordinates are (1D here)

$$v_k(\rho, \tau) = \frac{1}{\sqrt{4\pi\Omega_k}} \rho^{ik/\kappa} e^{-i\Omega_k \tau}.$$

If we consider the orbit with $g_{00} = \eta^\mu \eta_\mu = 1 = \rho^2\kappa^2$, we find $\rho = 1/\kappa$ and so indeed κ gives the proper acceleration.
The difference between the Minkowski vacuum previously $|0\rangle_M$ and the one on $\mathcal{M}$ $|0\rangle_{\mathcal{M}}$ is given by Bogoliubov transformations like before, for the Rindler coordinate operators $\hat{a}_k^R, \hat{a}_k^{R\dagger}$ in terms of the previous ones. After some work [], one gets

$$\langle 0|\hat{a}_k^{R\dagger} \hat{a}_k^R|0\rangle_{\mathcal{M}} \propto 1/(e^{2\pi\Omega_k/\kappa} - 1),$$

which shows, by comparison to the Bose-Einstein distribution for thermal photons, that an accelerated observer indeed experiences a Minkowski vacuum at temperature $T = \kappa/2\pi$![5]

Let's now go back to the DCE and warm up with an original example from Moore []: a uniformly moving mirror. So for $\ell(t) = \ell_0 + vt$, Moore's equation () can be exactly solved:

$$R(z) = \frac{1}{\tanh^{-1} v} \ln(z + \frac{\ell_0}{v}),$$

with eigenmodes

$$\phi_n(t, x) = \exp[-i\frac{n\pi}{\tanh^{-1} v} \ln(t + x + \frac{\ell_0}{v})] - \exp[-i\frac{n\pi}{\tanh^{-1} v} \ln(t - x + \frac{\ell_0}{v})].$$

In this case, it is found that N_{tot} diverges logarithmically, as from the Bogoliubov coefficients we find $\mathcal{N}_n^{out} \propto v^2/n$, for $n > 6$ and $v \ll 1 (= c)$.

Now let's do the oscillating mirror case: $\ell(t) = \ell_0[1 + \epsilon \sin \omega_{cav} t]$, $\epsilon \ll 1$ a small dimensionless amplitude. We will find that for a resonance $\omega_{cav} = 2\Omega_1^{in}$, the particle number increases quadratically for both very short timescales and very long ones. There is also resonant coupling for $\omega_{cav} = |\Omega_\ell^{in} \pm \Omega_k^{in}|$. Since the $\Omega_n \approx n\pi/\ell_0$ are equidistant there are infinitely many coupled modes. It turns out only odd modes are coupled and the even modes are not excited, so we have even multiples $2\Omega_1^{in} = |\Omega_\ell^{in} \pm \Omega_k^{in}|$.
For the main resonance $\omega_{cav} = 2\Omega_1^{in}$, we can find the particle numbers exactly:

$$\mathcal{N}_1(t) = \frac{2}{\pi^2} E(\kappa)K(\kappa) - \frac{1}{2}, \quad \mathcal{N}_{tot}(t) = \frac{1}{\pi^2}[(1 - \frac{1}{2}\kappa^2)K^2(\kappa) - E(\kappa)K(\kappa)],$$

[5] Note this gives the Unruh temperature putting back in c, G, and $\hbar$, which are =1 here.

where $\kappa \equiv \sqrt{1-\exp(-8\tau)}$ with a 'slow time' $\tau = \frac{1}{2}\epsilon\Omega_1^{in} t = \pi\epsilon t/2\ell_0$, and E and K are the complete elliptic integrals.
It is found that for $\tau \ll 1$, that $\mathcal{N}(\tau), \mathcal{N}_1(\tau) \propto \tau^2$. For very long times $\tau \gg 1$, we get $\mathcal{N}(\tau) \propto \tau^2$, $\mathcal{N}_1(\tau) \propto \tau$.

For the other (odd) resonant modes, it is found that

$$d\mathcal{N}_n(t)/dt = \frac{4\epsilon}{n\pi}, \quad \rightarrow \quad \mathcal{N}_{tot} \propto t,$$

if $\epsilon\tau \gg 1$ and $\tau \gg 1$,but also works for medium times $\tau \approx \frac{1}{2}$. For all these resonances, the energy is also calculated. For the main resonance

$$E(\tau) = \frac{1}{4}\Omega_1^{in} \sinh^2 2\tau,$$

and for the higher resonances $\omega_{cav} = 2\Omega_1^{in} = 2n\pi/\ell_0$, $\tau \ll 1$:

$$\mathcal{N}_k(\tau) = (2n-k)k\tau^2, \ k < n, \ n \in \frac{1}{2}\mathbb{N},$$

so that

$$\mathcal{N}(\tau) = \sum_{k=1}^{2n-1} (2n-k)k\tau^2 = \frac{n}{3}(4n^2-1)\tau^2.$$

It is also quadratic for $\tau \gg 1$, as it is found $\mathcal{N}(\tau) = \frac{8n^3}{\pi^2}\tau^2$. In either case, the energy grows exponentially,

$$\mathcal{E}(\tau) = \frac{4n^2-1}{12}\pi \sinh^2(2n\tau).$$

We can also look at detuning, for $\omega_{cav} = \omega_{res} + \Delta$, where $\Delta \equiv 2\pi\delta n/\ell_0$ small, and define $\gamma \equiv \delta n/n\epsilon$. We look at the following cases:

-For $\gamma < 1$: the energy increases exponentially in τ,

$$\mathcal{E}(\tau) = \frac{4n^2-1}{12}\pi\frac{\sinh^2(2na\tau)}{a^2}, \quad a \equiv \sqrt{1-\gamma^2}.$$

-For $\gamma = 1$: the energy increases quadratically in τ, using L'Hopital's rule,

$$\mathcal{E}(\tau) = \frac{4n^2-1}{3}\pi(n\tau)^2.$$

-For $\gamma > 1$: the cavity will no longer produce increasing energy, but it oscillates in simple harmonic motion,

$$\mathcal{E}(\tau) = \frac{4n^2-1}{12}\pi\frac{\sin^2(2n\tilde{a}\tau)}{\tilde{a}^2}, \quad \tilde{a} \equiv \sqrt{\gamma^2-1},$$

with period $T_n(\delta n, \epsilon) = 1/n\epsilon\tilde{a}$.

We now turn to a 3D resonant cavity with electromagnetic (vector) fields, and get on to photon and particle creation in the DCE. Recalling Maxwell's equations in vacuum with no charges or currents,

$$\nabla \cdot \vec{E} = 0 = \nabla \cdot \vec{B}, \quad \nabla \times \vec{E} = -\partial_t \vec{B}, \quad \nabla \times \vec{B} = \partial_t \vec{E},$$

with boundary conditions in a 3D cavity with one moving mirror (ideal so infinite conductivity $\sigma \to \infty$) in the $\hat{x}$-direction given by

$$\vec{E}_{||} = 0 \quad \Longleftrightarrow \quad \hat{x} \times \vec{E} = 0, \qquad B_\perp = 0 \quad \Longleftrightarrow \quad \hat{x} \cdot \vec{B} = 0.$$

We can split the modes into transverse electric (TE) and transverse magnetic (TM) components, defined from the vector field $\vec{A}$, so that:

$$\vec{E} = -\partial_t \vec{A}^{TE} + \nabla \times \vec{A}^{TM}, \qquad \vec{B} = \nabla \times \vec{A}^{TE} + \partial_t \vec{A}^{TM}.$$

The 4-vector $A^\mu = (\vec{A}, \phi)$, in Coulomb gauge $\nabla \cdot \vec{A}^{TE/TM} = 0$, solves the 4D wave equation, without charge or current sources, $\Box_{(4)} \vec{A}^{TE/TM}(t, \vec{x}) = 0$. Note that now $\vec{A}^{TE} \cdot \hat{x} = 0 = \vec{A}^{TM} \cdot \hat{x}$, so that it is invariant under Lorentz boosts in the $\hat{x}$-direction:

$$x = \gamma(x' + vt') + \ell(t_0), \quad t = \gamma(t' + vx') + t_0, \quad \vec{x}_{||} = \vec{x}'_{||},$$

thus for in the instantaneous reference frame of the mirror, with $\hat{x} \times \vec{E}'|0 = 0 = \hat{x} \cdot \vec{B}|0$ at the origin/ fixed mirror, at the moving mirror implies

$$0 = \gamma(\partial_t + v\partial_x)\vec{A}^{TE} t_0, x = \ell(t_0), \vec{x}_{||}) \quad \to \quad \vec{A}^{TE} t_0, x = \ell(t_0), \vec{x}_{||}) = const. \equiv 0,$$

as t_0 is arbitrary, and the derivative factor is a total derivative d/dt. Also,

$$0 = \gamma(v\partial_t + \partial_x)\vec{A}^{TM} t_0, x = \ell(t_0), \vec{x}_{||}),$$

is satisfied.

In 3D, we have the same results for one moving wall in the x-direction, $x = \ell(t) + \ell_0$, and $y = \ell_y,\ z = \ell_z$ fixed. Then

$$\Box_{(4)}\Phi(t, \vec{x}) = 0, \quad s.t. \;\; \Phi|_{all\ walls} = 0\,(TE{-}modes), \quad (v\partial_t + \partial_x)\Phi|_{x=\ell(t)} = \partial_x \Phi|_{static\ walls} = 0\,(TM{-}modes).$$

The transverse dimensions can be generalized to any number of dimensions, and can be brane or extra dimensional (bulk) compact spacelike or timelike extra dimensions (the timelike case is dealt with in a section below).
For TE-modes, we get an eigenmode decomposition

$$\Phi(t, \vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^3} q_{\vec{n}}(t) \phi_{\vec{n}}(t, \vec{x}),$$

with eigenfunctions

$$\phi_{\vec{n}}(t,\vec{x}) = \sqrt{\frac{2}{\ell(t)}} \sin\frac{n_x \pi x}{\ell(t)} \sqrt{\frac{4}{\ell_y \ell_z}} \sin\frac{n_y \pi y}{\ell_y} \sin\frac{n_z \pi z}{\ell_z},$$

and eigenvalues

$$\Omega_{\vec{n}}(t) = \pi\sqrt{(\frac{n_x}{\ell(t)})^2 + (\frac{n_y}{\ell_y})^2 + (\frac{n_z}{\ell_z})^2}, \quad k_{||}^2 \equiv (\frac{n_y}{\ell_y})^2 + (\frac{n_z}{\ell_z})^2,$$

with transverse components defining a mass, like a waveguide cutoff frequency, $k_{||} = m = M/\ell_0$, for a dimensionless mass, $M = \sqrt{2}n_{||}\pi/\ell$. The canonical variables satisfy the familiar euquations of motion as in the 1D case

$$\ddot{q}_{\vec{n}}(t) + \Omega^2_{\vec{n}}(t) q_{\vec{n}}(t) + 2\sum_{\vec{m}\in\mathbb{N}^3} M_{\vec{m}\vec{n}}(t)\dot{q}_{\vec{m}}(t) + \sum_{\vec{m}\in\mathbb{N}^3}[\dot{M}_{\vec{m}\vec{n}}(t) - N_{\vec{n}\vec{m}}(t)]q_{\vec{m}}(t) = 0,$$

with

$$\begin{aligned} M_{\vec{n}\vec{m}} &= \int_0^{\ell(t)} dx \int_0^{\ell_y} dy \int_0^{\ell_z} dz \dot{\phi}_{\vec{n}}(t,\vec{x})\phi_{\vec{m}}(t,\vec{x}) \\ &= \frac{\dot{\ell}(t)}{\ell(t)}(-1)^{n_x+m_x}\frac{2n_x m_x}{m_x^2 - n_x^2}\delta_{n_y m_y}\delta_{n_z m_z}, \end{aligned}$$

for $n_x \neq m_x$, and 0 elsewhere, and $N_{\vec{n}\vec{m}} = \sum_{\vec{k}} M_{\vec{n}\vec{m}} M_{\vec{m}\vec{k}}$ as before. One notes for a small-amplitude oscillating mirror, we have produced mode numbers

$$\mathcal{N}_{\vec{n}}(t) = \sinh^2(\vec{n}\cdot\vec{\gamma}_{\vec{n}}\epsilon t), \quad w/ \quad \vec{\gamma}_{\vec{n}} = \frac{\vec{n}\pi^2}{2\Omega^{in}_{\vec{n}}\ell_0^2},$$

and has $\Omega^{in}_k = (3+k)\Omega^{in}_{\vec{n}}$ and other similar resonant coupling modes as found before. Note again the 'cosmological' factor $\frac{\dot{\ell}(t)}{\ell(t)}$, in the coupling matrix.

For TM-modes, we use Neumann BCs with new variables $(x,t) \to (\eta\xi)$, with $\partial_\xi \Phi|_{\ell(\eta)} = 0$, giving a cosine factor in the eigenfunctions' x/ξ-component. The same analysis applies as in the TE-modes.

Let us now extend this to the FLRW metric (and others like RSI and RSII later),

$$ds^2 = -d\tau^2 + a^2(\tau)\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\phi d\phi^2],$$

and define a physical radius, and conformal time as

$$r_{phys}(\tau) = a(\tau)\int_0^r \frac{dr'}{\sqrt{1-Kr'^2}} = v_{phys}(\tau)/H(\tau),$$

(r is the comoving distance), and

$$\eta = \int^{\tau} d\tau'/a(\tau'),$$

so that $ds^2 = a^2(\eta)[-d\eta^2 + d\ell^2]$.

We can write Einstein's equations now as:

$$G_{\mu\nu} + \Lambda_4 g_{\mu\nu} = \kappa_4 T_{\mu\nu}, \quad w/ \quad \kappa_4 = 8\pi/M_{Pl}^2 = 8\pi G_4 = 1/M_4^2,$$

with Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$.
For vacuum, $\Lambda_4 = 0 = T_{\mu\nu}$, we get the Einstein-Hilbert action $S_{EH} = \frac{1}{2\kappa_4}\int \sqrt{-g}Rd^4x$, where the Einstein equations above result from varying with respect to the metric $g_{\mu\nu}$.
Friedmann's equations give the energy-momentum tensor,

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu),$$

which in a local frame $u^\mu = (1, \vec{0})$, gives $T^\mu_\nu = diag(-\rho, p, p, p)$, which is easily generalized to extra dimensions, as we do in the cosmology section later, and will mention some details now.

In extra compact dimensions, we have periodic boundary conditions for scalar and boson fields, while antiperiodic ones for fermionic modes: $\Phi(x^\mu, y^a) = \pm\Phi(x^\mu, y^a + 2\pi R_a)$ and satifying the Klein-Gordon equation

$$[\Box_{(D)} - m^2]\Phi(x^\mu, y^a) = 0, \quad w/ \quad \Box_{(D)} = -\partial_t^2 + \Delta_3 \pm \vec{\partial}_{\vec{y}}^2 = [\Box_{(4)} - m_{\vec{n}}^2]\phi_{\vec{n}}(x^\mu),$$

where the negative sign is used for the extra timelike dimensions, and the effective mass on the brane is

$$m_{\vec{n}} = \sqrt{m^2 \pm \vec{n}\cdot\vec{n}/R^2}$$

for mixed timelike and spacelike extra dimensions (more on this in the next Section). The solutions can be expanded into the eigenmodes above:

$$\Phi(x^\mu, \vec{y}) = \sum_{\vec{n}\in\mathbb{N}^{D-4}} \phi_{\vec{n}}(x^\mu)e^{i\vec{n}\cdot\vec{y}/R}.$$

These are generalized ADD braneworlds in $D = 3+1+n$ dimensions, with n 'large' compact extra dimensions, and $ds^2 = \eta_{AB}dx^A dx^B = \eta_{\mu\nu}dx^\mu dx^\nu \pm \delta_{ab}dy^a dy^b$. We note that the maximum size for one extra dimensions is $R_{max}^{n=1} \approx 10^{-18}\ m$, yet only $\approx 10\ \mu m$ for gravity!

We can look at a warped geometry with one extra dimension for now, with 'warp factor' $a^2(z)$, and $ds^2 = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu \pm dz^2$. We get

$$G_{AB} + \Lambda_5 g_{AB} = \kappa_5 T_{AB}, \quad w/ \quad \kappa_5 = 6\pi^2 G_5 = 1/M_5^2,$$

which $= 0$ in the bulk. For the brane at $y = 0$, $T_{AB} = -\mathcal{T}\delta^\mu_A\delta^\nu_B\eta_{\mu\nu}\delta(z)$, where $\mathcal{T}$ is the brane tension. For $T_{AB} = 0$ in the bulk, we get $(\partial_z a/a)^2 = -\Lambda_5/6$, which implies a negative bulk cosmological constant $\Lambda_5 < 0$!
We have solutions $a(z) = e^{\pm z/L}$, for $\mathcal{T} < 0$ and > 0, respectively, and $1/L = \mp\kappa_5\mathcal{T}/6$ gives the AdS curvature scale, from $\Lambda_5 = -6/L^2$. Now using

$$G_{\mu\nu} = 3\partial_z(a\partial_z a)\eta_{\mu\nu}, \quad \rightarrow \quad \int_{-\epsilon}^{+\epsilon} dz[G_{\mu\nu} + a^2\eta_{\mu\nu}] = -\mathcal{T}\kappa_5\eta_{\mu\nu},$$

for $1 \gg \epsilon > 0$. This gives a jump of $3a\partial_z a|_{-\epsilon}^{+\epsilon} = -\kappa_5\mathcal{T}$, crossing the brane.
Also, in Poincaré coordinates $y = Le^{z/L}$, with the brane now at $y = L$, leads to the conformally flat metric $ds^2 = (\frac{L}{y})^2[\eta_{\mu\nu}dx^\mu dx^\nu + d\vec{y}^2]$.
Also in Randall-Sundrum models, RSI and RSII, we have conformal time $d\eta = \sqrt{1 - (d\vec{y}_b/dt)^2} \equiv dt/\gamma$, for the brane at $\vec{y}_b$. We write the total energy-momentum tensor $T^\mu_{\nu,tot} = T^\mu_\nu - \mathcal{T}\delta^\mu_\nu$, leads to

$$\kappa_5(\rho + p) = -\frac{2L\partial_\eta H}{a\sqrt{1 + L^2H^2}}, \quad w/ \quad H = \partial_\eta a/a^2 = -\gamma v/L,$$

e $\gamma = 1/\sqrt{1 - v^2} = \sqrt{1 + L^2H^2}$, for a relativistic moving brane. This implies

$$H^2 = \frac{\kappa_5^2}{18}\mathcal{T}\rho(1 + \frac{\rho}{2\mathcal{T}}) + \frac{\kappa_5^2\mathcal{T}^2}{36} - \frac{1}{L^2}, \quad \rightarrow \quad \partial_\eta\rho = -3Ha(\rho + p).$$

This then gives $\Lambda_{4,brane} = \kappa_5^2\mathcal{T}^2/6 - 3/L^2 = 0$, by the fine-tuning condition, and so the 4D Friedmann equations are recovered if $\kappa_4 = \kappa_5^2\mathcal{T}/6$, and also $L = 6/\kappa_5\mathcal{T}$, which means $\kappa_4 = \kappa_5/L$, as expected. Hubble's parameter then has the dependence $H^2 = \frac{\kappa_4\rho}{3}(1 + \frac{\rho}{2\mathcal{T}})$. From current observations, the brane tension $\mathcal{T} > (1\ MeV)^4$ and $M_5 > 10^4\ GeV$. The solution has $a \propto \tau^{1/4}$ for early times and high energies, while $a \propto \tau^{1/2}$ for later times at lower energies.
One can also observe graviton production in this framework from the dynamical Casimir effect, as in [], but I'll leave this for another paper.

In a similar, but different, method, we can look at coupled modes between the piston's or mirror's vibrations due to the particle or photon field modes produced by their motions. This will involve other annihilation/creation operators for the piston's motion $\hat{b}_n, \hat{b}_n^\dagger$, with Hamiltonian the sum of the a's and b's oscillation terms and an interaction term between them arises. We look at this here, as another way to emerge an effective 4D QED from 6D, with and without extra time dimensions, with the advantages of extra time dimensions pointed out.

3.2 Resolving issues of extra time dimensions: maintaining causality, non-locality, ghosts and tachyons

Here

4 Casimir effect with extra time dimensions.

Following methods in [1], we describe techniques for regularizing Casimir energies in both spacelike and timelike dimensions.

The motivation for extra time dimensions, as explained above, will be not only in a possible resolution of the cosmological constant problem, but also many other Problems of Physics, along with its symmetry and simplicity, and mathematical and physical applications as well. I announce again, that those who refute extra time dimensions assume too much: time and space need not be treated in the same way. For example, basic constraints that are intuitive are that brane-world lines remain the same, just can follow curves with tangent vector taken as velocity, etc, and so no dimensional increase occurs. Secondly, extra time dimensions, just like spacelike ones if they exist, are small and only accessible by energies of order of the mass of the lightest Kuluza-Klein particle's mass, $M_{KK,1} \approx \pi/L$, where L is the size of the extra dimension. Problems of non-locally, causality, and unitarity can be prevented and rid of by the methods we present here. [6] Here, mostly, we take compactified extra dimensions of size L, or T of whatever space or timelike dimension, with periodic boundary conditions for bosons, and antiperiodic ones for fermionic modes, respectively,

$$\Phi(x^\mu, \vec{t}) = \Phi(x^\mu, \vec{t} + 2\pi\vec{\beta}),$$

$$\Psi(x^\mu, \vec{t}) = -\Psi(x^\mu, \vec{t} + 2\pi\vec{\beta})$$

where $x^\mu \in \mathcal{M}^{3,1}$, and $\vec{t}$ belongs to the torus bundle $T^2\mathcal{M}^{3,1}$, for periodic extra time dimensions of sizes $\vec{\beta} = (\beta_1, \beta_2)$, and similarly for spatial dimensions and extra time dimensions. The time/thermal modes are the Matsubara frequencies, $\omega_i = 2\pi n/\beta_i$, for the nth mode. This extra dimensional model can be pictured in Figure 1, if viewed in two spatial dimensions.

Periodic time is not a strange idea: temperature itself is an extra timelike variable, just inverse imaginary (compare Schrodinger's equation to the heat equation, or waves in space turn to thermal distributions from the partition function instead of the Fourier spectrum...). Indeed I wrote above β for the inverse temperature, as a (complex in general) time dimension size or scale

I will call light KK modes 'W bosons' as they have periodic boundary conditions, and acquire integer charge related to its coupling to electromagnetic fields (chromodynamic in general) in QED or Yang-Mills theory, with (including winos) or without supersymmetry (no gluinos either). Higher massive W's have mass $nM_W = 2\pi n/L$. We can always take $L \to \infty$ so it has nothing to do with size, it's just a smaller scale for the extra dimensions.

[6] I also have to emphasize that small extra dimensions does not mean bounded! On the small scale of higher and higher energy the dimensions are unbounded like for us, beyond locally. Do not confuse 'size' with 'scale'!.

Let's start with a simple case of a 5D space with an extra dimension of either signature $\epsilon = \pm 1$ (taking the convention of $-$ for timelike and $+$ for spacelike), with metric $g^{MN} = (\eta^{\mu\nu}, \epsilon)$, where $\eta^{\mu\nu} \approx (1,1,1,-1)$ is flat enough for now. I'll use uppercase Latin letters $M, N, ...$ for the bulk and Greek indices $\mu, \nu, ...$ for the brane ($\mathcal{M}^{3,3}$). Lowercase Latin indices $i, j, ...$ will be used for the extra dimensions. The spatial dimension in general will be d, while that of the bulk, $D > d$, leading to $D - d$ extra dimensions.

Now add a scalar field ϕ of mass m. It satisfies the Klein-Gordon equation (the equation of motion for the 5D action of the scalar field):

$$(\Box_D - m^2)\phi = 0, \tag{4.1}$$

where, in general, $\Box_D = g^{MN}\partial_N\partial_M = \eta^{\mu\nu}\partial_\mu\partial_\nu + \vec{\epsilon} \cdot \nabla_y^2$, for the case of more than one extra dimension. In this case the last term is just $\epsilon\partial_y^2$. The scalar field has periodic boundary conditions in the extra dimension, and so can be written as a Fourier series:

$$\phi(x^\mu, y) = \sum_{n\in\mathbb{Z}} \phi_n(x_n^\mu) e^{iny/L},$$

which makes (1) become

$$[\Box - (m^2 + \epsilon n^2/L^2)]\phi_n(x^\mu) = 0, \tag{4.2}$$

where $\Box$ is the standard 4D D'Alembertian from $\eta^{\mu\nu}$. One notes a spectrum of KK masses

$$m_n^2 = m^2 + \epsilon n^2/L^2$$

, which are quite massive for small extra spacelike dimensions. Note though the KK masses decrease and become imaginary (tachyonic) for any Matsubara mode higher than $k \in \mathbb{Z}$: $k^2/\beta^2 > m^2$. This isn't to worry, as we can integrate them out, or just throw them away [1]: they aren't observable, they don't cause any divergences like in QFT's estimate of the 'mass' of the vacuum being 120 orders of magnitude higher than observed from the cosmological constant. In a moment, we will see how this 'cosmological constant problem' can be resolved, possibly, using extra time dimensions. This is opposite the case of extra spatial dimensions magnifying even worse the discrepancy from theory to experiment.

Note the shift in energy just to note for later:

$$E_n - \vec{p}^2 - m^2 - \epsilon\frac{n^2}{L^2} = 0,$$

and the fear of tachyons can be resolved just by taking them as excitations of non-tachyonic modes: $E_n^{2'} \equiv E_n^2 + n^2/L^2$. Furthermore, we find that tachyonic modes decay quickly:
Say a tachyonic mode $\phi_n(x) \propto \exp(im_n t)$ and $\epsilon = -1$, and $|n| > Lm$ as above. Then for such a tachyonic mode $\phi_n(t) \propto \exp(-\sqrt{n^2/L^2 - m^2}t)$, has $|\phi(t)|^2 \to \emptyset$ decays into the vacuum and pure nothingness everywhere in time scale $\tau_n \approx L/\sqrt{n^2 - L^2m^2}$. And if taking the

'tachyonless' perspective, the modes remain unitary and $\phi_n \propto \exp(im_n t)$ certainly is unitary $|\phi_n(t)|^2 = 1$, and probability is conserved.

As [1] discusses, this reproduces 4D from the 5D Poisson equation (for potentials in both electromagnetism and gravity) $(\nabla^2 + \epsilon\partial_y^2)V(x^\mu, y) = 4\pi\rho(x^\mu, y)$, yields the modified photon propagator:

$$D_{\mu\nu} = -i\eta_{\mu\nu}/(p^2 - \epsilon n^2/L^2 + i0^+),$$

where $p^2 = E^2 - \vec{p}^2$ is the four-momentum. Unphysical poles exist for $\epsilon = -1$ for $p = \pm in/L$, and imaginary parts to gravity can result in dark energy or similarly 'antigravity'. The self action becomes non-static, but in $3+1$ dimensions Newton's laws still hold, $\nabla^2\phi(x^\mu) = 4\pi\rho(x^\mu)$. I won't discuss this further and refer the reader to [1]. The idea of the 'tachyonless' approach, we get the usual propagator

$$D^*_{\mu\nu} = -i\eta_{\mu\nu}/(p^{*2} + 0^+),$$

for $p^{*2} = E^{*2} - p^2$ and $E^{*2} = E^2 - \epsilon n^2/L^2$. I summarize the difference between extra space and extra time dimensions in this approach as:
Extra space dimensions $\Longleftrightarrow$ infinite tower of massive KK modes.
Extra time dimensions $\Longleftrightarrow$ excited states of the same particle, and a finite number of Matsubara modes.

In computing vacuum energies in pure spatial dimensions, a UV cutoff $\Lambda \approx M_P$ of the order of the Planck mass is introduced as one method of regularization. For 3D+1 extra (space or time dimension):

$$\langle\rho_v\rangle = \sum_{n\in\mathbb{Z}}(2\pi)^{-3}\int_0^\Lambda \frac{d^3p}{2}\sqrt{p^2 + m^2 + \epsilon n^2/L^2} \approx \Lambda^4/16\pi^2 = 2\times 10^{71}GeV^4,$$

where the $n = 0$ term was taken to be dominant for a spatial extra dimension, giving the quantum field theoretic value of the vacuum energy, and assuming $\Lambda >> m$. Zeta function regularization techniques can give the result for the summation over all modes, as in [2], [3], but these only make the cosmological constant problem worse. We are tempted, under the unitary, causal, tachyonless approach of extra time dimensions, we take $L \to \infty$ and replace the sum to an integral: $L\int d\xi$, where $n/L \to \xi$. For d extra multiple time dimensions, of sizes L_i, we get $d\ \xi_i \leftarrow n_i/L_i$. We choose a different cutoff for the ξ_i, λ, so as to keep $\langle\rho_v\rangle$ real. Hence it is instead an IR cutoff, giving a torus T-duality of space $\to$ time cycles, giving a UV-IR duality. And so the vacuum density in d extra time dimensions becomes

$$\langle\rho_v^*\rangle = \frac{L}{(2\pi)^3}\int_0^\Lambda d^3p\int_0^\lambda d^d\xi\sqrt{p^2 + m^2 - \vec{\xi}^2},$$

where $\lambda^2 = \sum_{i=1}^d \xi_i^2$.

Looking at the case of one extra time dimension, we perform the ξ-integral to obtain:

$$\langle \rho_v^* \rangle = \frac{L}{(2\pi)^3} \int_0^\Lambda \frac{d^3p}{2} [\lambda \sqrt{p^2 + m^2 - \lambda^2} + (p^2 + m^2) \sin^{-1}(\frac{\lambda}{\sqrt{p^2 + m^2}})],$$

where $\lambda \leq \sqrt{p^2 + m^2}$. The integral can be evaluated exactly shortly, but in the regime $\lambda \approx m << \Lambda$, the integral can be approximated as

$$\frac{L\lambda}{(2\pi)^3} \int_0^\Lambda d^3p \sqrt{p^2 + m^2} \approx L\lambda\Lambda^4/8\pi^2,$$

which can help resolve the cosmological constant problem for

$$\lambda^d \Lambda^4 \prod_{i=1}^d L_i \approx 10^{-118}.$$

So for small enough extra time dimensions L_i, and small enough $\lambda << \Lambda$, more than one time dimension seems to be supported.

In order to find the effective 4D dynamics from compactification from the $3+d+1$ theory, zeta function regularization can be used to find new properties, at higher energies, for extra spatial dimensions [] and special results for extra time dimensions. Doing the full regularization can give great insight into effective QED and QCD for any non-Abelian gauge group (see [] for dealing with Lie groups in general, and treating them in determining effective theories from compactification).

We begin with an example simple model to show the zeta function regularization method for extra spatial dimensions, and then extend to more exact results with extra timelike dimensions.

Let us begin with a toy model calculation or two for dynamics in $\mathcal{M}^{3,3}$.

4.1 Mechanics and Dynamics in $\mathcal{M}^{3,3}$ and new results

In classical mechanics with multiple time dimensions, multiple 'Hamiltonians' appear according to the canonical quantization and dual parameters (for each spatial coordinate q_i there is a dual momentum p_i that may be conserved possibly, and for each time coordinate there is an energy that may be conserved). We have in the case of 6D $SO(3,3)$:

$$\frac{dq_i}{dt_j} = \frac{\partial H_j}{\partial p_i}, \qquad \frac{dp_i}{dt_j} = -\{H_j, p_i\}_{PB} = -\nabla_i H_j,$$

where the bracket is the Poisson bracket and follows from the same Liouville theory argument in phase space density ρ evolution in any spacetime:

$$\frac{d\rho}{dt_i} = -\{H_i, \rho\}.$$

Now let's look at some examples:

Here we take a six dimensional spacetime coordinate vector $\hat{x}^\mu = (\vec{x}, c\vec{t})^{T\mu}$, with indices lowered with the $SO(3,3)$ metric with spatial indices negative. This is more consistent from a relativistic viewpoint with a symmetrization of space and time coordinates. The fact also that we perceive one time dimensions is the fact that the universe surrounding us locally all observed macroscopic bodies move along parallel time trajectories, and thus only one scalar parameter, say a length along these trajectories is enough to describe physical processes and evolutions in our world. However, for objects far apart and due to peculiarities of our Universe's evolution, it is possible that mutual correlation of times in objects can be lost and that they can have their own time directions on the large scale. We will look here at some examples of relativistic and gravitational phenomena in multitime space, so first, let us find the equations of motion of particles in multitime space via the least action principle

$$\delta S = \int \delta\mathcal{L}(\hat{x}, \hat{u}, t)dt = 0, \qquad \hat{u} = d\hat{x}/ds = \gamma c^{-1}(\vec{v}, c\vec{\tau})^T = \gamma\hat{V}/c,$$

where as usual $\gamma = 1/\sqrt{1 - v^2/c^2}$ and the parameter $ds = cdt/\gamma$, and $\vec{\tau}^2 \equiv \sum_i (dt_i/dt)^2 = 1$ in this parametrization, with scalar time t read along the trajectory of the particle $\vec{t} = \vec{t}(t)$. Performing the variation of the Lagrangian leads to the Euler-Lagrange equation of motion in multitime space:

$$\frac{\partial L}{\partial \hat{x}} - \frac{d}{ds}(\frac{\partial L}{\partial \hat{u}}) = 0,$$

where $\frac{\partial}{\partial\chi(t)}$ to each other. The equations of motion are then

$$d(m\vec{\beta})/dt = qc^{-1}\vec{E}(\vec{x})\chi, \qquad d(m\vec{\tau})/dt = qc^{-1}\vec{\beta}\cdot\vec{E}(\vec{x})\vec{\tau}, \qquad dm/dt = qc^{-1}\vec{\beta}\cdot\vec{E}(\vec{x})\chi,$$

where $\vec{\beta} = \vec{v}/c$ and $\vec{E} = -\nabla\phi$.
Multiplying these expressions respectively by $\vec{\beta}$ and $\vec{\tau}'$, gives a symmetric equation:

$$\vec{\beta}\cdot d(\gamma\vec{\beta})/dt = \chi d(\gamma\chi)/dt, \quad \to \quad d\vec{\beta}^2/(1-\vec{\beta}^2) = d\chi^2/(1-\chi^2),$$

where the relation $d\gamma/dt = \gamma^3\vec{\beta}d\vec{\beta}/dt$ is used. Integrating this, we get

$$1 - \chi^2 = \alpha(1 - \vec{\beta}^2) \qquad \gamma^2 = \gamma_0^2(1-\chi_0^2)/(1-\chi^2),$$

taking the integration constant α and the values of the variables at $\vec{x}_0 = 0$, $\alpha = \gamma_0^2(1-\chi_0^2)$. Thus, if $\chi_0 = 1$, and the time trajectories of the field and particle are initially aligned, they will stay that way, whether we are in one or three dimensional time. As the particle speeds up $\beta \to 1$, the particle's time vector cosine $\chi \to 1$ and the electric field time vector forces the particle to align its time vector with its $\vec{\tau}'$!

Now substituting the equation for $\chi(\vec{\beta})$ into the equation of motion, we get:

$$\frac{d\chi}{(1-\chi^2)^{3/2}} = \frac{qE}{mc^2}\frac{dx}{(1-\chi^2)^{1/2}},$$

with solution

$$\chi\sqrt{(1-\chi_0^2)/(1-\chi^2)} = \omega x + C,$$

where $\omega = qE/mc^2$, and the integration constant $C = \chi_0/\sqrt{1-\chi_0^2}$. Thus,

$$\chi(x) = (\chi_0 + \omega x)/\sqrt{1 + \omega x(\omega x + 2\chi_0)}.$$

We can also write in terms of γ,

$$\gamma^2 = \gamma_0^2[1 + \omega x(\omega x + 2\chi_0)], \qquad \chi = \frac{\gamma_0}{\gamma}(\omega x + \chi_0),$$

and the non-relativistic approximation:

$$\gamma \approx \gamma_0 + \chi_0 \omega x, \qquad \chi \approx \chi_0 + \omega x(1-\chi_0^2),$$

or the ultra relativistic approximation $\omega x \gg 1$:

$$\gamma \approx \gamma_0 \omega x, \qquad \chi \approx (1-\chi_0^2)/2\omega^2 x^2.$$

Again, as $\chi_0 \to 1$, the results of the one-time theory are obtained. Again, the expressions in three-time theory differ only by a decrease in effective charge $q \to q\chi_0$.

The time dependence $x(t)$ can be found from the relation $dx/dt = c\sqrt{\gamma(x)^2-1}/\gamma(x)$, implying an integral equation implicit for $x(t)$:

$$ct(\chi, x) = \int_0^x \gamma(x)/\sqrt{\gamma(x)^2-1}dx,$$

with the last expression $\Delta t/t(1,x) = [t(\chi,x) - t(1,x)]/t(1,x)$ showing the delay of the accelerated particle at point x in comparison with a one-time theory to be compared to an experiment, becomes significant for large deviations of time vectors ($\chi <\approx 0.5$), but also for larger electric fields and less massive particles, as the expressions for χ have distances occurring in the combination Ex/m. For example, in a proton beam the scale of distances x must be up to two thousand times and a significant delay can be observed up to several hundred meters.

For the case of a massive particle in a central field, say that for the Sun with a mass M and Sun time vector $\vec{\tau}_s$,

$$\hat{E} = \kappa M \vec{\tau}_s \hat{r}/r^2.$$

One can show that []

$$\chi = \frac{\gamma_0}{\gamma}\chi_0 + Q(r), \qquad Q(r) = \kappa c^{-2} M(1/r - \frac{\gamma}{\gamma_0 r_0}) \approx \kappa c^{-2} M(1/r - 1/r_0),$$

where χ_0 and γ_0 are taken at the perihelion r_0. We then determine $\chi(r)$ and $\gamma(r)$ as in the electric field case but with $\omega w \to Q(r)$ and a deviation of the planet's time trajectory from that of the Sun's results with an additional perihelion precession $\Delta\theta \approx 1 - \chi_0^2$. With the experimental value of $\Delta\theta$, we then know that $\chi_0 \approx 1$.

5 Electroweak unification on $\mathcal{M}^{3,3}$

6D theories, with or without extra compact time dimensions lead to a beautiful symmetric unification in electrodynamics (with magnetic monopoles treated on a dual setting) with useful generalities. From [] we can find how this works elegantly, and the consequences of extra dimensional fields present, and how extra time dimensions change these results (which remain the same up to a field redefinitions, complexifications, etc. Following [], we then extend this to full electroweak symmetry breaking from a 6D unifies theory, with or without Higgsing. The next Section 6 deals with supersymmetric unification in 6D, with it's usefulness in solving chirality problems, and reduction to 4D enhances the number of supersymmetries by at least one. This gives a predictive model for gauginos and gluinos, and their mass generation or prediction in terms of the ectra compact dimensions.

We begin with an $SO(3,3)$ unification of electrodynamics, in general and discuss the results in special cases, first turning to another method of effective QED from 6D models with non-perturbative objects (such as vortices) yielding the standard 4D model and modifications for experimental detection of extra dimensions.

5.1 Einstein-Maxwell theory in 6D (with vortex scenario) leads to effective 4D QED.

We consider Einstein-Maxwell gravity in 6D with a complex scaler field Φ, of cherge e, and two chiral fermions

$$\Psi_1 = \frac{1+\Gamma_7}{2}\Psi_1, \qquad \Psi_2 = \frac{1-\Gamma_7}{2}\Psi_1,$$

with $U(1)$ charges e_1 and e_2 in general, and to be determined later. These come from the 4_+ and 4_- representations of $SO(1,5)$ or $SO(3,3)$, and Γ_7 is the 6D chirality matrix. We then get a general action with Yukawa interactions in 6D:

$$S = \int d^6x\sqrt{-G}[\frac{1}{\kappa^2}R - \frac{1}{4}F_{MN}F^{MN} - (D_M\Phi)^\dagger D^M\Phi - U(\Phi) + \sum_{1,2}\bar{\Psi}_i\Gamma^A E_A^M\nabla_M\Psi_i + g\bar{\Psi}_1\Phi\Psi_2 + h.c.],$$

with a sechsbein E_A^M, a 6D Newton's constant κ, and the usual covariant derivative $D_M = \partial_M + ieA_M$ on scalars, but $\nabla_M = \partial_M - \Omega_M + ie_iA_M$ for the fermions Ψ_i respectively. Here $\Omega_M = \frac{1}{2}\Omega_{M[AB]}\Sigma^{AB}$ is the spin connection $\Sigma \in \mathfrak{so}(1,5)$ or $\mathfrak{so}(3,3)$, for $\mathfrak{so}(n,m) = Lie(SO(n,m))$ etc., with generators $\Sigma_{AB} = \frac{1}{4}[\Gamma_A,\Gamma_B]$, for six 8×8 curved-space Dirac matrices [] with anticommutation relations: $\{\Gamma_A,\Gamma_B\} = 2\eta_{AB}$, and $\Gamma_7 = diag(1_4,-1_4)$.

For an anomaly-free gauge theory, free from any combinations of gauge and gravitational anomalies, we require [] $e_1^2 = e_2^2$. We take here (as in the case of matter-antimatter) $e_1 = -e_2$.[7] Now the Yukawa coupling (real here) is non-zero only if $e_1 - e_2 = e \quad\rightarrow\quad e_1 = e/2 = -e_2$, and we obtain fractional charges for the fermions, in terms of the (quantized, as can be shown)

[7] $e_1 = e_2$ is the same case really, so extra time dimensions case is similar here, as $4_+ \approx 4_+^*$.

scalar field charge e.

We can add gauge-invariant mass terms for the fermions as

$$L_m = m\bar{\Psi}_1^c\Psi_2 + h.c.,$$

where $\Psi^c = C\bar{\Psi}^T$, for charge conjugation operator C. But if we wish for fermion number conservation (as must hold in the absence of magnetic monopoles or other non-perturbative objects), we need rather

$$L_m = m\bar{\Psi}^c\frac{1-\Gamma_7}{2}\Psi + h.c.,$$

where we have unified the spinors into one 8-component spinor $\Psi = \Psi_1 \oplus \Psi_2$, which has a well defined charge $\iff e_1 = e_2$, and so the true Dirac spinor must have charge conjugation and $\Psi_D = \Psi_1 \oplus \Psi_2^c$, and the fermion mass term is

$$L_m = m\bar{\Psi}_D\Psi_D + h.c.$$

Now let's look at a vortex solution in 6D, which we can localize on the 4D brane, whose bosonic solutions give the Nielsen-Olesen (NO) vortex solutions with metric

$$ds^2 = e^{A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + e^{B(r)}a^2d^2\theta,$$

for a cylindrical bulk, of radius a for $\theta \in [0, 2\pi]$, and fields taking solutions

$$\Phi = f(r)e^{in\theta}, \qquad aeA_\theta = (P(r) - n)d\theta,$$

with n is the vortex number (winding number), and the functions $f(r)$ and $P(r)$ satisfy the boundary conditions[8]:

$$f(0) = 0, \quad f(\infty) = f_0 \neq 0, \quad P(0) = n, \quad P(\infty) = 0,$$

where f_0 is (the modulus) of the minimum of $U(\Phi)$, which is approached as $r \to \infty$. Furthermore, in the metric,

$$A(0) = 1, \qquad A(r \to \infty) = B(r \to \infty) = -2cr, \qquad B(r \to 0) = 2\ln\frac{r}{a},$$

for a constant $c > 0$, which accompanied with a, depend on Newton's constant κ, the 6D cosmological constant $\approx U(\Psi_{min} = f_0)$, and Abelian-Higgs parameters (such as Higgs vacua). More details of 6D cosmology will be covered in a later Section. We note here that $r \to 0$ is a flat space with a vortex core $\approx a$, and gets curved as r increases away, and as $r \to \infty$ the bulk becomes anti-de-Sitter, AdS. From [] it is shown that gravity fluctuations are localized

[8] As before, Greek indices $\mu, \nu, \ldots$ are 4D spacetime indices.

on the 4D brane at the vortex core.

We can now look at the fields localized on the 4D brane:

$$V_\mu = \frac{1}{ae} P(r) W_\mu(x^\mu, r), \qquad h_{\mu\theta} = e^{B(r)} W_\mu(x^\mu, r),$$

with the latter a KK-vector field. We can insert these into the spin-1 part of the action to get the action for the W_μ-field ('photon' field):

$$S[W] = -\frac{1}{2a^2}\frac{2\pi a}{e^2}\int_0^\infty dr e^{B(r)/2}(P^2(r) + \frac{a^2e^2}{\kappa^2}e^{B(r)}\int d^4x[(\partial_\mu W_\nu)^2 + e^{-A(r)}(\partial_r W_\mu)^2].$$

For r-independent W's, we can write $S[W] = -\frac{1}{2g^2}\int d^4x(\partial_\mu W_\nu)^2$, for the 4D gauge coupling constant

$$\frac{1}{g^2} = \frac{2\pi a}{e^2}\int_0^\infty dr(e^{B(r)/2}(P^2(r) + \frac{a^2e^2}{\kappa^2}e^{B(r)})),$$

and the Maxwell and KK-fields are localized to $r \approx max\{1/c, 1/M_W, 1/M_H\}$, where M_W and M_H are the bulk vector and scalar masses without gravity. Note also $\sqrt{P^2(r) + \frac{a^2e^2}{\kappa^2}e^{B(r)}}W_\mu(x^\nu) \to W_\mu(x^\nu)$ for $r \to 0$, and vanishes as $r \to \infty$. More details can be found in [].

We now turn to the 4D fermionic sector. For $g \neq 0$ and $e_1 = -e_2 = e/2$, the fermionic equations of motion on the brane give the Dirac equation of the form

$$[e^{-A/2}\Gamma^\mu\partial_\mu + e^{-B/2}\Gamma^\theta(\frac{1}{a}\partial_\theta + i\frac{e}{2}\Gamma_7 A_\theta + g\frac{1-\Gamma_7}{2}\Psi + g\frac{1+\Gamma_7}{2}\Psi^*]\Psi = 0.$$

Note that also the sechsbein transformations ensure aperiodicity of the fermion in the θ-direction: $\Psi(\theta) = -\Psi(\theta + 2\pi)$, and we can give the 4D Γ-matrices in terms of the classic Dirac matrices γ_μ with $\gamma_5 = diag(1_2, -1_2)$:

$$\Gamma_\mu = \gamma_\mu \otimes \sigma_1, \qquad \Gamma_r = \gamma_5 \otimes \sigma_1, \qquad \Gamma_\theta = 1_2 \otimes \sigma_2.$$

We now propose a solution $\Psi^T = (\psi_1, \psi_2)$ (now periodic in θ) that satisfies

$$\Psi =$$

and substituting into the Dirac equation, for 4D zero-modes $\gamma^\mu\partial_\mu\psi_i = 0$, gives

$$e^{i\theta\gamma_5}(\partial_r + i\gamma_5 e^{-B/2}\partial_\theta/a)\psi_1 + g\Phi\gamma_5\psi_2 = 0$$

$$e^{-i\theta\gamma_5}(\partial_r - i\gamma_5 e^{-B/2}\partial_\theta/a)\psi_2 + g\Phi^*\gamma_5\psi_1 = 0.$$

The same result holds for chiral $\Psi_i = (\psi_1^L, \psi_i^R)^T$, where we get

$$e^{i\theta}(\partial_r - ie^{-B/2}\partial_\theta/a)\psi_1^L + g\Phi\gamma_5\psi_2^L = 0$$

$$e^{-i\theta}(\partial_r + ie^{-B/2}\partial_\theta/a)\psi_2^L + g\Phi^*\psi_1^L = 0,$$

and similarly for the right chirality,

$$e^{-i\theta}(\partial_r - ie^{-B/2}\partial_\theta/a)\psi_1^R - g\Phi\gamma_5\psi_2^R = 0$$

$$e^{i\theta}(\partial_r + ie^{-B/2}\partial_\theta/a)\psi_2^R - g\Phi^*\psi_1^R = 0.$$

Now we get the 4D fermionic action:

$$S[\Psi] = \int dr \int d\theta\sqrt{-G}\sum_{1,2}\bar{\Psi}_i\Gamma^A E_A^M\nabla_M\Psi_i =$$

or

$$= \int dr \int d\theta \sum_{1,2}[N_L(r)\bar{\psi}_i^L\gamma^\mu\partial_\mu\psi_i^L + N_R(r)\bar{\psi}_i^R\gamma^\mu\partial_\mu\psi_i^R]$$

where

$$N_{L(R)}(r) = \exp[-A(r)/2 + \int^r d\rho e^{-B(\rho)/2}(1/a + (-)eA_\theta(\rho))],$$

which $\to r$ as $r \to 0$, and $N_{L(R)}(r \to \infty) \to \exp(cr + \frac{1-(+)n}{ac}e^{cr})$.

Now pulling out the θ dependence, we can factor the fields into phases, radial wave functions, and the usual 2-component Weyl spinors $\chi_m^{L,R}(x^\mu)$ of $SO(1,3)$, with various 4D charges $q_m = (m-(n-1)/2)q$:

$$\psi_1^{L(R)} = e^{im\theta}u_m^{L(R)}(r)\chi_m^{L(R)}(x^\mu)$$

$$\psi_2^{L(R)} = e^{i(m+1-n)\theta}v_m^{L(R)}(r)\chi_m^{L(R)}$$

where the radial wave functions $u_m^{L(R)}(r)$ and $v_m^{L(R)}(r)$, and more details are found in [].

We can summarize the results obtained by compactification in this vortex geometry, beyond the localization of gravity, and possible charges one gets, some other details. We find an anomaly free 6D gauge theory yields 4D electrodynamics of charged particles interacting with photons and gravitons. Localized charged massless fermions and massive Majorana fermions interact with photons near the NO-vortex core. Later, we will extend this for 6D SYM (supersymmetric Yang-Mills) theory. But first, let us look to higher gauge group, and see how non-Abelian anomaly cancellation occurs from 6D to 4D compactification.

We now turn to a hopeful 6D approach to non-Abelian anomaly cancellation, leaving Abelian, mangeable, anomalies in 4D, before going to the 6D Einstein Maxwell theory, with non-perturbative objects, specifically vortices, and how an effective QED can result in 4D.

5.2 Non-Abelian anomaly cancellation in $6D \to 4D$ and consequences of compactification.

Here, in general, we investigate an even dimensional spacetime of dimension $m = 2\ell$, and add the point at infinity to look at even dimensional spheres, as in []: $\mathbb{R}^m \cup \{\infty\} \approx S^{2\ell}$, with a semisimple gauge group G that is simply connected (as is $SU(N)$ and most Lie groups), so $\pi_1(G) \approx 1$ like $\pi_1(SU(N)) \approx 1$. We then take a one-parameter family of gauge transformations, periodic such that $\forall g \in G : g(0, x) = g(2\pi, x) = e$, which can be generalized WLOG to any basepoint $x_0 \in S^m$ with $g(\theta, x_0) = e$ by rotations.

g here is now a map $g : S^1 \times S^m \to G$. These maps are characterized by the fundamental group $\pi_{m+1}(G)$ (giving the mapping classes). We have, also via the smash product $S^1 \wedge S^m \approx S^1 \times S^m / S^1 \vee S^m \approx S^{m+1}$, where the algebraic quotient is $S^1 \# S^m$, as a connected sum at general point (x_0, y_0), $\approx (\{x_0\} \times Y) \sqcup (X \times \{y_0\})$. The angle $\theta \in S^1$ can be thought just by rotating an S^m about (x_0, y_0), as can be visualized as an S^{m+1}, as easily seen from rotating a growing and shrinking circle an angle θ about (x_0, y_0).

See Figure (),

5.3 Maxwell in 6D: Electrodynamics over $SO(3,3)$ where electric and magnetic matter in 4D are free fields in 6D!

We turn to a simpler case: let us derive a form of Maxwell's equations in 6D with particularly simple pure (vacuum) dynamics, that reduce to Maxwell's equations in 4D with matter of electric and magnetic charge sources. This follows [210]. In vacuum the field equations of Maxwell come straight from Lie theory (Bianchi identity) and cohomology (closedness for no sources):

$$\partial_n F^{mn} = 0, \quad \partial_p F_{mn} + \partial_m F_{np} + \partial_n F_{pm} = 0,$$

where roman indices indicate indices over $\mathbb{R}^{3,3}$. 4D spacetime indices will be Greek letters as usual μ, and the metric is $g = diag(-1, -1, -1, 1, 1, 1)$ as stated in the Introduction, spatial indices first. The 6D Maxwell electromagnetic field tensor, contains the usuall 4D Maxwell field tensor, but has additional components: a 4-vector S^μ, a 4-pseudovector T^ν, and a 4-pseudoscalar η in the language of geometric algebra. It is given in the Figure below component-wise in matrix form. Using the notation of the Introduction, these equations in

$$F^{mn} = \begin{bmatrix} 0 & B_z & -B_y & -E_x & -S_x & -T_x \\ -B_z & 0 & B_x & -E_y & -S_y & -T_y \\ B_y & -B_x & 0 & -E_z & -S_z & -T_z \\ E_x & E_y & E_z & 0 & -\sigma & -\tau \\ S_x & S_y & S_z & \sigma & 0 & -\eta \\ T_x & T_y & T_z & \tau & \eta & 0 \end{bmatrix} = \begin{bmatrix} F^{\alpha\beta} & -S^\alpha & -T^\alpha \\ S^\beta & 0 & -\eta \\ T^\beta & \eta & 0 \end{bmatrix}$$

terms of 4D components read

$$F^{\mu\nu}_{,\nu} = \partial_u S^\mu + \partial_v T^\mu \equiv J^\mu, \quad S^\mu_{,\mu} = \partial_v \eta, \quad T^\mu_{,\mu} = -\partial_u \eta,$$

which gives 6 equations, whereas the 6D Bianchi identity leads to 20 equations:

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0,$$

$$\partial_u F_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu, \quad \partial_v F_{\mu\nu} = \partial_\mu T_\nu - \partial_\nu T_\mu,$$

$$\partial_\mu \eta = \partial_u T_\mu - \partial_v S_\mu \equiv K_\mu.$$

These 26 equations seem to overdetermine our only 15 field variables, however, some of these equations can be interpreted as initial conditions on the fields. For example, take $u = x^5$, and notice that the number of equations that don't involve derivatives with respect to it are $1 + \binom{6-1}{3} = 11$. One from the first set, and 10 from the second since we have only 5 indices not =5 to choose 3 out of in the Bianchi identity. 26-11=15 field variables and we are happy just by giving initial u conditions!

We wish also to keep proper transformations of time, and so time reversal must be accompanied by a u- or a v-reversal to keep the determinant of the transformation =1. Thus, if u were a pseudoscalar, then v would be a scalar. Why not keep this convention so the 'sources' J^μ and K^μ are 4-vectors, and 4-pseudovectors respectively, and thus η is in fact a 4-pseudoscalar. Geometric algebra is going to help us out a lot here! Notice what happens here is that from these time transformations, we have now a law of matter and antimatter on the same footing! Their sum is conserved and we have a zero energy vacuum always as opposed to Dirac's infinite energy vacuum 'sea'!

The total antisymmetry of the Bianchi identity implies that F_{mn} must be antisymmetric and derivable from a 6-potential: $F_{mn} = \partial_n A_m - \partial_m A_n$. We can assume a (Lorentz) gauge condition on the underdetermined potential, such as

$$\partial_m A^m = 0, \quad \text{and write} \quad A_m = (\vec{A}, \phi, \chi, \psi).$$

In terms of these components we can write the general 3+1-D Maxwell's equations!

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\nabla\phi - \dot{\vec{A}}$$

$$\vec{S} = -\nabla\chi - \partial_u \vec{A} \qquad \sigma = \dot{\chi} - \partial_u \phi$$

$$\vec{T} = -\nabla\psi - \partial_v \vec{A} \qquad \tau = \dot{\psi} - \partial_v \phi$$

$$\eta = \partial_u \psi - \partial_v \chi.$$

From the 6D vacuum Maxwell equations it is simple to derive the ultrahyperbolic wave equation for the potential A:

$$(\partial_t^2 + \partial_u^2 + \partial_v^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 \equiv \nabla_t^2 - \nabla_x^2) A = 0.$$

The currents J and K can also be written in terms of the potential,

$$J^\mu = -(\partial_u^2 + \partial_v^2)A^\mu + \partial^\mu(\partial_u\chi + \partial_v\psi) =$$

$$= (\partial_t^2 - \nabla^2)A^\mu - \partial^\mu(\nabla \cdot \vec{A} + \dot{\phi})$$

$$K^\mu = \partial^\mu(\partial_u\psi - \partial_v\chi),$$

and when the other time dimensions and scalar components are removed we get the usual full Maxwell wave equation with sources in 4D:

$$J^\mu = \Box A^\mu$$

in Lorentz gauge $\partial_\mu A_\mu = 0$! This was the result presented using geometric algebra an classical field theory in the previous Sections. Notice how in 6D we have a total vacuum and now we have in 4D all the sources we need from both electric and magnetic matter in general!!! This is a beautiful aspect of 6D simplicity and predictive power for 4D physics we experience and measure!

Maxwell's energy-momentum tensor can be derived from the symmetric and conserved 6-tensor

$$T^{mn} = F^{mp}F_p^n + \frac{1}{4}g^{mn}F^{ab}F_{ab}, \qquad \partial_n T^{mn} = 0,$$

stating 6D energy-momentum conservation! T is not traceless, however, as $Tr(T) = T^{mm} = \frac{1}{2}F^2$. This allows us to derive the nonhomogeneous Maxwell energy-momentum equation where fields are driven by currents, by repeated application of the field equations:

$$\partial_\mu T^{\mu\nu} = F^{\mu\rho}(\partial_u S_\rho + \partial_v T_\rho) = F^{\mu\rho}J_\rho \ !$$

This shows that the extra time dimensions are in fact physically real, as they provide the source for matter in 4D and its interactions with the fields!

Let us now look at the 6D Maxwell's equations in their simple form, and see where magnetic monopoles and magnetic charge come into play all of a sudden in 4D, not just charge and mass! Recall the 6-tensor F^{mn} above. Let us augment it to a dually-inclusive anti-symmetric tensor C^{mnp}, which satisfies the usual field equations from $S = \int d^6xC^2$:

$$\partial_p C^{mnp} = 0, \qquad \partial_{[a}C_{mnp]} = 0,$$

where the [] brackets indicate antisymmetric summation over all permutations (i.e. with sign). The first set has $\binom{6}{2} = 15$ equations, while the second has $\binom{6}{4} = 15$ equations, totalling 30 equations in 20 unknowns. 10 are, as above, redundant, as in the case of the bivectorial theory, we may ask how many equations do not involve one of the variables, say $x^4 = t$. In the first set, to avoid t-derivatives, one free index must be 4, leaving 5 choices for the other free index because of the total antisymmetry - while in the second set, to avoid t-derivatives, no free index can be 4, leaving $\binom{5}{4} = 5$ choices for the other free indices. We thus have 10 equations Let us derive a form of Maxwell's equations in 6D with particularly simple pure (vacuum) dynamics, that reduce to Maxwell's equations in 4D with matter of electric and magnetic charge sources. This follows [210]. In vacuum the field equations of Maxwell come straight from Lie theory (Bianchi identity) and cohomology (closedness for no sources):

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Our new generalized field tensor has two 4-bivector components which are $C^{\mu\nu5} = F^{\mu\nu}$, the usual Maxwell field strength tensor, and $C^{\mu\nu6} = G^{*\mu\nu}$ is the dual field tensor with $\vec{B} \to -\vec{D}$ and $\vec{E} \to \vec{H}$, and so it incorporates both induction and bare fields. What is new here is that we have two new 4-pseudovectors (the second is dual, so is actually a trivector, but I write its independent components as a 4-pseudovector):

$$C^{\mu 56} = M^\mu = [M_x \ M_y \ M_z \ m] = [\vec{M}, m],$$

and

$$C^{\mu\nu\rho} = \epsilon^{\mu\nu\rho\sigma}N_\sigma \approx [N_x \ N_y \ N_z \ n] = [\vec{N}, n].$$

Writing out the field equations in component form yield Maxwell-like equations, where u and v-derivative components represent electric and magnetic source terms when regarded in 4D!:

$$\nabla \wedge \vec{B} - \dot{\vec{E}} = \partial_v \vec{M}$$

$$-\nabla \wedge \vec{E} - \dot{\vec{B}} = \partial_u \vec{N}$$

$$\nabla \cdot \vec{E} = \partial_v m$$

$$\nabla \cdot \vec{B} = \partial_u n,$$

$$\nabla \wedge \vec{H} - \dot{\vec{D}} = -\partial_v \vec{N}$$

$$-\nabla \wedge \vec{D} - \dot{\vec{H}} = -\partial_u \vec{M}$$

$$\nabla \cdot \vec{D} = -\partial_v n$$

$$\nabla \cdot \vec{H} = -\partial_u m,$$

$$F^{\mu\nu} = \partial_u W^{\mu\nu} - \partial^\mu A^\nu + \partial^\nu A^\mu, \quad G^{\mu\nu} = -\partial_v W^{*\mu\nu} + \epsilon^{\mu\nu\rho\sigma}\partial_\rho V_\sigma$$

$$M^\mu = -\partial^\mu \rho + \partial_u V^\mu - \partial_v A^\mu, \qquad N^\nu = \partial_\mu W^{*\mu\nu},$$

or in terms of the 3D fields,

$$\vec{E} = -\nabla\phi - \dot{\vec{A}} + \partial_u \vec{P}$$

$$\vec{B} = \nabla \wedge \vec{A} + \partial_u \vec{Q}$$

$$\vec{D} = -\nabla \wedge \vec{V} - \partial_v \vec{Q}$$

$$\vec{H} = -\nabla\chi - \dot{\vec{V}} + \partial_v \vec{P}$$

$$\vec{M} = \nabla\rho + \partial_u \vec{V} - \partial_v \vec{A}$$

$$m = -\dot{\rho} + \partial_u \chi - \partial_v \phi$$

$$\vec{N} = -\nabla \wedge \vec{P} - \dot{\vec{Q}}$$

$$n = \nabla \cdot \vec{Q},$$

With the covariant gauge condition above, the potential can itself undergo a gauge transformation, for a 6-vector ω_m this can take up the six equations of the gauge condition to reduce the number of degrees of freedom of the gauge potential to nine:

$$W_{mn} \to W_{mn} + \partial_m \omega_n - \partial_n \omega_m.$$

These do not change the fields by cyclic differentiation. The six degrees of freedom in this gauge vector ω_m, we can remove the field vector $N^m = 0$ and all the currents it generates.

This leaves us with much simpler field equations, still with electric and magnetic sources in 4D, and with the more common

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu,$$

and

$$G^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\partial_\rho V_\sigma, \quad M^\mu = -\partial^\mu \rho + \partial_u V^\mu - \partial_v A^\mu.$$

The field equations reduce to

$$\partial_\nu F^{\mu\nu} = \partial_v M^\mu, \quad \partial_\nu G^{*\mu\nu} = -\partial_u M^\mu,$$

$$\partial_{[\mu} M_{\nu]} = \partial_v F_{\mu\nu} - \partial_u G^*_{\mu\nu}, \;\; \partial_v G_{\mu\nu} = \partial_u G^*_{\mu\nu}, \;\; \partial_\mu M^\mu = 0,$$

with G and F^* conserved (zero divergence) as well.

This has isolated the electric and magnetic phenomena, and by taking derivatives of the simplified gauge conditions we arrive at the general sources wave equations for both electric and magnetic charge-currents in 4D! They are denoted J and K, sources from F and G respectively:

$$J^\mu = -\partial^\mu(\nabla \cdot \vec{A} + \dot{\phi}) + (\partial_t^2 - \nabla^2)A^\mu,$$

and

$$K^\mu = -\partial^\mu(\nabla \cdot \vec{V} + \dot{\chi}) + (\partial_t^2 - \nabla^2)V^\mu,$$

as generalizations to electric and magnetic sources in any gauge to the usual $J^\mu = \Box A^\mu$. Further gauge conditions can be imposed if desired. Note the V-potential then propagates the magnetic current, while the usual photonic A-field propagates the electric sources. Note also that this gives a new way of viewing magnetic charge, charge and current in general, and even mass. In the next Section, after examining these wave equations, we will look at the Dirac equation in 6D and into the origin of mass itself into 4D!!

Just another thought for the origin of charge and mass from higher dimensions. Think of a waveguide in 6D, where two dimensions are compact of characteristic size R. The dispersion relation of a wave solution of the ultrahyperbolic wave equation in 6D, $\int d^3\vec{\omega} \int d^3\vec{k} \exp i(\vec{k}\cdot\vec{x} - \vec{\omega}\cdot\vec{t})$, perceived from someone living effectively in 4D at scales $\gg R$, will be $E^2 = \vec{\omega}^2 = \vec{k}^2 + 8/R^2$. Comparing this to Einstein's $E^2 = k^2 + m^2$ shows that in 4D we get a mass from the extra time dimensions depending on their scale, with $M \propto 1/R$! Furthermore, the 'charge wave' as it appears in 4D will move with speed less than c, as it must as it is massive, because it spends its time 'bouncing' around the extra dimensions as in a waveguide!!! By superposition of such charge waves as in Fourier spectra, one can generate any charge and current distribution in $\mathbb{R}^{3,1}$!

Solutions of the ultra-hyperbolic wave equation are described in the next subsection. We will also return to this waveguide extra dimensional approach to gravity later, and arrive at intriguing results from simple 6D to our 4D apparitions!

5.4 Electroweak $SU(2) \times U(1)$ unification from 6D

This section is an important one, stemming from the earlier research on 6D with three extra time dimensions. Establishing electroweak unification, and beyond, from compactification to 4D has led to many amazing results. I summarize the results from [] here, giving a context for this paper and new insights.

Let us discuss how 6D affine-Lorentz transformations act on different fields in $\mathcal{M}^{3,3}$. I use the notation $x^M = (t_x, t_y, t_z, x, y, z)$ so that the affine-Lorentz tranformations take the form:

$$x'^M = \Lambda^M_N x^N + a^M \in SO(3,3) \rtimes \mathbb{R}^{3,3}$$

where the 6D Lorentz transformation matrices Λ^μ_ν were discussed at length in Section 2.3.

The way these Lorentz transformations act on scalars, vectors, and tensor fields are described by $U(\Lambda)$ as follows:

$$U(\Lambda) \cdot \phi(x) = \phi(\Lambda^\mu_\nu x^\nu),$$
$$U(\Lambda) \cdot A^\mu(x) = \Lambda^\mu_\nu A^\nu(\Lambda^\mu_\nu x^\nu),$$
$$U(\Lambda) \cdot F^{\mu\nu}(x) = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}(\Lambda^\mu_\nu x^\nu).$$

I switched the notation to Greek indices, as in 4D, but the same holds for 6D capital Latin indices.

I will introduce a 6D Clifford algebra, to be used in later sections, as

$$\Gamma^\mu = \left(\begin{smallmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{smallmatrix}\right) = \sigma_z(\gamma^\mu) \in GL(8),$$

where the γ^μ are the usual 4D Pauli spin matrices, with anti-commutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} 1_{4\times 4}$. Also $\{\sigma_i, \sigma_j\} = \delta_{ij}$, and I introduce as well Γ-matrices in the timelike dimensions (I label as 01, 02, 03, and 1, 2, 3 for the spacelike dimensions),

$$\Gamma^{01} = \left(\begin{smallmatrix} 0 & 1_4 \\ 1_4 & 0 \end{smallmatrix}\right) = \sigma_x(1_4), \quad \Gamma^{02} = \left(\begin{smallmatrix} 0 & -i1_4 \\ i1_4 & 0 \end{smallmatrix}\right) = \sigma_y(1_4), \quad \Gamma^{03} = \sigma_z(\gamma^0),$$

and so $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$, for $g^{MN} = diag(1,1,1,-1,-1,-1)$ where capital indices run over 6D coordinates as $(01, 02, 03, 1, 2, 3)$.

The fermionic/matter content can now be written as 8D Dirac bi-spinors:

$$\psi = \left(\begin{smallmatrix} \psi_1 \\ \psi_2 \end{smallmatrix}\right) \approx \left(\begin{smallmatrix} e^- \\ \bar{\nu}_e \end{smallmatrix}\right),$$

and so on for the μ and τ families. We also write $\bar{\psi} = \psi^\dagger \Gamma_0 = (\bar{\psi}_1, -\bar{\psi}_2)$, where $\bar{\psi}_{1,2} = \psi^\dagger_{1,2}\gamma^0$. The 6D fermionic part of the Lagrangian then can be written as

$$\mathcal{L}(\psi) = i\bar{\psi}\Gamma^M \nabla_M \psi,$$

where the 6D gradient is written, taking the 'observable time' $t = t_z$, as

$$\nabla_M = (\nabla_{\vec{t}}, \nabla_{\vec{x}}) = (\nabla_{t\perp} = (\partial_{01}, \partial_{02}), \partial_{03} = \partial_t, \vec{\nabla}_{\vec{x}}).$$

We can then write the fermionic Lagrangian in component form, and see immediately how mass and charge arise!

$$\mathcal{L}(\psi) = i\bar{\psi}_1\gamma^\mu\partial_\mu\psi_1 + i\bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 +$$
$$+ i\bar{\psi}_1(\partial_{01} - i\partial_{02})\psi_2 - i\bar{\psi}_2(\partial_{01} + i\partial_{02})\psi_2,$$

and we find the electron and neutrino mass immediately:

$$(\partial_{01} - i\partial_{02})\psi_2 = im_e\psi_1 \qquad (\partial_{01} + i\partial_{02})\psi_1 = im_{\nu_e}\psi_1 = 0,$$

giving the mass terms. The latter assignment can be justified and $\nabla^2_{t\perp}\psi_2 = (\partial^2_{01} + \partial^2_{02})\psi_2 = 0$ can be solved with appropriate boundary conditions [].

Before going into electroweak unification, I'll mention the angular momentum generators for $SO(3,3)$, showing an $SU(2) \times U(1)$ invariance of the Lagrangian, and the emergence of gauge fields from the $\mathcal{M}^{3,3} \to \mathcal{M}^{1,3}$ compactification.
A basic (infinitessimal) spin angular momentum transformation for fermions $\psi' \to \psi + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}$, for antisymmetric $\omega_{\mu\nu} = -\omega_{\nu\mu}$, for angular momentum generators of $SO(3,3)$, $J^{\mu\nu} = \frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$. The timelike angular momentum operators are related to the spacelike ones:

$$J^{01,02} = J^{03} = \frac{1}{2}\sigma_z(1_4), \quad J^{02,03} = J^{01} = \frac{1}{2}\sigma_x(\gamma^0), \quad J^{03,01} = J^{02} = \frac{1}{2}\sigma_y(\gamma^0),$$

and $[J^{0i}, J^{0j}] = i\epsilon_{ijk}J^{0k}$ as angular momentum operators. For now $\vec{\sigma} = 2(J^{01}, J^{02}, J^{03})$, we have the operator rotating about direction $\hat{n}$ by angle θ,

$$U = \exp\frac{i}{2}\vec{\sigma}\cdot\hat{n}\theta = \cos\frac{\theta}{2}\hat{1} + i\vec{\sigma}\cdot\hat{n}\sin\frac{\theta}{2},$$

so that

$$\psi' \approx \psi + i\vec{\sigma}\cdot\hat{n}\frac{\theta}{2}\psi = \left(\begin{smallmatrix}\psi'_1 = \psi_1 + in_z\frac{\theta}{2}\psi_1 + i(n_x - in_y)\frac{\theta}{2}\gamma^0\psi_2 \\ \psi'_2 = \psi_2 - in_z\frac{\theta}{2}\psi_2 + i(n_x + in_y)\frac{\theta}{2}\gamma^0\psi_1\end{smallmatrix}\right),$$

giving the $\mathcal{L}$-invariance of $SU(2) \times U(1)$:

$$\psi' = \exp(i\chi + i\frac{\theta}{2}\vec{\sigma}\cdot\hat{n})\psi.$$

The first term in the exponent gives the $U(1)$-invariance as time translation, while the $SU(2)$-invariance comes from the $SU(2)$ operator invariance of the Pauli matrix and angular momentum invariance.

Now we turn the Lagrangian invariance under a local gauge choice:

$$D_\mu = \nabla_\mu + igX_\mu + ig'\vec{\sigma}\cdot\vec{W}_\mu$$

in $\mathcal{L} = i\bar{\psi}\Gamma^\mu D_\mu \psi$, giving interaction terms

$$\mathcal{L}_{int} = (\bar{\psi}_1, -\bar{\psi}_2)\gamma^\mu K_\mu \binom{\psi_1}{\psi_2},$$

where

$$K_\mu = \begin{pmatrix} -(gX_\mu + g' W_\mu^z) & -g'(W_\mu^x - iW_\mu^y)\gamma^0 \\ g'(W_\mu^x + iW_\mu^y)\gamma^0 & gX_\mu - g' W_\mu^z \end{pmatrix} =$$

which can be written in terms of the usual $SU(2)$ gauge fields:

$$Z_\mu = (gX_\mu - g' W_\mu^z)/\sqrt{g^2 + g'^2},$$

$$A_\mu = (g' X_\mu + gW_\mu^z)/\sqrt{g^2 + g'^2},$$

$$W_\mu^\pm = W_\mu^x \mp iW_\mu^y,$$

where the photon ($U(1)$) field A_μ only couples to the $e^\pm$, and we get the mixing angle $\sin\theta_W = g/\sqrt{g^2 + g'^2}$.

We then write the full electroweak unified Lagrangian (for $\psi = (\psi_1, \psi_2) = (e^-, \bar{\nu}_e)$, and $\bar{\psi}$ for the positron and neutrino):

$$\mathcal{L} = i\bar{\psi}_1\gamma^\mu\partial_\mu\psi_1 + i\bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 - m_e\bar{\psi}_1\psi_1 + \frac{g}{\sin\theta_W}\bar{\psi}_1\gamma^\mu(\cos(2\theta_W)Z_\mu - \frac{1}{2}\sin(2\theta_W)A_\mu)\psi_1 -$$

$$-\frac{g}{\sin\theta_W}\bar{\psi}_2\gamma^\mu Z_\mu\psi_2 - \frac{g}{\tan\theta_W}(\bar{\psi}_1\gamma^\mu W_\mu^+\gamma_0\psi_2 - \bar{\psi}_2\gamma^\mu W_\mu^-\gamma_0\psi_1),$$

as a Higgs-less achiral $SU(2)\times U(1)$ Lagrangian. The gauge field terms also result:

$$\mathcal{L}_{gf} = \frac{1}{4g^2}(G_{\mu\nu} = [D_\mu, D_\nu])^2 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{4}(\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu + g'\vec{W}_\mu \times \vec{W}_\nu)^2.$$

Again, the Greek indices run from 01, 02, 03, 1, 2, 3, and we find the photon A^μ is massless, while the Z_μ and $W_\mu^\pm$ acquire the proper mass without Higgsing, with massless neutrinos and massive $e^\pm$ correctly!

We will find later that the size of the extra dimensions can be determined from the mass of the X boson: $m_X = 10\pi\hbar/3L_5c$ as in the section on supersymmetric enhancement, giving also a gravitino mass $m_{3/2} = \pi\hbar/L_6c$. We turn to this now.

5.5 Supersymmetric enhancement from 6D compactifications.

From an $\mathcal{N} = 1$ supersymmetric theory in 6D, a $\mathcal{N} = 2$ supersymmetric theory in 4D, as well as a gauge-Higgs unification. This also resolves issues with chirality, extending results from [].

Let us begin with a typical general $\mathcal{N}=2$ supersymmetric SQCD Lagrangian, and see how it develops from a simpler $\mathcal{N}=1$ 6D SUSY theory, giving mass to the two SUSY particles depending on the size of the extra two dimensions. I write

$$\mathcal{L}_{4D}=\mathcal{L}_0+[S^*e^{2eV}S+T^*e^{-2eV}T+N^*N]_D+4e[T^*SN]_F,$$

where the D and F-terms are separated. The superfields are $V(V^\mu,\lambda(Majorana))$, $N(\xi(Majorana),a,b)$, $S(\psi_L,\varphi'')$ and $T(\psi_R,\varphi')$. Here a and b are the two extra dimensional components of V. The Weyl gaugino is $\Lambda_+=(\lambda_L,-i\xi_R)$, and the Weyl matter is $\psi_-=(\psi_R,\psi_L)$, along with scalar matter φ', φ''.

Writing the 4D Lagrangian in the superfield components above, yields

$$\mathcal{L}_{4D}=[-\frac{1}{4}V_{\mu\nu}V^{\mu\nu}-\frac{1}{2}\partial_\mu a\partial^\mu a-\frac{1}{2}\partial_\mu b\partial^\mu b-\frac{i}{2}\bar{\lambda}\not{\partial}\lambda-\frac{i}{2}\bar{\xi}\not{\partial}\xi]+$$
$$+[-i\bar{\psi}D\psi-ie\bar{\psi}(a-\gamma_5 b)\psi]+[-|D_\mu\varphi''|^2-|D_\mu\varphi'|^2-e^2(a^2+b^2)(\varphi''^\dagger\varphi''+\varphi'^\dagger\varphi']+$$
$$+ie\sqrt{2}[(\bar{\psi}_L\varphi''+\bar{\psi}_R\varphi')\lambda+\bar{\lambda}(\varphi''^\dagger\psi_L+\varphi'^\dagger\psi_R))]+$$
$$+e\sqrt{2}[(\bar{\psi}_L\varphi'+\bar{\psi}_R\varphi'')\xi+\bar{\xi}(\varphi'^\dagger\psi_L+\varphi''^\dagger\psi_R))]-\frac{e^2}{2}[\varphi''^\dagger\varphi''+\varphi'^\dagger\varphi']^2.$$

In contrast to [], we adopt the split (3,3) signature Clifford algebra of Γ matrices as we did in the previous Section. A simple change in sign, or rearranging of fields or chiralities makes the results thus far invariant physically. What changes will be discussed next, while examining the Γ matrices' actions on the Weyl fields. Doing this now, from a 6D perspective, we can arrive at a theory with chiral symmetry resolved, as well as a mechanism to give SUSY sparticles mass in terms of the size of the extra dimension, as in the previous subsection.

Looking at the Lie algebra $\mathfrak{so}(3,3)$, we have the Γ-matrices in the 8 dimensional representation $\subset GL(8)$ above, we have for a massive 6D Dirac spinor Ψ:

$$[\Gamma_{\hat{\mu}}(\partial^{\hat{\mu}}+ieA^{\hat{\mu}})+m]\Psi=0,$$

where its charge conjugate $\Psi^c=i\Gamma^7\Gamma^6\Psi^*=[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}]\Psi^*$, satisfies the above Dirac equation with $e\to -e$. It has the same chirality as Ψ, unlike in 4D! (so $\Lambda_+,\Lambda_+^c=[\begin{smallmatrix}-i\xi_L\\ \lambda_R\end{smallmatrix}]$ both have positive chirality, and similarly for their negative chirality counterparts.)

So let us look at the 6D Lagrangian in chiral decomposition:

$$\mathcal{L}_6=-\frac{1}{4}V_{\hat{\mu}\hat{\nu}}V^{\hat{\mu}\hat{\nu}}-i\bar{\Lambda}_+\not{\partial}\Lambda_+-i\bar{\psi}_-\not{D}\psi_--|D_{\hat{\mu}}\varphi'|^2-$$
$$|D_{\hat{\mu}}\varphi''|^2+ie\sqrt{2}[(\bar{\psi}_-\Lambda_+\varphi'+\varphi^{\dagger'}\bar{\Lambda}_+\psi_-)+(\bar{\psi}_-\Lambda_+^c\varphi''+\varphi^{\dagger''}\bar{\Lambda}_+^c\psi_-)]-\frac{1}{2}e^2[\varphi^{\dagger''}\varphi''+\varphi^{\dagger'}\varphi']^2,$$

which is easily generalizable to any number of families of generators $\{\psi_i, \varphi_i'', \varphi_i'\}$ of any charges Q_i, and for any gauge group (by inserting the Lie algebra generators T^a), adding a 6D potential

$$V_{6D}(\varphi_i', \varphi_i'') = \frac{1}{2}|\xi + \sum_i Q_i e(\varphi_i^{\dagger''}\varphi_i'' + \varphi_i^{\dagger'}\varphi_i')|^2 + \frac{1}{2}|\lambda + 2\sum_i Q_i e(\varphi_i^{\dagger'}\varphi_i'')|^2.$$

We can now look at spontaneous symmetry breaking, and determine the massive particle and sparticle mass spectrum, by adding an auxiliary FI term ξD, and eliminating the auxiliary field D by substituting in its equation of motion into the above 6D potential, and get (for one generation here)

$$V_{6D} = \frac{1}{2}|\xi + e(\varphi_i^{\dagger''}\varphi_i'' + \varphi_i^{\dagger'}\varphi_i')|^2 + \frac{1}{2}2e^2|\varphi_i^{\dagger'}\varphi_i''|^2,$$

where (for $\xi e < 0$, which is not a restriction!) we get a vanishing minimum for

$$\langle\varphi''\rangle = -v/\sqrt{2}, \quad \langle\varphi'\rangle = 0, \quad w/\ \ \xi + \frac{1}{2}ev^2 = 0.$$

Hence we have a single massive gauge multiplet with one massive vector, and 3 spin-0 Higgs bosons, with mass terms

$$\mathcal{L}_m^{6D} = -\frac{1}{2}m^2 V_{\hat{\mu}}V^{\hat{\mu}} - im(\bar{\psi}_-\Lambda_+^c + \bar{\Lambda}_+^c\psi_-) - m^2(|Re\varphi''|^2 + \varphi^{\dagger'}\varphi'),$$

for $m = ev$, giving the usual 4D mass terms

$$\mathcal{L}_m^{6D} = -\frac{1}{2}m^2(V_\mu V^\mu + a^2 + b^2) - im(\bar{\psi}_L\lambda_R + \bar{\lambda}_R\psi_L - i\bar{\psi}_R\xi_L + i\bar{\xi}_L\psi_R) - m^2(|Re\varphi''|^2 + \varphi^{\dagger'}\varphi'),$$

which leads to, in particular, a massive 6D spinor $\psi_- + \Lambda_+^c = \binom{\psi_R - i\xi_L}{\psi_L + \lambda_R}$, as two 4D massive Dirac spinors from two 6D Dirac winos and a Dirac zino, which gives 4 Dirac winos, and 2 Dirac zinos, in 4D, for example.

Now, as promised, we can find the masses in EW symmetry breaking, in terms of the sizes of the extra time dimensions. We take the masses m_W and m_Z we got in 6D, where (as in $SU(5)$), the $X^{\pm 4/3}$ is massless, and the $Y^{\pm 1/2}$ are degenerate with the $W^\pm$ (by the unbroken $SU(4)$ of 6D). We also take the two pairs of EW doublets in $SU(2) \times U(1)$ (or 2 pairs of quintuplets in the $SU(5)$ example) as the two Dirac pairs in general obtained above, and for any required number of generations i (2 here) and gauge group generators T_a, taken as the 24 generators of $SU(5)$ with normalization $Tr(T_aT_b) = \frac{1}{2}\delta_{ab}$, the 6D potential is as above

$$V_{6D} = \frac{1}{2}g^2|\sum_{i=1,2}(\varphi_i^{\dagger''}T_a\varphi_i'' - \varphi_i^{\dagger'}T_a\varphi_i') + 2g^2|\sum_{i=1.2}\varphi_i^{\dagger'}T_a\varphi_i''|^2,$$

with vanishing minimum for, eg,. $\langle\varphi_1^{0''}\rangle = \langle\varphi_2^{0'}\rangle = v/\sqrt{2}$. Here v is arbitrary until the $U(1)$ is gauged, so $SU(5) \to SU(4)$ is broken, giving Poincaré invariant $W^\pm$ and Z masses in 6D.

Returning to units in terms of $\hbar$ and c, we get the masses

$$m_X = \frac{10\pi\hbar}{3L_5 c}, \qquad m_{3/2} = \frac{\pi\hbar}{L_6 c},$$

the latter being a gravitino mass, and as promised!

6 Applications of 6D to cosmological models from Big Bang to the future of the universe.

In this Section, we look at cosmology with two extra dimensions, including when they are timelike. Many interesting consequences result, including stabilization of the extra dimensions during the evolution of the Universe, resolutions to the cosmological constant problem, as well getting deeper into our cosmology from the Big Bang to where it will end up.

We begin with a simple 6D geometry with three time dimensions, and look at the results for 4D, and then move on to showing how extra time dimensions do lead to a positive semi-definite action in a simple $4 + N = d$ dimensional universe.

6.1 Modelling a 6D general relativity.

Let us begin with a simple case of 6D general relativity: 3+3 FLRW Cosmology.

Assumptions:
1: A 6D spatially homogeneous and isotropic spacetime with flat vacuum 3D temporal space.
2: (a) spatial isotropy $\Rightarrow$ spherically symmetric coordinates (r, θ, φ),
(b) flat temporal coordinate (t, u, v)
3: homogeneity $\Rightarrow$ spatially conformal metric component independent of spatial coordinates
4: The 6D (3+3) FLRW line element is written in the form (with a reduced-circumference r):

$$ds^2 = R^2(t,u,v)\left[\frac{dr^2}{(1-kr^2)} + r^2 d\theta^2 + r^2\sin^2(\theta)d\varphi^2\right] - dt^2 - du^2 - dv^2$$

where $k \in \{-1, 0, +1\}$ measures spatial curvature

Compute Christoffel symbols of the second kind $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right]$

$$\Gamma^1_{11} = \frac{kr}{1-kr^2}; \qquad \Gamma^4_{11} = \frac{RR_t}{1-kr^2}; \qquad \Gamma^5_{11} = \frac{RR_u}{1-kr^2}; \qquad \Gamma^6_{11} = \frac{RR_v}{1-kr^2};$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}; \qquad \Gamma^1_{14} = \frac{R_t}{R}; \qquad \Gamma^1_{15} = \frac{R_u}{R}; \qquad \Gamma^1_{16} = \frac{R_v}{R};$$

$$\Gamma^1_{22} = r(kr^2 - 1); \qquad \Gamma^4_{22} = RR_t r^2; \qquad \Gamma^5_{22} = RR_u r^2; \qquad \Gamma^6_{22} = RR_v r^2;$$

$$\Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}; \qquad \Gamma^2_{24} = \frac{R_t}{R}; \qquad \Gamma^2_{25} = \frac{R_u}{R}; \qquad \Gamma^2_{26} = \frac{R_v}{R};$$

$$\Gamma^1_{33} = r(kr^2-1)\sin^2\theta; \qquad \Gamma^2_{33} = -\cos\theta\sin\theta;$$

$$\Gamma^4_{33} = RR_t r^2\sin^2\theta; \qquad \Gamma^5_{33} = RR_u r^2\sin^2\theta; \qquad \Gamma^6_{33} = RR_v r^2\sin^2\theta$$

$$\Gamma^3_{34} = \frac{R_t}{R}; \qquad \Gamma^3_{35} = \frac{R_u}{R}; \qquad \Gamma^3_{36} = \frac{R_v}{R}$$

The mixed form of the Einstein equations (with $c = G = 1$) are :

$$G^\mu_\nu = 8\pi T^\mu_\nu$$

where ρ and p are mass density and pressure respectively and contribute to 4D T^μ_ν.

$$G^\mu_\nu = \begin{bmatrix} G^1_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & G^2_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & G^3_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & G^4_4 & G^5_4 & G^6_4 \\ 0 & 0 & 0 & G^4_5 & G^5_5 & G^6_5 \\ 0 & 0 & 0 & G^4_6 & G^5_6 & G^6_6 \end{bmatrix} \qquad T^\mu_\nu = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

More explicitly the field equations can be written as:

$$-G^1_1 = -G^2_2 = -G^3_3 = \frac{2R_{tt}}{R} + \frac{2R_{uu}}{R} + \frac{2R_{vv}}{R} + \frac{R_t^2}{R^2} + \frac{R_u^2}{R^2} + \frac{R_v^2}{R^2} + \frac{k}{R^2} = -8\pi p \tag{1}$$

$$G^4_4 = \frac{3R_{uu}}{R} + \frac{3R_{vv}}{R} + \frac{3R_t^2}{R^2} + \frac{3R_u^2}{R^2} + \frac{3R_v^2}{R^2} + \frac{3k}{R^2} = 8\pi\rho \tag{2}$$

$$G^5_5 = \frac{3R_{tt}}{R} + \frac{3R_{vv}}{R} + \frac{3R_t^2}{R^2} + \frac{3R_u^2}{R^2} + \frac{3R_v^2}{R^2} + \frac{3k}{R^2} = 0 \tag{3}$$

$$G^6_6 = \frac{3R_{tt}}{R} + \frac{3R_{uu}}{R} + \frac{3R_t^2}{R^2} + \frac{3R_u^2}{R^2} + \frac{3R_v^2}{R^2} + \frac{3k}{R^2} = 0 \tag{4}$$

$$G^4_5 = G^5_4 = \frac{3R_{tu}}{R} = 0 \tag{5}$$

$$G^4_6 = G^6_4 = \frac{3R_{tv}}{R} = 0 \tag{6}$$

$$G^5_6 = G^6_5 = \frac{3R_{uv}}{R} = 0 \tag{7}$$

‘

I. The off-diagonal equations eqns(5-7) lead to an additive separation of variables of the form:

$$R(t,u,v) = T(t) + U(u) + V(v)$$

II. The equation (6)-(7) or $G^5_5 - G^6_6 = 0$ leads to:

$$R_{uu} = R_{vv}$$

Since the LHS is a function of u and the RHS a function of v in order for them to be equal both must equal the same constant, κ, therefore

$$U(u) = \frac{1}{2}\kappa u^2 + \alpha_1 u + \alpha_0 \qquad V(v) = \frac{1}{2}\kappa v^2 + \beta_1 v + \beta_0$$

These quadratic dependencies can make the scale factor finite at $t = u = v = 0$.
III. the equation (2)-(3) or $G^4_4 - G^5_5 = 8\pi\rho$ leads to:

$$R_{uu} - R_{tt} = \frac{8\pi}{3}\rho R$$

IV. the equation (2)-(4) or $G^4_4 - G^6_6 = 8\pi\rho$ leads to:

$$R_{vv} - R_{tt} = \frac{8\pi}{3}\rho R$$

and these two equations are consistent with $R_{uu} = R_{vv}$

V. the equation (2) - 3× (1) or $G^4_4 - 3G^1_1 = 8\pi(\rho + 3p)$

$$\frac{6R_{tt}}{R} + \frac{3R_{uu}}{R} + \frac{3R_{vv}}{R} = -8\pi(\rho + 3p)$$

replacing R_{uu} and R_{vv} from (III) and (IV) this becomes:

$$\frac{4R_{tt}}{R} + \frac{16\pi}{3}\rho = -\frac{8\pi}{3}(\rho + 3p)$$

or

$$\frac{R_{tt}}{R} = -2\pi(\rho + p) \tag{8}$$

Aside: for n extra time dimensions this equation can be generalized to:

$$(n+2)\frac{R_{tt}}{R} = -\frac{8\pi}{3}[(n+1)\rho + 3p]$$

so the extra time dimensions adds to the mass density. A form of dark matter? When $n = 0$ this is the same as the standard FLRW field equations.

The divergence of the energy momentum tensor must vanish $\nabla_\mu T^\mu_\nu = 0$ and this gives the time dependence of ρ and p

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\lambda\mu} T^\lambda_\nu - \Gamma^\lambda_{\mu\nu} T^\mu_\lambda$$

for $\nu = 1$

$$\begin{aligned}
\nabla_\mu T^\mu_1 &= \partial_1 T^1_1 + \Gamma^\mu_{1\mu} T^1_1 - \Gamma^\lambda_{\mu 1} T^\mu_\lambda \\
&= \partial_1 T^1_1 + \Gamma^1_{11} T^1_1 + \Gamma^2_{12} T^1_1 + \Gamma^3_{13} T^1_1 + \Gamma^4_{14} T^1_1 - \Gamma^1_{11} T^1_1 - \Gamma^2_{12} T^2_2 + \Gamma^3_{13} T^3_3 + \Gamma^4_{14} T^4_4 \\
&= 0
\end{aligned}$$

Since $T^1_1 = T^2_2 = T^3_3 = p$ and $\Gamma^2_{12} = \Gamma^3_{13} = r^{-1}$. For $\nu = 2$, 3, $\nabla_\mu T^\mu_\nu = 0$ identically as well.

for $\nu = 4$

$$\begin{aligned}
\nabla_\mu T^\mu_4 &= \partial_4 T^4_4 + \Gamma^\mu_{4\mu} T^4_4 - \Gamma^\lambda_{\mu 4} T^\mu_\lambda \\
&= \partial_4 T^4_4 + \Gamma^1_{41} T^4_4 + \Gamma^2_{42} T^4_4 + \Gamma^3_{43} T^4_4 + \Gamma^4_{44} T^4_4 - \Gamma^1_{41} T^1_1 - \Gamma^2_{42} T^2_2 + \Gamma^3_{43} T^3_3 + \Gamma^4_{44} T^4_4 \\
&= \partial_t \rho + 3\frac{R_t}{R}\rho + 3\frac{R_t}{R} p \qquad \text{multiply by} R^3 \\
&= \partial_t(\rho R^3) + p(\partial_t R^3) = 0
\end{aligned}$$

This is exactly the same as that for 4D FLRW equations.

Since $T^5_5 = T^6_6 = 0$ the cases $\nu = 5$, 6 lead to the same result.
for $\nu = 5$

$$\begin{aligned}
\nabla_\mu T^\mu_5 &= -\Gamma^\lambda_{5\mu} T^\mu_\lambda \\
&= -\Gamma^1_{51} T^1_1 - \Gamma^2_{52} T^2_2 - \Gamma^3_{53} T^3_3 = -3\frac{R_u}{R} p = 0
\end{aligned}$$

similarly for $\nu = 6$ $\qquad \nabla_\mu T^\mu_6 = -3\dfrac{R_v}{R} p = 0$

These last two equations present problems unless the matter content is dust: $p = 0$ which is ok for the current epoch.

For the current epoch, assume $p = 0$ then a first integral of eq (1) can be found.

After multiplying eq (1) by R^2 write it in the form:

$$2R_{tt}R + R_t^2 + k = -2R_{uu}R - 2R_{vv}R - R_u^2 - R_v^2$$

now multiply this by R_t

$$2R_{tt}R_t R + R_t^3 + kR_t = -4\kappa R R_t - (R_u^2 + R_v^2) R_t$$

Since R_u and R_t are independent of t one can integrate the above with respect to time to obtain:

$$R(R_t^2 + k) = -2\kappa R^2 - (R_u^2 + R_v^2) R + F$$

where F is a function of integration dependent on u and v. Isolating R_t^2 yields

$$R_t^2 = \frac{F}{R} - 2\kappa R - (k + R_u^2 + R_v^2) \tag{9}$$

is the first integral for R that is a "simple" first order ODE.

So far this is the simplest approach to the problem: assume that the extra dimensions are vacuum and that the extra time dependence behaves like the standard time coordinates and contributes only to the time dependence of the cosmological scale factor.

Alternative approaches can be assume that there are metric functions: $g_{uu} = A(t,u,v)$ and $g_{vv} = B(t,u,v)$ that keep the 6D spacetime metric diagonal but the stress-energy tensor is vacuum in the higher dimensions, or 2: assume T^u_u and T^v_v are non-zero keeping $g_{uu} = g_{vv} = -1$. What these non-zero components would be is not clear since they would also have to obey the conservation equation $\nabla\mu T^\mu_\nu = 0$ to be consistent with the higher order Bianchi identities for G^μ_ν. Finally 3: some combination of 1 and 2 which would be the most general (and complicated) approach.

Equation (2) reduces to the Friedmann equation in 4D and equation (8) is the Friedmann acceleration equation. The latter has an increase in mass density with increased number of time dimension but a decrease in the effective pressure. The cosmological acceleration is still negative. I was hoping that the extra dimensions would lead to a positive contribution and therefore act like a cosmological constant or quintescence source. This might come out of one of the more general cases.

Equation (9) leads to a first order ODE of the form:

$$\frac{dR}{dt} = \sqrt{\frac{A}{R} + B + CR}$$

Although this can be easily separated to obtain

$$dt = \frac{dR}{\sqrt{\frac{A}{R} + B + CR}} = \frac{\sqrt{R}\, dR}{\sqrt{A + BR + CR^2}}$$

This integral can be evaluated using elliptic integrals, It also must be real, so at least one of A, B, C is positive. Factor out one of the positive ones (which will then be determinable by the initial conditions), then requiring what's under the square root to be positive, gives a condition for the remaining two constants.

Finally, 3: some combination of 1 and 2 which would be the most general (and complicated) approach.

Let us now consider an AdS_4 space with two timelike extra dimensions and a cosmological constant of any sign in general. The brane couples to the bulk in 6D as well as in 10D string theories, with a crossover distances between these two regimes [] $R_c = M_{Pl}/2M_6^2$, and start with a general $(3,3)$-signature 6D metric:

$$ds^2 = -dt^2 + R(t)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] - \chi(t)^2[dy_1^2 + dy_2^2],$$

and we can have the extra time dimensions on $S^2/\mathbb{Z}_2$ with $dy_1^2 + dy_2^2 = d\rho^2 + \rho^2 d\varphi^2$.

Assume the 6D energy-momentum tensor satisfies Einstein's field equations in the early Universe, $G_{MN} = k_6^2 T_{MN}$, and can be written as

$$T_{MN} = -\Lambda g_{MN} + \delta(\rho)\delta(\varphi)(-\lambda q_{MN} + \tau_{MN}),$$

for G_{MN} the 6D Einstein tensor, for 4D brane vacuum energy τ_{MN} and energy-momentum tensor q_{MN}.
We can identify inflation as the separation between a brane and anti-brane [],

$$T_{\rho\rho} = -\Lambda g_{\rho\rho} + \delta(\rho)(-\lambda q_{\rho\rho} + \tau_{\rho\rho}),$$

and

$$T_{\varphi\varphi} = -\Lambda g_{\varphi\varphi} + \delta(\varphi)(-\lambda q_{\varphi\varphi} + \tau_{\varphi\varphi}).$$

For a dS 4-brane, we have $R(t) = e^{Ht}$, with $H = \sqrt{\Lambda_5/3}$ in terms of the 4-brane cosmological constant Λ_5. We get the following non zero components for Einstein's tensor, obtained from the Ricci tensor:

$$G_{tt} = 3(\frac{R'(t)}{R(t)})^2 + 6\frac{R'(t)\chi'(t)}{R(t)\chi(t)} + (\frac{\chi'(t)}{\chi(t)})^2,$$

$$\begin{aligned} G_{rr} &= -2(\frac{R''(t)}{R(t)}) - (\frac{R'(t)}{R(t)})^2 - 4\frac{R'(t)\chi'(t)}{R(t)\chi(t)} - 2(\frac{\chi''(t)}{\chi(t)})^2 - (\frac{\chi'(t)}{\chi(t)})^2 \\ &= G_{\theta\theta} = G_{\phi\phi}, \end{aligned}$$

$$G_{\rho\rho} = G_{\varphi\varphi} = -3(\frac{R''(t)}{R(t)}) - 3(\frac{R'(t)}{R(t)})^2 - 3\frac{R'(t)\chi'(t)}{R(t)\chi(t)} - \frac{\chi''(t)}{\chi(t)}.$$

Note that t here is to be seen as a time parameter in the 3D time space, and the prime is the derivative with respect to this parameter, and so this time dependence of the extra timelike dimensions, satisfy

$$G_{tt} + G_{rr} - 2G_{\rho\rho} = k_6^2(T_{tt} + T_{rr} - 2T_{\rho\rho}),$$

$$8(\frac{R'(t)}{R(t)})^2 + 8\frac{R'(t)\chi'(t)}{R(t)\chi(t)} + 4\frac{R''(t)}{R(t)} = -2k_6^2 T_{\rho\rho},$$

With $\delta(\rho) = 0 = \delta(\varphi)$ in the bulk (so $T_{MN} = -\Lambda g_{MN}$) and substituting $R(t) = e^{Ht}$ as before, we obtain the evolution of the $\chi(t)$ size parameter:

$$12H^2 + 8H\frac{\chi'(t)}{\chi(t)} = 2k_6^2\Lambda,$$

so

$$\frac{\chi'(t)}{\chi(t)} = -\frac{3}{2}H + k_6^2\Lambda/4H,$$

and for some pre-inflation value $\chi(0) = A_0$, we get a solution

$$\chi(t) = A_0 e^{-3Ht/2} e^{k_6^2\Lambda t/4H},$$

giving an exponent that can have either sign (inflation vs. decaying size post inflation, as we would expect is reasonable today). The cases for different sign of the bulk Λ.

For $\Lambda > 0$: We take the equation for $G_{\rho\rho}$) above, and get $\chi(t)$ satisfying

$$\chi''(t) + 3H\chi'(t) + (6H^2 - k_6^2\Lambda)\chi(t) = 0,$$

with solution

$$\chi(t) = A_0 e^{-3Ht/2} e^{\frac{1}{2}\sqrt{4k_6^2\Lambda - 15H^2}t},$$

since we would have at the end of inflation $t = \tau$ (where $H\tau \approx 100$), the extra term with a minus sign of the square root becomes vanishingly small, and can be neglected. Comparing the solution here to that previously, we have

$$k_6^2\Lambda = 2\sqrt{4k_6^2\Lambda - 15H^2},$$

and thus we have two solutions:

$$\Lambda_1 = 2\Lambda_5/k_6^2, \qquad \Lambda_2 = 10\Lambda_5/3k_6^2.$$

These can be easily shown to be consistent with $G_{\rho\rho}$ and $G_{\varphi\varphi}$.
For Λ_1, we get $\chi(t) = A_0$ is an unchanged constant!
For Λ_2, $\chi(t) = A_0 e^{Ht}$, and the extra time dimensions inflate as well in the way as the 4D spacetime brane. This makes sense for extra spatial dimensions (note the equations of motion/ Einstein's equations, are the same for any sign of χ^2, that is, spacelike or timelike! We can choose either for extra time dimensions, but usually the constant, non-inflating solution is best as, for example, masses of the winos and gravitinos above, would become arbitrarily smaller and smaller, at least in the dark energy dominated Universe epoch today, and we would have then found them in LHC particle searches.

For $\Lambda < 0$: The solution leads to an always negative exponent, and $\chi(t)$ becomes many orders of smaller than the Planck scale, which can be favourable for extra time dimensions. The size of the extra time dimensions get 'damped out' by the end of inflation:

$$\chi_-(t) = e^{-3Ht/2}(C_1\cos(\frac{1}{2}\sqrt{15H^2 - 4k_6^2\Lambda}t) + C_2\sin(\frac{1}{2}\sqrt{15H^2 - 4k_6^2\Lambda}t)).$$

Concluding, if we wish the extra dimensions to be small, or tightly curled up, this can hold a lot of potential energy, resulting in a dark energy source. For example,

$$R^{\rho}_{tt\rho} = R^{\varphi}_{tt\varphi} = |\frac{\chi''(t)}{\chi(t)}| = H^2 = \Lambda_5/3,$$

and so all together we get three equal contributions summing to the cosmological constant of the brane. (The Hubble parameter is also given from [] as $H = k_b^2 T^2/(10^{19} GeV)$.)

I'll end by giving the Ricci tensor components, leading to Einstein's equations above,

$$R_{tt} = -3\frac{R''(t)}{R(t)} - 2\frac{\chi''(t)}{\chi(t)}.$$

$$\begin{aligned} R_{rr} &= \frac{R''(t)}{R(t)} + 2(\frac{R'(t)}{R(t)})^2 + 2\frac{\chi'(t)R'(t)}{\chi(t)R(t)} \\ &= R_{\theta\theta} = R_{\phi\phi}, \end{aligned}$$

$$R_{\rho\rho} = R_{\varphi\varphi} = 3\frac{\chi'(t)R'(t)}{\chi(t)R(t)} + \frac{\chi''(t)}{\chi(t)} + (\frac{\chi'(t)}{\chi(t)})^2.$$

Now, let's do a more general metric structure...

6.2 Extra time dimensions leading to positive semi-definite actions.

Here we consider a universe with $d = 4 + N$ dimensions, and certainly when the extra dimensions are timelike. We assume a simple gravity-only Lagrangian in this d-dimensional space:

$$\mathcal{L}_{4+N} = \sqrt{-\hat{g}}\alpha\hat{R}^2,$$

for $\alpha > 0$, and look at compactifications to 4D and aspects of the action. We have $\hat{R}$ the 6D Ricci scalar (can be computed like in the previous section), ϕ is a scalar field, and the metric is $\hat{g} = diag(g_{ab}, \phi^{-1}\tilde{g}_{mn})$, where the early Latin indices $a, b, ...$ are for the 4D spacetime (brane), and the later Latin indices $m, n, ...$ are for the bulk, with metrics $g, \tilde{g}$, respectively.

Upon dimensional reduction to 4D, we find as in [],

$$\mathcal{L}_{4D} = \sqrt{-g}(\kappa_1\phi^{-(N-2)/2}R + \kappa_2\phi^{-(N-4)/2} + \phi^{-N/2}R^2),$$

with R being the 4D Ricci scalar, and $\kappa_{1,2}$ arbitrary constants. We show that this Lagrangian leads to a positive semi-definite action, avoiding the negative (Euclidean) action, for $\tilde{g}_{mn}$ time-like bulk metric. This also leads, as we will show, to the correct 4D dS spacetime we observe now, as well as all the past and future aspects of our Universe, and ϕ =constant.

Classically, this model is unstable for $t \approx 1/H$, time on the order of the Hubble time ($\approx$ Universe age), and a decrease in radius $\approx \phi^{-1/2}$ of the brane is stopped by quantum effects, and say by adding a potential $V(\phi)$ to allow for radius stabilization, which before we looked at from a Casimir stabilization [], and can allow for inflation, and other Universe evolution processes. This leads to $\mathcal{L}_4$ approaching that os usual Einsyein-Hilbert gravity, with a minute cosmological constant $\Lambda > 0$. Thus, we turh to 6D quantum cosmology now.

6D Quantum Cosmology.

Let's take a wavefunction for the Universe as Ψ, as

$$\Psi(h_{\alpha\beta}, \Phi) = \int \mathcal{D}g_{ab} \int \mathcal{D}\Phi \exp(-S_E[g_{ab}]),$$

where the Greek indices are for the 3 spatial dimensions, with metric $h_{\alpha\beta}$ on this spatial 3-fold Σ. The first path integral is over all Riemannian 4-metrics on Σ, and the second path integral is over all matter fields on manifolds M such that $\partial M = \Sigma$ and g_{ab} inducing $h_{\alpha\beta}$ on Σ.

Usually S_E is not usually positive semi-definite, which leads to a non well-defined path integral. Generally this is fixed by analytic continuations (a complex detour with $Re(S_E) \geq 0$), and uses resurgence theory, 'thimbles', and so on.

So let's state a *positive action theorem*:
An asymptotically Euclidean metric with positive action requires $R = 0$, and so $\Lambda = 0$.
This means that with $R_{ab} = \Lambda g_{ab}$ in dS space has no positive action analogue.
Similarly, then, a *negative action conjecture* can be stated:
For any metric on a 4-fold M such that $\partial M = \Sigma$ and $R = 4\Lambda$, and a negative action $S_E < 0$,one can do a Wick rotation $t \to i\tau$(instead of $\to -i\tau$ and $|S_E|$ can be used instead of S_E. It is possible that $S[\Phi] < 0$, so particle creation and fluctuations become unbounded, even classically! Thus the Universe wavefunction $\Psi \approx Ae^{-S_E} \to \infty$, where A comes from quantum effects, we see that classical fluctuations themselves diverge from the classical path, again classically!, and the path integral is undefined if there is any negative eigenvalue in the action's eigenbasis.

This can be resolved, firstly by adding higher order derivative terms [] ('higher order gravity'):

$$S_E = \int d^4x\sqrt{-g}[\alpha_1(R - 4\Lambda)^2 + \alpha_2 C_{abcd}C^{abcd}],$$

for C_{abcd} a fourth rank antisymmetric tensor with two derivatives (such as R_{abcd}, or that C in the electrodynamics section above), like in [], $\alpha_{1,2}$ some constants, with $256\pi\alpha_1\Lambda = M_{Pl}^2$, and is tachyon-free for $\alpha_2 < 0$ with a possible ghost at high energies.

We can explain the form of the action above by returning to an FRLW spacetime with metric from the line element

$$ds^2 = a^2(\eta)(d\eta^2 - dz^2) = dt^2 - a^2(t)d\vec{x}^2,$$

where $a(t) = R(t)/R_0$ is the radius size of the Universe, normalized to that at the current epoch R_0. This reduces to $S_E = \int d^4x\sqrt{-g}\alpha_1 R^2$, which is positive definite for $\Lambda = 0$. Varying this action with respect to g_{ab} gives a conserved (by Lagrangian invariance under coordinate transformations) quantity

$$H^{(1)}_{ab} = 2R_{;ab} - 2\Box R g_{ab} + \frac{1}{2}R^2 g_{ab} = 0.$$

For $\Lambda \neq 0$, we get

$$R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} - H^{(1)}_{ab}/8\Lambda = 0,$$

and contracting indices (by applying $g_{ac}g^{bc} = \delta^b_a$), we find that $R = 4\Lambda$! (as the trace of $H^{(1)}_{ab} = 0$ surely). Also from the 3-time=0 frame, we get $R_0^2 = R/4$ and $R_{ab} = \Lambda' g_{ab}$. We can then say that the universe wavefunction is the superposition of all classical d-dimensional spacetimes with $R > 12/a^2$.

Let us now specify this for the $d = 4 + N$ dimensional; spacetime. The action is

$$S = \frac{1}{16\pi\hat{G}}\int d^{4+N}x\sqrt{-\hat{g}}\hat{R}^2,$$

and, as above by varying with respect $\hat{g}_{AB}$, gives

$$\hat{H}^{(1)}_{AB} = 2\hat{R}_{;AB} - 2\hat{\Box}\hat{R}\hat{g}_{AB} + \frac{1}{2}\hat{R}^2\hat{g}_{AB} - 2\hat{R}\hat{R}_{AB} = 0,$$

where capital Latin indices A, B are $a, b \in \{0, 1, 2, 3\}$ and $m, n \in \{4, ..., 3 + N\}$.
A diagonal trial metric can be written as

$$\hat{g}_{AB}(x^A) = \left(\begin{smallmatrix} g_{ab}(x^a) & 0 \\ 0 & \phi^{-1}(x^a)\tilde{g}_{mn}(y^n) \end{smallmatrix}\right).$$

In the bulk N-dimensional sector, we look for $\tilde{R}_{mn} = \lambda\tilde{g}_{mn}$, and thus

$$\hat{\Box}\hat{R} - \hat{R}^2/4 + \lambda\phi\hat{R} = 0,$$

where now $\hat{R} = R + N\lambda\phi$ for such $\tilde{R}_{mn}$. This then gives

$$\hat{\Box}\hat{R} = -\frac{1}{2}N\phi^{-1}\phi_{ia}(R + N\lambda\phi)^2_i + \Box(R + N\lambda\phi),$$

and therefore,

$$\Box\phi - \frac{1}{2}N\phi^{-1}\phi_{ia}\phi^{ia} + \frac{1}{2\lambda}\phi^{-1}\phi_{ia}R^{ia} - \frac{N-2}{2N}\phi R - \frac{1}{4N\lambda}R^2 - \frac{N-4}{4}\lambda\phi^2 = 0,$$

whose brane (a, b) coordinates give

$$2R_{;ab} + N\lambda\phi_{;ab} + [N\phi^{-1}\phi_{;a}R^{;a} - 2\Box R - 2N\lambda\Box\phi + N\lambda\phi R + N^2\lambda\phi^{-1}\phi_{;a}\phi^{;a} +$$

$$+\frac{1}{2}R^2+\frac{1}{2}N^2\lambda^2\phi^2]g_{ab}-2(R+N\lambda\phi)R_{ab}=0,$$

which has a trace leading to

$$\Box R-(N-4)\lambda\phi R-R^2+4N\lambda^2\phi^2+\frac{1}{2}N\lambda\Box\phi=0.$$

We now look for a solution for a 'stationary epoch' with $R=R_0$ =constant and $\phi=\phi_0$ =constant, where we get an equation of the type (for $X=R_0/\lambda\phi_0$ and still $\tilde{R}_{mn}=\lambda\tilde{g}_{mn}$)

$$X^2+2(N-2)X+N(N-4)=0 \quad \& \quad X^2+(N-4)X-4N=0,$$

which is consistent $\iff$ $X=-N$, which implies precisely that $\hat{R}=0!$ $\Box$, and $R_{ab}=-N\lambda\phi_0 g_{ab}/4$.

Now if $\phi_0>0$, then $\tilde{R}/R<0$, and for bulk space metric $\tilde{g}_{mn}$ spacelike, compactification only occurs for $\lambda>0$ and we get instead an AdS spacetime. But for a dS spacetime, $\lambda<0$ and $\tilde{g}_{mn}$ is hyperbolic, and when timelike we get KK mode summation on compactification on a hyperbolic space. (For example, $\mathcal{L}_{4+N}=\sqrt{-\hat{g}}(\hat{r}-2\hat{\Lambda})$ does give a dS spacetime with timelike compactification,and is ghost free under certain conditions [].)

Let us now turn to the stability of the solution(s) above, taking $R=R_0+\delta R$ and $\phi=\phi_0+\delta\phi$ (knowing still that $R_0=-N\lambda\phi_0$ in static equilibrium), and a constant λ. By substitution and simplifying the equilibrium terms, we get:

$$\Box\delta\phi+N^{-1}(\phi_0\delta R-R_0\delta\phi)=0$$

$$\Box\delta R+(N+4)\lambda(\phi_0\delta R-R_0\delta\phi)+\frac{1}{2}N\lambda\Box\delta\phi=0,$$

which imply together that

$$\Box(\delta R-N(N+\frac{7}{2})\lambda\delta\phi)=0.$$

Let us now set $\delta R=N(N+\frac{7}{2})\lambda\delta\phi$, so we get the zero modes from the wave equation of the perturbations above, and get

$$\Box\delta\binom{R}{\phi}-m^2\delta\binom{R}{\phi}=0 \quad w/ \quad m^2=(1+\frac{9}{2N})R_0.$$

In Friedmann spacetimes, we have the above equal to

$$\delta\binom{\ddot{R}}{\ddot{\phi}}+3H\delta\binom{\dot{R}}{\dot{\phi}}-m^2\delta\binom{R}{\phi}=0,$$

where $H\equiv\dot{a}/a$ is Hubble's parameter (time varying in general) and the dot is a comoving time derivative.

Let us now look at the stability in different spacetimes.
For AdS spacetimes we have $R_0<0$ and $m^2<0$, and the fluctuations are stable, as well as

for dS spacetimes (with $R_0 > 0$ and $M^2 > 0$).

The solutions can be written [] as

$$\delta R = (\delta R_0) \exp[-\frac{3}{2}(1 \mp \sqrt{19/3 + 24/N} H_0 t)],$$

and the same for $\delta\phi$ with some $\delta\phi_0$. The positive root gives a possible growing solution, stable for times around the age of the Universe $t \approx 1/H_0$.
This can lead to an inflationary epoch solution, to solve the flat Universe horizon problem, for greater times $H_0 t >\approx 65$, and so []

$$H_0/M_{Pl} \approx \sqrt{3} \exp[-100(\sqrt{19/3 + 24/N} - 1)],$$

which is certainly not satisfactory.

And so we turn to $\lambda < 0$ for $m^2 > 0$, so $\delta\phi/\delta R < 0$. Thus, as R decreases, and the internal space radius $\phi^{-1/2}$ also decreases, and stops at some ϕ_1, say, by quantum effects from the scalar potential $V(\phi)$ as above, and inflation becomes possible in even the 6D compactification further compactified on $\mathcal{M}^{3,1}$. Note also this doubly compactified 10D theory gives an $\mathcal{L}_4$ an Einstein term $\propto R$ not present in 6D! We integrate out the bulk dimensions to get the result we initially began with:

$$\int \mathcal{D}\{y^m\} \quad \rightarrow \quad \mathcal{L}_4 = \sqrt{-g}[k_1 \phi^{-(N-2)/2} R + k_2 \phi^{-(N-4)/2} + \phi^{-N/2} R^2 + V(\phi)],$$

□

String theory and M-theory reduces to this case, but with assistance from extra degrees of freedom, with three time dimensions, in particular $SO(7,3)$. Specifically $E_8 \times E_8$ heterotic string theory, we have a 10D Lagrangian similar to the above with xcalar field ϕ as an inflaton:

$$\mathcal{L}_{10} = \sqrt{-\hat{g}}[\frac{\hat{R}}{2k^2} - \frac{(\hat{\nabla}\hat{\phi})^2}{k^2} - \frac{e^{-\hat{\phi}}}{4}(\gamma \hat{F}^2 - \hat{\mathcal{R}}^2) - \frac{3k^2}{2} e^{-2\hat{\phi}} \hat{H}^2],$$

where $\hat{\phi}$ is the inflaton field, $\hat{F}_{AB}$ is the Yang-Mills field tensor, and $\hat{H}_{ABC}$ is an antisymmetric tensor like before, and $\hat{\mathcal{R}}^2 = \hat{R}_{ABCD} \hat{R}^{ABCD}$.

For $\gamma = 1/30$, compactification onto a 6D Calabi-Yau X^6 is possible with signatures $(9,1)$, as in usually string theory, and $(3,7)$, with all extra dimensions on the 6D compactified space are timelike, and there are no ghosts! []

For $\gamma \neq 1/30$, a compactification of the 10D heterotic string theory exists in two ways: $\mathcal{M}^{10} \rightarrow \mathcal{M}^4 \times S^4 \times L^2$, and $\rightarrow \mathcal{M}^4 \times L^4 \times L^2$, with Minkowski 4D space $\mathcal{M}^4$, and L_n are the Lenz spaces, with signatures $(9,1)$ (as above in usual string theory), and the symmetric $(5,5)$!, but with $\hat{\mathcal{R}}^2$ replaced with the Euler number density:

$$\hat{\mathcal{N}}^2 = \hat{R}_{ABCD}\hat{R}^{ABCD} - 4\hat{R}_{AB}\hat{R}^{AB} + \hat{R}^2.$$

In conclusion, extra timelike dimensions are required to avoid problems involving the Wick rotations.

It is known, even from a High School International Physics Olympiad in 2017 [], that no potential $V(\phi)$ for the inflaton ϕ of polynomial form giving appropriate results. We need to consider non-perturbative contributions during such epochs as the inflationary one. This is considered next, as a summary, and was considered in the vortex scenario in the previous Section.

But let's first finish by showing how $V(\phi)$ cannot be perturbative/polynomial, from a High School contest, and so let's look at cosmological expansion and inflation covering all epochs, from a High School view:

Consider the Universe as a bubble of radius R_s, filled with space of density ρ and pressure P_0, expanding into a 'void' of the same density, and a pressure P. But a test mass m on the boundary of the universe, on the bubble surface $S^2_{R_s}$, so Newtonian gravity gives

$$m\ddot{R}(t) = -GmM_s/R^2(t),$$

which can be integrated by multiplying by $\dot{R}$:

$$\int \dot{R}\frac{d\dot{R}}{dt}dt = \frac{1}{2}\dot{R}^2 = GM_s/R + A,$$

where the mass inside the bubble Universe is $M_s = \frac{4}{3}\pi R^3(t)$ and we have $\dot{R} = \dot{a}R_s$, and so

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho(t) + 2A/R_s^2 a^2(t),$$

giving Friedmann's first equation.

Friedmann's second equation comes from the first law of thermodynamics[9],

$$dE = -pdV + dQ \qquad \rightarrow_{dQ=0} \qquad \dot{E} + p\dot{V} = 0,$$

for an adiabatic process. For a spherical Universe, $\dot{V} = 3V\frac{\dot{a}}{a}$, the total energy is $E = \rho(t)V(t)c^2$ thus,

$$\dot{E} = (\dot{\rho} + 3\frac{\dot{a}}{a})Vc^2 \qquad \rightarrow \qquad \dot{\rho} + 3(\rho + p/c^2)\frac{\dot{a}}{a} = 0.$$

[9]This law is also that 'we don't talk about thermodynamics'!

$\square$

Taking $\rho(t)c^2$ as energy density, and putting the equation of state, $p(t)/c^2 = w\rho(t)$, into Friedmann's second equation, gives

$$\dot{\rho} + 3\rho(1+w)\frac{\dot{a}}{a} = 0 \qquad \rightarrow \qquad \rho \propto a^{-3(w+1)},$$

generally, and for three possible states yields:
(i) For radiation[10]: the energy $E_{rad} = h\nu = hc/\lambda$ for a photon, gives an energy density $\rho_{rad} = E_{rad}/V \propto a^{-4}$, including redshifting, and thus $w_{rad} = 1/3$.
(ii) For non-relativistic matter: the energy density is mostly from its rest mass, $\rho_m \approx m_0c^2/V \propto a^{-3}$, and so $w_m = 0$.
(iii) For a constant energy density: call it ϵ_Λ =constant$\propto a^0$, and so $w_\Lambda = -1$. This has a negative pressure, or negative gravity if you like, and hence has the properties of dark energy.

We can look for the solutions to the Friedmann equations in each case above:

(i) In the case of zero curvature (like now) $k = 0$, we have $\rho_{rad}a^4(t)$ =constant= $\rho_{rad,0}a_0^4$, so then

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho_{rad,0}(\frac{a_0}{a})^4, \qquad \rightarrow \qquad \int a da = \frac{1}{2}a^2 + K = \sqrt{\frac{8\pi G}{3}\rho_{rad,0}(\frac{a_0}{a})^4}\ t,$$

and $K = 0$ since $a(t=0) = 0$, and thus

$$a(t) = \sqrt{2}(\frac{8\pi G}{3}\rho_{rad,0}(\frac{a_0}{a})^4)^{1/4}\sqrt{t} = \sqrt{2H_0t},$$

where $H_0 = \sqrt{\frac{8\pi G}{3}\rho_{rad,0}}$, with $a_0 \equiv 1$ today[11].

(ii) For non-relativistic matter, using $\rho_m a^3(t) = \rho_{m0}a_0^3$ =constant, we get similarly,

$$a(t) = (\frac{3}{2})^{2/3}(\frac{8\pi G}{3}\rho_{m0}(\frac{a_0}{a})^4)^{1/3}t^{2/3} = (\frac{3H_0}{2})^{2/3}t^{2/3},$$

where $H_0 = \sqrt{\frac{8\pi G}{3}\rho_{m0}}$.

(iii) For dark energy,

$$\ln a = H_0 t + C' \quad \rightarrow \quad a(t) = e^{H_0(t-t_0)},$$

where $H_0 = \sqrt{\frac{8\pi G}{3}\rho_\Lambda}$, and exponential, rather than algebraic, expansion occurs.

[10] or for relativisitic matter, like neutrinos.
[11] Note that $a(0) = 0$, not to be confused with $a_0 = 1$.

Using a critical energy condition $\rho_c(t) = 3H(t)^2/8\pi G$ and the definition of Ω in terms of it, Friedmann's equation can be written as

$$H^2(t) = H^2(t)\Omega(t) - \frac{kc}{R_0^2 a^2(t)} \qquad \rightarrow \qquad k = (\Omega - 1)R_0^2 H^2 a^2/c^2,$$

giving the curvature k. Since $R_0^2 H^2 a^2/c^2 > 0$, we find that

$$k = +1 \iff \Omega > 1$$

$$k = 0 \iff \Omega = 1$$

$$k = -1 \iff \Omega < 1.$$

After this basic overview of cosmology, let's turn to the necessity of an inflationary epoch.

Given $(\Omega - 1) = kc^2/R_0^2\dot{a}^2$ from before, a universe dominated by non-relativistic matter or radiation, write a scale factor p as $a = a_0(\frac{t}{t_0})^p$, we have $p < 1$ ($p = \frac{1}{2}$ for radiation, and $p = 2/3$ for non-relativistic matter), and we write $\Omega - 1 = \tilde{k}t^{2(1-p)}$.
For our current epoch and forward, a dark energy dominated Universe has $a(t) = e^{Ht}$, and $\dot{a} = He^{Ht}$, so that $\Omega - 1 = \frac{k}{H^2}t^{-2Ht}$.

So if an inflation period can be generated by dark energy alone, with equation of state $p = -\rho c^2$, differentiating Friedmann's equation $\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2}$ leads to

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho} + 2\rho a\dot{a}) = \frac{8\pi G}{3}(-3(\rho + p/c^2)a\dot{a} + 2\rho a\dot{a}),$$

and so

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p/c^2).$$

Now, as a result, $\ddot{a} = d(Ha)/dt > 0$ and $\ddot{a} = d(Ha)^{-1}/dt < 0$. The latter is the inflation condition, and

$$d(Ha)^{-1}/dt = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1 - \epsilon) < 0 \quad \rightarrow \quad \epsilon < 1,$$

in terms of the definition of ϵ.

Now using an inflaton field ϕ, with potential $V(\phi)$, as we considered before, taking the time derivative of the Lagrangian (a usual $\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m\phi^2 + V(\phi)$ gives an equation of motion

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi),$$

and the Hubble parameter satisfies

$$H^2 = (\frac{1}{2}\dot{\phi}^2 + V)/3M_{Pl}^2,$$

giving us

$$2H\dot{H} = \frac{1}{3M_{Pl}^2}[\dot{\phi}\ddot{\phi} + \frac{\partial V}{\partial \phi}\dot{\phi}],$$

and

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{Pl}^2} \quad \rightarrow \quad \epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{M_{Pl}^2 H^2}.$$

Inflation can only occur when the potential dominates the kinetic term (the particle's energy), and so $\dot{\phi}^2 \ll V$ so that $H^2 \approx V/3M_{Pl}^2$, which implies $\epsilon \approx \frac{1}{2}M_{Pl}^2(\frac{V'}{V})^2$. This is the 'slow rolling' approximation, with slow-roll parameters ϵ, $\eta_V = \delta + \epsilon$, and $\delta = -\ddot{\phi}/(H\dot{\phi})$.
We also have that

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi},$$

and so the parameters become

$$\delta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{V''}{3H^2} - \epsilon \quad \rightarrow \quad \eta_V \approx M_{Pl}^2\frac{V''}{V}.$$

We then get an N-folding number from

$$dN = Hdt = (H/\dot{\phi})d\phi \approx -\frac{1}{M_{Pl}^2}(V/V')d\phi \quad \rightarrow \quad dN/d\phi \approx -\frac{1}{M_{Pl}^2}(V/V').$$

Let's now look at cases for any simple potential $V(\phi) = \Lambda^4(\phi/M_{Pl})^n$, and inflation ends when $\epsilon = 1$, which means

$$\epsilon = \frac{1}{2}M_{Pl}^2(\frac{n}{\phi_f})^2 = 1 \quad \rightarrow \quad \phi_f = \frac{n}{\sqrt{2}}M_{Pl}.$$

From the equation for the N-folding number, we get for any perturbative or effective potential by integrating,

$$N = -(\frac{\phi}{M_{Pl}})^2/2n + B \quad \& \quad B = n/4,$$

as $N = 0$ at ϕ_f. Therefore the parameters above become now

$$\eta_v = n(n-1)(\frac{M_{Pl}}{\phi})^2 = \frac{2(n-1)}{n-4N},$$

$$\epsilon = \frac{1}{2}n^2(\frac{M_{Pl}}{\phi})^2 = \frac{n}{n-4N},$$

$$r = 16\epsilon = \frac{16n}{n-4N},$$

and finally.

$$n_s = 1 + 2\eta_v - 6\epsilon = 1 - \frac{2(n+2)}{n-4N}.$$

Here we see the constraint from current observation that $n_s = 0.968$, this gives $n = -5.93$, which is inconsistent with the constraint $r < 0.12$. No closest integer (and no polynomial potential in general as well) that satisfies this restriction. ($n = -6$ gives a contradiction $0 < -0.27$ and $n = -5$ gives $0 < -0.2$ as well.) $\square$

Let us now turn to the role of Casimir energy in extra dimensional cosmologies, and how it also determines the size of the extra dimensions through radius stabilization of Casimir energies.

6.3 Casimir energy in 6D spacetimes

Similar to the first case, in a general $D = d + 1 + n$-dimensional spacetime, where n is the number of extra spacelike or timelike extra dimensions to $\mathcal{M}^{d,1}$, we have a general metric

$$ds_D^2 = -dt^2 + a(t)^2 d\vec{x}^2 \pm b(t)^2 \vec{y}^2,$$

where the minus sign is for extra timelike dimensions, and in that case we can replace the parameter t in coefficient functions a, b as the time parameter τ in the $n+1$ time dimensional space.

Here, for the case of 6D,

$$(\nabla^A \nabla_A - m^2)\phi = 0 \qquad \rightarrow \qquad \phi \propto e^{i(k^\mu x_\mu + k_5 y_1 + k_6 y_2)},$$

where the first Klein-Gordon equation implies a quantization of momenta along the compact directions, $-k^\mu k_\mu = m^2 + n_1^2/b_1^2 + n_2^2/b_2^2$, and thus the vacuum energy, or Casimir energy, can be written as:

$$E_{Cas} = \frac{1}{2} L^3 \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{\vec{n}=(n_1,n_2)\in\mathbb{Z}^2} \sqrt{\vec{k}^2 + m^2 + n_1^2/b_1^2 + n_2^2/b_2^2}.$$

where we have taken this from the zero point energy result of $\frac{1}{2}\omega_k$, integrated over $\vec{k}$ and multiplied by the spatial volume L^3.
The sum can simply be done using the zeta function regularization in the case $m = 0$, and using the following formula the massive case can be done:

$$\sum_{n\in\mathbb{Z}} (q^2 + n^2)^{-s} = \frac{4\pi^s q^{1/2-s}}{\Gamma(s)} \sum_{k=1}^{\infty} k^{s-1/2} K_{1/2-s}(2\pi q k).$$

Here thus we get in the case of one extra dimension (5D):

$$\rho_{Cas}^{m=0} = \frac{E_{Cas}}{L^3 2\pi b} = \frac{2\Gamma(5/2)\zeta(5)}{\Gamma(-1/2)\pi^2(2\pi b)^5},$$

$$\rho_{Cas}^{m\neq 0} = \frac{-2(mb)^{5/2}}{(2\pi b)^5} \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} K_{5/2}(2\pi bmn).$$

Note that in both cases we have the vacuum energy density $\rho_{Cas} \propto 1/b^5$, and in general for D total dimensions it is always $\propto 1/b^D$, where b is the characteristic size of the extra dimensions. This provides a possible test for extra dimensions.

6.4 Dynamics of Casimir dark energy

Here we investigate a cosmological model for the Universe where dark energy and the cosmological constant arise purely from Casimir energies from Kaluza-Klein modes in extra compact dimensions of spacetime. We will focus on 6D spacetimes and will turn to timelike extra dimensions later.

In the general case of d dimensions with n extra dimensions, we begin our analysis by looking at the energy momentum tensor of the Casimir energy and put it into the Einstein field equations. This looks simply like

$$(T^\mu_\nu)_{Cas} = diag(-\rho_{Cas}, p_a, p_a, p_a, p_b, \dots, p_b),$$

where there are n p_b's, the pressures in the n extra dimensions. The pressures are defined thermodynamically as derivatives of Casimir energies with respect to volumes:

$$p_{a,b} = -\frac{\partial}{\partial V_{a,b}}(\rho_{Cas} V_{a,b}).$$

This gives the equation of state for the dimensional volumes, with $V_a \propto a^{d-n}$ and $V_b \propto b^n$. Since ρ_{Cas} depends on b only, as we saw in the last subsection, we get $p_a = -\rho_{Cas}$ but $p_b = -\rho_{Cas} - b\partial_b \rho_{Cas}$.[12] Note that p_a satisfies the same equation of state as the cosmological constant, $p_\Lambda = -\rho_\Lambda$ and so the bulk portion of the extra dimensional Universe with the Casimir energy behaves like a Universe with a finite cosmological constant.

The conservation of energy equation for the energy momentum tensor gives the equation

$$\nabla_\mu T^\mu_\nu = 0 = \dot{\rho}_{Cas} + 3H_a(\rho_{Cas} + p_a) + nH_b(\rho_{Cas} + p_b),$$

where the Hubble parameters are $H_a = \dot{a}/a$, $H_b = \dot{b}/b$.

The Einstein field equations governing the dynamics for the evolution of the dimensions of the Universe then become:

$$3H_a^2 + \frac{n(n-1)}{2}H_b^2 + 3nH_aH_b = M_*^{-(n+2)}\rho_{Cas} \tag{6.1}$$

$$n\frac{\ddot{b}}{b} + 2\frac{\ddot{a}}{a} + \frac{n(n-1)}{2}H_b^2 + H_a^2 + 2nH_aH_b = -M_*^{-(n+2)}p_a$$

$$(n-1)(\frac{\ddot{b}}{b} + \frac{3n-2}{2}H_b^2 + 3H_aH_b) + 3\frac{\ddot{a}}{a} + 3H_a^2 = -M_*^{-(n+2)}p_b,$$

where M_* is the characteristic mass scale.

[12] In general spacetime, matter and energy often obey $p = w\rho$ for some $w \in \mathbb{R}$, which can be found by putting $\rho = \rho_0 a^{-(1+w)/3}$ in the definition of pressure above.

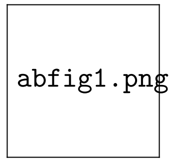

Figure 2. Extra dimensions stabilized (left) for relativistic matter Universe and bulk dimensions accelerated (right).

Simulations [] have shown that the size of the extra dimension b gets stabilized at a value of $b(t) \approx 0.0145$ with the scale factor $a(t)$ accelerated, as seen in our Universe, for $\lambda = m_b/m_f \in (0.516, 0.527)$, a narrow range. See Figure 4.8. It turns out that in the case of non-relativistic matter present, the Einstein field equations change and there is no possible way within the parameters of this model to stabilize the radius of the extra dimension in the case of an extra spatial dimension. See Figure 4.9. However, in [], adding a Gauss-Bonnet term can stabilize the extra dimension. In this case the action reads

$$S_{EGB} = \int d^D x \sqrt{-g} (\frac{M_*^{D/2}}{2} (-2\Lambda + R + k^2 \mathcal{G}) + L_{matter})$$

where the Gauss-Bonnet term is

$$\mathcal{G} = R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu}.$$

The field equations in this case are:

$$3H_a^2 + 3H_a H_b + 12k^2 H_a^3 H_b = M_*^{-3}(\rho_{Cas} + \rho_m)$$

$$\frac{\ddot{b}}{b} + 2\frac{\ddot{a}}{a} + H_a^2 + 2H_a H_b + k^2 (8H_a H_b \frac{\ddot{a}}{a} + 4H_a^2 \frac{\ddot{b}}{b}) = -M_*^{-3} p_a$$

$$3\frac{\ddot{a}}{a} + 3H_a^2 + k^2 H_a^2 \frac{\ddot{a}}{a} = -M_*^{-3} p_b.$$

Simulations [] show this model is stabilized for $k^2 \in (0.005, 5.0)$ for sufficiently small enough $\dot{b}(t = 0)$. See Figure 4.10.

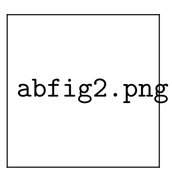

Figure 3. Extra dimensions not stabilized (left) for non relativistic matter Universe and bulk dimensions slowly accelerated (right).

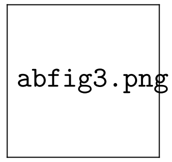

Figure 4. Extra dimensions stabilized (left) for non relativistic matter Universe with Gauss-Bonnet action, and bulk dimensions accelerated (right).

In 6D stabilization can also be achieved using this Einstein-Gauss-Bonnet model []. Here I write down the reduced action where I integrate out the extra two dimensions:

$$S_{(4)} = \int d^4x\sqrt{-\tilde{g}}[\frac{1}{2}M_{Pl}^2[\tilde{R} + k^2\tilde{\mathcal{G}}] - \frac{1}{2}(\partial\psi)^2 - V(\psi) +$$

$$+k^2 e^{\psi/\psi_0}(\tilde{G}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \frac{(\partial\psi)^2(\partial\psi)^2}{4\psi_0^2}) + e^{-2\psi/\psi_0}L^{(4)}_{matter}],$$

where $V(\psi) = (2\pi)^2 e^{-\psi/\psi_0}L^{(6)}_{Cas}(\psi) = e^{-2\psi/\psi_0}L^{(4)}_{Cas}(\psi)$ and $\psi_0 \equiv M_{Pl}$ was redefined. Note also that the Casimir energies with all different matter fields can be obtained in 6D as

$$\rho^{(6)}_{Cas} = -7\rho^{m=0}_{Cas} + 16\rho^{m\neq 0}_{Cas}(m_b) - 16\rho^{m\neq 0}_{Cas}(m_f),$$

as $-7 = -16 + 9$ is the contribution of the massless fermions and bosons (gravitons) respectively, noting that gravitons have 9 degrees of freedom in six dimensions, and other fermions and bosons have 16. Fermions contribute negatively to the energy in the Casimir as well.

One can compute the energies, pressures and V'_{eff} and find that simulations give stability of the extra dimensions again for λ near a lower bound of 0.456 and $k \in (0.01, 0.44)$.

Here we have used that

$$\zeta'(-4) = \frac{3}{4\pi^4}\zeta(5), \qquad \frac{\Gamma'(-2)}{\Gamma^2(-2)} = -2, \qquad \Gamma(-5/2) = -\frac{8}{15}\sqrt{\pi}$$

$$\zeta(-5) = -\frac{15}{4\pi^6}, \qquad \zeta(-6) = -1/252.$$

6.5 Radius stabilization in 6D on the torus

Let us look at a specific case again of 6D with two compact dimensions on a torus. This is used often as a model to describe composite and heavy Higgses, as will be shown in a later Section, and in the next Chapter we will show that 6D as such is useful in that 6D global anomaly cancellations forces fermion generations to be 3, or at least $n_{generations} = 0 \ mod \ 3$. The metric on the compactified 6D spacetime is then given by the interval

$$ds^2 = \mathcal{A}^{-1}\eta_{\mu\nu}dx^\mu dx^\nu + \mathcal{A}\gamma_{ij}dy^i dy^j,$$

where $y^i \in [0, L]$, and the metric on the torus is (with $\tau = \tau_1 + i\tau_2$)

$$\gamma_{ij} = \left(\begin{smallmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{smallmatrix}\right).$$

Note that $\det\gamma = 1$, so $\mathcal{A}$ is indeed the (dimensionless) area of the torus. With metric fluctuations, $\eta_{\mu\nu} \to g_{\mu\nu}$ and we note that

$$M_6^4 \int d^6x\sqrt{G}R(G) = M^2 \int d^4x\sqrt{g}[R(g) + g^{\mu\nu}\partial_\mu\tau\partial_\nu\bar{\tau}/2\tau_2^2 + g^{\mu\nu}\partial_\mu\mathcal{A}\partial_\nu\mathcal{A}/\mathcal{A}^2],$$

with $M^2 = L^2 M_6^4$.

We can now calculate the Casimir energy of a real scalar field in the background of a gauge field, beginning with the vacuum energy for a single massless scalar field (periodic boundary conditions) as done above and in []:

$$V^{scalar} = \frac{1}{2}\sum_{n,m=-\infty}^{\infty}\int\frac{d^4k}{(2\pi)^4}\log(R^2k^2 + \mathcal{A}^{-2}\tau_2^{-1}|m - n\tau|^2) = -\frac{d}{ds}\zeta^{scalar}(s)|_{s=0},$$

with $R = L/2\pi$, and again zeta function regularization is used as we have

$$\zeta^{scalar}(s) = \frac{1}{2}\sum_{0\neq n,m=-\infty}^{\infty}\int\frac{d^4k}{(2\pi)^4}(R^2k^2 + \mathcal{A}^{-2}\tau_2^{-1}|m - n\tau|^2)^{-s} =$$

$$= \frac{\pi^2\mathcal{A}^{2s-4}}{L^4(s-2)(s-1)}\sum_{m,n}{}^{'}\frac{\tau_2^{s-2}}{|m - n\tau|^{2s-4}}.$$

The prime in the summation means excluding the zero mode. A derivation done below evaluates the potential to be

$$V^{scalar} = -\frac{\pi^2}{2L^4}\mathcal{A}^{-4}[\frac{8\pi}{945}\tau_2^3 + \frac{3\zeta(5)}{\pi^4}\tau_2^{-2} +$$

$$\frac{4}{\pi^2}\sum_{m=1}^{\infty}(m^2 Li_3(q^m) + \frac{3}{2\pi}\tau_2^{-1} m Li_4(q^m) + \frac{3}{4\pi^2}\tau_2^{-2} Li_5(q^m) + h.c.)],$$

where the $Li_n(x)$ are the polylogarithm functions, and $q = e^{2\pi i(\tau_1 + i\tau_2)}$. Note that $\tau = e^{2\pi i/3}$ interestingly minimizes the potential for a fixed area! I also write down the contributions from the other bulk fields to the potential:

$$V^{graviton} = 9V^{scalar},$$

$$V^{Weyl\ fermion} = -4V^{scalar},$$

$$V^{vector} = 4V^{scalar},$$

$$V^{2-form} = 3V^{scalar},$$

$$V^{gravitino} = -12V^{scalar}.$$

Let us now look at the case with extra time dimensions.

6.5.1 Radius stabilization in extra time dimensions

Let us finally make some comments on the role of extra timelike dimensions, as we did in our simpler 6D model we began with, without Casimir and other effects.

We now go on to non-perturbative potentials, and their related solitons, like the vortices used above for 6D electroweak theory, as well as others that can be involved, such as monopoles, instantons, monopole-instantons, etc., and their role in 6D effective theories in 4D. Also, general gauge group will be considered, as used above in the 6D SUSY compactification leading to 4D and resolving many issues in current theories.

7 Perturbative and non-perturbative dynamics of (S)YM on $\mathcal{M}^{3,3}$ for general gauge group.

We now examine the non-perturbative sector of the theory. This includes the effects of magnetic monopole-instantons ((anti) self-dual objects) which are charged in the co-root lattice $\Lambda_r^\vee$ of the Lie algebra $\mathfrak{g}$, as well as exotic topological 'molecules': the neutral bions and the the magnetic bions (non self-dual objects). These enter into the path integral with action (3), and hence into the partition function of the theory. I begin by describing the zero temperature dynamics of such particles.

Due to the topology of gauge groups we find that for a gauge group G fully abelianized to $U(1)^r$ there are r BPS monopole solutions as $\pi_2(G/U(1)^r) \approx \pi_1(U(1)^r) \approx \mathbb{Z}^r$. Also, due to the compactness of the x^3-coordinate we obtain another solution called the twisted or KK monopole. These solutions are (anti) self-dual objects localized in space and time, and are dilute so we can ignore their internal structure and examine their long-range fields. The field of a single BPS ($B\bar{P}S$) monopole of type j, $j = 1, \ldots, r$, localized at the origin is, in the stringy gauge, [2] (Here $A_\mu = A_\mu^a H^a$)

$$
\begin{aligned}
A_0^{j,BPS,B\bar{P}S} &= \mp \frac{x_1}{r(r+x_2)} \alpha_j^\vee \cdot \vec{H}, \qquad (7.1)\\
A_1^{j,BPS,B\bar{P}S} &= \pm \frac{x_0}{r(r+x_2)} \alpha_j^\vee \cdot \vec{H}, \\
A_2^{j,BPS,B\bar{P}S} &= 0, \\
A_3^{j,BPS,B\bar{P}S} &= (\frac{\pi}{L} - \frac{1}{r}) \alpha_j^\vee \cdot \vec{H},
\end{aligned}
$$

where $r = \sqrt{x_0^2 + x_1^2 + x_2^2}$. These gauge field components are charged under the co-root lattice of the Lie algebra and so are multiplied by $\vec{\alpha}_a^\vee$[13]. These components of the gauge field

[13]The co-roots in other literature are sometimes written as α_j^*

give the correct asymptotics of the magnetic field at infinity as in the Appendix (C.11). The field for the KK ($\bar{K}K$) monopole similarly reads

$$\begin{aligned} A_0^{0,KK,\bar{K}K} &= \pm\frac{x_1}{r(r+x_2)}\alpha_0^\vee\cdot\vec{H}, \\ A_1^{0,KK,\bar{K}K} &= \mp\frac{x_0}{r(r+x_2)}\alpha_0^\vee\cdot\vec{H}, \\ A_2^{0,KK,\bar{K}K} &= 0, \\ A_3^{0,KK,\bar{K}K} &= (\frac{\pi}{L}+\frac{1}{r})\alpha_0^\vee\cdot\vec{H}. \end{aligned} \tag{7.2}$$

These gauge fields have an additional charge factor $\alpha_0^\vee$, the affine co-root of $\mathfrak{g}$. See Appendix C for more on monopole-instanton solutions. These monopole-instantons carry magnetic charge Q_m^a from Gauss' law

$$\int_{S^2_\infty} d^2\Sigma_\mu B_\mu^j = 4\pi Q_m^j,\ j=0,\ldots,r, \tag{7.3}$$

where $B_\mu^j = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda^j = Q_m^j\frac{x_\mu}{r^3}$ is the magnetic field. I will write $Q_m^i = q_m^i\vec{\alpha}_i^\vee$ as the monopole charges belong to the co-root lattice of the Lie algebra of the gauge group, $\Gamma_r^\vee$. q_m^i is the charge of the monopole and equals ±1. Monopoles of charge type a and b only interact when $\vec{\alpha}_a^\vee\cdot\vec{\alpha}_b^\vee\neq 0$, or, in other words, $a=b$ or they are nearby neighbours on the Dynkin diagram of $\mathfrak{g}$ (i.e. non-zero elements of the Cartan matrix of the Lie algebra). See Appendix A for the Dynkin diagrams for each (affine) Lie algebra. As mentioned in the introduction, there is also a long-range scalar field (from the A_3^a component of the gauge field) which can attract or repel these monopoles due to scalar charge interaction. There is further a topological charge of these monopole instantons Q_T defined by

$$Q_T^{(i)} = (32\pi)^{-1}\int_{\mathbb{R}^3\times S^1} F^a_{MN}F^{aMN},. \tag{7.4}$$

Using the solutions (3.1) and (3.2) I find the charges (Q_m, Q_T) for each monopole type, which for $SU(2)$ are:

$$BPS\ (+1,1/2)\ \ B\bar{P}S\ (-1,-1/2)\ \ KK\ (-1,1/2)\ \ \bar{K}K\ (+1,-1/2). \tag{7.5}$$

Note that in general, as is seen in [2], the values of the topological charge depend on the vacuum of the theory. There it is derived that the topological and magnetic charges, for monopoles of type i, are

$$Q_T^{(i)} = -\frac{L}{2\pi}\sum_{w\in\Delta_w^{adj}}(\vec{w}\cdot\vec{\phi}_0)(\vec{w}\cdot\vec{\alpha}_i^\vee), \qquad \vec{Q}_m^{(i)} = \vec{\alpha}_i^\vee.$$

The topological charge clearly depends on the vacuum $\vec{\phi}_0$ of the theory, and usually gives fractional charges.

Due to the presence of fermions and supersymmetry (our gaugino), the Callias index theorem [8], [10] on $\mathbb{R}^3 \times S^1$ implies the existence of two adjoint fermionic zero modes attached to each monopole-instanton. Appendix C.3 describes this index theorem in detail for general gauge group. Let us use the fields $\vec{\phi}$ instead of $\vec{A}_3^a$ and $\vec{\sigma}$ instead of $\vec{A}_\mu$ and attach fermionic zero modes to get the 't Hooft vertices (the field $\vec{z} = \vec{\phi} + i\vec{\sigma}$ is the lowest component of the chiral superfield $\vec{X}$ [2])

$$\mathcal{M}_{BPS,j} = e^{-4\pi^2/g^2} e^{-\vec{\alpha}_j^\vee \cdot \vec{z}} \vec{\alpha}_j^\vee \cdot \bar{\vec{\lambda}} \vec{\alpha}_j^\vee \cdot \bar{\vec{\lambda}}, \quad \mathcal{M}_{KK} = e^{-4\pi^2/g^2} e^{-\vec{\alpha}_0^\vee \cdot \vec{z}} \vec{\alpha}_0^\vee \cdot \bar{\vec{\lambda}} \vec{\alpha}_0^\vee \cdot \bar{\vec{\lambda}}. \tag{7.6}$$

The anti monopole vertices are just the complex conjugates of these. These are valid for arbitrary gauge group due to the supersymmetry of the theory, and are derived in detail from the superpotential in [2]. Inserting these into the partition function of the theory inserts the contribution of fermionic zero modes and the long range fields (the $e^{-\phi^a + i\sigma^a}$ factors). These in themselves due not alter the vacuum structure of the theory as they are attached to fermionic zero modes and do not generate a potential for the fields $\vec{\phi}$ and $\vec{\sigma}$ and no mass will be generated for the dual photons $\vec{\sigma}$. I consider then the effect of the neutral and magnetic bions in the non-perturbative potential and their role in the deconfinement phase transition.

Let us now consider the non self-dual 'molecules' these monopole constituents can form. Their charges and amplitudes of the so-called magnetic and neutral bions that form are summarized below in Table 2. In all cases the index $a \in \{0, \ldots, r\}$. A neutral KK bion would have $a = 0$ and is formed from a KK-anti-KK monopole pair. In the case of magnetic bions they are formed from two (BPS or KK) monopoles of charge types $a \neq b$ as long as $\vec{\alpha}_a \cdot \vec{\alpha}_b \neq 0$, that is the monopoles are Dynkin neighbours on the Dynkin diagram of G.

Molecule	vertex	$(Q_m.Q_T)$	amplitude
neutral bion	$\mathcal{M}_a \bar{\mathcal{M}}_a$	$(0,0)$	$e^{-8\pi^2/g^2} e^{-2\vec{\alpha}_a^\vee \cdot \vec{\phi}}$
magnetic bion	$\mathcal{M}_a \mathcal{M}_b$	$(2,0)$	$e^{-8\pi^2/g^2} e^{-(\vec{\alpha}_a^\vee + \vec{\alpha}_b^\vee)\cdot\vec{\phi} - i(\vec{\alpha}_a^\vee - \vec{\alpha}_b^\vee)\cdot\vec{\sigma}}$

Table 2: Magnetic molecule vertices, charges, and amplitudes for different molecules.

7.1 Non-perturbative structure at finite temperature

From [1] let us recall that the dual photon and ϕ fields have masses $m_{\sigma^a} = m_{\phi^a} \approx e^{-4\pi^2/g^2}/L$, and the fermions acquire a thermal mass $\approx T$. The light gauginos do not participate in the deconfinement phase transition as they carry no electric, magnetic or scalar charge, however they do allow the formation of the bions and so have an indirect role in the transition. The heavy fermions (the winos) do participate, however, in a way similar to the W-bosons. The deconfinement transition temperature is found to be of the order $T_c \approx g^2/8\pi L$ and Λ_{QCD} from both simulations and from setting the W-boson and magnetic bion fugacities to the same order [15]. This temperature is smaller then the inverse bion radius and so we need not worry about the dissolution of the bions before the deconfinement transition and our gas

of such particles exists beyond the deconfinement transition temperature. See [1] for more details.
The best way to study the finite temperature dynamics of our gas of all these particles is to map it to a double Coulomb gas and examine its partition function. This I derive in the next section.

Let me make a few comments about our result so far. Note already the usual Coulomb-Coulomb interactions between electric W-bosons and magnetic bions, as well as the Aharanov-Bohm interaction given by the Θ term as in [1]. One further point to consider is the dependency of the W-boson fugacity on the fields ϕ^a. In the case of circle compactification where the $\vec{\phi}$ field is absent (as in the zero temperature limit) we have a Kramiers-Wannier duality $32\pi LT/g^2 \to g^2/2\pi LT$ as in the sine-Gordon model which is the zero-temperature limit of our theory. The magnetic monopoles are not present in the partition function of our Coulomb gas, as these we ignore as they do not contribute to the dynamics of the deconfinement phase transition and the vacuum structure of the theory as they carry fermionic zero modes. These monopoles still interact with a potential similar to the W-bosons: $V_{m-m} = \frac{4\pi}{LTg^2}\sum_{i,j=1}^{N_m}\sum_{q_a,q_b=\pm}\sum_{a,b=0}^{r} q_i^a q_j^b \vec{\alpha}_a^\vee \cdot \vec{\alpha}_b^\vee \ln|\vec{x}_i^a - \vec{x}_j^b|$. Note the hierarchy of scales in the effective 2D Coulomb gas: $r_m \approx L << r_b \approx L/g^2 << d_{m-m} \approx Le^{2\pi^2/g^2} << d_{b-b} \approx Le^{4\pi^2/g^2}$ of monopole size, bion size, monopole-monopole separation distance, and bion-bion separation distance. This holds at weak coupling and shows that the vacuum partition function is truly that of an effective 2D dilute Coulomb gas of monopoles and bions. Note that the hierarchy fails at strong coupling and the Coulomb gas 'collapses', showing the importance of weak coupling to our duality.

To explain this one could go one step further and evaluate the path integral of the $\vec{\phi}$ field, but this proves difficult for general group. However I will do it for the case of $SU(2)$ in [1], where it was not done. The $SU(2)$ result was in [1]

$$\mathcal{Z}_{grand} = \sum_{N_{b\pm},q_i=\pm}\sum_{N_{W\pm},q_A=\pm}\frac{\beta\xi_b^{N_{b+}+N_{b-}}}{N_{b+}!N_{b-}!}\prod_i^{N_{b+}+N_{b-}}\int d^2x_i \frac{(\xi_W(\phi))^{N_{W+}+N_{W-}}}{N_{W+}!N_{W-}!}\prod_A^{N_{W+}+N_{W-}}\int d^2x_A\int\mathcal{D}\phi \tag{7.7}$$

$$\exp[\frac{32\pi LT}{g^2}\sum_{a>b}q_aq_b\log|\vec{x}_a-\vec{x}_b| + \frac{g^2}{2\pi LT}\sum_{A>B}q_Aq_B\log|\vec{x}_A-\vec{x}_B| + 4i\sum_{a,A}q_aq_A\Theta(\vec{x}_a-\vec{x}_A)+$$
$$+\int_{\mathbb{R}^2}d^2x(\frac{1}{2}\frac{g^2}{32\pi^2LT}(\partial_\mu\phi)^2+\frac{64\pi^2e^{-8\pi^2/g^2}}{TL^3g^6}\cosh 2\phi)].$$

One can go a step further and expand the cosh term using a result similar to (4.6), and solve the equations of motion for the ϕ field in the background of neutral bions. I present the details in Appendix B.2. The result for $SU(2)$ is

$$\mathcal{Z}_{grand} = \frac{Z_0}{L^2\beta}\sum_{N_b}\sum_{N_W}\sum_{N_{b'}}\sum_{q_X=\pm}\frac{\xi_b^{N_{b+}+N_{b-}}(2\xi_W)^{N_{W+}+N_{W-}}\xi_{b'}^{N_{b'+}+N_{b'-}}}{N_b^+!N_b^-!N_W^+!^2N_W^-!^2N_{b'+}!N_{b'+}!}\times \tag{7.8}$$

$$\times \prod_{a}^{N_b^+ + N_b^-} \prod_{A}^{N_W^+ + N_W^-} \prod_{\alpha}^{N_{b'}^+ + N_{b'}^-} \int d^{(2+1)} r_a \int d^{(2+1)} r_A \int d^{(2+1)} r_\alpha \times$$

$$\exp[-\frac{1}{\pi g^2} \sum_{a \neq b} q_a^m q_b^m \ln|r_a - r_b| - \frac{g^2}{4\pi(T)} \sum_{A \neq B} (q_A^e q_B^e - q_A^s q_B^s) \ln|r_A - r_B| - 4i \sum_{a,A} q_a^m q_A^e \Theta(r_a - r_A) +$$

$$+ \frac{16\pi}{g^2} \sum_{\alpha \neq \beta} q_\alpha^s q_\beta^s \ln|r_\alpha - r_\beta| + 4 \sum_{\alpha, A} q_\alpha^s q_A^s \ln|r_\alpha - r_A|].$$

From the interaction terms in the partition function (4.18) we see that like scalar charges attract, whereas like electric and magnetic charges repel. This is due to the different sign in their interaction. Similarly one could work out the propagator for the sinh-Gordon model and find that it is the same as the sine-Gordon model but with opposite sign. The W-bosons have a double nature attracting opposite electrically charged W-bosons, and attracting like scalar charged W-bosons and neutral bions. This Coulomb gas can be thought of as a 'bisexual-like' gas from these different interactions present. Let us summarize the components in the Coulomb gas and their charges, written as vectors $q_X = (q_{X,e}, q_{X,m}, q_{X,s})$. For $SU(2)$ there is only one root and co-root and so the charges are as-is numerically. For other gauge group a magnetic charge of 2, for example, corresponds not only to charges $2\vec{\alpha}_i^\vee$ but as to positive combinations of two charges $\vec{\alpha}_i^\vee + \vec{\alpha}_j^\vee$.

Coulomb gas constituent	$q_X = (q_{X,e}, q_{X,m}, q_{X,s})$
magnetic bions	$(0, \pm 2, 0)$
W-bosons	$(\pm 1, 0, \pm 1)$
neutral bions	$(0, 0, \pm 2)$

Table 3: Scalar, electric and magnetic charges of relevant Coulomb gas constituents.

This Coulomb gas can be subjected to lattice study as in [1] for the case of $SU(2)$, but for other gauge groups. Perhaps extending first the results to $SU(3)$ and $SU(N)$ would be a start in future research. See [1], [53], [54] for more on the Monte-Carlo simulations used in studying such Coulomb gases numerically. Another method of studying the deconfinement phase transition other than simulating the Coulomb gas is to map the Coulomb gas constituents to parameters of a dual spin model. The spin model that best suits the Coulomb gas at hand is a multiple component XY spin model with symmetry breaking perturbations and fugacities coupled to the scalar field $\vec{\phi}$. This we turn to now in the next section.

8 Experimental methods in detecting extra dimensions, spacelike and timelike.

Here

9 Conclusions and future work

It was found that $\mathcal{N} = 1$ super Yang-Mills on $\mathbb{R}^3 \times S^1$ has a dual description as a double Coulomb gas of various particles: W-bosons and their wino superpartners, monopole-instantons and neutral and magnetic bions and their anti-particles. The partition function was computed as well as the duality maps to the Coulomb gas of r such types of electric and magnetic charges, and several types of magnetic and neutral bions formed from combinations of BPS and KK monopoles (and their anti-monopoles). The electric charges are charged under the root lattice of the gauge group G, Λ_r, and the magnetic charges are charged under the co-root lattice, $\Lambda_r^\vee$. The elementary charges are the simple roots (co-roots), and their negatives. The interesting feature of this 'universal' Coulomb gas is that it presents a gas of particles of three charges: electric, magnetic and scalar. The first two interact with Coulomb-Coulomb interactions with particles of same charge type, or charge containing a root nearby on the Dynkin diagram. The scalar charges make the Coulomb gas unique as they interact such that like charges attract, and this introduces instability and exotic behaviour of the gas at different temperatures. The derivation of this exotic Coulomb gas for all gauge groups is the main result of this work and I hope that in the near future one will perform lattice Monte-Carlo simulations of this Coulomb gas, as done in [1] for the case of $SU(2)$, for all gauge groups.

A dual spin model was found to be of XY-model type with discrete symmetry breaking perturbations, and fugacities coupled to the scalar fields ϕ^a. The model has two coupled lattices (one for the W's which couples to a lattice of neutral and magnetic bions on the dual lattice) and the presence of the scalar fields ϕ^a. These are 'ferromagnetic' in nature as opposed to antiferromagnetic electric and magnetic charges, and introduce competition between lattices the W's couple to, and so the model can be viewed as a 'frustrated' XY-model with symmetry-breaking perturbations [49]. The W-bosons here are interpreted as the symmetry-breaking perturbations with strength proportional to the W-boson fugacity, which depends on the fields ϕ^a. The magnetic bions represent the vortices of the XY model instead of the W-bosons which were interpreted as vortices in [15]. The neutral bions introduce the 'frustration' and electromagnetic - scalar competition. As found in previous works [21], [23] the magnetic bions lead to mass gap for the dual photon fields σ^a allowing for confinement of electric charges, and the neutral bions lead to a centre-stabilizing potential. The magnetic monopole-instantons do not lead to a mass gap as they contain fermionic zero modes and so were not considered as they cannot contribute to the vacuum structure and effective potential of the theory. It is noted that in studying the supersymmetric theory on a torus is that the theory is not as simple as the non-supersymmetric version, due to the presence of the adjoint scalar fields ϕ^a, even though the GPY potential vanishes at zero temperature and partially cancels at $T > 0$. Nonetheless, the dualities derived here are interesting and have led us to new phenomena and new ways of studying Yang-Mills theory at finite temperature.

Future directions of study include the following pursuits:

1. Lattice studies, as done in [1], can be done in this general gauge group setting, even if for particular gauge group such as $SU(3)$ or G_2, in both the dual Coulomb gas model or the XY-spin model, in order to gain better understanding of the phase transition as found in [2]. A first order phase transition is expected as opposed to the second order transition in the $SU(2)$ theory [1]. This can also lead to further study of the continuity conjecture as mentioned in [2], [7] by comparing phase transitions in pure thermal Yang-Mills to the quantum phase transitions in mass deformed super Yang-Mills. Comparison can be made to previous lattice studies and new studies in general gauge group may be possible as well. One must still obtain a dual spin model for SYM in other gauge groups than $SU(2)$ as future work before simulations of the dual spin model can be done. However, from the methods presented in [1], one may be able to do simulations of the dual double Coulomb gas for all gauge groups in a soon future work.

2. It has also been of recent interest to consider finite density QCD-like theories, in particular super Yang-Mills, and their phase transitions. There is a known sign problem due to finite chemical potential and so imaginary chemical potentials have been studied instead [3], [35]. This leads to a theory with twisted boundary conditions for the adjoint fermions along the compact direction (or directions). Computing the Callias index as a function of the twist angle leads to a twist-dependent index, which equals the usual answer, 2, at the centre-symmetric and supersymmetric vacuum. This recent work can be generalized to general gauge group and dependence on the boundary conditions is quite interesting.

3. Mean field theory methods can be used for the XY spin models considered here, as well as related spin models in special cases. Although not exact, mean field theory can tell phase transitions and their orders, although at transition temperatures that are not always correct although within an order of magnitude. Studies of XY-models with symmetry breaking perturbations have been studied [44], [45] for different values of p in the $\cos p\theta$-term and phase diagrams mapped out. It would be curious to implement a mean field theory that takes into account vortices and can verify known results, and produce new ones for other gauge groups not studied before. This would be interesting even for the case of zero scalar fields $\phi^a = 0$. Cases with $\vec{\phi} \neq 0$ can be done as well in the mean field method. These cases are related to 'frustrated' XY models in the case that, on some lattices, bonds are ferromagnetic (like the scalar charged W's and neutral bions), while on others they are antiferromagnetic (like electric W's and magnetic bions). These competing interactions lead to 'frustration', that is a ground state that is degenerate and not at minimum possible energy without frustration. Models with competing F and AF interactions were studied in [44] and [46].

4. Renormalization group equations and flow can be determined from the partition function (4.18). Special cases for $SU(2)$ and $SU(3)$ have been done with good success in

[15] leading to known results of deconfinement, transition temperatures, and scaling parameters/critical exponents. In the cases of higher rank it was found that no fixed points appeared to exist for the RGEs and that the electric-magnetic duality no longer holds. It would be interesting to continue investigating the $SU(N)$ RGE cases and other groups of higher rank to see (possibly by going to higher order in the expansion) if there are fixed points and to find the nature of the critical points.

5. General compactifications on toroidal spaces such as $\mathbb{R}^D \to \mathbb{R}^d \times (S^1)^{\times D-d}$ can be done in this generic case, although the applications or interests may not be immediate.

It is hoped that this work has provided a framework for future study with the goal of simulating the Coulomb gas derived as a main result of this work for all gauge groups. It is hoped as well that correct spin models for $SU(N)$ and other gauge groups, both for YM and SYM, can be done and lattice simulations of them performed, in order to compare to the Coulomb gas results for any gauge group and to determine the nature of the deconfining phase transition.

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A Derivation of ζ-regulated summations

To present details in zeta function regularization summations, generalizations those in [], and my thesis [], can be summarized here for the calculations used in this paper.

Consider the sum

$$\Sigma(a;s) \equiv \sum_{(n,m)\in\mathbb{Z}}{}' [n^2 + a^2 m^2]^{-s} = \sum_{\vec{n}\neq 0} n^{-2s} + \sum_{m\neq 0}\sum_{n} [n^2 + a^2 m^2]^{-s},$$

where the prime in the sum means $m.n$ are not simultaneously zero. Using the Mellin transform, one obtains:

$$\Sigma(a;s) = 2\zeta(2s) + \sum_{m\neq 0}\sum_{n} \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-(n^2+a^2m^2)t} =$$

$$= 2\zeta(2s) + \sum_{m\neq 0}\sum_{n} \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty dt t^{s-3/2} e^{-a^2m^2t}(1 + 2\sum_{n=1}^{\infty} e^{-\pi^2 n^2 t}),$$

where using a Poisson resummation, leads to

$$= 2\zeta(2s) + 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}|a|^{1-2s}\zeta(2s-1) + \frac{8\pi^5}{\Gamma(s)}|a|^{\frac{1}{2}-s}\sum_{n,m>0}(\frac{n}{m})^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|a|mn),$$

where we can also use $\zeta'(-2n) = \frac{(-1)^n}{2(2\pi)^{2n}}\Gamma(2n+1)\zeta(2n+1)$.

For general 'anyons' (say α or $\beta = \frac{1}{2}$ for the fermionic/thermal case), we have a general sum:

$$\Sigma(a,\alpha,\beta;s) \equiv \sum_{(n,m)\in\mathbb{Z}}[(n+\alpha)^2+a^2(m+\beta)^2]^{-s} = \sum_m\sum_n\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}e^{-((n+\alpha)^2+a^2(m+\beta)^2)t} =$$

$$= \frac{\sqrt{\pi}}{\Gamma(s)}\sum_{m\neq 0}\int_0^\infty dt t^{s-3/2}e^{-a^2(m+\beta)^2t}(1+2\sum_{n=1}^\infty\cos(2\pi na)e^{-\pi^2n^2t}) =$$

$$= \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}|a|^{1-2s}[\zeta(2s-1,\beta)+\zeta(2s-1,1-\beta)] + \frac{4\pi^5}{\Gamma(s)}|a|^{\frac{1}{2}-s}\sum_{m=0}^\infty\sum_{n=1}^\infty n^{s-\frac{1}{2}}\times$$

$$\times\cos(2\pi na)[(m+\beta)^{-s+\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|a|n(m+\beta)a)+(m+1-\beta)^{-s+\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|a|n(m+1-\beta))],$$

and, using Hurwitz zeta functions for the half integer cases, yields the results from Sections 2,3. □

B Notes on Lie groups and Lie algebras

For a sufficiently self-contained description of the mathematical constructs in our theory let us review Lie groups and Lie algebras. The familiar reader can skip to A.2 for the notation of roots and weights.

B.1 Notes on general Lie theory

Let us begin by defining a Lie algebra and give its properties.
A *Lie algebra* $\mathfrak{g}$ is a vector space over a field F (which we take here to be either real, $\mathbb{R}$, or complex, $\mathbb{C}$) with a binary operation (called the Lie bracket) $[\cdot,\cdot]\to\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying the basic properties:
(i) bilinearity: $[ax+by,cz+dw] = ac[x,y]+ad[x,w]+bc[y,z]+bd[y,w],\ \forall a,b,c,d\in F$ and $\forall x,y,z,w\in\mathfrak{g}$.
(ii) assymetry: $[x,y]=-[y,x],\ \forall x,y\in\mathfrak{g}$
(iii) Jacobi identity: $[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0,\ \forall x,y,z\in\mathfrak{g}$.
A Lie algebra is equipped with a basis of generators $\{T^a\}_{a=1}^r$ where $r = dim(\mathfrak{g})$, and these satisfy the same relations above. The generators, forming a basis, have commutators which are linear combinations of generators, $[T^a,T^b]=f^{abc}T^c$, where the coefficients f^{abc} are the

structure constants of the algebra. In the fundamental representation this dimension is minimal and equal to the rank of its corresponding Lie group.

A Lie algebra is called *simple* if it is non-Abelian and has no non-zero proper ideals, and is *semi-simple* if it is non-Abelian and has no non-zero proper Abelian ideals. Hence a semi-simple Lie algebra $\mathfrak{g}$ can be written as a direct sum of simple Lie algebras $\mathfrak{g}_i$, $\mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_i$. We consider here just semi-simple Lie algebras.

A Lie algebra varies depending on its representation. A *representation* $\mathcal{R}$ is given a map $\pi_{\mathcal{R}} : \mathfrak{g} \to \mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is the enveloping associative Lie algebra of endomorphisms of a vector space V. The dimension of the representation $dim(\mathcal{R}) = dim(V)$ equals the dimension of the vector space V, if it is finite. For example, the fundamental representation has $dim(V) = rank(G)$. Also, in this paper, we use often the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ where the action is $ad(x)(y) = [x, y],\ \forall x, y \in \mathfrak{g}$.

A Lie group has a subgroup called the maximal torus $T \subset G$, whose elements commute with all other elements of the Lie group, and is topologically a torus $(S^1)^{\times r}$ where $r = dim(G)$, the topological dimension of the group. Its Lie algebra $\mathfrak{t} = Lie(T)$ is called the Cartan subalgebra of the Lie algebra and is of dimension r. Its generators $\{H^a\}_{a=1}^r$ with $[H^a, H^b] = 0]$ form an r-dimensional subspace of $\mathfrak{g}$ and satisfy the normalization $tr(H^a H^b) = \delta^{ab}$ here.

The other generators of the Lie algebra can be represented by $dim(G) - r$ raising and lowering operators, $\{E_\alpha\}$ and $\{E_{-\alpha} = E_\alpha^\dagger\}$, which satisfy the relations

$$\begin{aligned} [H^i, E_\alpha] &= \alpha^i E_\alpha \\ [E_\alpha, E_{-\alpha}] &= \alpha_i H^i \\ [E_\alpha, E_\beta] &= N_{\alpha\beta\gamma} E_\gamma, \end{aligned} \tag{B.1}$$

where the constants $N_{\alpha\beta\gamma}$ will not be needed later. The contravariant and covariant roots are related by the Cartan Killing form $g^{ij} = Tr[H^i H^j]$.

A *Lie group*, as a reminder, is a group that is also a differentiable manifold, and hence has a differential structure or derivation (that satisfies the Leibnitz rule). In fact, its Lie algebra corresponding to it is the tangent space to the Lie group, specifically to its covering space $\tilde{G}$. Figure 4 shows all possible simply-connected, semi-simple Lie algebras and their Dynkin diagrams. For more definitions and detailed theory see [9].

B.2 The roots and the weights

One way to define the roots of a Lie group G that will be useful later on is to consider it from the point of view of representations of its corresponding Lie algebra $\mathfrak{g}$. In general we

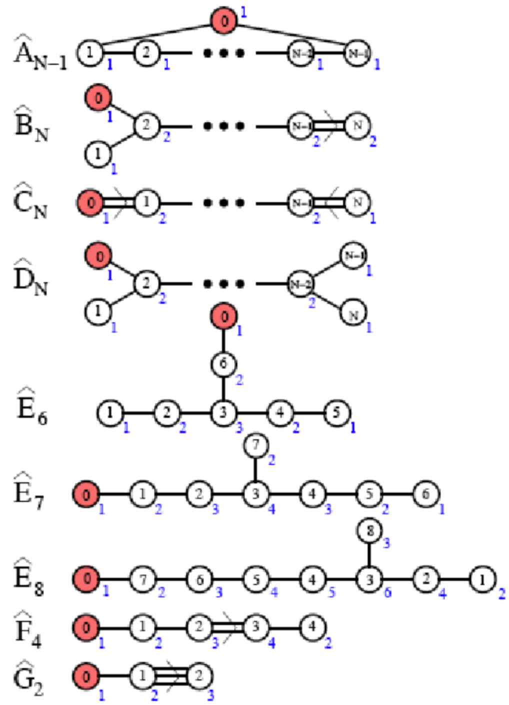

Figure 5. All simple Lie algebras and their affine Dynkin diagrams. The numbers inside the nodes denote the root, and the numbers bside each node are the Kac labels. The red circles are the lowest (affine) roots of the Lie algebra.

define the root $\vec{\alpha}_i$ as an eigenvalue. In fact it is a function valued on $\mathfrak{t} = LieT$, where T is the maximal torus of G

$$\vec{\alpha}_i : \mathbb{C}\mathfrak{t} \to \mathbb{C}$$

with its eigenspace $E_{\alpha_i} \in \mathbb{C}\mathfrak{g}$ defined by

$$[H, E_{\alpha_i}] = \vec{\alpha}_i(H) E_{\alpha_i}, \tag{B.2}$$

where $H \in \mathbb{C}\mathfrak{t}$, the Cartan subalgebra of $\mathfrak{g}$.
We can see how this works for $SU(N)$. Beginning with $SU(2)$, we have Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. It is clear that $\mathfrak{t} = span\{\lambda\sigma_3 = \left(\begin{smallmatrix}\lambda & 0 \\ 0 & -\lambda\end{smallmatrix}\right)\}_{\lambda\in\mathbb{C}}$ and that there are two root spaces, one with root the negative of the other: $E^+ = span\{\left(\begin{smallmatrix}0 & 1 \\ 0 & 0\end{smallmatrix}\right)\}$, $E^- = span\{\left(\begin{smallmatrix}0 & 0 \\ 1 & 0\end{smallmatrix}\right)\}$. It is easy to check that the roots satisfying equation (A.2) are $\vec{\alpha}_\pm(H(\lambda)) = \pm 2\lambda$.

This clearly generalizes to $SU(N)$ with Lie algebra $\mathfrak{sl}_N(\mathbb{C})$. The maximal torus is just $T \approx \mathbb{T}^{N-1} \approx \{diag(e^{i\theta_j})_{j=1}^N) | \prod^N e^{i\theta_j} = 1\}$. The Cartan subalgebra is the set of matrices with complex numbers λ_j along the diagonal, accompanied by their negatives $-\lambda_j$, so as to make the trace vanish (this is in fact for the adjoint representation). The root spaces are just the span of each E_{jk}, the $N \times N$ matrix with a 1 in the i, j-th position and zeroes elsewhere. One easily checks that the roots obey $\alpha_{jk}(H(\lambda)) = \lambda_j - \lambda_k$ along with their negatives from

$$[H(\lambda), E_{jk}] = (\lambda_j - \lambda_k)E_{jk}. \tag{B.3}$$

This is why in the adjoint representation, the roots take on values given by differences of Wilson line eigenphases θ_j.

We will need to know the weights in the adjoint representation, which I prove are the roots (the full set) of the Lie algebra. Let us describe representation theory in general a bit first.

A representation of a Lie algebra is a homomorphism from the Lie algebra $\mathfrak{g}$ into the endomorphism group of a certain vector space V,

$$\phi : \mathfrak{g} \to End(V),$$

and preserves the Lie bracket. The dimension of the representation is the dimension of the vector space V underlying the representation. The dimension of the Lie algebra itself is the numbers of independent generators of $\mathfrak{g}$. In the fundamental representation the dimension of $\mathfrak{g}$ equals the dimension of V and hence of the representation. The rank r, however, of a Lie algebra is the dimension of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The Cartan subalgebra has a set of Abelian generators in the Cartan-Weyl basis $\{H^i\}_{i=1}^r$ satisfying $[H^i, H^j] = 0$ and the roots, as mentioned before, satisfy eigenvalue-like expressions: $[H^i, E^\alpha] = \alpha_i E^\alpha$, and there are hence $r = rk(\mathfrak{g})$ simple positive roots of $\mathfrak{g}$. The Lie algebra then decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the eigenspace, spanned by E^α. We can also prove what the weights (eigenvalues of the H^i) are in fact the roots in the adjoint representation. Indeed,

$$\phi_{adj}(H^i)E^\alpha = ad_{H^i}E^\alpha \equiv [H^i, E^\alpha] = \alpha_i E^\alpha, \tag{B.4}$$

proving the claim. $\square$

Below I will list the positive roots for each Lie algebra, but one more point to make is the role of the affine root in spaces with a compact direction. The Lie algebra with the affine root included is the Lie algebra of the loop group LG of maps $\pi : S^1 \to G$, $Lie(LG) = L\mathfrak{g}$. Similarly on spaces with multiple compact directions there are more roots to be added and the resulting algebra is the toroidal Lie algebra. The affine roots are included below.

For all semi-simple Lie groups (described below) a choice of simple positive roots is given as follows:

$A_{N+1} \approx SU(N)$:

This is the group of rotations about the origin in $\mathbb{C}^N$. It preserves the lengths of vectors. Their sets of simple roots are:

$$\{\alpha_i = e_i - e_{i+1}\}_{i=1}^{N} \tag{B.5}$$

The affine root is $\alpha_0 = -\sum_{i=1}^{N} \alpha_i = e_N - e_1$.

$B_N \approx Spin(2N+1)$:

This is the double cover of the orthogonal group $SO(2N+1)$, the rotation group in $^{2N+1}$. Here, the set of simple roots is given by

$$\{e_i - e_{i+1}\}_{1<i<N-1} \cup \{e_N\} \tag{B.6}$$

The affine root is $-\alpha_0 = e_1 + e_2 = \alpha_1 + 2\sum_{i=2}^{N} \alpha_i$.

$C_N \approx Sp(2N)$:

This is the group of $2N \times 2N$ matrices preserving the antisymmetric scalar product $J = \left(\begin{smallmatrix} 0 & 1_N \\ -1_N & 0 \end{smallmatrix}\right)$, so $M^T J M = J \ \forall M \in Sp(2N)$. The simple roots are:

$$\{e_i - e_{i+1}\}_{1<i<N-1} \cup \{2e_N\} \tag{B.7}$$

The affine root is $-\alpha_0 = 2e_1 = \sum_{i=1}^{N-1} 2\alpha_i + \alpha_N$.

$D_N \approx Spin(4N), Spin(4N+2)$ (N even, odd respectively):

These are the double covers of the orthogonal groups $SO(4N)$ and $SO(4N+2)$ respectively. The simple roots are:

$$\{e_i - e_{i+1}\}_{1<i<N-1} \cup \{e_{N-1} + e_N\} \tag{B.8}$$

The affine root is $-\alpha_0 = e_1 + e_2 = \alpha_1 + 2\sum_{i=2}^{N-2} \alpha_i + \alpha_{N-1} + \alpha_N$.

E_6:

This is the rank 6 exceptional Lie group of dimension 78. The simple roots are:

$$\{(1,-1,0,0,0,0) \tag{B.9}$$

$$(0,1,-1,0,0,0)$$

$$(0,0,1,-1,0,0)$$

$$(0,0,0,1,1,0)$$
$$-\frac{1}{2}(1,1,1,1,1,-\sqrt{3})$$
$$(0,0,0,1,-1,0)\}$$

The affine root is $-\alpha_0 = e_1 - e_8 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$. This is in the 8 dimensional basis and we note all vectors are orthogonal to $\sum_{i=1}^{8} e_i$ and to $e_1 + e_8$ and so gauge fields are constrained by $\phi_1 + \phi_8 = \sum_{i=2}^{7} \phi_i = 0$.

E_7:

This is the rank 7 exceptional group of dimension 133. The simple roots are:

$$\{(1,-1,0,0,0,0,0) \tag{B.10}$$
$$(0,1,-1,0,0,0,0)$$
$$(0,0,1,-1,0,0,0)$$
$$(0,0,0,1,-1,0,0)$$
$$(0,0,0,0,1,1,0)$$
$$-\frac{1}{2}(1,1,1,1,1,1,-\sqrt{2})$$
$$(0,0,0,0,1,-1,0)\}$$

The affine root is $-\alpha_0 = e_2 - e_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7$. The fields are constrained to live on the plane orthogonal to $\sum_{i=1}^{8} e_i = 0$ in the 8 dimensional basis.

E_8:

This is the rank 8 exceptional group of dimension 248. The simple roots are:

$$\{(1,-1,0,0,0,0,0,0) \tag{B.11}$$
$$(0,1,-1,0,0,0,0,0)$$
$$(0,0,1,-1,0,0,0,0)$$
$$(0,0,0,1,-1,0,0,0)$$
$$(0,0,0,0,1,-1,0,0)$$
$$(0,0,0,0,0,1,1,0)$$
$$-\frac{1}{2}(1,1,1,1,1,1,1,1)$$
$$(0,0,0,0,0,1,-1,0)\}$$

The affine root is $-\alpha_0 = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$.

F_4:

This rank 4 exceptional Lie group has dimension 52. Its simple roots are given by

$$\{(0,1,-1,0) \tag{B.12}$$
$$(0,0,1,-1)$$
$$(0,0,0,1)$$
$$-\frac{1}{2}(-1,1,1,1)\}$$

The affine root is $-\alpha_0 = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$.

G_2: (embedded in 2D subspace of $\mathbb{R}^3$, the plane perpendicular to line $x + y + z = 0$)

This is the rank 2 exceptional Lie group of dimension 14. Its simple roots are:

$$\{(0,1,-1),(1,-2,1)\} \tag{B.13}$$

The affine root is $-\alpha_0 = e_1+e_2-2e_3 = 2\alpha_1+3\alpha_2$. All vectors are orthogonal to $e_1+e_2+e_3 = 0$.

Note that the coefficients k_i in the definition of the affine root $\vec{\alpha}_0 = -\sum_{i=1}^{r} k_i\vec{\alpha}_i$ are called the Kac labels of the Lie algebra. The Coxeter number of the Lie algebra is $h(G) = \sum_{i=1}^{r} k_i+1$.

We will also need co-roots in our future defined as

$$\vec{\alpha}^\vee \equiv \frac{2}{\vec{\alpha}^2}\vec{\alpha} \in \Lambda_r^\vee, \tag{B.14}$$

where they span the co-root lattice $\Lambda_r^\vee$, and the $\vec{\alpha}$'s are the $r = rank(G)$ simple roots given above and span the root lattice Λ_r.

The weights.

We also need to get to know the weight system, with lattice Λ_w, and its co-weight lattice. The weight vectors $\vec{w}_i$ for a set of simple roots $\vec{\alpha}_i$ are defined via

$$\vec{w}_j \cdot \vec{\alpha}^\vee = \delta_{ij}, \tag{B.15}$$

and the co-weights are defined as were the co-roots:

$$\vec{w}^\vee \equiv \frac{2}{\vec{w}^2}\vec{w} \in \Lambda_w^\vee. \tag{B.16}$$

Since we are dealing with affine Lie algebras we need to define the affine co-root in terms of simple co-roots (the affine roots were given above, $\vec{\alpha}_0 = -\sum_j^r k_j \vec{\alpha}_j$):

$$\vec{\alpha}_0^\vee = -\sum_j^r k_j^\vee \vec{\alpha}_j^\vee , \tag{B.17}$$

and the dual Coxeter number is defined from the coefficients $c_2 = \sum_{i=0}^r k_i^\vee$. For $\mathfrak{g} = \mathfrak{su}(r+1)$, $c_2 = r+1$ as all $k^\vee$'s are 1s, and all α's have norm $\sqrt{2}$. We will also soon need these data for $\mathfrak{g}_2$, where $c_2 = 4$ for $\{k_i^\vee\} = \{1,2,1\}$ and $\{\alpha_i^2\} = \{2,2,2/3\}$. For data such as these for all semi-simple Lie groups see [11]. Table 1 shows some such data including the Kac labels and (dual) Coxeter numbers. It is interesting to note that the Coxeter number h of a group is the number of roots divided by the rank of the group. Figure 4 shows all semi-simple Lie algebras as (affine) Dynkin diagrams with the Kac labels included. As a reminder a Dynkin diagram is (for our purposes) a graph with single, double or triple lines connecting nodes, represented by simple roots. The multiplicity of the lines (edges) will not concern us, but are related to the length of roots represented by the nodes the edge connects. The affine Dynkin diagram, shown in Figure 4, contains the affine root $\vec{\alpha}_0$.

Group, G	$r = rk(G)$	h	$c_2(G)$	$[k_0^\vee, \ldots, k_r^\vee]$	$[k_0, \ldots, k_r]$
$SU(N+1)$	N	$N+1$	$N+1$	$[1,1,\ldots,1]$	$[1,1,\ldots,1]$
$SO(2N+1)$	N	$2N$	$2N-2$	$[1,1,1,2,\ldots,2]$	$[1,1,2,\ldots,2]$
$SO(2N)$	N	$2N$	$2N-2$	$[1,1,1,1,2,\ldots,2]$	$[1,2,\ldots,2,1]$
$Sp(2N)$	N	$2N-2$	$N+1$	$[1,1,\ldots,1]$	$[1,1,2,\ldots,2,1,1]$
G_2	2	6	4	$[1,1,2]$	$[1,2,3]$
F_4	4	12	9	$[1,1,2,3,2]$	$[1,2,3,4,2]$
E_6	6	12	12	$[1,1,1,2,2,2,3]$	$[1,1,2,3,2,1,2]$
E_7	7	18	18	$[1,1,2,2,2,3,3,4]$	$[1,2,3,4,3,2,1,2]$
E_8	8	30	30	$[1,2,2,3,3,4,4,5,6]$	$[1,2,3,4,5,6,4,2,3]$

Table 4: (Dual) Kac labels and dual Coxeter numbers for semi-simple Lie groups. Note that these in general differ from the Kac labels found from the table of simple roots above.

The weights of a Lie algebra in a given irrep R represent the charges of particles possible for that irrep and hence are important Lie algebra data. The matrices $R(h)$ for any h in the Cartan subalgebra can be simultaneously diagonalized giving vectors $\vec{w} \in \mathfrak{t}^*$ of eigenvalues so that $\vec{w} \cdot h$ is an eigenvalue of $R(h)$. These vectors $\vec{w}$ belong to the set of weights of R Δ_w^R and their integral span $\mathbb{Z}[\Delta_w^R] = \Lambda_w^R$ is called the weight lattice of R. The group lattice $\Gamma_G = \cup_R \Lambda_w^R$ is the union of irrep weight lattices. At the level of Lie group, the eigenvalues of irrep R of an element $g \in T_G$, the maximal torus of G, are $\exp(2\pi i \vec{w} \cdot h)$. The periodicity of the maximal torus are given by shifts in the lattice of those h such that $\vec{w} \cdot h \in \mathbb{Z}$. The dual lattice of co-weights is defined by the lattice of such h, Λ_w^{R*}. The smallest arising group lattice is called the root lattice Λ_r, whereas the largest is called the weight lattice Λ_w.

For completeness I now present the weights of the adjoint representation (which are in fact the set of ALL roots as was shown above) for each Lie algebra. The number of weights is equal to the dimension of the representation minus the rank of the Lie algebra (the number of null weights of eigenvalue zero from the action of the Cartan generators.)

$A_{N+1} \approx SU(N)$:
There are $N^2 + N$ adjoint weights in all, $N(N+1)/2$ being positive. All are of length $\sqrt{2}$ with 1 in one entry i, -1 in position j, and zeros elsewhere. We denote them as $\vec{\alpha}_{ij}^{\pm}$ where the superscript is positive if the root is. The positive roots are taken to be the ones with a +1 occuring in an earlier position than -1, i.e. $i < j$.

$B_N \approx Spin(2N+1)$:
There are $2N^2$ weights of two types: $\vec{\alpha}_{ij}^{\pm}$ which are all integer vectors of length $\sqrt{2}$, and $\beta_i^{\pm,B}$ which are all integer vectors of length 1. The positive weights are those with a +1 occurring before a -1 as usual.

$C_N \approx Sp(N)$:
In all there are $2N^2$ roots including the $\vec{\alpha}_{ij}^{\pm}$ above, and with $\beta_i^{\pm,C} = \pm 2e_i$.Positive roots are as before.

$D_N \approx Spin(4N), Spin(N+2)$:
Here all roots are all integer vectors of length $\sqrt{2}$. These include the $\vec{\alpha}_{ij}^{\pm}$ above, but also those with 2 entries both -1 or both +1, called $\vec{\beta}_{ij}^{\pm,D}$. There are $2N(N-1)$ in all.

E_6:
The adjoint weights include the $4\times\binom{5}{2}$ permutations of the entries of the vectors $(\pm1, \pm1, 0, 0, 0, 0)$ keeping a zero in the last entry, plus the vectors of the form $\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm\sqrt{3})$ with an odd number of + signs. This gives a total of 72 weights.

E_7:
We have here $4 \times \binom{6}{2}$ permutations of $(\pm1, \pm1, 0, 0, 0, 0, 0)$ keeping a zero in the last entry, plus the vectors of the form $\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm1, \pm\sqrt{2})$ with an even number of + signs, plus the two vectors $(\vec{0}, \pm\sqrt{2})$. This gives a total of 126 weights.

E_8:
We have 112 roots as permutations of $(\pm1, \pm1, 0, 0, 0, 0, 0, 0)$, plus the 128 vectors of the form $\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm1, \pm1, \pm1)$ with an even number of - signs.

F_4:
We have here 48 roots: 24 as permutations of $(\pm 1, \pm 1, \vec{0})$ (call them type I), plus 8 roots as permutations of $(\pm 1, \vec{0})$ (type J), and 16 roots of the form $(\pm 1, \pm 1, \pm 1, \pm 1)/2$ (type K).

G_2:
Here there are 12 adjoint weights:

$$(1,-1,0),(2,-1,-1),(1,0,-1),(1,-2,1),(0,1,-1),(1,1,-2),$$

together with their negatives.

For more on Lie algebras, weights and representations, a great resource is [9].

Gauge cells and Weyl chambres.

The Weyl group $W(\mathfrak{g})$ is another group of gauge identifications on $\mathfrak{t}$, that acts as a group of linear transformations on $\mathfrak{t}$ that preserves the set of roots Δ_r (permutes them). It includes a Weyl reflection for each simple root α which acts on $h \in \mathfrak{t}^*$ by $\sigma_\alpha(h) = h - (h \cdot \vec{\alpha}^\vee)\vec{\alpha}$. It acts on $\varphi \in \mathfrak{t}$ by $\sigma_\alpha(h)[\vec{\varphi}] = h \cdot \sigma_\alpha(\vec{\varphi})$ and so $\sigma_\alpha(\vec{\varphi}) = \varphi - 2(\vec{\alpha} \cdot \vec{\varphi})\vec{\alpha}^\vee$ and is a reflection about the plane with normal vector $\vec{\alpha}$ passing through the origin. Allowing translations of the co-root lattice, the group of transformations is the semi-direct product $\hat{W}$ of W and $\Gamma_r^\vee$. A fundamental domain or gauge cell (or affine Weyl chambre) $\hat{\mathfrak{t}}$ for G is the quotient $\mathfrak{t}/\hat{W}$. A choice often used for the affine Weyl chambre is

$$\hat{\mathfrak{t}} = \{\vec{\varphi} \in \mathfrak{t} | 0 \leq \vec{\alpha} \cdot \vec{\varphi},\ \forall \vec{\alpha} \in \Delta_r^s,\ -\vec{\alpha}_0 \cdot \vec{\varphi} \leq 1\},$$

where Δ_r^s denotes the set of simple roots. This is the cell of interest here as it is the Cartan subalgebra modulo gauge equivalences. At points interior to $\hat{\mathfrak{t}}$ the unbroken gauge group is the maximal torus $U(1)^r$, while on the cell boundary, as mentioned previously, the gauge symmetry is enhanced due to elements being fixed by the gauge transformations $\hat{W}$, and the theory is no longer fully Abelianized. For explicit root systems and gauge cells see Appendix B of [4]. Figure 5 shows the gauge cells for rank 2 Lie algebras [4].

C Deriving the GPY potential, dual Coulomb gas and 'affine' XY model

C.1 GPY effective potential derivation

Here we present specific details of the zeta function regularization for perturbative effective potentials in various dimensions.

D Clifford algebras and group representations in split signature spacetimes

Here

E Monopole solutions for all simple groups

I present here monopole solutions used in section 3 and give their actions. Let us review the $SU(2)$ case first, where we have one BPS and one KK solution on $\mathbb{R}^3 \times S^1$. The solution exists due to the symmetry breaking $SU(2) \to U(1)$ so that $\pi_2(SU(2)/U(1)) \approx \pi_1(U(1)) \approx \mathbb{Z}$.

In general recall the Euclidean action of pure Yang-Mills splits into electric and magnetic field energy components.

$$S = \frac{1}{2g^2} \int_{\mathbb{R}^3 \times S^1} tr[F^a_{MN} T^a F^{bMN} T^b] = \frac{L}{g^2} \int d^3x tr[B^a_\mu T^a B^b_\mu T^b + E^a_\mu T^a E^b_\mu T^b], \tag{E.1}$$

where $B^a_\mu = \epsilon_{\mu\nu\lambda} F^a_{\nu\lambda}$ and $E^a_\mu = D_\mu A^a_3$, and the generators are taken in the fundamental representation. The action (78) can be rewritten as

$$S = \frac{L}{g^2} \int d^3x tr[(B^a_\mu \mp D_\mu A^a_3) T^a (B^b_\mu \pm D_\mu A^b_3) T^b] \pm 2tr[B^a_\mu T^a D_\mu A^b_3 T^b]. \tag{E.2}$$

The last term can be integrated by parts and the equation of motion $D_\mu B^a_\mu = 0$ so we have the action as

$$\begin{aligned} S = \frac{L}{g^2} \int d^3x tr[(B^a_\mu \mp D_\mu A^a_3) T^a (B^b_\mu \pm D_\mu A^b_3) T^b] \\ \pm 2 \frac{L}{g^2} \int_{S^2_\infty} d^2\Sigma_\mu tr[B^a_\mu T^a A^b_3 T^b]. \end{aligned} \tag{E.3}$$

Our monopoles satisfy self-dual or anti self-dual equations $B^a_\mu = \pm D_\mu A^a_3$ (equivalently, $F_{MN} = \pm \tilde{F}_{MN}$) which simplifies the action to

$$S = \pm \frac{L}{g^2} \int_{S^2_\infty} d^2\Sigma_\mu tr[B^a_\mu T^a A^b_3 T^b]. \tag{E.4}$$

The instanton number satisfies

$$\mathcal{K} = (16\pi)^{-1} \int_{\mathbb{R}^3 \times S^1} tr[F^a_{MN} T^a \tilde{F}^b_{MN} T^b] = \pm \frac{g^2}{8\pi^2} S, \tag{E.5}$$

where the last equality holds for self dual and anti self dual solutions respectively. Setting $\mathcal{K} = 1$ gives the usual monopole action as required: $S = 8\pi^2/g^2$.

E.1 $SU(2)$ monopole solution

For $SU(2)$ I present the solution as given in [2]. We take as generators $T^a = \tau^a/2$ where τ^a are the Pauli matrices, and satisfy $tr[T^a T^b] = \delta^{ab}/2$. The action is

$$S = \frac{1}{4g^2} \int_{\mathbb{R}^3 \times S^1} F^a_{MN} F^{aMN}. \tag{E.6}$$

In the hedgehog gauge the monopole solution is given by

$$A_\mu = A^c_\mu \tau^c = \epsilon_{\mu\nu c} \frac{x_\nu}{|x|^2} (1 - \frac{v|x|}{\sinh(v|x|)}) \tau^c, \tag{E.7}$$

$$A_3 = \Psi^c \tau^c = \frac{x_c}{|x|^2}(v|x| \coth(v|x|) + 1)\tau^c,$$

where $v =< A_3 >$. Putting these solutions (86) into the action (85) gives $S = 4\pi v L/g^2$ so that $v = 2\pi/L$ gives the usual monopole instanton action $8\pi^2/g^2$.

The magnetic field's asymptotics are found to be (in the string/singular gauge)

$$B_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}F_{\nu\lambda} \to_{|x|\to\infty} -\frac{x_\mu}{2|x|^3}\tau^3. \tag{E.8}$$

E.2 Monopole solutions for arbitrary gauge group G

To find the monopole solutions for general gauge group we can embed the $SU(2)$ solution into G, $SU(2) \subset G$, for each simple co-root $\vec{\alpha}_i^\vee$. The KK monopole solution will be given later to give $r+1$ monopole solutions as is consistent with the symmetry breaking $G \to U(1)^r$ (for the case of full Abelianization), which presents r BPS solutions as $\pi_2(G/U(1)^r) \approx \pi_1(U(1)^r) \approx \mathbb{Z}^r$ (since $\pi_2(G) \approx 0$ for covering spaces of Lie groups $\tilde{G}$, which we consider here as they allow all representations, in particular spin representations). The KK solution arises due to the compact direction and will be given later associated to the affine co-root $\vec{\alpha}_0^\vee$.

The $SU(2)$ embedding into G for each simple root $\vec{\alpha}_i$ is given by

$$t^1 = \frac{1}{\sqrt{2\vec{\alpha}_i^2}}(E_{\alpha_i} + E_{-\alpha_i}),\ t^2 = \frac{1}{\sqrt{2i\vec{\alpha}_i^2}}(E_{\alpha_i} - E_{-\alpha_i}),\ t^3 = \frac{1}{2}\vec{\alpha}_i^\vee \cdot \vec{H}, \tag{E.9}$$

which obey the $SU(2)$ algebra commutation relations $[t^a, t^b] = i\epsilon^{abc}t^c$. The solutions for the gauge field are the same as (20), (21) but with

$$A_3 = \Psi^c \tau^c + (\vec{\phi} - \frac{1}{2}\vec{\alpha}_i^\vee v) \cdot \vec{H}, \tag{E.10}$$

where $\vec{\phi}$ determines the asymptotics of the gauge field $v = \vec{A}_3 \cdot \vec{\alpha}_i = \vec{\alpha}_i \cdot \vec{\phi}/L$. The solution A_3 is as is to gaurantee these asymptotics since (in string gauge)

$$\Psi^c \tau^c|_{|x|\to\infty} = \frac{x^c}{|x|}t^c \frac{\vec{\alpha}_i \cdot \vec{\phi}}{L} = t^3 \frac{\vec{\alpha}_i \cdot \vec{\phi}}{L} = \frac{1}{2}\frac{\vec{\alpha}_i \cdot \vec{\phi}}{L}\vec{\alpha}_i^\vee \cdot \vec{H}.$$

The BPS magnetic monopole's magnetic field's asymptotics are given by

$$B_\mu^{\alpha_i} = -\frac{x_\mu}{|x|^3}\frac{\vec{\alpha}_i^\vee \cdot \vec{H}}{2},\ i = 1, \ldots, r. \tag{E.11}$$

Its action and instanton number are given by

$$S^{\alpha_i} = \frac{4\pi}{g^2}\vec{\alpha}_i^\vee \cdot \vec{\phi}, \quad \mathcal{K}^{\alpha_i} = \frac{\vec{\alpha}_i^\vee \cdot \vec{\phi}}{2\pi}, \tag{E.12}$$

respectively.

The other solution mentioned before, the KK monopole, can be found by a Weyl reflection as in [14]. Its asymptotic magnetic field is

$$B_\mu^{\alpha_0} = -\frac{x_\mu}{|x|^3}\frac{\vec{\alpha}_0^\vee \cdot \vec{H}}{2}. \tag{E.13}$$

Note that it has negative magnetic charge. (Also, since $\vec{\alpha}_0^\vee = -\sum_{i=1}^r k_i^\vee \vec{\alpha}_i^\vee$ an instanton can be formed from the collection of $2c_2(G)$ monopoles.) Its action and monopole number are found to be [2]

$$S^{\alpha_0} = \frac{4\pi}{g^2}(2\pi + \vec{\alpha}_0^\vee \cdot \vec{\phi}), \quad \mathcal{K}^{\alpha_0} = \frac{2\pi + \vec{\alpha}_0^\vee \cdot \vec{\phi}}{2\pi}. \tag{E.14}$$

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