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Raychaudhuri Equations, Tidal Forces, and the Weak-Field Limit in Schwarzshild–Finsler–Randers Spacetime

Alkiviadis Triantafyllopoulos [1,*], Emmanuel Kapsabelis [1] and Panayiotis C. Stavrinos [2]

[1] Section of Astrophysics, Astronomy and Mechanics, Department of Physics, National and Kapodistrian University of Athens, Panepistimiopolis, 15784 Athens, Greece; manoliskapsabelis@yahoo.gr
[2] Department of Mathematics, National and Kapodistrian University of Athens, Panepistimiopolis, 15784 Athens, Greece; pstavrin@math.uoa.gr
* Correspondence: alktrian@phys.uoa.gr

Abstract: In this article, we study the form of the deviation of geodesics (tidal forces) and the Raychaudhuri equation in a Schwarzschild–Finsler–Randers (SFR) spacetime which has been investigated in previous papers. This model is obtained by considering the structure of a Lorentz tangent bundle of spacetime and, in particular, the kind of the curvatures in generalized metric spaces where there is more than one curvature tensor, such as Finsler-like spacetimes. In these cases, the concept of the Raychaudhuri equation is extended with extra terms and degrees of freedom from the dependence on internal variables such as the velocity or an anisotropic vector field. Additionally, we investigate some consequences of the weak-field limit on the spacetime under consideration and study the Newtonian limit equations which include a generalization of the Poisson equation.

Keywords: Finsler geometry; modified theories of gravity; Raychaudhuri equation; geodesics deviation; cosmology; Weak field; tangent bundle

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1. Introduction

The evolution equation of the quantities that characterize the (gravitational) flow in a given background spacetime is the Raychaudhuri equation [1,2]. The flows are integral curves, geodesics, or they are generated by a vector field. The Raychaudhuri equation is of significant importance since it describes the dynamical evolution of a gravitational fluid, and it is produced by the structure of deviation of nearby geodesics which are dominated by the curvature of space. It was originated by A. Raychaudhuri [1]. When the metric structure of spacetime changes, the equation is modified. The deviation of geodesics and the tidal forces play a fundamental role in general relativity, gravitation, and cosmology because of the interaction of the curvature of spacetime with matter [3]. The profound role of the equation of geodesic deviation (EDG) on Riemannian spacetime has been recognized in general relativity for a long time. The observable deviation of two neighboring geodesics (time-like or null) brings to life an appearance of the curvature of spacetime, namely, the detection of curvature, expresses a property of the matter sector of spacetime that is given by means of EDG, and it is connected with the polarization of gravitational waves and their detection [4]. How does the small-length deviation vector between corresponding points of two nearby geodesics vary as they move along the geodesics? This is the problem of geodesics deviation, the solution of which provides a good insight into the nature and behavior of space. In cosmology, this problem can be connected with the tidal forces and the scale factor $\alpha(t)$ during the expansion of the universe between geodesic motions of two nearby galaxies (see [5]). The form of geodesics and their deviation depend on the spacetime metric, the connection, and the curvature. Raychaudhuri, in his articles [1,6–8], assumes that the universe can be represented by a time-dependent geometry but does not assume homogeneity or isotropy at early times. One of his aims is to see whether non-zero rotation (spin), anisotropy (shear), and/or a cosmological constant can succeed in circumventing the initial singularity [2]. Deviation of geodesics and the Raychaudhuri equation can be studied in a more general geometric framework than the Riemannian one. Finsler geometry consists of a natural metric generalization of Riemannian geometry. During the last years, rapid progress in the field of Finsler geometry and its applications to gravity and cosmology have extended the research in on corresponding topics; some recent works include [9–26]. In generalized metric spaces such as Finsler or Finsler-like spacetime, where the motion/velocity/direction are incorporated in the spacetime structure, internal anisotropy inherent in the EDG [27–29] and Raychaudhuri equations are attributed on the framework of a tangent bundle of spacetime manifold, thus extending the concept of volume θ, shear σ, and vorticity ω [30,31]. In our theory, the concept of volume Θ expresses the total volume on the tangent bundle which includes the standard form of volume θ and the internal anisotropic bulk that is caused by the geometrical structure and its coupling to the standard volume θ due to the additional degrees of freedom (Section 4.3). Additionally, the form of EDG is modified with extra terms that are originated by the connections, torsions, and anisotropic curvature tensors. In this geometric framework, anisotropic tidal phenomena arise from the internal and external structure of spacetime which are imprinted in the generalized EDG and Raychaudhuri equations. The appearance of extra terms in these equations plays the role of additional force fields or self-gravitating actions over spacetime which arise from the richer geometrical structure. The concept of a nonlinear connection in Finsler or Finsler-like spacetime can be interpreted as the interaction between the external and internal structures of spacetime. In a Finslerian gravitational theory on the tangent bundle of spacetime, curvature effects can be considered as total tidal forces which are produced by the external (horizontal) and internal (vertical) curvature tensors. Different considerations for EDG and Raychaudhuri equations on Finsler and Finsler-like spacetimes have been studied by the one of the authors in [30,31]. Einstein–Finsler-like gravitational field equations that govern the motion of matter have been derived in a generalized form of metric spaces on the Lorentz tangent bundle with Finsler-like geometrical structure [18,32,33]. These equations have also been given in a different form in [5,34–40]. In this article, as an additional motivation, we investigate the form of the equation of geodesics deviation, the Raychaudhuri equation in a Schwarzshild–Finsler–Randers (SFR) space adapted on the Lorentz tangent bundle of spacetime, thus extending the investigation on the SFR framework we have given in previous works [17,41,42]. Some physical consequences are also given in this article.

This work is organized as follows. In Section 2 we present the basic elements of the geometrical structure of the model. In Section 3, we study and derive the form of the deviation of geodesics and paths in completely generalized forms, we apply them to the SFR model, and we give some additional information for the deviation equation because of the extra degrees of freedom and the new geometrical concepts. The resulting anisotropic tidal acceleration is of great significance to the investigation of black hole phenomena. Additionally, we give the form of the weak-field limit of the deviation equations for the SFR model. In Section 4, we study the generalized Raychaudhuri equations in a general and special form for the model under consideration. We analyze the derived equations in the horizontal and vertical parts of the Lorentz tangent bundle and we give some interpretations to these equations. Finally, in the conclusion (Section 5) we discuss and summarize our results.

2. Geometrical Structure of the Model

In this section, we present some basic elements of the underlying geometry of the SFR gravitational model as well as the field equations that determine the relation between geometry and matter. A thorough study of this model can be found in [17,32].

2.1. The Lorentz Tangent Bundle

A Lorentz tangent bundle TM over a spacetime four-dimensional manifold M is a fibered eight-dimensional manifold with local coordinates $\{x^\mu, y^a\}$ where the indices of the spacetime variables x are $\kappa, \lambda, \mu, \nu, \ldots = 0, \ldots, 3$ and the indices of the fiber variables y are $a, b, \ldots, f = 0, \ldots, 3$. An extended Lorentzian structure on TM can be provided if the background manifold is equipped with a Lorentz metric tensor of signature $(-1, \ldots, 1)$.

Below, we present some basic geometrical structures of the model.

2.1.1. The Adapted Basis

In order to take a horizontal and vertical basis on a tangent bundle TM, we need to define a nonlinear connection **N** to unequivocally divide the bundle into a horizontal and vertical sub-bundles. The nonlinear connection defines a split of the total space TTM into a horizontal subspace $T_H TM$ and a vertical subspace $T_V TM$. The total space is the Whitney sum:

$$TTM = T_H TM \oplus T_V TM \tag{1}$$

We consider a vector field $X = X^\mu \frac{\partial}{\partial x^\mu}$ on a base Riemannian manifold M along a curve $x^\mu(t)$. If the vector field X^μ coincides with tangent vector of the curve $X^\mu = \dot{x}^\mu(t)$ then the geodesics equation on M is the standard equation using the Levi–Civita connection. It is written as:

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\kappa} \dot{x}^\nu \dot{x}^\kappa = 0 \tag{2}$$

The parallel transport of X is given by the equation of geodesics (2):

$$\frac{dX^\mu}{ds} + \Gamma^\mu_{\nu\kappa} X^\nu X^\kappa = 0 \tag{3}$$

where $\Gamma^\mu_{\nu\kappa}$ is the metrical connection.

We can extend the vector field X to the tangent bundle TM of M as

$$\overline{X} = X^\mu \frac{\partial}{\partial x^\mu} + \frac{dX^a}{dt} \frac{\partial}{\partial y^a} \tag{4}$$

where the coefficients X^a are one-to-one equal to the coefficients X^μ. Thus, X^μ and $\frac{dX^a}{dt}$ are the coefficients of the extended vector $\overline{X}$ on TM, the basis for the horizontal and vertical subspace of the tangent bundle.

According to the considerations of proposal (4.2) of p. 28 of [43] , we can substitute Equation (3) to Equation (4) and find:

$$\overline{X} = X^\mu \frac{\partial}{\partial x^\mu} - (\Gamma^a_{\nu\kappa} X^\nu X^\kappa) \frac{\partial}{\partial y^a} \tag{5}$$

By assuming a Cartan-type connection $\Gamma^a_{\mu b} y^b = N^a_\mu$ [44] in a Finsler connection, where N^a_μ are the coefficients of a nonlinear connection, we find:

$$\overline{X} = X^\mu \left(\frac{\partial}{\partial x^\mu} - N^a_\mu \frac{\partial}{\partial y^a} \right) = X^\mu \delta_\mu \tag{6}$$

where we define:

$$\delta_\mu(x, y) = \frac{\partial}{\partial x^\mu} - N^a_\mu \frac{\partial}{\partial y^a} \tag{7}$$

to be an adapted basis for the tangent bundle. Therefore, the nonlinear connection induces the basis $\{E_A\} = \{\delta_\mu, \dot{\partial}_a\}$ on the total space, with

$$\delta_\mu = \frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N^a_\mu(x, y) \frac{\partial}{\partial y^a} \tag{8}$$

and

$$\dot{\partial}_a = \frac{\partial}{\partial y^a} \tag{9}$$

2.1.2. Metric Structure on TM

A Sasaki-type metric $\mathcal{G}$ on TM is:

$$\mathcal{G} = g_{\mu\nu}(x,y)\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + v_{ab}(x,y)\,\delta y^a \otimes \delta y^b \tag{10}$$

A pseudo-Finslerian metric $f_{ab}(x,y)$ is defined as one that has a Lorentzian signature of $(-,+,+,+)$ and that also obeys the following form:

$$f_{ab}(x,y) = \pm\frac{1}{2}\frac{\partial^2 F^2}{\partial y^a \partial y^b} \tag{11}$$

where the function F satisfies the following conditions [43]:

1. F is continuous on TM and smooth on $\widetilde{TM} \equiv TM \setminus \{0\}$, i.e., the tangent bundle minus the null set $\{(x,y) \in TM | F(x,y) = 0\}$.
2. F is positively homogeneous to the first degree on its second argument:

$$F(x^\mu, ky^a) = kF(x^\mu, y^a), \qquad k > 0 \tag{12}$$

3. The form

$$f_{ab}(x,y) = \pm\frac{1}{2}\frac{\partial^2 F^2}{\partial y^a \partial y^b} \tag{13}$$

defines a non-degenerate matrix:

$$\det[f_{ab}] \neq 0 \tag{14}$$

where the plus–minus sign in (11) is chosen such that the metric has the correct signature.

2.1.3. Connection

In this work, we consider a distinguished connection (d-connection) D on TM. This is a linear connection with coefficients $\{\Gamma^A_{BC}\} = \{L^\mu_{\nu\kappa}, L^a_{b\kappa}, C^\mu_{\nu c}, C^a_{bc}\}$ that preserves the horizontal and vertical distributions using parallelism :

$$D_{\delta_\kappa}\delta_\nu = L^\mu_{\nu\kappa}(x,y)\delta_\mu, \quad D_{\dot{\partial}_c}\delta_\nu = C^\mu_{\nu c}(x,y)\delta_\mu \tag{15}$$

$$D_{\delta_\kappa}\dot{\partial}_b = L^a_{b\kappa}(x,y)\dot{\partial}_a, \quad D_{\dot{\partial}_c}\dot{\partial}_b = C^a_{bc}(x,y)\dot{\partial}_a \tag{16}$$

From these, the definitions for partial covariant differentiation follow as usual, e.g., for $X \in TTM$, we have the definitions for covariant h-derivative

$$X^A_{|\nu} \equiv D_\nu\, X^A \equiv \delta_\nu X^A + L^A_{B\nu}X^B \tag{17}$$

and covariant v-derivative

$$X^A|_b \equiv D_b\, X^A \equiv \dot{\partial}_b X^A + C^A_{Bb}X^B \tag{18}$$

In our consideration, the d-connection is metric-compatible:

$$D_\kappa\, g_{\mu\nu} = 0, \quad D_\kappa\, v_{ab} = 0, \quad D_c\, g_{\mu\nu} = 0, \quad D_c\, v_{ab} = 0 \tag{19}$$

The d-connection coefficients of our model have the following form:

$$L^{\mu}_{\nu\kappa} = \frac{1}{2}g^{\mu\rho}\left(\delta_k g_{\rho\nu} + \delta_\nu g_{\rho\kappa} - \delta_\rho g_{\nu\kappa}\right) \quad (20)$$

$$L^{a}_{b\kappa} = \dot{\partial}_b N^a_\kappa + \frac{1}{2}v^{ac}\left(\delta_\kappa v_{bc} - v_{dc}\,\dot{\partial}_b N^d_\kappa - v_{bd}\,\dot{\partial}_c N^d_\kappa\right) \quad (21)$$

$$C^{\mu}_{\nu c} = \frac{1}{2}g^{\mu\rho}\dot{\partial}_c g_{\rho\nu} \quad (22)$$

$$C^{a}_{bc} = \frac{1}{2}v^{ad}\left(\dot{\partial}_c v_{db} + \dot{\partial}_b v_{dc} - \dot{\partial}_d v_{bc}\right) \quad (23)$$

2.1.4. Curvature and Torsion

Curvatures and torsions on TM are defined by the multi-linear maps:

$$\mathcal{R}(X,Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z \quad (24)$$

and

$$\mathcal{T}(X,Y) = D_X Y - D_Y X - [X,Y] \quad (25)$$

where $X, Y, Z \in TTM$. We use the following definitions for the curvature components [43,45]:

$$\mathcal{R}(\delta_\lambda, \delta_\kappa)\delta_\nu = R^{\mu}_{\nu\kappa\lambda}\delta_\mu \quad (26)$$

$$\mathcal{R}(\delta_\lambda, \delta_\kappa)\dot{\partial}_b = R^{a}_{b\kappa\lambda}\dot{\partial}_a \quad (27)$$

$$\mathcal{R}(\dot{\partial}_c, \delta_\kappa)\delta_\nu = P^{\mu}_{\nu\kappa c}\delta_\mu \quad (28)$$

$$\mathcal{R}(\dot{\partial}_c, \delta_\kappa)\dot{\partial}_b = P^{a}_{b\kappa c}\dot{\partial}_a \quad (29)$$

$$\mathcal{R}(\dot{\partial}_\delta, \dot{\partial}_c)\delta_\nu = S^{\mu}_{\nu c\delta}\delta_\mu \quad (30)$$

$$\mathcal{R}(\dot{\partial}_\delta, \dot{\partial}_c)\dot{\partial}_b = S^{a}_{bc\delta}\dot{\partial}_a \quad (31)$$

In addition, we use the following definitions for the torsion components:

$$\mathcal{T}(\delta_\kappa, \delta_\nu) = \mathcal{T}^{\mu}_{\nu\kappa}\delta_\mu + \mathcal{T}^{a}_{\nu\kappa}\dot{\partial}_a \quad (32)$$

$$\mathcal{T}(\dot{\partial}_b, \delta_\nu) = \mathcal{T}^{\mu}_{\nu b}\delta_\mu + \mathcal{T}^{a}_{\nu b}\dot{\partial}_a \quad (33)$$

$$\mathcal{T}(\dot{\partial}_c, \dot{\partial}_b) = \mathcal{T}^{\mu}_{bc}\delta_\mu + \mathcal{T}^{a}_{bc}\dot{\partial}_a \quad (34)$$

The h-curvature tensor of the d-connection in the adapted basis and the corresponding h-Ricci tensor have, respectively, the components given from (26):

$$R^{\mu}_{\nu\kappa\lambda} = \delta_\lambda L^{\mu}_{\nu\kappa} - \delta_\kappa L^{\mu}_{\nu\lambda} + L^{\rho}_{\nu\kappa}L^{\mu}_{\rho\lambda} - L^{\rho}_{\nu\lambda}L^{\mu}_{\rho\kappa} + C^{\mu}_{\nu a}R^{a}_{\kappa\lambda} \quad (35)$$

$$R_{\mu\nu} = R^{\kappa}_{\mu\nu\kappa} = \delta_\kappa L^{\kappa}_{\mu\nu} - \delta_\nu L^{\kappa}_{\mu\kappa} + L^{\rho}_{\mu\nu}L^{\kappa}_{\rho\kappa} - L^{\rho}_{\mu\kappa}L^{\kappa}_{\rho\nu} + C^{\kappa}_{\mu a}R^{a}_{\nu\kappa} \quad (36)$$

where

$$R^{a}_{\nu\kappa} = \frac{\delta N^a_\nu}{\delta x^\kappa} - \frac{\delta N^a_\kappa}{\delta x^\nu} \quad (37)$$

are the non-holonomy coefficients, also known as the curvature of the nonlinear connection.

The v-curvature tensor of the d-connection in the adapted basis and the corresponding v-Ricci tensor have, respectively, the components (31):

$$S^{a}_{bcd} = \dot{\partial}_d C^{a}_{bc} - \dot{\partial}_c C^{a}_{bd} + C^{e}_{bc}C^{a}_{ed} - C^{e}_{bd}C^{a}_{ec} \quad (38)$$

$$S_{ab} = S^{c}_{abc} = \dot{\partial}_c C^{c}_{ab} - \dot{\partial}_b C^{c}_{ac} + C^{e}_{ab}C^{c}_{ec} - C^{e}_{ac}C^{c}_{eb} \quad (39)$$

The curvature tensor mixed coefficients are:

$$R^a_{b\kappa\lambda} = \delta_\lambda L^a_{b\kappa} - \delta_\kappa L^a_{b\lambda} + L^c_{b\kappa} L^a_{c\lambda} - L^c_{b\lambda} L^a_{c\kappa} + C^a_{bc} R^c_{\kappa\lambda} \tag{40}$$

$$P^\mu_{\nu\kappa c} = \dot{\partial}_c L^\mu_{\nu\kappa} - D_\kappa C^\mu_{\nu c} + C^\mu_{\nu b} \mathcal{T}^b_{\kappa c} \tag{41}$$

$$P^a_{b\kappa c} = \dot{\partial}_c L^a_{b\kappa} - D_\kappa C^a_{bc} + C^a_{bd} \mathcal{T}^d_{\kappa c} \tag{42}$$

$$S^\mu_{\nu cd} = \dot{\partial}_d C^\mu_{\nu c} - \dot{\partial}_c C^\mu_{\nu d} + C^\kappa_{\nu c} C^\mu_{\kappa d} - C^\kappa_{\nu d} C^\mu_{\kappa c} \tag{43}$$

The generalized Ricci scalar curvature in the adapted basis is defined as

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} + v^{ab} S_{ab} = R + S \tag{44}$$

where

$$R = g^{\mu\nu} R_{\mu\nu} \quad , \quad S = v^{ab} S_{ab} \tag{45}$$

2.1.5. Hilbert-like Action

A Hilbert-like action on TM can be defined as

$$K = \int_{\mathcal{N}} d^8\mathcal{U} \sqrt{|\mathcal{G}|}\, \mathcal{R} + 2\kappa \int_{\mathcal{N}} d^8\mathcal{U} \sqrt{|\mathcal{G}|}\, \mathcal{L}_M \tag{46}$$

for some closed subspace $\mathcal{N} \subset TM$, where $|\mathcal{G}|$ is the absolute value of the metric determinant, $\mathcal{L}_M$ is the Lagrangian of the matter fields, κ is a constant, and

$$d^8\mathcal{U} = \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^3 \wedge \mathrm{d}y^0 \wedge \ldots \wedge \mathrm{d}y^3 \tag{47}$$

where the eight-parallelepiped $d^8\mathcal{U}$ is considered an oriented compact element of volume.

2.2. *The SFR Model*

In the SFR model, the metric $g_{\mu\nu}$ is the classic Schwarzschild one:

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -f\, dt^2 + \frac{dr^2}{f} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{48}$$

with $f = 1 - \frac{R_s}{r}$ and $R_s = 2GM$ the Schwarzschild radius (we assume units where $c = 1$).

The metric v_{ab} is derived from a metric function F_v of the a-Randers type:

$$F_v = \sqrt{-g_{ab}(x) y^a y^b} + A_c(x) y^c \tag{49}$$

where $g_{ab} = g_{\mu\nu} \delta^\mu_a \delta^\nu_b$ is the Schwarzschild metric, and $A_c(x)$ is a covector that expresses a deviation from general relativity, with $|A_c(x)| \ll 1$. The nonlinear connection will take the form:

$$N^a_\mu = \frac{1}{2} y^b g^{ac} \partial_\mu g_{bc} \tag{50}$$

The metric tensor v_{ab} of (49) is derived from (11) after omitting the higher-order terms $O(A^2)$:

$$v_{ab}(x, y) = g_{ab}(x) + w_{ab}(x, y) \tag{51}$$

where

$$w_{ab} = \frac{1}{\tilde{a}} (A_b g_{ac} y^c + A_c g_{ab} y^c + A_a g_{bc} y^c) + \frac{1}{\tilde{a}^3} A_c g_{ae} g_{bd} y^c y^d y^e \tag{52}$$

with $\tilde{a} = \sqrt{-g_{ab} y^a y^b}$. The total metric defined from the steps above is called the Schwarzschild–Finsler–Randers (SFR) metric and the corresponding spacetime is called an SFR spacetime.

We remark that the term w_{ab} represents a deviation from the pseudo-Riemannian space. The above-mentioned term can be useful for studying gravitational waves in a locally anisotropic framework of an SFR spacetime.

Variation in the action (46) with respect to $g_{\mu\nu}$, v_{ab} and N^a_κ leads to the following field equations:

$$\overline{R}_{\mu\nu} - \frac{1}{2}(R+S)\, g_{\mu\nu} + \left(\delta^{(\lambda}_\nu \delta^{\kappa)}_\mu - g^{\kappa\lambda} g_{\mu\nu}\right)\left(D_\kappa \mathcal{T}^b_{\lambda b} - \mathcal{T}^c_{\kappa c}\mathcal{T}^b_{\lambda b}\right) = \kappa T_{\mu\nu} \tag{53}$$

$$S_{ab} - \frac{1}{2}(R+S)\, v_{ab} + \left(v^{cd} v_{ab} - \delta^{(c}_a \delta^{d)}_b\right)\left(D_c C^\mu_{\mu d} - C^\nu_{\nu c} C^\mu_{\mu d}\right) = \kappa Y_{ab} \tag{54}$$

$$g^{\mu[\kappa} \dot{\partial}_a L^{\nu]}_{\mu\nu} + 2\mathcal{T}^b_{\mu b} g^{\mu[\kappa} C^{\lambda]}_{\lambda a} = \frac{\kappa}{2}\mathcal{Z}^\kappa_a \tag{55}$$

with

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{|\mathcal{G}|}}\frac{\delta\left(\sqrt{|\mathcal{G}|}\,\mathcal{L}_M\right)}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\,\mathcal{L}_M\right)}{\delta g^{\mu\nu}} \tag{56}$$

$$Y_{ab} \equiv -\frac{2}{\sqrt{|\mathcal{G}|}}\frac{\delta\left(\sqrt{|\mathcal{G}|}\,\mathcal{L}_M\right)}{\delta v^{ab}} = -\frac{2}{\sqrt{-v}}\frac{\delta\left(\sqrt{-v}\,\mathcal{L}_M\right)}{\delta v^{ab}} \tag{57}$$

$$\mathcal{Z}^\kappa_a \equiv -\frac{2}{\sqrt{|\mathcal{G}|}}\frac{\delta\left(\sqrt{|\mathcal{G}|}\,\mathcal{L}_M\right)}{\delta N^a_\kappa} = -2\frac{\delta \mathcal{L}_M}{\delta N^a_\kappa} \tag{58}$$

and $\overline{R}_{\mu\nu} \equiv R_{\mu\nu} - C^\kappa_{\mu a} R^a_{\nu\kappa}$, where $\mathcal{L}_M$ is the Lagrangian of the matter fields, δ^μ_ν and δ^a_b are the Kronecker symbols, $|\mathcal{G}|$ is the absolute value of the determinant of the total metric (10), and

$$\mathcal{T}^a_{\nu b} = \dot{\partial}_b N^a_\nu - L^a_{b\nu} \tag{59}$$

are torsion components, where $L^a_{b\nu}$ is defined in (21). From the form of (10) it follows that $\sqrt{|\mathcal{G}|} = \sqrt{-g}\sqrt{-v}$, with g, v the determinants of the metrics $g_{\mu\nu}, v_{ab}$, respectively.

The local anisotropy can contribute to the energy–momentum tensors of the horizontal and vertical space $T_{\mu\nu}$ and Y_{ab}. As a result, the energy–momentum tensor $T_{\mu\nu}$ contains the additional information of local anisotropy of matter fields. Y_{ab}, on the other hand, is an extra concept with no equivalent in Riemannian gravity. It contains more information about local anisotropy which is produced from the metric v_{ab}, which includes additional internal structure of spacetime and can be connected to dark energy [33]. Finally, the energy–momentum tensor $\mathcal{Z}^\kappa_a$ reflects the dependence of matter fields with respect to the nonlinear connection N^a_μ, a structure which induces an interaction between internal and external spaces. This is a different form than of that $T_{\mu\nu}$ and Y_{ab}, which depend on just the external or internal structure, respectively.

Solving the field Equations (53)–(55) to the first order in $A_c(x)$ in a vacuum ($T_{\mu\nu} = Y_{ab} = \mathcal{Z}^\kappa_a = 0$), we obtain [17]:

$$A_c(x) = \left[\tilde{A}_0\left|1 - \frac{R_S}{r}\right|^{1/2}, 0, 0, 0\right] \tag{60}$$

where R_S is the Schwarzschild radius, r is the radial coordinate, and $\tilde{A}_0$ is a constant, where $|\tilde{A}_0| \ll 1$.

In the SFR model, the horizontal curvature Ricci tensor $R_{\mu\nu}$ is zero, but the internal vertical curvature v-Ricci tensor S_{ab} and the v-scalar S are different from zero. The components of the v-Ricci tensor are:

$$S_{00} = \frac{\tilde{A}_0^2 f \left(\tilde{a}^2 - f {y_t}^2\right)^2}{\tilde{a}^6} \tag{61}$$

$$S_{11} = \frac{\tilde{A}_0^2 \Big[Rs^2 y_\theta^2 \left(3\tilde{a}^2 - 4 f y_t^2\right) + Rs^2 y_\phi^2 \sin^2\theta \left(3\tilde{a}^2 - 4 f y_t^2\right) + 4(f-1)^2 f y_t^2 \left(f y_t^2 - \tilde{a}^2\right)\Big]}{4\tilde{a}^6 (f-1)^2 f} \tag{62}$$

$$S_{22} = \frac{\tilde{A}_0^2 Rs^2 \left[Rs^2 y_\theta^2 \left(4 f y_t^2 - 3\tilde{a}^2\right) - 3\tilde{a}^2 (f-1)^2 \left(\tilde{a}^2 - f y_t^2\right)\right]}{4\tilde{a}^6 (f-1)^4} \tag{63}$$

$$S_{33} = -\frac{\tilde{A}_0^2 Rs^2 \sin^2\theta \Big[Rs^2 y_\phi^2 \sin^2\theta \left(3\tilde{a}^2 - 4 f y_t^2\right) + 3\tilde{a}^2 (f-1)^2 \left(\tilde{a}^2 - f y_t^2\right)\Big]}{4\tilde{a}^6 (f-1)^4} \tag{64}$$

$$S_{01} = -\frac{\tilde{A}_0^2 y_t \left(f y_t^2 - \tilde{a}^2\right) \sqrt{f \Big[f y_t^2 - \tilde{a}^2 - \frac{Rs^2}{(f-1)^2} \left(y_\theta^2 + y_\phi^2 \sin^2\theta\right)\Big]}}{\tilde{a}^6} \tag{65}$$

$$S_{02} = \frac{\tilde{A}_0^2 f Rs^2 y_\theta y_t^2 \left(\tilde{a}^2 - f y_t^2\right)}{\tilde{a}^6 (f-1)^2} \tag{66}$$

$$S_{03} = \frac{\tilde{A}_0^2 f Rs^2 y_t y_\phi \sin^2\theta \left(\tilde{a}^2 - f y_t^2\right)}{\tilde{a}^6 (f-1)^2} \tag{67}$$

$$S_{12} = \frac{\tilde{A}_0^2 Rs^2 y_\theta y_r \Big[(f-1)^2 \left(\tilde{a}^2 f + 4 y_r^2\right) + 4 f Rs^2 y_\theta^2 + 4 f Rs^2 y_\phi^2 \sin^2\theta\Big]}{4\tilde{a}^6 (f-1)^4 f^2} \tag{68}$$

$$S_{13} = \frac{\tilde{A}_0^2 Rs^2 y_r y_\phi \sin^2\theta \Big[(f-1)^2 \left(\tilde{a}^2 f + 4 y_r^2\right) + 4 f Rs^2 y_\theta^2 + 4 f Rs^2 y_\phi^2 \sin^2\theta\Big]}{4\tilde{a}^6 (f-1)^4 f^2} \tag{69}$$

$$S_{23} = \frac{\tilde{A}_0^2 Rs^4 y_\theta y_\phi \sin^2\theta \left(4 f y_t^2 - 3\tilde{a}^2\right)}{4\tilde{a}^6 (f-1)^4} \tag{70}$$

and the scalar v-Ricci curvature is:

$$S = \frac{5\tilde{A}_0^2 \left[\tilde{a}^2 - f y_t^2\right]}{2\tilde{a}^4} \tag{71}$$

where $\tilde{a} = \sqrt{-g_{ab} y^a y^b}$, $f \equiv 1 - \frac{R_S}{r}$, and we have set $y^0 \equiv y_t, y^1 \equiv y_r, y^2 \equiv y_\theta, y^3 \equiv y_\phi$.

The internal curvature S^a_{bcd} can give a physical meaning to the anisotropy (dependence on direction) of gravitational waves on the SFR model.

We have derived [41] the nontrivial Kretschmann-like invariants of the metrics $g_{\mu\nu}$ and v_{ab} to the lowest non-vanishing order:

$$K_H \equiv R_{\kappa\lambda\mu\nu} R^{\kappa\lambda\mu\nu} = \frac{12 R_S^2}{r^6} \tag{72}$$

$$K_V \equiv S_{abcd} S^{abcd} = \left(\frac{3S}{5}\right)^2 \tag{73}$$

Finally, the mixed curvature coefficients are all zero in this model.

2.3. The Newtonial Limit

In this section, we will investigate the Newtonian limit of a Finsler-like metric space on TM. The metric on TM will take the form

$$G = \big(\eta_{\mu\nu} + h_{\mu\nu}(x)\big)\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} + (\eta_{ab} + w_{ab}(x,y))\delta y^{a} \otimes \delta y^{\beta} \tag{74}$$

where $h_{\mu\nu}(x)$ and $w_{ab}(x,y)$ are small perturbations over the flat Minkowski metrics $\eta_{\mu\nu}$ and η_{ab} on the horizontal and vertical space, respectively.

The field Equations (53) and (54) for the metric are written at first order as:

$$R_{\mu\nu} - \frac{1}{2}(R+S) = \kappa T_{\mu\nu} \tag{75}$$

$$S_{ab} - \frac{1}{2}(R+S) = \kappa Y_{ab} \tag{76}$$

or equivalently

$$\frac{1}{2}\Big(\partial_\mu\partial_\kappa h^\kappa_\nu + \partial_\nu\partial_\kappa h^\kappa_\mu - \partial^\kappa\partial_\kappa h_{\mu\nu} - \partial_\mu\partial_\nu h\Big) - \frac{1}{2}\eta_{\mu\nu}\Big(\partial_\kappa\partial_\lambda h^{\kappa\lambda} - \partial^\kappa\partial_\kappa h + \dot{\partial}_a\dot{\partial}_b w^{ab} - \dot{\partial}^a\dot{\partial}_a w\Big) = \kappa T_{\mu\nu} \tag{77}$$

$$\frac{1}{2}\Big(\dot{\partial}_c\dot{\partial}_a w^c_b + \dot{\partial}_c\dot{\partial}_b w^c_a - \dot{\partial}^c\dot{\partial}_c w_{ab} - \dot{\partial}_a\dot{\partial}_b\Big) - \frac{1}{2}\eta_{ab}\Big(\partial_\kappa\partial_\lambda h^{\kappa\lambda} - \partial^\kappa\partial_\kappa h + \dot{\partial}_a\dot{\partial}_b w^{ab} - \dot{\partial}^a\dot{\partial}_a w\Big) = \kappa Y_{ab} \tag{78}$$

where $h = h^\mu_\mu$ and $w = w^a_a$. The third field Equation (55) gives:

$$\dot{\partial}_a\big(\partial^\kappa h - \partial_\mu h^{\mu\kappa}\big) = \mathcal{Z}^\kappa_a \tag{79}$$

which in our case gives $\mathcal{Z}^\kappa_a = 0$. This means that the matter fields in our space do not directly depend on the nonlinear connection. An analytical approach on a weak-field metric over a Lorentz tangent bundle can be found in [33].

The horizontal metric is effectively a Riemannian one, so the usual symmetries apply to it. One can decompose this metric on a scalar part Φ, a vector a_j, a traceless spatial tensor b_{ij}, and the trace Ψ of the spatial part, where all these parts transform independently under spatial rotations. The metric then takes the form:

$$\begin{aligned} G =& \Big\{-(1+2\Phi)\mathrm{d}t^2 + a_i(\mathrm{d}t\mathrm{d}x^i + \mathrm{d}x^i\mathrm{d}t) + \big[(1-2\Psi)\delta_{ij} + 2b_{ij}\big]\mathrm{d}x^i\mathrm{d}x^j\Big\}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \\ &+ (\eta_{ab} + w_{ab}(x,y))\delta y^a \otimes \delta y^\beta \end{aligned} \tag{80}$$

Additionally, we can take advantage of the gauge degrees of freedom of the Riemannian metric to set $\partial_i b^{ij} = 0$ and $\partial_i a^i = 0$. Finally, in our setting, the matter content of spacetime is assumed to be dust in its rest frame:

$$T_{\mu\nu} = \rho u_\mu u_\nu \tag{81}$$

where u^μ is the four-velocity field of the matter fluid. We consider a static spacetime, so all the time derivatives will vanish.

Under these assumptions, the field equations are written as:

$$\nabla^2\Psi + \frac{1}{4}S = \frac{\kappa}{2}\rho \tag{82}$$

$$\nabla^2 a_i = 0 \tag{83}$$

$$(\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \nabla^2 b_{ij} - \frac{1}{2}S\eta_{ij} = 0 \tag{84}$$

where ∇ is the three-dimensional spatial grad operator. Taking the trace of (84) yields:

$$\nabla^2(\Phi - \Psi) = \frac{3}{4}S \tag{85}$$

Substituting (85) to (82) gives:

$$\nabla^2\Phi = \frac{\kappa}{2}\rho + \frac{S}{2} \tag{86}$$

This equation is a direct generalization of the Poisson equation of Newtonian physics. It has been shown in [33] that S can describe the effect of a vacuum energy density, so it can be considered to be a dark energy candidate.

Substituting (86) to (84) gives:

$$\frac{1}{4}S\delta_{ij} - \partial_i\partial_j(\Phi - \Psi) - \nabla^2 b_{ij} = 0 \tag{87}$$

Finally, (83) for a well-behaved field gives:

$$a_i = 0 \tag{88}$$

Equations (86)–(88), together with (76) (or (78)), determine the metric (80) up to the boundary conditions. The vertical energy–momentum tensor can be approximated by its GR limit value, i.e.,

$$Y_{ab} = \frac{1}{2}T^{\mu}_{\mu}\eta_{ab} \tag{89}$$

(see also [17]).

We remark that in the Newtonian limit of GR, only the scalar Φ is nonzero, while in our case, more degrees of freedom survive, such as the trace Ψ, the traceless tensor b_{ij}, and the vertical curvature S.

3. Generalized Deviation of Geodesics and Paths

In this section, we derive a completely generalized deviation equation using all forms of torsions for the geodesics and the paths. When there are forces acting on particles, they are not moving on geodesics and are accelerating. In our framework, we present some applications to the SFR model. The deviation equation of geodesics for different types of generalized locally anisotropic spacetime has been studied for a long time [27,29]. Here, we also present the weak deviation equation for this space.

3.1. General Equations

We assume that these geodesics and paths take the general form:

$$\frac{dy^a}{d\lambda} + 2G^a = 0, \quad y^a = \delta^a_\mu \frac{dx^\mu}{d\lambda} \tag{90}$$

Geodesics for the SFR model have been derived in a previous work [42]:

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + g^{\kappa\lambda}\Phi_{\kappa\mu}\dot{x}^\mu = 0 \tag{91}$$

where $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols of Riemann geometry, $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$, $\Phi_{\kappa\mu} = \partial_\kappa A_\mu - \partial_\mu A_\kappa$, and A_μ is the solution Equation (60). We notice that from the definition of $\Phi_{\kappa\mu}$, we obtain a rotation form of geodesics. If A_μ is a gradient of a scalar field, $A_\mu = \dfrac{\partial\Phi}{\partial x^\mu}$, then $\Phi_{\kappa\mu} = 0$, and the geodesics of our model are identified with the Riemannian ones.

These geodesics are a specific case of (90) for

$$G^\lambda = \frac{1}{2}\left(\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + g^{\kappa\lambda}\Phi_{\kappa\mu}\dot{x}^\mu\right) \tag{92}$$

The geodesics can be explicitly written in the form:

$$\ddot{t} + \frac{1-f}{rf}\dot{r}\dot{t} = -\tilde{A}_0\dot{r}\frac{f^{-3/2}(1-f)}{2r} \tag{93}$$

$$\ddot{r} + \frac{f(1-f)}{2r}\dot{t}^2 - \frac{1-f}{2rf}\dot{r}^2 - rf\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = -\tilde{A}_0\dot{t}\frac{f^{1/2}(1-f)}{2r} \tag{94}$$

$$\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \frac{1}{2}\sin 2\theta\,\dot{\phi}^2 = 0 \tag{95}$$

$$\ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0 \tag{96}$$

The deflection angle of the SFR model has been studied in a previous work [42] and has been calculated for the model in hand,

$$\delta\phi_{SFR} \approx \left(1 + \frac{a^2}{2}\right)\frac{4GM}{b} \tag{97}$$

with $a = \tilde{A}_0 b/J$, where $b = J/ER$ is a composite constant formed by the ratio of the angular momentum J divided by the energy ER of the particle moving along the geodesic. The deflection angle $\delta\phi$ of GR is $\delta\phi_{SFR} = \delta\phi_{GR}$ when $\lim_{\tilde{A}_0\to 0}$. Connecting the geometrical concept of the curvature $\kappa_\phi = \frac{d\phi}{d\tau}$ with the quantity $\kappa_\phi = \frac{\delta\phi_{SFR}}{d\tau}$, we obtain the deflection curvature of the SFR model. Comparing this result with that of the deflection angle parameter in Shapiro et al. [46], we find the value

$$2\left(1 + \frac{a^2}{2}\right) \simeq 1 + \gamma_{Shap} \tag{98}$$

or $a \sim 0.0141421$.

The small difference in the deflection angle of the SFR model in comparison to the GR deflection angle can be attributed to the Lorentz violations [9] or on the small amount of energy which is added to the gravitational potential of SFR model.

Subsequently, we will calculate the eight-velocity tangent vector to the geodesics (90) and define a deviation vector of the geodesics. We obtain:

$$U = y^\mu\partial_\mu - 2G^a\dot{\partial}_a = y^\mu\delta_\mu + (y^\nu N^a_\nu - 2G^a)\dot{\partial}_a \tag{99}$$

We write the decomposition of U to its horizontal and vertical components as $\{U^A\} = \{u^\mu, v^a\}$.

We define the deviation vector J such that, together with U, they form a local coordinate basis:

$$[U, J] = 0 \tag{100}$$

After some calculations, the above commutator relation gives

$$U^C D_C J^B = J^C D_C U^B + U^A J^C \mathcal{T}^B_{AC} \tag{101}$$

The second covariant derivative of the deviation vector is

$$\begin{aligned} D^2_U J^I &= U^A D_A(U^B D_B J^I) \\ &= U^A D_A\left(J^B D_B U^I + U^B J^C \mathcal{T}^I_{BC}\right) \\ &= \left(J^A D_A U^B + U^A J^C \mathcal{T}^B_{AC}\right) D_B U^I + U^A J^B([D_A, D_B] + D_B D_A)U^I + U^A D_A(U^B J^C \mathcal{T}^I_{BC}) \end{aligned} \tag{102}$$

We can write

$$U^A D_A u^\mu = \frac{du^\mu}{d\lambda} + u^\lambda u^\nu L^\mu_{\nu\lambda} + v^a u^\nu C^\mu_{\nu a}$$
$$= u^\lambda u^\nu L^\mu_{\nu\lambda} + v^a u^\nu C^\mu_{\nu a} - 2G^\mu \tag{103}$$

where we used (90). We define

$$\Delta^\mu \equiv u^\nu u^\lambda L^\mu_{\nu\lambda} + u^\nu v^a C^\mu_{\nu a} - 2G^\mu \tag{104}$$

so we obtain

$$U^A D_A u^\mu = \Delta^\mu \tag{105}$$

We can consider Δ^μ to be a vector expressing the failure of u^μ being parallel transported along the geodesic.

Similarly, we find the covariant derivative of the vertical part of U:

$$U^A D_A v^d = \frac{dv^d}{d\lambda} + u^\lambda v^a L^d_{a\lambda} + v^a v^b C^d_{ab}$$
$$= u^\lambda v^a L^d_{a\lambda} + v^a v^b C^d_{ab} + \frac{du^\nu}{d\lambda} N^a_\nu + u^\nu \frac{dN^a_\nu}{d\lambda} - 2\frac{dG^a}{d\lambda}$$
$$\equiv \Delta^d \tag{106}$$

The commutator of the covariant derivatives is given by the following known relation:

$$[D_A, D_B]U^I = \mathcal{R}^I_{CBA} U^C U^A J^B + \mathcal{T}^C_{BA} D_C U^I \tag{107}$$

where $\mathcal{R}^I_{CBA}$ is the generalized curvature tensor of the connection D on the tangent bundle. Taking the spacetime part of (102), and after some straightforward calculations, we obtain the result:

$$D^2_U J^\mu = \left(\mathcal{R}^\mu_{BCA} + D_A \mathcal{T}^\mu_{BC} + \mathcal{T}^\mu_{BD} \mathcal{T}^D_{AC}\right) U^A U^B J^C + 2U^{[A} J^{C]} \mathcal{T}^\mu_{BC} D_A U^B + J^B D_B \Delta^\mu \tag{108}$$

where we used the relations (104) and (107).

Similarly, the fiber part of (102) gives:

$$D^2_U J^d = \left(\mathcal{R}^d_{BCA} + D_A \mathcal{T}^d_{BC} + \mathcal{T}^d_{BD} \mathcal{T}^D_{AC}\right) U^A U^B J^C + 2U^{[A} J^{C]} \mathcal{T}^d_{BC} D_A U^B + J^B D_B \Delta^d \tag{109}$$

Equations (108) and (109) denote the deviation equation of horizontal and vertical paths between two nearby time-like paths on the Lorentzian tangent bundle. If the vectors Δ^μ and Δ^d are equal to zero, the trajectories of nearby observers are geodesics. Equation (108) is reduced to the standard geodesic deviation equation of general relativity when all the torsion components and the vector Δ^μ are equal to zero, and in that case, the curvature tensor coincides with the Riemannian one of the Levi–Civita connection. The torsion terms in (108) come from the geometry of our space; they play the role of a perturbation for the geodesic deviation of GR. From a physical point of view, perturbations of geodesics are affected by extra terms in their equations, e.g., because of additional mass, gas, or dark matter which interact gravitationally during their motion [25]. Tidal acceleration phenomena and anisotropic tidal-field perturbations can appear to be coming from different sources of spacetime.

3.2. Application of the Weak-Field Limit

We investigate first-order perturbations of the deviation equation in a weak Finslerian framework on the tangent bundle of the Riemannian space for an SFR space. A weak-field metric takes the form

$$G = \left[g_{\mu\nu}(x) + h_{\mu\nu}(x,y)\right] \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \left[g_{ab}(x) + w_{ab}(x,y)\right] \delta y^a \otimes \delta y^b \tag{110}$$

with $|h_{\mu\nu}| \ll 1$ and $|w_{ab}| \ll 1$. From relations (40)–(43), it is straightforward to see that the mixed term curvatures $R^{a}_{b\kappa\lambda}$, $P^{\mu}_{\nu\kappa c}$, $P^{a}_{b\kappa c}$, and $S^{\mu}_{\nu cd}$ as well as the vertical curvature S^{a}_{bcd} are first-order on the perturbations $h_{\mu\nu}$ and w_{ab}.

At first order on $h_{\mu\nu}$ and w_{ab}, the horizontal deviation equation is:

$$D^2_U J^\mu = R^{\mu}_{\kappa\lambda\nu} u^\kappa u^\nu J^\lambda + \tilde{H}^\mu \tag{111}$$

where $\tilde{H}^\mu$ is a weak correction on the deviation equation, linear on $h_{\mu\nu}$ and w_{ab}:

$$\begin{aligned}\tilde{H}^\mu = {} & \mathcal{T}^{\mu}_{\nu b}\mathcal{T}^{b}_{\kappa\lambda} u^\kappa u^\nu J^\lambda + P^{\mu}_{\nu\kappa c} u^\nu v^c J^\kappa + \left(S^{\mu}_{\nu bc} + D_c \mathcal{T}^{\mu}_{\nu b}\right) u^\nu v^c J^b + D_\lambda \mathcal{T}^{\mu}_{\nu b} u^\lambda u^\nu J^b \\ & + \mathcal{T}^{\mu}_{\nu b}\left[\left(u^\lambda J^b - v^b J^\lambda\right) D_\lambda u^\nu + \left(v^c J^b - v^b J^c\right) D_c u^\nu\right] + J^B D_B \Delta^\mu\end{aligned} \tag{112}$$

The first-order vertical deviation equation is:

$$D^2_U J^a = D_\lambda \mathcal{T}^{a}_{\nu\kappa} u^\nu u^\lambda J^\kappa + \mathcal{T}^{a}_{\nu\kappa}\left[\left(u^\lambda J^\kappa - u^\kappa J^\lambda\right) D_\lambda u^\nu + (v^c J^\kappa - u^\kappa J^c) D_c u^\nu\right] + J^B D_B \Delta^a + \tilde{V}^a \tag{113}$$

with

$$\begin{aligned}\tilde{V}^a = {} & S^{a}_{bcd} v^b v^d J^c + P^{a}_{b\kappa c} v^b v^c J^\kappa + (R^{a}_{b\kappa\lambda} + D_b \mathcal{T}^{a}_{\lambda\kappa}) v^b u^\lambda J^\kappa + D_c \mathcal{T}^{a}_{\kappa b} u^\kappa v^c J^b \\ & + (D_\lambda \mathcal{T}^{a}_{\nu b} + \mathcal{T}^{a}_{\nu\kappa}\mathcal{T}^{\kappa}_{\lambda b}) u^\nu u^\lambda J^b + \mathcal{T}^{a}_{\nu b}\mathcal{T}^{b}_{\lambda\kappa} u^\nu u^\lambda J^\kappa\end{aligned} \tag{114}$$

the perturbation on the vertical deviation equation.

We apply the above equations for the SFR model and the geodesics (91) and we obtain:

$$\nabla^2_U J^\mu = R^{\mu}_{\nu\kappa\lambda} u^\kappa u^\nu J^\lambda + 2u^\lambda u^\nu \nabla_\lambda\left(N^a_\nu \dot{\partial}_a J^\mu\right) + J^\lambda g^{\kappa\mu}\, \nabla_\lambda(\Phi_{\nu\kappa} u^\nu) \tag{115}$$

where ∇ is the Levi–Civita connection on the base manifold, $R^{\mu}_{\nu\kappa\lambda}$ is the Riemann curvature tensor of the classic Schwarzschild spacetime, and we have assumed that the y-dependence of J^μ on y is weak, i.e., $|\dot{\partial}_a J^\mu| \ll 1$. Equation (115) is the first-order generalization of the deviation equation on the SFR model.

We remark that a variation in $\Phi_{\nu\kappa} u^\nu$ along the deviation vector J^μ can induce an anisotropy of J^μ which varies along the geodesics. In the GR limit, the two last terms in (115) vanish and we obtain the classical deviation equation.

The vertical deviation equation in the SFR spacetime is:

$$D^2_U J^a = J^B D_B \Delta^a \tag{116}$$

Relations (115) and (116) show the rate of change of anisotropic deviation equation (tidal fields) at first order.

3.3. The Schwarzschild–Finsler–Randers spacetime

In SFR spacetime, the h-Ricci curvature tensor $R_{\mu\nu}$ and the h-Ricci curvature scalar R, defined in (36) and (45), respectively, are both zero. Consequently, the non-zero components of the h-Riemann curvature tensor, defined in (35), are equal to the ones of [3]:

$$R^t{}_{rrt} = 2R^\theta{}_{r\theta r} = 2R^\phi{}_{r\phi r} = \frac{R_S}{r^2(R_S - r)} \tag{117}$$

$$2R^t{}_{\theta\theta t} = 2R^r{}_{\theta\theta r} = R^\phi{}_{\theta\phi\theta} = \frac{R_S}{r} \tag{118}$$

$$2R^t{}_{\phi\phi t} = 2R^r{}_{\phi\phi r} = -R^\theta{}_{\phi\phi\theta} = \frac{R_S \sin^2\theta}{r} \tag{119}$$

$$R^{r}{}_{trt} = -2R^{\theta}{}_{t\theta t} = -2R^{\phi}{}_{t\phi t} = c^2 \frac{R_S(R_S - r)}{r^4} \tag{120}$$

The non-zero components of the h-Riemann curvature tensor are the following:

$$R^{r'}{}_{t'r't'} = -R^{\theta'}{}_{\phi'\theta'\phi'} = -\frac{R_S}{r^3} \tag{121}$$

$$R^{\theta'}{}_{t'\theta't'} = R^{\phi'}{}_{t'\phi't'} = -R^{r'}{}_{\theta'r'\theta'} = -R^{r'}{}_{\phi'r'\phi'} = \frac{R_S}{2r^3}. \tag{122}$$

It is fundamental that the geodesic deviation equation shows the tidal acceleration between two observers who are separated by J^{μ}. It takes the form $D^2 J^{\mu'}/D\tau^2 = -R^{\mu'}{}_{t'\nu't'}J^{\nu'}$. The observable acceleration $(R_S/r^3)c^2L$ of a body of length L in radial direction extends it and shrinks it in the lateral direction by $-(R_S/(2r^3))c^2L$. Moving a body to the direction of a black hole causes spaghettification on the sizes of the body. Additionally, the anisotropic curvature S contributes to an anisotropic deformation of the body in our space.

The vertical curvature depends on the position x and the direction y; it is related to the intrinsic mechanism of spacetime where the gravitational field is extended on the total space of the SFR bundle. If we accept that the Schwarzschild spacetime takes an anisotropic structure with a force field (one form) on its metric, the additional energy originates from the internal (vertical) curvature and increases the form of tidal field around of a black hole. Consequently, we consider that both the horizontal and vertical (anisotropic) curvatures may affect the radial and lateral motion of an observer.

4. Generalized Raychaudhuri Equations

In this section, we investigate the generalized Raychaudhuri equations originated by our model and we give the equations for the horizontal and vertical parts of the Lorentz tangent bundle.

4.1. Horizontal Equations

We define the divergence tensor B^{μ}_{ν} as:

$$B^{\mu}_{\nu} = D_{\nu}u^{\mu} \tag{123}$$

where D_{ν} denotes the horizontal covariant derivative, and u^{μ} is a horizontal vector tangent to the geodesic congruence. We calculate the acceleration vector along the direction of u^{μ} as:

$$\frac{Du^{\mu}}{d\tau} = u^{\nu}D_{\nu}u^{\mu} = u^{\nu}B^{\mu}_{\nu} \tag{124}$$

In order to find the deviation of B^{μ}_{ν} we can use Equation (124):

$$\frac{DB^{\mu}_{\nu}}{d\tau} = u^{\sigma}D_{\sigma}B^{\mu}_{\nu} = u^{\sigma}D_{\sigma}(D_{\nu}u^{\mu}) \tag{125}$$

We can use the commutator of D as:

$$[D_{\sigma}, D_{\nu}]u^{\mu} = R^{\mu}_{\lambda\sigma\nu}u^{\lambda} - \mathcal{T}^{\lambda}_{\sigma\nu}D_{\lambda}u^{\mu} - R^{a}_{\sigma\nu}D_{a}u^{\mu} \tag{126}$$

where $R^{\mu}_{\lambda\sigma\nu}$ is the horizontal curvature tensor, $\mathcal{T}^{\lambda}_{\sigma\nu}$ is the torsion tensor, and $R^{a}_{\sigma\nu}$ is the curvature of the nonlinear connection. If we use Equations (125) and (126), we have:

$$\frac{DB^{\mu}_{\nu}}{d\tau} = u^{\sigma}[D_{\sigma}, D_{\nu}]u^{\mu} + u^{\sigma}D_{\nu}D_{\sigma}u^{\mu} \Rightarrow$$

$$\frac{DB^{\mu}_{\nu}}{d\tau} = u^{\sigma}\left[R^{\mu}_{\lambda\sigma\nu}u^{\lambda} - \mathcal{T}^{\lambda}_{\sigma\nu}D_{\lambda}u^{\mu} - R^{a}_{\sigma\nu}D_{a}u^{\mu}\right] + \left[D_{\nu}(u^{\sigma}D_{\sigma}u^{\mu}) - (D_{\nu}u^{\sigma})(D_{\sigma}u^{\mu})\right] \Rightarrow$$

$$\frac{DB^{\mu}_{\nu}}{d\tau} = R^{\mu}_{\lambda\sigma\nu}u^{\lambda}u^{\sigma} - \mathcal{T}^{\lambda}_{\sigma\nu}u^{\sigma}B^{\mu}_{\lambda} - R^{a}_{\sigma\nu}u^{\sigma}B^{\mu}_{a} - B^{\sigma}_{\nu}B^{\mu}_{\sigma} \tag{127}$$

where we have used Equation (123) and $u^{\sigma}D_{\sigma}u^{\mu} = 0$ since u^{μ} is tangent to the geodesics.

We define the horizontal projection tensor as:

$$P^{\mu}_{\nu} = \delta^{\mu}_{\nu} + u^{\mu}u_{\nu} \tag{128}$$

The divergence tensor B^{μ}_{ν} can be separated in two parts: a symmetric part and an anti-symmetric part. The symmetric part can also be separated into a part with a trace and a traceless part. We can write the decomposition by using the projection tensor as follows:

$$B^{\mu}_{\nu} = \frac{1}{3}\theta P^{\mu}_{\nu} + \sigma^{\mu}_{\nu} + \omega^{\mu}_{\nu} \tag{129}$$

Using the projection tensor in Equation (127) we obtain:

$$P^{\nu}_{\mu}\frac{DB^{\mu}_{\nu}}{d\tau} = -R_{\mu\nu}u^{\mu}u^{\nu} - \mathcal{T}^{\lambda}_{\sigma\mu}u^{\sigma}B^{\mu}_{\lambda} - R^{a}_{\sigma\mu}u^{\sigma}B^{\mu}_{a} - B^{\sigma}_{\mu}B^{\mu}_{\sigma} \tag{130}$$

and if we use the decomposition from Equation (129) we find:

$$\frac{d\theta}{d\tau} = -R_{\mu\nu}u^{\mu}u^{\nu} - \mathcal{T}^{\lambda}_{\sigma\mu}B^{\mu}_{\lambda}u^{\sigma} - R^{a}_{\sigma\mu}B^{\mu}_{a}u^{\sigma} - \frac{1}{3}\theta^{2} - \sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} \tag{131}$$

The Equation (131) is the generalized Raychaudhuri equation for the horizontal space.

As we can see, Equation (131) disturbs the rate of the volume because of the presence of the nonlinear connection N^{a}_{μ}, and the torsion functions affect the evolution of the gravitational fluid for possible singularities/conjugate points in the universe. In the framework of a given congruence of time-like geodesics, the expansion Θ and shear $\sigma_{\mu\nu}$ are described in a generalized form, provided that the generalized type of Raychaudhuri Equation (131) gives additional information on the kinematics. This is possible due to the perturbation of the deviation equation of nearby geodesics or trajectories, which was described in Section 3.

4.2. Vertical Equations

As with the horizontal space, we define the divergence tensor as:

$$B^{a}_{b} = D_{b}v^{a} \tag{132}$$

where D_{b} denotes the vertical covariant derivative, and v^{a} is a vertical vector tangent to the geodesic congruence. In order to find the deviation of B^{a}_{b}, we have:

$$\frac{DB^{a}_{b}}{d\tau} = v^{c}D_{c}B^{a}_{b} = v^{c}D_{c}(D_{b}v^{a}) \tag{133}$$

We can use the commutator of D as:

$$[D_{c}, D_{b}]v^{a} = S^{a}_{dcb}v^{d} - S^{d}_{cb}D_{d}v^{a} \tag{134}$$

where S^a_{dcb} is the vertical curvature tensor, and S^d_{cb} is the commutator of the connection coefficients. If we use Equations (133) and (134) we have:

$$\frac{DB^a_b}{d\tau} = u^c[D_c, D_b]v^a + v^c D_b D_c v^a \Rightarrow$$
$$\frac{DB^a_b}{d\tau} = v^c\left[S^a_{dcb}v^d - S^d_{cb}D_d v^a\right] + [D_b(u^c D_c v^a) - (D_b v^c)(D_c v^a)] \Rightarrow$$
$$\frac{DB^a_b}{d\tau} = S^a_{dcb}v^c v^d - S^d_{cb}B^a_d v^c - B^c_b B^a_c \tag{135}$$

where we have used Equation (132) and $v^c D_c v^a = 0$ since v^a is tangent to the geodesics.

As with the horizontal part, we decompose the divergence tensor as follows:

$$B^a_b = \frac{1}{3}\tilde{\theta}P^a_b + \tilde{\sigma}^a_b + \tilde{\omega}^a_b \tag{136}$$

where the projection tensor is written as:

$$P^a_b = \tilde{\delta}^a_b + v^a v_b \tag{137}$$

If we use the projection tensor in Equation (135) we obtain:

$$P^b_a\frac{DB^a_b}{d\tau} = -S_{ab}v^a v^b - S^d_{ca}B^a_d v^c - B^c_a B^a_c \tag{138}$$

and by using the separation from Equation (136), we find:

$$\frac{d\tilde{\theta}}{d\tau} = -S_{ab}v^a v^b - S^d_{ca}B^a_d v^c - \frac{1}{3}\tilde{\theta}^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + \tilde{\omega}^{ab}\tilde{\omega}_{ab} \tag{139}$$

Relation (139) is the vertical Raychaudhuri equation.

4.3. Application to the SFR Model

The non-holonomy coefficients of the nonlinear connection $R^a_{\nu\kappa}$ is given by:

$$R^a_{\nu\kappa} = \delta_\kappa N^a_\nu - \delta_\nu N^a_\kappa \tag{140}$$

If we use Equations (8) and (50) we find:

$$\delta_\kappa N^a_\nu = \partial_\kappa N^a_\nu - N^e_\kappa \dot{\partial}_e N^a_\nu$$
$$\delta_\kappa N^a_\nu = \frac{1}{2}y^b\partial_\kappa(g^{ac}\partial_\nu g_{bc}) - \frac{1}{2}y^f g^{ed}\partial_\kappa g_{df}\dot{\partial}_e(\frac{1}{2}y^b g^{ac}\partial_\nu g_{bc})$$
$$\delta_\kappa N^a_\nu = \frac{1}{2}y^b\left(\partial_\kappa g^{ac}\partial_\nu g_{bc} + g^{ac}\partial_\kappa\partial_\nu g_{bc} + \frac{1}{4}\partial_\kappa g_{bc}\partial_\nu g^{ac}\right) \tag{141}$$

So, the non-holonomy coefficients of the nonlinear connection from (140) become:

$$R^a_{\nu\kappa} = \frac{1}{4}y^b(\partial_\kappa g^{ac}\partial_\nu g_{bc} - \partial_\nu g^{ac}\partial_\kappa g_{bc}) \tag{142}$$

The non holonomy coefficients of the vertical connection C^a_{bc} are:

$$S^a_{bc} = C^a_{bc} - C^a_{cb} = 0 \tag{143}$$

because from Equation (23), C^a_{bc} is symmetric.

The horizontal component of the torsion tensor is given by:

$$\mathcal{T}^{\lambda}_{\sigma\mu} = L^{\lambda}_{\sigma\mu} - L^{\sigma}_{\mu\sigma} = 0 \tag{144}$$

because the from Equation (20) $L^{\lambda}_{\sigma\mu}$ is symmetric.

For simplicity, we will take σ^{a}_{b} and ω^{a}_{b} to be zero in both the horizontal and the vertical Raychaudhuri equations. So, Equations (131) and (139) can be written as:

$$\frac{d\theta}{d\tau} = -R_{\mu\nu}u^{\mu}u^{\nu} - R^{a}_{\sigma\mu}B^{\mu}_{a}u^{\sigma} - \frac{1}{3}\theta^{2} \tag{145}$$

$$\frac{d\tilde{\theta}}{d\tau} = -S_{ab}v^{a}v^{b} - \frac{1}{3}\tilde{\theta}^{2} \tag{146}$$

where $R_{\mu\nu} = 0$ because it is identified with the GR case in first-order approximation. Furthermore, if we use Equation (142) and take the vertical vector $v^{a} = (-1, 0, 0, 0)$ we find:

$$\frac{d\theta}{d\tau} = -\frac{1}{4}y^{b}\left(\partial_{\mu}g^{ac}\partial_{\sigma}g_{bc} - \partial_{\sigma}g^{ac}\partial_{\kappa}g_{bc}\right)B^{\mu}_{a}u^{\sigma} - \frac{1}{3}\theta^{2} \tag{147}$$

$$\frac{d\tilde{\theta}}{d\tau} = -\frac{2\tilde{A}_{0}^{2}}{\tilde{a}^{2}}f\left(1 - f\frac{y_{t}^{2}}{\tilde{a}^{2}}\right)^{2} - \frac{1}{3}\tilde{\theta}^{2} \tag{148}$$

Here, we have used that the time component of the vertical curvature is given by (61), which can be interpreted as the evolution of anisotropic expansion $\tilde{\theta}$. By adding Equations (145) and (146) we find:

$$\frac{d\theta}{d\tau} + \frac{d\tilde{\theta}}{d\tau} + \frac{1}{3}\theta^{2} + \frac{1}{3}\tilde{\theta}^{2} = -R_{\mu\nu}u^{\mu}u^{\nu} - R^{a}_{\sigma\mu}B^{\mu}_{a}u^{\sigma} - S_{ab}v^{a}v^{b} \Rightarrow$$

$$\frac{d}{d\tau}(\theta + \tilde{\theta}) + \frac{1}{3}(\theta + \tilde{\theta})^{2} - \frac{2}{3}\theta\tilde{\theta} = -R_{\mu\nu}u^{\mu}u^{\nu} - R^{a}_{\sigma\mu}B^{\mu}_{a}u^{\sigma} - S_{ab}v^{a}v^{b} \tag{149}$$

The above-mentioned Equation (149) represents the volumes and their changes, and θ and $\tilde{\theta}$ denote the standard volume from the horizontal background part of the tangent bundle and the internal anisotropic bulk which is caused by the anisotropic structure. Likewise, $\theta\tilde{\theta}$ can be considered to be the coupling of the background volume with its anisotropic bulk during the evolution of world lines, and the quantity $\theta + \tilde{\theta}$ expresses the total volume.

5. Discussion and Conclusions

The fully developed equations that characterize the flow in a given background spacetime are the Raychaudhuri equations, which are fundamental since they describe the dynamical evolution of the gravitational fluid. They are produced by the structure of deviation of nearby geodesics, which is dominated by the curvature of space. In general, the correspondence between fluids and gravity can be represented in a realistic way to understanding current theoretical and observational problems.

In this article, we examine and derive the deviation equation of geodesics and paths as well as the form of a Raychaudhuri equation in a completely generalized framework and we apply them to a Schwarzschild–Finsler–Randers model in which we showed that the extra terms in (108), (109), (131), and (139) anisotropically affect the acceleration tidal vector field and the variation in the volume (expansion) during the evolution of fluid lines (geodesics and paths). From a physical point of view, the generalized deviation geodesics equation is influenced by extra degrees of freedom, e.g, because of additional mass, gas, dark matter, etc. [18,25], which interact gravitationally during their motion. Tidal acceleration phenomena and anisotropic tidal-field perturbations can appear from different sources of spacetime. It is remarkable to mention here that acceleration geometrical concepts constitute intrinsic properties on a tangent bundle as, e.g, a vector field. The extended geometrical structure of the SFR model includes the Schwarzschild spacetime and gives us additional information on the kinematics because of the extra degrees of freedom reflected in additional terms of torsion, nonlinear connection and S-Kretschmann-like curvature invariants that are imprinted on the corresponding equations.

Moreover, we investigate the weak-field limit of a Finslerian perturbation on a Riemannian spacetime in the cases of a deviation equation. We also study the Newtonian limit of the model and derive a generalized Poisson equation. Additionally, we presented an interesting application for the deviation angle on our generalized framework and compared the result with that of GR. We also study the Raychaudhuri equation, which is extended in the horizontal and vertical parts of the SFR spacetime. In particular, the concept of nonlinear connection in Finsler or Finsler-like spacetime can be geometrically interpreted as an interaction between external and internal structures on the Lorentz tangent bundle spacetime. In a more specific case, a physical interpretation of nonlinear connection can relate internal scalar fields with the matter sector of spacetime, as, for instance, in [18,25]. It can be understood from all the derived equations of the SFR model that they reduce to standard GR when all the extra terms of generalized geometrical structure are omitted. The anisotropic S-curvature is significant in our cosmological model since it can provide addition information for the evolution of gravitational flow lines and the expansion of the universe as well as singularities (focusing/defocusing) of spacetime. This curvature expresses a very small source of anisotropy as is evident from the Equations (60)–(71) in which all the terms are multiplied by a constant $\tilde{A}_0 \ll 1$ (Equation (60)); this means that all the values of S are very small. Consequently, from a physical point of view, it is possible that in a very early period of the universe, the anisotropies of CMB influenced the geometry during cosmological evolution.

It is of special interest that one investigates the weak-field limit in more detail and connect it to the anisotropic polarization of gravitational waves. This research will be the motivation for a future work.

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