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6 Quantum Gates and Quantum Algorithms with Clifford Algebra Technique

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Abstract. We use the Clifford algebra technique [1,2] for representing in an elegant way quantum gates and quantum algorithms needed in quantum computers. We express the phase gate, Hadamard's gate and the C-NOT gate as well as the Grover's algorithm in terms of nilpotents and projectors—binomials of the Clifford algebra objects ^{a} with the property $\{^a, ^b\}_+ = 2^{ab}$, identifying n-qubits with the spinor representations of the group $SO(1,3)$ for the system of n spinors expressed in terms of products of projectors and nilpotents.

6.1 Introduction

It is easy to prove (and it is also well known) that any type of a quantum gate, operating on one qubit and represented by an unitary operator, can be expressed as a product of the two types of quantum gates—the phase gate and the Hadamard's gate—while the C-NOT gate, operating on two quantum bits, enables to make a quantum computer realizable, since all the needed operations can be expressed in terms of these three types of gates. In the references[3,4] the use of the geometrical algebra to demonstrate these gates and their functioning is presented.

In this paper we use the technique from the ref. [1,2], which represents spinor representations of the group $SO(1,3)$ in terms of projectors and nilpotents, which are binomials of the Clifford algebra objects ^{a}. We identify the spinor representation of two one spinor states with the two quantum bits $|0\rangle$ and $|1\rangle$ and accordingly n spinors' representation of $SO(1,3)$ with the n-qubits. The three types of the gates can then be expressed in terms of projectors and nilpotents in a transparent and elegant way. We express one of the known quantum algorithms, the Grover's algorithm, in term of projectors and nilpotents to see what properties does it demonstrate and to what new algorithms with particular useful properties might it be generalized.

6.2 The technique for spinor representations

We define in this section the basic states for the representation of the group $SO(1,3)$ and identify the one qubit with one of the spinor states. We distinguish between the chiral representation and the representation with a well defined parity. We shall at the end make use of the states of well defined parity, since they seem to be more appropriate for the realizable types of quantum computers. However, the proposed gates work for the chiral representation of spinors as well. We identify n-qubits with states which are superposition of products of n one spinor states. We also present some relations, useful when defining the quantum gates.

The group $SO(1,3)$ has six generators S^{ab}: $S^{01}, S^{02}, S^{03}, S^{23}, S^{31}, S^{12}$, fulfilling the Lorentz algebra $\{S^{ab}, S^{cd}\}_- = i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac})$. For spinors can the generators S^{ab} be written in terms of the operators γ^a fulfilling the Clifford algebra

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}, \quad \mathrm{diag}(\eta) = (1,-1,-1,-1),$$
$$S^{ab} = \frac{i}{2}\gamma^a\gamma^b, \text{ for } a \neq b \text{ and } 0 \text{ otherwise}. \tag{6.1}$$

They define the spinor (fundamental) representation of the group $SO(1,3)$. Choosing for the Cartan subalgebra set of commuting operators S^{03} and S^{12} the spinor states

$$|0\rangle_L = \overset{03}{[-i]}\overset{12}{(+)}, \quad |1\rangle_L = \overset{03}{(+i)}\overset{12}{[-]},$$
$$|0\rangle_R = \overset{03}{(+i)}\overset{12}{(+)}, \quad |1\rangle_R = \overset{03}{[-i]}\overset{12}{[-]} \tag{6.2}$$

with the definition

$$\overset{03}{(\pm i)} := \frac{1}{2}(\gamma^0 \mp \gamma^3), \quad \overset{12}{(\pm)} := \frac{1}{2}(\gamma^1 \pm i\gamma^2),$$
$$\overset{03}{[\pm i]} := \frac{1}{2}(1 \pm \gamma^0\gamma^3), \quad \overset{12}{[\pm]} := \frac{1}{2}(1 \pm i\gamma^1\gamma^2), \tag{6.3}$$

are all eigenstates of the Cartan subalgebra set S^{03} and S^{12}, since $S^{03}\overset{03}{(\pm i)} = \pm\frac{i}{2}\overset{03}{(\pm i)}$, $S^{03}\overset{03}{[\pm i]} = \pm\frac{i}{2}\overset{03}{[\pm i]}$ and similarly $S^{12}\overset{12}{(\pm)} = \pm\frac{1}{2}\overset{12}{(\pm)}$, $S^{12}\overset{12}{[\pm]} = \pm\frac{1}{2}\overset{12}{[\pm]}$, what can very easily be checked, just by applying S^{03} and S^{12} on the particular nilpotent or projector and using Eq.(6.1). The states $|0\rangle_L$ and $|1\rangle_L$ have handedness $\Gamma = -4iS^{03}S^{12}$ equal to -1, while the states $|0\rangle_R$ and $|1\rangle_R$ have handedness equal to 1. We normalize the states as follows [1]

$$\langle i_\alpha | j_\beta \rangle = \delta_{ij}\delta_{\alpha\beta}, \tag{6.4}$$

where i,j denote 0 or 1 and α, β left and right handedness.

When describing a spinor in its center of mass motion, the representation with a well defined parity is more convenient

$$|0\rangle = \frac{1}{\sqrt{2}}(\overset{03}{[-i]}\overset{12}{(+)} \pm \overset{03}{(+i)}\overset{12}{(+)}),$$
$$|1\rangle = \frac{1}{\sqrt{2}}(\overset{03}{(+i)}\overset{12}{[-]} \pm \overset{03}{[-i]}\overset{12}{[-]}). \tag{6.5}$$

Nilpotents and projectors fulfill the following relations [1,2] (which can be checked just by using the definition of the nilpotents and projectors (Eq.6.3) and by taking into account the Clifford property of ^{a}'s (Eq.6.1))

$$\overset{ab}{(k)}\overset{ab}{(k)} = 0; \quad \overset{ab}{(k)}\overset{ab}{(-k)} = {}^{aa}\overset{ab}{[k]}; \quad \overset{ab}{[k]}\overset{ab}{[k]} = \overset{ab}{[k]}; \quad \overset{ab}{[k]}\overset{ab}{[-k]} = 0;$$
$$\overset{ab}{(k)}\overset{ab}{[k]} = 0; \quad \overset{ab}{[k]}\overset{ab}{(k)} = \overset{ab}{(k)}; \quad \overset{ab}{(k)}\overset{ab}{[-k]} = \overset{ab}{(k)}; \quad \overset{ab}{[k]}\overset{ab}{(-k)} = 0 \qquad (6.6)$$

We then find that the operators

$$^{L} = -(\;^{03}\; i\;^{12}\;); \quad ^{R} = (\;^{03}\; i\;^{12}\;); \qquad (6.7)$$

transform the states of the same representation, left and right correspondingly, one into another or annihilate them, while they annihilate the states of the opposite handedness

$$^{L-}|0\rangle_L = |1\rangle_L; \quad ^{L+}|1\rangle_L = |0\rangle_L;$$
$$^{R-}|0\rangle_R = |1\rangle_R; \quad ^{R+}|1\rangle_R = |0\rangle_R; \qquad (6.8)$$

all the other applications ^{L} and ^{R} give zero.

We also find that the operators

$$:= \;^{L} + \;^{R} = -(\;^{03}\; i\;^{12}\;) + (\;^{03}\; i\;^{12}\;) \qquad (6.9)$$

transform the states of well defined parity (Eq.6.5) into one another or annihilate them

$$^{-}|0\rangle = |1\rangle; \quad ^{+}|1\rangle = |0\rangle \qquad (6.10)$$

while the rest of applications give zero, accordingly $(\;^{+} + \;^{-})|0\rangle = |1\rangle; (\;^{+} + \;^{-})|1\rangle = |0\rangle.$

We present the following useful properties of , valid for ^{L} and ^{R} as well so that we shall skip the index $L;R$;,

$$(\;)^2 = 0;$$
$$(\;)^{y} = \;;$$
$$^{+}\;^{-} = \overset{12}{[+]}; \quad ^{-}\;^{+} = \overset{12}{[-]};$$
$$(\;^{+} + \;^{-})^2 = I;$$
$$^{+}\overset{12}{[+]} = 0; \quad ^{-}\overset{12}{[-]} = 0; \quad \overset{12}{[+]}\;^{-} = 0; \quad \overset{12}{[-]}\;^{+} = 0;$$
$$^{+}\overset{12}{[-]} = \;^{+}; \quad ^{-}\overset{12}{[+]} = \;^{-}; \quad \overset{12}{[+]}\;^{+} = \;^{+}; \quad \overset{12}{[-]}\;^{-} = \;^{-}: \qquad (6.11)$$

A n-qubit state can be written in the chiral representation as

$$|i_1 i_2 \;_1\; i_n\rangle = \prod_{l=1;n} |i_l\rangle \;; \quad = L;R; \qquad (6.12)$$

while in the representation with well defined parity we similarly have

$$|i_1 i_2 \;_1\; i_n\rangle = i \prod_{l=1;n} |i_k\rangle: \qquad (6.13)$$

$|i_l\rangle$ stand for $|0_l\rangle$ or $|1_l\rangle$. All the raising and lowering operators $_{l}$, $= L;R$ or $_{l}$ carry the index of the corresponding qubit manifesting that they only apply on the particular k state, while they do not "see" all the other states. Since they are made out of an even number of the Clifford odd nilpotents, they do not bring any sign when jumping over one-qubit states.

6.3 Quantum gates

We define in this section three kinds of quantum gates: the phase gate and the Hadamard's gate, which apply on a particular qubit l and the C-NOT gate, which applies on two qubits, say l and m. All three gates are expressed in terms of projectors and an even number of nilpotents.

i. The phase gate R_{l} is defined as

$$R_{l} = \overset{12}{[+]}_l + e^{i\,_l}\,\overset{12}{[-]}_l \, . \tag{6.14}$$

Statement: The phase gate R_{l} if applying on $|0_l\rangle$ leaves it in state $|0_l\rangle$, while if applying on $|1_l\rangle$ multiplies this state with e^{i}. This is true for states with well defined parity $|i_l\rangle$ and also for the states in the chiral representation $|i_l\rangle_L$ and $|i_l\rangle_R$.

Proof: To prove this statement one only has to apply the operator R_{l} on $|i_l\rangle$, $|i_l\rangle_L$ and $|i_l\rangle_R$, with i_l equal 0 or 1, taking into account equations from Sect. 6.2.

ii. The Hadamard's gate H_l is defined as

$$H_l = \frac{1}{\sqrt{2}}\Big[\overset{12}{[+]}_l - \overset{12}{[-]}_l - \overset{03}{(+i)}_l\overset{12}{(-)}_l + \overset{03}{(-i)}_l\overset{12}{(-)}_l - \overset{03}{(-i)}_l\overset{12}{(+)}_l + \overset{03}{(+i)}_l\overset{12}{(+)}_l\Big]; \tag{6.15}$$

or equivalently in terms of (Eq.(6.9))

$$H_l = \frac{1}{\sqrt{2}}\Big[\overset{12}{[+]}_l - \overset{12}{[-]}_l + {}^{-}_{l} + {}^{+}_{l}\Big]. \tag{6.16}$$

Statement: The Hadamard's gate H_l if applying on $|0_l\rangle$ transforms it to

$$\Big(\frac{1}{\sqrt{2}}(|0_l\rangle + |1_l\rangle)\Big);$$

while if applying on $|1_l\rangle$ it transforms the state to $(\frac{1}{\sqrt{2}}(|0_l\rangle - |1_l\rangle))$. This is true for states with well defined parity $|i_l\rangle$ and also for the states in the chiral representation $|i_l\rangle_L$ and $|i_l\rangle_R$.

Proof: To prove this statement one only has to apply the operator H_l on $|i_l\rangle$, $|i_l\rangle_L$ and $|i_l\rangle_R$, with i_l equal 0 or 1, taking into account equations from Sect. 6.2.

iii. The C-NOT gate C_{lm} is defined as

$$C_{lm} = \overset{12}{[+]}_l + \overset{12}{[-]}_l\Big[- \overset{03}{(+i)}_m\overset{12}{(-)}_m + \overset{03}{(-i)}_m\overset{12}{(-)}_m - \overset{03}{(-i)}_m\overset{12}{(+)}_m + \overset{03}{(+i)}_m\overset{12}{(+)}_m\Big]; \tag{6.17}$$

or equivalently

$$C_{lm} = \overset{12}{[+]}_l + \overset{12}{[-]}_l\,[\,{}^{-}_{m} + {}^{+}_{m}\,]: \tag{6.18}$$

Statement: The C-NOT gate C_{lm} if applying on $|\;_l\; 0_m\; 0\;\rangle$ transforms it back to the same state, if applying on $|\;_l\; 0_m\; 1\;\rangle$ transforms it to back to the same state. If C_{lm} applies on $|\;_l\; 1_m\; 0\;\rangle$ transforms it to $|\;_m\; 11\;\rangle$, while it transforms the state $|\;_l\; 1_m\; 1\;\rangle$ to the state $|\;_m\; 10\;\rangle$.

Proof: To prove this statement one only has to apply the operator C_{lm} on the states $|\;_l\; i_m\; i\;\rangle$, $|\;_m\; i\rangle_L$, $|\,i\;_l\; i_m\; i\,\rangle_R$, with $i_l; i_m$ equal 0 or 1, taking into account equations from Sect. 6.2.

Statement: When applying $\prod_i^n H_i$ on the n qubit with all the qubits in the state $|0_i\rangle$, we get the state $|\;_0\rangle$

$$|\;_0\rangle = \prod_i^n H_i|0_i\rangle = \prod_i^n (|0_i\rangle + |1_i\rangle): \tag{6.19}$$

Proof: It is straightforward to prove, if the statement ii. of this section is taken into account.

6.4 Useful properties of quantum gates in the technique using nilpotents and projectors

We present in this section some useful relations.
i. One easily finds, taking into account Eqs.(6.14,6.16,6.18,6.9,6.11), the relation

$$R\;_l H_l R\;_l H_l = \frac{1}{2}\{(\overset{12}{[+]}_l + e^{i\;_l}\overset{12}{[-]}_l)(1 + e^{i\;_l}) + (\;^{+}_{l} + \;^{-}_{l})(1 - e^{i\;_l})\}; \tag{6.20}$$

which transforms $|1_l\rangle$ into a general superposition of $|0_l\rangle$ and $|1_l\rangle$ like

$$e^{-i\;_l=2}R_{(\;_l+\;=2)}H_l R\;_l H_l|0_l\rangle = \cos(\;_l=2)|0_l\rangle + e^{i\;_l}\sin(\;_l=2)|1_l\rangle;$$
$$e^{-i(\;_l-\;)=2}R\;_l H_l R\;_l H_l|1_l\rangle = \sin(\;_l=2)|0_l\rangle - e^{i\;_l}\cos(\;_l=2)|1_l\rangle: \tag{6.21}$$

ii. Let us define the unitary operator $\hat{O}_p$

$$\hat{O}_p = I - 2\prod_{l_i=1}^{p}[i_0]_{l_i} = I - 2\hat{R}_p;$$
$$\hat{O}_p^2 = (I - 2\hat{R}_p)^2 = I; \tag{6.22}$$

where $[i_0]_{l_i}$ projects out of the i-th qubit a particular state $|i_0\rangle$ (with $i_0 = 0;1$) and where p is the number of qubits taken into account, while $\hat{R}_p$ is defined as follows

$$\hat{R}_p = \prod_{l_i}^{p}[i_0]_{l_i}$$
$$\hat{R}_p^2 = \hat{R}_p = \hat{R}_p^{\dagger}: \tag{6.23}$$

We define also the unitary operator $\hat{D}_p$

$$\hat{D}_p = -I + \frac{2}{2^p}\left[\prod_{l_i=1}^{p} (I_{l_i} + {}^{+}_{l_i} + {}^{-}_{l_i})\right] = 2\hat{S}_p - I,$$

$$\hat{S}_{l_i} = \frac{1}{2}(I_{l_i} + {}^{+}_{l_i} + {}^{-}_{l_i}), \quad \hat{S}_p = \prod_{l_i=1}^{p} \hat{S}_{l_i}. \tag{6.24}$$

We find that $\hat{S}_p$ is a projector

$$(\hat{S}_{l_i})^k = \hat{S}_{l_i},$$
$$(\hat{S}_p)^k = \hat{S}_p,$$
$$(\hat{S}_p)^k (\hat{R}_p)^l = \hat{S}_p \hat{R}_p. \tag{6.25}$$

Consequently it follows

$$\hat{S}_p \hat{R}_p \hat{S}_p = \frac{1}{2^p}\hat{S}_p,$$
$$\hat{R}_p \hat{S}_p \hat{R}_p = \frac{1}{2^p}\hat{R}_p,$$
$$\hat{S}_p \hat{R}_p \hat{S}_p \hat{R}_p = \frac{1}{2^p}\hat{S}_p \hat{R}_p,$$
$$(\hat{D}_p)^2 = I. \tag{6.26}$$

Let us simplify the notation

$$\{i_0\}_{SR} = \frac{1}{2}\begin{cases} [+]_{l_i} + {}^{-}_{l_i}, & \text{if } [i_0]_{l_i} = [+]_{l_i} \\ [-]_{l_i} + {}^{+}_{l_i}, & \text{if } [i_0]_{l_i} = [-]_{l_i} \end{cases} \tag{6.27}$$

$$\{i_0\}_{RS} = \frac{1}{2}\begin{cases} [+]_{l_i} + {}^{+}_{l_i}, & \text{if } [i_0]_{l_i} = [+]_{l_i} \\ [-]_{l_i} + {}^{-}_{l_i}, & \text{if } [i_0]_{l_i} = [-]_{l_i} \end{cases} \tag{6.28}$$

$$\{0\}_{SR} = \prod_i \{i_0\}_{SR} \tag{6.29}$$

$$\{0\}_{RS} = \prod_i \{i_0\}_{RS}. \tag{6.30}$$

then we can write

$$\hat{R}_p \hat{S}_p = \{0\}_{RS},$$
$$\hat{S}_p \hat{R}_p \hat{S}_p = \hat{S}_p \{0\}_{RS},$$
$$\hat{R}_p \hat{S}_p \hat{R}_p = \{0\}_{RS} \hat{R}_p,$$
$$\hat{S}_p \hat{R}_p \hat{S}_p \hat{R}_p = \frac{1}{2^p}\{0\}_{SR}. \tag{6.31}$$

Let us recognize that for $p = n$, where n is the number of qubits,

$$
\begin{aligned}
\{0\}_{RS}|\psi_0\rangle &= R_p S_p|\psi_0\rangle = \frac{1}{\sqrt{2^p}}\prod_p |\psi_i\rangle ;\\
\{0\}_{SR}|\psi_0\rangle &= S_p R_p|\psi_0\rangle = \frac{1}{2^p}|\psi_0\rangle ;\\
\hat{S}_p|\psi_0\rangle &= |\psi_0\rangle ;\\
\hat{R}_p|\psi_0\rangle &= \frac{1}{\sqrt{2^p}}\prod_p |\psi_i\rangle .
\end{aligned} \tag{6.32}
$$

6.5 Grover's algorithm

Grover's quantum algorithm is designed to search a particular information out of a data base with n qubits. It enables us to find the desired information in $O(\sqrt{2^n})$ trials, with a certain probability.

Let us define the operator G_p

$$\hat{G}_p = \hat{D}_p\hat{O}_p ; \tag{6.33}$$

with $\hat{O}_p$ and $\hat{D}_p$ defined in Eqs.(6.22, 6.24) in sect. 6.4. We find, if using notation from equations (6.25,6.23)

$$
\begin{aligned}
(\hat{G}_p)^k &= (D_p O_p)^k = \left((2\hat{S}_p - I)(I - 2\hat{R}_p)\right)^k \\
&= \left(2\hat{S}_p + 2\hat{R}_p - 4\hat{S}_p\hat{R}_p - I\right)^k
\end{aligned} \tag{6.34}
$$

where $\hat{S}_p$ and $\hat{R}_p$ do not commute.

We can further write

$$(\hat{G}_p)^k = (-)^k I + N_1\{0\}_{SR} + N_2\{0\}_{RS} + N_3\hat{S}_p + N_4\hat{R}_p ; \tag{6.35}$$

where $N_i; i \in \{1,4\}$ are integers, which depend on p and k.

For $n = p$ and $k = 1$ we find $N_i; i \in \{1,4\}$ are $-4;0;2;2$, for $k = 2$ we find $N_i; i \in \{1,4\}$ are $2^{4-p} - 2^2; 2^2; -2^{3-p}; -2^{3-p}$.

Since, according to Eq.(6.32) the application of $\{0\}_{SR}; \{0\}_{RS}; \hat{S}_p$ and $\hat{R}_p$ on the state $|\psi_0\rangle$ gives $\frac{1}{2^p}|\psi_0\rangle$, $\frac{1}{\sqrt{2^p}}\prod_i^p|\psi_i\rangle$, $|\psi_0\rangle$, $\frac{1}{\sqrt{2^p}}\prod_i^p|\psi_i\rangle$, the application of the operator $\hat{G}_p$ k times, lead to the state

$$G^k|\psi_0\rangle = \alpha_k|\psi_0\rangle + \beta_k\prod_i|\psi_i\rangle ; \tag{6.36}$$

where $\alpha_k = \frac{1}{2^p}|\psi_0\rangle = [(-)^k + N_1 2^{-p} + N_3]|\psi_0\rangle$ and $\beta_k = 2^{-\frac{p}{2}}[N_2 + N_4]$.

One can find the $\alpha_k; \beta_k$ by recognizing that

$$\hat{G}_p(\alpha_j|\psi_0\rangle + \beta_j\prod_i|\psi_i\rangle) = \alpha_{j+1}|\psi_0\rangle + \beta_{j+1}\prod_i|\psi_i\rangle . \tag{6.37}$$

It follows that

$$ {}_{j+1} = {}_{j} \; 1 - 2^{2-p} \; - 2^{1-\frac{p}{2}} \; {}_{j} \quad (6.38)$$
$$ {}_{j+1} = 2^{1-\frac{p}{2}} \; {}_{j} + {}_{j}: \quad (6.39)$$

Let us calculate a few values for and

$$ {}_{0} = 1; \quad {}_{0} = 0 \quad (6.40)$$
$$ {}_{1} = 1 - 2^{2-p}; \quad {}_{1} = 2^{1-\frac{p}{2}} \quad (6.41)$$
$$ {}_{2} = (1 - 2^{2-p})^2 - (2^{1-\frac{p}{2}})^2; \quad {}_{2} = 2^{2-\frac{p}{2}}(1 - 2^{1-p}) \quad (6.42)$$

6.6 Concluding remarks

We have demonstrated in this paper how can the Clifford algebra technique [1,2] be used in quantum computers gates and algorithms. Although our projectors and nilpotents can as well be expressed in terms of the ordinary projectors and the ordinary operators, the elegance of the technique seems helpful to better understand the operators appearing in the quantum gates and quantum algorithms. We shall use the experience from this contribution to try to generate new quantum algorithms.

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