PROCEEDINGS OF SCIENCE

Three-hadron dynamics from lattice QCD

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Three-hadron spectroscopy is a key frontier in our understanding of the hadron spectrum. In recent years, significant formal and numerical advances have paved the way for studying three-hadron processes directly from lattice QCD, with outstanding applications including the Roper resonance and the doubly charmed tetraquark. This requires theoretical frameworks that relate finite-volume energies to infinite-volume three-particle scattering amplitudes. In this contribution, I discuss recent progress in formulating such frameworks for generic three-hadron systems, and present numerical results for three-meson systems at maximal isospin with physical quark masses, as well as our recent investigation of the three-body dynamics of the doubly charmed tetraquark, T_{cc}.

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1. Introduction

There are many reasons to study three-hadron dynamics. First, it is essential to understand how the complexity of the hadron spectrum emerges from Quantum Chromodynamics (QCD). This is especially relevant in light of recent discoveries of exotic hadrons at experiments such as LHCb [1] and BESIII [2], see Ref. [3] for a review. A notable example is the recently discovered doubly-charmed tetraquark $T_{cc}(3875)$ [4, 5], which has only one decay channel, $T_{cc} \to DD\pi$. Another interesting corner of the hadron spectrum are baryon excitations, whose description generally requires the inclusion of three-body effects. For instance, the Roper resonance, which decays to both $N\pi$ and $N\pi\pi$, remains poorly understood [6, 7]. Similarly, excited hyperons such as $\Lambda(1405)$ and $\Lambda(1520)$ may also require the inclusion of their three-body decay channels, such as $\Lambda\pi\pi$, for a complete theoretical description.

Beyond understanding the hadron spectrum for its own sake, there are numerous applications in nuclear and high-energy physics. For instance, three-nucleon forces are crucial for accurately describing atomic nuclei [8], and three-baryon interactions involving hyperons have been highlighted for their impact on neutron star properties [9, 10]. In the context of precision tests of the Standard Model, experimental studies of weak kaon decays into three pions provide some evidence for CP violation [11, 12].

Motivated by these applications, a decade of theoretical efforts has provided the necessary tools to study three-hadron dynamics using lattice QCD [13–51]; see also recent reviews [52–55]. The ultimate goal of these approaches is to provide first-principles predictions of three-hadron scattering amplitudes with controlled uncertainties. By identifying poles in these amplitudes, one can predict the masses, decay widths and couplings of resonances. Indeed, there are already many applications [56–74], including pioneering works on three-body resonances [75–77]. However, the field is relatively new, and some theoretical issues and many potential applications remain to be explored.

Here, I review the methods and some recent results in three hadron spectroscopy. First, I will discuss the background on obtaining three-particle scattering amplitudes from lattice QCD. Then, I will present two applications: the study of three-meson amplitudes at maximal isospin [78, 79], and the three-body dynamics of the doubly charmed tetraquark [45, 80]. I will also provide a brief outlook of the remaining challenges in three-hadron spectroscopy.

2. Three-hadron scattering amplitudes from lattice QCD

While lattice QCD calculations are performed in a finite volume and Euclidean spacetime, scattering amplitudes are defined in infinite volume and real time. Therefore, the study of multihadron systems using lattice QCD relies on an indirect method, as originally proposed by Lüscher for two-particle systems [81, 82]. The central idea is that finite-volume energies obtained from lattice QCD can be used as constraints on scattering amplitudes. Specifically, quantum field theories in finite volume with periodic boundary conditions only have a discrete spectrum due to quantized momentum modes, as $\boldsymbol{p} = (2\pi/L)\boldsymbol{n}$, where L is the box size, and $\boldsymbol{n} \in \mathbb{Z}^3$. Stationary energies deviate from the non-interacting case due to multihadron interactions. Intuitively, particles in a large but finite box are mostly separate, but undergo scatterings that shift the energy away from the energy spectrum of a non-interacting theory.

For two-body scattering, the two-particle quantization condition (QC2) relates finite-volume energies to the two-particle K matrix, $\mathcal{K}_2$:

$$\det_{\ell m}\left[F_2(E_2^*, \boldsymbol{P}, L)^{-1} + \mathcal{K}_2(E_2^*)\right] = 0\,, \tag{1}$$

where F_2 is a known kinematic function that depends on the box-size L and the total momentum of the system $\boldsymbol{P}$ (referred to as the "Lüscher zeta function"). In the QC2, the determinant runs over indices that label two-body partial waves, and values of the two-body center-of-mass (CM) energy, E_n^*, for which the determinant vanishes correspond to the quantized finite-volume energies.

The two-body formalism has been applied to many systems and has become an established tool in lattice QCD spectroscopy, see e.g. the recent discussion in Refs. [83, 84].[1] Remarkably, direct computation of two-body scattering amplitudes directly at the physical point have been conducted [65, 86–91], and certain calculations are approaching full control of systematic errors, including discretization effects [92].

Three-particle tools are based on the same principle: three-body finite-volume energies are determined by three-particle interactions. However, there are some additional difficulties due to the increased complexity of three-particle scattering. First, three-particle amplitudes can be divergent in certain kinematics due to one-particle exchange processes where the exchanged particle goes on shell. Second, three-hadron interactions, both in finite and infinite volume, depend on off-shell two-body interactions via pairwise rescattering. However, a separation between two and three-body effects is not unique. Because of this, the three-hadron formalism needs to introduce an intermediate, cutoff-dependent quantity to parametrize three-body short-range interactions. In the relativistic field-theoretic (RFT) three-particle formalism [17, 18], this is denoted as the divergence-free three particle K matrix, $\mathcal{K}_{\mathrm{df},3}$.[2]

Due to this additional complexity, the three-body formalism is structured as a two-step process. The first step employs the three-body quantization condition (QC3) to constrain the three-particle K matrix from lattice QCD energy levels:

$$\det_{k\ell m}\left[F_3(E_3^*, \boldsymbol{P}, L)^{-1} + \mathcal{K}_{\mathrm{df},3}(E_3^*)\right] = 0\,, \tag{2}$$

where F_3 is a known matrix incorporating kinematic effects and two-particle interactions via the two-particle K matrix. The determinant is taken over indices describing three on-shell particles, specifically the two-body angular momentum indices of the "interacting pair" and the finite-volume momentum of the third "spectator" particle.[3] The matrices in the QC3 remain finite because we truncate two-body interactions at a maximal partial wave, $\ell \leq \ell_{\max}$, and impose a cutoff on the spectator momentum, $k \leq k_{\max}$. This cutoff dependence explicitly reflects the unphysical nature of $\mathcal{K}_{\mathrm{df},3}$, as noted earlier. The second step involves solving integral equations that relate the two- and three-particle K matrices to the physical scattering amplitude. These integral equations remove the cutoff dependence from the three-body K matrix, yielding a physical amplitude that is both

[1] The HalQCD method is another tool to study two-hadron scattering, see e.g. Ref. [85].

[2] See Refs. [20, 21] for the non-relativistic EFT approach and Ref. [22] for the finite-volume unitarity approach.

[3] If the three particles are not identical, an additional index indicates the choice of spectator.

cutoff-independent and consistent with unitarity [26, 28]. Solutions to these integral equations in the RFT framework have been presented in Refs. [43, 93–95].

In a typical lattice QCD calculation, one has access to a finite number of energy levels to constrain the functional forms of the K matrices. Thus, a practical remark on the finite-volume formalism is that parametrizations of the K matrices must be assumed to fit the finite-volume spectrum in terms of a few parameters. In typical fits, 5 to 10 parameters are constrained through correlated χ^2 minimization. Several approaches to do this optimization have been discussed in Ref. [72], for instance, a combined fit of two- and three-particle energies.

The two-particle K matrix is often parametrized using simple polynomial or rational functions of the CM momentum q^2. For instance, in the case of two pions, the s-wave can be described using

$$\frac{q}{M_\pi} \cot \delta_0(q) = \frac{M_\pi E_2^*}{E_2^{*2} - 2z^2 M_\pi^2} \sum_{n=0}^{n_{\max}} B_n \left(\frac{q^2}{M_\pi^2}\right)^n , \tag{3}$$

where B_n and z^2 are fit parameters, the latter corresponding to the position of the Adler zero. For the three-particle case, additional considerations are required to parametrize $\mathcal{K}_{\mathrm{df},3}$. One approach that has been employed is a threshold expansion of $\mathcal{K}_{\mathrm{df},3}$ [27], in which terms are organized in powers of the distance to the three-particle threshold. The number of allowed terms is constrained by particle-exchange as well as C, P, and T symmetries. For instance, for three identical scalar particles, it takes the form

$$M^2 \mathcal{K}_{\mathrm{df},3} = \mathcal{K}_0 + \mathcal{K}_1 \Delta + \mathcal{K}_2 \Delta^2 + \mathcal{K}_\mathrm{A} \Delta_\mathrm{A} + \mathcal{K}_\mathrm{B} \Delta_\mathrm{B}, \tag{4}$$

where $9M_\pi^2 \Delta = P^2 - 9M_\pi^2$, and Δ_A and Δ_B are kinematic functions of the four-momenta of the incoming and outgoing particles—see Ref. [27].

3. Two and three mesons at maximal isospin

The three-body formalism is a relatively recent tool, with the first papers appearing only a decade ago [15, 16]. Consequently, many of its features remain unexplored and require careful numerical investigations. The long-term objective is to apply these tools to systems with three-body resonances, and even in the context of three-body electroweak transitions. However, it is essential to begin with a well-controlled setup, e.g. weakly interacting systems, such as $3\pi^+$. In these systems, no resonances are present, but statistical signals are typically strong and require relatively lower computational cost, making them an excellent testing ground. Moreover, in certain cases, comparisons with effective theories are possible, providing valuable benchmarks.

In our most recent works [78, 79], we focus on such systems, specifically, three-body systems built of pions or kaons: $3\pi^+, 3K^+, \pi^+\pi^+K^+$, and $K^+K^+\pi^+$. We had already explored such systems at heavier-than-physical quark masses [69, 72], ranging from 200-340 MeV. In the latest update, we included an ensemble with physical quark masses—the E250 CLS ensemble [96].

The calculation begins with the extraction of energies from the lattice QCD Euclidean correlation functions. This is achieved by using a large set of operators with the quantum numbers of the system we aim to investigate. The matrix of correlation functions is analyzed as a generalized eigenvalue problem [97].[4] From the exponential decay of the generalized eigenvalues in Euclidean-time, the low-lying energies of the system can be determined. An example of the outcome of such calculation is provided in Fig. 1 for a system of three kaons with physical quark masses.

We simultaneously fit the two- and three-body finite-volume energy levels. In the case of Fig. 1, this means that both $2K^+$ and $3K^+$ energy levels are analyzed together. The extracted parameters are then used to generate predictions from the quantization condition, also displayed in Fig. 1. Although the formalism is strictly valid only below the first inelastic threshold, its breakdown is not immediate, and the first few energy levels above this inelastic threshold are still well-reproduced.

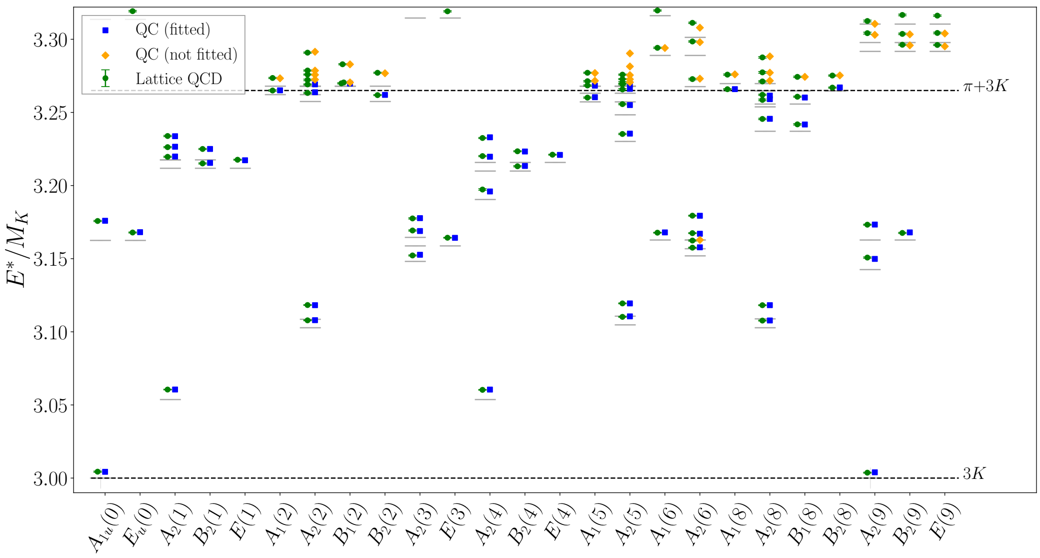

Figure 1: Finite-volume energies of the $3K^+$ system with physical quark masses in different irreducible representations and finite-volume frames [78, 79]. Green circles are lattice QCD determinations, while blue squares indicate predictions from the three-particle finite-volume formalism for the levels included in the fit, and orange diamonds for levels not included in the fit. The elastic and first inelastic thresholds are shown as black dashed horizontal lines.

Given that two- and three-meson energies have been analyzed together, a byproduct is the precise determination of two-body interactions. This is summarized in Fig. 2. Since these results correspond to the physical point, they can be compared to dispersive analyses for $\pi^+\pi^+$ and π^+K^+ [103, 104]. We generally find good agreement with these analyses and, notably, achieve even lower statistical uncertainties in some cases. Additionally, we obtain some constraints on higher partial waves, although only in the K^+K^+ system does the d-wave deviate from zero by more than one sigma.

From the fit results, we can examine the values of the K-matrix in the $3\pi^+$ system, for which Chiral Perturbation Theory (ChPT) predictions are available at leading and next-to-leading orders (LO and NLO) [105–108]. This comparison is meaningful despite the scheme dependence of the K matrix, as the ChPT calculation can be performed within the same scheme. Figure 3 presents the results for the four ensembles used in this study, including a physical-point result from the ETMC collaboration [65] and the ChPT predictions. For the NLO ChPT curve, phenomenologically

[4]For alternative novel methods of energy extraction, see Refs. [98–102].

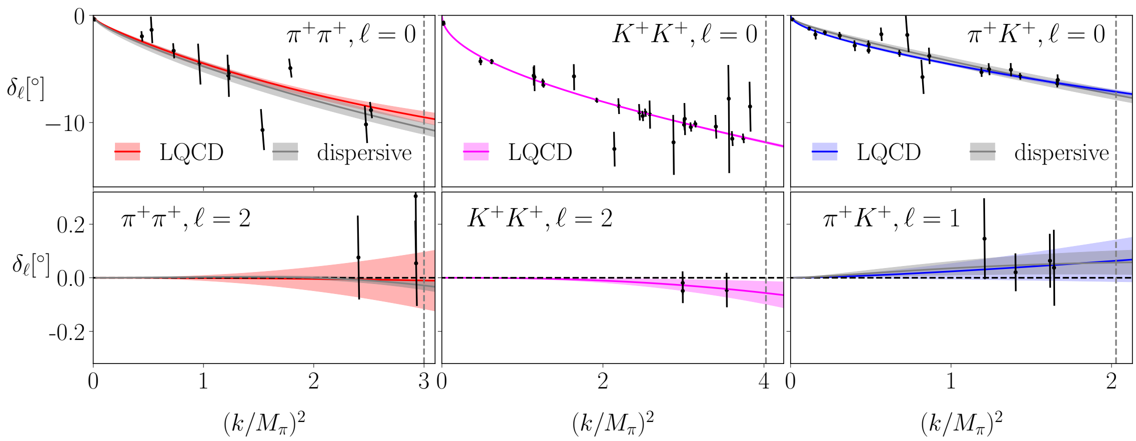

Figure 2: Phase shift as a function of the squared scattering momentum in the lowest two partial waves for maximal-isospin two-meson systems with physical quark masses [78, 79]. Dispersive results [103, 104] are also shown. The vertical grey dashed lines mark the lowest inelastic threshold. Black points are mapped from two-meson energy levels under the assumption that only the lowest partial wave contributes.

determined low-energy constants (LECs) are used as inputs, see Ref. [107]. As shown in the figure, qualitative agreement is found between the lattice QCD results and NLO ChPT. A striking feature, however, is the significant shift from LO to NLO. Whether this poor convergence persists at higher orders remains an open question, requiring further higher-order calculations.

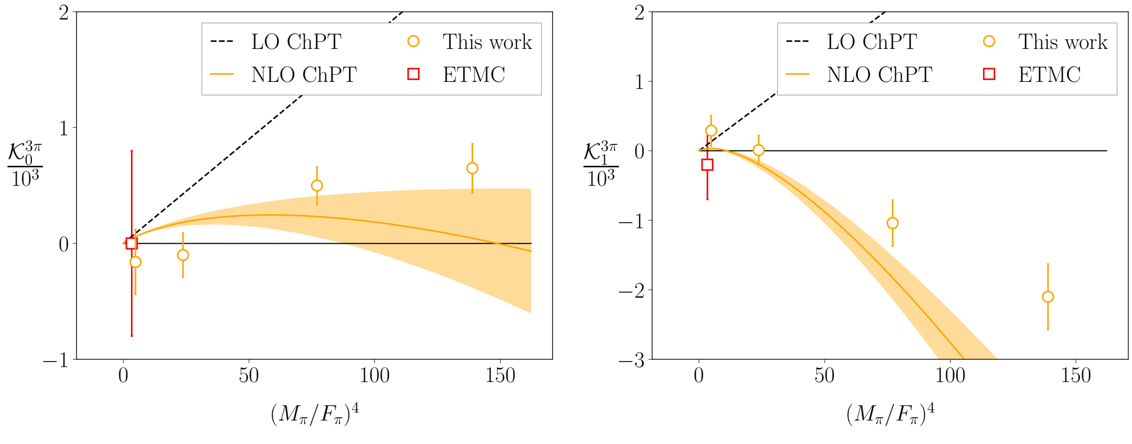

Figure 3: Results for the lowest two coefficients, $\mathcal{K}_0$ (left) and $\mathcal{K}_1$ (right), in the threshold expansion of the $3\pi^+$ $\mathcal{K}_{\text{df},3}$. Orange circles correspond to our latest results [69, 78, 79], while red squares represent results from Ref. [65]. Leading-order (LO) and next-to-leading-order (NLO) ChPT predictions are also shown [107]. The error bands reflect the uncertainties in the low-energy constants (LECs), as detailed in Ref. [107].

After computing the two- and three-meson K matrices from the lattice QCD spectrum, the corresponding scattering amplitudes are obtained by solving integral equations projected onto definite angular momentum and parity, J^P. However, visualizing these results is challenging, as three-meson scattering amplitudes depend on eight kinematic variables after accounting for Poincaré invariance. To display the results in a two-dimensional plot, we choose kinematic configurations of incoming and outgoing momenta. One such choice is the equilateral configuration, where the momenta of the incoming and outgoing particles form equilateral triangles. Then, the scattering amplitude can be plotted as a function of energy. Figure 4 presents results for systems of three mesons composed of either π^+ or K^+. Two key features are visible. First, all amplitudes diverge at threshold, due to one-particle exchange processes. Second, the $3K^+$ amplitude is significantly larger than the others, consistent with the naive chiral expectation that kaon interactions are stronger due to the heavier strange-quark mass.

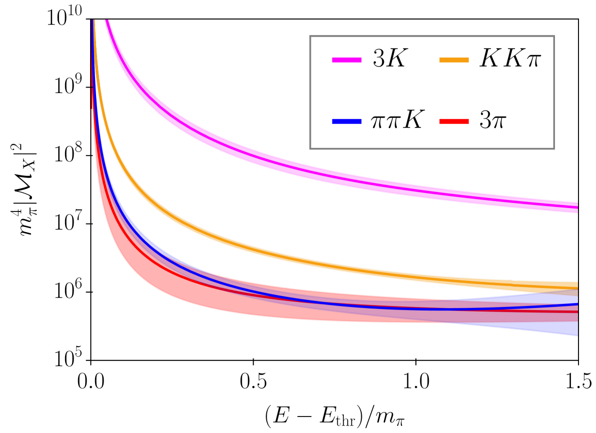

Figure 4: $J^P = 0^-$ three-hadron scattering amplitude as function of the energy difference to threshold for systems of pions and kaons at maximal isospin. The incoming and outgoing particles lay on an equilateral kinematic configuration, so that fixing the energy completely defines all kinematic variables.

To conclude the discussion of these systems, we investigate the chiral dependence of the $3\pi^+$ amplitude. Figure 5 presents the results for the $J^P = 0^-$ amplitude on the equilateral kinematic configuration and on all four ensembles. As can be seen, pion interactions become stronger with increasing pion mass. The full amplitude can also be compared to NLO ChPT predictions, showing good agreement at the physical point. However, the agreement deteriorates at heavier pion masses and higher energies, which is consistent with the power counting of the chiral expansion.

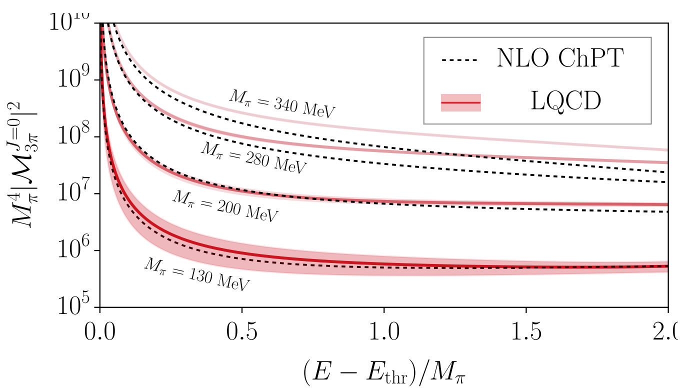

Figure 5: Three-pion scattering amplitude at maximal isospin for four different ensembles with quark masses ranging from approximately physical to around 340 MeV. The ChPT predictions [105, 107] are also shown.

4. Towards a three-body description of the doubly-charmed tetraquark

Motivated by the discovery of the doubly-charmed tetraquark, T_{cc}, by LHCb via its three-hadron decay mode $T_{cc} \to DD\pi$ [4, 5], there has been a growing effort to predict its properties using lattice QCD [109–117] and phenomenological analyses [118–124]. Since its decay involves three hadrons, a rigorous theoretical treatment requires a three-body formalism at physical quark masses.

In many lattice QCD studies, where the light quark mass is heavier than physical, the D^* meson becomes stable. In such cases, the extraction of T_{cc} properties can be formulated in terms of two-body DD^* scattering. However, the DD^* amplitude features a left-hand cut, which complicates the determination of the pole position and cannot be neglected—see Fig. 6a. To address this issue, several approaches have been proposed, including explicitly incorporating pion exchange in DD^* scattering [119, 125–129] (see also Ref. [130]).

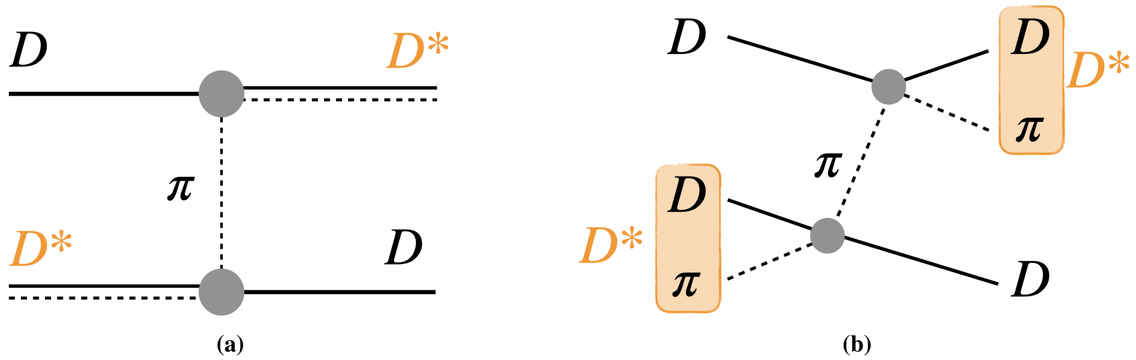

Figure 6: Left: one-pion exchange diagram in DD^* scattering responsible for the left-hand cut. Right: equivalent diagram in the context of $DD\pi$ scattering, where the D^* meson is viewed as pole in the $D\pi$ amplitude.

Our proposal, however, is to treat the system as a genuine three-meson system when the D^* meson is stable, using the formalism derived in Ref. [45]. In this approach, inspired by Refs. [93, 94], we incorporate the D^* meson as a bound-state pole in the $I = 1/2$ $D\pi$ scattering amplitude. This allows the finite-volume energies of the DD^* system to be computed using the three-body quantization condition. This approach is motivated by the explicit inclusion of the diagram in Fig. 6b in the three-body formalism. An important advantage of this method over two-body treatments is that it enables a smooth connection between the regimes where the D^* is stable and unstable, both in finite and infinite volume. However, it is also qualitatively more involved, as a full analysis requires input for both the DD and $D\pi$ scattering amplitudes.

In a recent study [80], we applied the three-body formalism to analyze the finite-volume energies from Ref. [109]. Our analysis requires as input the s-wave $I = 1$ DD scattering amplitude, the $I = 1/2$ $D\pi$ scattering amplitude in both s- and p-waves, and a parametrization of the three-particle K matrix. For the three-body K matrix, we adopt a parametrization based on the first term in the threshold expansion that couples to $J^P = 1^+$:

$$\mathcal{K}_{\text{df},3} = \mathcal{K}_E (p_\pi - k_\pi)^2, \tag{5}$$

where $\mathcal{K}_E$ is a constant (i.e., an adjustable parameter), and p_π and k_π denote the momenta of the incoming/outgoing pions in $DD\pi$ scattering. However, since Ref. [109] does not provide DD or $D\pi$ energy levels, we use input inspired by other lattice QCD results [131] and effective field theories [132, 133] for the two-body amplitudes. In practice, we neglect DD scattering (as it is weakly interacting), and employ parametrizations for the $D\pi$ amplitude that incorporate the D^* meson as a stable particle in the p-wave, and the D_0^* resonance in the s-wave.

The results of this analysis are shown in Fig. 7, where we plot the phase shift in the "$q\cot\delta$" form as a function of energy. The values of $\mathcal{K}_E$ have been tuned such that the resulting curves reproduce the phase shifts extracted from the above-threshold energies of Ref. [109] using the two-body formalism. As evident from the plot, the $q\cot\delta$ function develops a non-zero imaginary part on the left-hand cut—a sign of the non-analyticity at the branch point. The existence of this imaginary component at real energies rules out the presence of bound or virtual state poles for real energies on the left-hand cut for this choice of parameters.

The integral equations also allow for an analytic continuation of the DD^* scattering amplitude into the complex energy plane, enabling the search for poles. In Fig. 8, we display the location of the $T_{\rm cc}$ pole in the complex plane as a function of $\mathcal{K}_E$, keeping the two-body parameters fixed. At the value of $\mathcal{K}_E$ that best describes the finite-volume data (corresponding to Fig. 7), we find two poles with non-zero imaginary parts. These poles, which have been observed in other analyses [124], correspond to a so-called "subthreshold resonance." As $\mathcal{K}_E$ is increased, the poles move to higher energies and approach the real axis. Eventually, they transition into a pair of virtual state poles, and with further increase of $\mathcal{K}_E$, one of them becomes a bound-state pole.

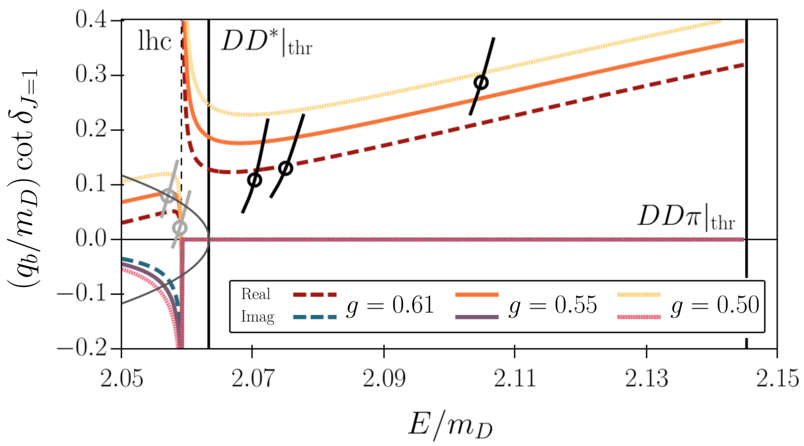

Figure 7: DD^* s-wave phase shift as a function of energy. Points with error bars come from the energies in Ref. [109] using the two-body formalism. The ones in grey indicate that the left-hand cut invalidates the applicability of the original two-body approach. Three curves are shown for different choices of the $DD^*\pi$ coupling in the $D\pi$ amplitude, g. In all cases, $m_D^2\mathcal{K}_E = 1.9\cdot10^5$. As can be seen, the phase shift becomes complex on the left-hand cut.

To conclude this topic, we perform a finite-volume analysis of the DD^* spectrum using the three-body quantization condition. As input, we use the same parametrizations of the K matrices employed for the central curve in Fig. 7. The resulting QC3 spectra for several irreps and moving frames are shown in Fig. 9, where they are compared to the finite-volume energies from Ref. [109].

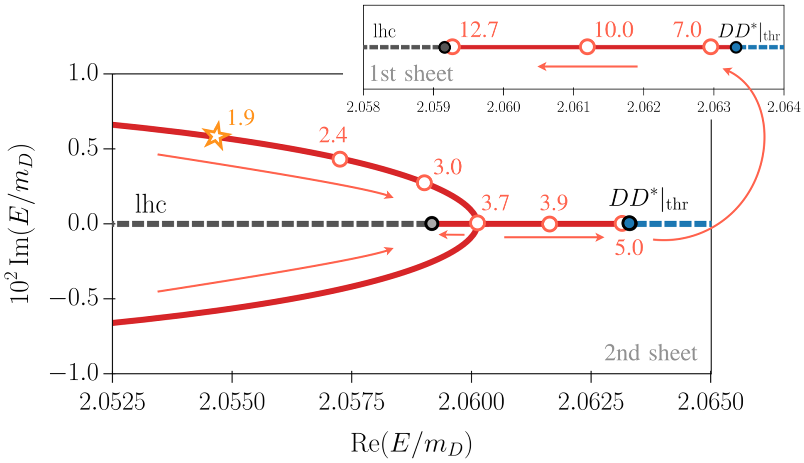

Figure 8: T_{cc} pole position in the complex plane as a function of the parameter in the three -body K matrix. The numbers of the plot indicate the value of $\mathcal{K}_E$ in units $m_D^2 \mathcal{K}_E/10^5$. The one matching the setup in Fig. 7 is marked with a star. The positions of the left-hand cut (lhc) and of the threshold (thr) are also displayed.

Overall, we find good agreement. However, a notable discrepancy is present in the $A_{1u}(0)$ irrep, where the QC3 predicts an energy level in the smaller volume that was not observed in the lattice QCD calculation. We interpret this level qualitatively as a "$DD\pi$"-like state, shifted downward by the attractive threshold interactions in the s-wave $D\pi$ channel due to the proximity of the D_0^* resonance. We suspect that this level was missed in the lattice calculation due to the absence of $DD\pi$-type interpolating operators. This further highlights the importance of a genuine three-body analysis even in cases where the D^* meson is stable.

5. Conclusion and Outlook

In this contribution, I have highlighted some recent advances in three-hadron spectroscopy. Specifically, I discussed recent calculations of the three-meson scattering amplitude at physical quark masses and our three-body study of the doubly-charmed tetraquark.

The first topic marks a significant milestone in three-body studies, demonstrating that lattice QCD techniques are now capable of physical-point calculations and that subleading three-body effects can be resolved in certain systems [78, 79].

For the second topic, we have fully developed a strategy to perform lattice QCD calculations of the T_{cc}, incorporating both three-body effects and left-hand cuts. Our results successfully reproduce the existing lattice QCD spectrum from Ref. [109], and show qualitative agreement with other approaches to DD^* scattering. However, we have argued that it is crucial to perform a three-body analysis even if the D^* meson is bound, as the proximity of the $DD\pi$ threshold can have effects in energy determinations, as well as finite-volume effects.

Progress in this subfield has been remarkable over the past decade. Formalism exists to deal with complicated three-hadron systems, such as coupled-channel three-meson systems [50]. However, several open challenges remain in three-particle spectroscopy. First, many existing theoretical tools still need to be applied to lattice QCD systems, specifically considering three-hadron resonances [32], and even three baryons [46] or electroweak decays [35, 40, 41, 49]. Beyond

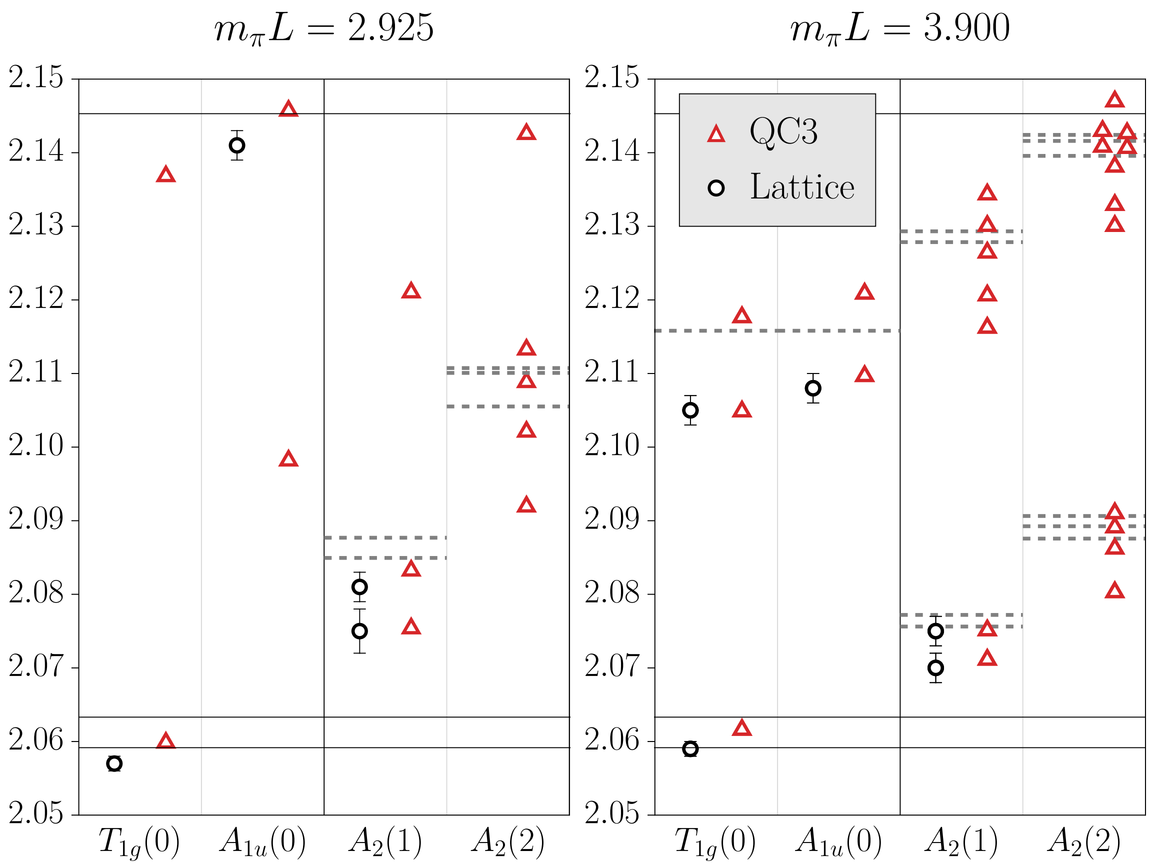

Figure 9: Finite-volume spectrum of the DD^* system in several irreps and frames. The results from Ref. [109] are shown as black circles. The energies predicted by the QC3, using the same setup as Fig. 7, are shown as red triangles. Non-interacting levels are marked as horizontal grey dashed lines.

numerical applications, further advancements in formalism are essential, particularly in addressing coupled-channel systems that include baryons. On a more theoretical level, the analytic continuation of three-hadron scattering amplitudes remains a challenging problem, especially due to the smooth coupling inherent in the RFT formalism [94].

The study of hadron properties and interactions remains a significant challenge within the Standard Model, and advances in three-hadron spectroscopy are providing new theoretical perspectives and valuable phenomenological insights.

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