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Renormalization of Electroweak Gauge Interactions

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Abstract

The mathematical and physical aspects of gauge theory radiative corrections are reviewed, with application to quantum electrodynamics and the full SU(2)×U(1) electroweak theory. The special role of broken gauge and global symmetries is stressed, along with non-decoupling of heavy particles and high precision data fits constraining standard and non-standard electroweak interactions. A technical appendix on electroweak corrections is included.

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CONTENTS

Natural units $\hbar = c = 1$ and Minkowski metric $q^2 = q_0^2 - \mathbf{q}^2$ used throughout.

I. Introduction: A Bird's-Eye View

I.1 Electroweak Gauge Interactions

The Standard Model of electroweak interactions (EW SM) is one of the great achievements of twentieth century physics[1]. It reproduces the remarkably successful quantum theory of electrodynamics (QED)[2,3] and the Fermi low-energy four-point theory [4], and successfully predicted the existence and properties of the massive intermediate vector bosons that carry the "weak force" responsible for β-decay and other weak interactions.

The properties of the EW SM as a *gauge* or *Yang-Mills theory,* a quantum field theory based on the principle of local gauge invariance[5], account, more than any others, for its quantitative success. Gauge theories have a definite structure that follows as a consequence of gauge symmetry, giving them great predictive power. The EW SM is a weakly coupled gauge theory, allowing perturbative calculations carried to arbitrary order and saved from self-inconsistency by the twin properties of *renormalizability* and *unitarity.* These properties require gauge symmetry for their implementation.

The subject of these lectures is the perturbative expansion, renormalization, and precise testing of the EW SM and its extensions as a quantum field theory, restricted here to the flavor-conserving weak charged and weak and electromagnetic neutral currents, as manifested in four-fermion processes. The technical aspects of gauge interactions are introduced through quantum electrodynamics[2,3], the prototype of later gauge theories and a subtheory of the full electroweak theory. The phenomenology of precision tests of EW interactions forms the last part of the lectures, which draw on the advances in electroweak measurements of the last three years made available by the new high-energy colliders and atomic parity violation. The radiative corrections techniques presented here are based on the work of Kennedy and Lynn, and others. This formalism is geared towards complete radiative calculations at a given perturbative order and near or on the gauge boson poles, while taking advantage of the special properties of gauge theories. The student is assumed to have a background in field theory and particle physics at the level of the books of Bjorken and Drell[6] and of Quigg[7], with more advanced preparation (such as the book of Cheng and Li[8]) being helpful. General presentations of the Standard Model and the Higgs sector can be found in the lectures of Ed Farhi and Jon Bagger in this school. Electroweak corrections have been the subject of a number of previous TASI lecturers as well[9,10].

From the point of view of electroweak interactions, the known elementary particles can be divided into two broad groups. The first are particles of "matter," the three generations of ordinary Dirac fermions:

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}, \tag{I.1}$$

grouped into doublets under the *left-handed weak isospin* SU(2). "Left" here refers to *weak chirality,* the generalized chiral symmetry of the Standard Model. For ordinary fermions, this chirality is coincident with the usual Lorentz chirality of Weyl spinors. The full EW group is $SU(2)_L \times U(1)_Y$, a product of two simple Lie algebras; the electric charge $Q = I_3^L + Y$ generates a closed subgroup of the full EW group that produces electrodynamics. Y is the *weak hypercharge.* Each multiplet under $SU(2)_L$ carries a hypercharge assignment. The groups $U(1)_Y$ and $U(1)_Q$ are Abelian groups; that is, $[Y,Y] = [Y,Q] = [Q,Q] = 0$. $SU(2)_L$ is non-Abelian. The second group of elementary particles contains the particles of "force", the spin-1 quanta of gauge fields that mediate the electroweak interactions: $W^\pm, Z^\circ, \gamma$. The first three are massive and mediate the weak charged and neutral currents (CC, NC) respectively. The fourth is the massless photon. Although $SU(2)_L \times U(1)_Y$ is not a simple group and thus is not really unified, a quantum field theory of the weak charged current nonetheless clearly requires electrodynamics because the $W^\pm$ are charged[1,11].

Consider the EW SM at the classical or tree level to start. The Lagrangian $\mathcal{L}$ is invariant under the separate $SU(2)_L$ and $U(1)_Y$ local gauge transformations. This invariance fixes the form of the gauge interactions by the covariant derivative:

$$D_\mu = \partial_\mu + igW_\mu^i I_i^L + ig' B_\mu Y,$$

where I_i^L and Y are the $SU(2)_L$ and $U(1)_Y$ generators, and g, g' the universal weak isospin and hypercharge couplings. Although $\mathcal{L}$ and its associated dynamics are gauge invariant, the vacuum or ground state of the theory need not be and clearly is not: the full EW symmetry is not manifest in Nature and is said to be broken or hidden. That is, the gauge charges do not annihilate the vacuum: $I_i^L|0\rangle, Y|0\rangle \neq 0$. Since there is a gauge field associated with each group generator, the total number of gauge bosons is four; however, one linear combination of generators remains unbroken, the electric charge: $Q|0\rangle = (I_{L_3} + Y)|0\rangle = 0$. Correspondingly, a linear combination of the neutral (W^3, B) is massless and identified with the photon. $W^{1,2}$ and the orthogonal combination of (W^3, B) become massive, as their generators are broken. The relation between the simple fields (W^3, B) and the orthogonal mass eigenstates is given by the *weak mixing angle* θ_W :

$$\tan\theta_W = g'/g,$$

$$Z^\circ = c_\theta W_3 - s_\theta B,$$

$$\gamma = s_\theta W_3 + c_\theta B,$$

with the notation: $s_\theta = \sin\theta_W$, $c_\theta^2 = 1 - s_\theta^2$, used throughout these lectures. The other two massive states are the $W^\pm = (W^1 \mp iW^2)/\sqrt{2}$, carrying electric charge ± 1. The appearance of non-zero gauge boson masses from the broken vacuum is called the *Higgs mechanism*[12].

The vacuum cannot be simply broken "just so" – renormalizability and unitarity require that the symmetry breaking (SB) be an outcome of some dynamics in $\mathcal{L}$ itself. SB can be implemented in many ways; the simplest is to introduce one elementary scalar doublet Φ[1]:

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{I.2}$$

with $Y = 1/2$, then arrange $\mathcal{L}$ so that the ground state of the theory includes a non-zero neutral vacuum expectation value (VEV) of Φ :

$$\langle\Phi\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{I.3}$$

More complicated scenarios are possible: many elementary scalar multiplets with arbitrary EW quantum numbers and/or effective composite scalars. Such composites would be fermion-antifermion pairs, new fermions (technicolor)[13] or perhaps top quark-antiquark pairs[14]. Whatever the condensate, bosons or fermion pairs, the only general requirements are that it be electrically neutral and a Lorentz scalar, preserving $\mathrm{U}(1)_Q$ and Lorentz invariance. The sector of the theory producing the vacuum condensates is called the *Higgs sector*.⋆ Fields acquiring VEVs are called *Higgs scalars* or *Higgses*.

Symmetry breaking then produces gauge boson masses via the Higgs mechanism. The condition $Q|0\rangle = 0$ automatically leads to $M_\gamma = 0$. In the simple case of (I.3), $M_W^2 = g^2v^2/4, M_Z^2 = (g^2 + g'^2)v^2/4$, the general form $M^2 \sim g^2 \cdot v^2$ being a consequence of the gauge symmetry. More generally, if there are many Higgs VEVs Φ_i, the gauge boson masses are related by the *rho parameter:* $M_W^2 = \rho M_Z^2 c_\theta^2$, where:

$$\rho = 1 + \frac{\sum_i \langle \Phi_i^\dagger (\mathbf{I}_L^2 - 3{I_3^L}^2)\Phi_i\rangle}{\sum_i \langle \Phi_i^\dagger (2{I_3^L}^2)\Phi_i\rangle}, \tag{I.4}$$

for any type of multiplet VEVs $\langle\Phi_i\rangle$, depending only on their EW symmetry proper-

⋆ For historical reasons, SB by elementary fields is referred to as "spontaneous SB," while SB by composite fields is called "dynamical SB." The distinction is amorphous, however, since the condensation of elementary scalars is as much as product of dynamics as that of composite scalars, only a different kind of dynamics. In general, SB here refers to either case, any kind of vacuum with non-trivial gauge quantum numbers. Symmetry breaking in the Lagrangian, not in the vacuum, is called "explicit SB."

ties. The EW SM with $\rho \equiv 1$ and the minimal fermion and gauge content I call the *Minimal SM (MSM),* the same with arbitrary ρ the *Extended Vacuum SM (EVSM);* since, for the study of gauge interactions, the explicit dynamics of the Higgs sector doesn't matter: only the vacuum structure does. The condition $\rho \equiv 1$ can be obtained in many ways; if the Φ_i's are $\mathrm{SU}(2)_L$ doublets only, then $\rho = 1$ automatically. These statements need modification in the presence of radiative corrections.

A massless gauge boson has two degrees of freedom (d.o.f.'s), the two transverse states, while a massive one has three, with the additional longitudinal d.o.f. The $W^\pm, Z^\circ$ acquire their longitudinal d.o.f.'s by "eating" three of the appropriate d.o.f.'s from the Higgs sector[12]. These scalar d.o.f.'s are the *would-be Goldstone bosons,* because, had the broken symmetry been only a global one, they would have appeared as the usual massless Goldstone states. In a gauge theory, they instead disappear from the physical spectrum and appear "digested" in the longitudinal gauge boson d.o.f.'s. In general, the physical Higgs bosons consist of whatever scalar d.o.f.'s are left over after this meal. In the simple case of (I.2) and (I.3), $\phi^- = (\phi^+)^*$ and $\phi^+ = \varphi_1 + i\varphi_2$ are entirely eaten up by the $W^\pm$; while of $\phi^0 = \varphi_0 + i\varphi_3, \varphi_3$ is eaten up by the Z°. Left behind is φ_0, which acquires the VEV. The quantum of φ_0 excitations is then referred to as *the* Higgs boson, in this case being the only physical scalar left.

Gauge symmetry in general is also broken by fermion and Higgs masses. They are similar to the form of gauge boson masses (dimensionless coupling times VEV), but with a key difference: in the gauge case, the coupling is the universal gauge coupling that also governs gauge interactions. No one has yet experimentally established a principle for the Yukawa and Higgs couplings analogous to the gauge principle for vector bosons.

I.2 Radiative Corrections

The gauge sector at tree level contains three arbitrary parameters to be fixed by data so that the theory becomes predictive: g, g', and v^2, where the couplings are dimensionless, and v^2 has canonical mass dimension two ($dim(v^2) = +2$) and sets the mass scale. In the EVSM, ρ is added as a fourth parameter. These parameters can be re-expressed in terms of experimental measurables in many ways; e.g.,

$$\begin{gathered} \frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}, \quad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \\ M_W = \frac{gv}{2}, \quad M_Z = \frac{M_W}{c_\theta}. \end{gathered} \tag{I.5}$$

$e^2 = 4\pi\alpha$ is the electric charge squared in units of proton charge. G_F is Fermi's constant from β-decay: $v^2 = 1/\sqrt{2}G_F = (246 \text{ GeV})^2$. M_Z and M_W are the gauge boson masses.

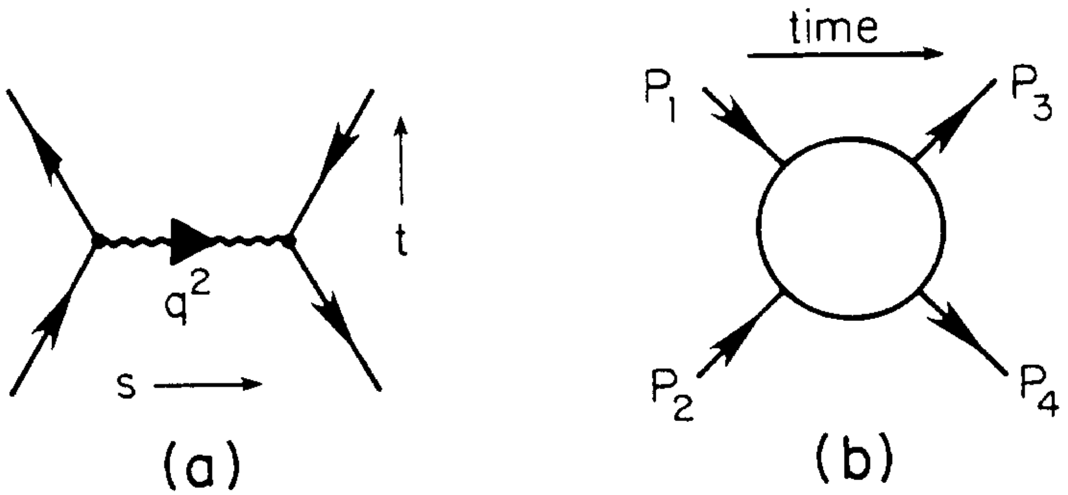

Figure 1. (a) Tree-level four-fermion interaction via gauge boson exchange, with invariant square four-momentum $q^2 = s$ or t. (b) Mandelstam variables with p_1, p_2 in, p_3, p_4 out: $s = (p_1+p_2)^2$, $t = (p_3-p_1)^2$, and $u = (p_3-p_2)^2$ are Lorentz-invariant. Note: $p_1 + p_2 = p_3 + p_4$, and $s + t + u = \sum_{i=1}^{4} m_i^2$.

But the classical theory generated by $\mathcal{L}$ is not the full theory, merely its zeroth approximation. The four-fermion tree diagram is shown in Figure 1, with invariant square momentum q^2 flowing through the gauge boson line. The quantum corrections to this tree-level approximation are obtained by carrying out a perturbative expansion with the usual apparatus of quantized field theory[10]. The EW perturbation theory is a double expansion in g and g', but since $s_\theta^2 \simeq 0.23$, it can without difficulty be expressed more simply in powers of α. This expansion is naively justified by $\alpha \ll 1$. However, the collection of higher-order terms beyond tree level, the *quantum* or *radiative corrections* to the theory, such as the loop diagrams of Figure 2, complicates the interpretation of the theory and its parameters in an essential way. Since experiment makes no distinction between classical tree and quantum loop levels, the classical parameters never occur alone, but always in combination with corrections. The gauge symmetry, even when broken, restricts the form and occurrences of these combinations in a very special way. So the classical theory, more correctly called the *bare* theory, is not directly observable but is always "dressed" or "renormalized" by corrections. Bare quantities are denoted here by "nought" subscript: $\mathcal{L}_0 = \mathcal{L}_0(\vec{p}_0; \phi_0)$ is shorthand for the classical bare Lagrangian as a function of bare parameters $\vec{p}_0 = \{g_0, g_0', v_0^2\}$ and bare fields. The Feynman rules spell out how the perturbative expansion is constructed from $\mathcal{L}_0$.

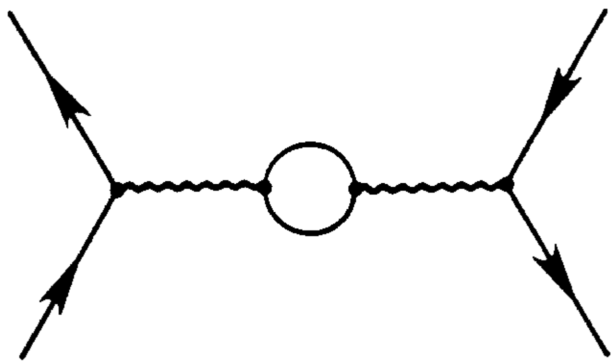

Figure 2. Typical one-loop correction to four-fermion process.

The dynamical structure of the full quantum theory is considerably more complicated than that of the classical theory because of the corrections. A zeroth-order matrix element such as Figure 1 typically has a simple dependence on momentum transfer q^2 and the bare parameters $\vec{p}_0 : \mathcal{M}_0 = \mathcal{M}_0[q^2;\vec{p}_0]$; the full amplitude depends on the same parameters *and also* on all of the corrections relevant to the process in question. The loops have their own internal dependence on q^2 and couple to known and unknown particles. The complete theory may be viewed as the set of relationships among all observables (Figure 3). We need three observables to fix the tree-level theory numerically, and each observable is defined in terms of some process with given kinematics. These relationships are modified by the corrections, reflecting the modified dynamics. For example, the simple definitions of (I.5) are changed:

$$M_{W_0}^2 = c_0^2 \cdot M_{Z_0}^2 \rightarrow c_\theta^2 \cdot M_Z^2 \cdot [1 + \mathcal{O}(\alpha)], \qquad (I.6)$$

where c_0 is the *bare* c_θ and where M_W, M_Z and c_θ need to be operationally defined from matrix elements. It proves convenient to choose three parameters $\vec{p}$ to replace the unobservable bare $\vec{p}_0$ in the perturbative expansion. This reparametrization is called *renormalization of parameters* and a particular choice of such parameters is called a *renormalization scheme (RS)*. These new parameters $\vec{p}$ should be unambiguously defined in terms of observables, although this relationship does not have to be a simple one.

The full quantum EW gauge theory thus forms a rich structure, with almost every aspect of quantum field theory coming into play. EW radiative corrections typically change tree-level relations by about 1% or so, but larger corrections occur in certain cases. Figures 4 and 5 show the dependence of some EW observables on different types of corrections. As EW experiments have now reached the precision of tenths of a percent, radiative corrections are inescapable.

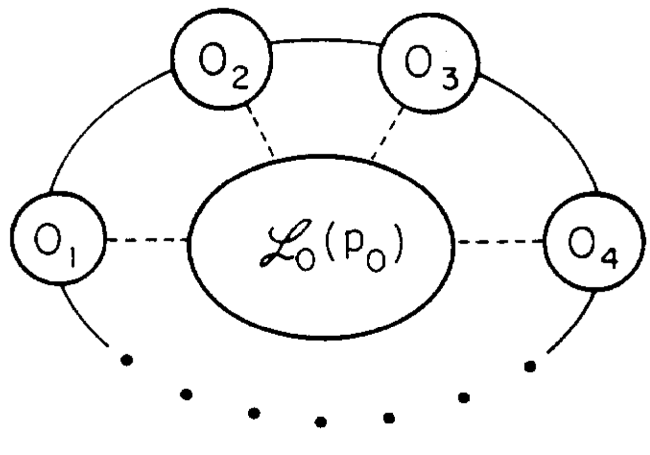

Figure 3. A field theory is the set of all relationships among observables, controlled by the form and parameters of the bare Lagrangian $\mathcal{L}_0(\vec{p}_0)$.

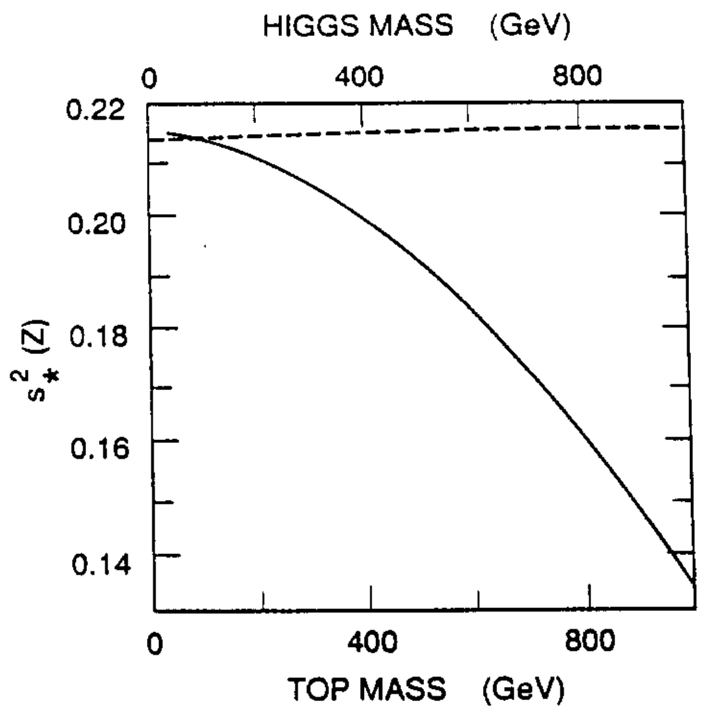

Figure 4. Effective weak mixing $s_*^2(Z)$ at the Z pole as a function of heavy virtual top quark and Higgs boson masses; $M_Z = 94$ GeV[39].

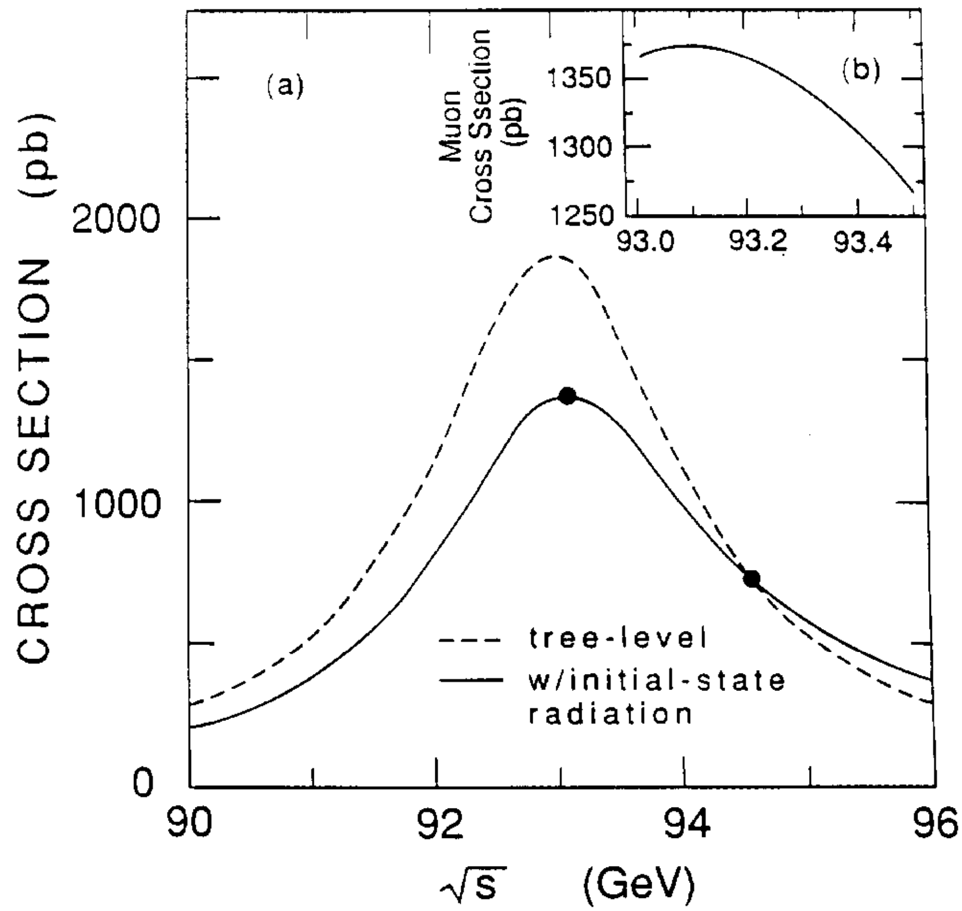

Figure 5. Effect of initial state radiation on the $e^+e^- \to Z \to \mu^+\mu^-$ *annihilation cross section peak;* $M_Z = 93$ *GeV*[78].

I.3 Renormalizability

☞*There is a concept which corrupts and upsets all others. I refer not to Evil whose limited realm is that of ethics; I refer to the Infinite.*

– Borges, *Avatars of the Tortoise*[15]

A quantum field theory built from fields that are functions of a single spacetime point must satisfy not only the requirements of Lorentz invariance, microcausality — (anti)commutators vanish outside the light cone — and quantum mechanics, but also of locality. That is, $\mathcal{L}_0$ should consist only of products of operators evaluated at only one point in spacetime. A non-local theory would have implicitly propagating but not dynamically explicit d.o.f.'s smeared out over a spacetime region. The operational meaning of the bare parameters is then clear: they are the parameters of the theory measured at zero spacetime intervals or infinite momentum transfer. Such a formulation can only be an idealization, however, as such conditions are unphysical. At any time in the history of physics, $|q^2|$ is always limited by some definite bound, today approximately $(100\ \text{GeV})^2$. Moreover, if we stick to this approach, the formulation of particle physics at and below $|q^2| \sim M_Z^2$ (which I label "light physics") requires, in principle, knowing *all* particle physics, the particle spectrum and the dynamics, to and beyond the Planck scale. The renormalization of parameters now takes on fuller meaning. Unless we are attempting to construct a "theory of everything" such as superstrings, it is obviously preferable to have local theories which, when the bare parameters are replaced by a RS of parameters with a well-defined relation to measurable quantities (light physics), manifest no sensitivity to physics at arbitrarily high energies not explicitly specified in the theory. Such dynamically complete theories are said to be *renormalizable* [16].*

In field theory calculations, sensitivity to arbitrarily high energies appears in a well-known and more definite fashion, in the form of loop corrections arising from local interactions at arbitrarily small spacetime separations. The loop diagram of Figure 2, for example, is logarithmically divergent. When the unconstrained integral over the internal loop momentum is evaluated, it diverges as:

$$\int^{|l|<\Lambda} \frac{d^4 l}{l^4} \sim \ln(\Lambda^2), \tag{I.7}$$

with the cutoff $\Lambda \to \infty$, therefore more exactly called *ultraviolet (UV) divergent.* In $n = 4$ spacetime dimensions, infinite loops generically diverge logarithmically, quadratically, or quartically:

$$\int \frac{d^4 l}{l^4}, \quad \int \frac{d^4 l}{l^2}, \quad \int \frac{d^4 l}{1}, \quad l^2 \to \infty. \tag{I.8}$$

The quartic divergences are contributions to vacuum energy density or the cosmological constant, which, since we measure everything relative to the vacuum, we simply ignore. Their true fate awaits a quantum theory of gravity. Gauge interactions proper, with broken or unbroken gauge symmetry, exhibit only logarithmic divergences. We learn why in the next chapter.

Radiative corrections thus introduce two troublesome features into a theory that potentially render it meaningless. One is the formally divergent character of corrections due to light and heavy particles equally. It finds its solution if the theory is renormalizable, which means: when the bare parameters are replaced by a RS, the infinities disappear. This implies the divergent amplitudes must occur in one-to-one correspondence with each bare parameter, each one becoming finite when

* This includes scalar field theories, which are renormalizable, although they can require fine-tuning. The renormalization would then be unstable to further corrections, but could still be carried out.

re-expressed with the renormalized parameter. That is:

$$\mathcal{M} = \mathcal{M}[q^2; \vec{p}_0; loops(q^2, \Lambda)] \doteq \mathcal{M}[q^2; \vec{p}; \Delta loops(q^2)], \tag{I.9}$$

where the reparametrized form of $\mathcal{M}$ now depends only on finite *differences* of loops, and the dependence on the cutoff Λ has cancelled. The exact form of these finite corrections depends on the RS choice $\vec{p}$. The relation between $\vec{p}$ and $\vec{p}_0$ explicitly depends on the divergences, so that the perturbative expansion in $\vec{p}_0$ has no more than formal meaning. The infinities should be seen as a mathematical shorthand for the asymptotic ($|q^2| \to \infty$) behavior of the theory, which is in fact determined by the form of the divergences. Renormalizability means that $\vec{p}_0$ and Λ are equivalent to $\vec{p}$ and a set of inputs; that is, $\vec{p}_0$ and Λ always occur together in such a way that Λ is redundant. Gauge theories like the EW SM are renormalizable in this sense [17].

The sensitivity to *dynamics* of arbitrarily high *energies* and the associated divergences have disappeared, but we are still left with the other problem, radiative corrections to low-energy processes that may depend on *particles* of arbitrary *mass*. These potentially make the theory untestable by introducing unknown parameters. Gauge interactions have the property of decoupling, which is closely related to the absence of quadratic divergences[18]. In unbroken gauge theories, after renormalization, the radiative corrections due to a particle of mass m_i at $|q^2| \ll m_i^2$ are suppressed by powers of q^2/m_i^2. Later we shall see that, in broken gauge theories, such corrections can occur unsuppressed as m_i^2, $\ln(m_i^2)$, or $\mathcal{O}(1)$, *if* the heavy mass in question $m_i^2 \sim$ (variable dimensionless coupling)$\times$(fixed v^2), the same v^2 that breaks the gauge symmetry[19]. This feature of *non-decoupling* in broken gauge theories introduces uncertainties into the precision testing of the EW SM, but it plays a major role in electroweak phenomenology by allowing limits to be placed on the masses and couplings of particles heavier than the Z.

Non-renormalizable theories are not useless, but they must be interpreted as *effective field theories*[16,20]. Their sensitivity to unknown high-energy physics remains after renormalization:

$$\mathcal{M} = \mathcal{M}[q^2; \vec{p}_0; loops(\Lambda, q^2)] \doteq \mathcal{M}[q^2; \vec{p}; \Delta loops(\Lambda, q^2)]. \tag{I.10}$$

The crucial thing is the relation of observables to observables, not the cancellation of divergences. In an effective field theory, the cutoff has physical significance and is an observable. It marks a lower cutoff in spacetime separation and thus leads to an incomplete and non-local theory that must be "saved" by new dynamics stepping in at some scale $\geq \Lambda$. The scale of the new dynamics can be estimated by increasing Λ until the corrections become as large as the tree-level amplitudes. In Fermi's four-point theory, whose only parameter is G_F, explicit cutoff dependence appears in the perturbative corrections as powers of $G_F\Lambda^2$. This indicates the presence of new dynamics at $\Lambda \sim G_F^{-1/2} \simeq 250$ GeV to complete the theory, as is indeed the case. This kind of reasoning played an important role in development of the EW SM[21]. Reversing the logic, non-renormalizable theories are dynamically incomplete remnants of complete, renormalizable theories.

To sum up: renormalizable theories are dynamically complete, although not closed. Additional or non-standard "new" physics carrying EW quantum numbers may exist, but is not necessary to make the gauge interactions self-consistent. Non-renormalizable theories must be effective field theories, incomplete fragments of a larger theory. Divergences in a field theory indicate sensitivity to arbitrarily high energies, but, in renormalizable theories, they disappear when the theory is reparametrized in terms of quantities related to observables. Gauge theories are renormalizable and, if unbroken, exhibit the property of decoupling: the finite corrections due to heavy particles are suppressed by inverse powers of their masses. In broken gauge theories, like the EW SM, decoupling is violated under certain conditions.

I.4 Lightning Review of Perturbation Theory[10,8]

Unless explicitly stated otherwise, I assume the MSM: $\rho_0 \equiv 1$, with one Higgs doublet (I.2,3) and a single physical Higgs with unknown mass m_H. There are three generations of quarks and leptons, with known properties (except for the top quark mass m_t), and the four gauge bosons. The gauge group is $SU(2)_L \times U(1)_Y$, the gauge interaction properties of particles given by their quantum numbers under these groups. Under $SU(2)_L$, the left-handed fermions f_L transform as a **2** and the right-handed f_R fermions as a **1**; the primitive gauge fields W^i and B as a **3** and a **1**, respectively. The restriction to the MSM is partly lifted in chapters IV and V.

Unless we need the Lorentz tensor/spinor structure explicitly, I write the interactions and matrix elements in a schematic, shorthand form. The QED, weak neutral- and charged-current interactions read:

$$\mathcal{L}_{QED} = J_\gamma A_0, \quad \mathcal{L}_{NC} = J_Z Z_0, \quad \mathcal{L}_{CC} = (J_+ W_0^- + J_- W_0^+)/\sqrt{2}, \tag{I.11}$$

in terms of the bare fields and the electroweak currents:

$$J_\gamma = e_0 J_Q, \quad J_Z = \frac{e_0}{s_0 c_0}(J_3^L - s_0^2 J_Q), \quad J_\pm = \frac{e_0}{s_0}(J_1^L \pm i J_2^L), \tag{I.12}$$

in reduced form, with the couplings shown explicitly. "L" for fermions stands for the left-handed Weyl state projected by $\frac{1}{2}(1-\gamma_5)$, thus reproducing the standard V-A form for the charged current with maximal parity violation. The neutral current contains this left-handed current, but also the purely vector-coupled electromagnetic J_Q.

The quantization procedure via canonical quantization or the path integral starts with $\mathcal{L}_0$ as the input and results in the full quantum theory, expressible either in terms of Green's functions and the S-matrix, or in terms of an effective Lagrangian $\mathcal{L}_{\rm eff}$. Both are explicit functions of the bare parameters prior to renormalization, and, in perturbation theory at least, the two descriptions are completely equivalent. I use the S-matrix approach throughout because of its transparent connection to quantities measured in the laboratory, cross sections and rates. The Feynman rules can be used to construct the perturbative expansion diagrammatically, but EW calculations also require some extensions of perturbation theory. These extensions, as well as the choices of experimental inputs and renormalization scheme, are partly an art and not completely determined by the theory: they are constrained by experimental considerations, certain general features of the gauge theory (especially the need for gauge invariance), and by compromises between keeping the calculations simple and making them complete. The formalism sketched in these lectures is guided by the desire to do justice to these opposing requirements. Since $\mathcal{L}_0$ and $\vec{p}_0$ are unique, the full theory, the set of relationships among observables (Figure 3), is in principle independent of these choices and RS-independent in particular. However, RS-invariance is only respected if the theory is solved exactly. No one knows how to do this, so the perturbative series must be truncated at some order of α, introducing an unknown but bounded error of higher order. Matrix elements parametrized by alternative RS's differ by this error — renormalization and perturbation theory do not "commute." RS's are moreover not equivalent to one another in perturbation theory: some are better representations of the Taylor series than others, making it converge more quickly. The higher-order error can thus be minimized by a judicious choice of RS and, since α is small, be made negligible. The appropriate choices vary depending on the nature of the process in question and of its loops. This feature of perturbation theory goes under the confusing name of *RS-dependence* and is of purely mathematical, not physical, significance [22].*

The bare Lagrangian has $dim(\mathcal{L}_0) = +4$, a sum of products of parameters p_0^i and operators O_0^i :

$$\mathcal{L}_0 = \sum_i p_0^i O_0^i.$$

Terms in $\mathcal{L}_0$ with $dim(O_0^i) \leq 4$, $dim(p_0^i) \geq 0$ are renormalizable; other terms are not and so must be excluded. Renormalizability with spin-1 vector fields in general requires a gauge theory and $\mathcal{L}_0$ to be gauge invariant.† The full theory expressed by $\mathcal{L}_{\rm eff}$ has operators of all dimensions. Those having the same Lorentz and gauge properties as terms in $\mathcal{L}_0$ renormalize the bare interactions; they can and generally

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⋆ The renormalized perturbation series, although finite order by order in α, is also not convergent at very high orders. For low orders, it is convergent enough, taking the form of an *asymptotic expansion*[29,23].

† The only exception is a theory with massive vector bosons and a conserved global U(1) charge.

do have divergences, but divergences are absorbed into the corresponding bare p_0^i in any case. Operators in $\mathcal{L}_{\text{eff}}$ not having any analogue in $\mathcal{L}_0$ have $dim > 4$ and must have finite coefficients, since divergent coefficients in these operators have no bare parameters to be combined with. In renormalizable theories like gauge theories, the finiteness of such $dim > 4$ operators is automatic. Any $\mathcal{L}_{\text{eff}}$ has an infinite number of operators. An $\mathcal{L}_{\text{eff}}$ derived from a renormalizable $\mathcal{L}_0$ has an infinite number of coefficients expressible in terms of a finite number of parameters, equal in number to the number of bare parameters[24]. These statements can be translated easily into S-matrix language, comparing the Lorentz and gauge form of the tree-level $\mathcal{M}_0$ to the terms in the fully corrected $\mathcal{M}$.

In these lectures, I cover only the class of *flavor-conserving* four-fermion NC/CC processes: low-energy electrodynamics; atomic parity violation (APV); β-decay; νe, νN and lepton-nucleon scattering; $e^+e^- \to f\bar{f}$ annihilation; W and Z masses, and Z pole properties. Another very important set of processes is generated by *flavor-changing* neutral currents, but the details are quite different. The same general connections occur there between gauge invariance and renormalizability, and between broken symmetry and non-decoupling of heavy particle effects in radiative corrections, but the subject has its own technology, folklore and body of experimental results distinct from what I present in these lectures. Please see the lectures of Yosef Nir and Howard Georgi in this school for more about this topic[25].

The four-fermion Feynman diagram of Figure 1 serves as our starting point. Figure 1b shows the kinematics and the Mandelstam variables s, t and u. I assume that the external fermion masses m_f are kinematically negligible ($m_f^2 \ll |q^2|$), except for a brief foray into low-energy QED results in chapter V. The Higgs sector concerns us only as far as it affects the vacuum and loop corrections. The effects of flavor mixing and CP-violation are negligible for these processes, so set the CKM flavor mixing matrix to unity. Let us then start with the unbroken subtheory $U(1)_Q$ of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, quantum electrodynamics.

II. Quantum Electrodynamics: First Part

II.1 Irreducible Corrections – Lorentz Properties

Electrodynamics makes a good starting point for learning about the EW SM, because it is a subtheory of the SM and one of its historical forerunners, and because it introduces many of the features of gauge theories in a simple way. QED is an unbroken Abelian gauge theory with scalars and vector-coupled Dirac fermions[2,3,27,28]. Our major concerns are with the photon propagator, the renormalization of the electric charge e_0, and radiative corrections to four-fermion processes. Figure 1 shows the tree-level or Born diagram for fermion line f interacting with fermion line f'. Momentum transfer q^2 runs through the photon line, and the tree-level gauge-matter coupling $e_0 Q$ occurs at each vertex with the appropriate dimensionless charge Q :

$$\mathcal{M}_0 = e_0 Q \cdot \frac{1}{q^2} \cdot e_0 Q'. \qquad (II.1)$$

This *Born approximation* has the form *gauge current* $\times$ *propagator* $\times$ *gauge current* and is the relativistic generalization of the Coulomb potential. The electric charge Q is normalized in units of the proton charge, so that $Q_e = -1$. In a physical process, the amplitude can be in either the scattering ($q^2 = t$) or the annihilation ($q^2 = s$) channel, or in both (sum of both channels). My approach is schematic, stressing the important concepts and quoting results to illustrate the properties of QED, assuming that you have seen at least parts of it before. The focus is on decorating the $\mathcal{M}_0$ of (II.1) with corrections.

A convenient first step is to introduce the distinction between *proper* and *improper* diagrams[27,29]. *Proper,* or *one-particle irreducible,* diagrams cannot be disconnected by cutting one internal line.* It is clear that a complete four-fermion matrix element can be built up from irreducible graphs (Figure 6). The sums of all irreducible graphs with given sets of external legs represent the elementary Green's functions of the theory, from which the elements of the S-matrix can be assembled. These sums are the pictorial shorthand for the solution of the coupled integro-differential *Schwinger-Dyson equations* for the Green's functions[28,29]. These equations require the irreducible parts as inputs. The resulting Green's functions are then functionals of these irreducible parts and usually must be solved for perturbatively. The perturbation theory can then be defined completely in terms of the irreducible graphs only, the expansion method used in these lectures. Only one-proper-loop calculations are shown explicitly.

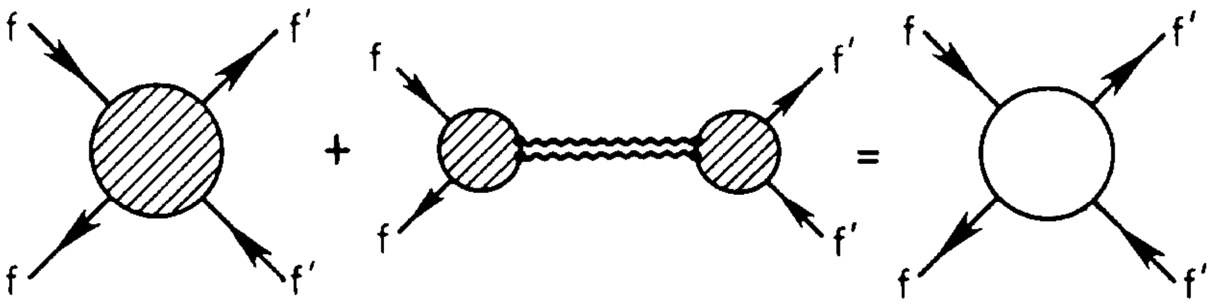

Figure 6. Four-fermion process as sum of irreducible and reducible four-fermion Green's functions.

⋆ I use "irreducible" and "proper" as synonyms from this point.

In our case, there are three irreducible Green's functions[27, 28]. The first is the *photon two-point function* or *photon propagator* $D(q^2)$. For its definition, the bare uncorrected propagator $D_0(q^2)$ and the irreducible *vacuum polarization* or *photon self-energy* graphs $\Pi(q^2)$ are necessary (Figure 7a). The full or *dressed* propagator $D(q^2)$ is a *Dyson sum*, the solution of the relevant Schwinger-Dyson equation for Figure 7b:

$$\begin{aligned} D = D_0 + D\Pi D_0 \quad &= \frac{1}{D_0^{-1} - \Pi} \\ &= \frac{1}{q^2 - \Pi}, \end{aligned} \tag{II.2}$$

since $D_0(q^2) = 1/q^2$. Π is the *kernel* of this Schwinger-Dyson equation. In perturbation theory, Π itself is a series of irreducible graphs carried out to a given order in α (Figure 7a).

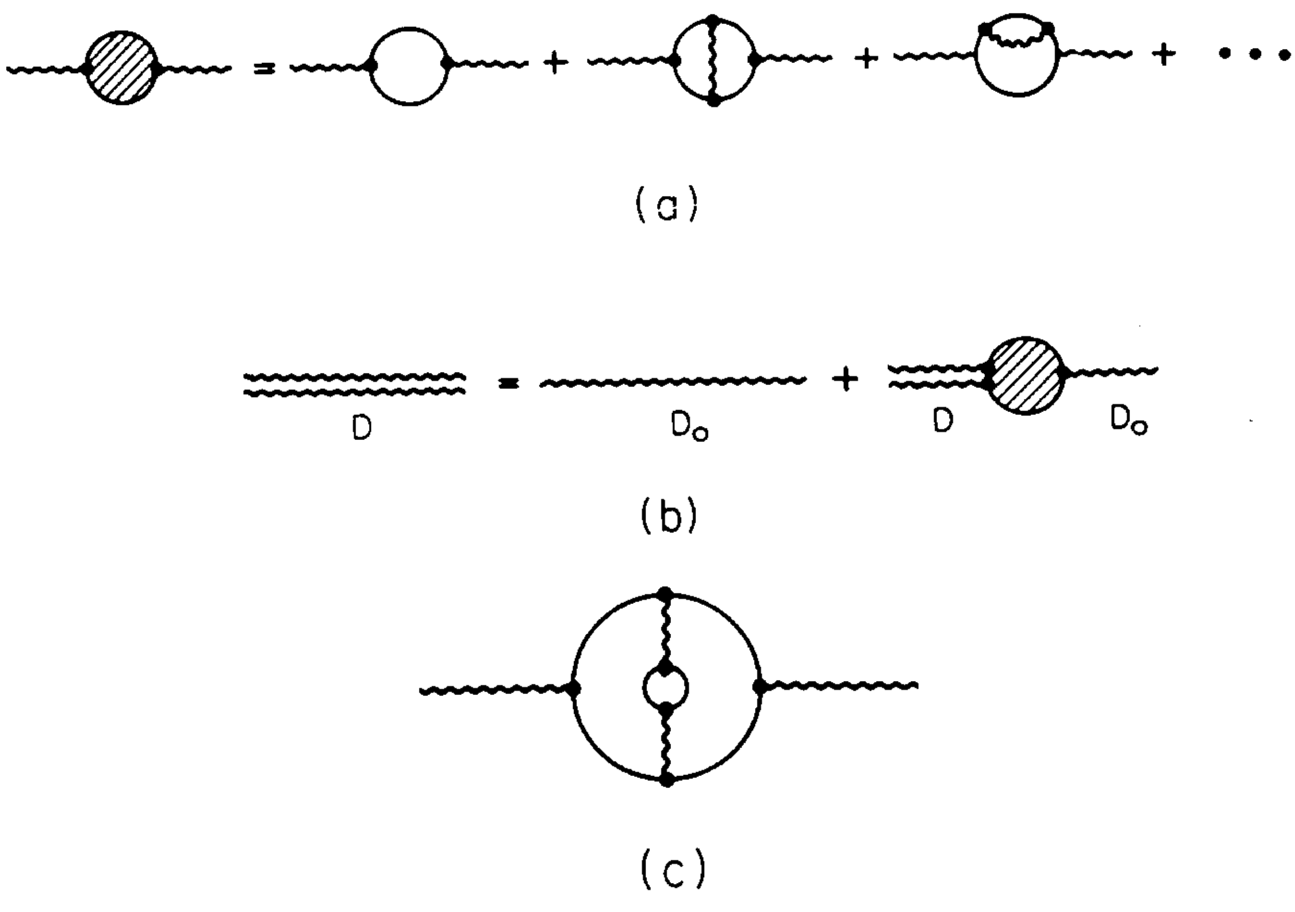

Figure 7. (a) Perturbative expansion of irreducible photon self-energy. (b) Dyson-Schwinger equation for the dressed, fully corrected photon propagator. (c) Three-loop photon self-energy, with first embedding of multiple fermion loops.

The second Green's function is the *three-point function* or *proper vertex function* $\Gamma(q^2)$. It too is a series in α of irreducible vertex diagrams. In addition, it is necessary to include the corresponding external fermion self-energies in this series (Figure 8). The third Green's function is the *irreducible box function* Θ (Figure 9). These three are sufficient to define the four-fermion matrix element. Note particularly that the perturbative series is rearranged by carrying out the Dyson sum for the photon propagator. This Dyson sum is automatic in these lectures for gauge boson propagators, although non-trivial in chapter IV for the non-Abelian theory. The perturbation theory is now redefined in terms of the irreducible parts alone. The Dyson sum for the gauge propagators is physically necessary in order to examine the properties of the dressed gauge bosons, especially the position of the poles in $D(q^2)$ that mark the physical gauge boson masses. The Dyson sum is our first extension of perturbation theory.

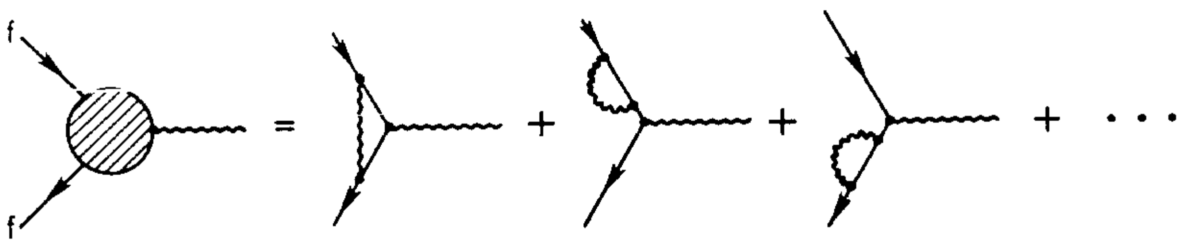

Figure 8. Irreducible vertex and external self-energy graphs for an on-shell fermion.

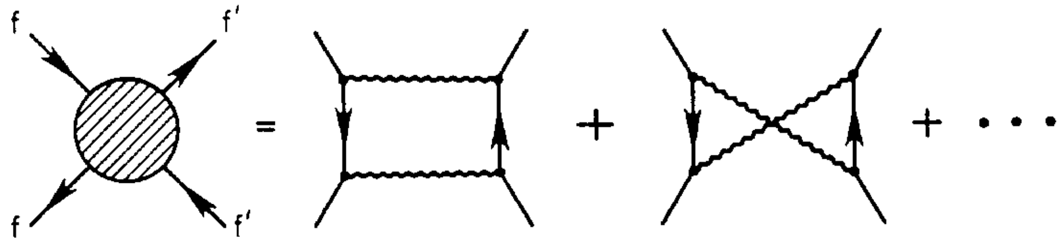

Figure 9. Irreducible box graphs for four-fermion interaction.

Let us make a momentary digression to look at the Lorentz properties of the irreducible Green's functions, displaying explicitly tensor and spinor structures and factors of i[27]. $iD_{\mu\nu}(x,y) = \langle 0|TA_{0\mu}(x)A_{0\nu}(y)|0\rangle$ and $-i\Pi_{\mu\nu}(x,y) = \langle 0|TJ^{\gamma}_{\mu}(x)J^{\gamma}_{\nu}(y)|0\rangle$ in real space. In momentum space:

$$D_{\mu\nu}(q^2) = \frac{-i[g_{\mu\nu} - q_\mu q_\nu(1-\xi)/q^2]}{q^2 - \Pi}, \tag{II.3}$$

where Π is the reduced scalar self-energy and ξ a gauge parameter, both discussed in more detail in the next section. The tree-level propagator is just $D_{\mu\nu}(q^2)$ with Π set to zero. In a matrix element, the $\mu\nu$ indices contract with the corresponding indices at vertices at either end of the photon line. A QED theorem tells us that the the $q_\mu q_\nu$ terms in the propagator can be omitted if the fermion lines at one or both of the vertices are on-shell, a result of vector coupling and applying the Dirac equation to the external fermions. This justifies the suppression of tensor indices in (II.2). Furthermore, note the canonical mass dimensions $dim(D) = -2$ and $dim(\Pi) = +2$.

The proper vertex function requires one vector index for the photon line and two spinor indices, one for each external fermion line: $-i\Gamma(q^2)_{\mu,\alpha\beta}$. The most general Lorentz tensor/spinor decomposition of Γ consistent with Lorentz, C and P symmetries is:

$$\Gamma_\mu = F_1(q^2)\gamma_\mu - \frac{1}{2m_f}F_2(q^2)\sigma_{\mu\nu}q^\nu \quad , \tag{II.4}$$

where $F_1(q^2)$ and $F_2(q^2)$ are the *electric* and *magnetic form factors* of the fermion in question[27]. $F_1(q^2)$ generalizes the tree-level vertex. Note that $dim(\Gamma) = dim(F_1) = 0$. $F_2(0)$ is proportional to the anomalous magnetic moment of the fermion, beginning at $\mathcal{O}(\alpha)$, since $\sigma_{\mu\nu}$ reduces non-relativistically to the spin operator σ. Note also that the magnetic term in (II.4) has dimension -1, corresponding to the dimension-5 operator $\bar{\psi}_f\sigma_{\mu\nu}\psi_f F^{\mu\nu}$ in $\mathcal{L}_{\text{eff}}$. Thus $F_2(q^2)$ is finite. It is also a helicity-changing operator, suppressed at high energies by m_f^2/q^2, and is hence ignored, except for a brief reappearance when we look at low-energy QED. The box function $\Theta(q^2)_{\alpha\beta\gamma\delta}$ carries four spinor indices. Note that $dim(\Theta) = -2$, so that Θ is finite.

II.2 Gauge Invariance – Choice of Gauge

Gauge symmetry is the single principle that holds a gauge theory together, both on the classical and quantum levels, in a number of different ways. In the classical theory, it means that $\mathcal{L}_0$ is invariant under the local gauge transformations:

$$\psi_0 \to e^{-ie_0 Q_\psi \theta(x)}\psi_0, \quad A_{0\mu} \to A_{0\mu} + \frac{1}{e_0}\partial_\mu\theta(x),$$

which requires the covariant derivative coupling of matter to gauge fields: $D_\mu = \partial_\mu + ie_0 QA_{0\mu}$, and the conservation of charge $\partial_\mu J^\mu = 0$. At the quantum level, the gauge invariance reappears in the form of relations between, and restrictions on, Green's functions, in electrodynamics called *Ward-Takahashi identities*[27,28,30]. Gauge invariance in $\mathcal{L}_0$ holds even if the vacuum breaks the gauge symmetry; the dynamics is gauge-invariant even if the ground state isn't. Gauge-invariant dynamics helps to guarantee renormalizability by reducing the number of infinities to the few that can be absorbed by reparametrization.

At the level of photon properties, gauge invariance means that of the four degrees of freedom (d.o.f.'s) in the vector field A_μ, two are spurious, associated with the freedom of arbitrary gauge transformations. Because of this unphysical ambiguity, quantization requires us to *fix a gauge;* that is, to restrict A_μ in some way to reduce its functional arbitrariness. Gauge invariance of the theory implies that the results for physical processes should be independent of this choice. In covariant perturbation theory, the common choice for an unbroken theory is the Lorentz gauge, $\partial_\mu A^\mu = 0$, which can be implemented in the Lagrangian by adding a Lagrange multiplier term:

$$\mathcal{L}_0 \rightarrow \mathcal{L}_0 - \frac{1}{2\xi}(\partial_\mu A_0^\mu)^2,$$

the simplest version of the general renormalizable ξ or R_ξ gauges used in most gauge theory calculations[28]. Physically, the Lorentz gauge projects out of A_μ only transverse d.o.f.'s, covariantly generalizing the frame-dependent notions of transverse and longitudinal d.o.f.'s. Define the covariant transverse and longitudinal projection operators:

$$P^T_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \quad P^L_{\mu\nu} = \frac{q_\mu q_\nu}{q^2}.$$

They sum to unity in the sense that $P^T_{\mu\nu} + P^L_{\mu\nu} = g_{\mu\nu}$. Note then that the covariant propagator of (II.3) takes the form $D_{\mu\nu} = -i[P^T_{\mu\nu} + \xi P^L_{\mu\nu}]D(q^2)$, where the unphysical longitudinal P^L carries the gauge-dependence. In the context of R_ξ gauges, gauge invariance means that matrix elements between physical on-shell states are independent of ξ and thus independent of the spurious longitudinal modes. The cancellation of the $q_\mu q_\nu$ terms enforces this ξ-independence and is the momentum space form of current conservation at the vertex, $\partial_\mu J^\mu_\gamma = 0$.

There are two key Ward identities for our case. The *first* Ward identity requires that $F_1(0) = e_0 Q$. That is, the complete proper vertex function reduces back to the tree-level form at zero momentum transfer. The vertex radiative corrections disappear or *decouple* as $q^2 \rightarrow 0$. This is our first, but not last, encounter with the connection between unbroken gauge invariance and decoupling required by a Ward identity, which in this instance also leads to the "non-renormalization" of the tree-level vertex. The *second* Ward identity requires that $\Pi_{\mu\nu}(q^2) = +iP^T_{\mu\nu}\Pi(q^2)$, to any order; that is, $\Pi_{\mu\nu}$ is purely transverse. Futhermore, $\Pi(0) = 0$, so that we can write

$\Pi(q^2) = \Pi_{\gamma\gamma} = e_0^2 \cdot \Pi_{QQ} \equiv e_0^2 \cdot q^2 \cdot \Pi'_{QQ}$ (recalling (I.12)). It then follows that:

$$D = \frac{1}{q^2[1 - e_0^2 \Pi'_{QQ}]}. \qquad (II.5)$$

In other words, *if* gauge invariance is good and the photon mass zero at tree level, they remain so at any order. These statements are necessarily true only in perturbation theory.*

From the Ward identities and cancellation of the ξ's, it is straightforward to show that Π, Γ and Θ form separately gauge-invariant contributions to a complete matrix element. Discussing each class of corrections represented by these Green's functions separately thus has physical meaning, as each is gauge independent.

II.3 Divergences – Some Technical Tools

☞*It is easy to speak of the infinite, as every theologian knows, but it is difficult to speak of it meaningfully.*

– J. D. North, *The Measure of the Universe*[31]

We need to look into these black boxes Π, Γ, Θ now to get some idea of how they, particularly their divergences, behave. Two broad approaches are available to tackle the infinities. One is to extract *only* the divergent parts to prove renormalizability at an arbitrary order, using an arbitrary renormalization scheme. A powerful method using *counterterms* is available to carry out this program[8,28,32,33]. Instead of beginning with the classical $\mathcal{L}_0$, one starts with a $\mathcal{L}_{\rm ren}$ of the same form, defined in some scheme $\vec{p}$. Then one demonstrates that the infinities occur in the perturbative series only in such a way that they can be cancelled by subtracting counterterms $\mathcal{L}_{\rm ct}$ from $\mathcal{L}_{\rm ren}$ and recovering the bare $\mathcal{L}_0$:

$$\mathcal{L}_{\rm ren} = \mathcal{L}_0 + \mathcal{L}_{\rm ct}, \quad \mathcal{L}_{\rm rc} = \mathcal{L}_1 - \mathcal{L}_{\rm ct},$$

$$\mathcal{L}_{\rm eff} = \mathcal{L}_0 + \mathcal{L}_1 = \mathcal{L}_{\rm ren} + \mathcal{L}_{\rm rc},$$

where $\mathcal{L}_1$ represents the proper one-loop operators, and $\mathcal{L}_0 = \mathcal{L}_{\rm ren} - \mathcal{L}_{\rm ct}$, implying that the terms necessary in $\mathcal{L}_{\rm ct}$ to cancel divergences in $\mathcal{L}_1$ have only tree-level forms.

⋆ Breaking of the gauge symmetry is always non-perturbative in some coupling. If we failed to notice it in $\mathcal{L}_0$, it would show up as a non-perturbative pole in $\Pi'_{QQ} \to v_0^2/q^2$ as $q^2 \to 0$, so that $M_{\gamma 0}^2$ would be $e_0^2 v_0^2$.

The other approach is the one used here: compute the full matrix elements for specific processes with $\mathcal{L}_0$, including the full corrections, then renormalize by reparametrizing the matrix elements[29]. Counterterms are usable here to separate divergences as well, but are unnecessary, as the matrix elements can be reparametrized directly. On the other hand, it would not be easy to prove renormalizability at any order using this method: the finite parts of the loops would constitute a superfluous complication.

There are three paradigmatic one-loop corrections in electrodynamics. Figure 7a shows a divergent contribution to $\Pi_{\mu\nu}$, which upon extracting $-iq^2P^T_{\mu\nu}$ to isolate Π', is dimensionless and logarithmically divergent. Figure 8 shows a contribution to Γ_μ. Without the external self-energies, the reduced function F_1 is logarithmically divergent. The box diagram in Figure 9 has $dim(\Theta) = -2$ and is finite. The fact that virtual corrections in gauge interactions are no more than logarithmically divergent in $n = 4$ spacetime dimensions is directly connected to the fact that $dim(\Pi', \Gamma) = 0$ and, in turn, that $dim(e_0^2) = 0$ as well.

How does one evaluate such loop integrals in perturbation theory? Let me use the fermionic loop of Figure 7a as a prop to sketch the method and to introduce tools that are used in such calculations[28].

$$-i\Pi^{\gamma\gamma}_{\mu\nu} = -(-ie_0Q_i)^2 \int \frac{d^4l}{(2\pi)^4} \mathrm{Tr}[\frac{i}{\not{l} - m_i + i\epsilon}\gamma_\mu \frac{i}{(\not{l} + \not{q}) - m_i + i\epsilon}\gamma_\nu]. \qquad (II.6)$$

The *zeroth* step is to rationalize the fermion propagator denominators and to remove $e_0^2 P^T_{\mu\nu}$ and isolate the reduced scalar function Π_{QQ}. The evaluation of Π_{QQ} can take a number of paths, all of which lead to the same results for finite differences of loops. Let me map out one commonly used path.

The *first* step is make the divergent integrals meaningful by the formal redefinition of infinities as mathematical limits, so that they can be manipulated as if they were finite. This step is called *regularization.* A number of regularization methods are available. The important thing is to make sure that the method does not inadvertently break a good classical symmetry. The naive regularization of simply cutting off the integral at some upper bound $l^2 \leq \Lambda^2$ breaks Lorentz symmetry. The *Pauli-Villars* method[28,34] preserves gauge invariance in QED and Lorentz invariance: replace

$$\frac{1}{l^2 - m^2} \rightarrow \frac{1}{l^2 - m^2} - \frac{1}{l^2 - \Lambda^2},$$

and let $\Lambda^2 \gg m_i^2, q^2$. Pauli-Villars does break gauge invariance in non-Abelian theories, however. This fact led Veltman and 't Hooft to introduce a new method that keeps non-Abelian symmetries intact, *dimensional regularization*[35,33,28]. Analytically continue in the number of spacetime dimensions away from four to n, an arbitrary complex number. That is,

$$
\begin{aligned}
l_\mu = (l_0; l_1, l_2, l_3) &\rightarrow (l_0; l_1, ..., l_{n-1}) \\
g_{\mu\nu} g^{\mu\nu} = \delta^\mu_\mu = 4 \rightarrow n, \qquad Tr[spinors] &= 4 \rightarrow 2^{n/2} \qquad (II.7) \\
d^4 l \rightarrow d^n l, \qquad (2\pi)^4 &\rightarrow (2\pi)^n,
\end{aligned}
$$

and so on, consistently throughout the integral. Divergences reappear as we continue back $n \rightarrow 4$.*

The *second* step is to Euclideanize the space of integration, so as to convert the integral from the unwieldy Minkowski metric to a more manageable integral in $\mathcal{R}^n$. In perturbation theory, this step is legitimate because of the analytic properties of loop integrands as functions of p_μ. The tree-level inverse propagators $(l^2 - m_i^2 + i\epsilon)$ have an infinitesimal imaginary part in their zeros that reflects the correct (Feynman) boundary conditions for physical on-shell states: positive-energy particles propagating forward in time, negative-energy particles propagating backward in time as positive-energy antiparticles propagating forward in time. For real loop masses, the loop integrands are analytic functions of l_0 in the first and third quadrants of complex l_0 plane, for certain values of the external momenta, and their poles occur in the second and fourth quadrants only (Figure 10). Therefore, Minkowski integrals over l_0 on the interval $(-\infty, +\infty)$ can be transformed into integrals on the interval $(-i\infty, +i\infty)$ by using Cauchy's theorem and the contour shown in Figure 10. The contour or Wick rotation is implemented equivalently by changing variables: $l_0 = il_n$, $l_E = (l_1, ...l_n)$, $l^2 = l_0^2 - \mathbf{l}^2 = -l_E^2$, $d^n l = i d^n l_E$, $q_0 = iq_n$, and so on. After the integration is performed, rotate back: $q_n = -iq_0$, $q_E^2 = -q^2$, etc., and extend to all values of the external momenta by analytic continuation.

The *third* step is to combine separate denominators of propagators in the integrand into a single denominator using the *Feynman parameters* x_i[37]:

$$
\begin{aligned}
\frac{1}{A_1^{a_1} \cdots A_k^{a_k}} &= \frac{\Gamma[a_1 + \cdots + a_k]}{\Gamma[a_1] \cdots \Gamma[a_k]} \\
&\times \int_0^1 dx_1 \cdots \int_0^1 dx_k \frac{\delta(1 - \sum_j x_j) x_1^{a_1 - 1} \cdots x_k^{a_k - 1}}{[A_1 x_1 + \cdots + A_k x_k]^{a_1 + \cdots + a_k}} \quad , \qquad (II.8)
\end{aligned}
$$

where $\Gamma[z] = (z-1)!$ is the gamma function; then change variables $l_E \rightarrow l'_E$ so as to

* If no regularization method preserves the symmetries of the classical theory, at least one of the symmetries can be *anomalous*, broken by quantum effects. An anomalous local gauge symmetry destroys renormalizability, and the theory no longer exists. The SM gauge symmetries are anomaly-free, although some SM *global* symmetries are anomalous, giving rise to new non-classical effects[28,36].

eliminate all terms in the combined denominator linear in l'_E. Only terms quadratic in l'_E remain now.

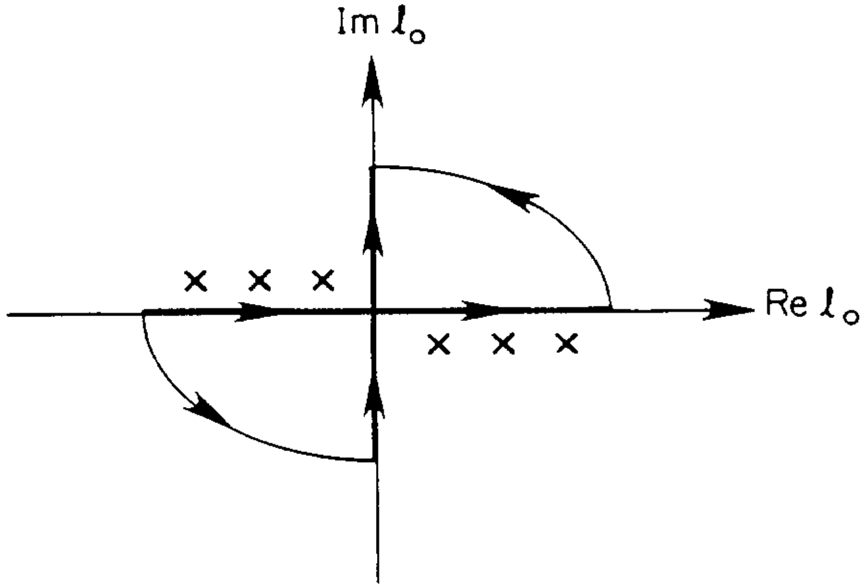

Figure 10. Contour (Wick) rotation for Feynman integral over virtual momentum l, from Minkowski to Euclidean metric.

The *fourth* step is to switch the order of integration, pulling the Feynman parameter integrals outside and performing the l'_E integral first. This step is allowed because the integral is formally finite. Since the integrand is a function of ${l'_E}^2$ only now, the $(n-1)$-dimensional solid angle integration in $\mathcal{R}^n$ can be carried out. Dropping primes, $d^n l_E = l_E^{n-1}\, dl_E\, d\Omega_n$, and:

$$\int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{II.9}$$

This handy formula can be derived using the trick of evaluating the Gaussian integral $\int d^n x\ e^{-x^2}$ in radial, then Cartesian, co-ordinates. The full $\mathcal{R}^n$ integrals take the form[33,35]:

$$\int d^n l_E \frac{(l_E^2)^a}{(l_E^2+\beta^2)^b} = \frac{i}{(\beta^2)^{b-a}} \frac{(\pi\beta^2)^{n/2}}{\Gamma[n/2]} \frac{\Gamma[a+\frac{n}{2}]\Gamma[b-a-\frac{n}{2}]}{\Gamma[b]}, \tag{II.10}$$

where all the dynamics — the dependence on the external momenta q^2 and the loops masses — is buried in β^2.

The *fifth* and last step is to continue back to Minkowski space and let $n \to 4$. Divergences now show up in the gamma functions, but we must be careful to consistently expand out the $n \to 4$ limit everywhere. Let $\epsilon \equiv (4-n)/2 \to 0$ — not to be confused with the imaginary parts of propagators! The *degree of divergence* $\mathcal{D}$ of the integral is $\mathcal{D} = n + 2(a-b)$, and the divergence, if one occurs, appears in $\Gamma[b-a-\frac{n}{2}] = \Gamma[-\frac{\mathcal{D}_4}{2}+\epsilon]$. In $n=4, \mathcal{D}_4 = 0,2$, corresponding to logarithmic and quadratic divergences, respectively. In gauge interactions, $\mathcal{D}_4 \leq 0$; that is, loops are either finite or logarithmically divergent, because Π' and the vertex Γ are dimensionless.† The gamma function near the non-positive integers can be expanded as:

$$\Gamma[-m+\epsilon] = \frac{(-1)^m}{m!}[\frac{1}{\epsilon}+1+\frac{1}{2}+\cdots+\frac{1}{m}-\gamma+\mathcal{O}(\epsilon)], \qquad (II.11)$$

where γ is the Euler constant: $\gamma = (d/dz)(\ln\Gamma[z])_{z=1} = 0.57721566...$, and $m = 0,1,2,....$ For $\mathcal{D}_4 < 0$, the integral is finite; for $\mathcal{D}_4 = 0$, $\Gamma[-0+\epsilon] = 1/\epsilon - \gamma + \mathcal{O}(\epsilon)$. Now re-express all n's in the integral in terms of ϵ. Various quantities occur taken to the power ϵ, all of which must be expanded: $z^\epsilon = 1+\epsilon\ln(z)+\mathcal{O}(\epsilon^2)$. The $\mathcal{O}(\epsilon)$ terms, when multiplied by the $1/\epsilon$, come out as $\mathcal{O}(1)$, including the finite parts buried in β. The divergent integrals in the end look like:

$$\Pi', F_1 \sim \sum_i \int d[x_j]\delta(1-\sum_j x_j)[\Delta - \ln\beta^2(x_j;q^2,m_i^2)], \qquad (II.12)$$

with logarithmic divergences in dimensional regularization consistently appearing in the combination $\Delta \equiv 1/\epsilon - \gamma + \ln 4\pi$, and throwing away terms of $\mathcal{O}(\epsilon)$ and higher. To get these results reliably and to reproduce the unrenormalized Ward identities, it is crucial to carry out the ϵ expansion universally; e.g., including the $(2\pi)^n = (2\pi)^4(2\pi)^{-2\epsilon}$, and so on. Finite integrals can be evaluated directly using (II.10) without the ϵ expansion.

An optional but very convenient step is to define a set of standard one-loop form factors[38]. For example, for two-point loops:

$$\{B_0, B_\mu, B_{\mu\nu}\}(q_E^2;m_1,m_2) = \int \frac{d^n l_E}{i\pi}\frac{\{1, l_\mu, l_\mu l_\nu\}_E}{[l_E^2+m_1^2-i\epsilon][(l_E+q_E)^2+m_2^2-i\epsilon]}, \qquad (II.13)$$

where all two-point integrals (that is, contributions to Π) can be reduced to combinations of $B's$. Three-point functions C with three propagators and four-point functions D with four propagators can similarly be defined. The properties of the loop functions as functions of external momenta can be studied by studying these form factors.

† Quadratic divergences can appear in the intermediate steps of calculating Π and Γ, but they cancel when all terms are added up.

To summarize, we need five steps to carry out the loops integrals, after simplifying the integrand in tensor/spinor structure: (1) introduce regularization; (2) continue from Minkowski to Euclidean space; (3) combine separate propagator denominators using Feynman parameters; (4) switch the order of integration and perform the Euclidean integration; (5) continue back to Minkowski space and separate the divergences. Standardized form factors are convenient for simplifying the calculation. For the collection of all good regularization methods — that is, those that preserve good classical symmetries — the identification of the divergent parts is essentially unique, and they can be translated from one regularization to another. The simplest "good" method that works generally for gauge theories is the dimensional regularization of Veltman and 't Hooft. There are intricate Feynman parameter integrals left to do at the end, with complicated dependences on external momenta and loops masses. Some generalities are presented in the next few sections.

II.4 Born-like Corrections: Reparametrization and Finiteness

Let us begin examining the physical properties of the corrections by considering the photon self-energy and propagator[27,28]. Because the propagator corrections reproduce the tree-level Born matrix element form, with linear charge dependence Q, Q' on the external legs and no dependence on the external masses m_f, let us call the corrections incorporated into the full propagator *Born-like* and the matrix element so corrected the *improved Born approximation (IBA)*. The other corrections, vertices and boxes, are *non-Born-like,* as they depend non-linearly on the external gauge charges Q, Q'. The improved Born matrix element $\mathcal{M}_{IBA}$ is:

$$\mathcal{M}_{IBA} = \frac{e_0^2 QQ'}{q^2[1 - e_0^2 \Pi'_{QQ}]}. \qquad (II.14a)$$

The Born-like corrections can be put into a compact form by defining[39]:

$$\frac{1}{e_*^2(q^2)} = \frac{1}{e_0^2} - \Pi'_{QQ}, \qquad (II.15)$$

so that:

$$\mathcal{M}_{IBA} = \frac{e_*^2(q^2) QQ'}{q^2}. \qquad (II.14b)$$

$\mathcal{M}_{IBA}$ can be put into this form because the universal corrections Π can be combined with the bare gauge coupling e_0^2 to define a q^2-dependent or running function $e_*^2(q^2)$ (II.15), denoted with a star subscript, that appears in $\mathcal{M}_{IBA}$ the way e_0^2 appears in $\mathcal{M}_0$ (II.1). The fact that propagator corrections in gauge theories always take this form is a consequence of gauge symmetry. The star function $e_*^2(q^2)$ serves as a running coupling.

To obtain usable predictions from $\mathcal{M}_{IBA}$, we need to reparametrize $e_*^2(q^2)$ by replacing e_0^2 with a finite parameter and define that parameter in terms of some experiment. The renormalization scheme (RS) is arbitrary, but we can simplify our task by taking advantage of the decoupling properties of QED. The full decoupling theorem states that, as $q^2 \to 0$:

$$q^2 \mathcal{M}_{complete} \to e_*^2(0) Q Q', \tag{II.16}$$

including *all* loop corrections (self-energy, vertices and boxes) to *any* order in perturbation theory. On photon shell ($q^2 = 0$), only the Born-like or propagator corrections are left. So an obvious RS is the *QED on-shell scheme*[29] with $e^2 \equiv e_*^2(0)$ as the parameter, solving for e_0^2 as a function of e^2 :

$$\frac{1}{e^2} = \frac{1}{e_0^2} - \Pi'_{QQ}(0), \tag{II.17}$$

and re-expressing

$$\begin{aligned} e_*^2(q^2) &= \frac{e^2}{1 - e^2 \Delta_Q(q^2)}, \\ \Delta_Q(q^2) &= \Pi'_{QQ}(q^2) - \Pi'_{QQ}(0), \end{aligned} \tag{II.18}$$

in terms of e^2.

It is important to make a clear distinction at this point between the *mathematical parameter* e^2 used to redefine the perturbative series — a renormalization scheme — and the *physical experiment* used to define numerically the value of that parameter. Let us call the latter choice an *input scheme (IS)*. A finite parameter is defined as an arbitrary combination of bare parameters and divergent corrections, but since QED reduces to the tree-level Maxwellian form as $q^2 \to 0$, the on-shell definition of e^2 is clearly attractive for its simplicity. However the parameter is defined, experimentally measurable quantities can then be expressed in terms of the parameter and its value fixed by one of the experiments. An alternative experiment with the same scheme or definition of e^2 (II.16) is Compton scattering (Figure 12)[27,28]. Since the photon lines are external, they are automatically on-shell ($k^2 = k'^2 = 0$). The virtual fermion line is not, however, so the decoupling theorem does not apply here. However, if the photon *energy* is much smaller than the fermion mass ($k_0^2 \ll m_f^2$), then the virtual fermion line is arbitrarily close to being on-shell. Decoupling then applies, the corrections disappear, and the tree-level Compton amplitude is recovered. In this case, we have in addition $k_0 \to 0$, so that the process passes to the non-relativistic limit of Thomson scattering:

$$\sigma_{Compton} \rightarrow \sigma_{Thomson} = \frac{8\pi\alpha^2 Q_f^4}{3m_f^2}$$

to all orders in α, as proven by Thirring in 1950[40]. Modern measurements of α typically use solid-state experiments, as discussed in section IV.5.

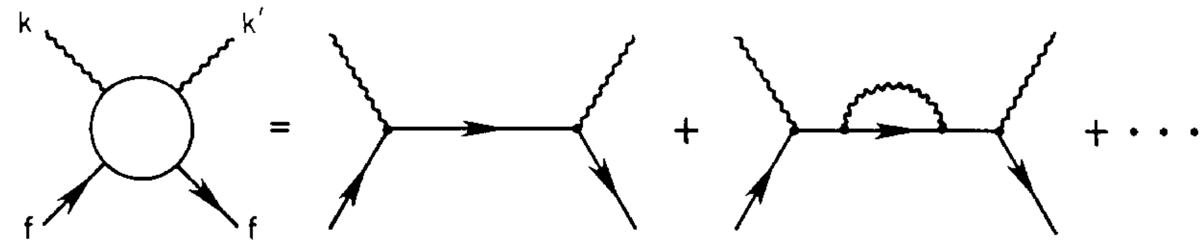

Figure 11. Perturbative expansion for Compton scattering $\gamma f \rightarrow \gamma f$.

Since divergences occur in $\Pi'_{QQ}(q^2)$ as $(\sum_i C_i)\Delta$ with a coefficient C_i for each contributing particle, the function $\Delta_Q(q^2)$ is obviously finite. The corrections, once reparametrized, take the form of finite differences of loop functions. The subtraction of $\Pi'_{QQ}(q^2)$ at $q^2 = 0$ is due to the use of e^2 as the renormalized parameter. Proving the finiteness of QED to all orders is not as obvious[29]. At two loops, for example, the irreducible one-loop diagrams appear embedded in the second outer loop (Figure 7c). A general proof of renormalizability must proceed iteratively. Having shown that the one-loop amplitudes are finite upon the renormalization of parameters, one needs to demonstrate that this *same* renormalization renders the $(n+1)^{th}$-loop amplitudes finite if the n^{th}-order ones are. The renormalizability of QED was conjectured by the founders of modern covariant quantum field theory, Tomonaga[41], Schwinger[42], and Feynman[43] in the late 1940's, sketched by Dyson in 1949-50[29] and proven rigorously by Weinberg in 1960[30,44].

III. Quantum Electrodynamics: Second Part

III.1 Born-like Corrections: Running Charge – Asymptotia – Decoupling(I)

The function $e_*^2(q^2)$ acts as an effective running charge or a dielectric function for the polarizable vacuum. The contributions to $\Delta_Q(q^2)$ are a sum over all contributing particles (II.12):

$$\Delta_Q(q^2) = \sum_i C_i Q_i^2 f_i(q^2; m_i^2), \tag{III.1}$$

where f_i has one form for fermion loops, another for scalars. For $|q^2| \gg m_i^2$, $f_i(q^2; m_i^2) = \ln(|q^2|/m_i^2) + \mathcal{O}(m_i^2/q^2)$, the asymptotic limit. In the low-energy limit, $|q^2| \ll m_i^2$, $f_i(q^2; m_i^2) = \mathcal{O}(q^2/m_i^2)$[27,28].

The asymptotic running of the vacuum polarization is logarithmic:

$$\frac{1}{e_*^2(q^2)} \simeq \frac{1}{e^2} - \sum_i C_i Q_i^2 \ln(|q^2|/m_i^2), \tag{III.2}$$

for all $m_i^2 \ll |q^2|$, reflecting the fact that the ultraviolet divergences are also logarithmic. The presence of vacuum polarization gives $\mathcal{M}_{IBA}$ non-trivial scaling properties as $|q^2| \to \infty$. In higher orders, each proper diagram of n loops with *one* fermion loop contributes $\mathcal{O}(\alpha^n \ln(|q^2|/m_i^2))$ to $e^2\Delta_Q(q^2)$[28]. The two-loop diagrams contribute "subleading" or "next-to-leading" logarithms, and so on. The functional behavior is always logarithmic, like the one-loop functions, as the divergences are logarithmic at any order — only the power of α and the numerical coefficient change. This result, a special case of Weinberg's theorem (1960)[44], can be used to estimate the error made in ignoring higher loops. With lower-order vacuum polarization inserted into the higher-order diagrams (more than one fermion loop, Figure 7c), we start to obtain higher powers of logarithms. For fermions and scalars, $C_i > 0$ at any order, so that $e_*^2(q^2)$ increases with $|q^2|$, an asymptotically unfree theory. The effective charge increases as we penetrate the cloud of virtual vacuum polarization. Eventually, it diverges at a very high q^2 : $[\sum_i C_i Q_i^2 \ln(|q^2|/m_i^2)]^{-1} \simeq e^2$. This *Landau pole*[27,28] is actually unphysical, because QED merges into an asymptotically free non-Abelian theory long before $e_*^2(q^2)$ blows up. The connection between the ultraviolet divergences of the theory and its asymptotic behavior is the starting point for the *renormalization group,* which provides a means of directly calculating asymptotic Green's functions equivalent to the Dyson sum at any order of irreducible perturbation theory[45]. The renormalization group can also be applied to very complicated amplitudes where simple formulas like the Dyson sum are not available[46].

The small q^2 behavior of vacuum polarization exhibits decoupling explicitly, since $\Delta_Q(q^2) = \sum_i C_i Q_i^2 \mathcal{O}(|q^2|/m_i^2)$ as $q^2 \to 0$, for all $m_i^2 \gg |q^2|$. Each contribution to Δ_Q thus rises linearly with q^2 for $|q^2| \ll m_i^2$, then softens to logarithmic behavior as $|q^2| \gg m_i^2$. The point $q^2 = (m_i + m_i)^2 = 4m_i^2$, for $q^2 > 0$, is called the *threshold* for that particular loop and divides the so-called threshold region $q^2 \lesssim m_i^2$ from the asymptotic regime $q^2 \gtrsim m_i^2$. Threshold contributions are small compared with asymptotic logarithms. For time-like $q^2 > 0$, the threshold point marks the onset of imaginary contributions to vacuum polarization[27,28]. $\mathrm{Im}\Pi'$ and $\mathrm{Im}\Delta_Q \neq 0$ signals the decay of the virtual photon, beginning with decay into physical on-shell pairs at lowest order (Figure 12). $\mathrm{Im}\Pi'$ is proportional to the photon decay rate, a consequence of unitarity (the optical theorem). The real and imaginary parts are related to each other by causality, which restricts the analytic properties of Π as a function of complex q^2. Causality requires that Π be analytic for $\mathrm{Im}q^2 > 0$. Cauchy's theorem then implies a *dispersion relation*[27,28] for Π analogous to the Kramers-Kröning relations of optics[26]:

$$\Pi'(z) = \frac{1}{2\pi i} \oint_C dq'^2 \, \frac{\Pi'(q'^2)}{q'^2 - z}, \tag{III.3}$$

with $z = q^2 + i\epsilon$, $\epsilon \to 0^+$, and the contour C taken in the upper half of the q'^2 plane. The imaginary parts of the self-energy arise in perturbation theory from the the infinitesimal imaginary parts of loop propagators. Each state in the theory starts contributing as its threshold is passed. $\mathrm{Im}\Pi_{\gamma\gamma}$ can be rewritten as a photon width in the photon propagator: $\mathrm{Im}\Pi_{\gamma\gamma} = \sqrt{q^2}\Gamma_\gamma$. After this section, imaginary parts in amplitudes are not displayed, except for the decays widths of the W and Z bosons. Note that the widths need not be put by hand into the propagators: they come out of perturbation theory automatically by not forgetting the imaginary parts of the self-energies.

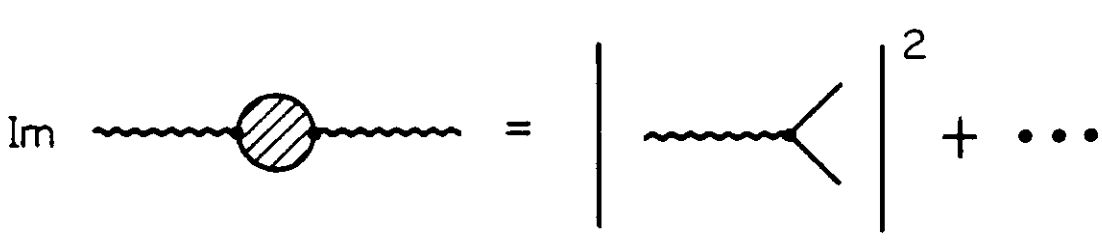

Figure 12. Perturbative expansion of imaginary part of photon self-energy as sum over virtual photon decays.

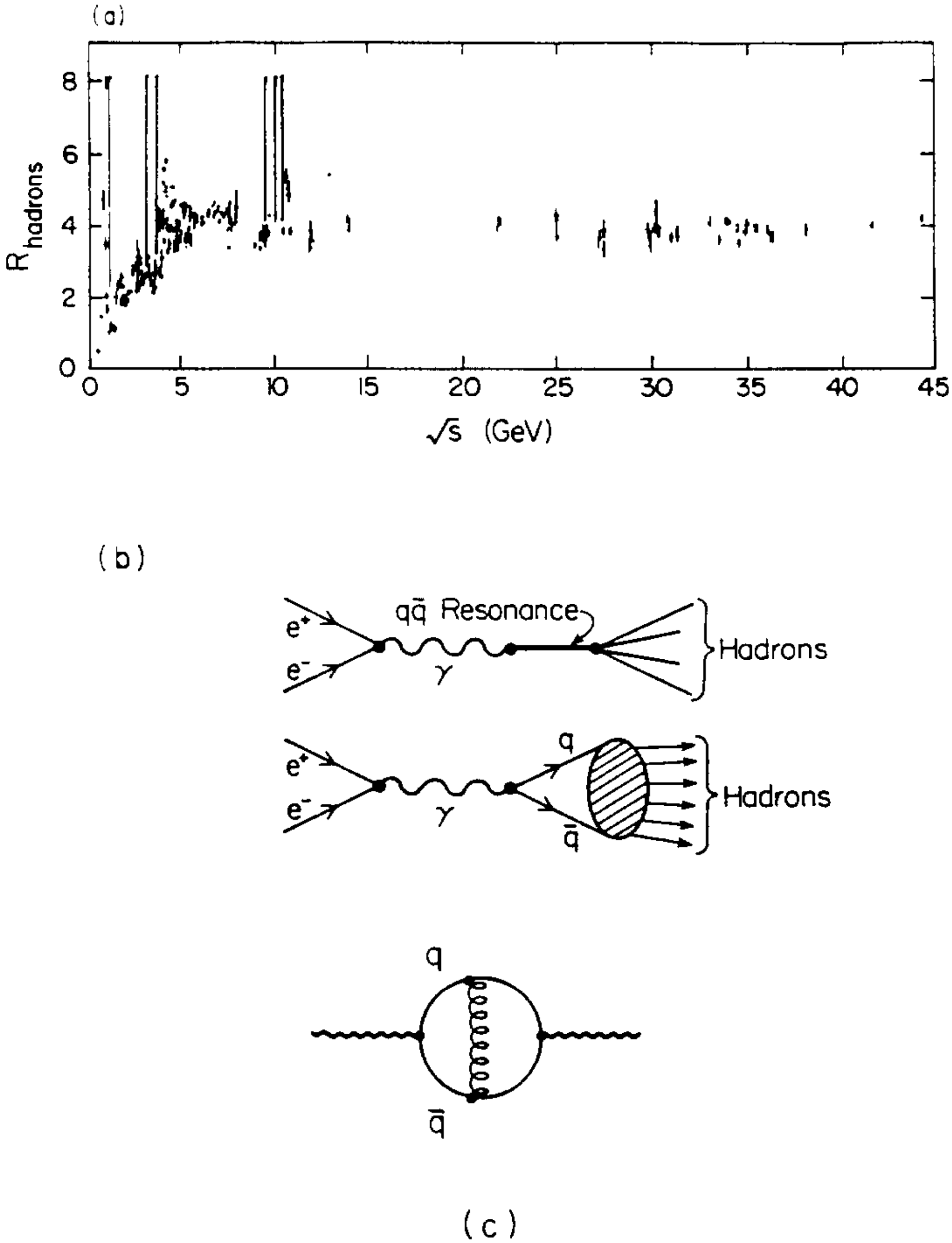

Figure 13. (a) $R_{had}(s)$ from e^+e^- annihilation in the non-perturbative resonance and perturbative regions. (b) e^+e^- annihilation into hadronic resonances and quasi-free $q\bar{q}$ pairs. (c) Perturbative QCD $\mathcal{O}(\alpha\alpha_s)$ corrections to the photon self-energy[49,47,48].

The dispersion relation (III.3) is valid independent of perturbation theory and can be used to relate the real part of vacuum polarization to the imaginary part:

$$\mathrm{Re}\Delta_Q(s) = \frac{s}{\pi} \int_0^\infty \frac{ds'}{s'} \cdot \frac{\mathrm{Im}\Pi'_{QQ}(s')}{s' - s} \quad , \qquad (III.4)$$

with $s = q^2 > 0$ in the annihilation channel. In practical calculations, relation (III.4) is the only way to evaluate the contributions of $q\bar{q}$ pairs to vacuum polarization and the running $e_*^2(q^2)$. QCD perturbation theory cannot be used for $q\bar{q}$ loops coupled to the virtual photon until q^2 is well above the region of non-perturbative hadronic resonances (Figure 13a,b). Thus the vacuum polarization for the *udcsb* quarks is evaluated instead using the ratio $R_{had}(s) = \sigma(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$ from e^+e^- colliders and the theoretically assumed muon pair production cross section, $\sigma(e^+e^- \to \mu^+\mu^-) = 4\pi\alpha^2/3s$ at center-of-mass energy $\sqrt{s}$[47]. Then:

$$\mathrm{Re}\Delta_Q(s)_{had} = \frac{s}{12\pi^2} \int\limits_{4m_\pi^2}^{\infty} \frac{R_{had}(s')}{s'(s'-s)} ds', \qquad (III.5)$$

integrating the data from the pion pair threshold through the hadron resonances (Figure 13a) until the quasi-free quark region is reached. At high q^2, perturbative QCD can be used to compute the $\mathcal{O}(\alpha\alpha_s)$ strong corrections to the vacuum polarization (Figure 13c)[48]. To carry out Z pole calculations, $\mathrm{Re}\Delta_Q(q^2 = M_Z^2)$ is needed and depends on this hadronic contribution. The use of experimental data then introduces a small uncertainty into the computed value of $\Delta_Q(Z)$ and $e_*^2(Z)$[47].

III.2 Non-Born-like Corrections: Reparametrization – Decoupling(II)

Let us now turn to the non-Born-like corrections, the vertices and boxes[27,28]. These have the same Lorentz structure as the Born amplitude (II.1), but not the same gauge structure: higher powers of Q and Q' appear. Recall $\Gamma_\mu = F_1(q^2)\gamma_\mu$, where the *first* QED Ward identity requires decoupling and non-renormalization of the tree-level vertex at $q^2 = 0$: $F_1(0) = e_0 Q$. Following Figure 8, divide F_1 into true vertex and external self-energy contributions, $F_1 = F_1^V + F_1^{ext}$. Then $F_1^V(q^2 \to 0) = e_0 Q[1 + e_0^2 Q^2 \mathcal{O}(q^2/m_f^2) + D]$, where the second term in brackets contains no ultraviolet divergences. D is a UV-divergent constant.

To understand F_1^{ext}, we need to digress briefly into the parts of QED renormalization not associated with the photon propagator — the renormalization of the fermion mass and field[27,28]. If Σ is the irreducible fermion self-energy and m_{f_0} the fermion's bare mass, then the fully corrected or "dressed" fermion propagator is:

$$S = \frac{i}{\not{p} - m_{f_0} - \Sigma}, \qquad (III.6)$$

and the tree-level propagator is (III.6) with $\Sigma = 0$. S is in exact analogy with the dressed photon propagator, where $dim(S) = -1$ and $dim(\Sigma) = +1$. The irreducible self-energy can be decomposed as $\Sigma = m_{f_0}A + (\not{p} - m_{f_0})B + (\not{p} - m_{f_0})\Sigma_f(\not{p} -$

m_{f_0}), where A, B are dimensionless logarithmically divergent constants. Σ_f is finite. A, B, Σ_f all start at $\mathcal{O}(\alpha)$ in perturbation theory. The physical fermion mass m_f is the pole of the propagator, so $m_f = (1+A)m_{f_0}$. In perturbation theory, we can replace the m_{f_0} in the second and third terms of Σ with m_f. Then for $p^2 \to m_f^2$, the dressed propagator reads:

$$S \to \frac{i(1+B)^{-1}}{\not{p} - m_f}, \qquad (III.7)$$

with a nontrivial residue at the pole. In real space, $iS(x,y) = \langle 0|T\psi_0(x)\bar{\psi}_0(y)|0\rangle$. A physical on-shell fermion ψ would have a residue of unity at the pole, so that $\psi_0 = \psi/\sqrt{1+B}$. This rescaling of bare to physical fields is the *field* or *wavefunction renormalization.* A complete matrix element has bare fermion wavefunctions attached to the external legs, which then must be rewritten in terms of physical on-shell states: $\psi_0 = (1 - \frac{1}{2}B)\psi$ at this order.

The contribution of the external leg self-energies is $F_1^{ext} = 2B$ before wavefunction renormalization. After ψ_0 is replaced by $(1-\frac{1}{2}B)\psi$, $F_1^{ext} = 2B - \frac{1}{2}B - \frac{1}{2}B = B$. Then $F_1(0) = e_0 Q[1 + D + B]$, and the *first* QED Ward identity implies in perturbation theory that $D + B = 0$ to any order. That is, the external fermion self-energies cancel the UV divergence D of the true irreducible vertex after wavefunction renormalization, leaving no infinite or finite dressing of the vertex at $q^2 = 0$[27,28,30]. This Ward identity points to a non-trivial relationship between the vertex and the fermion self-energy. In all, QED has three true or *primitive* UV divergences, which in four-fermion process show up as: photon self-energy (electric charge renormalization), fermion mass renormalization and fermion field or wavefunction renormalization. The fermion mass and residue corrections, like the vacuum polarization, can be absorbed in many different schemes, but for physical on-shell fermions, they are all equivalent to the mass and residue shifts shown above. The fourth divergence D is spurious. Henceforth, the external self-energies and external wavefunction renormalizations are not explicitly shown.

Recall the box diagram has $dim(\Theta) = -2$ and is finite. Decoupling applies to boxes also: $q^2\Theta \to 0$, as $q^2 \to 0$. The box loop also depends on *two* kinematical variables – s and t, say – while the vertex corrections depend only on one, $q^2 = s$ *or* t.

The non-universal corrections are still parametrized using e_0. Clearly, we need to replace the bare charge with something meaningful. This requires considering the higher-order corrections to vertices and boxes, such as those shown in Figure 14, and developing reasonable but simple approximations to capture the dominant part of the higher-order corrections. This is a controlled truncation of the perturbative series, an example of RS or scheme dependence[39,44]. The higher-order corrections of Figure 14 are convolutions of the lower-order corrections embedded in loops. To estimate the dominant effects, we need only determine which regions of the internal loop momenta dominate the integral. For example, in the vertex integral, the internal photon can be dressed by replacing two factors of e_0 with $e_*^2(l^2)$:

$$\Gamma_\mu \sim \int d^4l \cdot \frac{e_*^2(l^2)}{l^2} \cdots \quad . \tag{III.8}$$

The convolution can then be estimated by noting that the integral is dominated by the $l^2 \to 0$ region — so set $e_*^2(l^2) \simeq e_*^2(0) = e^2$ and pull it out of the integral. The third factor of e_0 in the vertex goes with the q^2-exchange photon line (see next paragraph). The same procedure can be applied to the box Θ. The box integral is dominated by the regions where the internal photon momenta are either $l^2 \simeq 0$ or q^2, the external momentum transfer. Therefore, the higher-order vacuum polarization insertions in Θ can be estimated by replacing the factor e_0^4 in the box by $e_*^2(0)e_*^2(q^2) = e^2 e_*^2(q^2)$. This procedure can be refined to greater accuracy if need be, by expanding $e_*^2(l^2)$ in powers of l^2, for example.

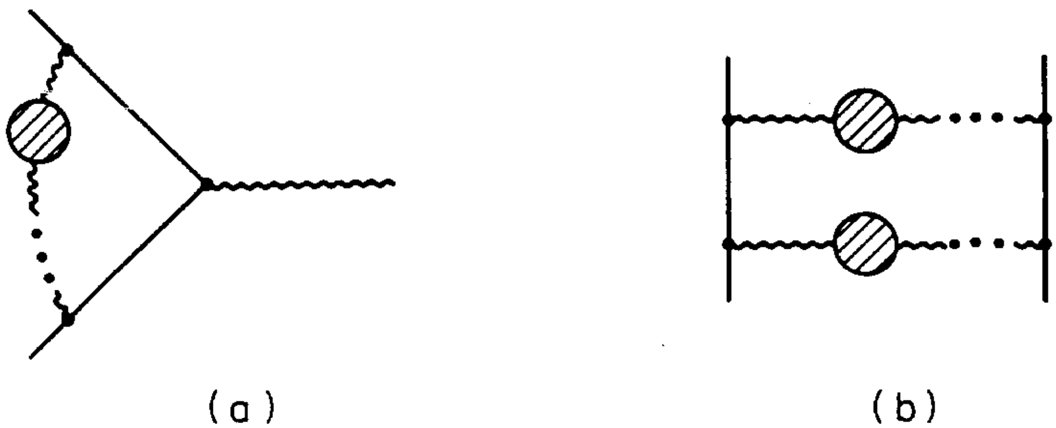

Figure 14. (a) Higher-order insertions in the irreducible vertex function. (b) Higher-order insertions in the irreducible box function.

With the non-universal corrections sensibly reparametrized with renormalized couplings, it is possible to jazz up the IBA by modifying its tree-level vertices with corrections: $e_0 Q \to e_0 Q \cdot \Gamma(q^2)$. The *modified IBA* has the form:

$$\begin{aligned} \mathcal{M}_{MIBA} &= \mathcal{M}_{IBA} + vertices \\ &\doteq \frac{Q \cdot \Gamma(q^2) \cdot e_*^2(q^2) \cdot Q' \cdot \Gamma(q^2)}{q^2}, \end{aligned} \tag{III.9}$$

where we have taken a second step beyond perturbation theory by combining the vertex and propagator corrections together into a single matrix element. We have also gone beyond the true Born structure, because the vertex corrections, while having the same Lorentz properties as the tree-level vertex, carry two additional powers of Q for each loop: $\Gamma = e_0 Q[1 + e^2 Q^2 \cdot function(q^2) + \cdots]$. The non-Born-like box graph can also be added to obtain the matrix element with the complete loop corrections:

$$\mathcal{M}_{all\ loops} = \mathcal{M}_{MIBA}(q^2) + Q^2 \cdot Q'^2 \cdot \Theta(s,t), \qquad (III.10)$$

where Θ has the couplings factor $e^2 \cdot e_*^2(q^2)$ and a non-Born-like gauge charge structure at one-loop order.

III.3 Non-Born-like Corrections: Radiation – Infrared Divergences

Having constructed this elegant modified Born approximation, we must now tear it up, for there is another kind of infinity lurking in the non-Born-like corrections, *infrared (IR) divergences.* These are peculiar to theories with massless particles like the photon, and are closely connected with the low-energy, long-distance properties of the theory. The IR-divergent loop contributions always have the logarithmic form $\ln(|q^2|/M_\gamma^2)$, where M_γ^2 is a vanishingly small fictitious photon mass inserted to regulate the divergence. Parallel divergences also occur in the type of radiative corrections we have hitherto ignored, the radiation of real, on-shell photons or *bremsstrahlung*[27,28,50].

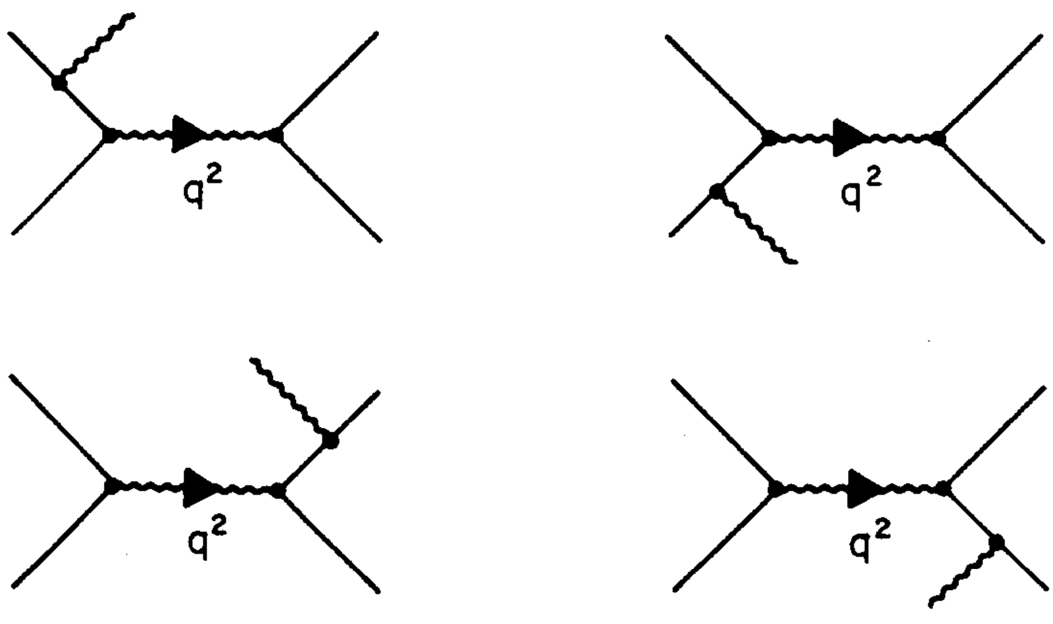

Figure 15. One-photon radiation from a neutral-current four-fermion process.

Figure 15 shows the set of one-photon radiation graphs for our process. The insertion of radiated photons into $\mathcal{M}_0$ actually produces the amplitude and probability for a different process, four fermions with one or more real photons in the final state. However, the process $4f$ + *photons* is degenerate with the process $4f$ in the limit that the energy-momentum lost from the fermions to radiation goes to zero. This is kinematically possible because the photon is massless. In a physical situation, the total probability for the process $4f$ is the sum of the probabilities for $4f$, $4f+\gamma$, $4f+2\gamma$, etc., for soft photons — not amplitudes, because these are still distinct states in the theory. What "soft" means specifically depends on the experimental set-up — whatever the lower limit is on detecting the photons either indirectly, by their effect on the fermions through missing energy and momentum, or directly. There is always some such limitation to any experimental apparatus, because energy-momentum measurements of infinite precision are not possible without infinite amounts of time and space and zero thermal noise. Note also that the electric charge e_0 accompanying each real photon at its point of emission is exactly renormalized to the on-shell e once the photon self-energy is inserted into the radiated photon line, because these photons are real.

Working to a given order of α fixes the number of emitted photons, as Figure 15 makes clear. The differential probability for the emission of one soft photon from a radiation-less process with probability P is:

$$\frac{dP(1\gamma)}{d\omega}=\frac{1}{\omega}\cdot\frac{2\alpha}{\pi}[\ln(\frac{|q^2|}{m_f^2})-1]\cdot P(0\gamma), \qquad (III.11a)$$

where ω is the photon energy, for $|q^2|\gg m_f^2$. (III.11a) exhibits the typical $1/\omega$ form of the soft photon spectrum, and the typical large ultraviolet logarithm $\ln(|q^2|/m_f^2)-1$ for the high-energy case. The total probability integrated over photon energies from zero to the physical cutoff ω_c logarithmically diverges! This is an infrared divergence, for if we put in a small photon mass M_γ, the divergence is regulated:

$$\begin{aligned} P(1\gamma)&=\beta\cdot\ln(\omega_c/M_\gamma)\cdot P(0\gamma),\\ \beta&=\frac{2\alpha}{\pi}[\ln(|q^2|/m_f^2)-1]. \end{aligned} \qquad (III.11b)$$

This logarithmic form of infrared divergences occurs in all QED processes and, in the high-energy case, is accompanied by an additional ultraviolet logarithmic enhancement[27,28,50].

The term $P(0\gamma)$, once corrected with loops, contains IR-divergent vertices and boxes, and the two divergences — one due to massless virtual photons in loops, the other due to radiated massless photons, both emitted from the external legs — cancel, and M_γ disappears. The divergences cancel order by order in perturbation theory. They are due to the perturbative separation of virtual from real photon emission — the two processes actually run smoothly into each other as the loop photons approach their mass-shell. The cancellation of infrared divergences was first shown by Bloch and Nordsieck in 1937 for the special case of classical matter currents and quantized radiation[51], then extended to fully quantized electrodynamics by Yennie, Frautschi and Suura in 1961[50]. The QED result is in turn a special case of a general theorem due to Kinoshita, Lee and Nauenberg: for quantum field theories with massless particles, appropriate sums of probabilities over degenerate states — states degenerate but distinct because of the presence of an arbitrary number of soft massless particles — are infrared-finite[52].

In the case at hand, we need to disassemble the MIBA and compute the probability to the appropriate order:

$$|\mathcal{M}_{0\gamma}|^2 = |\mathcal{M}_0 + \mathcal{M}_{vertices} + \mathcal{M}_{boxes}|^2 \doteq |\mathcal{M}_0|^2 + 2\mathrm{Re}(\mathcal{M}_0)^*(\mathcal{M}_{vertices} + \mathcal{M}_{boxes}). \tag{III.12a}$$

On the other hand:

$$|\mathcal{M}_{1\gamma}|^2 \doteq |\mathcal{M}_{vertex\ Q}|^2 + |\mathcal{M}_{vertex\ Q'}|^2 + 2\mathrm{Re}(\mathcal{M}_{vertex\ Q})^*(\mathcal{M}_{vertex\ Q'}). \tag{III.12b}$$

The loop divergence of the vertex loop at Q, with the form Q^2, cancels the radiation divergence from the same vertex; the analogous cancellation occurs for the vertex at Q'; and the loop divergences of the boxes, with the form $Q \cdot Q'$, cancel the radiation divergences of the same form, the interference cross terms. Diagrammatically, this is intuitively clear from studying Figures 8 and 9, imagining that one cuts each virtual photon line in turn and matches it up with the corresponding radiation diagram in the square of Figure 15. Notice that the replacement of e_0^2 in the vertex and e_0^4 with appropriate renormalized couplings, as outlined in the last section, does not interfere with the cancellation of infrared divergences — in fact, such approximations take advantage of precisely the infrared properties of the vertex and box graphs, which are dominated by the photon mass-shell $l^2 = 0$ or the hard exchange momentum $l^2 = q^2$. The vertex radiation is exactly of $\mathcal{O}(e^2)$ and the interference radiation exactly of $\mathcal{O}(e^2 \cdot e_*^2(q^2))$, the same order as the leading higher-order corrections to the vertex and box loops!

The fact that the infrared divergences are artifacts of the separation of virtual and real photon emission suggests a different approach, treating both kinds of photons together. Staring at (III.11) carefully, we notice that while the photon emission probability and therefore the total number of emitted photons diverges, the total *energy* emitted does not:

$$E_{emitted} \sim \int d\omega\ \omega \cdot \frac{1}{\omega}.$$

This fact indicates that soft photon emission occurs in a semiclassical regime where photon number loses it meaning[28]. Photon number is not conserved in QED in any case. An alternative treatment could take advantage of this fact and ignore photons explicitly, calculating the effects of radiation solely in terms of the energy-momentum lost. The photon approach can be converted into a semiclassical regime by summing over photon number in the total probability. This sum in the soft limit has the form:

$$P = \{1 + \beta \ln(\omega_c/M_\gamma) + \frac{1}{2!}[\beta \ln(\omega_c/M_\gamma)]^2 + \cdots\} \cdot P(0\gamma);$$

because, in this limit, each photon emission is independent, and the contribution of n photons is divided by the Bose symmetry factor $n!$. The soft series sums to an exponential. The IR-divergences of the vertices and boxes can also be shown to exponentiate, so that the first-order divergence cancellation carries through to all orders[50]. The effect of radiation sums to a power form $(\omega_c^2/|q^2|)^\beta$. The corresponding differential spectrum thus behaves as $\omega^{\beta-1}$, rather than ω^{-1}, and has acquired an *anomalous dimension* β after resummation. Note that the probability of *no radiation* ($\omega \to 0$) is zero once β is summed into the exponent — a physically sensible answer — while the one-photon formula would have given a meaningless divergent answer. By carrying out this sum, we have freed the calculation of radiation from any dependence on photon number. We have also overcome the numerical inadequacy of perturbation theory for soft photons, which is actually an expansion in $\beta \ln(\omega_c^2/|q^2|)$ in the high-energy case[27]. The one-photon approximation is inadequate for radiation near or on resonances, such as the Z; the product of infrared and ultraviolet logarithms is large, and some higher-order emission is needed to make the calculation accurate[50]. This summation or exponentiation of radiation is our third extension of perturbation theory.

An elegant way to implement these ideas is through the use of *structure functions* that describe the spectrum of energy-momentum loss due to the emission of radiation and the effect of the virtual soft photons emitted in vertex loops. Starting from the simple version of this idea, first proposed by Williams and Weizsäcker for photon emission[53], Altarelli and Parisi developed a set of integro-differential evolution equations for the structure functions of quarks and gluons in QCD[54]. This modern gauge theory version has been successfully reimported into QED[55]. Structure functions are valid only for photon emission from vertices, where the same external fermion line runs continuously through the diagram. Such contributions are "partonic," in analogy with QCD. Diagrams not satisfying this criterion, such as box diagrams and the interference of radiation from two different fermion lines, are "non-partonic" and cannot be treated this way. Fortunately, the box/interference contributions are usually not important. Consider the initial vertex radiation in e^+e^- annihilation at $s = q^2$ (Figure 16). There is an arbitrary number of soft on-shell or virtual loop photons at the e^+e^- vertex. We only care about the total energy-momentum lost to radiated real photons and the effect that real and virtual photon emission has on the "hard" annihilation vertex into the virtual photon line q^2. By the time the fermions get to the annihilation vertex, the total four-momentum of the e^+e^- system has been modified by the loss of four-momentum to radiation. The radiation loss needs four variables for a complete description.* In the center-of-mass frame, the e^+ and e^- lie along a line described by their equal and opposite three-momenta: $p_- = (E; \mathbf{p})$, $p_+ = (E; -\mathbf{p})$, with $s = 4E^2$ and $p_\pm^2 = 0$. The loss to radiation is most simply written in terms of *collinear* reduced three-momenta: $\mathbf{p} \rightarrow x_-\mathbf{p}$ for the electron, $-\mathbf{p} \rightarrow -x_+\mathbf{p}$ for the positron; and the *transverse* momentum $\mathbf{p}_\perp$, where $\mathbf{p}_\perp \cdot \mathbf{p} = 0$. The effective annihilation energy is given by $s' = x_+x_-s - \mathbf{p}_\perp^2$, and the cross section with radiation is just a convolution of the radiation-less cross section with a structure function $F(x_+, x_-, \mathbf{p}_\perp)$:

$$\sigma_{e^+e^-}(s) = \int dx_+ dx_- d\mathbf{p}_\perp \; F(x_+, x_-, \mathbf{p}_\perp) \; \sigma_{e^+e^-}^{IBA}(s'), \qquad (III.13)$$

with s and $\mathbf{p}_\perp$ evaluated in the center-of-mass frame. In this approximation, the IBA can be used for the radiation-less cross section, the vertex corrections are included in F and the boxes have been ignored. The *factorization* of the soft photons shown in (III.13) is valid only for the infrared-like $\ln(\omega_c^2/|q^2|)$ partonic contributions and only if the masses of the external fermions are negligible, $m_f^2 \ll |q^2|$. The Altarelli-Parisi equations for F can be solved to a fixed order in α, or solved non-perturbatively, exponentiating the soft logarithms. For example, if we integrate out the transverse momentum, the purely infrared exponentiated part of F takes the form:

$$F(x_+, x_-) \doteq \beta(1 - x_+x_-)^{\beta-1},$$

where $s' = x_+x_-s$. This is a covariant generalization of the infrared-summed emission probability, with the same anomalous dimension.

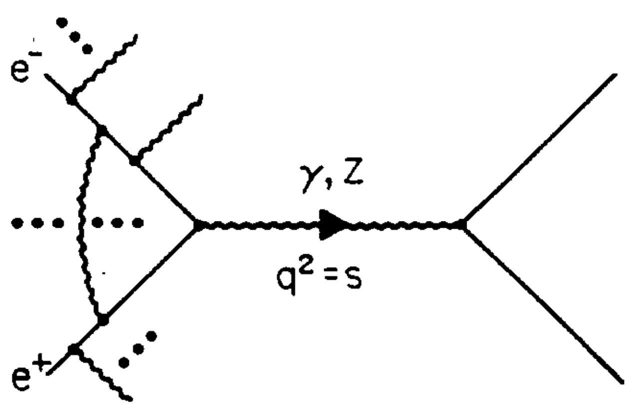

Figure 16. Arbitrary real and virtual photon emission from e^+e^- *annihilation vertex, summable via Altarelli-Parisi equations*[55].

* Note that the energy-momentum of the emitted radiation does *not* satisfy the dispersion relation of a single photon. The radiation really does require four variables, not three.

III.4 Putting It All Together – Higher-Order Corrections

IBA/radiation approximations, such as (III.12), are more than adequate for most purposes. The Born-like corrections are taken care of by using $e_*^2(q^2)$ with $q^2 = s'$, remembering that the effective kinematics flowing into the hard q^2 photon line is modified by radiation loss. The structure function approach can take care of the real and virtual emission of photons from one or both vertices, if $m_f^2 \ll |q^2|$. Fortunately, the box/interference terms are usually not important. Instead of (III.13), we do have the alternative of (III.12), computing the radiation and vertex and box loops in ordinary perturbation theory. Along the way, we encountered three extensions of perturbation theory: (1) the Dyson sum and the IBA, which includes Born-like corrections; (2) combining universal corrections and vertex loops into a single matrix element, the MIBA matrix element; and (3) the exponentiated form of the IR-like real and virtual photon emission. These extensions involve selective summation of certain corrections and require as inputs the irreducible parts, themselves a perturbative series in proper loops. The MIBA is not valid in QED under any circumstance, because it leaves the infrared divergences uncancelled. It must be replaced by either the order-by-order radiation calculation or, when valid, the structure function method. These extensions of perturbation theory are often necessary to sum large corrections typical of high-energy processes into a better-behaved form — in particular the large UV logarithms found in $e_*^2(q^2)$ and the large combined IR-UV logarithms of soft radiation.*

So far, we have only worked through one loop of the proper corrections. A full calculation would, of course, require all the higher-order terms as well[44]. Because QED and the EW SM are renormalizable and α is small, the higher-order terms as a practical matter fall into two classes: (1) corrections that can be put into a form that corrects the parameters occurring in one-loop expressions; and (2) corrections that cannot be put into such forms. An example of the first type of correction is the infinite series of fermion self-energy bubbles correcting the mass of a fermion already appearing in a one-loop photon self-energy (Figure 17). The fermion mass that should appear in the one-loop expression is the bare one; the subset of higher-order diagrams made of all the embedded fermion self-energies can be approximated very well by just changing the bare fermion mass to the physical one in the one-loop expression. We have already seen an example of this type of higher-order correction when we shifted the bare coupling e_0^2 to meaningful effective couplings in the vertex and box diagrams (see above). Note in this case that the form of the one-loop expressions is unchanged; only the parameters are changed, and the higher-order corrections are implicitly absorbed into the new parameters.

* These three extensions of perturbation theory really are *extensions* and not non-perturbative constructions, since they are derived from selective summation of the perturbative series. True non-perturbative effects, such as bound states (negative powers of α) or tunneling (exponentiated negative powers of α), are non-analytic in the coupling.

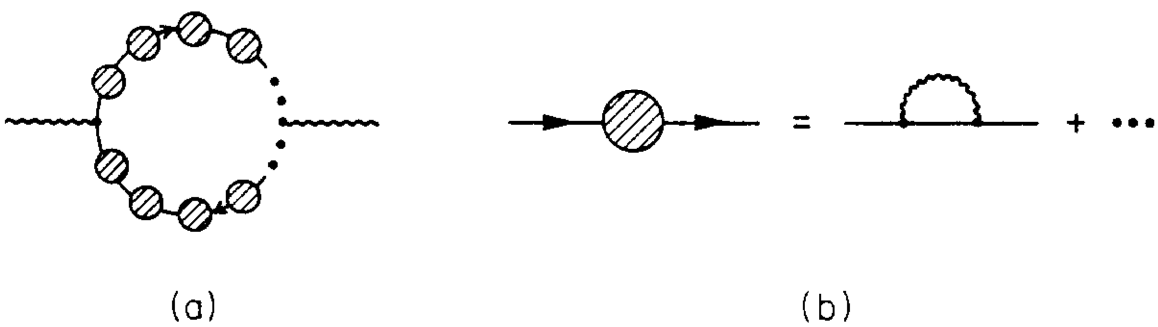

Figure 17. (a) Higher-order insertions of fermion self-energy in the photon self-energy. (b) Perturbative expansion of irreducible fermion self-energy.

In the second case, we cannot fold the higher-order terms into shifts of the parameters that occur in the one-loop terms. In this case, we have to express the corrections explicitly in the proper functions. A simple example is the two-loop vacuum polarization[28]:

$$\Pi_{\gamma\gamma} = q^2[e_0^2 \cdot {}^1\Pi'_{QQ}(q^2) + e_0^4 \cdot {}^2\Pi'_{QQ}(q^2)],$$

as in Figure 7a. Part of the two-loop correction falls into the first class, above, and can be absorbed by shifting the internal loop fermion masses. Some part is left over, however, as ${}^2\Pi'_{QQ}$. When inserted into the photon propagator, we obtain the function $e_*^2(q^2)$ to two loops. If we now renormalize, for example, using the QED on-shell scheme, we have to solve a more complicated equation for e_0^2 in terms of e^2 :

$$\frac{1}{e^2} = \frac{1}{e_0^2} - {}^1\Pi'_{QQ}(0) - e_0^2 \cdot {}^2\Pi'_{QQ}(0).$$

In practice, this is done by means of successive approximation; i.e., ignoring the two-loop term at first, finding e_0^2 to $\mathcal{O}(e^2)$, then substituting back in to find e_0^2 to $\mathcal{O}(e^4)$.

In any case, since we cannot carry out the perturbation theory to all orders, the practical approach is to develop some systematic and intuitive way of approximating the higher-order contributions so as to capture the dominant effects, without making the calculation impossibly complicated. The goal is to strike a compromise between simplicity (the one-loop expressions) and completeness (all orders). Fortunately, in QED and the EW SM, higher-order corrections of the second type are usually negligible, while corrections of the first type, while small, are often necessary to keep the calculations up to a reasonable numerical accuracy. "Reasonable" in this context means more than accurate enough for the current and next generation of experiments, just to be conservative. Experimental limitations and convergence of the series are the eventual arbiters of where exactly to stop worrying about higher-order terms.

The higher-order corrections of the second type (not of one-loop form) are numerically significant only in exceptional cases, occurring when terms otherwise suppressed by higher powers of α are systematically enhanced by large dimensionless factors. We have already met one example, the radiation corrections of the last section. Two more classic instances of this possibility are the large logarithms of the running couplings for grand unification, and the summation of ladder diagrams in QED bound states. In the GUT case[56], the UV logarithms that run the couplings are taken to very high energies, so that two-loop contributions to the gauge boson self-energies, while small $\mathcal{O}(\alpha^2 \ln(M_X^2/M_W^2))$, are not negligible with the precision available in present experiments. In the QED bound state case[27,28], the naive expectation that successive terms in the ladder expansion (Figure 18) are smaller by a factor of α is upset by the fact that each diagram is enhanced by a momentum exchange factor of $\mathcal{O}(1/\alpha)$. The appearance of inverse powers of α signals a non-perturbative bound state, and the momentum exchange between the constituents is fixed to be of $\mathcal{O}(\alpha m_f)$. This kind of non-perturbative effect is not important in high-energy EW SM processes.

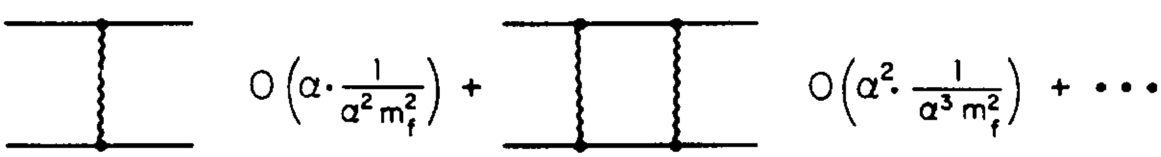

Figure 18. Ladder expansion of fermionic bound state Bethe-Salpeter equation, each term of order $\mathcal{O}(\alpha^{-1})$[27].

III.5 Renormalization Schemes – the $\overline{MS}$ Scheme

The error induced by not carrying out the perturbative expansion exactly and approximating the higher-order terms instead, usually goes under the name of *renormalization scheme-dependence* in the literature[22]. The reason follows from the last section: the important higher-order corrections in EW SM processes are usually the ones that effectively amount to replacing the bare parameters occurring in one-loop expressions by appropriate physical ones. This choice of parametrization is not unique, and is something of an art, but a good choice for any specific case is guided by the size of the coupling (how fast the series converges), the nature of the loops, the kinematical regime, and the experimental accuracy available.

Even the fundamental reparametrization of section II.4 needed already at one loop is merely a special case of this more general problem of reparametrizing the complete amplitude at higher orders. The reparametrization as a practical matter is always based on the renormalization of bare parameters by propagator corrections (section II.4) in a renormalizable theory, because of the special correspondence of bare parameters and divergent propagator corrections in such theories. At this point, it is important to distinguish carefully between *the order to which an amplitude is computed* and *the order to which the renormalized parameters that parametrize the amplitude are defined.* Given a physical process and the perturbative calculation carried to compute its amplitude, we are free to reparametrize that matrix element in any way we please *as long as the reparametrization is carried out consistently to or beyond the order to which the amplitude is computed.* On the other hand, it is certainly least painful to choose a renormalization scheme that makes the matrix element as simple as possible!

The paradigmatic example of the simplification available from the freedom to reparametrize in more than one way occurs in asymptotic QED[45, 57]. Considering only the Born-like corrections in this example, we know that $e_*^2(q^2)$ in the QED on-shell scheme runs with the UV logarithmic corrections of the form $\alpha \ln(q^2/m_i^2)$ from loops of mass m_i, neglecting the non-logarithmic terms $\mathcal{O}(m_i^2/q^2)$. The simple scaling properties of tree-level QED are violated by the presence of the loop masses m_i in the running. To compute $e_*^2(q^2)$ from e^2 requires knowing all the relevant loop masses. Suppose now that instead of parametrizing $e_*^2(q^2)$ in terms of the on-shell e^2, we use $e^2(\mu^2) \equiv e_*^2(\mu^2)$ at some $|\mu^2|, |q^2| \gg$ all m_i^2. Then $e_*^2(q^2)$, neglecting terms of $\mathcal{O}(m_i^2/q^2, m_i^2/\mu^2)$, runs in this new scheme with logarithms of the form $\alpha \ln(q^2/\mu^2)$. Simple scaling has reappeared, albeit in a new, non-trivial form: the running is now independent of the masses m_i of the loop particles. Asymptotic amplitudes simplify when we chose a better renormalization scheme. This scheme is usable if we then have a way to measure $e^2(\mu^2)$. In the asymptotic regime, once past the particle thresholds, QED becomes simple again, although not as simple as the tree-level or low-energy ($q^2 \to 0$) cases. Recall that the logarithmic asymptotic behavior is connected with the logarithmic divergences and these, finally, with the fact that the gauge coupling is dimensionless in four spacetime dimensions.

This asymptotic simplification to "leading logs only" suggests an alternative renormalization scheme, where the parameters have the logarithms built-in. In the context of dimensional regularization, this approach is known as the *modified minimal subtraction* or $\overline{MS}$ *scheme*[33]. To understand $\overline{MS}$, first notice that in $n \neq 4$ spacetime dimensions, gauge couplings are no longer dimensionless: $e_0 \to e_0 \cdot \kappa^\epsilon$, where $\epsilon = (4-n)/2$ as before, κ is a new parameter of mass dimension one, and e_0 has been redefined to be dimensionless. When we continue back to $n = 4$, the κ disappears in the classical theory. In the quantum theory, however, κ superficially remains in unrenormalized amplitudes. Noting that $e_*^2(q^2)$ in $n \neq 4$ dimensions is also dimensioned and inserting the correct factors of κ into (II.15):

$$\begin{aligned} \frac{1}{e_*^2(q^2)} &= \frac{1}{e_0^2} - \kappa^{2\epsilon}\Pi'_{QQ}(q^2) \\ &= \frac{1}{e_0^2} - [\sum_i C_i Q_i^2(\Delta + \ln\kappa^2 + finite\ terms)], \end{aligned} \tag{III.14}$$

where the displayed e_*^2 is redefined to be dimensionless and the $\epsilon \to 0$ expansion has been carried out. Notice now that along with each divergence in Π'_{QQ} an additional $\ln(\kappa^2)$ occurs: $\Pi'_{QQ} = C_i Q_i^2[\Delta + \ln(\kappa^2) + \cdots]$. The attentive reader will have noticed that the unrenormalized Π's contained logarithms of dimension-2 quantities all along. These logarithms are now properly dimensionless with κ present. Since after renormalization, only differences of logarithms remain in $e_*^2(q^2)$, the unphysical κ cancels. In order for this cancellation to hold in (III.14), e_0^2 itself must depend on κ! In terms of a "truly bare" coupling e_B, the bare coupling must read:

$$\frac{1}{e_0^2} = \frac{1}{e_B^2} + \sum_i C_i Q_i^2(\ln\kappa^2 - \ln m_i^2).$$

Note that e_0^2 then depends on *all* the particles carrying gauge charges Q_i.

The $\overline{MS}$ scheme takes advantage of the unphysical κ terms to simplify the parametrization of $e_*^2(q^2)$. First set $\kappa^2 = q^2$. Divide up all the particles i in loops into sets j and k, such that $m_j^2 < q^2$ and $m_k^2 > q^2$. Then define the $\overline{MS}$ coupling $\hat{e}^2(\kappa^2)$:

$$\frac{1}{\hat{e}^2(\kappa^2)} = \frac{1}{e_B^2} - \sum_i C_i Q_i^2\Delta + \sum_j C_j Q_j^2\ln(\kappa^2/m_j^2). \tag{III.15a}$$

Then $e_*^2(q^2)$ can be re-expressed in the $\overline{MS}$ scheme:

$$\frac{1}{e_*^2(q^2)} = \frac{1}{\hat{e}^2(\kappa^2)} - [\sum_j C_j Q_j^2 \mathcal{O}(m_j^2/q^2) + \sum_k C_k Q_k^2 \mathcal{O}(q^2/m_k^2)]. \tag{III.15b}$$

That is, $\hat{e}^2(\kappa^2)$ with $\kappa^2 = q^2$ absorbs the leading logarithms due to light particles

m_j, leaving only the subleading $\mathcal{O}(m_j^2/q^2)$ and heavy threshold $\mathcal{O}(q^2/m_k^2)$ terms. Denote the terms in brackets in (III.15b) by $\hat{\Pi}'_{QQ}(q^2)$, the self-energy subtracted in the $\overline{MS}$ scheme. The $\overline{MS}$ scheme is called *modified,* because in the original minimal subtraction (MS) scheme, only the $1/\epsilon$ parts were absorbed into the renormalized coupling, not the full $\Delta = 1/\epsilon - \gamma + \ln 4\pi$, as in (III.15a).

IV. Electroweak Gauge Theory

☞*...my point of view at the time was, 'This is the way that theories of the weak interactions must be.' But not, 'I know that SU(2)× U(1) is right.' I regarded that theory as illustrative, and I still – I can't get over the fact that it turned out to be right.*

– Steven Weinberg[58]

IV.1 Symmetry-Breaking – Vacuum Structure

The full electroweak theory[1,7,8,10] differs from QED by three features: (1) the gauge group SU(2)×U(1) is not simple; (2) the group is non-Abelian – that is, its generators are not all mutually commuting[5]; and (3) the gauge symmetry is spontaneously broken in the vacuum or ground state of the theory, although the gauge symmetry is still a good symmetry of the Lagrangian. The bare theory requires three parameters: the dimensionless gauge couplings g_0 and g_0' and, in the MSM, the VEV $\langle\phi\rangle_0 = v_0/\sqrt{2}$ of a single Higgs scalar $SU(2)_L$ doublet to set the mass scale. Extensions of the Higgs sector are possible and even desirable. Let us defer discussion of the non-Abelian aspects of the theory to the next section and consider the broken symmetry first.

The key point about *spontaneous* SB is that the full gauge symmetry, which is not manifest at low energies, can be broken by the ground state, while the dynamics in $\mathcal{L}_0$ remains gauge-invariant, renormalizable and unitary[12,17]. The chief requirement on the ground state is that it be electrically neutral, so as to preserve unbroken the U(1) of QED. The photon is then massless, and the long-range limit of electrodynamics is described by Maxwell's equations. Since $Q = I_3^L + Y$, the condition $Q|0\rangle = 0$ imposes a relation between the left isospin and the hypercharge of the VEV. For arbitrary Higgs multiplets Φ_i, recall from section I.1 that the bare gauge boson masses are related by:

$$M_{W_0}^2 = \rho_0 M_{Z_0}^2 \cos^2\theta_{W_0}, \qquad (IV.1)$$

where the bare "ρ parameter" is:

$$\rho_0 = 1 + \frac{\sum_i \langle \Phi_{i_0}^\dagger (\mathbf{I}_L^2 - 3{I_3^L}^2)\Phi_{i_0}\rangle}{\sum_i \langle \Phi_{i_0}^\dagger (2{I_3^L}^2)\Phi_{i_0}\rangle}. \qquad (IV.2)$$

The parameter ρ_0 is arbitrary in general, unmeasurable, and renormalized by the Higgs interactions. It is replaced after renormalization by an arbitrary tree-level parameter ρ. The EW SM gauge interactions then require four tree-level parameters for renormalization. Clearly, if all our VEV multiplets satisfy $\mathbf{I}_L^2 = 3{I_3^L}^2$, then $\rho_0 \equiv 1$. This holds, for example, if all the multiplets are $SU(2)_L$ doublets. In this case, ρ_0 is replaced after renormalization by a finite combination of radiative corrections, and there is no fourth independent tree-level parameter. As we shall see in section V.3, the measured $\rho \simeq 1$, so it is plausible to assume the deviation of ρ from unity is due to loop effects alone and that $\rho_0 \equiv 1$. The $\rho_0 \equiv 1$ case is the *Minimal Standard Model (MSM)*, and the arbitrary ρ case the *Extended Vacuum Standard Model (EVSM)*. The condition $\rho_0 \equiv 1$ does not have to be *ad hoc*, but is a consequence of a global symmetry that can be imposed on the Higgs VEV's; that is, a global symmetry that remains *after* the Higgses acquire VEV's and the gauge symmetry is broken.

The Higgs couplings to fermions (Yukawa couplings) are proportional to the fermion masses; in the limit of massless external fermions, the tree Higgs exchange is not important in four-fermion processes. The Higgs sector is important for us only in how the Higgs couples to the gauge bosons. This determines how the gauge bosons acquire their masses from the static VEV's and the ρ parameter. The Higgses are not important dynamically, except as they occur in the gauge boson self-energy loops. In the R_ξ gauge, the spontaneous SB requires also the inclusion of the would-be Goldstone bosons in intermediate states, although they are not physical and never appear as external legs[8,28].

The major interest in radiative corrections lies in the gauge boson self-energies that occur universally in all gauge-exchange interactions.* From section I.4, recall that there are three physical gauge currents, associated with the photon, Z, and the W; but that these can be broken down into left isospin and electric charge currents. Define $\Pi_{AB} = \langle T[J_A J_B]\rangle$ as shorthand for self-energy functions. The four independent physical self-energy functions are:

$$\begin{aligned}
\Pi_{ZZ} &= \frac{e_0^2}{s_0^2 c_0^2}[\Pi_{33} - 2s_0^2\Pi_{3Q} + s_0^4\Pi_{QQ}] \\
\Pi_{Z\gamma} &= \frac{e_0^2}{s_0 c_0}[\Pi_{3Q} - s_0^2\Pi_{QQ}] \\
\Pi_{\gamma\gamma} &= e_0^2\Pi_{QQ} \\
\Pi_{WW} &= \frac{e_0^2}{s_0^2}\frac{[\Pi_{11} + \Pi_{22}]}{2} = \frac{e_0^2}{s_0^2}\frac{[\Pi_{+-} + \Pi_{-+}]}{2} = \frac{e_0^2}{s_0^2}\Pi_{\pm};
\end{aligned} \qquad (IV.3)$$

* In much of the literature, the Born-like electroweak radiative corrections are referred to as "oblique" corrections and the non-Born-like as "direct," following ref. 59. In the limit of zero external fermion masses, the relevant gauge boson self-energies are proportional to $g_{\mu\nu}$.

and since $\Pi_{11} = \Pi_{22}$, $\Pi_{-+} = \Pi_{+-} \equiv \Pi_{\pm}$ by the residual isospin symmetry left after SB, there are four equivalent simple reduced self-energies: $\Pi_{QQ}, \Pi_{3Q}, \Pi_{33}, \Pi_{\pm}$. As $Q = I_3^L + Y$ and $J_Q = J_3^L + J_Y$, we could use Π_{3Y} and Π_{YY} instead of Π_{QQ} and Π_{3Q}. But because J_Q^μ is a conserved current and $U(1)_Q$ unbroken, the latter set is more convenient. The complete set of neutral- and charged-current Dyson's equations, with the tree-level propagators and Π's as inputs, are written out in Appendix A[39,59,63].

The tree-level consequence of SB is that the bare gauge boson masses are not zero. The breaking has a further consequence at loop level; namely, that the Ward identities of the gauge theory, called *Slavnov-Taylor identities* [60,61] in non-Abelian cases, are broken. A new set of modified Slavnov-Taylor identities emerges, whose form is controlled by the gauge-invariant couplings and the exact pattern of symmetry-breaking. Since left isospin is completely broken, the Slavnov-Taylor identity corresponding to the *second* QED Ward identity — that $\Pi_{QQ}(q^2 = 0) = 0$ — is changed to so that $\Pi_{33}(0)$ and $\Pi_{\pm}(0) \neq 0$. This causes no problem with the propagato rs since the corresponding tree-level Z and W masses are not zero to start with! On the other hand, $\Pi_{QQ}(0) = 0$ still holds, as QED is unbroken. (Π_{3Q} almost has this property, except for a subtle peculiarity discussed in the next section.) Roughly speaking, the "broken" self-energies take the form $\Pi \sim q^2 \cdot A(q^2) + M_i^2 \cdot B(q^2)$, where M_i is a loop mass and the functions A, B are dimensionless. Their divergences continue to be exclusively logarithmic. The functions $A(q^2)$, as in QED, run the gauge couplings, while the functions $B(q^2)$ shift the gauge boson masses. The terms $M_i^2 \cdot B(q^2)$ appear in the Π's *if and only if* the mass of the loop particle M_i arose from the *same* Higgs mechanism that breaks $SU(2)_L \times U(1)_Y$. That is, $M_i \sim \lambda \times v$, where λ is a generic mass-generating coupling to the Higgs sector. Actually, this condition is necessary but not sufficient for these terms to have physical effects; the precise requirements are spelled out in section IV.3. Nevertheless, these terms open a new world of "non-decoupling" radiative corrections[19, 59], corrections that do not respect the decoupling theorems of unbroken gauge theories and form a major aspect of weak interaction physics. "Heavy" physics, above the Z or whatever the exact point we reach, can have effects on low-energy processes, and low-energy measurements can place limits on particles too heavy to produce directly.

IV.2 Non-Abelian Properties – Running Couplings

To start our analysis of the non-Abelian properties of the EW SM gauge interactions, it is useful to revert to the original $SU(2)_L$ and $U(1)_Y$ gauge couplings, g_0 and g_0', for the moment. From the properties of QED, we should expect that the bare couplings are replaced in the Born-like corrections by effective running couplings, $g_*^2(q^2)$ and $g'^2_*(q^2)$, driven by the gauge boson self-energy functions. This expectation is fulfilled, but the procedure outlined in sections II.4/III.1 for QED suffers some essential modifications, because of the symmetry-breaking and because of the non-Abelian properties of the theory[39,62].

The key difference between Abelian and non-Abelian gauge theories is that the gauge bosons of the latter themselves carry charge under the gauge group. This feature leads to gauge boson self-interactions, something that cannot occur in QED, because the photon is neutral[5,7,8, 10,28]. In mathematical terms, the generators of the gauge group T^a (in any representation, with adjoint index a) do not commute with themselves: $[T^a, T^b] = if^{abc}T^c$ in general, where the antisymmetric f^{abc} are called the *structure constants* of the group and are identically zero in the Abelian case. This "non-Abelian-ness" changes the properties of gauge theories in many ways. Of specific interest here is the appearance of gauge boson loops in the gauge boson self-energies (Figure 19). Their presence in the expansion requires the addition of the *Fadeev-Popov ghosts* (Figure 19c) for a consistent quantization[5,8,28,60,61].

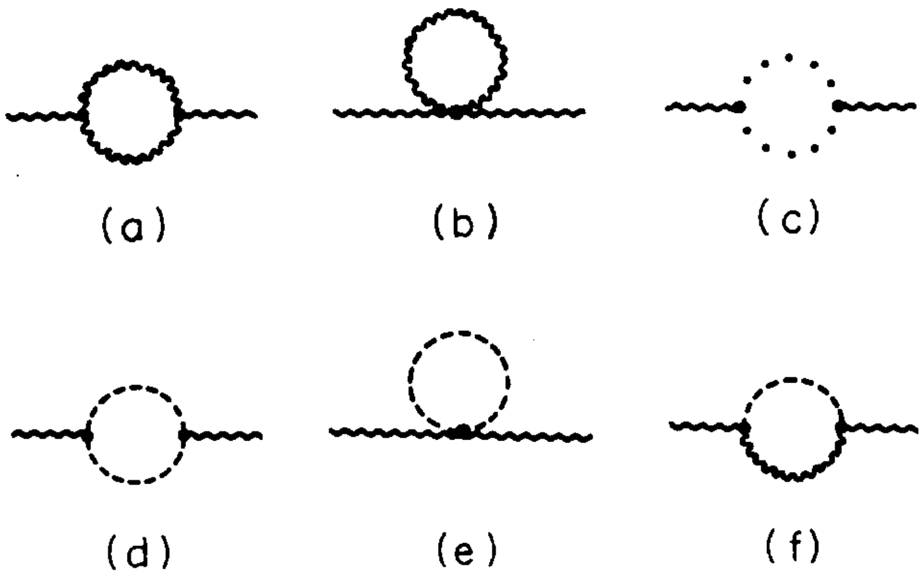

Figure 19. Non-Abelian contributions to the gauge boson self-energies: (a,b) gauge boson loops; (c) Fadeev-Popov ghost loops; (d,e,f): would-be Goldstone and Higgs boson loops[8,28].

The contributions of scalar and fermion loops to the gauge boson self-energies are gauge-invariant (ξ-independent), for each multiplet separately. The contribution of the gauge boson loops (including the ghosts and would-be Goldstones) to the gauge boson self-energies is *not* gauge-invariant. This contribution is proportional to the adjoint Casimir operator $C(V)\delta^{ad} = f^{abc}f^{dbc}$. Because the gauge charges for any particles transforming under the group (for example, the external fermions in our processes) do not commute, new terms with no QED analogue appear in the vertex and box corrections[39]. In our case, two generic "non-Abelian" contributions appear in the vertices and one in the boxes. They can be isolated by writing out the products of gauge generators (fermion charges) running along a continuous fermion line

(Figure 20), re-expressing the pairwise products as commutators and anticommutators, then simplifying by using the symmetry properties of the (anti)commutators. The "non-Abelian" parts of vertices and boxes are proportional to at least one gauge generator commutator and have no analogue in QED; while the remaining "Abelian" vertex and box corrections are proportional to only anticommutators and are just generalizations of the vertex and box corrections of QED, which are proportional to products of the external fermion electric charges. These Abelian vertex and box loops are separately gauge-invariant, exactly as in QED; the non-Abelian corrections are not, but they are proportional to $C(V)$. If we avoided carrying out a Dyson sum, we could just add up the non-Abelian self-energy, vertex and box loops to lowest order and discover that the sum proportional to $C(V)$ *is* gauge-invariant. A related feature of these non-Abelian terms is their group transformation properties. All of the non-Abelian vertex and part of the non-Abelian box terms are Born-like, reproducing the Born Lorentz/gauge current-current form, with a single gauge generator at each vertex and the correct kinematical dependence. Inserted into the matrix elements, these non-Abelian terms really belong with the propagator corrections[63].

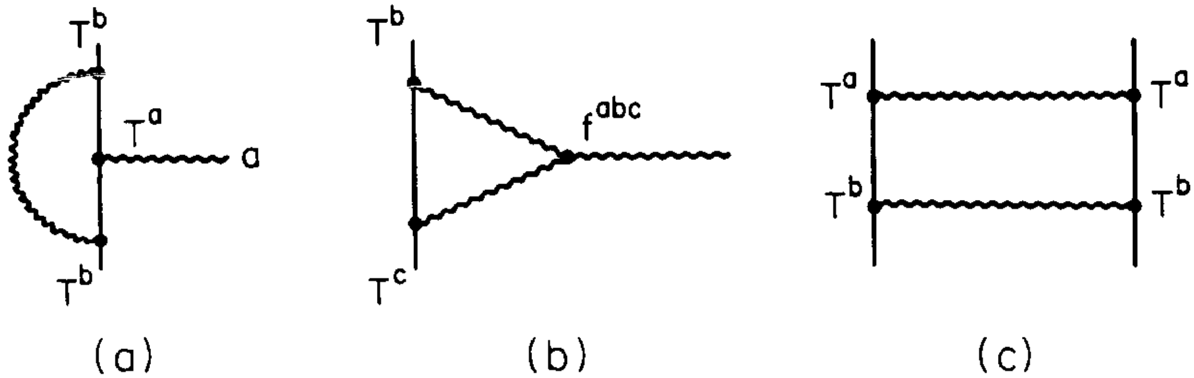

Figure 20. Vertex and box graphs with Abelian contributions (a,c) and non-Abelian (a,b,c) contributions[39].

Related to the gauge non-invariance of the Π's is the presence of new divergences and leading logarithms in the non-Abelian vertex terms. The non-Abelian vertex terms Γ^{nAb} do not satisfy the *first* QED Ward identity, meaning that the non-Abelian gauge coupling undergoes an "extra" renormalization in addition to the one generated by the self-energies. A purely renormalization group analysis picks up these divergences and leading logarithms from the corresponding counterterms[8, 28]. The parallel technique in the S-matrix approach starts with defining new, gauge-invariant kernels for the Dyson equations. This step can be stated in terms of a set of gauge-invariant effective self-energies Π^*, which are the appropriate combinations of self-energies, non-Abelian vertices and non-Abelian boxes. The result of solving the new Dyson equations is then a set of gauge-invariant effective propagators that make up a gauge-invariant electroweak IBA. This IBA can be re-expressed using a system of star functions analogous to $e_*^2(q^2)$ for QED gauge interactions[62]. Non-Born-like corrections, such as the Abelian vertices and boxes, are grouped separately in the matrix elements[64].

The unavoidable drawback of this procedure is that the Born-like star system varies depending on what types of particles are attached to the external legs of the process. For example, two-fermion/two-gauge boson processes (such as $e^+e^- \to W^+W^-$) have a set of effective gauge boson propagators different from those in four-fermion processes, because the vertices and boxes take a different form in this case with gauge bosons on the external legs. However, gauge invariance and renormalizability guarantee that we can always construct effective propagators from one class of processes to another that differ only by gauge-invariant, non-divergent, non-leading logarithmic terms. Even within the class of four-fermion processes, we have some freedom in what to put in the effective self-energies, as long as they end up gauge invariant[62]. Some residual non-Abelian corrections remain in the non-Born-like part of the matrix element, unabsorbed into the effective propagator. In four-fermion processes, an additional constraint this procedure must satisfy is that whatever residual non-Abelian vertex is left in the neutral-current matrix element vanish at $q^2 = 0$, so that the full EW matrix element precisely reproduces the QED matrix element in the $q^2 \to 0$ limit and the electric charge suffers no new renormalization on the photon shell. The assembly of the effective gauge boson propagators and the corresponding system of star functions for the IBA is a distinct step in perturbation theory prior to renormalization; let me call this choice a *propagator construction.* A natural construction that respects the necessary constraints is outlined in Appendix A.

Using the gauge-invariant effective self-energies Π^*, we can define the two running gauge couplings $e_*^2(q^2)$ and $g_*^2(q^2)$:

$$\begin{aligned}
\frac{1}{e_*^2(q^2)} &= \frac{1}{e_0^2} - \Pi^{*\prime}{}_{QQ}(q^2),\\
\frac{1}{g_*^2(q^2)} &= \frac{1}{g_0^2} - \Pi^{*\prime}{}_{3Q}(q^2),
\end{aligned} \qquad (IV.4)$$

with $s_*^2(q^2) = e_*^2(q^2)/g_*^2(q^2)$. $\Pi^*_{QQ}(q^2)$ and $\Pi^*_{3Q}(q^2)$ enjoy the property of being divisible by q^2 and therefore respect the decoupling theorem. This should come as no surprise since the electromagnetic gauge current J_Q^μ is still exactly conserved.* If we run a gauge coupling from μ^2 to q^2, say, the effect of particles with masses $m_i^2 \ll |q^2|, |\mu^2|$,

* The properties of Π^*_{3Q} are somewhat subtle. $\Pi_{3Q}(0) \neq 0$, in spite of the fact that J_Q^μ is conserved[59]. Now $\Pi^*_{3Q} = \Pi_{3Q}$ plus some other terms. These other terms are not unique, but

decouples from the relation between these two couplings as $(q^2,\mu^2)/m_i^2$. Thus the *running* of $e_*^2(q^2)$ and $g_*^2(q^2)$ respects decoupling, although the *normalization* of a coupling at a given q^2 may have non-decoupled corrections if the coupling is computed from some other quantity. For example, the canonical reference point for $s_*^2(q^2)$ is the Z pole, $q^2 = M_Z^2$. Given $s_*^2(Z)$, we can compute $s_*^2(0)$ from:

$$\begin{aligned} \frac{1}{e_*^2(0)} - \frac{1}{e_*^2(Z)} &= -\Pi^{*\prime}{}_{QQ}(0) + \Pi^{*\prime}{}_{QQ}(Z) \equiv +\Delta_Q(Z), \\ \frac{1}{g_*^2(0)} - \frac{1}{g_*^2(Z)} &= -\Pi^{*\prime}{}_{3Q}(0) + \Pi^{*\prime}{}_{3Q}(Z) \equiv -\Delta_{3Q}(0), \end{aligned} \tag{IV.5a}$$

where two finite functions have been introduced:

$$\begin{aligned} \Delta_Q(q^2) &= \Pi^{*\prime}{}_{QQ}(q^2) - \Pi^{*\prime}{}_{QQ}(0), \\ \Delta_{3Q}(q^2) &= \Pi^{*\prime}{}_{3Q}(q^2) - \Pi^{*\prime}{}_{3Q}(Z), \end{aligned} \tag{IV.5b}$$

containing leading logarithms for particles of mass $m_i \ll M_Z$ and threshold terms $\sim M_Z^2/m_i^2$ for particles of mass $m_i \gtrsim M_Z$. These functions observe decoupling. Their leading logarithms have the coefficients:

$$\begin{aligned} QQ &: -\frac{1}{48\pi^2}(11N - \frac{32}{3}N_f - 1), \\ 3Q &: -\frac{1}{48\pi^2}(11N - 4N_f - \frac{1}{2}), \end{aligned} \tag{IV.5c}$$

above all thresholds; $N = 2$ for SU(2), N_f is the number of fermion families, and the last constants are due to the single Higgs doublet. The simplest determination of $s_*^2(Z)$ itself is from M_Z :

$$s_*^2(Z)c_*^2(Z) = \frac{e_*^2(Z)}{4\sqrt{2}G_F M_Z^2}[1 + other\ terms], \tag{IV.6}$$

which leads to a soluble quadratic equation for $s_*^2(Z)$ when $c_*^2(Z) = 1 - s_*^2(Z)$ is inserted. The "other terms" contain heavy, non-decoupled contributions. This is

---

depend on the propagator construction. However, if we choose a gauge-invariant construction that satisfies the sensible Ward identity constraint; i.e., that the residual $\Gamma_3^{nAb}(q^2)$ left over in the non-Born-like corrections vanish at $q^2 = 0$, then $\Pi_{3Q}^*(0) = 0$ automatically. It is important that this condition hold, so that a sensible definition of $g_*^2(q^2)$, such as (IV.4), can be made by dividing $\Pi_{3Q}^*(q^2)$ by $q^{\cdot}$ Otherwise, a new pole of the form $\Pi_{3Q}^*(0)/q^2$ would show up in the matrix elements as $q^2 \to 0$. Notice how the condition of "no new charge renormalization at $q^2 = 0$" implies "no new gauge boson poles at $q^2 = 0$" - the uniqueness of e^2 implies the uniqueness of the photon and vice versa.

because M_Z is a consequence of symmetry-breaking; expressing $s_*^2(Z)$ in terms of M_Z then automatically introduces corrections that do not respect decoupling. Note that $e_*^2(Z)$ does observe decoupling, as QED is unbroken; computed in terms of $e^2 = e_*^2(0)$, $e_*^2(Z)$ requires loop contributions only from particles between $q^2 = 0$ and $q^2 = M_Z^2$, up to the effect of unknown particles suppressed by inverse powers of their masses.

IV.3 Global Symmetries and Non-Decoupling

☞*'All right,' said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone. 'Well! I've often seen a cat without a grin,' thought Alice; 'but a grin without a cat! It's the most curious thing I ever saw in all my life!'*

– Lewis Carroll, *Alice in Wonderland*

We have already seen one necessary condition for the non-decoupling of heavy particles in loops: the local gauge symmetry must be broken, allowing new terms in the gauge boson self-energies proportional to squares of loop masses M_i^2. The loop masses in question furthermore have to be generated by the same Higgs mechanism that spontaneously breaks the gauge symmetry. The leading relative effects of these corrections then take the form:

$$\begin{aligned} \frac{\Pi^*}{M_{gauge}^2} &\sim g^2 \cdot \frac{(\lambda \times v)^2}{(g \times v)^2} \\ &= \lambda^2. \end{aligned} \qquad (IV.7)$$

Once the gauge boson masses are taken into account, the non-decoupled effect of the heavy particles in gauge boson self-energies is actually a function of the heavy mass-generating coupling alone and is not connected with gauge interactions at all. The larger this dimensionless coupling, the larger the effect of the heavy particle loop[19,59].

We can understand this fact by remembering that the extra degree of freedom each gauge boson acquires along with its mass comes from a would-be Goldstone boson eaten up from the Higgs sector. The properties of this longitudinal gauge degree of freedom associated with the gauge boson mass are essentially the properties of the would-be Goldstone somewhat disguised by its mixing with the true transverse gauge degrees of freedom[65]. Now the Goldstones, being scalars, would have had quadratic divergences in their self-energies, and their self-energies would have been proportional to powers of the Higgs couplings with heavy particles that endow those particles with $SU(2)_L \times U(1)_Y$-breaking masses. Because of the underlying gauge symmetry, the gauge boson self-energies have only logarithmic divergences, and the quadratically divergent self-energies are in effect cutoff at the heavy particle mass. Hence the gauge boson self-energies $\propto M_i^2$. It is also clear why, for such non-decoupling effects to appear in the Π^*'s, the masses of the heavy loop particles must come from the same $SU(2)_L \times U(1)_Y$-breaking Higgs sector as the gauge boson masses: otherwise, these corrections could never mix with the gauge boson masses in the first place.

However, this condition is necessary but not sufficient for a physical effect. In general, the non-decoupled terms are absorbed upon renormalization of parameters and their effect erased from observable relationships. The loop effects must violate relationships assumed to be valid at tree level, so that any deviation from these relationships in measured data can be attributed to the presence of radiative corrections. If the deviations are small, it is plausible to assume that they are perturbatively calculable loop effects; turning the argument around means that experimental limits on these deviations translate into limits on non-decoupled loop effects. The only way to have fixed relationships valid at tree level without fine-tuning, in turn, is to assume that these relationships are consequences of a global symmetry respected by the gauge interactions at tree level, violated by some other sector of the EW SM, and thus violated in the gauge interactions at loop level by the gauge boson coupling to that other sector. The global symmetry-violating sector must contribute to the gauge boson self-energies. On the other hand, sectors whose masses violate the global symmetry but not the gauge symmetry still decouple in their loop effects from the gauge sector[66].

The necessity for a good tree-level global symmetry that is then violated by loop effects is not unique to the processes we are considering in these lectures, flavor-conserving electroweak interactions. Probably the best-known example in the EW SM is the Glashow-Iliopoulos-Maiani (GIM) family symmetry, which prevents flavor-changing neutral currents (FCNC's) in the gauge sector; i.e., processes such as $e^+e^- \to \mu^+e^-$, or strangeness-violating $K^0 - \bar{K}^0$ mixing[7,8,25, 67]. Some FCNC's actually do occur experimentally, but at rates suppressed relative to other weak interactions. The EW SM has a natural mechanism for small but non-zero FCNC's. The GIM family symmetry rotates all the up-type quarks into each other, and analogously for the down-type quarks:

$$\begin{array}{c} \underbrace{u \quad c \quad t} \\ \\ \overbrace{d \quad s \quad b} \end{array} \tag{IV.8}$$

forcing all neutral currents to be flavor-diagonal. The symmetry is respected by the gauge sector, but violated by the Higgs-quark sector: the Yukawa couplings to fermions are different for each fermion, and so we expect loop-level flavor-changing effects (such as the box diagrams of Figure 21) to have strength $\mathcal{O}(g^4 m_q^2 m_{q'}^2 / M_W^4)$; that is, $\mathcal{O}(G_F^2 m_q^4)$. The FCNC terms are finite in the SM, because the logarithmic divergences generated by the gauge sector still respect the GIM symmetry (the divergences are mass-independent) and cancel each other when summed over by the unitarity of GIM family rotations, leaving the finite mass-dependent terms. This example illustrates the importance of the underlying global symmetry in cancelling unrenormalizable divergences (Figure 21 is an unrenormalizable $dim = 6$ operator), even while it is still violated by finite broken symmetry effects. The lectures of Nir and Georgi in this school have more about the GIM symmetry and FCNC's. Here we just note that the pattern "good tree-level global symmetry, violated by finite radiative effects" is a general one and applies to other than just gauge theories[68].

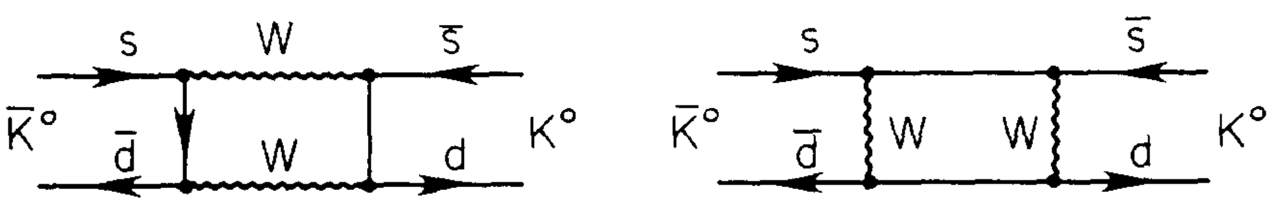

Figure 21. Prototypical GIM-violating flavor-changing neutral current in Standard Model: $\mathcal{O}(G_F^2 m_q^4)$ $K^0 - \bar{K}^0$ *mixing at one loop*[8].

The relevant global symmetry for flavor-conserving interactions must operate independently on each family. This symmetry is $G = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_\Lambda$, of which the subgroup $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is gauged[69]. The $\mathrm{SU}(2)_R$ group is the parity mirror of $\mathrm{SU}(2)_L$ and completes the weak chirality symmetry group. For ordinary fermions, L and R refer to the left and right Weyl states, and $\Lambda = (B - L)/2$, where B and L are the vector-like ordinary baryon and lepton number, respectively. The weak hypercharge is $Y = I_3^R + \Lambda$, while the electric charge is $Q = I_3^L + I_3^R + \Lambda$: electrodynamics is parity-conserving. The $L \times R$ subgroup of G has as its maximal subgroup the combined $\mathrm{SU}(2)_{L+R}$. Let $\mathbf{I}_V = \mathbf{I}_L + \mathbf{I}_R$ and $\mathbf{I}_A = \mathbf{I}_L - \mathbf{I}_R$, with:

$$
\begin{aligned}
[I_i^V, I_j^V] &= i\epsilon_{ijk} I_k^V , \\
[I_i^V, I_j^A] &= i\epsilon_{ijk} I_k^A , \qquad (IV.9) \\
[I_i^A, I_j^A] &= i\epsilon_{ijk} I_k^V .
\end{aligned}
$$

Because it is not closed, the $\mathbf{I}_A$ algebra generates a coset of G, not a group. The $\mathrm{SU}(2)_{L+R} \equiv \mathrm{SU}(2)_V$ subgroup of G is usually called the weak *custodial* or *vector* *isospin*[19,69]. It is the analogue of the total rotation group for the addition of two angular momenta $\mathbf{I}_L$ and $\mathbf{I}_R$, where $\mathbf{I}_V$ is the total angular momentum. Both the full group G and the subgroup $\mathrm{SU}(2)_V \times \mathrm{U}(1)_\Lambda$ have their own complete Hilbert space bases, the set $|i_L m_L i_R m_R\rangle_\Lambda$ and the set $|i_V m_V i_L i_R\rangle_\Lambda$, respectively, related by Clebsch-Gordon coefficients. All electroweak multiplets have definite G quantum numbers or can be decomposed into irreducible representations (irreps) of G, the $|i_L m_L i_R m_R\rangle_\Lambda$ states. Denote a multiplet's G quantum numbers by $(2i_L+1, 2i_R+1)_\Lambda$. Any multiplet can then be rewritten in terms of the $\mathrm{SU}(2)_V$ states $|i_V m_V i_L i_R\rangle_\Lambda$. Denote the $\mathrm{SU}(2)_V$ quantum number $2i_V+1$ in boldface. That is, an $\mathrm{SU}(2)_V$ singlet is a $\mathbf{1}$, a doublet is a $\mathbf{2}$, etc. The Wigner-Eckart theorem tells us that the operator $\mathbf{I}_V$ is the only independent vector ($\mathbf{3}$) operator, and all other vector operators must therefore be proportional to $\mathbf{I}_V$. From the Lie algebra is easy to see that the operators $\mathbf{I}_L, \mathbf{I}_R$ transform as $\mathrm{SU}(2)_V$ triplets; therefore, $\mathbf{I}_{L,R} = a_{L,R}\mathbf{I}_V$ when acting on definite $\mathrm{SU}(2)_V$ $|i_V m_V i_L i_R\rangle_\Lambda$ states[66,70].

The $\mathrm{SU}(2)_V$ symmetry group gives us a deeper grasp on the meaning of the ρ parameter. The minimal Higgs doublet of the MSM is not only a doublet under $\mathrm{SU}(2)_L$, but under $\mathrm{SU}(2)_R$ as well: its G quantum numbers are $(2,2)_0$. The parallel L and R doublet transformations of the minimal Higgs field can best be seen by arranging Φ and its conjugate Φ^c into a 2×2 matrix:

$$H = (\Phi \; \Phi^c), \qquad \Phi^c = i\sigma_2 \Phi^*, \tag{IV.10}$$

where the $\mathrm{SU}(2)_L$ rotations work vertically and the $\mathrm{SU}(2)_R$ horizontally. The ungauged Higgs Lagrangian is G-invariant:

$$\mathcal{L}_{\mathrm{Higgs}} = \frac{1}{2}(\partial_\mu H)^\dagger(\partial^\mu H) - \frac{1}{4}\lambda_H (H^\dagger H - v^2/2)^2. \tag{IV.11}$$

The minimal Higgs $(2,2)_0$ decomposes into $\mathbf{1}\oplus\mathbf{3}$ under $\mathrm{SU}(2)_V$. When the Higgs acquires its VEV, it is the $\mathbf{1}$ part, not the $\mathbf{3}$, that does so: $\langle H\rangle$ is proportional to the identity. The VEV has broken the good global symmetry from G to $\mathrm{SU}(2)_V \times \mathrm{U}(1)_\Lambda$, and it is the remaining good $\mathrm{SU}(2)_V$ symmetry that forces $\rho_0 \equiv 1$. The condition for the MSM then is ***that the vacuum respect SU(2)$_V$***; the minimal doublet model with the $(2,2)_0$ Higgs is merely the simplest way to implement this symmetry. Now switch on the $\mathrm{SU}(2)_L$ gauge interactions. Gauging $\mathrm{SU}(2)_L$ does not break this group; it continues to be good global symmetry, and G is unbroken in $\mathcal{L}_{\mathrm{Higgs}}$, although not in the vacuum. But now switch on the $\mathrm{U}(1)_Y$ gauge interactions. This gauges the subgroup of $\mathrm{SU}(2)_R$ generated by I_3^R without gauging the whole group, and therefore explicitly – in the Lagrangian – breaks G and the subgroup $\mathrm{SU}(2)_V$. Switching on hypercharge $g' \neq 0$ means $\sin\theta_W = g'/\sqrt{g^2 + g'^2} \neq 0$ and $M_W \neq M_Z$. If the VEV's respected $SU(2)_V$ before hypercharge is gauged, they continue to do so, and $\rho_0 = 1$ still, at tree level. But we should expect some loop effects involving $M_W \neq M_Z$ to radiatively break the $\rho = 1$ condition after renormalization[66,70].

The non-decoupling effects in flavor-conserving gauge interactions can thus break the full G or the subgroup $SU(2)_V$. Now the full G is broken already by the Higgs VEV at tree level, but this is a static or q^2-independent breaking. This breaking is given by the VEV v (or multiple VEV's); equivalently by G_F, measured at $q^2 = 0$ in beta decay. G-breaking loop effects can still appear as dynamical (q^2-dependent) effects and escape being absorbed into the renormalization of G_F. Furthermore, if $SU(2)_V$ is a good symmetry of the vacuum, then there can also be static and dynamical loop effects breaking this subgroup. Hence, in the MSM case, there should be three distinct ways in which non-decoupling enters through loop corrections. In the EVSM case, where $SU(2)_V$ is not a good symmetry of the vacuum and ρ_0 is arbitrary, one of these loop effects (the static breaking of $SU(2)_V$) is replaced by a tree-level ρ parameter, leaving only two general non-decoupled loop effects[66].

The two gauge boson self-energies exhibiting non-decoupling effects are Π^*_{33} and Π^*_{11}. It is not hard to prove that the three independent functions:

$$\begin{aligned}
\Delta_\rho(q^2) &= \Pi^*_\pm(q^2) - \Pi^*_{33}(q^2), \\
\Delta_3(q^2) &= \Pi^*_{33}(0) + \Pi^*_{3Q}(q^2) - \Pi^*_{33}(q^2), \\
\Delta_\pm(q^2) &= \Pi^*_\pm(0) + \Pi^*_{3Q}(q^2) - \Pi^*_\pm(q^2),
\end{aligned} \qquad (IV.12)$$

or some equivalent set, exhaust the possible finite combinations of loop functions involving Π^*_{33} and Π^*_{11}, in the MSM. $\Delta_3(0) = \Delta_\pm(0) \equiv 0$ is the result of using G_F measured at $q^2 = 0$ as an input, subtracting Π^*_{33} and $\Pi^*_\pm$ at $q^2 = 0$. This step renders the parts of $\Pi^*_{33,\pm}$ proportional to M_i^2 finite. The subtraction of Π^*_{3Q} in both cases arises from its use in the running of the $SU(2)_L$ coupling $g_*^2(q^2)$. Now the divergences in Π^*_{33} must equal those in $\Pi^*_\pm$ by the underlying $SU(2)_L$ gauge symmetry, which is still respected by the dynamics of the theory. Furthermore, the divergences in the parts of $\Pi^*_{33} \propto q^2$ must equal those in Π^*_{3Q}. This is because the divergences are logarithmic, independent of the loop masses and $\propto Tr(I_3^L I_3^L)$ over each multiplet in one case, $\propto Tr(I_3^L Q)$ in the other; then $Tr(I_3^L Q) = Tr(I_3^L I_3^L) + Tr(I_3^L Y)$, and the second term is zero because Y is constant and the SU(2))L generators traceless for each multiplet. In the EVSM, only $\Delta_3(q^2)$ and $\Delta_\pm(q^2)$ are independent heavy particle functions – two as promised – with $\Delta_\rho(q^2)$ redundant:

$$\Delta_\rho(q^2) = \Delta_3(q^2) - \rho^{-1}\Delta_\pm(q^2). \qquad (IV.13)$$

ρ then replaces $\Delta_\rho(0)$ as the static $SU(2)_V$-breaking parameter[19,39,59,66,70].

In the MSM, we can use the identity $\Delta_\rho(q^2) - \Delta_\rho(0) = \Delta_3(q^2) - \Delta_\pm(q^2)$ to replace $\Delta_1(q^2)$ by $\Delta_\rho(q^2)-\Delta_\rho(0)$. The breakings of global symmetries are then: static $SU(2)_V$-breaking $\Delta_\rho(0)$, dynamical $SU(2)_V$-breaking $\Delta_\rho(q^2)-\Delta_\rho(0)$, and dynamical G-breaking $\Delta_3(q^2)$. G-breaking but $SU(2)_V$-preserving corrections contribute to the last function, but not the first two; $SU(2)_V$-breaking corrections contribute to all three. As a practical matter, the Δ functions cannot be measured for all q^2; the gauge interactions simplify in special ways at the gauge boson poles $q^2 = 0, M_W^2, M_Z^2$, and these points provide the natural places for measuring the effects of heavy loop effects. It is convenient to define dimensionless parameters S, T, U[66,71]:

$$\begin{aligned} \alpha T &= 4\sqrt{2} G_F \Delta_\rho(0) \quad , \\ S &= -16\pi \Delta_3(Z)/M_Z^2 \quad , \\ S + U &= -16\pi \Delta_\pm(W)/M_W^2 \quad . \end{aligned} \tag{IV.14}$$

Notice that Δ_3 and $\Delta_\pm$ have to be measured away from $q^2 = 0$. These Δ functions must be computed for all SM contributions (top and Higgs), whether heavy or not. Furthermore, for any given non-Standard contributions (such as supersymmetry), the Δ's can be computed to arbitrary accuracy as functions of the new physics. In the EVSM case, simply replace $\alpha T \to 1 - \rho^{-1}$ and note that S and U are unchanged in terms of $\Delta_{3,\pm}(q^2)$, but that $\Delta_\rho(q^2)$ is determined by (IV.13) instead.

If we have no definite model of new physics, and so cannot compute heavy loop effects, we would have to represent the unknown loop effects of this non-SM physics by an infinite series in q^2. The new physics would contribute to $e_*^2(q^2)$ and $s_*^2(q^2)$ as well, although only by powers of q^2/M_{heavy}^2. The alternative to (IV.14) is to ignore all coefficients proportional to inverse powers of M_{heavy}, thus deleting the effect of heavy loops on the gauge couplings and reducing the contributions to the Δ functions to three coefficients equivalent to S, T, U[66,71]:

$$\begin{aligned} \alpha T &= 4\sqrt{2} G_F \Delta_\rho(0), \\ S &= -16\pi \left[\frac{d}{dq^2} \Delta_3(q^2) \right|_{q^2=0}, \\ U &= +16\pi \left[\frac{d}{dq^2} \Delta_\rho(q^2) \right|_{q^2=0}. \end{aligned} \tag{IV.15}$$

A q^2 expansion in the amplitudes is equivalent to a derivative expansion in the effective Lagrangian. Effective Lagrangian methods are discussed in Jon Bagger's lectures in this school. The EVSM case follows from the previous discussion.

Two star functions are necessary to account for the non-decoupled Δ functions:

$$\frac{1}{4\sqrt{2}G_{F*}(q^2)} = \frac{1}{4\sqrt{2}G_F} - \Delta_{\pm}(q^2),$$
$$\frac{1}{\rho_*(q^2)} = 1 - 4\sqrt{2}G_{F*}(q^2)\Delta_\rho(q^2), \qquad (IV.16)$$
$$\frac{1}{2\sqrt{2}G_{F*}(q^2)\rho_*(q^2)} = \frac{1}{4\sqrt{2}G_F} - [\Delta_3(q^2) + \Delta_\rho(0)].$$

In the EVSM case, where ρ_0 is arbitrary, $\rho_*(q^2)$ is independent, and $\rho \equiv \rho_*(0)$ is the natural choice for the additional necessary tree-level parameter. Then $\rho_*(q^2)$ and $\Delta_\rho(q^2)$ change:

$$\frac{1}{\rho_*(q^2)} = \frac{1}{\rho} - 4\sqrt{2}G_{F*}(q^2)\Delta_\rho(q^2),$$
$$\frac{1}{4\sqrt{2}G_{F*}(q^2)\rho_*(q^2)} = \frac{1}{4\sqrt{2}G_F\rho} - \Delta_3(q^2). \qquad (IV.17)$$

We have now covered the whole set of general possibilities with non-decoupled loop effects in the MSM and EVSM cases alike. We have four star functions $e_*^2(q^2)$, $s_*^2(q^2)$, $G_{F*}(q^2)$, $\rho_*(q^2)$ that incorporate the Born-like corrections for electroweak gauge interactions. And, along the way, we have renormalized the gauge sector of the EW SM to one loop. There are no UV divergences left — all cancel when we re-express the theory in terms of three (EVSM: four) tree-level parameters and five (EVSM: four) finite and gauge-invariant combinations of effective self-energies Π^* : α, G_F, M_Z, for example, and $\Delta_Q, \Delta_{3Q}, \Delta_3, \Delta_\pm$, and $\Delta_\rho(0)$ (or ρ). The three heavy physics Δ functions contain all the possible non-decoupled Born-like radiative corrections to four-fermion processes[39].

IV.4 Non-Born-like Corrections – Renormalization Schemes

With the Born-like corrections expressible in terms of the star system, the neutral- and charged-current matrix elements in the electroweak improved Born approximation (IBA) are:

$$\mathcal{M}_{NC}^{IBA} = \frac{e_*^2(q^2)QQ'}{q^2}$$
$$+ \frac{e_*^2(q^2)}{s_*^2(q^2)c_*^2(q^2)} \frac{[I_3^L - s_*^2(q^2)Q][I_3^L - s_*^2(q^2)Q]'}{q^2 - e_*^2(q^2)/s_*^2(q^2)c_*^2(q^2)4\sqrt{2}G_{F*}(q^2)\rho_*(q^2) + i\sqrt{q^2}\Gamma_{Z*}(q^2)},$$

$$\mathcal{M}_{CC}^{IBA} = \frac{e_*^2(q^2)}{2s_*^2(q^2)} \frac{[I_+^L I_-^{L'} + I_-^L I_+^{L'}]}{q^2 - e_*^2(q^2)/s_*^2(q^2)4\sqrt{2}G_{F*}(q^2) + i\Gamma_{W*}(q^2)}, \tag{IV.18}$$

taking on the Born-like form *current – propagator – current*, with the currents and propagator in the tree-level form, only the bare parameters replaced by the corresponding star functions. This form of the matrix elements is discussed in Appendix A. Note that the star functions are not a renormalization scheme, but simply a compact shorthand for the bare perturbation theory[39,62]. In order to introduce renormalization schemes, we must first define some useful parameters.

At $q^2 = 0$, the matrix elements simplify considerably. The photon pole dominates $\mathcal{M}_{NC}$, and, as in pure QED, we define $e^2 = 4\pi\alpha = e_*^2(0)$ from $q^2\mathcal{M}_{NC}(q^2) = e_*^2(0)QQ'$ as $q^2 \to 0$. The charged-current amplitude reduces to a function of a single parameter, conventionally expressed as $G_F = G_{F*}(0)$. In the EVSM case, we also need $\rho = \rho_*(0)$ as an independent parameter for the neutral current. The other convenient parameters we can obtain from the massive gauge boson poles. The gauge-invariant mass and width of a resonance are defined from the real and imaginary parts of the complex pole of the matrix element[73]. The definition of the Z and W mass used in the literature is the so-called "on-shell" definition, the zeros of the real parts of the inverse propagators[10,72], not gauge invariant in general. Through $\mathcal{O}(\alpha)$ (one loop) in the denominators, the two definitions coincide, and we use the on-shell definition here:

$$\begin{aligned}
M_Z^2 &= \frac{e_*^2(Z)}{s_*^2(Z)c_*^2(Z)} \frac{1}{4\sqrt{2}G_{F*}(Z)\rho_*(Z)} \\
&= \frac{e_*^2(Z)}{s_*^2(Z)c_*^2(Z)} \frac{1}{4\sqrt{2}G_F}[1 - 4\sqrt{2}G_F(\Delta_3(Z) + \Delta_\rho(0))], \\
M_W^2 &= \frac{e_*^2(W)}{s_*^2(W)} \frac{1}{4\sqrt{2}G_{F*}(W)} \\
&= \frac{e_*^2(W)}{s_*^2(W)} \frac{1}{2\sqrt{2}G_F}[1 - 4\sqrt{2}G_F\Delta_\pm(W)].
\end{aligned} \tag{IV.19}$$

The widths at the poles are defined by:

$$
\begin{aligned}
\Gamma_{Z,W} &= \frac{\Gamma_{Z,W*}(Z,W)}{1-\kappa_{Z,W*}}, \\
\kappa_{Z,W*} &= \frac{\partial}{\partial q^2} M^2_{Z,W*}(q^2),
\end{aligned}
\qquad (IV.20)
$$

where $M^2_{Z,W*}(q^2)$ are the effective q^2-dependent square masses in (IV.18). The Γ_*'s are defined from the imaginary parts of the Z and W self-energies. The Γ_*'s are themselves proportional to $\sqrt{q^2}$, so that $\Gamma_{Z,W}$ are proportional to $M_{Z,W}$[39,72,73].

From these parameters, we can construct the three renormalization schemes in common use in the literature in order to parametrize the star functions. The first scheme in wide use was the *on-shell scheme* of Sirlin and Marciano (1980)[72,74]. The three parameters it uses are α, M_Z, and M_W, with the on-shell weak mixing $\sin^2\theta_W \equiv 1 - M^2_W/M^2_W$ to all orders defined as an auxiliary quantity. G_F is a calculable function of parameters in this scheme. A second scheme was introduced by *Lynn, Peskin,* and *Stuart (LPS)* by modifying the on-shell scheme, with the parameters α, G_F, and M_Z (1985)[59]. This scheme has the advantage of being related to the canonical experimental inputs in a simple way. In this scheme, M_W is a calculable function of parameters. The third scheme was introduced recently by Sirlin and others, the $\overline{MS}$ scheme (1989)[75]. It uses G_F and the $\overline{MS}$ couplings $\hat{e}^2(\mu^2)$ and $\hat{g}^2(\mu^2) \equiv \hat{e}^2(\mu^2)/\hat{s}^2(\mu^2)$ as parameters. $\overline{MS}$ is especially convenient for running the gauge couplings and relating the electroweak measurements to grand unified theories. Because the $\overline{MS}$ couplings absorb only the divergences and leading logarithms, they are truly universal, gauge-invariant couplings that are independent of process and propagator construction. M_Z and M_W are calculable functions of parameters in this scheme. In all three schemes, the gauge boson widths are also calculable functions of parameters, and ρ must be introduced as a fourth independent parameter in the EVSM case.

The details of these three schemes and their mutual relationships are presented in Appendix B, along with the SM gauge boson self-energies. The calculations and the schemes are given through one loop, and at this level, the different schemes can be freely exchanged for one another and even mixed in the same calculation. The problem of RS-dependence is always present, however; the one-loop functions are themselves strictly functions of the bare parameters, and the embedded higher-order corrections must be somehow included[22,64]. The only serious higher-order dependence necessary for these calculations is the type that can be taken care of to good approximation by replacing the bare parameters in the loops by the appropriate renormalized ones. In the gauge boson self-energies, particle masses in loops should be replaced by the physical on-shell masses, and bare s_0^2, c_0^2 by the on-shell $\sin^2\theta_W, \cos^2\theta_W = 1 - \sin^2\theta_W$ for loops with gauge bosons. Notice again that one particular scheme (on-shell, in this case) is favored to implicitly sum up the largest higher-order terms, this choice being the best of the three at making this set of corrections converge. In the EVSM case, ρ is lumped with the radiative parameters for convenience, even though it is a tree-level parameter, for clarity in comparing the EVSM and MSM cases. The value of ρ is not extracted from a single special experiment (like the three other tree parameters), but is instead derived from global fits to many experiments in the same way the radiative parameters are.

Here I use, and *advocate in all future electroweak work* the universal use of, the combination of the $\overline{MS}$ and LPS schemes, with the canonical fixed inputs (α, G_μ, M_Z). The LPS heavy physics Δ functions and the (S, T, U) use $q^2 = 0$ as their reference point, the natural static limit or vacuum state. This "metascheme" is independent of propagator construction and independent of vertex and box corrections, except for the known sum of Born-like and non-Born-like vertex and box corrections to muon decay at $q^2 = 0$ (through G_μ). Because of the limitations imposed by perturbation theory, it is impossible to construct a truly renormalization scheme-invariant parametrization of electroweak interactions, but this combination of $\overline{MS}$, LPS, and canonical inputs seems optimal.

For the input scheme, a variety of experiments can serve, but the theoretical simplification available at the gauge poles and the present experimental accuracies lead uniquely to the canonical set consisting of: $q^2 \to 0$ QED, to define α[29,76]; the muon lifetime τ_μ, to define G_F[76,111]; and $e^+e^- \to Z$ annihilation, to define M_Z[72, 74,77]. Generally, the matrix elements require non-Born-like corrections to be complete, the details depending on process measured, so the relation of α, G_F, M_Z to experiment involves non-Born-like terms. The best current determinations of α are made in macroscopic condensed matter experiments (quantum Hall effect and AC Josephson junction) — purely Maxwell plus quantum mechanics — yielding[76]:

$$\alpha^{-1} = 137.0359895 \pm 0.0000061.$$

The determination of M_Z from the e^+e^- annihilation cross sections (Figure 5) does involve non-Born-like terms: boxes, vertices, and especially the combined effect of radiation from the initial e^+e^- beams and the Z-photon interference in the cross section[78]. The on-shell definition fit to the e^+e^- data gives:

$$M_Z = 91.174 \pm 0.021 \text{ GeV},$$

as of March 1991[77]. The measured value of the muon lifetime against beta decay is conventionally quoted in terms of a muonic Fermi's constant G_μ. Its definition from the lifetime includes a a gauge-invariant subset of non-Born-like corrections arising from QED corrections to the beta decay[111]:

$$\tau_\mu^{-1} = \frac{G_\mu^2 m_\mu^5}{192\pi^3}[1 - 8\frac{m_e^2}{m_\mu^2}]\left\{1 + \frac{\alpha}{2\pi}(\frac{25}{4} - \pi^2)[1 + \frac{2\alpha}{3\pi}\ln(\frac{m_\mu}{m_e})]\right\}.$$

The measured value is[76]:

$$G_\mu = (1.16637 \pm 0.00002) \times 10^{-5}\ \mathrm{GeV}^{-2}.$$

To extract a universal value of G_F requires removing the remaining non-Born-like corrections (Appendix C).

The matrix elements for all processes begin with the IBA form of (IV.18), but require the addition of the non-Born-like terms relevant for each process. With the massive gauge bosons present in the loops, not all the non-Born-like corrections are infrared-divergent. Consider the neutral-current vertices and boxes first (Figures 20 and 15 for both γ and Z channels)[39,62]. The *QED set,* loops and radiation with at least one photon, contains *vertex loop/radiation* and *box loop/interference radiation* graphs and is purely Abelian. As in section III.3, the appropriately combined corrections — i.e., vertex loop plus its radiation, and so on — are separately gauge-invariant and IR-finite. The *weak set,* loops with only heavy gauge bosons, is IR-finite and contains Abelian and non-Abelian terms. Our propagator construction (section IV.2/Appendix A) absorbs all of the non-Abelian vertex Γ_3^{nAb} and part of the non-Abelian box loops $\bar{\Theta}_3^{nAb}$ into the gauge-invariant effective self-energies $\Pi^*_{QQ}, \Pi^*_{3Q}, \Pi^*_{33}$. The non-Born-like neutral-current corrections consist of six separately gauge-invariant classes of corrections (Figure 22): the two vertex and one box/interference QED sets, the two Abelian weak vertex and one Abelian weak box classes, and the residual non-Abelian weak box terms. To assemble the complete matrix element, one adds the boxes to the IBA of (IV.18), then multiplicatively modifies the tree-level vertices with the Abelian weak vertex corrections. The neutral-current matrix element is now a modified IBA: $\mathcal{M}_{NC}^{MIBA} = \mathcal{M}_{NC}^{IBA+Ab\ ver} + \Theta_{NC}^{Ab} + \bar{\Theta}_{NC}^{nAb}$. Only the QED sets are left. They can be included either by simply adding the loops to the MIBA, squaring, then adding the squared radiation matrix element (III.12); or, if the box loop/interference radiation contributions are negligible, by convolving the MIBA cross section with the structure functions appropriate for each vertex[78]: $\sigma = \int D(f) \cdot D(f') \cdot |\mathcal{M}_{NC}^{MIBA}|^2$.

- •• QED vertices/radiation both fermion legs.
- •• Abelian weak vertices both fermion legs.
- • QED boxes/interference.
- • Abelian weak boxes.
- • residual non-Abelian weak boxes.

Figure 22. Gauge-invariant classes of non-Born-like corrections[39,62,64] to the neutral current.

Now consider the non-Born like corrections for the charged-current matrix element (Figures 15 and 20 with W channel and Figure 23)[62]. The *QED sets*, with one photon and radiation, are like the neutral-current case, except that they contain both Abelian and non-Abelian terms: the charge-changing generators $I_\pm^L$ do not commute with Q. There is a fifth, purely non-Abelian radiation graph (Figure 23, photon emitted from the W). The Abelian QED subsets, with combined radiation and loops, are gauge invariant and IR-finite. The weak sets also include Abelian and non-Abelian parts. The effective W propagator construction is analogous to the neutral-current case: all of the non-Abelian vertex $\Gamma_\pm^{nAb}(q^2)$ and part of the non-Abelian box $\bar{\Theta}_\pm^{nAb}(q^2)$ are absorbed into the gauge-invariant effective self-energy $\Pi_\pm^*$. The charged-current case requires an additional step, however: the non-Abelian self-energy, vertex and box terms contain IR divergences due to the vanishing of the photon mass in the loops. These appear in $\Pi_\pm^*$ in a gauge-invariant way, proportional to $\ln(M_\gamma^2/M_W^2)$ and $\ln(q^2/M_\gamma^2)$. These terms must then be restored to the non-Born-like corrections and combined with the non-Abelian radiation terms. The upshot for the charged-current non-Born-like corrections is eight gauge-invariant and IR-finite classes (Figure 24): the two vertex and one box/interference QED classes, the two Abelian weak vertex and one Abelian weak box classes, the residual non-Abelian weak box terms, and the non-Abelian QED radiation/non-Abelian IR loop terms. If the box/interference radiation corrections are negligible (as they often are), then the QED vertex and radiation contributions can be incorporated instead by using structure functions. The charged-current structure functions are more complicated, having non-Abelian contributions[79].

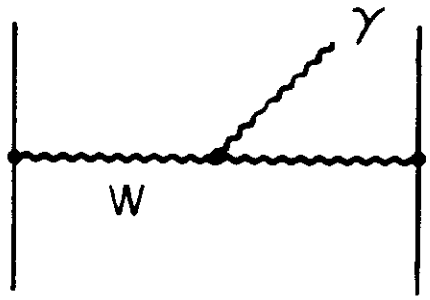

Figure 23. Purely non-Abelian radiation graph in the charged current: photon emission from the $W^{\pm}$.

The non-Born-like corrections to four-fermion processes are further discussed in Appendix C.

- • Abelian QED vertices/radiation
- Abelian QED boxes/interference.

- • Abelian weak vertices
- Abelian weak boxes

- non - Abelian IR vertices and boxes
- residual non-Abelian weak boxes

Figure 24. Gauge-invariant classes of non-Born-like corrections to the charged current.

V. Precision Tests of Electroweak Interactions

V.1 Precision Tests of QED – Hadronic Uncertainties

Low-energy precision tests of quantum electrodynamics[3] are the exception to our rule of massless external fermions. The external masses cannot be neglected in this case, and such processes involve helicity-changing operators also. QED has passed its low-energy tests with stunning precision, leaving it as the most successful physical theory ever devised, the very model of a modern major field theory.

First, consider the anomalous magnetic moment of a fermion f. The vertex function in this case is coupled to a macroscopic magnetic field, instead of a photon. The moment is quoted in terms of the gyromagnetic ratio $g = 2 + 2a$, where a is the anomalous deviation of g from the tree-level Dirac value of $g = 2 : a = F_2(0)$. a for leptons is calculable in perturbation theory, with some higher-order hadronic uncertainty; contributions to a are usually separated into pure QED (no hadronic or weak contributions), hadronic contributions to the self-energy of the photon in the vertex loop, and weak contributions (W, Z, Higgs, new physics). QED being unbroken, decoupling applies, so that heavy physics is suppressed by powers of m_f^2/M_{heavy}^2.* The one-loop contribution is mass-independent, $a_{one-loop} = \alpha/2\pi$. The pure QED contributions to a_e and a_μ have now been calculated to four loops. Contributions from physics at or beyond the weak scale are $\Delta a_{weak} \lesssim 10^{-9}$. The comparison of the predicted a_e and a_μ :

$$a_e^{theo} = 1\ 159\ 652\ 140\ (27) \times 10^{-12}$$

$$a_\mu^{theo} = 1\ 165\ 919\ 18\ (191) \times 10^{-11}$$

with the measured values:

$$a_e^{exp} = 1\ 159\ 652\ 188\ (4) \times 10^{-12}$$

$$a_\mu^{exp} = 1\ 165\ 937\ (12) \times 10^{-9}$$

is a triumph for quantum electrodynamics[3]. (The particle and antiparticle moments should be equal by CPT invariance.) The next generation of muon moment experiments, with an order of magnitude or better improvement in precision, will place some constraints on extensions of the MSM.

The second set of low-energy QED tests comes from measurements of bound states: hydrogen (e^-p), muonium ($e^-\mu^+$), and positronium (e^-e^+), with small nonzero momentum transfer of $\mathcal{O}(\alpha m_e)$ in each case. Some of the field-theoretic effects can be couched in Hamiltonian form and inserted into ordinary quantum mechanical perturbation theory. A more consistent approach is to develop the field theory of bound states, a difficult and beautiful subject worked out over the last four decades by Bethe and Salpeter [80], Lepage[81] and others. The most precise tests are based on the small splittings of otherwise degenerate states, these degeneracies being the result of good tree-level symmetries. The classic example is the Lamb shift, the radiative splitting of the $2S_{1/2}$ and $2P_{1/2}$ states of hydrogen; the tree-level Dirac

* Note that $q^2 = 0$ here, so that the fermion mass is not small. A more general decoupling theorem applies then, allowing contributions in powers of m_f^2/M_{heavy}^2 as $q^2 \to 0$. The moment in physical units, $\mu_f = ge/2m_f$, exhibits the overall decoupling of an dimension-5 operator from the inverse fermion mass.

energy levels depend only on the total angular momentum j, not on the orbital quantum number l[3].

$$\Delta E(2S_{1/2} - 2P_{1/2})^{theo} = 1\ 057\ 853\ (13)\ \text{kHz}$$

$$\Delta E(2S_{1/2} - 2P_{1/2})^{exp} = 1\ 057\ 851\ (2)\ \text{kHz}.$$

The prediction of this shift was an early success for renormalized perturbative QED, but, ironically, most of the shift is due to infrared effects[2]. These are not divergent in discrete bound states because there is a minimum virtual photon energy set by the differences of quantized levels. In muonium and positronium, the ground state hyperfine splittings (spin-spin coupling) are an exceptionally good measure of radiative effects. Other precision QED tests in these and similar bound states have also been performed[3].

A third important low-energy test of QED is bounding the photon mass M_γ. Limits on M_γ come from electronic tests, satellite measurements of the Earth's magnetic field, and, with the best limit, the inferred properties of the galactic magnetic field: $M_\gamma < 3 \times 10^{-27}$ eV[76,82].

The best high-energy tests of QED cannot compare with low-energy measurements in precision, but are still important checks of the asymptotic behavior of electrodynamics. Bhabha scattering $e^+e^- \to e^+e^-$[27,28], which receives both t-channel scattering and s-channel annihilation contributions, is currently the standard high-energy test of QED, with a precision of about a percent. Figure 25 shows the theoretical vs. measured angular dependence of Bhabha scattering as measured by the Mark J at DESY[83]. Following the discussion in section III.1, the measured $e^+e^- \to$ *hadrons* cross section, while not a test of QED, is an essential input for computing the running of vacuum polarization to the Z pole[47]. The computed value of the running charge at the Z, in the $\overline{MS}$ scheme, is[75]:

$$\frac{1}{\hat{\alpha}(Z)} = 127.8 \pm 0.1, \tag{V.1}$$

for $m_t = M_Z$, where the uncertainty is due to the measured R_{had}. Other contributions to $e_*^2(Z)$, including threshold dependence on M_W and m_t, are calculable. The hadronic uncertainty propagates to computed quantities, such as the Z pole properties and the W mass, that depend on $e_*^2(Z)$. The other strong corrections to EW processes in Π_{3Q}, Π_{33}, and $\Pi_\pm$ can be accounted for using perturbative QCD to $\mathcal{O}(\alpha\alpha_s)$ (Figure 13c)[48], since they all involve very high energies. These additional $\mathcal{O}(\alpha\alpha_s)$ effects are quite small.

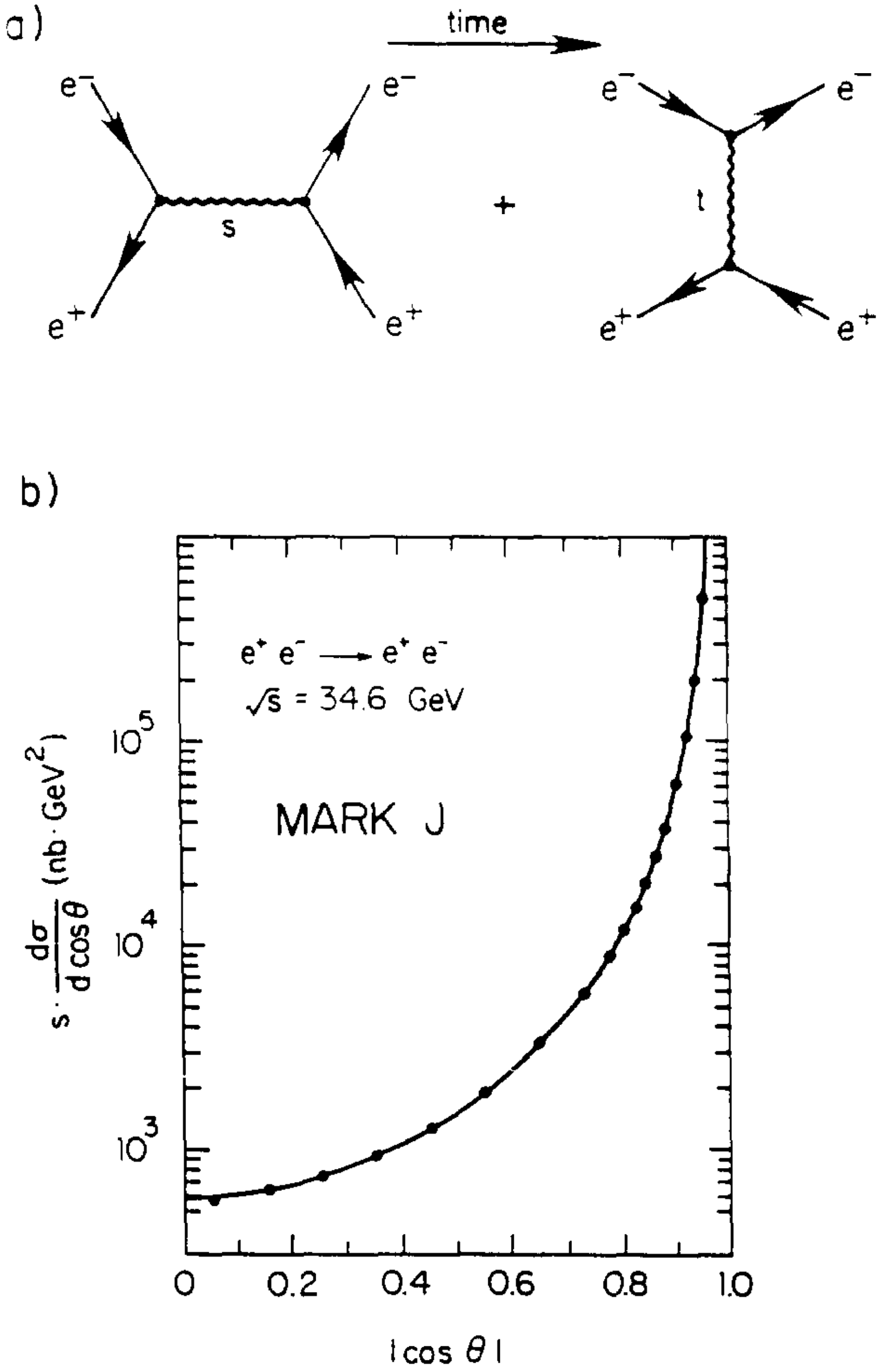

Figure 25. Bhabha scattering $e^+e^- \rightarrow e^+e^-$: (a) Feynman diagrams for s and t channels; (b) QED prediction (solid line) vs. Mark J measurement (dots) of Bhabha cross section[27, 83].

V.2 Electroweak Processes – Choice of Models, Parameters and Inputs

To make predictions from the EW SM as a function of free parameters, we need an input scheme — here, the canonical α, G_μ, and M_Z — and two assumptions. The first is to specify which EW SM we are testing. The MSM includes all one-loop corrections, assumes $\rho_0 = 1$ and has two free parameters (m_t, m_H) as the only heavy physics. Direct searches already limit $m_t > 89$ GeV [84] and $m_H > 42$ GeV[85]. The Extended Vacuum SM (EVSM) is the same as the MSM, except it assumes an arbitrary ρ as a fourth free tree-level parameter. We can also assume two other parallel Standard Models: MSM with the Born-like heavy corrections parameters (S, T, U) replacing (m_t, m_H) as the free loop parameters; and EVSM with (S, U) as the heavy physics parameters and T replaced by the free ρ. These four distinct model choices are summarized in Table I. Notice that we make the fixed global assumption that $SU(2)_L \times U(1)_Y$ is the correct gauge structure, described by two running couplings $s_*^2(q^2)$ and $e_*^2(q^2)$, and that any new physics consists of scalars and fermions only[39]. Searches for the presence of new gauge interactions[86] in our set of four-fermion processes would require additional free tree-level parameters beyond the three (or four) needed for the tree-level EW SM.

MSM(m_t, m_H) $\rho_0 = 1$	EVSM$(\rho; m_t, m_H)$ $\rho_0 \neq 1$
MSM(S, T, U) $\rho_0 = 1$	EVSM$(\rho; S, U)$ $\rho_0 \neq 1$

Table I. Four variations of the electroweak Standard Model discussed in the text.

The second necessary assumption is that the computed value of $e_*^2(Z)$ is correct, an assumption we really cannot check once q^2 is close to M_Z^2. The inputs to the fits then are (1) the two assumptions, (2) the three free tree-level parameters, and (3) the four-fermion data. The outputs always include a value for $s_*^2(Z)$, which I quote in terms of the $\overline{MS}$ value $\hat{s}^2(M_Z^2)$. This quantity is actually overdetermined in the fits, because there are a large number of experiments sensitive to it.* The other outputs are some combination of heavy physics and ρ in the EVSM cases. The (S, T, U) are always defined relative to the reference point MSM$(m_t = m_H = M_Z)$, with only departures explicitly displayed. Thus each parameter is a sum of top and Higgs mass deviations and non-Standard contributions: $S = S^{NS} + \Delta S^{MSM}$, and so on.

* The fits presented in the next two sections[87,88] leave $s_*^2(Z)$ as another free parameter, even though it is determined by $M_Z, e_*^2(Z), G_F$, and the heavy physics parameters. This procedure serves as a consistency check on the fits and a partial check on the $SU(2)_L \times U(1)_Y$. gauge structure.

The specific flavor-conserving four-fermion processes now available for the precise testing of the EW SM fall into three distinct kinematical regimes. The first regime is $q^2 \sim \mathcal{O}(\alpha^2 m_e^2)$: atomic parity violation (APV)[89]. The second is the low-energy accelerator regime with $|q^2| < M_{W,Z}^2$[90]: neutrino-nucleon (νN) and neutrino-electron (νe) scattering of various types, lepton-nucleon scattering, and e^+e^- annihilation below the Z pole. The third regime is the gauge boson pole region, $q^2 = M_W^2$ and M_Z^2[87, 88]: the W and Z masses, the total and partial Z decay widths, and various asymmetries at the Z pole.

Atomic parity violation has been a topic of active interest since it was realized in the 1970s that the parity-violating Z exchange between nuclei and atomic electrons would mix electronic states of opposite parity, such as S and P states. The present success of APV studies is due to the use of one element, cesium, which is hydrogenic and heavy, allowing accurate theoretical evaluation of many-electron effects. The weak contribution stems essentially from the neutral-current exchange:

$$\mathcal{M}_{NC}^{IBA}(q^2 = 0) = \frac{4\sqrt{2}G_F}{1-\alpha T}[I_3^L - s_*^2(0)Q] \cdot [I_3^L - s_*^2(0)Q]'. \qquad (V.2)$$

$s_*^2(0)$ is determined from $s_*^2(Z)$ by $\Delta_{3Q}(0)$, and from M_Z in turn. The form (V.2) is the universal IBA for low-energy neutral-current processes, depending only on $s_*^2(0)$ and T. The use of M_Z as an input induces an implicit dependence in $s_*^2(0)$ on S and T. In the special case of heavy hydrogenic (alkali) atoms such as cesium, with $s_*^2(0) \simeq 1/4$ and the proton/neutron ratio $\simeq 2/3$ (because of Coulomb repulsion), the T dependence fortuitously cancels from $\mathcal{M}_{NC}^{IBA}(q^2 = 0)$ when the exchange is summed over all the nucleons. Hence, the only heavy physics dependence is due to S. The recent beautiful experiments of Wieman and collaborators at NIST/Colorado (Boulder) measure the APV effect using parity-violating asymmetries derived from observed radiative transitions of cesium in crossed electric and magnetic fields. The result is usually quoted is terms of the *weak charge* Q_W^{Cs}, essentially the electron axial-vector coupling times the nuclear vector coupling: $Q_W^{Cs} = -71.04 \pm 1.58 \pm 0.88$, the first error being experimental, the second due to atomic theory[89]. In the purely electroweak IBA, the heavy alkali effective matrix element takes the form:

$$\mathcal{M}_{NC}^{APV} = -\frac{4\sqrt{2}G_F}{1-\alpha T}N[1 - (1 - 4s_*^2(0))(Z/N)],$$

where $Z(N)$ = proton (neutron) number ($Z = 55$ for cesium). Because it is proportional to nucleon spins, the sum of nucleon axial-vector couplings is $\mathcal{O}(1)$, while that of the vector current is $\mathcal{O}(N)$. The full APV interaction requires non-Born-like electroweak corrections and atomic physics terms as well[89].

The form (V.2) applies generally to all of the low-energy accelerator measurements of the neutral current, between pairs of fermions. A large body of data is available from the late 1970s to the present that includes [90,91]: $\nu N, \bar{\nu} N, \nu e, \bar{\nu} e$ scattering with different neutrino flavors and both elastic and inelastic final states; inelastic polarized lepton-nucleus scattering; and e^+e^- annihilation for $s < M_Z^2$. These experiments all measure $s_*^2(0)$ and T (or ρ), and thus $s_*^2(Z)$, if M_Z is used as an input.†

The fundamental gauge pole measurement is the Z mass[77]. From there, the line shape and peak cross section measurements determine the total and partial Z widths. The four collaborations at CERN LEP (ALEPH, DELPHI, L3, and OPAL) together yield: $\Gamma_Z = 2.487 \pm 0.009$ GeV, the universal partial width to each charged lepton pair $\Gamma_Z^{l^+l^-} = 83.3 \pm 0.4$ MeV, and the normalized Z width to hadrons $R = \Gamma_Z^{had}/\Gamma_Z^{l^+l^-} = 20.94 \pm 0.12$[77]. Assuming the SM width for light neutrinos, the invisible width is in good agreement with three light neutrino flavors: $N_\nu = 2.89 \pm 0.10$[85].

The Z pole asymmetries are defined from normalized differences of cross sections for various pairs of opposite states:

$$A_{a\bar{a}}(Z) = \frac{\sigma(Z)_a - \sigma(Z)_{\bar{a}}}{\sigma(Z)_a + \sigma(Z)_{\bar{a}}}. \qquad (V.3a)$$

For example, the forward-backward asymmetry $A_{FB}(Z)$ measures the difference of final-state fermions going in the e^- (forward) and e^+ (backward) directions at the point of annihilation. $A_{FB}(Z)$ is thus a measure of C (charge conjugation) violation and depends on the quantum numbers of the final state. The best currently established measurement is the asymmetry to muons $A_{FB}^{\mu^+\mu^-}(Z) = 0.0154 \pm 0.0048$[77]. Other fermion final states are becoming available at LEP for other measurements of $A_{FB}(Z)$. The polarization asymmetry $A_{LR}(Z)$, the difference between cross sections of left- and right-handed electrons at the Z pole, will be of great importance, if and when it is measured[92]. $A_{LR}(Z)$ is essentially independent of the final state and is a measure of P (parity) violation. In the EW SM, C and P violation are directly linked in such a way that CP is conserved (in the gauge sector) and their strengths controlled by $s_*^2(q^2)$. Thus $A_{LR}(Z)$ and $A_{FB}(Z)$ are essentially measures of $s_*^2(Z)$. In the IBA, $A_{LR}(Z)$ can written as:

$$A_{LR}(Z) = [\frac{g_L^2 - g_R^2}{g_L^2 + g_R^2}]_e = \frac{2[1 - 4s_*^2(Z)]}{1 + [1 - 4s_*^2(Z)]^2} \qquad (V.3b)$$

where $g_L = I_3^L - s_*^2(Z)Q$, $g_R = -s_*^2(Z)Q$. (The Abelian vertex corrections can be inserted using the results quoted in Appendix C; radiation has a small effect on

† The low-energy processes listed here have their own non-Born-like corrections which are small but not negligible, but which are also calculable to the necessary accuracy with only "light" $< Z$ physics.

$A_{LR}(Z)$[39,78].) The forward-backward asymmetry to fermion-antifermion f is then formally:

$$A_{FB}^{f\bar{f}}(Z) = \frac{3}{4} A_{LR}(Z)_e A_{LR}(Z)_f, \qquad (V.3c)$$

with explicit dependence on the final-state quantum numbers through the second factor. (The Abelian vertex corrections again can be easily inserted; radiation has a relatively large effect on $A_{FB}(Z)$[39, 78].) Shifts due to heavy physics are treatable by using $\delta A_{LR}(Z) \simeq -8\delta s_*^2(Z)$. Please see the lectures of Alain Blondel in this school for more about Z pole physics with e^+e^- annihilation and the LEP experiments.

The W mass[93] has been determined accurately in the recent $p\bar{p}$ collider results at the Tevatron CDF (Fermilab) and Sp$\bar{\text{p}}$S UA2 (CERN) collaborations. They measure the M_W/M_Z ratio, then use the LEP value of M_Z to obtain M_W. The best current value of the W mass is $M_W = 79.91 \pm 0.39$ GeV.*

V.3 Minimal Standard Model Fits – Grand Unification

In this section, let us consider only the models denoted by MSM(m_t, m_H) and EVSM($\rho; m_t, m_H$). The global fits of the neutral- and charged-current matrix elements to all the available data produce limits on the two (or three) free parameters, the finite volume of free parameter space being determined ultimately by the precision of the experimental data. The value of $\hat{s}^2(M_Z^2)$ is calculable once M_Z and the heavy physics are known. Since there are more experiments than free parameters, the latter are actually overdetermined by a global fit; this overdetermination is an important check on the self-consistency of EW measurements as a whole.† The parametrization of various observables in terms of the gauge couplings and heavy physics functions is discussed in Appendix B. The global fit results of this section are taken from the work of Langacker and Luo[87]. Other groups have obtained similar results[94].

The heavy physics sensitivities are quite different for the top quark and minimal

* The gauge pole measurements quoted here are current to March 1991.

† Leaving $s_*^2(Z)$ as a free parameter is a further check since it can then be compared with the value derived from the Z mass constraint (IV.19).

Higgs masses:

$$
\begin{aligned}
\alpha T &= \frac{3G_F M_Z^2}{8\sqrt{2}\pi^2}[(m_t/M_Z)^2 - 1] \\
&- \frac{3G_F}{8\sqrt{2}\pi^2}(M_Z^2 - M_W^2)\ln(m_H^2/M_Z^2), \\
S &= -\frac{1}{6\pi}\ln(m_t^2/M_Z^2) + \frac{1}{12\pi}\ln(m_H^2/M_Z^2), \\
U &= \frac{1}{2\pi}\ln(m_t^2/M_Z^2),
\end{aligned}
\tag{V.4}
$$

displaying only the leading quadratic and logarithmic terms. In particular, the Higgs mass dependencies are so weak that no useful limits can be derived from present data. The MSM(m_t, m_H) model fit to all data yields:

$$
\begin{aligned}
\hat{s}^2(M_Z^2) &= 0.2334 \pm 0.0008, \\
m_t &= 124^{+26+20}_{-34-15}\ \mathrm{GeV} \\
&< 174\ \mathrm{GeV}\ (90\%),
\end{aligned}
\tag{V.5}
$$

with the second uncertainty in the central value of m_t due to varying the unknown Higgs mass over the physically reasonable range of 42 to 1000 GeV. Figure 26 shows the simultaneous constraints on $\hat{s}^2(M_Z^2)$ and m_t from all experiments. In the EVSM($\rho; m_t, m_H$) model, we lose the quadratic top mass dependence in T, but a quadratic m_t dependence still remains in the non-Born-like Abelian corrections to one specific interaction, the $Z \to b\bar{b}$ vertex (Figure 20a,b with W's in the loops, and Appendix C). The logarithmic m_t dependence still appears universally in S. The EVSM($\rho; m_t, m_H$) fit to all data yields:

$$
\begin{aligned}
\hat{s}^2(M_Z^2) &= 0.2333 \pm 0.0008, \\
\rho &= 0.992 \pm 0.011, \\
m_t &< 294\ \mathrm{GeV}\ (90\%).
\end{aligned}
\tag{V.6}
$$

Note how close ρ is to unity.

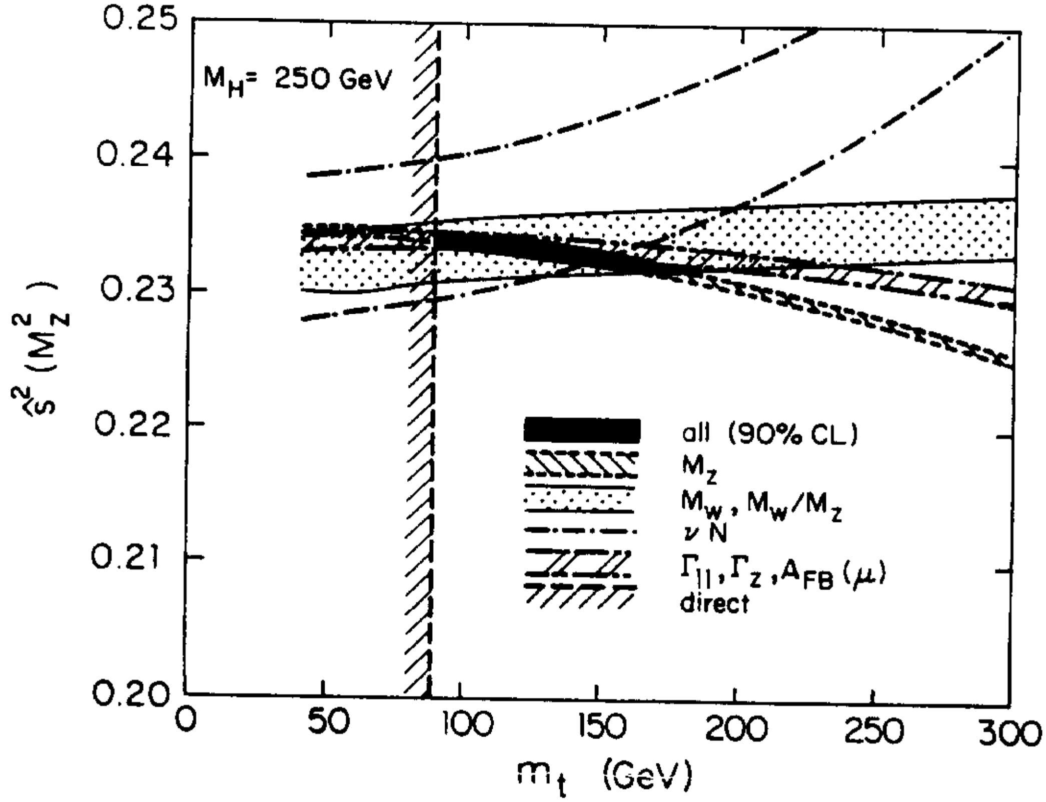

Figure 26. $\overline{MS}$ electroweak mixing $\hat{s}^2(M_Z^2)$ vs. top quark mass, as constrained by various experiments, assuming MSM and Higgs boson of mass 250 GeV (March 1991)[87].

The alert reader may have noticed that there is a potentially serious problem with the non-decoupled physics of very heavy particles; for example, if $m_t \to \infty$. As we saw in section IV.3, the effect of the heavy non-decoupled loop corrections relative to the tree-level gauge boson masses is actually a perturbative series purely in powers (or logarithms) of the dimensionless couplings that produce the $SU(2)_L \times U(1)_Y$-breaking masses, not necessarily connected with the gauge couplings. Thus, the top mass effect in T depends only on the top quark Yukawa coupling h_t :

$$\alpha T = \frac{3G_F m_t^2}{8\sqrt{2}\pi^2} = \frac{3}{32\pi^2} h_t^2. \qquad (V.7)$$

At higher loops, (V.7) actually becomes a series in h_t, so for large top mass $m_t = h_t \cdot v/\sqrt{2}$ (v fixed), we might wonder about how good the one-loop result is and even whether the electroweak gauge theory breaks down with arbitrarily large corrections to low-energy processes. In fact, this never happens, because these dimensionless mass-generating couplings to the Higgs sector can be freely tuned only for small couplings, in the perturbative regime[95]. As the couplings become stronger — roughly, $h_t^2/4\pi \to 1$ — the coupled Higgs-top sector develops strong non-linear effects of various types. Perturbative approximations such as (V.7) are no longer valid. The decay widths of the strongly-coupled particles become comparable to their masses. New kinds of symmetry breaking can appear, such as $t\bar{t}$ condensates. Also, the effective running couplings, such as $\hat{h}_t(\mu)$, develop Landau poles below the particle mass m_t for large enough m_t, rendering the strongly-coupled sector meaningless and imposing the so-called "triviality bound" on particle masses. The upshot of this non-linear strong-coupling limit is that such theories cannot have arbitrarily large couplings. For example, h_t is effectively prevented from increasing beyond a certain upper bound roughly of order $\sqrt{4\pi} \simeq 3.5$. Thus the non-decoupled corrections in the gauge sector never blow up, because the heavy particle couplings to the Higgs sector and thus their masses are always limited. These limits are typically (for the top quark and Higgs) around 600-800 GeV, but the basic idea applies more generally, the details varying with the model content.

The measurement of the weak mixing function $s_*^2(q^2)$ has no further implication in the $SU(2)_L \times U(1)_Y$ gauge theory, but is crucial for testing grand unified theories (GUTs), where it is not a free parameter [56]. In a GUT, instead of three independent couplings for the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge interactions and the VEV of the MSM, we have two mass scales, the EW VEV and the GUT VEV, and a single gauge coupling. One of the SM gauge couplings is thus redundant, if the GUT satisfies the self-consistency condition that all three running couplings unify at a single scale. Figure 27a shows the runnings of the three $\overline{MS}$ gauge couplings $\hat{\alpha}_i(\mu^2)$ as functions of μ in the MSM. This assumes no new particles until the GUT scale M_X (the "great desert"), where the model unifies into the minimal SU(5) GUT, the simplest GUT. The couplings clearly do not all run together, so that minimal SU(5) is ruled out. On the other hand, the minimal supersymmetric (SUSY) SM unifies into the SU(5) SUSY GUT quite nicely (Figure 27b), with a superheavy gauge boson mass of $M_X = (1.9 \pm 1.4) \times 10^{16}$ GeV. The extra SUSY particles, with masses somewhere between M_Z and 1 TeV, change the running of the couplings enough to force unification. The extra SUSY particles can be introduced into the EW SM radiative corrections without significantly disturbing the heavy physics limits, as we shall see in the next section.*

---

* The less precise runnings available before the LEP Z measurements had implied too low an SU(5) GUT scale, $M_X \simeq 10^{14}$ GeV, and too short a proton lifetime $\tau_p \propto M_X^4/m_p^5$, to fit the present limit, $\tau_p > 5 \times 10^{32}$ years. The larger SUSY SU(5) GUT scale alleviates this problem, at least in the dimension-6 low-energy effective operators for proton decay[56,76].

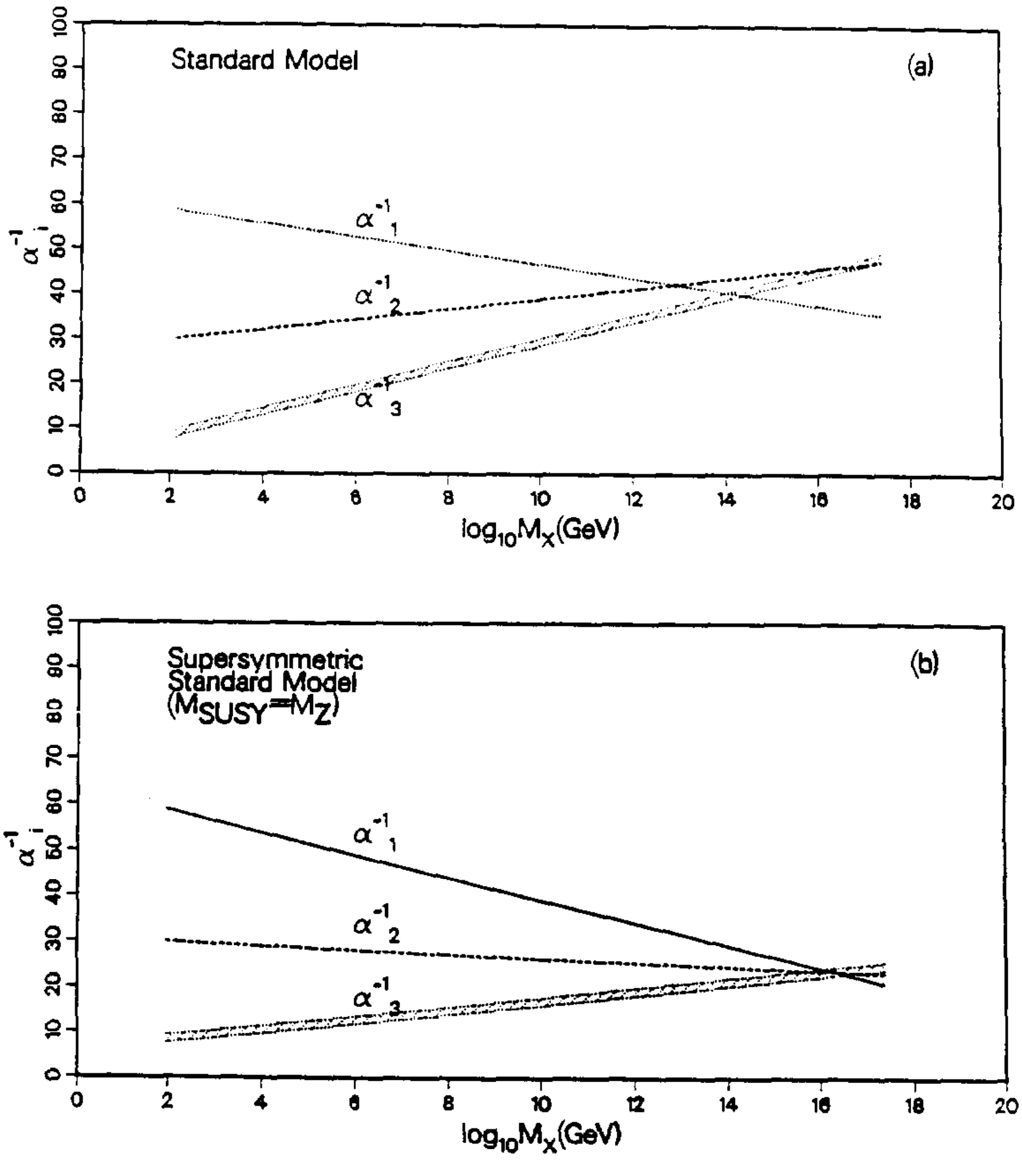

Figure 27. Simultaneous asymptotic runnings of Standard Model $\overline{MS}$ couplings: (a) MSM with 'Great Desert' fails to unify into minimal SU(5) GUT; (b) SUSY version of MSM with superpartners of mass M_Z- TeV self-consistently unifies into SUSY SU(5)[87].

V.4 Heavy Physics: Global Fits – Limits on Non-Standard Models

Let us now attack the two alternative Standard Models, MSM(S,T,U) and EVSM($\rho;S,U$). In these cases, we again obtain a value of $s_*^2(Z)$ and limits on the three parameters. The results of this section are drawn from the work of Kennedy and Langacker[88].

First, consider the MSM(S,T,U) case. Since there is no large custodial symmetry breaking in electroweak gauge forces (recall that $\rho \simeq 1$), it is useful to carry out first a constrained fit for MSM(S,T,U) with $U=0$ imposed by hand. Only the W mass depends on U in any case. The resulting fit for S,T is shown in Figure 28, with the regions allowed by different types of experiments, as well as the global result for *all data.* The numerical results are:

$$
\begin{aligned}
\hat{s}^2(M_Z^2) &= 0.231 \pm 0.002, \\
S &= -1.0 \pm 0.9, \\
T &= -0.3 \pm 0.5.
\end{aligned}
\tag{V.8}
$$

The unconstrained fit yields:

$$
\begin{aligned}
S &= -0.9 \pm 0.9, \\
T &= -0.3 \pm 0.5, \\
U &= +0.3 \pm 1.1,
\end{aligned}
\tag{V.9}
$$

with the same value of $\hat{s}^2(M_Z^2)$. The large uncertainty in U is due to the fact that the W mass is not yet measured as well as the other, neutral-current, data. The EVSM$(\rho; S, U)$ can be read off from the MSM(S,T,U) fits, just by replacing $\alpha T \to 1-\rho^{-1}$:

$$
\rho = 0.998 \pm 0.004, \tag{V.10}
$$

indicating again that the assumption of tree-level custodial symmetry in the VEV(s) is natural. The self-consistency of these fits, particularly in the value of $s_*^2(Z)$, is some check on the $SU(2)_L \times U(1)_Y$ gauge symmetry and indicates that this minimal gauge structure is the correct one at currently available energies.

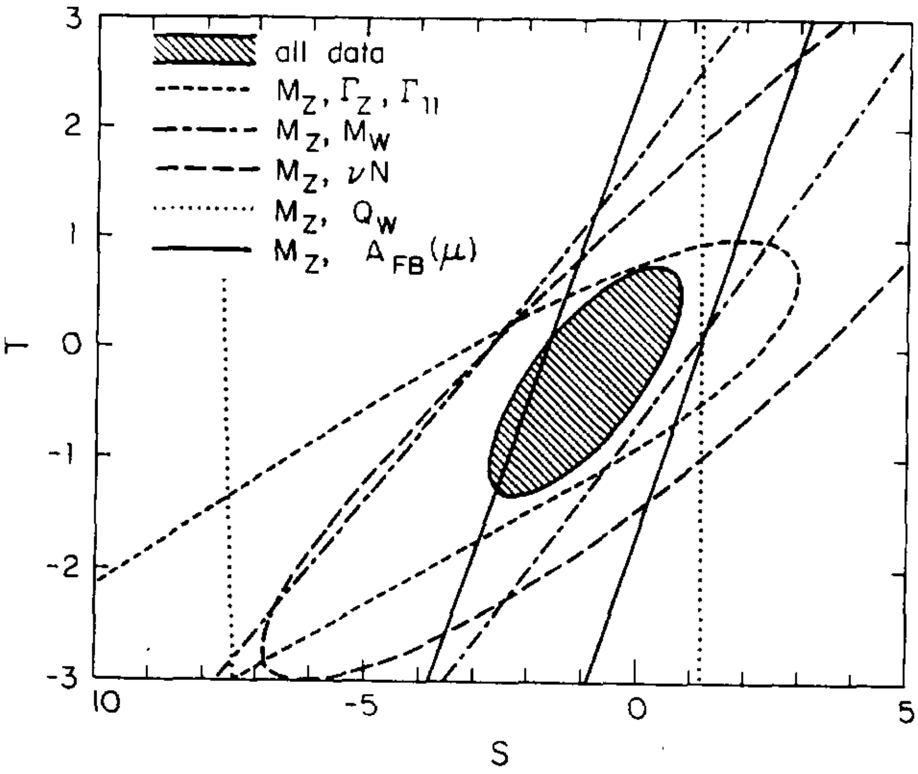

Figure 28. Present constraints on electroweak heavy physics parameters S and T from various experiments and common 90% region (shaded), with $U \equiv 0$ (March 1991)[88].

What are the general expectations for the radiative effects of new particles and how do the precision data constrain these expectations? We must note at this point that, in general, the (S, T, U) can receive contributions of opposite signs, depending on the exact model. This might be of some help if the trend toward negative values seen above continues with more precise data. Furthermore, the current values are easily consistent with zero. Apart from fine-tuning cancellations, the two natural ways of obtaining this result are to have no significant heavy physics apart from the heavy top quark, or to have non-Standard models that automatically respect the gauge or the full global chiral $SU(2)_L \times SU(2)_R$ symmetry.

The generic contribution to the custodial-breaking T comes from mass splittings in weak iso-multiplets. For example, a split doublet of Dirac fermions or non-Higgs scalars contributes[19]:

$$\alpha T = \frac{N_C G_F}{8\sqrt{2}\pi^2}[m_u^2 + m_d^2 - \frac{m_u^2 m_d^2}{m_u^2 - m_d^2}\ln(\frac{m_u^2}{m_d^2})], \qquad (V.11)$$

with N_C = number of colors (three for quarks). (V.11) is positive semi-definite, vanishing if $m_u = m_d$. In the SM, we already have one built-in significant contribution of this type from the top-bottom quark doublet. On the other hand, there is also the Higgs contribution (V.4), of opposite sign (generic to loops with gauge bosons) and vanishing if $M_W = M_Z$, but numerically quite small for physically meaningful values of the Higgs mass, $m_H <$ TeV. Negative contributions to T can arise from heavy Majorana neutrinos[96] or from pseudoscalars in some cases[97]. It is convenient to define a mass-squared splitting parameter Δm^2 from T, using the currently positive 90% upper bound (1.28 times one standard deviation above the central value): $T < 0.33$, and

$$\alpha T = \frac{N_c G_F}{8\sqrt{2}\pi^2} \cdot \Delta m^2, \tag{V.12}$$

yielding $\Delta m^2 < (152 \text{ GeV}/\sqrt{N_c})^2$. Thus one-loop splitting limits are now smaller than the basic VEV $v = 246$ GeV.

The parameter S[19,39,66,70], which receives both custodial-conserving and -breaking contributions, was not available before the Z pole measurements, since $\Delta_3(q^2)$ disappears from low-energy $q^2 \to 0$ observables. For degenerate multiplets heavier than the Z, S is always proportional to the sum over each multiplet of custodial isospin weight i_V, $\sum I_{3V}^2 = i_V(i_V+1)(2i_V+1)/6$. For example, for each complete new generation of heavy, degenerate ordinary Dirac fermions, $i_V = 1/2$, because ordinary fermions are equal mixtures of $i_L = 1/2, i_R = 0$ and $i_L = 0, i_R = 1/2$ states. The sum is $(1/2)(3/2)(1+1)/6 = 1/4$, and the contribution to S is:

$$S = \frac{2}{3\pi} \cdot \frac{1}{4} \cdot (1+3) = \frac{2}{3\pi}, \tag{V.13}$$

the first term for leptons ($N_C = 1$) and the second for quarks ($N_C = 3$). Thus $S = 0.21$ per generation; the 90% upper limit for S is 0.27, allowing no more than one degenerate heavy generation of fermions if we saturate the bound. However, S can receive negative contributions from both custodial-conserving and custodial-breaking corrections. For example, a split doublet of fermions with $\Lambda = (B-L)/2$ contributes:

$$S = \frac{N_C}{6\pi}[1 - 2\Lambda \ln(m_u^2/m_d^2)]. \tag{V.14}$$

Unfortunately, the negative contributions to S in general tend to be small, so they are not much help in cancelling positive contributions unless multiplied by large group representation factors. In (V.14), for example, we cannot split m_u and m_d by much without creating problems with T.

The major interest in constraining new particles with electroweak couplings revolves around testing specific non-Standard models which are taken seriously for other reasons. The most important motivation for such models is stabilizing the Higgs sector[8,13,14,97]. The Higgs potential of elementary scalars is sensitive to quadratically divergent radiative corrections with particles in loops of arbitrary masses: there is no decoupling for scalars, in general. The large gauge hierarchy of a GUT VEV $\gtrsim 10^{15}$ GeV and the electroweak VEV $\simeq 250$ GeV is unstable; the small VEV naturally floats up to the GUT VEV once radiative corrections to the Higgs potential are introduced. Even with a fixed electroweak VEV, the physical Higgs mass is unstable by radiative corrections against floating up to a TeV, implying a strongly-coupled Higgs sector. This instability arises from the weak logarithmic corrections to $\lambda_H \sim m_H^2/v^2$. If we wish to avoid the gauge hierarchy problem and possibly also demand a light physical Higgs spectrum without fine-tuning, a new symmetry must be introduced into the Higgs sector to prevent quadratically-divergent corrections from appearing at all and possibly also forcing the logarithmic corrections to λ_H to be small. One approach is not to have elementary scalars at all, but rather fermion-antifermion condensates. The Higgs mass corrections are naturally controlled in this scenario by the chiral symmetry of the fermions. The canonical examples are $t\bar{t}$ condensates[14] induced by suppressed higher-dimensional operators in $\mathcal{L}_{\rm eff}$, and technicolor[13]. In the $t\bar{t}$ models, the physical Higgs is a tightly bound top-antitop state, with $m_t \lesssim m_H \lesssim 2m_t$. The top mass in these models is related to the W mass. The resulting low-energy effective theory is essentially the same as the MSM, with these mass relations and the same radiative corrections. The minimal models require top masses $m_t \gtrsim 200$ GeV, higher than the allowed bounds (see last section), but this may be curable in variant models.

Technicolor models[8,13] produce electroweak symmetry breaking with the condensates of *technifermions,* fermions with electroweak couplings and also couplings to a new strong force mediated by technicolor (TC) gauge bosons. The TC gauge coupling is asymptotically free and tuned to become strong at $\Lambda_{TC} \simeq 250$ GeV. The resulting condensates break the chiral $SU(2)_L \times SU(2)_R$ down to the $SU(2)_V$ in a manner analogous to the spontaneous breaking of hadronic chiral symmetry by QCD. Three of the resulting Goldstone technimesons (analogues of pions) are eaten by the W and Z. The minimal TC scenario cannot produce masses for ordinary fermions, however, requiring the introduction of a larger gauge group, extended technicolor (ETC)[8,98], that interacts with both fermions and technifermions. The technifermion condensates give mass to the fermions via two-fermion/two-technifermion interactions. Until recently, the best limits on technicolor models came from bounds on FCNC's produced by the GIM- and custodial-violating ETC interactions that give rise to fermion masses. The new measurements of gauge boson properties, however, test directly for the first time the fundamental TC idea itself, in the gauge boson masses. The custodial-preserving S parameter in particular can be estimated with some confidence in TC theories, because it arises from the overall chiral symmetry breakdown, while the custodial-violating T is more difficult to compute. We can ignore U in TC theories, as it is typically of order $T \cdot (M_Z^2/\Lambda_{TC}^2)$. The implementation of TC proceeds by assuming a TC gauge group $SU(N_{TC})$, relating its properties to QCD ($N = 3$) by shifting the mass scale and using the $1/N$ approximation, then reading off the appropriately rescaled numbers from known hadronic interactions. QCD becomes an analog computer for TC models. These models also assume N_{TF}

flavors of technifermions, in analogy with ordinary quarks. Taking this approach, Peskin and Takeuchi found[71]:

$$S \simeq 0.3\frac{N_{TF}}{2}\frac{N_{TC}}{3} - 0.13, \qquad (V.15)$$

where the final shift comes from their use of $m_H = 1$ TeV as a reference. The 90% upper limit $S < 0.27$ is consistent only with the smallest TC models. Other groups have carried out TC radiative corrections estimates using chiral Lagrangian techniques[99]. However, the validity of the QCD analogy and chiral symmetry is unclear for the realistic ETC models. The ETC gauge group is assumed to break at some Λ_{ETC} to the SM plus TC gauge interactions; the latter then breaks down at Λ_{TC}. In the QCD-like models, $\Lambda_{ETC} \simeq 30$ TeV, which is not large enough to suppress FCNC's with a large top quark mass. To raise Λ_{ETC} to the needed value $\simeq 100$ TeV, a number of technitheorists have proposed "walking TC"[100]. Here the TC gauge coupling, instead of dropping rapidly as the QCD coupling does, runs slowly ("walks") from Λ_{TC} to Λ_{ETC}, pushing the latter up. Whether this new dynamics affects the chiral symmetry breaking properties of TC so as to invalidate the QCD analogy and chiral perturbation theory is unknown[101]. This situation will change with a better understanding of the properties of walking technicolor.

The other possibility for saving the Higgs sector is to retain elementary scalars, but then relate them with a symmetry to some fermions, whose masses are then protected by chiral symmetry. This symmetry is *supersymmetry* [97,102]. It implies superpartners for all SM particles: scalar squarks and sleptons, and fermionic Higgsinos and gauginos (winos, zinos, photinos, and gluinos). The minimal SUSY SM requires two Higgs doublets. Unbroken SUSY would imply that particles and their superpartners have the same masses. Clearly, supersymmetry must be broken. That breaking can be arranged in a number of ways. The key to this breaking for our purposes is that the SUSY-breaking mechanism by itself does not break the $SU(2)_L \times U(1)_Y$ gauge symmetry. Thus SUSY extensions of the SM have no *new* fundamental sources of EW symmetry breaking. The masses of the SUSY particles are combinations of $SU(2)_L \times U(1)_Y$-breaking and SUSY-breaking terms. The presence of the new states contributes new loops to the (S, T, U), but only in a way that depends on $SU(2)_L \times U(1)_Y$-breaking physics already present in the SM: the Higgs VEVs, the gauge boson masses M_W and M_Z, and the fermion masses, especially the large top mass. If the SUSY-breaking scale is large compared to the weak scale $v = 246$ GeV, then the superpartner masses are dominated by the $SU(2)_L \times U(1)_Y$-invariant SUSY-breaking terms, and the effects of the superpartners decouple from electroweak gauge interactions by inverse powers of the SUSY-breaking scale. Thus, the precision EW data places only weak bounds on supersymmetric extensions of the EW SM. These decoupling contributions have been computed by a number of authors[103]. As we saw in the last section, SUSY can have dramatic implications for the running of gauge couplings in GUTs, but this effect involves a large new scale, M_X, and large leading logarithms of the form $\ln(M_X^2/M_{SUSY}^2)$. To stabilize the Higgs sector, the SUSY-breaking scale M_{SUSY} cannot be much larger than a TeV.

V.5 Future Prospects

☞*Alice laughed. 'There's no use trying,' she said; 'one* can't *believe impossible things.' 'I dare say you haven't had much practice,' said the Queen. 'When I was your age I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast.'*

– Lewis Carroll, *Through the Looking Glass*

The basic message of these significant constraints on non-SM physics is that whatever extension of the MSM we consider for theoretical reasons or personal taste, we are faced with three choices: (1) the non-SM physics does not participate in or add to the $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$-breaking sector, thus decoupling from the gauge interactions; (2) the new physics satisfies the more restrictive global chiral symmetry $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ to good approximation; or (3) the non-standard physics is minimal or non-existent. It seems natural, that is, to assume that one or more custodial-preserving Higgs VEVs and the heavy top quark are able to account for all the gauge and global symmetry-breaking indicated by precise measurements of electroweak gauge interactions. The suppression of flavor-changing neutral currents suggests the GIM family symmetry (or a similar mechanism) as another approximately good global symmetry. On the other hand, a Higgs sector made of elementary scalars is unstable against quadratic divergences (such as the GUT-electroweak gauge hierarchy), making modification or extension of the Higgs sector necessary if we are to avoid fine-tuning. This requirement and the two global symmetry conditions place rather general restrictions on what sort of new physics we might find beyond the Z. The extended Higgs sectors based on supersymmetry and $t\bar{t}$ condensates are illustrative of the first and third possibilities listed above. Technicolor models that satisfy the second case seem possible, at least if the number of techniflavors is kept small.

Short of measuring the spectrum and dynamics of elementary particles all the way to a TeV with the SSC, an impressive array of experiments in this decade will make further progress in restricting the possibilities for non-standard physics. An important milestone will be the discovery of the top quark by direct production, with a reasonable determination of its mass[104]. This will remove the major MSM uncertainty in the heavy physics (S, T, U) and also in the flavor-changing neutral currents. Because the top-Higgs Yukawa coupling is large, measurement of the top's decay modes, branching ratios and total width may also shed light on the Higgs sector. And, of course, any direct production of non-Standard states, such as the superpartners or technimesons, would change the situation immediately.

Meanwhile, limits on radiative corrections of greater precision will steadily eat away at the allowed space of (S, T, U). Better low-energy limits will come from neutrino scattering experiments being planned for the next few years[105], and from another round of cesium APV experiments[89]. The latter will involve comparisons of parity violation in different isotopes of cesium, a procedure that can eliminate many of the systematic experimental and atomic theory uncertainties. Also, low-energy experiments generally have better sensitivity to new gauge bosons than Z pole measurements, suppressed by $M_Z^2/M_{Z'}^2$, not $\Gamma_Z M_Z/M_{Z'}^2$, so low-energy electroweak physics is not dead yet!

Gauge boson measurements will improve in the near future. High-statistics remeasurement of the currently known Z properties, the Z partial widths and the forward-backward asymmetries $A_{FB}(Z)$ especially, would improve the measured $s_*^2(Z)$. Potentially the greatest improvement in the value of $s_*^2(Z)$ can come from the Z polarization asymmetry $A_{LR}(Z)$[92], which is not subject to the same systematic and theoretical uncertainties that the forward-backward asymmetries are. Two other important measurements are the W mass and width from the $e^+e^- \to W^+W^-$ production cross section threshold[106]. Figure 29 is a repeat of Figure 28; but with two different Z polarization asymmetry measurements, the W mass measured to ± 100 MeV, and an improved cesium APV measurement, giving some idea of how much the present heavy physics limits might be bettered within the decade.

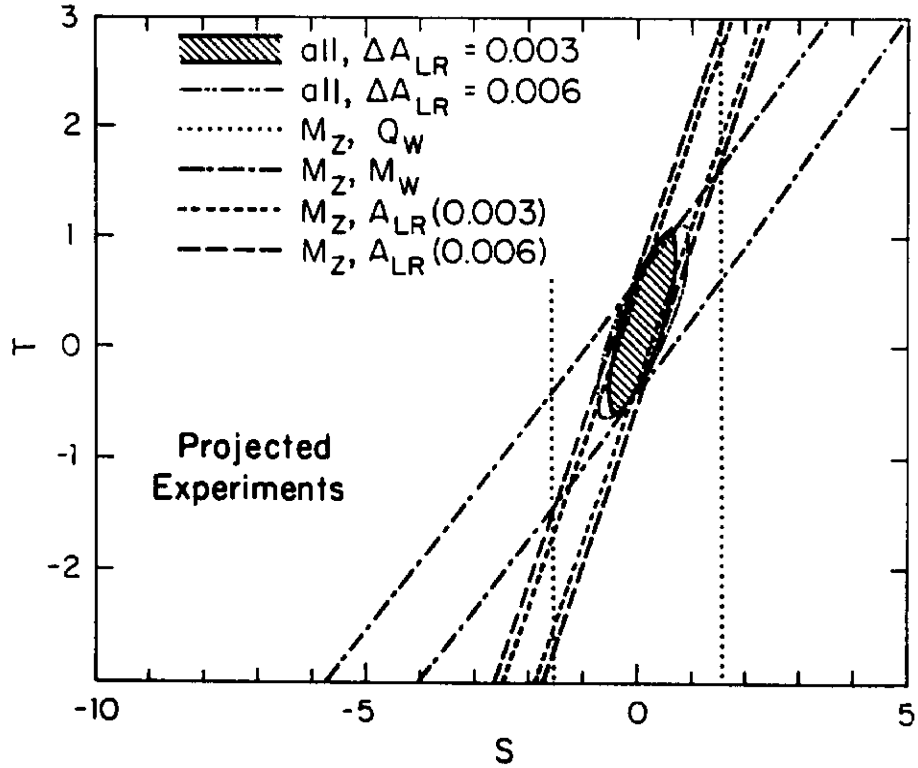

Figure 29. Possible future constraints on electroweak heavy physics parameters S and T ($U \equiv 0$), assuming new Z polarization asymmetry, W mass, and APV measurements described in text, and no heavy physics except top quark of mass 124 GeV and Higgs boson of mass 250 GeV [88].

A qualitatively new kind of electroweak interaction is measured by the $e^+e^- \to W^+W^-$ process, the three-boson vertex (Figure 30a). The W pair production experiment will directly measure this purely non-Abelian effect for the first time in the $SU(2)_L \times U(1)_Y$ theory[107]. At energies well above threshold, the total tree-level cross section falls as s^{-1}, enforced by a tree-level gauge symmetry cancellation between Figures 30a and 30b. The presence of massive particles in the three gauge boson vertex loop corrections (Figure 30c) upsets this cancellation until much higher energies. Such an effect is another example of breaking of the global chiral $SU(2)_L \times SU(2)_R$ symmetry by radiative corrections, but is a new effect independent of the gauge propagator heavy physics Δ functions — it is a non-Born-like correction to the three-boson vertex analogous to the heavy top correction to the $Z \to b\bar{b}$ vertex.

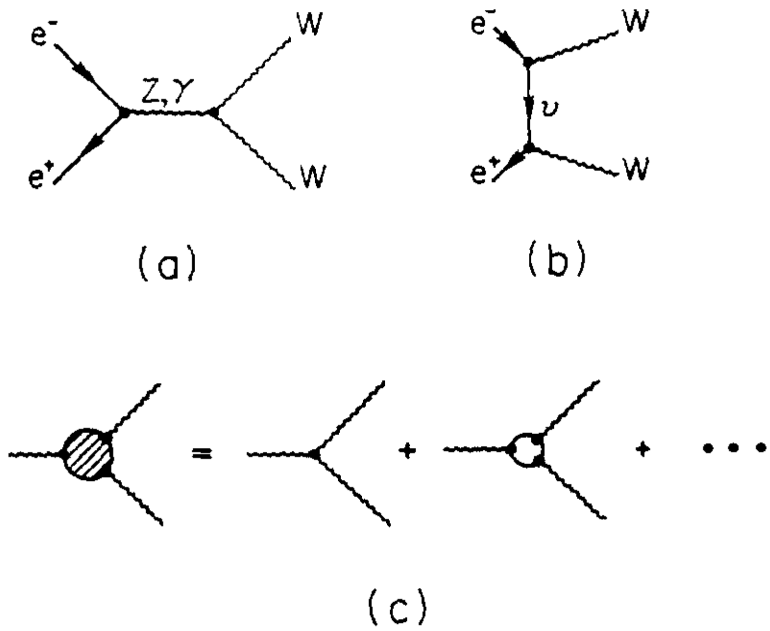

Figure 30. Tree and loop graphs for $e^+e^- \to W^+W^-$: *(a) Z and photon annihilation channels; (b) neutrino exchange channel; (c) loop corrections to three-gauge-boson vertex*[107].

Thus it seems clear that in 1992, particle physics has solid experimental proof for the validity of the gauge principle and of its specific Standard Model implementation in electroweak interactions. What is lacking is equally deep insight into the Higgs sector. Why the breaking of electroweak gauge symmetry and the associated global chiral symmetry is so well accounted for by the minimal Higgs doublet VEV, a heavy top quark, and little or nothing else, is a mystery. Reconciling this fact with the need for extensions of the Higgs sector to cure its radiative instabilities and experimentally establishing what that extension is, has now become the central question of electroweak phenomenology.

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Appendix A: Neutral- and Charged-Current Matrix Elements in the IBA[39,74,109]

The tree-level or Born matrix elements for massless external fermions f and f' have the simple form:

$$\begin{aligned}
\mathcal{M}_{NC} &= \frac{e_0^2 QQ'}{q^2} + \frac{e_0^2}{s_0^2 c_0^2} \frac{[I_3^L - s_0^2 Q][I_3^L - s_0^2 Q]'}{q^2 - M_{Z_0}^2}, \\
\mathcal{M}_{CC} &= \frac{e_0^2}{2s_0^2} \frac{[I_+^L I_-^{L'} + I_-^L I_+^{L'}]}{q^2 - M_{W_0}^2}.
\end{aligned} \tag{A.1}$$

The matrix elements have the Born-like *current-current* form, where the matter-gauge vertices are purely vector or axial-vector in the limit of massless external fermions. In this same limit and at tree level, the effect of the would-be Goldstones in the R_ξ gauge can be ignored, and the physical Higgs channel decouples from the gauge channels because its CP properties are different from those of the gauge interactions. The gauge symmetry property of the matter-gauge vertex is simply of the form *gauge coupling · gauge group generator* : $e_0 Q$ for the photon, $(e_0/s_0 c_0)(I_3^L - s_0^2 Q)$ for the Z, and $(e_0/s_0)I_\pm^L$ for the W. All radiative corrections to the $\mathcal{M}$'s that come with these same Lorentz and gauge symmetry properties and depend only on q^2, are Born-like[7,8,39,59,62,74].

The inclusion of the loop corrections to (A.1) is non-trivial in the EW gauge interactions. We cannot simply calculate the $\mathcal{M}$'s to a fixed order of perturbation theory, because the Z and W masses are shifted by the radiative corrections. Alternatively, the propagators become quasi-singular as q^2 approaches the gauge poles. We must therefore carry out a Dyson sum of the irreducible self-energies to obtain the modified propagator that describes the physical gauge bosons. However, the gauge boson self-energies in a non-Abelian gauge theory are not gauge invariant, because the gauge bosons' contributions to their own self-energies are not gauge invariant. If we could obtain the exact vertex, box and self-energy corrections to the Born matrix elements (A.1), we could, in principle, continue to use the gauge non-invariant propagator, because the full matrix elements would still be gauge invariant. One way to apply this approach would be to express the matrix elements in terms of the exact irreducible corrections[73]:

$$\frac{e_0^2}{s_0^2 c_0^2}\frac{[I_3^L - s_0^2 Q](1+\Gamma(q^2))[I_3^L - s_0^2 Q]'(1+\Gamma(q^2))}{q^2 - M_{Z_0}^2 + \Pi(q^2)} + \Theta(q^2), \qquad (A.2)$$

is the neutral-current matrix element keeping only the Z channel. Then, we could Laurent expand $\mathcal{M}_{NC}$ or the cross sections in negative and positive powers of $s - s_p$, where s_p is the complex pole of the matrix element. The coefficient of each term in this expansion must be gauge invariant; each coefficient could itself be re-expanded in powers of α, and the subcoefficients are also gauge invariant[73]. Since the theory is solvable only in perturbation theory, this method must at some point make contact with the α expansion. An alternative method, the one used in these lectures, is to rearrange the corrections at any order in the irreducible expansion into gauge-invariant classes. The corrections fall into Born-like and non-Born-like types: the former is a gauge-invariant class by construction; the latter is made up of a number of separately gauge-invariant classes, as discussed in section IV.4. The Born-like corrections are inserted into modified Dyson equations for the propagators. We then obtain gauge-invariant effective propagators for all q^2[39,62]. The effective propagators are essentially the analytic continuations of matrix elements such as (A.2) from the pole regions to all kinematical regimes, but only order by order in perturbation theory. To a given order, the continuation is unique[116].

The gauge boson self-energy corrections Π are obviously Born-like, since they are connected to the outer legs by bare propagators at any order (Figures 6,7). The matter currents coupled to the physical gauge bosons can be decomposed according to their electric charge and left isospin quantum numbers:

$$J_\gamma = e_0 J_Q, \quad J_Z = (e_0/s_0 c_0)[J_3^L - s_0^2 Q], \quad J_W^{\pm} = (e_0/s_0)[J_1^L \pm i J_2^L], \qquad (A.3)$$

The physical gauge boson self-energies can then be decomposed analogously:

$$
\begin{aligned}
\Pi_{\gamma\gamma} &= e_0^2\Pi_{QQ}, \quad \Pi_{\gamma Z} = (e_0^2/s_0c_0)[\Pi_{3Q} - s_0^2\Pi_{QQ}], \\
\Pi_{ZZ} &= (e_0^2/s_0^2c_0^2)[\Pi_{33} - 2s_0^2\Pi_{3Q} + s_0^4\Pi_{QQ}], \\
\Pi_{WW} &= (e_0^2/s_0^2)[\Pi_{11} + \Pi_{22}]/2 \\
&= (e_0^2/s_0^2)[\Pi_{+-} + \Pi_{-+}]/2 = (e_0^2/s_0^2)\Pi_{\pm},
\end{aligned}
\tag{A.4}
$$

There are four independent self-energies: $\Pi_{QQ}, \Pi_{3Q}, \Pi_{33}$, and $\Pi_{\pm}$. Note that, by the unbroken residual isospin symmetry, $\Pi_{+-} = \Pi_{-+}$.

All of the vertex and box loops have the Born Lorentz properties, but not always the Born gauge structure. The non-commutation of the gauge group generators in a non-Abelian gauge theory gives rise to Born-like vertex and box corrections. Consider first the vertex corrections (Figures 20a, b). Denote a gauge group generator by T^a. Let the vertex carry index a, and let other letters $b, c, \ldots$ denote dummy indices. The loop of Figure 20a has the group structure:

$$
T^bT^aT^b = (C_F - \frac{1}{2}C_V) \cdot T^a,
\tag{A.5}
$$

where $C_F = T^bT^b$ is the external fermion Casimir operator. The second term is proportional to the gauge boson (adjoint) Casimir operator $C_V\delta^{bc} = f^{bde}f^{cde}$ and to the original tree-level generator T^a. This factorizable non-Abelian correction to the tree-level vertex is therefore Born-like, not depending on the quantum numbers of the external fermions. The second term is also a correction to the tree vertex, but is *not* Born-like, as it depends on the quantum numbers of the external fermions. This term is cubic in external quantum numbers and is thus Abelian. The loop of Figure 20b is purely non-Abelian, being proportional overall to the non-Abelian three-gauge boson vertex. Its group structure is:

$$
T^bT^cf^{abc} = \frac{i}{2} \cdot C_V \cdot T^a.
\tag{A.6}
$$

The Abelian and non-Abelian parts of the vertex corrections to the charged- and neutral-current matrix elements are uniquely isolated this way. The non-Abelian vertex corrections are completely absorbed into the Born-like corrections; the Abelian vertex corrections are non-Born-like (Appendix C).

The box loops also fall into Abelian and non-Abelian forms. Consider the direct (Figure 20c) and crossed (Figure 20c, with the gauge lines crossed) box loops. The group structures take either form:

$$(T^aT^b)_f(T^aT^b)_{f'}, \quad (T^aT^b)_f(T^bT^a)_{f'}, \tag{A.7}$$

where the two generators running along each fermion line act on the same fermion indices, but each pair acts separately. These forms can be expressed in terms of commutators and anticommutators, and the commutators then re-expressed using the Lie algebra. If the gauge channels are all summed over, (A.7) is transformed to:

$$\frac{1}{4}[\mp C_V(T^a)_f(T^a)_{f'} + \{T^a, T^b\}_f\{T^a, T^b\}_{f'}], \tag{A.8}$$

with the upper (lower) sign for the direct (crossed) box. For the EW gauge theory, where the gauge boson masses are different, the sum over gauge channels cannot be performed. If the gauge channels are not summed over, (A.8) also contains mixed terms (not shown), products of one commutator and one anticommutator. (Mixed terms cannot occur in the vertex, because there is only one virtual gauge boson in the loop.) The pure anticommutator term is Abelian, while the mixed terms are non-Abelian; both are non-Born-like. But again, we notice a pure non-Abelian term proportional to the gauge boson Casimir operator C_V and reproducing the Born current-current gauge form. This pure non-Abelian term, however, cannot be fully absorbed into the Born-like corrections, because it lacks the simple kinematical dependence (only q^2). The tree-level, self-energy and vertex graphs depend on $q^2 = s$ or t, while the box graphs depend on s and t simultaneously. But $\Theta^{nAb}(s,t)$ can always be separated into parts that depend only on $q^2 = s$ or t, and everything else. The first type of corrections are then Born-like. The terms left from $\Theta^{nAb}(s,t)$ carrying both s and t dependence are non-Born-like. This separation of non-Abelian box corrections is not unique. Here we use the following separation method. Note that the three kinematical Mandelstam variables sum to zero for massless external particles: $s+t+u=0$. Set $u=0$ to isolate the Born-like part. That is, $\Theta^{nAb}(s,t) = \bar{\Theta}^{nAb}(q^2) + \tilde{\Theta}^{nAb}(s,t)$; where $\bar{\Theta}^{nAb}(q^2) = \Theta^{nAb}(s,t;u=0)$, so that $q^2 = s = -t$ or $q^2 = t = -s$. The residual $\tilde{\Theta}^{nAb}$ is the non-Born-like remainder. There are three advantages to this particular approach. First, in the kinematically physical region, s is positive semi-definite ($s \geq 0$) and t negative semi-definite ($t \leq 0$). s is the square center-of-mass energy of the process, while t is minus the square center-of-mass three-momentum transfer. Thus, setting $u = 0$ is always kinematically physical. Second, the condition $u = 0$ treats s and t symmetrically, which is convenient for processes such as Bhabha scattering (Figure 25) where both s and t channels occur. Third, there are no poles in physical amplitudes at $u = 0$; while $s = 0$ and $t = 0$, for example, are photon channel poles for $q^2 = s$ and $q^2 = t$, respectively.

The next step is to organize the different gauge-invariant effective self-energies Π^* from the self-energies $\Pi(q^2)$, non-Abelian vertices $\Gamma^{nAb}(q^2)$, and the Born-like non-Abelian boxes $\bar{\Theta}^{nAb}(q^2)$[62]. The full reducible four-point matrix element $\mathcal{M}$ is generically made up of the full propagator D, the full vertex correction Γ, and the irreducible Θ. Schematically,

$$\mathcal{M} = (1+\Gamma)D(1+\Gamma) + \Theta, \tag{A.9a}$$

where D conventionally satisfies the Dyson equation:

$$D = D_0 + D\Pi D_0. \tag{A.9b}$$

We could put $\mathcal{M}$ into a pure propagator form by collecting Γ and Θ into an effective Π^* :

$$\mathcal{M} = D_0 + D\Pi^* D_0. \tag{A.9c}$$

To linear order in the irreducible loop functions, the effective self-energy is then:

$$\Pi^* = \Pi + 2D_0^{-1}\Gamma + D_0^{-2}\Theta. \tag{A.9d}$$

This form is valid if the irreducible Π, Γ, and Θ are computed to one loop; the form of Π^* in terms of these functions changes at higher order. At one loop, Π^* must be truncated at linear order in the irreducibles. Note, however, that the effective propagator D is always a non-linear function of Π^*, at any order. This absorption of vertex and box corrections into an effective propagator is valid only for the Born-like vertex and box corrections. This same form (A.9d) for Π^* can be obtained by computing the $\mathcal{M}$ to first order with no Dyson sum, rearranging the terms in the matrix element to form an effective self-energy from Π, Γ, and Θ. Since we can eliminate other gauge-invariant classes of corrections from the matrix element (the non-Born-like terms), the Born-like corrections, so defined, must be separately gauge invariant. All gauge-dependent results quoted here are computed in the $\xi = 1$ gauge[39,62].

The neutral- and charged-current matrix elements in terms of the full propagators are[39,59]:

$$\begin{aligned}\mathcal{M}_{NC} = {} & e_0 Q \cdot D_{\gamma\gamma}(q^2) \cdot e_0 Q' \\ & + \frac{e_0}{s_0 c_0} \cdot e_0 \cdot [Q(I_3^L - s_0^2 Q)' + Q'(I_3^L - s_0^2 Q)] \cdot D_{\gamma Z}(q^2) \\ & + \frac{e_0}{s_0 c_0}(I_3^L - s_0^2 Q) \cdot D_{ZZ}(q^2) \cdot \frac{e_0}{s_0 c_0}(I_3^L - s_0^2 Q)',\end{aligned} \tag{A.10a}$$

and:

$$\mathcal{M}_{CC} = \frac{e_0}{s_0} \cdot [\frac{I_+^L I_-^{L'} + I_-^L I_+^{L'}}{2}] \cdot D_{WW}(q^2) \cdot \frac{e_0}{s_0}, \tag{A.10b}$$

The Dyson equations for the full propagators are[39,74, 109]:

$$\begin{aligned} D_{\gamma\gamma} &= D^0_{\gamma\gamma} + D^0_{\gamma\gamma} \cdot \Pi^*_{\gamma\gamma} \cdot D_{\gamma\gamma} + D^0_{\gamma\gamma} \cdot \Pi^*_{\gamma Z} \cdot D_{\gamma Z} \\ D_{ZZ} &= D^0_{ZZ} + D^0_{ZZ} \cdot \Pi^*_{ZZ} \cdot D_{ZZ} + D^0_{ZZ} \cdot \Pi^*_{\gamma Z} \cdot D_{\gamma Z} \\ D_{\gamma Z} &= D^0_{ZZ} \cdot \Pi^*_{ZZ} \cdot D_{\gamma Z} + D^0_{ZZ} \cdot \Pi^*_{\gamma Z} \cdot D_{\gamma\gamma} \\ D_{WW} &= D^0_{WW} + D^0_{WW} \cdot \Pi^*_{WW} \cdot D_{WW}. \end{aligned} \tag{A.11}$$

With the following effective self-energies[62]:

$$\begin{aligned} \Pi^*_{QQ} &= \Pi_{QQ} + 2q^2 \Gamma_3^{nAb} - 2q^4 \bar{\Theta}_3^{nAb} \\ \Pi^*_{3Q} &= \Pi_{3Q} + 2q^2 \Gamma_3^{nAb} - 2q^4 \bar{\Theta}_3^{nAb} \\ &\quad - \frac{M_W^2}{\rho}[\Gamma_3^{nAb} - 2q^2 \bar{\Theta}_3^{nAb}] \\ \Pi^*_{33} &= \Pi_{33} + 2q^2 \Gamma_3^{nAb} - 2q^4 \bar{\Theta}_3^{nAb} \\ &\quad - \frac{2M_W^2}{\rho}[\Gamma_3^{nAb} + (-2q^2 + \frac{M_W^2}{\rho})\bar{\Theta}_3^{nAb}] \\ \Pi^*_{\pm} &= \Pi_{\pm} + 2(q^2 - M_W^2)\Gamma_{\pm}^{nAb} + 2(q^2 - M_W^2)^2 \bar{\Theta}_{\pm}^{nAb}, \end{aligned} \tag{A.12}$$

the matrix elements with the full Born-like corrections can be brought into the forms[39,62]:

$$\begin{aligned} \mathcal{M}_{NC} &= \frac{e_*^2(q^2)QQ'}{q^2} \\ &\quad + \frac{e_*^2(q^2)}{s_*^2(q^2)c_*^2(q^2)} \frac{[I_3^L - s_*^2(q^2)Q][I_3^L - s_*^2(q^2)Q]'}{q^2 - \frac{e_*^2}{s_*^2 c_*^2}\frac{1}{4\sqrt{2}G_{F*}\rho_*} + i\sqrt{q^2}\Gamma_{Z*}(q^2)} \end{aligned} \tag{A.13a}$$

$$\mathcal{M}_{CC} = \frac{e_*^2(q^2)}{2s_*^2(q^2)} \frac{[I_+^L I_-^{L'} + I_-^L I_+^{L'}]}{q^2 - \frac{e_*^2}{s_*^2}\frac{1}{4\sqrt{2}G_{F*}} + i\sqrt{q^2}\Gamma_{W*}(q^2)},$$

if we solve the Dyson equations (A.11) and substitute them into the matrix elements (A.10), with the four star functions defined as follows[39,62]:

$$\begin{aligned}\frac{1}{e_*^2(q^2)} &= \frac{1}{e_0^2} - \Pi^{*\prime}{}_{QQ}(q^2)\\ \frac{1}{g_*^2(q^2)} &= \frac{1}{g_0^2} - \Pi^{*\prime}{}_{3Q}(q^2)\\ \frac{1}{4\sqrt{2}G_{F*}(q^2)\rho_*(q^2)} &= \frac{1}{4\sqrt{2}G_{F_0}\rho_0} - [\Pi_{3Q}^*(q^2) - \Pi_{33}^*(q^2)]\\ \frac{1}{4\sqrt{2}G_{F*}(q^2)} &= \frac{1}{4\sqrt{2}G_{F_0}} - [\Pi_{3Q}^*(q^2) - \Pi_{\pm}^*(q^2)].\end{aligned} \tag{A.14}$$

Then $s_*^2(q^2) = e_*^2(q^2)/g_*^2(q^2)$. In the Minimal SM case, set $\rho_0 = 1$ in (A.14) and $\rho = 1$ on the *right*-hand side of (A.12). The width functions for massless fermions are:

$$\begin{aligned}\Gamma_{Z*}(q^2) &= \frac{e_*^2}{12\pi s_*^2 c_*^2}\sqrt{q^2}\sum_f [(\frac{1}{2}I_{3f}^L - s_*^2 Q_f)^2\\ &\quad + (\frac{1}{2}I_{3f}^L)^2]\cdot C_{QCD}\end{aligned} \tag{A.13b}$$

$$\Gamma_{W*}(q^2) = \frac{e_*^2}{48\pi s_*^2}\sqrt{q^2}\sum_D C_{QCD},$$

for all above-threshold fermions f and fermion doublets $D(f, f')$, where $q^2 > 0$, and:

$$C_{QCD} = [1 + \frac{3\alpha(Q_f^2 + Q_{f'}^2)}{8\pi}]\begin{cases}1, & \text{leptons}\\ 3[1 + \frac{\alpha_s(q^2)}{\pi} + \mathcal{O}(\alpha_s^2)], & \text{quarks,}\end{cases} \tag{A.13c}$$

and $f' = f$ in the Z case. CKM mixing has been ignored in the W width. The QCD and QED modifications to the imaginary parts of the Π's are included. The present value of the $\overline{MS}$ strong coupling at the Z pole from LEP is: $\hat{\alpha}_s(Z) = 0.119 \pm 0.006$[110]. The modifications to the real parts are also perturbatively calculable to $\mathcal{O}(\alpha\alpha_s)$[48], say, but are small (Figure 13c) — except for the effect of strong corrections in the hadronic contributions to Π_{QQ}, included by use of the dispersion relation from e^+e^- data (sections III.1/V.1)[47]. I show no non-Born-like strong corrections to the outer legs, since these generally require specialized QCD techniques[8,114]. *Total* cross sections to quark pairs are corrected by the simple C_{QCD} factor.

The "prime" superscript indicates that the self-energy has been divided by q^2; that is, $\Pi^*_{QQ} = q^2 \cdot \Pi^{*\prime}_{QQ}$, etc. That $\Pi_{QQ}(q^2)$, and thus $\Pi^*_{QQ}(q^2)$ in (A.12), is divisible by q^2, is known from the QED Ward identities[30]. That Π^*_{3Q} is also divisible by q^2, is due to the non-trivial fact that[59]:

$$\Pi^*_{3Q}(0) = \Pi_{3Q}(0) - \frac{M_W^2}{\rho}\Gamma_3^{nAb}(0) = 0. \qquad (A.15)$$

This electroweak Slavnov-Taylor identity is essential for preserving the structure of the theory beyond tree-level, because $\Pi_{3Q}(0) \neq 0$. The occurrence of $\Pi^*_{3Q}(0) \neq 0$ would produce spurious pole as $q^2 \to 0$, of the form $\Pi^*_{3Q}(0)/q^2$, in the Z-photon mixing. That $\Pi_{3Q}(0) \neq 0$ indicates that there is a new non-diagonal mass mixing between the Z and the photon at the loop level. This mixing is actually a vertex diagram in disguise (Figure 31). The non-Abelian vertex of the external legs (and it doesn't matter what type of particles are on the legs!) identically cancels this mixing in Π^*_{3Q} as $q^2 \to 0$, and rediagonalizes the neutral-current interactions in the S-matrix[39].

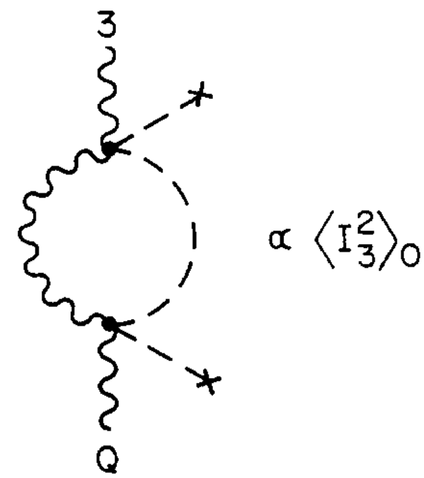

Figure 31. One-loop γZ mixing induces neutral-current mass mixing and is actually a two-gauge-boson/two-Higgs vertex in disguise, with $\langle \Phi I_3^{L\,2} \Phi \rangle$ summed over Higgs VEVs Φ[39].

The self-energies Π are compactly expressed in terms of the form factors B_0, B_1, and B_{21} of Passarino and Veltman[38]:

$$
\begin{aligned}
B_0(q^2;m_1^2,m_2^2) &= +\Delta - \int_0^1 dz f(z,q^2;m_1^2,m_2^2) \\
B_1(q^2;m_1^2.m_2^2) &= -\frac{1}{2}\Delta + \int_0^1 dz z f(z,q^2;m_1^2,m_2^2) \\
B_{21}(q^2;m_1^2,m_2^2) &= +\frac{1}{3}\Delta - \int_0^1 dz z^2 f(z,q^2;m_1^2,m_2^2), \\
f(z,q^2;m_1^2,m_2^2) &= \ln[m_1^2 + (m_2^2 - m_1^2 - q^2)z + z^2q^2 - i\epsilon].
\end{aligned}
\tag{A.16}
$$

Also define $B_3 = B_{21} + B_1$. The divergences in dimensional regularization always appear in the combination:

$$
\Delta = \frac{2}{4-n} - \gamma + \ln 4\pi, \tag{A.17}
$$

with $n \to 4$. Then, to one loop in the Standard Model, the self-energies in the $\xi = 1$ gauge are as follows[39,74].

Gauge bosons/Higgs

$$
\begin{aligned}
16\pi^2\Pi_{33}(q^2) = &-q^2[9B_3 - \frac{7}{4}B_0 - \frac{2}{3}](W,W) + 2M_W^2 B_0(W,W) \\
&- \frac{1}{4}(M_Z^2 - m_H^2)[2B_1 + B_0](Z,H) \\
&- \frac{1}{4}q^2[4B_3 + B_0](Z,H) - M_Z^2 B_0(Z,H),
\end{aligned}
\tag{A.18a}
$$

$$16\pi^2\Pi_\pm(q^2) = \frac{2}{3}q^2 - s_\theta^2[q^2(8B_3 - 2B_0) + 2M_W^2(2B_1 + B_0)](W,\gamma)$$
$$- c_\theta^2[q^2(8B_3 - 2B_0) + 2(M_W^2 - M_Z^2)(2B_1 + B_0)](W,Z)$$
$$- [q^2(B_3 + \frac{1}{4}B_0) + \frac{1}{4}(M_W^2 - M_Z^2)(2B_1 + B_0)](W,Z)$$
$$- (M_Z^2 - 3M_W^2)B_0(W,Z) - \frac{1}{4}(M_W^2 - m_H^2)[2B_1 + B_0](W,H)$$
$$- \frac{1}{4}q^2[4B_3 + B_0](W,H) - M_W^2 B_0(W,H), \tag{A.18b}$$

$$16\pi^2\Pi_{3Q}(q^2) = q^2[-10B_3 + \frac{3}{2}B_0 + \frac{2}{3}](W,W) + 2M_W^2 B_0(W,W), \tag{A.18c}$$

$$16\pi^2\Pi_{QQ}(q^2) = q^2[-12B_3 + B_0 + \frac{2}{3}](W,W), \tag{A.18d}$$

where $W = M_W$, $Z = M_Z$, $\gamma = M_\gamma = 0$, and $H = m_H$. In $\xi = 1$ gauge, the $\Pi_\pm$ self-energy contains no $M_\gamma \to 0$ IR divergences. For $\Pi_\pm$ and the charged-current vertices and boxes below, take $s_\theta^2 = \sin^2\theta_W = 1 - M_W^2/M_Z^2$, and $c_\theta^2 = 1 - s_\theta^2$.

Fermions

$$16\pi^2\Pi_{33}(q^2) = 2\sum_f {I_{3f}^L}^2[2q^2B_3 + m_f^2 B_0](f,f), \tag{A.19a}$$

$$16\pi^2\Pi_\pm(q^2) = \sum_D [2q^2B_3(1,2) - m_1^2B_1(2,1) - m_2^2B_1(1,2)], \tag{A.19b}$$

$$16\pi^2\Pi_{3Q}(q^2) = 4q^2\sum_f Q_f I_{3f}^L B_3(f,f), \tag{A.19c}$$

$$16\pi^2\Pi_{QQ}(q^2) = 8q^2\sum_f Q_f^2 B_3(f,f), \tag{A.19d}$$

where f stands for all fermions; D for all doublets of fermions with masses $1 = m_1$ and $2 = m_2$.

The neutral- and charged-current non-Abelian vertices are:

$$
\begin{aligned}
16\pi^2\Gamma_3^{nAb}(q^2) &= -[\Phi(q^2;W) + \Lambda(q^2;W,W) - 2B_0(0;W,W)], \\
16\pi^2\Gamma_\pm^{nAb}(q^2) &= -[s_\theta^2\Phi(q^2;\gamma) + c_\theta^2\Phi(q^2;Z) \\
&\quad + s_\theta^2\Lambda(q^2;W,\gamma) + c_\theta^2\Lambda(q^2;W,Z) - 2B_0(0;W,W)].
\end{aligned} \tag{A.20}
$$

The neutral- and charged-current Born-like non-Abelian boxes are:

$$
\begin{aligned}
16\pi^2\bar{\Theta}_3^{nAb}(q^2) = \frac{1}{M_W^2(2-u)}\Big\{ &-\frac{5}{2} - \frac{u\ln(u-i\epsilon)}{2} + \frac{7u}{6} + \frac{u^2}{12} \\
&+ \frac{2+2u-u^2}{2(1+2u)}(l_W\sqrt{1-4/u} + 2 - \frac{u}{6}) \\
&+ \frac{-1-2u+u^2}{2(1+2u)}(\Lambda(q^2;W,W) - \frac{u}{2}) \\
&- \frac{1-u}{u^2}[Sp(u-i\epsilon) - u - \frac{u^2}{4} \\
&+ \ln(u-i\epsilon)(\ln(1-u+i\epsilon) + u + \frac{u^2}{2})]\Big\},
\end{aligned} \tag{A.21a}
$$

$$16\pi^2\bar{\Theta}^{nAb}_{\pm}(q^2) = \frac{s_\theta^2}{M_W^2(1-u)}\Big\{\frac{(u-1)(1-4u^2)}{2u(4u+1)}[1+(\frac{1}{u}-1)l_\gamma]$$

$$+\frac{u-1}{2u}[\ln(u-i\epsilon)-1]$$

$$+\frac{2u(u-1)}{4u+1}[\Lambda(W,\gamma)+\frac{1}{2}]$$

$$+\frac{u-1}{2u^2}[Sp(u-i\epsilon)+\ln(1-u-i\epsilon)\ln(u-i\epsilon)]\Big\}$$

$$+\frac{c_\theta^4}{M_W^2(1+c_\theta^2-uc_\theta^2)}$$

$$\times\Big\{\frac{B(u,c_\theta^2)}{2uc_\theta^2(4uc_\theta^2+1+c_\theta^2)(1+c_\theta^2)}[x_+l_+ + x_-l_- + 1 - \frac{c_\theta^2}{s_\theta^2}\ln c_\theta^2]$$

$$+\frac{uc_\theta^2-1-c_\theta^2}{2uc_\theta^2}[\ln(uc_\theta^2-i\epsilon)-1+\frac{c_\theta^2}{s_\theta^2}\ln c_\theta^2]$$

$$+\frac{2(u^2c_\theta^4-uc_\theta^2-uc_\theta^4-1)}{(4uc_\theta^2+1+c_\theta^2)(1+c_\theta^2)}[\Lambda(W,Z)+\frac{1}{2}]$$

$$+\frac{1-u^2c_\theta^4+uc_\theta^4+uc_\theta^2}{uc_\theta^2}\frac{\ln c_\theta^2}{s_\theta^2}$$

$$+\frac{u-1}{2u^2}[Sp(u-i\epsilon)+\ln(1-u+i\epsilon)\ln(u-i\epsilon)]$$

$$+\frac{uc_\theta^2-1}{2u^2c_\theta^4}[Sp(uc_\theta^2-i\epsilon)+\ln(1-uc_\theta^2+i\epsilon)\ln(uc_\theta^2-i\epsilon)]\Big\},$$

$$B(u,c_\theta^2) = 4u^2c_\theta^4+uc_\theta^2-7uc_\theta^2-(1-c_\theta^4)(1-c_\theta^2)$$

$$+4u^2c_\theta^6+uc_\theta^4+uc_\theta^6-4u^3c_\theta^6,$$

$$(A.21b)$$

with $u = q^2/M_W^2$. The special functions $\Phi(q^2;V)$ and $\Lambda(q^2;W,V)$ are:[39,38,74]:

$$\Phi(q^2;V) = \frac{1}{2} - \ln(-v-i\epsilon) + 4(1+\frac{1}{2v})[\ln(-v-i\epsilon) - 1]$$

$$- 2(1+\frac{1}{v})^2[\frac{\pi^2}{6} - Sp(1+v+i\epsilon)],$$

$$\Lambda(q^2;W,W) = \frac{-5}{2} + \frac{2}{u} + (1+\frac{2}{u})l_W\sqrt{1-\frac{4}{u}} - (1+\frac{1}{2u})\frac{4}{u}l_W^2,$$

$$\Lambda(q^2;W,\gamma) = \frac{1}{2} + \frac{1}{u} + (\frac{1}{u^2} - 1)l_\gamma$$

$$- (2+\frac{1}{2u})\frac{1}{u}l_\gamma^2,$$

$$\Lambda(q^2;W,Z) = -\frac{1}{2} + \frac{(uc_\theta^2 + c_\theta^2 + 1)}{uc_\theta^2}[-\frac{c_\theta^2}{s_\theta^2}\ln c_\theta^2 + 1 + x_+l_+ + x_-l_-] \qquad (A.22)$$

$$+ \frac{(4uc_\theta^2 + c_\theta^2 + 1)}{2uc_\theta^2}(1+c_\theta^2)[\frac{l_+l_-}{uc_\theta^2} + \frac{\ln c_\theta^2}{s_\theta^2}],$$

$$l_W = \ln[\frac{\sqrt{1-4/u} - 1}{\sqrt{1-4/u} + 1} + i\epsilon]$$

$$l_\gamma = \ln[1-u-i\epsilon]$$

$$l_\pm = \ln[\frac{x_\pm - 1}{x_\pm} \pm i\epsilon]$$

$$x_\pm = \frac{1}{2u}[u - s_\theta^2/c_\theta^2 \pm \sqrt{(u - s_\theta^2/c_\theta^2)^2 - 4u}],$$

with $v = q^2/M_V^2$. The branch cuts of the logarithms are taken along the negative real axis of their respective arguments. $Sp(x)$ is the Spence or dilogarithm function[38,74, 115]:

$$Sp(x) = -\int_0^x \frac{dz}{z}\ln(1-z),$$

with $Sp(0) = 0$ and $Sp(1) = \pi^2/6$. The IR divergences of the Born-like charged-current vertices and boxes can be treated by the method outlined in section IV.5. The infrared-divergent parts (proportional to $\ln M_\gamma^2$) have been removed from $\Lambda(W,\gamma)$ and $\bar{\Theta}_\pm^{nAb}$; the divergences are gauge invariant. The gauge-invariant function $\Phi(q^2;\gamma)$ is infrared divergent and should be removed from $\Gamma_{\pm}^{nAb}$. These IR-divergent parts must then be combined with the non-Abelian parts of the radiation graphs[39,62,74].

Appendix B: Commonly Used Renormalization Schemes

We must define some standard parameters. The first ones are the gauge boson masses and widths. The correct gauge-invariant definition of the masses and widths is obtained from the complex poles of the propagators:

$$D_{ZZ,WW} \to \frac{1}{s - s_p}, \qquad (B.1)$$

as $s \to s_p$ in the s channel, for example. This pole cannot be reached for any physical (real) value of s, but is nevertheless uniquely specified by the analytic properties of the S-matrix[108]. From the complex s_p, the mass and width can be defined in a number of ways. A convenient choice is: $s_p \equiv M^2 - iM\Gamma$[73].* That is, $M^2 = \mathrm{Re}s_p$, and $\Gamma = -\mathrm{Im}s_p/M$. The definition of gauge boson masses that has been standard in the literature is the *on-shell* definition M_{OS}, defined as the zero of the real part of the inverse propagator, for real s. These two definitions are *not* equivalent, and the on-shell definition is *not* gauge invariant in general. For $q^2 > 0$, the corrected propagator takes the form:

$$\frac{1}{q^2 - M_*^2(q^2) + i\sqrt{q^2}\Gamma_*(q^2)} \to \frac{1}{(q^2 - M_{OS}^2)(1-\kappa) + iq^2(\Gamma_{OS}/M_{OS})}, \qquad (B.2)$$

with $M_{OS}^2 = M_*^2(M_{OS}^2)$, $\Gamma_{OS} = \Gamma_*(M_{OS}^2)$, and $\kappa = [\partial M_*^2(q^2)/\partial q^2|(q^2 = M_{OS}^2)$. Then the pole and on-shell definitions are related by:

$$s_p = M_{OS}^2[1 - \frac{(\Gamma_{OS}/M_{OS})^2}{(1-\kappa)^2}] - i\frac{M_{OS}\Gamma_{OS}}{1-\kappa} + \mathcal{O}(\alpha^3), \qquad (B.3)$$

differing by terms of $\mathcal{O}(\Gamma_{OS}^2/M_{OS}^2)$ – that is, $\mathcal{O}(\alpha^2)$ – and higher. The on-shell Z mass and width are $M_Z^{OS} = 91.174$ GeV and $\Gamma_Z^{OS}/(1-\kappa_Z) = 2.49$ GeV. The gauge-invariant pole Z mass is then $M_Z = 91.140$ GeV, a shift downward of 34 MeV. The results and analysis presented in these lectures are carried only through $\mathcal{O}(\alpha)$; to this order, the two definitions are equivalent, and I use only the on-shell definition forthwith and drop the OS sub(super)script. The photon mass is zero in anybody's definition, because the photon's "width" (its imaginary self-energy) vanishes as $q^2 \to 0$. The Z and W widths quoted from data *always* include the pole residues $(1-\kappa)^{-1}$.

---

* An alternative choice is: $s_p \equiv (M - i\Gamma/2)^2$[73,76].

The on-shell conditions for the gauge boson masses enjoy the special property that the extra vertex and box terms necessary to convert the Π's into the gauge-invariant Π^*'s *cancel out,* but *only at the poles.* This is not surprising, if we remember from (A.9d) that the extra terms in the Π^*'s are proportional to one or two powers of the tree-level inverse propagator $(q^2 - M^2)$.† It is convenient to use $s_*^2(Z)$ as an intermediate quantity for many purposes:

$$s_*^2(Z)c_*^2(Z) = \frac{e_*^2(Z)}{4\sqrt{2}M_Z^2 G_{F*}(Z)\rho_*(Z)}, \tag{B.4}$$

where $c_*^2(q^2) = 1 - s_*^2(q^2)$. Defined in this way, $s_*^2(Z)$ depends only on the Π's evaluated at $q^2 = M_Z^2$, not on the full Π^*'s. The relationship:

$$M_W^2 = \frac{e_*^2(W)}{s_*^2(W)} \frac{1}{4\sqrt{2}G_{F*}(W)} \tag{B.5}$$

also requires only the Π's. To run $s_*^2(q^2)$ away from the Z pole, we need the full $\Pi^{*\prime}_{3Q}(q^2) : \Delta_{3Q}(q^2) \equiv \Pi^{*\prime}_{3Q}(q^2) - \Pi^{*\prime}_{3Q}(Z)$; and the full $\Pi^{*\prime}_{QQ}(q^2)$. But the $g_*^2(W)$ needed for (B.5) and the determination of the W mass requires only $\Pi_{3Q}(W)$.

We also define:

$$\begin{aligned} \frac{1}{4\sqrt{2}G_{F*}(q^2)} &= \frac{1}{4\sqrt{2}G_F} - \Delta_\pm(q^2) \\ \frac{1}{\rho_*(q^2)} &= 1 - 4\sqrt{2}G_{F*}(q^2)\Delta_\rho(q^2) \\ \frac{1}{4\sqrt{2}G_{F*}(q^2)\rho_*(q^2)} &= \frac{1}{4\sqrt{2}G_F} - \Delta_3(q^2) - \Delta_\rho(0), \end{aligned} \tag{B.6a}$$

† The pole derivatives κ *do* depend on propagator construction; they depend on the Born-like non-Abelian vertices, but not on the Born-like boxes, because the latter enter the Π^*'s with two powers of inverse propagators. The Abelian vertex corrections can be inserted into the widths by hand by changing the width couplings (Appendix C). Thus the total corrections to the widths at the poles include all vertices, but no boxes. The Born-like neutral-current box is the non-resonant WW, while the Born-like charged-current boxes include the resonant and IR-divergent $W\gamma$ box: in the prescription discussed in section IV.4, this IR-divergent part is removed and put back with the non-Born-like terms. The non-Born-like boxes of course include resonant $Z\gamma$ and $W\gamma$ contributions also.

with:

$$\Delta_{\pm}(q^2) = \Pi^*_{\pm}(0) + \Pi^*_{3Q}(q^2) - \Pi^*_{\pm}(q^2)$$

$$\Delta_3(q^2) = \Pi^*_{33}(0) + \Pi^*_{3Q}(q^2) - \Pi^*_{33}(q^2) \qquad (B.6b)$$

$$\Delta_\rho(q^2) = \Pi^*_{\pm}(q^2) - \Pi^*_{33}(q^2)$$

and $\Delta_3(q^2) - \Delta_{\pm}(q^2) = \Delta_\rho(q^2) - \Delta_\rho(0)$. In the $\rho_0 \neq 1$ EVSM case, let $\rho \equiv \rho_*(0)$. Then replace (B.6) with:

$$\frac{1}{\rho_*(q^2)} = \frac{1}{\rho} - 4\sqrt{2}G_{F*}(q^2)\Delta_\rho(q^2)$$

$$\frac{1}{4\sqrt{2}G_{F*}(q^2)\rho_*(q^2)} = \frac{1}{4\sqrt{2}G_F\rho} - \Delta_3(q^2), \qquad (B.7a)$$

and:

$$\Delta_\rho(q^2) = \Delta_3(q^2) - \rho^{-1}\Delta_{\pm}(q^2). \qquad (B.7b)$$

The tree-level ρ replaces $[1 - 4\sqrt{2}G_F\Delta_\rho(0)]^{-1}$ in the EVSM case[62].

The on-shell electric charge is derived from $e^2 = e^2_*(0)$ at the photon pole, where

$$q^2\mathcal{M}^{exact}_{NC}(q^2) \to e^2_*(0)QQ', \quad q^2 \to 0, \qquad (B.8)$$

exactly to all orders[40,76]. $e^2_*(0) = e^2$ is a gauge-invariant relationship requiring only $\Pi_{QQ}(0)$. Running $e^2_*(q^2)$ away from $q^2 = 0$ requires ${\Pi^*}'_{QQ}(q^2) : \Delta_{QQ}(q^2) \equiv {\Pi^*}'_{QQ}(q^2) - \Delta'_{QQ}(0)$. However, the computed value of $e^2_*(Z)$ needed for the Z pole condition (B.4) and the determination of $s^2_*(Z)$ requires only $\Pi_{QQ}(Z)$. For the W mass (B.5), $e^2_*(W)$ is not really there, since $e^2_*/s^2_* = g^2_*$. The universal Fermi constant G_F is defined from the charged current:

$$\mathcal{M}^{IBA}_{C}C(q^2 = 0) = 4\sqrt{2}G_F \cdot \frac{1}{2}[I^L_+ I^{L'}_- + I^L_- I^{L'}_+] \qquad (B.9)$$

and should not be confused with the muon decay constant G_μ[76]. Note that $G_{F*}(0) = G_F$. G_μ and G_F differ by the $\mathcal{O}(\alpha)$ non-Born-like corrections to muon decay (Appendix C). Because G_F is not defined from the W pole, we cannot replace the $\Pi^*_{\pm}(0)$ by $\Pi_{\pm}(0)$ in this relationship, and the relationship between G_F and G_μ depends on the exact choice of propagator construction. In the Z and W pole conditions (B.4,5), $\Pi^*_{33}(Z)$, $\Pi^*_{3Q}(Z)$, $\Pi^*_{3Q}(W)$ and $\Pi^*_{\pm}(W)$ are all replaced by the respective Π's. The dependence on the full $\Pi_{\pm}^{*}(0)$ remains, when G_F is used as an input, and includes the Born-like vertices and boxes $\Gamma_{\pm}^{nAb}(0)$ and $\bar{\Theta}_{\pm}^{nAb}(0)$. However, when G_F is re-expressed in terms of G_μ, the pole conditions then also depend on the non-Born-like vertices and boxes relating G_F and G_μ. Thus, the pole conditions, and any quantities derived from the pole relations, expressed in terms of G_μ depend on the vertex and box corrections to muon decay, but only on the *sum* of the Born-like and non-Born-like terms, a sum independent of the propagator construction[62,72].

On-Shell Scheme

The *on-shell (OS) scheme* was introduced by Sirlin and Marciano in 1980[72,74]. It uses the three parameters α, M_Z and M_W defined above to renormalize the gauge interactions. The OS weak mixing is defined as an auxiliary quantity:

$$\sin^2\theta_W \equiv 1 - \frac{M_W^2}{M_Z^2} \tag{B.10}$$

to all orders. (At higher orders, we would have to make sure to use a gauge-invariant definition of the masses[73].) At one loop, $\sin^2\theta_W$ is related to $s_*^2(Z)$ by:

$$\sin^2\theta_W = s_*^2(Z) - c_*^2(Z)\{\frac{e_*^2(Z)}{s_*^2(Z)}\Delta_{3Q}(W) + 4\sqrt{2}G_F[\Delta_3(Z) - \Delta_\pm(W) + \Delta_\rho(0)]\}. \tag{B.11}$$

In this scheme, G_F is defined through:

$$M_W^2 = \frac{\pi\alpha}{\sqrt{2}\sin^2\theta_W G_F(1-\Delta r)}, \tag{B.12a}$$

and

$$\begin{aligned}\Delta r = {} & e^2\Delta_Q(Z) + \frac{e_*^2(Z)}{s_*^2(Z)}\Delta_{3Q}(W)[1 - \frac{c_*^2(Z)}{s_*^2(Z)}] \\ & - 4\sqrt{2}G_F\{\Delta_\pm(W) + \frac{c_*^2(Z)}{s_*^2(Z)}[\Delta_3(Z) + \Delta_\rho(0) - \Delta_\pm(W)]\}.\end{aligned} \tag{B.12b}$$

This expression for Δr includes only Born-like corrections. To calculate, the G_F in (B.12a) is re-expressed in terms of the muon decay G_μ. Δr then includes in addition the non-Born-like corrections that relate G_F to G_μ, discussed in Appendix C. The complete $q^2 = 0$ charged-current vertex and box contribution to Δr, the sum of Born-like and non-Born-like terms, is:

$$\Delta r_G = \frac{e^2}{16\pi^2 s_\theta^2}[6 + \frac{7 - 12s_\theta^2}{2s_\theta^2}\ln c_\theta^2]. \tag{B.12c}$$

Lynn-Peskin-Stuart (LPS) Scheme

This scheme was introduced in 1985 and replaces M_W by G_F as the third tree-level parameter[59,39]. The LPS scheme is convenient in particular because it uses as renormalized parameters the three most accurately known electroweak observables, α, M_Z, and G_F. The Z pole condition (B.4) determines $s_*^2(Z)$:

$$s_*^2(Z)c_*^2(Z) = \frac{e_*^2(Z)}{4\sqrt{2}G_F M_Z^2}\{1 - 4\sqrt{2}G_F[(\Delta_3(Z) + \Delta_\rho(0)]\}. \qquad (B.13a)$$

In this scheme, M_W is calculable —

$$\begin{aligned} M_W^2 &= \frac{e_*^2(W)}{s_*^2(W)} \cdot \frac{1}{4\sqrt{2}G_F} \cdot [1 - 4\sqrt{2}G_F\Delta_\pm(W)] \\ &= \frac{e_*^2(Z)}{s^2(Z)} \cdot \frac{1}{4\sqrt{2}G_F} \cdot [1 + \frac{e_*^2(Z)}{s_*^2(Z)}\Delta_{3Q}(W) - 4\sqrt{2}G_F\Delta_\pm(W)]. \end{aligned} \qquad (B.13b)$$

The heavy physics functions Δ_ρ, Δ_3, and $\Delta_\pm$, evaluated at $q^2 = 0$, M_Z^2, and M_W^2, respectively, find their most natural place in this scheme, with G_F enforcing the subtraction of Δ_3 and $\Delta_\pm$ at $q^2 = 0$. Introduce dimensionless heavy physics parameters[39,71]:

$$\begin{aligned} \alpha T &= 4\sqrt{2}G_F\Delta_\rho(0) \\ S &= -16\pi\Delta_3(Z)/M_Z^2 \\ S + U &= -16\pi\Delta_\pm(W)/M_W^2. \end{aligned} \qquad (B.14)$$

Modified Minimal Subtraction ($\overline{MS}$) Scheme

The $\overline{MS}$ scheme, long in use in the QCD literature, was introduced for electroweak corrections by Sirlin in 1989[75]. Unlike the OS and LPS schemes, the $\overline{MS}$ scheme uses two scale-dependent gauge couplings $\hat{e}^2(\mu^2)$ and $\hat{s}^2(\mu^2)$, as well as G_F, as renormalized parameters. The $\overline{MS}$ gauge coupling are not physical, but are related to the full running gauge couplings $e_*^2(q^2)$ and $s_*^2(q^2)$ by the $\overline{MS}$ prescription (section III.5). Let $\hat{\Pi}_{QQ}^*(q^2)$ and $\hat{\Pi}_{3Q}^*(q^2)$ be the effective self-energies subtracted in the $\overline{MS}$ procedure; they contain the above-threshold sublogarithmic and below-threshold terms.

$$\begin{aligned} \frac{1}{e_*^2(q^2)} &= \frac{1}{\hat{e}^2(\mu^2)} - \hat{\Pi}_{QQ}^{*\prime}(q^2) \\ \frac{1}{\hat{e}^2(\mu^2)} &= \frac{1}{e_B^2} - \sum_i C_i Q_i^2 \Delta + \sum_j C_j Q_j^2 \ln(\mu^2/m_j^2), \end{aligned} \qquad (B.15)$$

where $\mu^2 = q^2$, and $\forall m_j^2 < \mu^2$; similarly for $g_*^2(q^2)$, where $\hat{s}^2(\mu^2) = \hat{e}^2(\mu^2)/\hat{g}^2(\mu^2)$.

The relationships between the star coupling functions and the $\overline{MS}$ couplings depend on the choice of propagator construction (the Π^*'s), but the $\overline{MS}$ couplings do not: they depend only on the divergences and leading logarithms, which are universal. The convenience of the $\overline{MS}$ scheme is that the $e_*^2(q^2)$ and $s_*^2(q^2)$ are expressible in terms of the $\overline{MS}$ couplings and purely non-logarithmic terms. If we use the $\overline{MS}$ couplings in place of the full star gauge couplings, the resulting *leading logarithm approximation* is reasonably accurate for "light" radiative corrections (*not* including S, T, and U). At the gauge boson poles, the Π^*'s reduce to the Π's anyway, and it proves convenient to define two auxiliary $\overline{MS}$ functions $\Delta\hat{r}$ and $\Delta\hat{r}_W$:

$$\hat{s}^2(M_Z^2) = \frac{e^2}{4\sqrt{2}G_F M_W^2(1-\Delta\hat{r}_W)}, \quad \hat{s}^2(M_Z^2)\hat{c}^2(M_Z^2) = \frac{e^2}{4\sqrt{2}G_F M_Z^2(1-\Delta\hat{r})}, \tag{B.16}$$

and then note that the re-expression of G_F in terms of G_μ leads to the same additional contribution to $\Delta\hat{r}_W$ and $\Delta\hat{r}$ that appears in Δr (B.12c).

Relationships among Schemes

By equating the various expressions of the star functions, we obtain relationships among the different schemes. First, note that the functions $\Delta_Q(q^2)$ and $\Delta_{3Q}(q^2)$ defined above:

$$\begin{aligned} \frac{1}{e_*^2(q^2)} &= \frac{1}{e^2} - \Delta_Q(q^2) \\ \frac{1}{g_*^2(q^2)} &= \frac{1}{g_*^2(Z)} - \Delta_{3Q}(q^2), \end{aligned} \tag{B.17}$$

where $s_*^2 = e_*^2/g_*^2$, allow us to run e_*^2 from $q^2 = 0$ and g_*^2 from $q^2 = M_Z^2$.

Relating the OS and LPS schemes is straightforward: we need to relate G_F to M_W and express $\sin^2\theta_W$ in terms of G_F and M_Z. From (B.12), at one loop:

$$M_W^2 = \frac{1}{2}M_Z^2\Big[1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}G_F M_Z^2(1-\Delta r)}}\Big] \tag{B.18a}$$

$$\begin{aligned} \Delta r = {} & 4\pi\alpha\Delta_Q(Z) + \frac{4\pi\alpha(Z)}{s_\theta^2}\Delta_{3Q}(W) \\ & - \frac{\alpha(Z)}{4s_\theta^2 c_\theta^2}[-2c_\theta^2 S + \frac{c_\theta^2}{s_\theta^2}(1-2s_\theta^2)U] - \frac{c_\theta^2}{s_\theta^2}\alpha T, \end{aligned} \tag{B.18b}$$

with convenient one-loop replacements of parameters $s_\theta^2 = 1 - c_\theta^2 = \sin^2\theta_W$, $\alpha(Z) \simeq \hat{\alpha}(Z) \simeq (128)^{-1}$. Including the non-Born-like vertex and box corrections to replace

G_F by G_μ :

$$\Delta r = 0.0602 + (0.0169)S - (0.0196)U - (0.0242)T, \qquad (B.18c)$$

where the (S,T,U) are subtracted relative to the Standard Model point MSM($m_t = m_H = M_Z$), as discussed in the section IV.3. Then the OS weak mixing angle is expressible as:

$$\begin{aligned}\sin^2\theta_W \cdot \cos^2\theta_W &= \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2(1-\Delta r)} \\ &= \frac{0.1779}{1-(0.0160)S+(0.0185)U+(0.0228)T},\end{aligned} \qquad (B.19)$$

where in the second line, the Δr of (B.18c) has been used. M_W can then be computed from (B.19), using the definition of the weak mixing (B.10). The W mass at the MSM($m_t = m_H = M_Z$) reference point is $M_{WO} = 79.93$ GeV. Again, these numerical results (B.18 and B.19) do *not* depend on the choice of propagator construction[59,72,75].

The LPS and $\overline{MS}$ schemes are trickier to relate, because of the non-physical nature of the $\overline{MS}$ couplings. The masses M_Z and M_W are related to $\hat{s}^2(M_Z^2)$ by (B.16), where:

$$\begin{aligned}\Delta\hat{r} &= 4\pi\alpha[\Delta_Q(Z) - (1-\frac{s_\theta^2}{c_\theta^2})\hat{\Pi}'_{QQ}(Z)] + \frac{4\pi\alpha(Z)}{s_\theta^2}(1-\frac{s_\theta^2}{c_\theta^2})\hat{\Pi}'_{3Q}(Z) \\ &\quad - \alpha T + \frac{\alpha(Z)}{4s_\theta^2c_\theta^2}S \\ \Delta\hat{r}_W &= 4\pi\alpha[\Delta_Q(Z) - \hat{\Pi}'_{QQ}(Z)] + \frac{4\pi\alpha(Z)}{s_\theta^2}[\Delta_{3Q}(W) + \hat{\Pi}'_{3Q}(Z)] \\ &\quad + \frac{\alpha(Z)}{4s_\theta^2}(S+U).\end{aligned} \qquad (B.20)$$

Note that $\hat{s}^2(M_Z^2)$ computed from M_W ($\Delta\hat{r}_W$) contains no quadratic top mass dependence (T); while computed from M_Z ($\Delta\hat{r}$), it does. By convention, the top quark is taken as "light" in carrying out the $\overline{MS}$ subtraction of Π^*_{QQ} and Π^*_{3Q}, even if $m_t > M_Z$. This convention makes GUT calculations with $\hat{s}^2$ and $\hat{\alpha}^2$ easier. The threshold logarithms of the "heavy" top must be re-expressed: $\ln m_t^2 = \ln M_Z^2 + \ln(m_t^2/M_Z^2)$, with the former absorbed into the couplings $\hat{e}^2$ and $\hat{g}^2$, the latter retained in $\hat{\Pi}_{QQ}(Z)$ and $\hat{\Pi}_{3Q}(Z)$. The heavy physics parameters S and U also have $\ln(m_t^2/M_Z^2)$ dependence. Including the vertex and box corrections for G_F and displaying all $\ln(m_t^2/M_Z^2)$ dependence explicitly:

$$\begin{aligned}
\Delta\hat{r} &= 0.0659 + \frac{\alpha}{\pi}[\frac{1}{4s_\theta^2 c_\theta^2}(1 - \frac{8}{3}s_\theta^2 + \frac{32}{9}s_\theta^4) - \frac{8}{9}]\ln(m_t/M_Z) \\
&- \alpha T + \frac{\alpha(Z)}{4s_\theta^2 c_\theta^2}\hat{S} \\
\Delta\hat{r}_W &= 0.0678 + \frac{\alpha}{\pi}[\frac{1}{2s_\theta^2} - \frac{8}{9}]\ln(m_t/M_Z) \\
&+ \frac{\alpha(Z)}{4s_\theta^2}(\hat{S} + \hat{U}),
\end{aligned} \tag{B.21}$$

with the heavy parameters subtracted again at the reference point MSM($m_t = m_H = M_Z$), and the $\ln(m_t/m_Z)$ dependences of S and U removed, thus producing $\hat{S}$ and $\hat{U}$. The latter two still have weak sublogarithmic top mass dependences. Then:

$$\begin{aligned}
\hat{s}^2(M_Z^2) &= \frac{\pi\alpha}{\sqrt{2}G_F M_W^2(1 - \Delta\hat{r}_W)} \\
&= \frac{0.2333}{1 - (0.0032)\ln(m_t/M_Z) + (0.0091)(\hat{S} + \hat{U})},
\end{aligned} \tag{B.22a}$$

using the reference $M_{WO} = 79.93$ GeV. On the other hand:

$$\begin{aligned}
\hat{s}^2(M_Z^2)\hat{c}^2(M_Z^2) &= \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2(1 - \Delta\hat{r})} \\
&= \frac{0.1790}{1 - (0.0002)\ln(m_t/M_Z) - (0.0078)T + (0.0112)\hat{S}}.
\end{aligned} \tag{B.22b}$$

The reference value is $\hat{s}^2(M_Z^2) = 0.2333$. The running of $\hat{\alpha}(\mu^2)$ begins at $\mu^2 = m_e^2$, with matching condition $\hat{\alpha}(m_e^2) = \alpha$. At the Z pole:

$$\frac{1}{\hat{\alpha}(M_Z^2)} = 127.8 \pm 0.1 + \frac{8}{9\pi}\ln(m_t/M_Z), \tag{B.23}$$

where the uncertainty is due to the hadronic vacuum polarization (see section V.1) and the top quark is taken to be "light", as discussed[59,75]. The OS and $\overline{MS}$ schemes can be related easily, using the formulas above[75].

Let the subscript "O" (not to be confused with "0", bare) denote quantities computed with the heavy physics parameters set to zero — that is, including all "light" radiative corrections and subtracting the heavy physics at the MSM reference point. Then

$$M_Z^2 = M_{ZO}^2[\frac{1-\alpha T}{1-\frac{\alpha}{4s_\theta^2 c_\theta^2}S}]$$

$$M_W^2 = M_{WO}^2[\frac{1}{1-\frac{\alpha}{4s_\theta^2}(S+U)}] \tag{B.24a}$$

are the OS masses and

$$\Gamma_Z = \frac{G_F M_Z^3 \gamma_Z}{1-\alpha T}$$

$$\Gamma_W = G_F M_W^3 \gamma_W \tag{B.24b}$$

are the OS widths renormalized by the pole residues $(1-\kappa)^{-1}$. The functions γ_Z and γ_W are the dimensionless factors quoted above (A.13b,c). γ_Z is a function of $s_*^2(Z)$, which has heavy physics dependence:

$$\begin{aligned} s_*^2(Z) &= s_*^2(Z)_O - \frac{s_*^2(Z)_O c_*^2(Z)_O}{1-2s_*^2(Z)_O} \cdot 4\sqrt{2}G_F[\Delta_3(Z)+\Delta_\rho(0)] \\ &= 0.2322 + (0.0037)S - (0.0024)T, \end{aligned} \tag{B.25}$$

derivable from the pole condition (B.4) and including the threshold terms; the electric charge is: $4\pi/e_*^2(Z)_O = 127.6\pm0.1$. The top threshold dependence in $e_*^2(Z)$ and $s_*^2(Z)$ from $\Pi_{QQ}(Z)$ and $\Pi_{3Q}(Z)$ is negligible for $m_t > M_Z$. The value of $e_*^2(Z)$ used in the pole condition is computed with only the self-energies Π. The value of $s_*^2(Z)$ depends in the usual way on the charged-current vertices and boxes from muon decay[39,59,71,88].

The low-energy neutral-current matrix element is:

$$\mathcal{M}_{NC}^{IBA}(q^2=0) = \frac{4\sqrt{2}G_F}{1-\alpha T}[I_3^L - s_*^2(0)Q][I_3^L - s_*^2(0)Q]'. \tag{B.26}$$

When M_Z is used as an input, the relation of $s_*^2(0)$ to $s_*^2(Z)$ involves no heavy physics, depending only on $\Delta_{3Q}(0)$; but the relation of $s_*^2(Z)$ to M_Z does, given by (B.25).

Thus:

$$s_*^2(0) = s_*^2(Z) \cdot \frac{1 - \frac{e_*^2(Z)}{s_*^2(Z)}\Delta_{3Q}(0)}{1 + e_*^2(Z)\Delta(Z)}, \tag{B.27}$$

where the second factor on the r.h.s. contains "light" physics only, and all the heavy physics is in $s_*^2(Z)$. (B.27) depends on the choice of propagator construction, through $\Delta_{3Q}(0)$, because $s_*^2(0)$ is not on Z pole. The full $\mathcal{M}_{NC}(q^2 = 0)$, including the non-Born-like corrections not displayed in (B.26), does not depend on this choice, but only on the full sum of all relevant corrections[39,59,71, 88]. Reference values for many electroweak observables are published in a number of works; for example, the first work of ref. [87].

All computed electroweak observables quoted in these lectures are calculated to $\mathcal{O}(\alpha)$ plus the hadronic dispersion relation in $\Delta_Q(Z)$. The additional non-vacuum polarization strong effects, $\mathcal{O}(\alpha\alpha_s)$ in QCD[48], are negligible, the possible exception being the one-gluon exchange corrections to T (or $\Delta_\rho(0)$) for large top mass. These lower the W mass by about 40 MeV for $m_t = 200$ GeV, for example, and by less than 10 MeV for $m_t = M_Z$.

Appendix C: Non-Born-like Corrections – Muon Lifetime

All four-fermion processes contain non-Born-like corrections specific to each process, in gauge-invariant classes outlined in section IV.4. The QED corrections, because of their infrared-sensitive nature, depend not only on the external fermion quantum numbers and masses, but on the details of the macroscopic experimental apparatus, such as photon detection cuts. For this reason, a general discussion of QED corrections is beyond the scope of these lectures. Only the Abelian vertex corrections are presented here.

The neutral-current Abelian weak vertices for massless external fermions can be inserted into the IBA matrix element by modifying the fermion-gauge couplings[39,74]:

$$\begin{aligned} Q &\to Q[1 + \frac{e_*^2(Z)}{s_\theta^2}\Gamma_3^{Ab}(q^2)] \\ I_3^L - s_*^2(q^2)Q &\to (I_3^L - s_*^2(q^2))[1 + \frac{e_*^2(Z)}{s_\theta^2}\Gamma_3^{Ab}(q^2)], \end{aligned} \tag{C.1}$$

resulting in the neutral-current modified IBA (MIBA) matrix element. It is accurate to approximate $e_*^2(Z) \doteq 4\pi\hat{\alpha}(M_Z^2)$ in (C.1). The function $\Gamma_3^{Ab}(q^2)$ is:

$$16\pi^2\Gamma_3^{Ab}(q^2) = P_L[(\frac{I_{3f}^L - s_\theta^2 Q_f}{c_\theta})^2\Phi(q^2;Z) + \frac{1}{2}\Phi(q^2;W)]$$

$$+ P_R[(\frac{-s_\theta^2 Q_f}{c_\theta})^2\Phi(q^2;Z)] \qquad (C.2a)$$

$$+ (s_\theta Q_f)^2\Phi(q^2;\gamma),$$

where $P_{L,R}$ are L, R Dirac projectors for the external fermion f. The function $\Phi(q^2;\gamma)$ is the IR-divergent QED part and should be removed from $\Gamma_3^{Ab}(q^2)$ and combined with the external radiation. In the special case of the $Z \to b\bar{b}$ vertex, with a heavy virtual top quark in the vertex loops, the vertex and box functions are changed, with new gauge-invariant contributions. The Z exchange vertices and boxes have a modified overall form, with new terms proportional to the Z-fermion axial-vector couplings times the square of the top mass. This leading quadratic dependence occurs in the vertices and arises in the R_ξ gauge from the axial-vector W-fermion couplings and the extra graphs with would-be Goldstone bosons (both couplings to fermions proportional to the fermion masses). The result depends on the Higgs sector, but since $\rho \simeq 1$, the effect must be close to or exactly the MSM result — the leading quadratic and threshold dependence[112] can be incorporated by adding to the r.h.s. of the second line in brackets of (C.1) the extra term:

$$[\cdots + \frac{e_*^2(Z)}{s_\theta^2}P_L F_L^b(q^2)],$$

$$16\pi^2 F_L^b(q^2 = M_Z^2) = \frac{1}{4}\{\frac{m_t^2}{M_W^2} + (\frac{8}{3} + \frac{1}{6c_\theta^2})[\ln(m_t^2/M_Z^2) - 2\beta(\tan^{-1}\beta - \frac{\pi}{2})]\}, \qquad (C.2b)$$

with $\beta^2 = 4m_t^2/M_Z^2 - 1$. These corrections affect only the L external bottom quark states, because they involve W exchange exclusively. There are similar modifications to the $Z \to t\bar{t}$ vertex, but massive external fermions are beyond the scope of these lectures; massive external quarks also involve QCD bound state threshold effects, as well as physical Higgs exchange[113].

The charged-current case, as discussed in the text, is more complicated, because of the non-Abelian QED terms. Again, only the Abelian vertices are displayed:

$$I_\pm^L \to I_\pm^L[1 + \frac{e_*^2(Z)}{s_\theta^2}\Gamma_\pm^{Ab}]. \qquad (C.3)$$

For a charged-current vertex with fermions f and f' :

$$16\pi^2\Gamma_{\pm}^{Ab}(q^2) = \frac{1}{2}\Big[(\frac{I_{3f}^L - s_\theta^2 Q_f}{c_\theta})^2 + (\frac{I_{3f'}^L - s_\theta^2 Q_{f'}}{c_\theta})^2\Big]\Phi(q^2; Z)$$
$$+ \frac{1}{2}[s_\theta^2\Phi(q^2;\gamma) + c_\theta^2\Phi(q^2; Z)] \tag{C.4}$$
$$+ \frac{1}{2}s_\theta^2[Q_f^2 + Q_{f'}^2]\Phi(q^2;\gamma).$$

Again, the $\Phi(q^2;\gamma)$ terms should be removed to be combined with the Abelian radiation terms[62].

The three canonical inputs α, G_F and M_Z are each inferred from processes that include non-Born-like corrections. The electromagnetic α is exceptional, since the definition (B.8) holds true to any order of perturbation theory. The Z mass is inferred from the peak of the e^+e^- annihilation cross section. This cross section requires the neutral-current non-Born-like weak corrections, but these have negligible effect on the peak position. The QED corrections, on the other hand, have the important property of smearing out the distribution of energy in the initial state (because of radiation: Figure 16 with Z in place of the photon) and moving the peak of cross section to an center-of-mass energy higher than the actual OS mass in the Born-like propagator (Figure 5). The presence of the photon channel in the neutral current also has a small effect on the peak[78].

The last quantity is G_F, defined from the muon lifetime τ_μ for the β-decay $\mu \rightarrow e\nu\bar{\nu}$. The radiative corrections to this process must include the QED effects. These were largely taken care of by Kinoshita and Sirlin in 1958 by using the Fermi four-fermion low-energy effective Lagrangian, then inserting the photons where possible on the tree-level graph[111], producing a gauge-invariant and IR-finite subset of non-Born-like corrections. Fermi's constant from muon decay G_μ has been conventionally defined from their result:

$$\tau_\mu^{-1} \equiv \frac{G_\mu^2 m_\mu^5}{192\pi^3}(1 - \frac{8m_e^2}{m_\mu^2})\{1 + \frac{\alpha}{2\pi}(\frac{25}{4} - \pi^2)[1 + \frac{2\alpha}{3\pi}\ln(m_\mu/m_e)]\}. \tag{C.5}$$

The Kinoshita-Sirlin result can be obtained from the full theory by leaving out the $Z - W$ box and Z vertex corrections and dividing the photon propagator in the photon vertices into two parts: $D_{\gamma\gamma}(k^2) = D_<(k^2) + D_>(k^2)$, with $D_{\gamma\gamma}(k^2) = k^{-2}$ and $D_>(k^2) = (k^2 - M_W^2)^{-1}$, and using only $D_<(k^2)$ in the vertex. Then G_F and

G_μ are related by:

$$\mathcal{M}_{CC}^{exact}(\mu \to e\nu_\mu\bar{\nu}_e) = 4\sqrt{2}G_F[1 + Kinoshita - Sirlin - \delta_G]$$

$$G_F = G_\mu(1+\delta_G),$$

where

$$\begin{aligned}\delta_G &= \Delta r_G \\ &\quad - 4\sqrt{2}G_\mu M_W^2[2(\Gamma_\pm^{nAb}(0) - \Gamma_3^{nAb}(0)) - 2M_W^2\bar{\Theta}_\pm^{nAb}(0)]\end{aligned} \tag{C.6}$$

at order $\mathcal{O}(\alpha)$.

A Guide to the Literature

A somewhat arbitrary selection of helpful books, reviews, lectures and articles.

Quantum Field Theory — Some introductory texts:

J. J. Sakurai, *Advanced Quantum Mechanics* (Reading, Mass.: Addison-Wesley, 1967). M. D. Scadron, *Advanced Quantum Theory* (New York: Springer-Verlag, 1979).

More advanced books:

J. D. Bjorken, S. D. Drell, *Relativistic Quantum Mechanics* and *Relativistic Quantum Fields* (New York: McGraw-Hill, 1965). C. Itzykson, J.-B. Zuber, *Quantum Field Theory* (New York: McGraw-Hill, 1980). P. Ramond, *Field Theory: A Modern Primer* (Redwood City, California: Addison-Wesley, 2^{nd} ed., 1989).

A classic compendium of perturbation theory:

G. 't Hooft, M. Veltman, *Diagrammar,* CERN 73-9 (1973).

On renormalization and the renormalization group:

S. Coleman, "Dilatations" and "Renormalization and Symmetry" in *Aspects of Symmetry* (Cambridge University Press, 1985). J. Collins, *Renormalization* (Cambridge University Press, 1984).

Excellent introductions to renormalization:

K. G. Wilson, "Problems in Physics with Many Scales of Length," *Sci. Am.* **241**(2) (August 1979) 158. S. Weinberg, "Why the Renormalization Group is a Good Thing," in *Asymptotic Realms of Physics,* A. H. Guth, K. Huang, R. L. Jaffe, eds. (Cambridge, Mass.: MIT Press, 1983) 1. G. Peter Lepage, "What is Renormalization?" in *Proc. 1989 TASI Summer School,* T. DeGrand, D. Toussaint, eds. (Singapore: World Scientific, 1990) 483.

Quantum Electrodynamics — Standard sources:

J. Schwinger, ed., *Quantum Electrodynamics* (N.Y.: Dover, 1958). J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Berlin: Springer-Verlag, 1976). V. B. Berestetskii, E. M. Lifshitz, L. P. Pitaevskii, *Quantum Electrodynamics* (Oxford: Pergamon Press, 1982). T. Kinoshita, ed., *Quantum Electrodynamics* (Singapore: World Scientific, 1990).

Gauge Theories and Phenomenology — Introductory level:

C. Quigg, *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions* (Redwood City, California: Addison-Wesley, 1983). K. Huang, *Quarks, Leptons and Gauge Fields* (Singapore: World Scientific, 1982).

Indispensible:

T.-P. Cheng, L.-F. Li, *Gauge Theories of Elementary Particle Physics* (New York: Oxford University Press, 1984).

More technical, especially quantization, perturbation theory, renormalization, Ward identities:

E. S. Abers, B. W. Lee, *Phys. Rep.* **9** (1973) 1. L. D. Fadeev, A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory* (Reading, Mass.: Addison-Wesley, 1980).

Electroweak Standard Model: Basics —

J. C. Taylor, *Gauge Theories of Weak Interactions* (Cambridge: Cambridge University Press, 1976). E. D. Commins, P. Bucksbaum, *Weak Interactions of Leptons and Quarks* (Cambridge: Cambridge University Press, 1983). Cheng and Li, above, chapters 10 and 11.

Electroweak Standard Model: Radiative Corrections —

Renormalization in general: see ref. [17]. One-loop technology: see ref. [38].

On-shell scheme, and calculations therein: see refs. [72,74].

Lynn-Peskin-Stuart scheme: see ref. [59].

$\overline{MS}$ scheme: see ref. [75].

Bare perturbation theory and star system: see refs. [39,62].

Non-decoupling of radiative corrections: see refs. [19,59,66].

Electroweak Standard Model: Data Analysis —

Pre-1988 world of electroweak data: see U. Amaldi *et al.*, ref. [90].

Recent electroweak analysis: see refs. [77,85,87,88,94].

☞ *That is infinite, this is infinite; from that infinity emanates this infinity. Taking away this infinity from that infinity, infinity still remains behind.*

– *Ishavasya Upanishad*

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☞ *The end of a thing is better than its beginning....to making books there is no end, and much study wearies the body.*

– *Ecclesiastes*