AdS/CFT beyond the $\mathcal{N} = 4$ SYM paradigm

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Abstract of the Dissertation

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In this thesis we present studies in the AdS/CFT correspondence that intend to push the present knowledge beyond the $\mathcal{N} = 4$ super Yang-Mills (SYM) paradigm.

The first part is concerned with the study of non-supersymmetric deformations of $\mathcal{N} = 4$ SYM (which still are in the $\mathcal{N} = 4$ universality class). For non-supersymmetric CFT's at Large N we explore the correspondence between string theory tachyons in the bulk and instabilities on the boundary effective action. The operators dual to AdS tachyons have anomalous dimensions that are purely complex numbers. We give a prescription for calculating the mass of the tachyon from the field theory side. Moreover, we apply this general dictionary to the case of intersecting $D7$ flavor branes in $AdS_5 \times S^5$ and obtain the mass of the open string tachyon that is dual to the instability in the mesonic sector of the theory.

In the second part we present work aiming at finding string theory duals for gauge theories beyond the $\mathcal{N} = 4$ universality class, *i.e.* theories that have genuinely less supersymmetry and unquenched flavor. Arguably the next simplest example after $\mathcal{N} = 4$ SYM is $\mathcal{N} = 2$ $SU(N_c)$ SYM coupled to $N_f = 2N_c$ fundamental hypermultiplets. The theory admits a Veneziano expansion of large N_c and large N_f, with N_f/N_c and $\lambda = g^2 N_c$ kept fixed. The topological structure of large N diagrams invites a general conjecture: the flavor-singlet sector of a gauge theory in the Veneziano limit is dual to a closed string theory. We present the one-loop Hamiltonian for the scalar sector of $\mathcal{N} = 2$ superconformal QCD and study this integrability of the theory. Furthermore, we explore the chiral spectrum of the protected operators of the theory using the one-loop anomalous dimensions and, additionally, by studying the index of the theory. We finally search for possible AdS dual trying to match the chiral spectrum. We conclude that the string dual is a sub-critical background containing both an AdS_5 and an S^1 factor.

Στον πατερούλη μου,
στην μανούλα μου
και στην Αναστασία.

Contents

List of Figures

List of Tables

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Chapter 1

Introduction

1.1 Motivation - Overview

The AdS/CFT correspondence [1–3] (see [4] for a review) has been one of the most important advancements in string theory in the last decade. It is an example of gauge theory/gravity duality and provides tools to explore the strongly coupled gauge theories using weekly coupled gravity (string theory) and, maybe, to even learn about the non-pertubative regime of string theory using weekly coupled gauge theory. The basic paradigm is the duality between $\mathcal{N}=4$ super-Yang Mills (SYM) and type IIB string theory on $AdS_5 \times S^5$. Unfortunately, $\mathcal{N}=4$ SYM is far from realistic. It has maximal supersymmetry and all its fields are in the adjoint representation of the color group. Even though there are many "phenomenological" models inspired by the AdS/CFT (which give for example a qualitative geometric understanding of confinement and chiral symmetry breaking), only for a sparse set of four-dimensional gauge theories do we have quantitative evidence of an "exact" duality.

Most of the other (besides $\mathcal{N}=4$) conjectured "exact" dualities are close cousins of this basic paradigm. Some examples are obtained by deforming the $\mathcal{N}=4$ paradigm by turning on relevant deformations (to flow to a new CFT preserving fewer supersymmetries). Another approach is to break supersymmetry by changing the topology of the string theory background replacing the $AdS_5 \times S^5$ by some $AdS_5 \times X_5$[1]. These relatively well-understood dualities are constructed by considering stacks of $D3$ branes in critical (*i.e* ten dimensional) string theory in $\mathbb{R}^{3,1} \times \mathcal{M}_6$. Given the choice of the manifold X_5 (the isometries of X_5 correspond[2] to the R-symmetry of the gauge theory) $\mathcal{N}=4,2,1$, or

[1] see for example the review by Klebanov [5]

[2] More accurately, the isometries of X_5 should include the R-symmetry of the gauge

even zero supersymmetry is preserved. Moreover, a closely related set of well-understood dual geometries is constructed by adding a small number $N_f \ll N_c$ of probe, flavor branes in the $AdS_5 \times X_5$ backgrounds, neglecting their back-reaction. These are gravity duals to gauge theories with fundamental quarks in the quenched approximation. The way the flavor branes are placed in the $AdS_5 \times X_5$ backgrounds, preserving some of the isometries of the original geometry, can lead to further breaking of supersymmetry. In the cases where some supersymmetry is preserved, the gravity backgrounds are better understood. When supersymmetry is completely broken, tachyons might appear on the string theory side with their holographic image being the instabilities of the effective potential on the field theory side.

For example in non-supersymmetric orbifolds of $\mathcal{N}=4$ SYM, conformal invariance is broken by the logarithmic running of double-trace operators – a leading effect at large N. A tachyonic instability in AdS_5 has been proposed as the bulk dual of double-trace running. In the first part of this thesis we make this correspondence more precise. In chapter 2, using standard field theory methods, we show that the double-trace beta function is quadratic in the coupling to all orders in planar perturbation theory. Tuning the double-trace coupling to its (complex) fixed point, we find conformal dimensions of the form $2 \pm i\, b(\lambda)$, as formally expected for operators dual to bulk scalars that violate the stability bound. We also show that conformal invariance is broken in perturbation theory if and only if dynamical symmetry breaking occurs. Our analysis is applicable to a general large N field theory with vanishing single-trace beta functions. This work was originally presented in [6].

In chapter 3 we consider an instance of the AdS/CFT duality where the bulk theory contains an open string tachyon, and study the instability from the viewpoint of the boundary field theory. We focus on the specific example of the $AdS_5 \times S^5$ background with two probe $D7$ branes intersecting at general angles worked out in [7]. For generic angles supersymmetry is completely broken and there is an open string tachyon between the branes. The field theory action for this system is obtained by coupling to $\mathcal{N}=4$ SYM two $\mathcal{N}=2$ hyper multiplets in the fundamental representation of the $SU(N)$ gauge group, but with different choices of embedding of the two $\mathcal{N}=2$ subalgebras into $\mathcal{N}=4$. On the field theory side we find a one-loop Coleman-Weinberg instability in the effective potential for the fundamental scalars. We identify a mesonic operator as the dual of the open string tachyon. By AdS/CFT, we predict the tachyon mass for small 't Hooft coupling (large bulk curvature) and confirm that it

theory. There are examples where the Sasaki-Einstein spaces have $U(1)^3$ isometries that correspond to the $U(1)_R \times U(1)_F^2$ global symmetry with the $U(1)_F^2$ being a non-beryonic flavor symmetry group of certain quiver gauge theories.

violates the AdS stability bound.

For the set of dual pairs constructed using critical string theory, the six transverse dimensions correspond to extra scalar fields in the field theory. While it is possible to break supersymmetry, the resulting gauge theory still "remembers" the maximal supersymmetry of the "mother" $\mathcal{N} = 4$ theory. A basic open question is whether we can find exact dual pairs outside the "universality class" of $\mathcal{N} = 4$ SYM. One of the research directions explored in this dissertation is to find string theory duals of gauge theories that have "genuinely" less supersymmetry. A related direction is to look for duals of gauge theories that include a large number of quark fields in the fundamental representation of the gauge group. In chapter 4 we present some progress made in both directions [8]. We study what is arguably the simplest (most symmetric) gauge theory outside a $\mathcal{N} = 4$ universality class, namely $\mathcal{N} = 2$ superconformal QCD (SCQCD), the $\mathcal{N} = 2$ super Yang Mills theory with gauge group $SU(N_c)$ and $N_f = 2N_c$ fundamental hyper multiplets. $\mathcal{N} = 2$ SQCD for $N_f = 2N_c$ has a vanishing beta function and, thus, should have an AdS dual description. The theory admits a Veneziano expansion of large N_c and large N_f, with N_f/N_c and $\lambda = g^2_{YM}N_c$ kept fixed. The topological structure of large N diagrams motivates a general conjecture: the *flavor-singlet sector* of a gauge theory in the Veneziano limit is dual to a closed string theory. Single closed string states correspond to "generalized single-trace" operators, where adjoint letters and flavor-contracted fundamental/antifundamental pairs are stringed together in a closed chain. We look for the string dual of $\mathcal{N} = 2$ SCQCD from two fronts. From the bottom-up, we perform a systematic analysis of the protected spectrum using superconformal representation theory and the the one-loop dilation operator that we derive in chapter 5. From the top-down, we consider the decoupling limit of known brane constructions. In both approaches, more insight is gained by viewing the theory as the degenerate limit of the $\mathcal{N} = 2$ $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM, as one of the two gauge couplings is tuned to zero. A consistent picture emerges. We conclude that the string dual is a sub-critical background with seven "geometric" dimensions, containing both an AdS_5 and an S^1 factor. The supergravity approximation is never entirely valid, even for large λ, since the field theory has an exponential degeneracy of exactly protected states with higher spin, which must be dual to a sector of light string states.

Integrability techniques have been proven an extremely powerful tool in studying $\mathcal{N} = 4$ SYM in the large N_c limit. The complete determination of the exact operator spectrum of the planar theory seems within reach. The same integrability structures arise both on the field theory and on the dual string theory side, allowing a very precise check of the AdS/CFT correspondence. In fact, although historically it happened otherwise, the AdS/CFT correspondence for $\mathcal{N} = 4$ SYM could have been discovered "from the bottom-up", that is from the perturbative analysis of the field theory dilation operator. Already the one-loop spin chain gives a strong hint of the duality, clearly suggesting a string worldsheet propagating in the $AdS_5 \times S^5$ target space [10–12]. In chapter 5, we consider the one-loop spin chain for $\mathcal{N} = 2$ SCQCD, in order to collect "bottom-up" clues about the string dual, finding nice qualitative agreement with the "top-down" string theory approach. We, finally, study the magnon excitations of the spin chain and their bound states and investigate whether this spin chain is integrable [9].

1.2 AdS/CFT: History - Status Report

String theory was originally discovered as the dual resonance model of the strong interactions prior to the development of gauge theories and QCD. In the mid 70's the interest in string theory was revived after the spin 2 massless particle discovered in its spectrum was recognized as the graviton. This suggested that string theory is a theory of quantum gravity. Naturally containing general relativity and gauge theories, string theory emerged as a strong candidate for the unified theory of all particle interactions. In the last decade, with the discovery of the AdS/CFT correspondence, string theory has partly gone back to its original roots.

String theory contains closed strings, whose low energy effective theory is Einstein's gravity, and open strings that end on (non-pertubative objects) Dp Branes, whose low energy theory is electromagnetism (gauge theory). Simply by observing the fact that under interchanging the worldsheet time and space coordinates the open string one loop diagram becomes a closed tree level propagating string one expects that there should be some kind of open/closed string duality (gauge theory/gravity duality).

AdS/CFT is an example of such a duality. There are, by now, several examples of theories that have two equivalent descriptions, one in terms of open strings (gauge theory), the other in terms of closed strings (gravity). Open/closed duality provides both a powerful insight into the dynamics of strongly coupled physical systems, and a key to the fundamental nature of string theory. Through the AdS/CFT correspondence the gravitational descriptions give insights into strongly coupled gauge theories, while gauge theories provide in principle a reformulation of gravity as an emergent phenomenon.

A beautiful argument that there must be a relation between string theory and gauge theories originates back in the work of 't Hooft [13]. The perturbative expansion of $U(N)$ gauge theories in the *large* N limit can be reorganized in terms of the genus expansion of the surface spanned by the Feynman diagrams, like the perturbative expansion of string theory. The parameter $1/N$ counts the genus of the Feynman diagram, while the 't Hooft coupling

$$\lambda = g_{YM}^2 N \tag{1.1}$$

enumerates the quantum loops.

Holography is an other old idea in the direction of gauge/gravity duality (again originally suggested by 't Hooft [14] and further by Susskind [15]). The *Holographic principle* states that the degrees of freedom of a gravitational theory that lives in some space $\mathcal{M}$ are equal to the degrees of freedom of a non-gravitational theory that lives on the boundary $\partial\mathcal{M}$. This can be understood using the fact that the entropy of a black hole given by the Bekenstein Hawking formula [16]

$$S = \frac{A}{4G} \tag{1.2}$$

The argument goes as follows: if some setup had more degrees of freedom than a black hole and it was not a black hole, we could throw more matter to it and turn it into a black hole. But this new configuration would have bigger entropy that the initial one. And thus more than (1.2) which is wrong.

1.2.1 The $\mathcal{N} = 4$ paradigm

The canonical example of AdS/CFT is between Type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory in four dimensional Minkowski space. It was conjectured by Maldacena in [1] after studying the low energy limit (decoupling limit) of ten dimensional sting theory on the background of N $D3$ branes. Closed strings are excitations of the empty space (the bulk) while open strings end on the D-branes and describe their excitations. After taking the low energy limit ($\alpha' \to 0$) the two theories decouple. $\mathcal{N} = 4$ SYM is the low energy limit of the theory that lives on the $D3$ branes[3]. The $AdS_5 \times S^5$ geometry arises if we study the supergravity solution around $D3$ branes and take the near horizon limit. The AdS scale of the solution is $R^4 = 4\pi g_s \alpha'^2 N$. The

[3]The action for the theory of open strings that live on the worldvolume of a brane is the Dirac-Born-Infeld action (DBI). It is multiplied by the tension $\mathcal{T}_{Dp} \sim 1/g_s \ell_s^{p+1}$ of the brane and, in low energy, it is reduced to the SYM action which has $1/g_{YM}^2$ as a prefactor. This leads to the identification $g_s \ell_s^{p-3} \sim g_{YM}^2$ which for the special case of $D3$ gives a dimensionless coupling constant.

conjecture states that a ten dimensional theory of gravity, type IIB superstring in $AdS_5 \times S^5$ (with N units of five-form flux on S^5), and a four dimensional gauge theory, $\mathcal{N} = 4$ SYM, are dual to each other. The fact that all the ten dimensional dynamical degrees of freedom can somehow be encoded in a four dimensional theory living at the boundary of AdS_5 suggests that the gravity bulk dynamics result from a *holographic image* generated by the dynamics of the boundary theory. The string theory is characterized by two parameters: the string coupling constant g_s and the effective string tension $R^2/\alpha' = R^2/\ell_s^2$. The gauge theory on the other hand is parametrized by the rank N of the gauge color group and the coupling constant g_{YM}. The two theories are dual to each other with the following identifications between their parameters,

$$4\pi g_s = g_{YM}^2 \qquad\qquad \frac{R^4}{\ell_s^4} = \lambda \tag{1.3}$$

The identification of the parameters is achieved by inspecting the supergravity solution around N coincident $D3$ Branes. The conjecture is to hold for all values of N and of $4\pi g_s = g_{YM}^2$.

A very essential indication for the validity of the AdS/CFT correspondence was the observation that the global symmetries of the two theories match. The $SU(4)_R \sim SO(6)_R$ R-symmetry of $\mathcal{N} = 4$ SYM is identified with the isometries of the 5-sphere and the conformal symmetry $SO(2,4)$ with the isometries of AdS_5. Furthermore, the combination of $\mathcal{N} = 4$ supersymmetry, Poincaré and conformal invariance produces an even larger superconformal symmetry given by the supergroup $PSU(2,2|4)$. What is more, the β-function of $\mathcal{N} = 4$ SYM theory is identically zero due to non renormalization theorems. The theory is exactly scale invariant at the quantum level, and the superconformal group $SU(2,2|4)$ is a fully quantum mechanical symmetry.

$\mathcal{N} = 4$ SYM also has electromagnetic (S-duality) symmetry, realized on the complex coupling constant $\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}$ by Möbius transformations in $SL(2,\mathbb{Z})$. On the AdS side, this symmetry is a global discrete symmetry of type IIB string theory with $\tau = \frac{i}{g_s} + \frac{\chi}{2\pi}$ (g_s and χ are the expectation values of the NSNS and the RR scalars – the dilaton and the axion). It is unbroken by the $D3$ brane solution, in the sense that it maps non-trivially only the dilaton and axion expectation values. It must be noted, however, that S-duality is a useful symmetry only for finite N since S-duality transformations do not commute with the large N limit with λ fixed.

Witten and Gubser, Klebanov, and Polyakov [2, 3] used holography to make the AdS/CFT "dictionary" precise. The holographic map between the two theories was discovered by studying the possible boundary conditions for the string fields living in the AdS space. Gauge invariant operators, which are the observable quantities on the boundary, are dual to string fields in the AdS bulk with the same quantum numbers. The string theory fields $\phi(\boldsymbol{z}, \boldsymbol{y})$ that live in $AdS_5 \times S^5$ have to first be decomposed in a series on S^5,

$$\phi(\boldsymbol{z}, \boldsymbol{y}) = \sum_{\Delta=0}^{\infty} \phi_I(\boldsymbol{z}) Y_I(\boldsymbol{y}) \tag{1.4}$$

where Y_I is a basis of spherical harmonics on S^5. Just as fields on a circle receive a mass contribution from the momentum mode on the circle, so do fields compactified on S^5 receive a contribution to the mass. Once we have the full spectrum of the bulk fields that live in AdS space only, we have to consider solving their field equations. Solving for example the Klein-Gordon equation for a scalar in AdS, we discover that there are two solutions as it is a second order differential equation: a non-normalizable and a normalizable one that correspond to Dirichlet and Neumann boundary conditions respectively. Near the boundary [4],

$$\phi(\boldsymbol{x}, z) \to \phi_0(\boldsymbol{x})\, z^{4-\Delta} + (\partial\phi)_0\, (\boldsymbol{x})\, z^{\Delta}\,. \tag{1.5}$$

ϕ_0 is a prescribed "source" function ($\phi_0 = J$) and $(\partial\phi)_0$ describes a physical fluctuation that ends up (after a short calculation using (1.8)) being related to the vacuum expectation value of the dual operator

$$(\partial\phi)_0 = \frac{1}{2\Delta - 4} \langle \mathcal{O} \rangle\,. \tag{1.6}$$

A scalar field with mass m will have the asymptotic behavior (1.5) for $m^2R^2 = \Delta(\Delta-4)$. Similarly, for the various types of fields we get that the massess and the conformal dimensions are connected through the following map:

$$\begin{array}{cc} \text{scalars} & m^2R^2 = \Delta(\Delta-4) \\ \text{spin } \ 1/2, \ 3/2 & |m|R = \Delta - 2 \\ p- \ \text{form} & m^2R^2 = (\Delta-p)(\Delta+p-4)\ . \\ \text{spin } 2 & m^2R^2 = \Delta(\Delta-4) \end{array}$$

In order to prove the AdS/CFT correspondence the two Hilbert spaces of the two theories have to be identified using (1.7) for all values of N and

[4]Here we are using Poincaré coordinates: $\boldsymbol{x}$ are the coordinates in $\mathbb{R}^4$ and z is the radial direction of AdS_5.

λ. That is to say, the partition function of the string theory with boundary conditionds ϕ_0 should be equal to the gauge theory partition function that contains all the correlators of single trace operators $\mathcal{O}$ with sources $\phi_0 = J$,

$$Z_{string}[\phi_0] \equiv Z_{\mathcal{N}=4}[J] = \langle e^{\int_{\partial\mathcal{M}} J\,\mathcal{O}} \rangle\,. \tag{1.7}$$

This expression is understood to hold order by order in a perturbative expansion. On the AdS side, we assume that we have an action $S_{string}[\phi]$ that summarizes the dynamics of type IIB string theory[5] on AdS_5. In the supergravity approximation ($\alpha' \to 0$), $S[\phi]$ is the very well understood type IIB supergravity action on AdS_5. If we first take the large N limit, $g_s \to 0$ (or equivalently[6] $G_N \to 0$), we do not have to compute any string theory (quantum gravity) corrections. Then, we can further take the strong coupling $\lambda \to \infty$ limit which in the gravity side is $\alpha' \to 0$ and we can just use type IIB supergravity. Doing so, the path integral of the string theory side is reduced to

$$e^{-S[\phi_0]} \equiv \langle e^{\int_{\partial M} \phi_0\,\mathcal{O}} \rangle \tag{1.8}$$

and we can easily compute correlation functions for operators in the strongly coupled gauge theory, just by taking functional derivatives with respect to the sources $J = \phi_0$. To go beyond the supergravity approximation (*i.e.* to calculate $1/\sqrt{\lambda}$ corrections for the field theory correlators), $S[\phi]$ should also include α' corrections due to massive string effects.

Having the basic dictionary set up, the simplest check of the correspondence we can perform is to match the chiral spectrum of the gauge theory with the KK spectrum of supergravity. This is basically a superconformal representation theory multiplets matching (group theory plus field content) argument. Studying superconformal representation theory, we discover that there are short multiplets, some of which are protected (*i.e.* their anomalous dimension is always zero in all orders perturbation theory). For $\mathcal{N} = 4$ SYM there is one half-BPS multiplet that is protected. The conformal dimensions and quantum numbers of its members are directly matched to the one and only gravity multiplet of type IIB supergravity. This multiplet starts with ϕ_1^ℓ, ϕ_2^ℓ, or ϕ_3^ℓ scalars and contains, among their other descendants, the stress energy tensor of $\mathcal{N} = 4$ and the supercurrents. These are dual to scalars with all the indices in the 5-sphere, the graviton and the gravitino respectively. This check was originally performed in [3, 17] but can also be easily achieved with a novel tool, "the index" [18], which is nothing but the partition function (up to

[5]This action is of course not known. We do not even know how to write the full spectrum of string theory in $AdS_5 \times S^5$ – apart from the special case of the plane wave limit.

[6]Newton's constant $G_N \sim R^8/N^2$ goes to zero as N goes to infinity.

signs for the fermions) for only the protected operators. We describe in detail this very powerful tool and employ it to check the chiral spectrum of $\mathcal{N} = 2$ SCQCD in chapter 4.

The next step is to consider the large N limit but calculate corrections in λ. The first reasults in this direction were obtained using the BMN limit (Berenstein - Maldacena - Nastase [10]). Classical rotating strings in the 5-sphere or the AdS_5 are mapped to operators with high R-charges or many derivatives, for all λ but with $J \to \infty$. The geometry seen by these fast rotating strings is the so-called plane wave geometry and string theory in this background is exactly solvable. In other words, we know the excitations spectrum which is comprised by massive bosons/fermions.

Although the aforementioned results offered essential checks of the AdS/CFT correspondence, they are yet limited in large J type of operators. The real breakthrough came only after the observation, by Minahan and Zarembo [11], that the calculation of an operator's anomalous dimension can be mapped to a spin chain type of problem that is integrable. Using integrability techniques, the complete determination of the planar theory's exact operator spectrum, for all λ, seems only a matter of time. The dilatation operator at higher loops (all loops) was calculated and found to correspond to a long range Hamiltonian that is integrable to all orders in λ [19, 20]. Shortly after that, the all loop Scattering matrix was obtained (up to a phase) using the symmetry algebra [21]. Finally, the Thermodynamic Bethe Ansatz (TBA) was used to account for the finite size corrections (wrapping) for small operators [22–24].

Even though in this dissertation we will only consider the planar theory, we also report for completeness the status of checking the AdS/CFT for finite N. The relevant perturbative calculations are rather difficult and, hence, there has not been much progress in this direction. However, $1/N$ corrections have been calculated for some quantities like the c and a anomalies for $\mathcal{N} = 4$ SYM and orbifolds of it, finding that the field theory and gravity results match [25–27]. The most significant check of the AdS/CFT for finite N was achieved by Aharony and Witten [28]. In their work, they show that upon compactification on a circle, $\mathcal{N} = 4$ SYM acquires a Z_N global symmetry (the center of the gauge group is Z_N) that happens to be a "topological symmetry". Finally, an alternative check of the AdS/CFT for finite N is provided by the giant gravitons. Giant gravitons are $D3$ branes wrapping an $S^3 \subset S^5$ with large angular momentum on the S^5. There are dynamically stable solutions of type IIB supergravity that preserve half of the supersymmetries. Their dual operators on the boundary are not made gauge invariant by taking the trace, but the determinant in the color space. For these operators non-planar diagrams dominate over planar diagrams and, thus, the $1/N$ check.

Finally, although in this thesis we will only be concerned with dual pairs for which the boundary theory is four dimensional (with the exception of chapter 4 where the arguments are more general), we wish to mention that there are many other examples of AdS/CFT correspondence for dimensions different than four. First of all, using $M2$ and $M5$ branes was already suggested in [1]. M-theory in $AdS_4 \times S^7$ and $AdS_7 \times S^4$ should be dual to supersymmetric three dimensional conformal field theory and $(2,0)$ superconformal field theory in six dimensions respectively. Recently, a lot of progress was made after the discovery of the Lagrangian for $M2$ branes [29]. ABJM showed that string theory on $AdS_4 \times \mathbb{CP}^3$ is dual to $\mathcal{N}=6$ superconformal Chern-Simons [30]. There has also considerable progress for theories in two dimensions. $AdS_3 \times S^3 \times M^4$ is dual to the two dimensional symmetric orbifold theory (the field theory that lives on $D1/D5$ intersections). For AdS_8 or higher dimensions there are no more dual pairs as there is no simple AdS supergroup.

1.2.2 Beyond $\mathcal{N}=4$ SYM

As discussed in the previous section most of the other conjectured "exact" dualities, such as $\mathcal{N}=4$ orbifolds and orientifolds, the conifold, $AdS_5 \times X^5$ where X^5 is any Sasaki Einstein manifold $Y^{p,q}$ or $L^{a,b,c}$, are all close cousins of the $\mathcal{N}=4$ SYM standard paradigm. They are obtained by considering stacks of $D3$ branes in *critical string theory* in $\mathbb{R}^{3,1} \times \mathcal{M}_6$ with

$$ds^2_{\mathcal{M}_6} = dr^2 + r^2 dX_5^2\,. \tag{1.9}$$

A choice of $\mathcal{M}_6$ with less isometries than $\mathbb{R}^6$ will lead to a field theory with fewer supersymmetries preserved. If, for example, $\mathcal{M}_6$ is Ricci flat (*i.e.* a Calabi-Yau 3-fold) $\mathcal{N}=1$ supersymmetry is preserved. More dual pairs are obtained by deforming the $\mathcal{N}=4$ SYM with gamma deformations or with marginal and relevant deformations. Further examples are constructed by adding flavor branes in the probe approximation in the $AdS_5 \times X_5$ backgrounds, neglecting their backreaction. The open strings that stretch between the color $D3$ branes and the $N_f \ll N_c$ flavor branes lead to fundamental quarks. Again, the relative position of the flavor branes to the $D3$s can lead to further breaking of supersymmetry.

There is also the famed Sakai-Sugimoto model that is arguably the best (the closest) string-theoretical holographic realization of the "real" QCD in the infrared [32, 33]. It provides a beautiful geometric realization of chiral symmetry breaking and has been used to study the spectrum of mesons and baryons. [7]

[7] In this direction of getting a nice string model of holographic QCD there is a new

In all the aforementioned examples, when supersymmetry is completely broken our work [6] is applicable. Specifically, in chapters 2 and 3, we study the tachyons that appear on the string theory side from the instabilities of the effective potential on the field theory side.

An orthogonal direction is instead of using critical string theory in which the resulting gauge theory always "remembers" the maximal supersymmetry due to extra fields, to work with *non-critical string theory* . This was originally suggested by Polyakov [35, 36]. He argues that pure Yang-Mills theory in four dimensions should be described by a sub-critical string theory in AdS_5 space. The physical meaning of the fifth dimension is that of the renormalization scale represented by the Liouville field. Building on this idea, theories with genuinely lower supersymmetry should be dual to non-critical string theories, according to the pattern: $\mathcal{N} = 4$, $d = 10$ (critical case); $\mathcal{N} = 2$: $d = 8$; $\mathcal{N} = 1$: $d = 6$; $\mathcal{N} = 0$: $d = 5$. In chapter 4 we present our work on $\mathcal{N} = 2$ SCQCD that is arguably the next simplest case beyond the $\mathcal{N} = 4$ SYM universality class.

example constructed with intersecting $D4$ branes , that was recently suggested by Van Raamsdonk and Whyte [34]. This exaple have very similar features to our intersecting $D7$ branes model presented in chapter 5.

Chapter 2

Large N Field Theory and AdS Tachyons

2.1 Introduction

Conformal invariant quantum field theories in four dimensions are interesting both theoretically and for potential phenomenological applications. While perturbatively finite supersymmetric QFTs have been known for a long time [37] and a vast zoo of non-perturbative supersymmetric examples was discovered during the duality revolution of the 1990s, only few non-supersymmetric, interacting CFTs in $d = 4$ are presently known.[1]

The AdS/CFT correspondence [1–3] seems to offer an easy route to several more examples. A well-known construction [39, 40] starts by placing a stack of N D3 branes at an orbifold singularity $\mathbb{R}^6/\Gamma$. In the decoupling limit one obtains the duality between an orbifold of $\mathcal{N} = 4$ SYM by $\Gamma \subset SU(4)_R$ and Type IIB on $AdS_5 \times S^5/\Gamma$. Supersymmetry is completely broken if $\Gamma \not\subset SU(3)$, but since the AdS factor of the geometry is unaffected by the orbifold procedure, conformal invariance appears to be preserved, at least for large N. However, in the absence of supersymmetry one may worry about possible instabilities [41].

On the string theory side of the duality, one must draw a distinction [42] according to whether the orbifold action has fixed points or acts freely on S^5. If Γ has fixed points, there are always closed string tachyons in the twisted sector. If Γ acts freely, the twisted strings are stretched by a distance of the order of the S^5 radius R; the would-be tachyons are then massive for large enough R (strong 't Hooft coupling λ), but it is difficult to say anything definite about small R.

[1] Large N Bank-Zaks [38] fixed points come to mind.

On the field theory side, a perturbative analysis at small λ reveals that conformal invariance is *always* broken, regardless of whether the orbifold is freely acting or not [43, 44]. The inheritance arguments of [41, 45] guarantee that the orbifold theory is conformal in its single-trace sector: at large N, all couplings of marginal single-trace operators have vanishing beta functions. However, even at leading order in N, there are non-zero beta functions for double-trace couplings of the form

$$\delta S = f \int d^4x \, \mathcal{O}\bar{\mathcal{O}} \,, \tag{2.1}$$

where $\mathcal{O}$ is a twisted single-trace operator of classical dimension two [42–44, 46, 47]. Conformal invariance could still be restored, if all double-trace couplings f_k had conformal fixed points. It turns out that this is never the case in the one-loop approximation [43, 44]. So for sufficiently small λ, all non-supersymmetric orbifolds of $\mathcal{N} = 4$ break conformal invariance.

It is natural to associate this breaking of conformal invariance with the presence of tachyons in the dual AdS theory [43]. By an AdS tachyon, we mean a scalar field that *violates* the Breitenlohner-Freedman bound [48]:

$$\text{For a tachyon}\,, \quad m^2 < m_{BF}^2 = -\frac{4}{R^2}\,. \tag{2.2}$$

One is then led to speculate [43] that even for freely acting orbifolds, some of the twisted states must become tachyonic for λ smaller than some critical value λ_C. The conjectural behavior of $m^2(\lambda)$ for a "tachyon" in a freely acting orbifold theory is shown in Figure 2.1. A related viewpoint [42] links the tachyonic instability in the bulk theory with a perturbative Coleman-Weinberg instability in the boundary theory. From this latter viewpoint however, it seems at first that whether Γ is freely acting or not makes a difference even at weak coupling [42]: if Γ has fixed points, the quantum-generated double-trace potential destabilizes the theory along a classical flat direction; if Γ is freely acting, the symmetric vacuum appears to be stable, because twisted operators have zero vevs along classical flat directions.

In chapter 2 of this dissertation we make the correspondence between double-trace running and bulk tachyons more precise. Taken at face value, an AdS_5 tachyon would appear to be dual to a boundary operator with *complex* conformal dimension of the form

$$\Delta = 2 \pm i\, b\,, \quad b = \sqrt{|m^2R^2 + 4|}\,. \tag{2.3}$$

We are going to find a formal sense in which this is correct, and a prescription

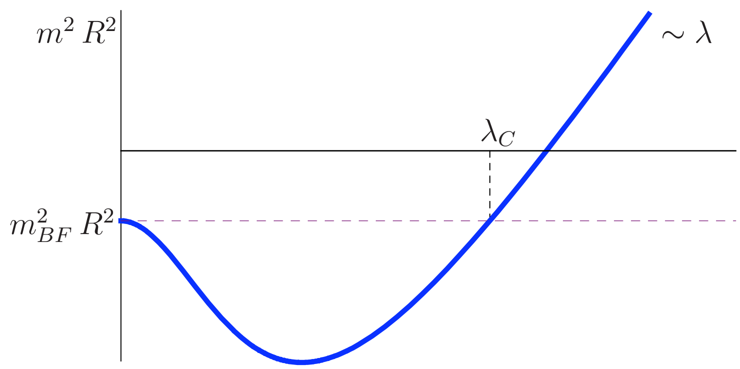


Figure 2.1: Proposal for the qualitative behavior of a "tachyon" mass in a freely acting orbifold, as a function of the 't Hooft coupling λ. The field is an actual tachyon (violating the BF stability bound) for $\lambda < \lambda_C$. See section 2.4.1 for more comments.

to compute the tachyon mass $m^2(\lambda)$ from the boundary theory. In principle this prescription could be implemented order by order in λ and allow to test the conjectural picture of Figure 2.1. We also show that the perturbative CW instability is present if and only if conformal invariance is broken, independently of the tree-level potential, and thus independently of whether the orbifold is freely acting or not.

Our analysis applies to the rather general class of large N theories "conformal in their single-trace sector". We consider non-supersymmetric, classically conformal field theories with lagrangian of the standard single-trace form $\mathcal{L} = N\,\mathrm{Tr}\,[\dots]$. Denoting collectively by λ the single-trace couplings that are kept fixed in the large N limit,[2] we assume that $\beta_\lambda \equiv \mu\frac{\partial}{\partial\mu}\lambda = 0$ at large N. Generically however, perturbative renormalizability forces the addition of double-trace couplings of the form (2.1), where $\mathcal{O} \sim \mathrm{Tr}\,\phi^2$ is a single trace operator of classical dimension two. Thus it is essential to compute the double-trace beta functions β_f to determine whether or not conformal invariance is maintained in the quantum theory. Our main technical results are expressions for β_f, for the conformal dimension $\Delta_{\mathcal{O}}$ and for the effective potential $\mathcal{V}(\varphi)$, valid to all orders in planar perturbation theory.

Besides orbifolds of $\mathcal{N} = 4$ SYM, other examples of large N theories conformal in their single-trace sector are certain non-supersymmetric continuous deformations of $\mathcal{N} = 4$ SYM [49–51]. One can also contemplate theories with

[2]In the example of an orbifold of $\mathcal{N} = 4$ SYM, $\lambda = g^2_{YM}N$ is the usual 't Hooft coupling.

adjoint *and* fundamental matter, where the instability arises in the mesonic sector and is dual to an *open* string tachyon. A detailed analysis of such an "open string" example will appear in the next chapter 3 of this thesis and it was originally presented in [7]. Somewhat surprisingly, conformal invariance turns out to be broken in all concrete cases of non-supersymmetric "single-trace conformal" theories that have been studied so far. There is no a priori reason of why this should be the case in general. A more systematic search for conformal examples is certainly warranted.

We should also mention from the outset that independently of the perturbative instabilities which are the focus of this paper, non-supersymmetric orbifold theories may exhibit a non-perturbative instability akin to the decay of the Kaluza-Klein vacuum [52] (see also [53]). For a class of freely acting $\mathbb{Z}_{2k+1}$ orbifolds, at large coupling λ the decay-rate per unit volume scales as [52]

$$\Gamma_{decay} \sim k^9 e^{-N^2/k^8} \Lambda^4 \,, \tag{2.4}$$

where Λ is a UV cut-off. This instability is logically distinct and parametrically different from the tree-level tachyonic instability. It is conceivable that a given orbifold theory may be stable in a window of couplings $\lambda_C < \lambda < \lambda_{KK}$ intermediate between a critical value λ_C where the "tachyon" is lifted (Figure 2.1) and another critical value λ_{KK} where the the non-perturbative instability sets in.

Multitrace deformations in the context of the AdS/CFT correspondence have been investigated in several papers, beginning with [54–58].

This chapter is organized as follows. In section 2.2 we study the renormalization of a general field theory conformal in the single-trace sector and derive expressions for β_f and $\Delta_{\mathcal{O}}$ valid to all orders in planar perturbation theory. In section 2.3 we study the behavior of the running coupling $f(\mu)$ and the issue of stability of the quantum effective potential $\mathcal{V}(\varphi)$. In section 2.4 we make our proposal for the computation of the tachyon mass $m^2(\lambda)$ from the dual field theory. We illustrate the prescription in a couple of examples and make some remarks on flat directions in freely acting orbifold theories. We conclude this chapter with section 2.5 discussing a few open problems.

2.2 Renormalization of double-trace couplings

We are interested in large N, non-supersymmetric field theories in four dimensions. We start with a conformally invariant classical action of the standard single-trace form. Schematically,

$$S_{ST}[N,\lambda] = \int d^4x\, N\, \mathrm{Tr}\, [(D\phi)^2 + \psi D\psi + (DA)^2 + \lambda\, \phi^4 + \dots] \,, \tag{2.5}$$

where ϕ, ψ, A are $N \times N$ matrix-valued scalar, spinor and gauge fields. We have written out the sample interaction term $N\lambda \mathrm{Tr}\, \phi^4$ to establish our notation for the couplings: we denote collectively by λ the couplings in S_{ST} that are kept fixed in the large N limit.

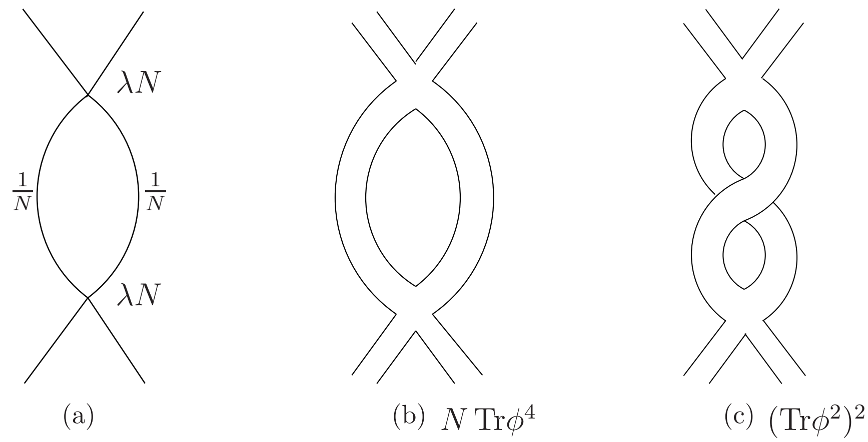


Figure 2.2: One-loop contributions to the effective action from a diagram with two quartic vertices. Each vertex contributes a factor of λN and each propagator a factor of $1/N$, as indicated in (a). There are two ways to contract color indices: a single-trace structure (b), or a double-trace structure (c).

Generically, the action (2.5) is not renormalizable as it stands, because extra double-trace interactions are induced by quantum corrections. It is an elementary but under-appreciated fact that double-trace renormalization is a *leading* effect at large N. For example, consider the contribution to the effective action from one-loop diagrams with two quartic scalar vertices (Figure 2.2). Schematically,

$$\begin{aligned} &\int d^4x\, N\, \lambda \mathrm{Tr}\, \phi^4(x)\ \int d^4y\, N\, \lambda \mathrm{Tr}\, \phi^4(y) \\ &\sim \lambda^2\, \log\Lambda\, \int d^4z\, [\, N\, \mathrm{Tr}\, \phi^4 + (\,\mathrm{Tr}\, \phi^2)^2\,]\ . \end{aligned} \tag{2.6}$$

The single-trace term $N\,\mathrm{Tr}\phi^4$ renormalizes a coupling already present in the action (2.5). The double-trace term $(\mathrm{Tr}\phi^2)^2$ forces the addition of an extra piece to the bare action,

$$S = S_{ST} + S_{DT}\,, \qquad S_{DT} = \int d^4x\, f_0\,(\mathrm{Tr}\phi^2)^2\,, \qquad f_0 \sim \lambda^2 \log\Lambda\,. \tag{2.7}$$

It is crucial to realize that S_{ST} and S_{DT} are of the same order in the large N limit, namely $O(N^2)$. For S_{ST}, one factor of N is explicit and the other arises from the trace; for S_{DT}, each trace contributes one factor of N.

In the following, we specialize to theories for which the single-trace couplings do not run in the large N limit, $\beta_\lambda = \mu\frac{\partial}{\partial\mu}\lambda = O(1/N)$. In particular the single-trace contribution in (2.6) is canceled when we add all the relevant Feynman diagrams. This is what happens in orbifolds of $\mathcal{N} = 4$ SYM. Twisted single-trace couplings cannot be generated in the effective action, since they are charged under the quantum symmetry, while untwisted single-trace couplings are not renormalized, since they behave as in the parent theory by large N inheritance. However, neither argument applies to double-trace couplings of the form $f\,\mathcal{O}_g\,\mathcal{O}_g^\dagger$, where $\mathcal{O}_g = \mathrm{Tr}(g\phi^2)$ is a twisted single-trace operator of classical dimension two.[3] Such double-trace couplings will be generated in perturbation theory.

In this rest of this section, we analyze the general structure of double-trace renormalization.

2.2.1 Double-trace renormalization to all orders

The beta function for the double-trace coupling (2.1) was computed at one loop in [43],

$$\beta_f \equiv \mu\frac{\partial}{\partial\mu}f = v^{(1)}f^2 + 2\gamma^{(1)}\lambda f + a^{(1)}\lambda^2\,. \tag{2.8}$$

This result applies to any theory conformal in its single-trace sector. Here $v^{(1)}$ is the normalization of the single-trace operator $\mathcal{O} \sim \mathrm{Tr}\,\phi^2$, defined as

$$\langle\mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle = \frac{v^{(1)}}{2\pi^2(x-y)^4}\,. \tag{2.9}$$

The quantity $\gamma^{(1)}\lambda$ is the one-loop contribution to the anomalous dimension of $\mathcal{O}$ from the single-trace interactions. The double-trace interaction also contributes to the renormalization of $\mathcal{O}$, so that the full result for its one-loop

[3] Here $\mathrm{Tr} = \mathrm{Tr}_{SU(|\Gamma|N)}$ and $g \in \Gamma$.

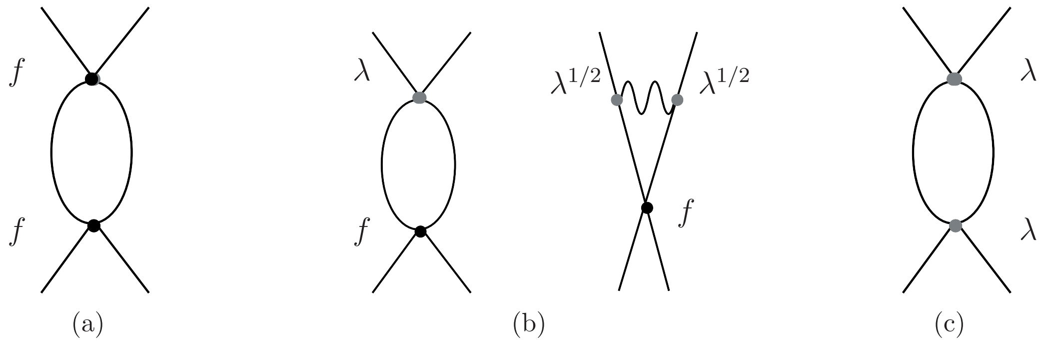


Figure 2.3: Sample diagrams contributing to β_f at one loop: (a) $v^{(1)}f^2$; (b) $2\gamma^{(1)}\lambda f$; (c) $a^{(1)}\lambda^2$.

anomalous dimension is

$$\gamma_{\mathcal{O}} = \gamma^{(1)}\lambda + v^{(1)}f\,. \tag{2.10}$$

Some representative Feynman diagrams contributing to β_f are shown in Figure 2.3. Our goal is to generalize these results to all orders in planar perturbation theory.

The $\lambda = 0$ case

Let us first practice with the simple situation where the single-trace part of the action is free.[4] The total lagrangian is

$$\mathcal{L} = \mathcal{L}_{ST}^{free} + \mathcal{L}_{DT}\,, \qquad \mathcal{L}_{DT} = f\,\mathcal{O}\bar{\mathcal{O}}\,. \tag{2.11}$$

The discussion of the large N theory is facilitated by a Hubbard-Stratonovich transformation. We introduce the auxiliary complex scalar field σ and write the equivalent form for the double-trace interaction,[5]

$$\mathcal{L}_{DT} = -f\sigma\bar{\sigma} + f\sigma\bar{\mathcal{O}} + f\bar{\sigma}\mathcal{O}\,. \tag{2.12}$$

The obvious Feynman rules are displayed in Figure 2.4. The renormalization program is carried out as usual, by adding to the tree-level lagrangian (2.12) local counterterms, which we parametrize as

$$\delta\mathcal{L}_{DT} = -(Z_2 - 1)f\sigma\bar{\sigma} + (Z_3 - 1)(f\sigma\bar{\mathcal{O}} + f\bar{\sigma}\mathcal{O})\,. \tag{2.13}$$

[4]The calculation of β_f for this case already appears in [56].
[5]For ease of notation we suppress possible flavor indices for $\mathcal{O}$ and σ.

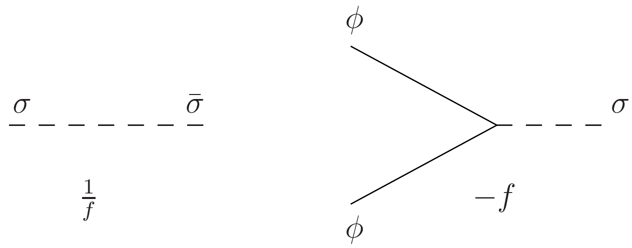


Figure 2.4: Feynman rules for (2.12).

The one-particle irreducible structures that may contain divergences are $\Gamma_{\sigma\bar{\sigma}}$, $\Gamma_{\sigma\phi\phi}$ and $\Gamma_{\phi\phi\phi\phi}$. The quartic vertex $\Gamma_{\phi\phi\phi\phi}$ is in fact subleading in the large N limit, as illustrated in Figure 2.5 in a one-loop example. The leading contributions to the scalar four-point function contain cuttable σ propagators. This is an example of a general fact that we will use repeatedly: 1PI diagrams with internal σ propagators are subleading for large N. Indeed, adding internal σ lines increases the number of ϕ propagators, which are suppressed by $1/N$.

The upshot is that while for finite N (2.12) is not renormalizable as written (we need to add an explicit $\mathcal{O}\bar{\mathcal{O}}$ counterterm), for large N it is.

From the Feynman rules, we immediately find

$$\Gamma_{\sigma\bar{\sigma}}(x,y) = f Z_2\, \delta(x-y) + Z_3^2\, f^2 \,\langle\, \mathcal{O}(x)\bar{\mathcal{O}}(y)\,\rangle_{f=0}\,, \tag{2.14}$$

$$\Gamma_{\sigma\phi\phi}(x;y,z) = -f\, Z_3 \,\langle\, \mathcal{O}(x)\phi(y)\phi(z)\,\rangle^{1PI}_{f=0}\,. \tag{2.15}$$

Since we are assuming for now that the single-trace action is free, the $f=0$ correlators appearing above are given by their tree-level expressions. The three-point function $\langle \mathcal{O}\phi\phi\rangle^{1PI}_{f=0}$ is simply a constant,

$$\Gamma_{\sigma\phi\phi} = -f Z_3 \cdot \mathrm{const}\,. \tag{2.16}$$

Clearly no renormalization of the $\sigma\phi\phi$ vertex is needed and we can set $Z_3 = 1$. On the other hand, the two-point function

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(0)\rangle_{f=0} \equiv \frac{v}{2\pi^2 x^4} \tag{2.17}$$

requires renormalization, since its short-distance behavior is too singular to admit a Fourier transform. We adopt the elegant scheme of differential renormalization [59, 60]. The singularity is regulated by smearing the scalar prop-

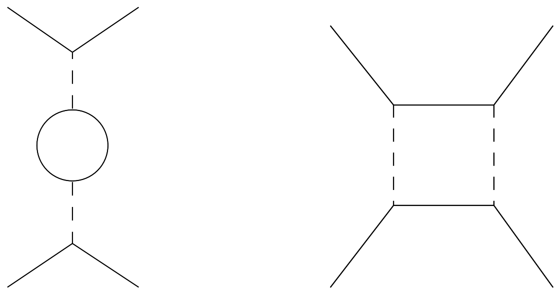


Figure 2.5: Diagram (a) is leading at large N, of order $O(1)$, but it is reducible. Diagram (b) is irreducible but it is subleading at large N, of order $O(1/N^2)$.

agator,

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(0)\rangle_{f=0} = \frac{v}{2\pi^2}\frac{1}{(x^2+\epsilon^2)^2}\,, \tag{2.18}$$

where ϵ is a short distance cutoff. Introducing a dimensionful constant μ, one may separate out the divergence as follows,

$$\frac{v}{2\pi^2}\frac{1}{(x^2+\epsilon^2)^2} \xrightarrow{\epsilon\to 0} -\frac{v}{8\pi^2}\Box\frac{\ln x^2\mu^2}{x^2} - v\ln\mu\epsilon\,\delta(x)\,. \tag{2.19}$$

The first term is the renormalized two-point function: it is finite (Fourier transformable) if one interprets the Laplacian as acting to the left under the integral sign. The constant μ plays the role of the renormalization scale. Back in (2.14), we take the Z-factors to be

$$Z_2 = 1 + vf\log\mu\epsilon\,, \qquad Z_3 = 1\,, \tag{2.20}$$

and find the renormalized correlator

$$\Gamma_{\sigma\bar{\sigma}}(x,y) = f\delta(x-y) - \frac{vf^2}{8\pi^2}\Box\frac{\ln\mu^2(x-y)^2}{(x-y)^2}\,. \tag{2.21}$$

We are now in the position to calculate β_f and the anomalous dimension $\gamma_{\mathcal{O}}$ of the single-trace operator.[6] The renormalized two-point function satisfies the

[6]Note that $\gamma_{\mathcal{O}}$ coincides with γ_σ, since connected correlation functions of σ are equal (for separated points) to connected correlation functions of $\mathcal{O}$.

Callan-Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_f\frac{\partial}{\partial f} - 2\gamma_{\mathcal{O}}\right]\Gamma_{\sigma\bar{\sigma}} = 0\,. \tag{2.22}$$

Recalling the identity

$$\mu\frac{\partial}{\partial\mu}\left[-\frac{1}{8\pi^2}\Box\frac{\ln\mu^2x^2}{x^2}\right] = \delta(x)\,, \tag{2.23}$$

we see that the CS equation implies

$$2f\beta_f - 2\gamma_{\mathcal{O}}f^2 = 0 \tag{2.24}$$
$$\beta_f - 2\gamma_{\mathcal{O}}f + vf^2 = 0\,, \tag{2.25}$$

the first condition arising for $x \neq y$ and the second from the delta function term. Incidentally, the CS equation for $\Gamma_{\sigma\phi\phi}$, namely

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_f\frac{\partial}{\partial f} - \gamma_{\mathcal{O}} - 2\gamma_\phi\right]\Gamma_{\sigma\phi\phi} = 0\,, \quad \gamma_\phi = 0\,, \tag{2.26}$$

immediately gives $\beta_f = f\gamma_{\mathcal{O}}$, equivalent to (2.24). Solving the linear system, we find

$$\beta_f = vf^2\,, \qquad \gamma_{\mathcal{O}} = vf\,. \tag{2.27}$$

These are exact results (all orders in f) in the large N theory. The essential point, borne out by the auxiliary field trick, is that the for $\lambda = 0$ the only primitively divergent diagram is the one-loop renormalization of the σ propagator.

The general case

As we take $\lambda \neq 0$, we face the complication that the version of the theory with the auxiliary field, equation (2.12), is not renormalizable as it stands, since an explicit quartic term $\mathcal{O}\bar{\mathcal{O}}$ is regenerated by the interactions. We are led to consider the two-parameter theory

$$\mathcal{L}^{(2)}(g,h) \equiv \mathcal{L}_{ST} - g\sigma\bar{\sigma} + g\sigma\bar{\mathcal{O}} + g\bar{\sigma}\mathcal{O} + h\mathcal{O}\bar{\mathcal{O}}\,. \tag{2.28}$$

Comparing with the original form of the lagrangian without auxiliary field,

$$\mathcal{L}^{(1)}(f) \equiv \mathcal{L}_{ST} + f\mathcal{O}\bar{\mathcal{O}}\,, \tag{2.29}$$

we have the equivalence

$$\mathcal{L}^{(1)}(g+h) \sim \mathcal{L}^{(2)}(g,h)\,. \tag{2.30}$$

(We leave implicit the dependence of $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ on the single-trace couplings λ and on N.) Clearly,

$$\beta_f(g+h) = \beta_g(g,h) + \beta_h(g,h)\,, \tag{2.31}$$

where β_f is the beta function for the coupling f in theory (2.29), and β_g and β_h are the beta functions for the couplings g and h in theory (2.28). It may appear that not much is gained by considering the more complicated lagrangian $\mathcal{L}^{(2)}(g,h)$, but in fact the auxiliary field trick still provides a useful reorganization of large N diagrammatics. Our strategy is to work in the theory defined by $\mathcal{L}^{(2)}(g,h)$, but *in the limit that the renormalized quartic coupling* $h \to 0$.

We need not discuss explicitly the renormalization of the single-trace part of the action. For large N, the 1PI diagrams that renormalize the couplings in $\mathcal{L}_{ST}(\lambda)$ are independent of g, because leading diagrams at large N do not contain internal σ lines. Since we are also taking $h \to 0$, this implies that the renormalization of $\mathcal{L}_{ST}(\lambda)$ proceeds independently of $\mathcal{L}^{(2)}_{DT}$. We recall that by assumption, $\mathcal{L}_{ST}(\lambda)$ is such that $\beta_\lambda = 0$ for large N.

To discuss the renormalization of $\mathcal{L}^{(2)}_{DT}(g, h \to 0)$, we parametrize the counterterms as

$$\delta\mathcal{L}^{(2)}_{DT} = -(Z_2-1)g\sigma\bar{\sigma} + (Z_3-1)(g\sigma\bar{\mathcal{O}} + g\bar{\sigma}\mathcal{O}) + (Z_4-1)h\mathcal{O}\bar{\mathcal{O}}\,. \tag{2.32}$$

As we have emphasized, even for $h \to 0$ a quartic counterterm $(Z_4-1)h\mathcal{O}\bar{\mathcal{O}}$ is needed in order to cancel the divergence of $\Gamma_{\phi\phi\phi\phi}$. We can use again the fact that for large N, $\Gamma_{\phi\phi\phi\phi}$ is independent of g (recall Figure 2.5). Hence for $h \to 0$ the quartic counterterm can only depend on the single-trace coupling λ,

$$\lim_{h\to 0}(Z_4-1)h = f(\lambda,\epsilon,\mu)\,. \tag{2.33}$$

It follows that the corresponding beta function is only a function of λ,

$$\beta_h(g,h=0) = a(\lambda)\,. \tag{2.34}$$

In orbifolds of $\mathcal{N}=4$ SYM, λ is the usual 't Hooft coupling, and $a(\lambda)$ has a perturbative expansion of the form

$$a(\lambda) = \sum_{L=1}^{\infty} a^{(L)} \lambda^{L+1} , \tag{2.35}$$

where L is the number of loops.

The analysis of the two remaining primitively divergent structures, $\Gamma_{\sigma\bar{\sigma}}$ and $\Gamma_{\sigma\phi\phi}$, proceeds similarly as in the $\lambda = 0$ case, with a few extra elements. We have (for $h = 0$),

$$\Gamma_{\sigma\bar{\sigma}}(x,y) \;=\; g Z_2\, \delta(x-y) + Z_3^2\, g^2 \,\langle\, \mathcal{O}(x) \bar{\mathcal{O}}(y) \,\rangle_{g=h=0} \,, \tag{2.36}$$

$$\Gamma_{\sigma\phi\phi}(x;y,z) \;=\; -g\, Z_3 \,\langle\, \mathcal{O}(x)\phi(y)\phi(z) \,\rangle^{1PI}_{g=h=0} \,. \tag{2.37}$$

From the last equation, we see that the factor Z_3 has the role of renormalizing the composite operator $\mathcal{O}$ in the theory with $g = h = 0$,

$$\mathcal{O}^{ren}_{g=h=0} \equiv Z_3(\lambda, \mu, \epsilon)\, \mathcal{O} \,. \tag{2.38}$$

The dependence of $\mathcal{O}^{ren}_{g=h=0}$ on the renormalization scale μ is given by

$$\mu \frac{\partial}{\partial \mu}\, \mathcal{O}^{ren}_{g=h=0} = -\gamma(\lambda)\, \mathcal{O}^{ren}_{g=h=0} \,, \tag{2.39}$$

where $\gamma(\lambda)$ is, by definition, the anomalous dimension of the single-trace operator in the theory where we set to zero the double-trace couplings. The two-point function of $\mathcal{O}^{ren}_{g=h=0}$ takes then the standard form

$$\langle \mathcal{O}^{ren}(x) \mathcal{O}^{ren}(0) \rangle_{g=h=0} = \frac{v(\lambda)}{2\pi^2}\, \frac{\mu^{-2\gamma(\lambda)}}{x^{4+2\gamma(\lambda)}} \,, \quad x \neq 0 \,. \tag{2.40}$$

We have indicated that the normalization v will in general depend on λ. In orbifolds of $\mathcal{N} = 4$, $v(\lambda)$ and $\gamma(\lambda)$ have perturbative expansions of the form

$$v(\lambda) = \sum_{L=1}^{\infty} v^{(L)} \lambda^{L-1} \,, \qquad \gamma(\lambda) = \sum_{L=1}^{\infty} \gamma^{(L)} \lambda^{L} \,. \tag{2.41}$$

The expression (2.40) is not well-defined at short distance and needs further renormalization, which we perform again in the differential renormalization scheme. We first expand

$$\frac{\mu^{-2\gamma}}{x^{4+2\gamma}} = \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{n!} \frac{\log^n \mu^2 x^2}{x^4} \,, \tag{2.42}$$

and then renormalize each term of the series using the substitutions [61]

$$\frac{\log^n \mu^2 x^2}{x^4} = -\frac{n!}{4} \,\Box \sum_{k=1}^{n+1} \frac{1}{k!} \frac{\log^k \mu^2 x^2}{x^2} \,. \tag{2.43}$$

These are exact identities for $x \neq 0$ and provide the required modification of the behavior at $x = 0$, if one stipulates that free integration by parts is allowed under the integral sign.

Back in (2.36), we have[7]

$$\begin{aligned} \Gamma_{\sigma\bar{\sigma}}(x,0) &= g Z_2 \,\delta(x) + g^2 \langle \mathcal{O}^{ren}(x) \mathcal{O}^{ren}(0) \rangle_{g=h=0} && (2.44) \\ &= g\,\delta(x) - g^2 \frac{v}{8\pi^2} \sum_{n=0}^{\infty} (-\gamma)^n \,\Box \sum_{k=1}^{n+1} \frac{1}{k!} \frac{\log^k(\mu^2 x^2)}{x^2} \,. && (2.45) \end{aligned}$$

The CS equation,

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} - 2\gamma_{\mathcal{O}} \right] \Gamma_{\sigma\bar{\sigma}} = 0 \,, \tag{2.46}$$

gives as before two conditions, one for $x \neq 0$ and one from the delta function term. For $x \neq 0$, we may simply use the naive expression (2.40) for the correlator, and we find

$$2g\beta_g - 2\gamma_{\mathcal{O}} g^2 - 2\gamma g^2 = 0 \,. \tag{2.47}$$

It is easy to check that the same condition follows from the CS for $\Gamma_{\sigma\phi\phi}$. On the other hand, terms proportional to $\delta(x)$ in (2.46) arise either from the explicit $g\delta(x)$ in $\Gamma_{\sigma\bar{\sigma}}$, or when the μ derivative hits the $k = 1$ terms of the series,

$$0 = \beta_g - 2\gamma_{\mathcal{O}} g + g^2 v \sum_{n=0}^{\infty} (-\gamma)^n = \beta_g - 2\gamma_{\mathcal{O}} g + \frac{g^2 v}{1+\gamma} \,. \tag{2.48}$$

[7] The value of Z_2 is defined implicitly by this equation. As in the $\lambda = 0$ case, we could introduce a short-distance cutoff ϵ and then choose $Z_2(\epsilon, \mu)$ such that the final result (2.45) for the fully renormalized correlator is obtained.

The solution of the linear system (2.47, 2.48) is

$$\gamma_{\mathcal{O}} = \gamma + \frac{vg}{1+\gamma}\,, \quad \beta_g = \frac{vg^2}{1+\gamma} + 2g\gamma\,. \tag{2.49}$$

We can finally evaluate β_f in the original theory (2.29). From

$$\beta_f(f) = \beta_g(g = f, h = 0) + \beta_h(f, h = 0)\,, \tag{2.50}$$

we find

$$\boxed{\beta_f = \frac{v(\lambda)}{1+\gamma(\lambda)}\, f^2 + 2\,\gamma(\lambda)\, f + a(\lambda)\,.} \tag{2.51}$$

This is the sought generalization of the one-loop result (2.8) originally found in [43]. The expression for the full conformal dimension of the single-trace operator is

$$\boxed{\Delta_{\mathcal{O}} = 2 + \gamma_{\mathcal{O}}(f,\lambda) = 2 + \gamma(\lambda) + \frac{v(\lambda)}{1+\gamma(\lambda)}\, f\,.} \tag{2.52}$$

The boxed equations are valid to all orders in large N perturbation theory.

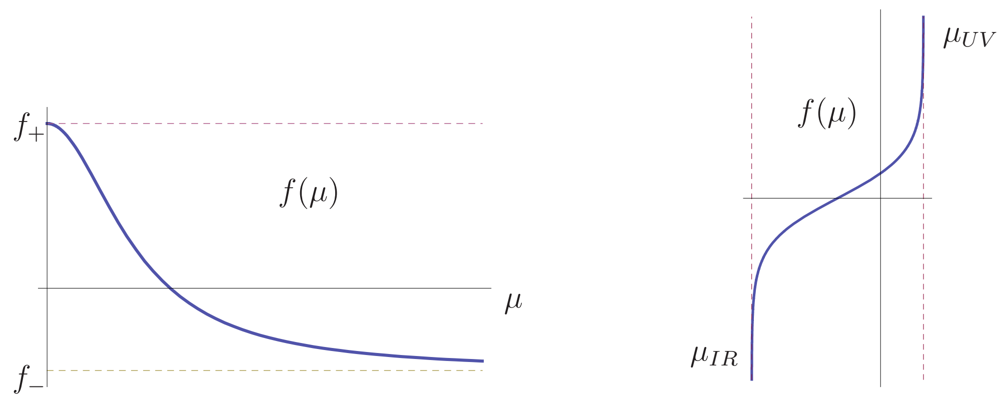
2.3 Double-trace running and dynamical symmetry breaking

The beta function of the double-trace coupling remains quadratic in f, to all orders in planar perturbation theory. This simplification allows to draw some general conclusions about the behavior of the running coupling and the stability of the Coleman-Weinberg potential. While the essential physics is already visible in the one-loop approximation, it seems worthwhile to pursue a general analysis.

2.3.1 Running coupling

We need to distinguish two cases, according to whether the quadratic equation

$$\beta_f = \frac{v(\lambda)}{1+\gamma(\lambda)}\, f^2 + 2\,\gamma(\lambda)\, f + a(\lambda) = 0 \tag{2.53}$$



(a) $D > 0$ (b) $D < 0$

Figure 2.6: The two qualitative behaviors of the running coupling $f(\mu)$ for $D > 0$ and $D < 0$.

has real or complex zeros. We define the discriminant $D(\lambda)$,

$$D(\lambda) \equiv \gamma(\lambda)^2 - \frac{a(\lambda)v(\lambda)}{1+\gamma(\lambda)}, \tag{2.54}$$

and the square root of $|D|$,

$$b(\lambda) \equiv \sqrt{|D(\lambda)|}\,. \tag{2.55}$$

From (2.35, 2.41), $b(\lambda)$ has a perturbative expansion of the form

$$b(\lambda) = b^{(1)}\lambda + b^{(2)}\lambda^2 + \dots\,. \tag{2.56}$$

- *Positive discriminant*

If $D > 0$, (2.53) has real solutions

$$f_\pm = -\frac{\gamma}{\tilde{v}} \pm \frac{b}{\tilde{v}}, \quad \tilde{v} \equiv \frac{v}{1+\gamma}\,. \tag{2.57}$$

In this case we can maintain conformal invariance in the quantum theory by tuning f to one of the two fixed points. Since $v > 0$ (the two-point function of $\mathcal{O}$ is positive by unitarity), we see that f_- is UV stable and f_+ IR stable. The differential equation for the running coupling,

$$\mu\frac{\partial}{\partial\mu}f(\mu) = \beta_f(f(\mu))\,, \tag{2.58}$$

is easily solved to give

$$f(\mu) = \frac{\left(\frac{\mu}{\mu_0}\right)^{2b} f_- + f_+}{\left(\frac{\mu}{\mu_0}\right)^{2b} + 1} \,. \tag{2.59}$$

The function $f(\mu)$ is plotted on the left in Figure 2.6. The running coupling interpolates smoothly between the IR and the UV fixed points.

- *Negative discriminant*

If $D < 0$ there are no fixed points for real f and conformal invariance is broken in the quantum theory. The solution of (2.58) is

$$f(\mu) = -\frac{\gamma}{\tilde{v}} + \frac{b}{\tilde{v}} \tan\left[\frac{b}{\tilde{v}} \ln(\mu/\mu_0)\right] \,, \quad \tilde{v} \equiv \frac{v}{1+\gamma} \,. \tag{2.60}$$

There are Landau poles both in the UV and in the IR, at energies

$$\mu_{IR} = \mu_0 \exp\left(-\frac{\pi\tilde{v}}{2b}\right) \cong \mu_0 \exp\left(-\frac{\pi v^{(1)}}{2b^{(1)}\lambda}\right) \tag{2.61}$$

$$\mu_{UV} = \mu_0 \exp\left(\frac{\pi\tilde{v}}{2b}\right) \cong \mu_0 \exp\left(\frac{\pi v^{(1)}}{2b^{(1)}\lambda}\right) \,. \tag{2.62}$$

The behavior of $f(\mu)$ is plotted on the right in Figure 2.6.

2.3.2 Effective potential

The running of the double-trace coupling f and the generation of a quantum effective potential for the scalar fields are closely related. We wish to make this relation precise.

Let us consider a spacetime independent vev for the scalars,

$$\langle \phi_a^{i\,b} \rangle = \varphi\, T_a^{i\,b} \,. \tag{2.63}$$

We have picked some direction in field space specified by the tensor $T_a^{i\,b}$, where i is a flavor index and $a, b = 1, \dots N$ are color indices. We need not assume that it is a classical flat direction. With no loss of generality we take $\varphi \geq 0$.

We now go through the textbook renormalization group analysis of the quantum effective potential $\mathcal{V}(\varphi)$. The RG equation reads

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_f\frac{\partial}{\partial f} - \gamma_\phi\,\varphi\frac{\partial}{\partial\varphi}\right]\mathcal{V}(\varphi,\mu,f,\lambda) = 0\,, \tag{2.64}$$

where $\gamma_\phi(\lambda)$ is the anomalous dimension of the scalar field ϕ. Note that for large N, $\gamma_\phi(\lambda)$ is independent of f. Writing (A.97) as

$$\mathcal{V}(\varphi,\mu,f,\lambda) \equiv \varphi^4\,U(\varphi/\mu,f,\lambda)\,, \quad \left[\varphi\frac{\partial}{\partial\varphi} - \frac{\beta_f}{1+\gamma_\phi}\frac{\partial}{\partial f} + \frac{4\gamma_\phi}{1+\gamma_\phi}\right]U = 0\,, \tag{2.65}$$

one finds that the most general solution takes the form

$$\mathcal{V}(\varphi,\mu,f,\lambda) = \varphi^4\left(\frac{\varphi}{\mu}\right)^{-\frac{4\gamma_\phi}{1+\gamma_\phi}} U_0(\hat{f}(\varphi),\lambda)\,, \tag{2.66}$$

where $\hat{f}(\mu)$ satisfies

$$\mu\frac{\partial}{\partial\mu}\,\hat{f}(\mu) = \frac{\beta_f(\hat{f})}{1+\gamma_\phi}\,. \tag{2.67}$$

In general, the arbitrary function $U_0(\hat{f},\lambda)$ is found order by order by comparing with explicit perturbative results. In our case, because of large N, the double-trace coupling contributes to the effective potential only at tree-level. This is again a consequence of the fact that 1PI diagrams with internal σ lines are suppressed. Moreover, by assumption the single-trace quartic term $N\lambda\mathrm{Tr}\phi^4$ is not renormalized at large N, so that the *explicit* λ dependence of $U_0(\hat{f},\lambda)$ is also exhausted by the tree-level contribution. There is of course an implicit λ dependence in $\hat{f}$, as clear from (2.67, 2.51). The full tree-level contribution to the effective potential is

$$\mathcal{V}_{tree}(\varphi) = N\lambda\mathrm{Tr}\phi^4 + f\,\mathcal{O}\bar{\mathcal{O}} = N^2(C_{ST}\lambda + C_{DT}f)\,\varphi^4\,, \tag{2.68}$$

where C_{ST} and C_{DT} are some non-negative proportionality constants of order one.[8] If the vev is taken along a classical flat direction of the single-trace lagrangian, then $C_{ST}=0$, but we need not assume this is the case. Thus

$$U_0(\hat{f},\lambda) = N^2(C_{ST}\lambda + C_{DT}\hat{f})\,. \tag{2.69}$$

[8] We are suppressing flavor indices: $N\lambda\mathrm{Tr}\phi^4$ in (A.96) is a shortcut for the scalar potential of the single-trace lagrangian $\mathcal{L}_{ST}$, which we require to be bounded from below. Then $C_{ST}\geq 0$. On the other hand, positivity of C_{DT} is clear from (A.96), since $\mathcal{O}\bar{\mathcal{O}}$ is a positive quantity.

The final result for the large N effective potential is

$$\boxed{\mathcal{V}(\varphi) = N^2 \, \mu^{\frac{4\gamma_\phi}{1+\gamma_\phi}} \, \left[C_{ST}\,\lambda + C_{DT}\,\hat{f}(\varphi)\right] \varphi^{\frac{4}{1+\gamma_\phi}} \, .} \tag{2.70}$$

Ordinarily, at a fixed order in perturbation theory the RG improved effective potential can be trusted in the range of φ such that the running coupling $\hat{f}(\varphi)$ is small. In our case, $\mathcal{V}(\varphi)$ receives no higher corrections in $\hat{f}$, so it appears that (2.70), being the full non-perturbative answer, may have a broader validity.

Let us make contact with the explicit one-loop expression of the effective potential. To this order,

$$\tilde{v}(\lambda) \cong v^{(1)} \, , \quad \gamma(\lambda) \cong \gamma^{(1)}\lambda \, , \quad a(\lambda) \cong a^{(1)}\lambda^2 \, , \quad \gamma_\phi \cong \gamma_\phi^{(1)}\lambda \, , \tag{2.71}$$

and the expansion of (2.70) gives

$$\begin{aligned} \mathcal{V}_{1-loop}(\varphi) = N^2\varphi^4 \log\left(\frac{\varphi}{\mu}\right) \times \qquad \qquad & (2.72) \\ \left[v^{(1)} f^2 C_{DT} + 2f\lambda(\gamma^{(1)} - 2\gamma_\phi^{(1)}) C_{DT} + \lambda^2 (a^{(1)} C_{DT} - 4\gamma_\phi^{(1)} C_{ST}) \right] \, . & \end{aligned}$$

Each term has an obvious diagrammatic interpretation.

2.3.3 Stability versus conformal invariance

Armed with the general form (2.70) of the large N effective potential, we can investigate the stability of the symmetric vacuum at $\varphi = 0$. Since the single-trace coupling λ does not run, we can treat it as an external parameter. For given λ, the functions $a(\lambda)$, $\tilde{v}(\lambda)$, $\gamma(\lambda)$ and $\gamma_\phi(\lambda)$ are just constant parameters that enter the expression for $\mathcal{V}(\varphi)$.

The qualitative behavior of $\mathcal{V}(\varphi)$ is dictated by the discriminant $D(\lambda)$. Comparing (2.67) with (2.58), we see that $\hat{f}(\varphi)$ behaves just as $f(\varphi)$, up to some trivial rescaling of coefficients by $1/(1+\gamma_\phi)$. We consider again the two cases:

- *Positive discriminant*

For $D > 0$, the running coupling is given by

$$\hat{f}(\varphi) = \frac{\left(\frac{\varphi}{\mu}\right)^{2\hat{b}} f_- + f_+}{\left(\frac{\varphi}{\mu}\right)^{2\hat{b}} + 1}\,, \qquad \hat{b} \equiv \frac{b}{1+\gamma_\phi}\,. \tag{2.73}$$

The constant solutions $\hat{f}(\varphi) = f_\pm$ are obtained as degenerate cases for $\mu \to 0$ and $\mu \to \infty$. In the generic case, the effective potential is bounded by the two functions (we set $\mu \equiv 1$)

$$N^2 \left(C_{ST}\lambda + C_{DT} f_+\right) \varphi^{\frac{4}{1+\gamma_\phi}} \le \mathcal{V}(\varphi) \le N^2 \left(C_{ST}\lambda + C_{DT} f_-\right) \varphi^{\frac{4}{1+\gamma_\phi}}\,, \tag{2.74}$$

where the lower bound is attained for $\varphi \to 0$ and the upper bound for $\varphi \to \infty$. Recall from (2.57) that $f_- < f_+$, with f_- always negative. If

$$C_{ST}\lambda + C_{DT} f_+ > 0\,, \tag{2.75}$$

then $\varphi = 0$ is at least a local minimum, otherwise it is a global maximum and the potential is unbounded from below. Condition (2.75) is simply the requirement that the tree-level potential (A.96) be bounded from below when f is set to its IR fixed point f_+. If (2.75) holds, it is also permissible to simply pick the constant solution $\hat{f}(\varphi) = f_+$. Then $\mathcal{V}$ is monotonically increasing and $\varphi = 0$ is the global minimum. In the generic case (2.73), we need the stronger condition

$$C_{ST}\lambda + C_{DT} f_- > 0 \tag{2.76}$$

to ensure that the potential is bounded from below. Then $\varphi = 0$ is the global minimum.

In view of the comments below (2.70), we believe that this analysis has general validity. It is certainly valid for $\lambda \ll 1$, since then $f_\pm \sim \lambda + O(\lambda^2)$, and the effective coupling $\hat{f}(\varphi) \ll 1$ for every value of φ.

In summary, barring pathological cases where the potential is unbounded from below, for $D > 0$ the vacuum $\varphi = 0$ is stable and dynamical symmetry breaking does not occur.

- *Negative discriminant*

If $D < 0$, the effective potential reads, in units $\mu \equiv 1$,

$$\mathcal{V}(\varphi) = N^2 \left[C_{ST}\lambda + C_{DT}\hat{f}(\varphi)\right] \varphi^{\frac{4}{1+\gamma_\phi}}\,, \quad \hat{f}(\varphi) = -\frac{\gamma}{\tilde{v}} + \frac{b}{\tilde{v}} \tan\left(\frac{b}{\tilde{v}} \log\varphi\right) \tag{2.77}$$

The theory only makes sense as an effective field theory for energy scales intermediate between the two Landau poles, $\mu_{IR} = e^{-\frac{\pi}{2b}} \ll \varphi \ll \mu_{UV} = e^{+\frac{\pi}{2b}}$. The potential ranges between minus infinity at μ_{IR} and plus infinity at μ_{UV}. A little algebra shows that $\mathcal{V}(\varphi)$ is either a monotonically increasing function, or it admits a local maximum and a local minimum. Local extrema exist if

$$\lambda \frac{C_{ST}}{C_{DT}} - \frac{\gamma}{\tilde{v}} < \frac{1}{1+\gamma_\phi} - \frac{b^2(1+\gamma_\phi)}{4\tilde{v}^2}\,, \tag{2.78}$$

with the potential always negative at the local minimum,

$$\mathcal{V}(\varphi_{min}) < 0\,. \tag{2.79}$$

From (2.71, 2.55), we see that (2.78) is always obeyed for sufficiently small λ. The value of the running coupling at the minimum can be expanded for $\lambda \ll 1$,

$$\hat{f}(\varphi_{min}) = -\alpha\,\lambda + \left(-\frac{a^{(1)}}{4} + \gamma_\phi^{(1)}\,\alpha - \frac{v^{(1)}}{4}\,\alpha^2\right)\lambda^2 + O(\lambda^3)\,, \quad \alpha \equiv \frac{C_{ST}}{C_{DT}} \tag{2.80}$$

For small λ, $\hat{f}(\varphi_{min})$ is also small, the local minimum can be trusted, and dynamical symmetry breaking occurs. If the vev is taken along a flat direction for the single-trace potential, namely if $C_{ST} = 0$, then the double-trace coupling at the new vacuum is of order $O(\lambda^2)$, which is perhaps the more familiar behavior – as in the original analysis of massless scalar electrodynamics [62]. From (2.78, 2.79, 2.80), we find that for small λ symmetry breaking occurs even if the tree level single-trace potential does not vanish ($C_{ST} \neq 0$).

We take the liberty to belabor this conclusion, giving an alternative derivation. One can first expand the effective potential to lowest non-trivial order,

$$\mathcal{V}(\varphi) \cong N^2[C_{ST}\lambda + C_{DT}\hat{f}(\mu)] + \mathcal{V}_{1-loop}(\varphi)\,, \tag{2.81}$$

with $\mathcal{V}_{1-loop}$ given by (2.72). In looking for the minimum, $\mathcal{V}'(\varphi_{min}) = 0$, $\mathcal{V}''(\varphi_{min}) > 0$, it is convenient to set the renormalization scale $\mu \equiv \varphi_{min}$. Then we just solve for $\hat{f}(\varphi_{min})$ and easily reproduce (2.80). This is a consistent procedure provided we can *find* a renormalization trajectory where $\hat{f}(\varphi_{min})$ takes the value (2.80). A glance at Figure 2.6 shows that yes, we can set $\hat{f}$ to any prescribed value. Finally, since (2.80) happens to be small for λ small, the whole analysis can be trusted in perturbation theory.

The inequality (2.78) can be satisfied also if λ is of order one, in which case $\hat{f}(\varphi)$ is of order one. In view of our remarks about the non-perturbative validity of $\mathcal{V}(\varphi)$, it seems plausible that the local minimum can also be trusted in this case.

2.4 AdS/CFT

We have used standard field theory arguments to characterize the two possible behaviors for a large N theory conformal in its single-trace sector. Either all double-trace beta functions admit real zeros, and then the symmetric vacuum is stable and conformal invariance is preserved; or at least one beta function has no real solutions, and then conformal invariance is broken and dynamical symmetry breaking occurs.

We now give a reinterpretation of these results in light of the AdS/CFT correspondence. Even for negative discriminant, we insist in solving for the zeros of the double-trace beta function,

$$f_\pm = -\frac{\gamma}{\tilde{v}} \pm \frac{\sqrt{D}}{\tilde{v}}\,. \tag{2.82}$$

Setting $f = f_\pm$, the full conformal dimension (2.52) of the single-trace operator $\mathcal{O}$ reads

$$\Delta_{\mathcal{O}} = 2 + \gamma + \tilde{v} f_\pm = 2 + \gamma - \gamma \pm \sqrt{D} = 2 \pm \sqrt{D}\,. \tag{2.83}$$

So at the fixed point, the anomalous dimension of $\mathcal{O}$ is either real if $D > 0$ or *purely imaginary* if $D < 0$. This is just as expected from the AdS/CFT formula

$$\Delta_{\mathcal{O}} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2} = 2 \pm \sqrt{4 + m^2 R^2}\,, \tag{2.84}$$

where m is the mass of the AdS_5 scalar field dual to $\mathcal{O}$, if we identify

$$\boxed{m^2(\lambda) R^2 = m^2_{BF} R^2 + D(\lambda) = -4 + D(\lambda)\,.} \tag{2.85}$$

For $D > 0$, we are in the standard situation of real coupling constant, real anomalous dimension and dual scalar mass above the stability bound, $m^2 > m^2_{BF}$. We propose to take (2.85) at face value even when $D < 0$. If $m^2 < m^2_{BF}$, the AdS bulk vacuum is unstable. Similarly, if $D < 0$, the field theory conformal-invariant vacuum is unstable. Equation (2.85) gives the precise relation between the two instabilities. The proper treatment of both the bulk and the boundary theory would be to expand around the stable minimum. But in stating that the AdS scalar has a certain mass $m^2 < m_{BF}^2$, we are implicitly quantizing the bulk theory in an AdS invariant way. The dual statement is to formally quantize the boundary theory in a conformal invariant way, around the symmetric minimum $\varphi = 0$, by tuning the coupling to the complex fixed point $f = f_+$ (or f_-). At either fixed point, the operator dimension is complex,

$$\Delta_{\mathcal{O}} = 2 \pm i\, b\,. \tag{2.86}$$

The discriminant $D(\lambda) = \gamma(\lambda)^2 - a(\lambda)\tilde{v}(\lambda)$ is a purely field-theoretic quantity. In principle (2.85) is a prescription to compute the tachyon mass from the field theory, at least order by order in perturbation theory. It would be interesting to see if integrability techniques [63] are applicable to this problem, though the fact that $\mathcal{O}$ is a "short" operator may represent a challenge. For now we may compare field theory results at weak coupling with the strong coupling behavior predicted by the gravity side. Let us look at a couple of examples.

2.4.1 Two examples

Expanding (2.85) to one-loop order,

$$m^2(\lambda)R^2 = -4 + D(\lambda) = -4 + \left[(\gamma^{(1)})^2 - a^{(1)}v^{(1)}\right]\lambda^2 + O(\lambda^3)\,. \tag{2.87}$$

The coefficients $v^{(1)}$, $\gamma^{(1)}$, and $a^{(1)}$ were computed in [43, 44] for several orbifolds of $\mathcal{N} = 4$ SYM. Obtaining the corresponding m^2 is an exercise in arithmetic.

As a first illustration, take the $\mathbb{Z}_2$ orbifold theory that arises on a stack of N electric and N magnetic D3 branes of Type 0B string theory. There are twisted scalars in the $\mathbf{20'}$ and $\mathbf{1}$ representations of $SU(4)_R$. From the results in [43, 46], one finds

$$m^2_{\mathbf{20'}}R^2 \cong -4 - \frac{\lambda^2}{8\pi^4} + O(\lambda^3)\,, \qquad m^2_{\mathbf{1}}R^2 \cong -4 - \frac{23\lambda^2}{64\pi^4} + O(\lambda^3)\,. \tag{2.88}$$

Since this orbifold has fixed points on the S^5 (it fixes the whole sphere), we expect these masses to remain negative below the stability bound for all λ, with the asymptotic behavior

$$m^2(\lambda)R^2 \sim -\frac{R^2}{\alpha'} = -\lambda^{1/2}\,, \quad \lambda \to \infty\,. \tag{2.89}$$

Let us also consider a simple class of non-supersymmetric freely acting orbifold,

$\mathbb{Z}_k$ orbifold with $SU(3)$ global symmetry [43]. The $\mathbb{Z}_k$ action is

$$z_i \to \omega_k^n \, z_i \,, \quad \omega_k \equiv e^{\frac{2\pi i}{k}} \,, \quad n = 1, \dots k \,, \tag{2.90}$$

where z_i, $i = 1, 2, 3$ are the three complex coordinates of $\mathbb{R}^6 = \mathbb{C}^3$. The orbifold is freely acting for k odd, and breaks supersymmetry for $k > 3$. Let us focus on the $\mathbb{Z}_5$ case. There are twisted operators $\mathcal{O}_{\mathbf{8},n}$ $\mathcal{O}_{\mathbf{1},n}$, with $n = 1, 2$, in the octet and singlet of the $SU(3)$ flavor group. It turns out that in the one-loop approximation the $n = 1$ operators have positive discriminant, while the $n = 2$ operators have negative discriminant. From the results of [43], one calculates

$$\begin{aligned} m^2_{\mathbf{8},2} R^2 &\cong -4 - \frac{\sqrt{5} - 1}{640\pi^4} \lambda^2 + O(\lambda^3) \,, \\ m^2_{\mathbf{1},2} R^2 &\cong -4 - \frac{7\sqrt{5} - 1}{1600\pi^4} \lambda^2 + O(\lambda^3) \,. \end{aligned} \tag{2.91}$$

The conjectural behavior of $m^2(\lambda)$ for freely acting orbifolds is plotted in Figure 2.1 in the introduction. The one-loop calculation (2.91) gives the second derivative at $\lambda = 0$. For large λ, these states correspond to highly stretched strings on the S^5. The asymptotic behavior should thus be

$$m^2(\lambda) R^2 \sim \frac{R^4}{\alpha'^2} \sim \lambda \,, \quad \lambda \to \infty \,. \tag{2.92}$$

Figure 2.1 plots the simplest interpolation between the small and large λ limits. It would be very interesting to compute the $O(\lambda^3)$ corrections to (2.91): this picture suggests that they should be positive.

2.4.2 Classical flat directions and instability

The $\mathbb{Z}_{2k+1}$ freely-acting orbifolds serve as an illustration of another point – classical flat directions are immaterial in our context. The classical moduli space of the theory is $(\mathbb{C}^3/\mathbb{Z}_{2k+1})^N/S_N$. In the brane picture this corresponds to the positions of the N D3 branes on the orbifold space $\mathbb{C}^3/\mathbb{Z}_{2k+1}$. The flat directions are parametrized by vevs for the bifundamental scalars (there are no adjoints). Along the flat directions, all twisted operators have zero vev.

As emphasized in [42], this is the case in general for freely acting orbifolds: they have no adjoint scalars and hence no classical branch along which the twisted operators could develop a vev. However, this does not imply that the symmetric vacuum is stable. On the contrary, we have seen in section 2.3.3 that dynamical symmetry breaking occurs at small coupling whenever $D < 0$, irrespective of the classical potential. Since one can always find a double-trace coupling with $D < 0$, whether the orbifold is freely acting or not [44], we conclude that freely acting orbifolds also have a CW instability which drives into condensation a twisted operator, $\langle \mathcal{O} \rangle \neq 0$. The instability occurs away from the flat directions.

This reconciles the proposal of [43], which relates bulk tachyons with the breaking of conformal invariance, with the general viewpoint of [42], which relates them to the Coleman-Weinberg instability. A detailed analysis of the CW instability in some examples of freely acting orbifolds has been pursued by [64].

2.5 Discussion

The logarithmic running of double-trace couplings $f\mathcal{O}\bar{\mathcal{O}}$, where $\mathcal{O} \sim \mathrm{Tr}\phi^2$, is a general feature of large N field theories that contain scalar fields. In this chapter we have studied the renormalization of double-trace couplings in theories that have vanishing single-trace beta functions at large N. We have derived general expressions for the double-trace beta function β_f, the conformal dimension $\Delta_{\mathcal{O}}$ and the effective potential $\mathcal{V}(\varphi)$. The main point is that β_f is a quadratic function of f (and $\Delta_{\mathcal{O}}$ a linear function of f), to all-orders in planar perturbation theory, with coefficients that depend on the single-trace couplings λ.

Double-trace running plays an important role in non-supersymmetric examples of the AdS/CFT correspondence. We have related the discriminant $D(\lambda)$ of β_f to the mass $m^2(\lambda)$ of the bulk scalar dual to the single-trace operator $\mathcal{O}$. If $D(\lambda) < 0$, the bulk scalar is a tachyon; on the field theory side, conformal invariance is broken and dynamical symmetry breaking occurs.

The authors of [44] considered orbifolds of $\mathcal{N} = 4$ SYM, realized as the low energy limit of the theory on N D3 branes at the tip of the cone $\mathbb{R}^6/\Gamma$. They found a one-to-one correspondence between double-trace couplings with negative discriminant and twisted tachyons in the tree-level spectrum of the type IIB background before the decoupling limit, namely $\mathbb{R}^{3,1} \times \mathbb{R}^6/\Gamma$. (Note that these flat-space tachyons are conceptually distinct from the tachyons in the curved $AdS_5 \times S^5/\Gamma$ background that have been the focus of this paper.)[9] It turns out that for *all* non-supersymmetric examples in this class, at least one double-trace coupling has negative discriminant, and conformal invariance is broken.

[9] The correspondence between twisted sector tachyons and field theory instabilities was first observed in [65] in the context of non-commutative field theory.

It will be interesting to investigate more general constructions to see if conformal examples exist, both as a question of principle and in view of phenomenological applications.[10] One possibility, suggested by the correspondence found in [44], is to add discrete torsion in a way that removes the tree-level tachyons [68]. Another is to add appropriate orientifold planes. A promising candidate for a conformal orientifold theory is the $U(N)$ gauge theory with six scalars in the adjoint and four Dirac fermions in the antisymmetric representation of the gauge group [69].

Another important question, which is being investigated by [64], is to analyze the IR fate of non-supersymmetric orbifolds of $\mathcal{N} = 4$ SYM, by expanding their lagrangian around the local minimum of the effective potential. This is a well-posed field theory problem because the minimum can be trusted for small coupling. It would also be very interesting to extend the calculations of [43, 44] to two loops. At one-loop, there is no obvious distinction between freely acting and non-freely acting examples. This distinction may arise at two loops, with the freely acting cases beginning to show the behavior of Figure 2.1.

Finally, it would be nice to find a more detailed AdS interpretation for the individual terms appearing in the double-trace beta function. For $\lambda = 0$, when only the term vf^2 is present, β_f can be reproduced by a simple bulk calculation [56], using the interpretation [56, 57] of the double-trace deformation as a mixed boundary condition for the bulk scalar. There should be a bulk interpretation for the other terms of β_f as well, in particular for the coefficient $a(\lambda)$ which drives the instability.

[10] See *e.g.* [66, 67] for an approach to conformal phenomenology.

Chapter 3

Intersecting Flavor Branes

3.1 Introduction

Open string tachyon condensation has been studied from many viewpoints, see [70] for a review. Here we consider a holographic (AdS/CFT) setup where the bulk theory contains an open string tachyon, and ask what is the counterpart of tachyon condensation in the boundary field theory. We will identify a sector of the boundary theory as a "holographic open string field theory" capturing the tachyon dynamics. Since the bulk is weakly coupled when the boundary is strongly coupled, and viceversa, we are bound to learn something new from their comparison.

We introduce the open string tachyon by adding to the $AdS_5 \times S^5$ background two probe $D7$ branes intersecting at general angles. Probe branes are the familiar way to include a small number of fundamental flavors in the AdS/CFT correspondence [71]. If the closed string background is supersymmetric, it is possible, and often desirable, to consider configurations of probe branes that preserve some supersymmetry, as *e.g.* in [71–86]. Instead, we are after supersymmetry breaking and the ensuing tachyonic instability. Another way to motivate our work is then as a natural susy-breaking generalization of the standard supersymmetric setup of [71]. This generalization is technically challenging, and the technical aspects have some interest of their own. Intersecting brane systems have many other applications in string theory, from string phenomenology to string cosmology, and the technical lessons learnt in our problem may be useful in those contexts as well.

The system that we study is as an open string analogue of the AdS/CFT pairs involving closed string tachyons considered in [6, 42–44] and described in chapter 2 of this dissertation. For general angles the bulk theory is unstable via condensation of an open string tachyon, or at least this is the picture for large λ where we can calculate the string spectrum. In this chapter we focus on the the field theory analysis at small λ, with the goal of detecting the expected instability.

The first challenge is to write down the Lagrangian of the dual field theory. As is well-known, adding N_f parallel $D7$ branes to $AdS_5 \times S^5$ corresponds to adding to $\mathcal{N} = 4$ SYM action N_f extra $\mathcal{N} = 2$ hyper multiplet in the fundamental representation of the $SU(N)$ gauge group. The resulting action preserves an $\mathcal{N} = 2$ subalgebra of the original $\mathcal{N} = 4$ supersymmetry algebra – which particular $\mathcal{N} = 2$ being a matter of convention so long as it is the same for all the hyper multiplets. Introducing relative angles between the $D7$ branes corresponds to choosing *different* embeddings for the $\mathcal{N} = 2$ subalgebras of each different hyper multiplet. In general supersymmetry will be completely broken, while for special angles $\mathcal{N} = 1$ susy is preserved. When $\mathcal{N} = 1$ is preserved we can use $\mathcal{N} = 1$ superspace to write the Lagrangian. When supersymmetry is completely broken the determination of the Lagrangian turns out to be a difficult technical problem that we are unable to solve completely. We cannot fix the quartic terms $\sim Q^4$ where Q are the hyper multiplet scalars. The difficulty is related to the lack of an off-shell superspace formulation of $\mathcal{N} = 4$ SYM. Nevertheless, by making what we believe is a mild technical assumption, we can fix the *sign* of the classical quartic potential. This is sufficient to argue that the theory is indeed unstable from the renormalization of "double-trace" terms $f \int d^4x\, \mathcal{O}^2$, where now $\mathcal{O} \sim \bar{Q}^{\mathfrak{a}} Q_{\mathfrak{a}}$ with $\mathfrak{a} = 1, \ldots N$ a color index. We are now using "double-trace" in quotes since of course the fields Q are not matrices but vectors, but the logic is much the same. The renormalization of f has the same twofold interpretation as above. We identify the mesonic operator $\mathcal{O}$ as the dual of the open string tachyon between the two $D7$ branes. The Coleman-Weinberg potential for Q plays the role a holographic effective action for the tachyon.

3.2 AdS/CFT with Flavor Branes Intersecting at General Angles

We begin with a review the Karch-Katz setup [71], where parallel probe $D7$ branes are used to engineer an $\mathcal{N} = 2$ supersymmetric field theory with flavor. We then break supersymmetry by introducing a relative angle between the $D7$ branes. We derive the dual Lagrangian, up to an ambiguity in the quartic potential for the fundamental scalars. We end the section with a review of the basic bulk-to-boundary dictionary.

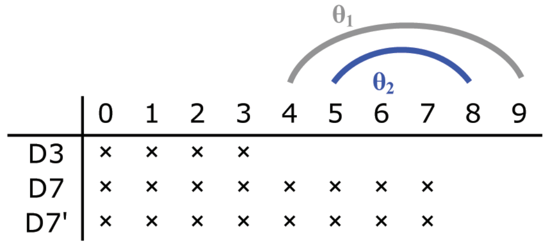


Figure 3.1: The brane configuration with Flavor Branes Intersecting.

3.2.1 Parallel flavor branes

We start with the familiar $D3/D7$ supersymmetric brane configuration with N "color" $D3$s and N_f "flavor" $D7$s, arranged as shown in Figure 3.1. For now $\theta_1 = \theta_2 = 0$, that is, all $D7$ branes are parallel to one another. Taking the decoupling limit on the $D3$s wordvolume, the $D3$ branes are replaced by their near-horizon geometry. If $N_f \ll N$, we can treat the $D7$ branes as probes in the $AdS_5 \times S^5$ background, neglecting their backreaction [71]. This background preserves $\mathcal{N} = 2$ supersymmetry in four dimensions.

The dual field theory is $\mathcal{N} = 4$ $SU(N)$ SYM coupled to N_f $\mathcal{N} = 2$ hyper multiplets in the fundamental representation of the $SU(N)$ color group, arising from the $D3$-$D7$ open strings. We are interested in the case of massless hyper multiplets, corresponding to the brane setup where the $D7$s coincide with the $D3$s (at the origin of the 89 plane). After decoupling, the probe $D7$s fill the whole AdS_5 and wrap an $S^3 \subset S^5$.

Let us briefly recall the field content of the boundary theory. A more detailed treatment and the full Lagrangian can be found in Appendix A.1. The $\mathcal{N} = 4$ vector multiplet consists of the gauge field A_μ, four Weyl spinors λ^A_α, $A = 1, \ldots, 4$ and six real scalars X_m, $m = 4, \ldots, 9$ corresponding to the six transverse directions to the $D3$ branes. It is convenient to represent the scalars as a self-dual antisymmetric tensor X^{AB} of the R-symmetry group $SU(4)_R \cong Spin(6)$,

$$(X^{AB})^\dagger = \bar{X}_{AB} \equiv \frac{1}{2}\epsilon_{ABCD}\, X^{CD} \,. \tag{3.1}$$

The explicit change of variables is

$$X^{AB} = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 0 & X_8 + iX_9 & X_6 + iX_7 & X_4 + iX_5 \\ -X_8 - iX_9 & 0 & X_4 - iX_5 & -X_6 + iX_7 \\ \hline -X_6 - iX_7 & -X_4 + iX_5 & 0 & X_8 - iX_9 \\ -X_4 - iX_5 & X_6 - iX_7 & -X_8 + iX_9 & 0 \end{array} \right) . \tag{3.2}$$

Each $\mathcal{N} = 2$ flavor hyper multiplet consists of two Weyl spinors and two complex scalars,

$$\begin{array}{ccc} & \psi^i_\alpha & \\ q^i & & (\tilde{q}_i)^\dagger \\ & \left(\tilde{\psi}_{i\,\alpha}\right)^\dagger & \end{array} \tag{3.3}$$

Here $i = 1, \ldots, N_f$ is the flavor index. The scalars form an $SU(2)_R$ doublet,

$$Q^{\mathcal{I}} \equiv \begin{pmatrix} q \\ \tilde{q}^\dagger \end{pmatrix} , \quad \mathcal{I} = 1, 2 . \tag{3.4}$$

The flavor hyper multiplets are minimally coupled to the $\mathcal{N} = 2$ vector multiplet that sits inside the $\mathcal{N} = 4$ vector multiplet. This coupling breaks the R-symmetry $SU(4)_R$ to $SU(2)_L \times SU(2)_R \times U(1)_R$, where $SU(2)_R \times U(1)_R$ is the R-symmetry of the resulting $\mathcal{N} = 2$ theory. There is a certain arbitrariness in the choice of embedding $SU(2)_L \times SU(2)_R \times U(1)_R \subset SU(4)_R \cong Spin(6)$. This corresponds to the choice of orientation of the whole stack of $D7$ branes in the 456789 directions (we need to pick an $\mathbb{R}^4 \subset \mathbb{R}^6$). For example if we choose the configuration of Figure 3.1, we identify $SU(2)_L \times SU(2)_R \cong SO(4)$ with rotations in the 4567 directions and $U(1)_R \cong SO(2)$ with a rotation on the 89 plane. A short calculation using our parametrization of the scalars (5.10) shows that this corresponds to the following natural embedding of $SU(2)_L \times SU(2)_R \times U(1)_R \subset SU(4)_R$:

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \left(\begin{array}{c|c} SU_R(2) \times U(1)_R & \\ & \\ \hline & \\ & SU_L(2) \times U(1)^*_R \end{array} \right) . \tag{3.5}$$

Of course, any other choice would be equivalent, so long as it is performed simultaneously for all $D7$ branes. With the choice (B.23), the $\mathcal{N}=4$ vector multiplet splits into the $\mathcal{N}=2$ vector multiplet

$$\begin{array}{ccc} & A_\mu & \\ \lambda^1_\alpha & & \lambda^2_\alpha \\ & \frac{X_8+iX_9}{\sqrt{2}} & \end{array} , \tag{3.6}$$

and the $\mathcal{N}=2$ hyper multiplet

$$\begin{array}{ccc} & \lambda^3_\alpha & \\ \frac{X_4+iX_5}{\sqrt{2}} & & \frac{X_6+iX_7}{\sqrt{2}} \\ & \lambda^4_\alpha & \end{array} . \tag{3.7}$$

The two Weyl spinors in the vector multiplet form an $SU(2)_R$ doublet

$$\Lambda_{\mathcal{I}} \equiv \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} , \quad \mathcal{I}=1,2\,, \tag{3.8}$$

while the two spinors in the hyper multiplet form an $SU(2)_L$ doublet,

$$\hat{\Lambda}_{\hat{\mathcal{I}}} \equiv \begin{pmatrix} \lambda_3 \\ \lambda_4 \end{pmatrix} , \quad \hat{\mathcal{I}}=1,2\,. \tag{3.9}$$

We use $\mathcal{I}\,,\mathcal{J}\,\cdots=1,2$ for $SU(2)_R$ indices and $\hat{\mathcal{I}}\,,\hat{\mathcal{J}}\,\cdots=1,2$ for $SU(2)_L$ indices. To make the $SU(2)_L\times SU(2)_R$ quantum numbers of the scalars more transparent we also introduce the 2×2 complex matrix $\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}}$, defined as the off-diagonal block of X^{AB},

$$\mathcal{X}^{\hat{\mathcal{I}}\mathcal{I}} = \begin{pmatrix} X_6+iX_7 & X_4+iX_5 \\ X_4-iX_5 & -X_6+iX_7 \end{pmatrix} . \tag{3.10}$$

Note that $\mathcal{X}^{\hat{\mathcal{I}}\mathcal{I}}$ obeys the reality condition

$$\left(\mathcal{X}^{\hat{\mathcal{I}}\mathcal{I}}\right)^* = -\mathcal{X}_{\hat{\mathcal{I}}\mathcal{I}} = -\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\epsilon_{\mathcal{I}\mathcal{J}}\mathcal{X}^{\hat{\mathcal{J}}\mathcal{J}}\,. \tag{3.11}$$

We summarize in the following table the transformation properties of the fields:

	$SU(N)$	$SU(N_f)$	$SU(2)_L$	$SU(2)_R$	$U(1)_R$
A_μ	Adj	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0
X^{12}	Adj	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	+2
$\mathcal{X}^{\mathcal{I}\hat{\mathcal{I}}}$	Adj	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	0
$\Lambda_{\mathcal{I}}$	Adj	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	+1
$\hat{\Lambda}_{\hat{\mathcal{I}}}$	Adj	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	–1
$Q^{\mathcal{I}}$	$\square$	$\square$	$\mathbf{1}$	$\mathbf{2}$	0
ψ	$\square$	$\square$	$\mathbf{1}$	$\mathbf{1}$	–1
$\tilde{\psi}$	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	+1

Table 3.1: Quantum numbers of the fields for $\mathcal{N} = 4$ $SU(N)$ SYM coupled to N_f hyper multiplets.

3.2.2 Rotating the flavor branes

We now describe a non-supersymmetric open string deformation of this background. For simplicity we consider the case $N_f = 2$. While keeping the two $D7$ branes coincident with the $D3$s in the 0123 directions, we rotate them with respect to each other in the transverse six directions, see Figure 1. There are two independent angles, so without loss of generality we may perform a rotation of angle $\theta_1 = \theta_{49}$ in the 49 plane and a rotation of angle $\theta_2 = \theta_{85}$ in the 58 plane. For generic angles supersymmetry is completely broken; for $\theta_1 = \theta_2$ it is broken to $\mathcal{N} = 1$. As we rotate the branes, some $D7$-$D7'$ open string modes become tachyonic. The main goal of this paper is to study this tachyonic instability from the viewpoint of the dual field theory.

On the field theory side, rotating the second brane $D7'$ amounts to choosing a different embedding of $SU(2)_R \subset SU(4)$ for the second hyper multiplet, while keeping the standard embedding (B.23) for the first. In the Lagrangian, we must perform an $SU(4)$ rotation of the $\mathcal{N} = 4$ fields that couple to the second hyper multiplet, leaving the ones that couple to the first unchanged. The rotation is of the form

$$X'_m = \mathcal{R}^{(\mathbf{6})\,n}_{m}(\theta_1, \theta_2)\, X_n\,, \qquad \lambda'_A = \mathcal{R}^{(\mathbf{4})\,B}_{A}(\theta_1, \theta_2)\, \lambda_B\,. \tag{3.12}$$

The explicit form of the rotation matrices $\mathcal{R}^{(\mathbf{6})}$ and $\mathcal{R}^{(\mathbf{4})}$ is given in Appendix A.2.

Naively, the Q^4 terms are not affected by the rotation, but this is incorrect. This is seen clearly in $\mathcal{N} = 1$ superspace. The $\mathcal{N} = 4$ multiplet is built out of three chiral multiplets Φ^a, $a = 1, 2, 3$ and one vector multiplet V. The Q^4 terms arise from integrating out the auxiliary fields F^a $(a = 1, 2, 3)$ of the chiral multiplets and D of the vector multiplet, which transform under the $SU(4)_R$ rotation. For example, a rotation that preserves $\mathcal{N} = 1$ supersymmetry $(\theta_1 = \theta_2)$ corresponds to a matrix $\mathcal{R}^{(\mathbf{4})} \subset SU(3)$, which acts on F^a leaving D invariant. The correct Lagrangian is obtained by performing the rotation on the X_m, λ_A *and* F^a fields that couple to the primed hyper multiplet, and only then can the auxiliary fields be integrated out. The Q^4 terms get modified accordingly.

Under a more general $SU(4)_R$ rotation, the F^a and D auxiliary fields are expected to mix in a non-trivial fashion. There exists a formalism developed in [87, 88] that provides the generic R-symmetry transformations action in $\mathcal{N} = 1$ superspace. Unfortunately, for $\mathcal{N} = 4$ supersymmetry we cannot rely on this formalism because the transformations do not close off-shell. This technically involved point is explained in detail in Appendix A.3. There we also provide an $\mathcal{N} = 2$ supersymmetry toy example where the formalism works perfectly since the $\mathcal{N} = 2$ R-symmetry algebra closes off-shell.

To proceed, we parametrize our ignorance of the Q^4 terms. The exact form of the full Lagrangian, including the parametrized Q^4 potential, is spelled out in Appendix A.2. Schematically, we write the Q^4 potential as

$$V_{Q^4} = Q_1^4 + Q_2^4 + (Q_1 Q_2)_F^2 \ f(\theta_1, \theta_2) + (Q_1 Q_2)_D^2 \ d(\theta_1, \theta_2) \ , \tag{3.13}$$

for some unknown functions $f(\theta_1, \theta_2)$ and $d(\theta_1, \theta_2)$. Here Q_1 and Q_2 are shorthands for the scalars in the first and second hyper multiplets and the subscripts F and D refer to different ways to contract the indices, see (A.49) for the exact expressions. The letters F and D are chosen as reminders of the (naive) origin of the two structures from integrating out the "rotated" F and D $\mathcal{N} = 1$ auxiliary fields, but this form of the potential follows from rather general symmetry considerations, as we explain in Appendix A.2. When $\theta_1 = \theta_2$, $\mathcal{N} = 1$ supersymmetry is preserved and $\mathcal{N} = 1$ superspace allows to fix the two functions,

$$f(\theta, \theta) = \cos\theta \, , \qquad d(\theta, \theta) = 1 \, . \tag{3.14}$$

For general angles, we can constrain f and d somewhat, using bosonic symmetries (see Appendix A.2), but unfortunately we are unable to fix them uniquely. The most important assumption we will make in the following is *positivity* of the classical potential, $V_{Q^4} \geq 0$, implying $f(\theta_1, \theta_2) \leq 1$ and $d(\theta_1, \theta_2) \leq 1$ for all θ_1, θ_2. Positivity would follow from the mere existence of any reasonable off-shell superspace formulation, as the scalar potential would always be proportional to the square of the auxiliary fields, even when supersymmetry is broken by the relative R-charge rotation between the two hyper multiplets.[1] Note also that the classical potential V_{Q^4} is a homogeneous function of the Qs, so it is everywhere positive if and only if it is bounded from below, which is another plausible requirement.

3.2.3 Bulk-boundary dictionary

The basic bulk-to-boundary dictionary for the parallel brane case has been worked out in [89, 90]. A brief review is in order.

In the closed string sector, Type IIB closed string fields map to single-trace operators of $\mathcal{N} = 4$ SYM, as usual. In the open string sector, open string fields on the $D7$ worldvolume map to gauge-singlet mesonic operators, of the schematic form $\bar{Q}X^nQ$, where Q stands for a generic fundamental field and X for a generic adjoint field.

The massless bosonic fields on the $D7$ worldvolume are a scalar Φ and a gauge field $(A_{\hat{\mu}}, A_{\hat{\alpha}})$, where $\hat{\mu}$ are AdS_5 indices and $\hat{\alpha}$ are S^3 indices. Kaluza Klein reduction on the S^3 generates the following tower of states, labeled in terms of $(j_1\,,\, j_2)_s$ representations of $SU(2)_L \times SU(2)_R \times U(1)_R$:

$$\begin{aligned} &\Phi \to \Phi^\ell = \left(\frac{\ell}{2}\,,\,\frac{\ell}{2}\right)_2 \,, \quad A_{\hat{\mu}} \to A^\ell_{\hat{\mu}} = \left(\frac{\ell}{2}\,,\,\frac{\ell}{2}\right)_0 \,, \\ &A_{\hat{\alpha}} \to A^\ell_\pm = \left(\frac{\ell\pm 1}{2}, \frac{\ell \mp 1}{2}\right)_0 \,. \end{aligned} \tag{3.15}$$

(The longitudinal component of $A_{\hat{\alpha}}$ is not included because it can be gauged away). These states (and their fermionic partners, which we omit) can be organized into short multiplets of the $\mathcal{N} = 2$ superconformal algebra,

$$\left(A^{\ell+1}_-, A^\ell_{\hat{\mu}}, \Phi^\ell, A^{\ell-1}_+\right) \,, \quad \ell = 0, 1, 2, \ldots \tag{3.16}$$

of conformal dimensions

$$(\ell+2, \ell+3, \ell+3, \ell+4)\,. \tag{3.17}$$

For $\ell = 0$ the A_+ state is absent. Note that all states in a given multiplet have the same $SU(2)_L$ spin, indeed the $\mathcal{N} = 2$ supercharges are neutral under $SU(2)_L$.

The lowest member of each multiplet, namely $A^{\ell+1}_-$, is dual to the chiral

[1] To illustrate how this would work we consider in section A.3.2 $\mathcal{N} = 2$ SYM theory coupled to two fundamental $\mathcal{N} = 1$ chiral multiplets, with different choices of the two $\mathcal{N} = 1$ subalgebras.

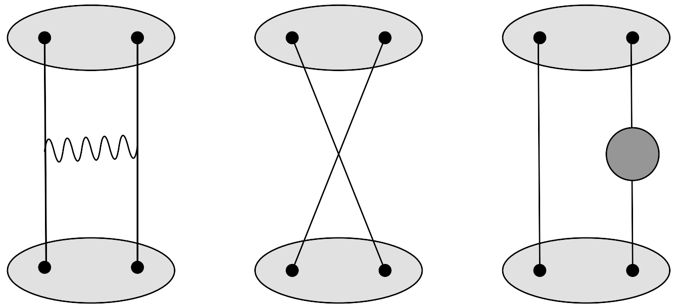


Figure 3.2: Diagrams contributing to the one-loop renormalization of the mesonic operators.

primary operator

$$\bar{Q}_{\{\mathcal{I}}\mathcal{X}_{\mathcal{I}_1\hat{\mathcal{I}}_1}\ldots\mathcal{X}_{\mathcal{I}_\ell\hat{\mathcal{I}}_\ell}Q_{\mathcal{J}\}}\,, \tag{3.18}$$

where $Q^{\mathcal{I}}$ is the $SU(2)_R$ doublet of complex fundamental scalars. In (3.18) the $SU(2)_L$ and $SU(2)_R$ indices are separately symmetrized. In particular for $\ell = 0$, we have the triplet of mesonic operators

$$\mathcal{O}_{\mathbf{3}} \equiv \bar{Q}_{\{\mathcal{I}}Q_{\mathcal{J}\}} = \bar{Q}_{\mathcal{I}}Q^{\mathcal{J}} - \frac{1}{2}\bar{Q}_{\mathcal{K}}Q^{\mathcal{K}}\delta^{\mathcal{J}}_{\mathcal{I}}\,. \tag{3.19}$$

The singlet operator

$$\mathcal{O}_{\mathbf{1}} \equiv \bar{Q}_{\mathcal{I}}Q^{\mathcal{I}}\,, \tag{3.20}$$

is not a chiral primary and maps to a massive open string state.

In Appendix A.4 we compute the one-loop dilatation operator acting on the basis of states $\mathcal{O} \equiv \bar{Q}_{\mathcal{I}}Q^{\mathcal{J}}$, evaluating the diagrams schematically drawn in Figure 3.2. We find

$$\Gamma^{(1)} = \frac{\lambda}{4\pi^2}\,\mathbb{K}\,, \quad \mathbb{K} \equiv \delta^{\mathcal{I}}_{\mathcal{J}}\delta^{\mathcal{K}}_{\mathcal{L}}\,. \tag{3.21}$$

The eigenstates are the triplet and the singlet, with eigenvalues

$$\gamma_{\mathbf{3}} = 0\,, \quad \gamma_{\mathbf{1}} = \frac{\lambda}{2\pi^2}\,. \tag{3.22}$$

As expected, the chiral triplet operator has protected dimension. At one-loop, this result does not change as we turn on non-zero angles θ_1 and θ_2.

So far we have considered the case of a single $D7$ brane, or a single flavor. For multiple $D7$ branes (multiple flavors) the Chan-Paton labels of the open strings are interpreted as the bifundamental flavor indices of the mesonic operators, $\mathcal{O}^{ij}$, $i, j = 1, \ldots N_f$. In our setup, with $N_f = 2$, the lowest mode of the open string with off-diagonal Chan-Paton labels, which is the massless gauge field for parallel branes, becomes tachyonic as we turn on a relative angle between the $D7$s. In the dual field theory we expect to find an instability associated with the operator $\mathcal{O}^{12}_{\mathbf{3}}$, the lowest dimensional operator dual to the off-diagonal open string mode.

3.3 "Double-trace" Renormalization and the Open String Tachyon

Our setup is an open string version of the phenomena studied in [6, 42–44] and chapter 2 of this dissertation. We now generalize this story to AdS/CFT dual pairs containing an open string sector. In the presence of flavor branes in the AdS bulk, the dual field theory contains extra fundamental matter. An *open* string tachyon corresponds to an instability in the *mesonic* sector of the boundary theory. In this chapter we illustrate this phenomenon in the example of the intersecting $D7$ brane system. The classical Lagrangian of the boundary theory takes the schematic form

$$\mathcal{L} = \mathcal{L}_{adj} + \mathcal{L}_{fund} = -\mathrm{Tr}\left[F^2 + (DX)^2 + \ldots\right] - (DQ)^2 - \frac{\lambda}{N}\mathcal{O}^{ij}\bar{\mathcal{O}}^{ij} + \ldots \quad (3.23)$$

where $\mathcal{O}^{ij} = Q^{i\,\mathfrak{a}}\bar{Q}_{j\,\mathfrak{a}}$ are the gauge-invariant mesonic operators made from the fundamental scalars and for simplicity we have ignored the $SU(2)_R$ structure, which will be restored shortly.[2] The whole $\mathcal{L}_{fund}$ is $1/N$ suppressed with respect to $\mathcal{L}_{adj}$, in harmony with the fact that the classical D-brane effective action arises from worldsheets with disk topology, and is thus suppressed by a power of $g_s \sim 1/N$ with respect to the classical closed string effective action, arising from worldsheets with sphere topology. Nevertheless, as always in the tachyon condensation problem, it makes perfect sense to focus on classical open string field theory. The classical open string dynamics is dual to the quantum planar dynamics of the mesonic sector of the field theory. The 't Hooft coupling λ does not run at leading order in N, indeed the hyper multiplet contribute to β_λ at order $O(1/N)$. For generic angles, the term in the Q^4 potential that mix the two flavors run, so perturbative renormalizability forces the introduction

[2] To avoid cluttering in some expressions below we always write the flavor indices as upper indices in $\mathcal{O}^{ij}$.

of a new coupling constant f,

$$\delta\mathcal{L}_{fund} = -\frac{f}{N}\mathcal{O}^{12}\bar{\mathcal{O}}^{12}\,. \tag{3.24}$$

Note on the other hand that no extra terms diagonal in flavor (namely $\mathcal{O}^{11}\bar{\mathcal{O}}^{11}$ and $\mathcal{O}^{22}\bar{\mathcal{O}}^{22}$ are induced at one-loop. For the first flavor this is immediate to see: the diagrams contributing to the term $\mathcal{O}^{11}\bar{\mathcal{O}}^{11}$ of the effective potential are independent of θ_1, θ_2 (they do not involved any coupling mixing the two flavors) and thus their sum must vanish, as it does in the $\mathcal{N}=2$ supersymmetric theory with $\theta_1 = \theta_2 = 0$. For the second flavor this follows by symmetry, since the two flavors are of course interchangeable.[3]

This extra "double-trace" term (3.24) arise at the same order in N order as the classical $\mathcal{L}_{fund}$, indeed inspection of the Feynman diagrams shows that the one-loop bare coupling f_0 behaves as

$$f_0 \sim \lambda^2 \log\Lambda\,. \tag{3.25}$$

The analysis of chapter 2 can be applied in its entirety to this "open string" case. The "double-trace" beta function β_f takes again the form (2.51), and its discriminant $D(\lambda)$ computes now (through (2.85) the mass of the *open* string tachyon dual to mesonic operator $\mathcal{O}^{12}$. Let us turn to explicit calculations.

3.3.1 The one-loop "double-trace" beta function

To proceed, we need to be more precise about the structure of the"double-trace" terms induced at one-loop, restoring their $SU(2)_R$ structure. For general angles θ_1 and θ_2, there are three independent structures,

$$\delta\mathcal{L}_{fund} = -\frac{1}{N}\left[f_{\mathbf{3}^\pm}\left(\mathcal{O}^{12}_{\mathbf{3}^+}\mathcal{O}^{21}_{\mathbf{3}^-} + \mathcal{O}^{12}_{\mathbf{3}^-}\mathcal{O}^{21}_{\mathbf{3}^+}\right) + f_{\mathbf{3}^0}\mathcal{O}^{12}_{\mathbf{3}^0}\mathcal{O}^{21}_{\mathbf{3}^0} + f_{\mathbf{1}}\mathcal{O}^{12}_{\mathbf{1}}\mathcal{O}^{21}_{\mathbf{1}}\right]\,. \tag{3.26}$$

We have imposed neutrality under the Cartan of $SU(2)_R$, since this is an exact symmetry for generic angles, corresponding geometrically to rotations in the 67 plane (more precisely a 67 rotation is a linear combination of the Cartan of $SU(2)_L$ and $SU(2)_R$, but the hyper multiplets are neutral under $SU(2)_L$). When one of the angles is zero, say $\theta_2 = 0$, rotations in the 567 directions a symmetry (again a diagonal combination of $SU(2)_L$ and $SU(2)_R$), implying

[3] In more detail, the diagrams contributing to $\mathcal{O}^{22}\bar{\mathcal{O}}^{22}$ do not involve the first flavor, which could then be set to zero as the calculation of this terms of the effective potential is concerned. The Lagrangian with the first flavor set to zero is $\mathcal{N}=2$ supersymmetric, only with an unconventional choice of $SU(2)_R$ embedding into $SU(4)_R$ – it can be turned into the standard Lagrangian by an R-symmetry rotation of the $\mathcal{N}=4$ fields.

$f_{\mathbf{3}^\pm} = f_{\mathbf{3}^0} \equiv f_{\mathbf{3}}$. We focus on the triplet mesons, which are dual to the open string tachyon. For a single non-zero angle there is one beta function $\beta_{f_{\mathbf{3}}}$ to compute, since the three components of the triplet are related by symmetry. For generic angles there are in principle two distinct beta functions $\beta_{f_{\mathbf{3}^\pm}}$ and $\beta_{f_{\mathbf{3}^0}}$; we will illustrate our method computing the first, which is a slightly simpler calculation. At one-loop, the "double-trace" beta function takes the form

$$\beta_f = v^{(1)} f^2 + 2\gamma^{(1)} \lambda f + a^{(1)} \lambda^2 \,. \tag{3.27}$$

We have seen that $\gamma^{(1)} = 0$ at one loop for the triplet mesons. The normalization coefficient $v^{(1)}$ is easily evaluated by free Wick contractions,

$$\langle \mathcal{O}^{12}_{\mathbf{3^0}}(x) \mathcal{O}^{21}_{\mathbf{3^0}}(y) \rangle = \langle \mathcal{O}^{12}_{\mathbf{3}^+}(x) \mathcal{O}^{21}_{\mathbf{3}^-}(y) \rangle = \langle \mathcal{O}^{12}_{\mathbf{3}^-}(x) \mathcal{O}^{21}_{\mathbf{3}^+}(y) \rangle = \frac{1}{16\pi^4 |x-y|^4} \tag{3.28}$$

implying

$$v^{(1)}_{\mathbf{3}^+} = v^{(1)}_{\mathbf{3}^-} = v^{(1)}_{\mathbf{3^0}} = \frac{1}{8\pi^2} \,. \tag{3.29}$$

It remains to evaluate the coefficient $a^{(1)}$. We are going to extract $a^{(1)}$ from the one-loop Coleman-Weinberg potential along the "Higgs branch" of the gauge theory, $\langle X_{AB} \rangle = 0$, $\mathcal{Q}^{\mathcal{I}} \neq 0$. We put "Higgs branch" in quotes because for general angles it is in fact lifted already at the classical level. Let us first recall the analysis for the $\mathcal{N} = 2$ supersymmetric theory corresponding to two parallel flavor branes are parallel ($\theta_1 = \theta_2 = 0$).

As always in a supersymmetric theory, flat directions are parametrized by holomorphic gauge-invariant composite operators. In our case the relevant operators are the mesons

$$\mathcal{O}^{ij} = q^i \cdot \tilde{q}_j \,, \quad i, j = 1, 2 \tag{3.30}$$

The dot stands for color contraction $q \cdot q^* \equiv q^{\mathfrak{a}}\, q^*_{\mathfrak{a}}$ and i, j are the flavor indices. The holomorphic, gauge invariant mesons that parameterize the Higgs flat directions are subject to F-flatness conditions

$$q^{\mathfrak{a} i} \tilde{q}_{\mathfrak{b} i} = 0 \Leftrightarrow \operatorname{tr} \mathcal{O} = \det \mathcal{O} = 0 \,, \tag{3.31}$$

thus there are $4 - 2 = 2$ complex parameters for the moduli space of the supersymmetric theory ($\theta_1 = \theta_2 = 0$). We may parameterize the flat directions by

$$Q_1 = U \left(\begin{array}{c} q \\ 0 \end{array} \right) , \quad Q_2 = U \left(\begin{array}{c} 0 \\ -q \end{array} \right) , \quad U \in SU(2) \quad \text{and} \quad q \in \mathbb{R} \, . \tag{3.32}$$

Color indices are kept implicit. In color space we may take $q^{\mathfrak{a}=1} = q$ and $q^{\mathfrak{a}\neq 1} = 0$. For generic θ_1, θ_2 supersymmetry is explicitly broken in the classical Lagrangian and the Higgs branch is completely lifted.

To select $\beta_{f_{\mathbf{3}^+}}$ (which is of course equal to $\beta_{f_{\mathbf{3}^-}}$), we calculate the effective potential around a classical background such that $\langle \mathcal{O}^{12}_{\mathbf{1}} \rangle = \langle \mathcal{O}^{12}_{\mathbf{3}^0} \rangle = \langle \mathcal{O}^{12}_{\mathbf{3}^-} \rangle = 0$, but $\langle \mathcal{O}^{12}_{\mathbf{3}^+} \rangle \neq 0$, namely

$$Q_1 = \left(\begin{array}{c} q \\ 0 \end{array} \right) , \quad Q_2 = \left(\begin{array}{c} 0 \\ -q \end{array} \right) , \qquad q \in \mathbb{C} \tag{3.33}$$

This choice corresponds to the flat direction for the $\mathcal{N} = 1$ susy case $\theta_1 = \theta_2$. The F-terms of the classical potential vanish for general angles, but for $\theta_1 \neq \theta_2$ the D-terms do not, $\mathcal{V}^D_{Q^4} = g^2 |q|^4 (1 - d\,(\theta_1, \theta_2))$.

In Appendix A.5 we evaluate the one-loop contribution to the effective potential along this background (at large N). With the help of the Callan-Symanzik equation we find

$$\begin{aligned} a^{(1)}_{\mathbf{3}^\pm} \; &= \; \frac{1}{16\pi^2} \Bigg[\Big(1 - d(\theta_1, \theta_2) \Big) + \frac{1}{2} \Big(1 - d(\theta_1, \theta_2) \Big)^2 \\ & \qquad + 4 \, \sin^2 \left(\frac{\theta_1 + \theta_2}{2} \right) \sin^2 \left(\frac{\theta_1 - \theta_2}{2} \right) \Bigg] \, . \end{aligned} \tag{3.34}$$

From our (mild) assumption that the classical potential be positive we have $d(\theta_1, \theta_2) \leq 1$, has the crucial implication

$$a^{(1)}_{\mathbf{3}^\pm} \geq 0 \, . \tag{3.35}$$

In the supersymmetric case ($\theta_1 = \theta_2$), $a^{(1)}_{\mathbf{3}^\pm} = 0$, as it must. For $\theta_2 = 0$, the $SU(2)$ symmetry is restored, so

$$a^{(1)}_{\mathbf{3}^\pm} = a^{(1)}_{\mathbf{3}^0} = \frac{1}{16\pi^2} \left[\Big(1 - d(\theta, 0) \Big) + \frac{1}{2} \Big(1 - d(\theta, 0) \Big)^2 + 4 \, \sin^4 \left(\frac{\theta}{2} \right) \right] \geq 0 \tag{3.36}$$

The one-loop triplet beta function (let us focus on the single-angle case)

$$\beta_{f_{\mathbf{3}}} = v^{(1)}_{\mathbf{3}} \, f^2_{\mathbf{3}} + a^{(1)}_{\mathbf{3}} \lambda^2 \tag{3.37}$$

does not admit real fixed points for $f_{\mathbf{3}}$, so conformal invariance is inevitably broken in the quantum theory.[4] The running coupling

$$\bar{f}(\mu) = \frac{a^{(1)}}{\sqrt{v^{(1)}_{\mathbf{3}}}}\,\lambda^2\,\tan\left[\frac{a^{(1)}\lambda^2}{\sqrt{v^{(1)}_{\mathbf{3}}}}\ln(\mu/\mu_0)\right] \tag{3.38}$$

is a monotonically increasing function interpolating between IR and UV Landau poles, at energies

$$\mu_{IR} = \mu_0\,\exp\left(-\frac{\pi\sqrt{v^{(1)}_{\mathbf{3}}}}{\lambda\sqrt{a^{(1)}_{\mathbf{3}}}}\right)\,, \quad \mu_{UV} = \mu_0\,\exp\left(\frac{\pi\sqrt{v^{(1)}_{\mathbf{3}}}}{\lambda\sqrt{a^{(1)}_{\mathbf{3}}}}\right)\,. \tag{3.39}$$

For small coupling $\lambda \to 0$, the Landau poles are pushed respectively to zero and infinity.

3.3.2 The tachyon mass

As reviewed above, the mass of the field dual to $\mathcal{O}^{12}_{\mathbf{3}^{\pm}}$ is directly related to the discriminant of $\beta_{\mathbf{3}^{\pm}}$,

$$m^2_{\mathbf{3}^{\pm}}R^2 = m^2_{BF}R^2 + D_{\mathbf{3}^{\pm}}(\lambda;\theta_1,\theta_2) = -\,4\,-\,\frac{\lambda^2}{16\pi^4}\,\mathcal{D}^{(1)}_{\mathbf{3}}\,(\theta_1,\theta_2) + \mathcal{O}(\lambda^3)\,, \tag{3.40}$$

where

$$\begin{aligned} \mathcal{D}^{(1)}_{\mathbf{3}} \;&=\; \Big(1-d(\theta_1,\theta_2)\Big) + \frac{1}{2}\Big(1-d(\theta_1,\theta_2)\Big)^2 \\ &\quad +4\,\sin^2\left(\frac{\theta_1+\theta_2}{2}\right)\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\,. \end{aligned} \tag{3.41}$$

For $\theta_1 \neq \theta_2$ the discriminant is negative, implying that the bulk field violates the BF stability bound. Whenever supersymmetry is broken, the bulk field dual to $\mathcal{O}^{12}_{\mathbf{3}^{\pm}}$ is a true tachyon. For $\theta_2 = 0$ the $\pm$ and 0 components of the

[4] Conformal invariance is already broken in the adjoint ("closed string") sector by the hyper multiplet contribution to β_λ, but this is subleading effect (of order $O(1/N)$) with respect to the classical Lagrangian. In the fundamental ("open string") sector the breaking of conformal invariance is at leading order in N (quantum effects arise as the same order as the classical Lagrangian). Of course the *whole* fundamental sector is $O(1/N)$ with respect to the adjoint sector, but we can meaningfully separate the effect we are interested in. This is the field theory counterpart of focussing on the classical open string dynamics of the D-branes, while ignoring the backreaction of the branes on the bulk background.

triplet are related by the $SU(2)$ symmetry and are all tachyonic. We expect the field dual to $\mathcal{O}^{12}_{\mathbf{3}^0}$ to be tachyonic for general angles.

For small angles, the $O(\lambda^2)$ tachyon mass depends on a single unknown parameter α (which enters the parametrization of the classical Q^4 potential, see Appendix A.2),

$$R^2 m^2_{\mathbf{3}}(\lambda) = -4 - \alpha \, \frac{\lambda^2}{16\pi^4} \, (\theta_1 - \theta_2)^2 + \mathcal{O}(\lambda^3) \, , \qquad \theta_1 \, , \theta_2 \ll 1 \, . \tag{3.42}$$

This expression applies to all three components of the triplet. For the $\pm$ components it is just the expansion of (3.40) for small angles. For the 0 component it follows by imposing the symmetry constraints $m^2_{\mathbf{3}^0}(\theta, \theta) = 0$ and $m^2_{\mathbf{3}^0}(\theta, 0) = m^2_{\mathbf{3}^\pm}(\theta, 0)$. By AdS/CFT, we get an interesting prediction for the mass of the open string tachyon for large AdS curvature (small λ).

Conversely, for large λ (small AdS curvature) we can compute the mass of the open string tachyon using the dual string picture. The open string spectrum of branes intersecting at small angles in flat space is well-known. The lowest tachyon mode has mass (see *e.g.* [91] for a review),

$$m^2 = -\frac{|\theta_1 - \theta_2|}{\pi^2 \alpha'} \, , \qquad \theta_1 \, , \theta_2 \ll 1 \, . \tag{3.43}$$

This becomes a good approximation to the mass in the exact AdS sigma model in the limit $\alpha'/R^2 \sim \lambda^{-1/2} \to 0$. Thus

$$\lim_{\lambda \to \infty} R^2 m^2_{\mathbf{3}}(\lambda) = -\frac{|\theta_1 - \theta_2|}{\pi^2} \frac{R^2}{\alpha'} = -\frac{|\theta_1 - \theta_2|}{\pi^2} \lambda^{1/2} \, , \qquad \theta_1 \, , \theta_2 \ll 1 \, . \tag{3.44}$$

This can be regarded as a prediction for the large λ behavior of the discriminant $D_{\mathbf{3}}(\lambda)$, which is a purely field-theoretic quantity. Note that apart from the λ dependence, which could have been anticipated on general grounds, the weak coupling result (3.42) and the strong coupling result (3.44) differ in their angular dependence.

3.4 Discussion

The main technical question that we leave answered is the precise form of the classical Q^4 potential for generic angles. As we have emphasized, a superspace formulation of $\mathcal{N} = 4$ SYM with manifest $SU(4)_R$ symmetry would offer a solution. It would be interesting to see whether the new off-shell formalism for $\mathcal{N} = 1$ SYM in ten dimensions introduced in [92, 93] could be applied to our problem. In principle, another way to obtain the Q^4 potential is by taking the decoupling limit of the intersecting brane effective action. This would first require the calculation of a four-point function of twist fields, two twist fields corresponding to $D3$-$D7$ open strings and two twist fields corresponding to $D3$-$D7'$ open strings. This problem has been solved for branes intersecting at right angles (see *e.g.* [94–97]). The generalization to arbitrary angles is an interesting and difficult problem in boundary conformal field theory. Taking the decoupling limit may also be challenging in the presence of tachyons – it is not clear to us whether the result would be unambiguous or it would require some renormalization prescription.

Even without a complete knowledge of the classical Q^4 potential, by making a plausible positivity assumption we argued that the field theory is unstable at the quantum level. By AdS/CFT, we obtained a non-trivial prediction for the tachyon squared mass $m^2_{\mathbf{3}}(\lambda)$ at small λ. Its behavior at large λ is known from flat-space string theory. There must exist an interpolating function $m^2_{\mathbf{3}}(\lambda)$ valid for all λ. It would be extremely interesting to apply integrability techniques to find the whole function. There is a large literature on open spin chains arising in the calculation of anomalous dimensions of mesonic operators, see in particular [98–101] for our system in the $\mathcal{N} = 2$ supersymmetric case $\theta_1 = \theta_2 = 0$. It remains to be seen whether the susy-breaking rotation preserves integrability.

Another direction for future work is to study the actual tachyon condensation process on the field theory side. In the bulk, after tachyon condensation the intersecting $D7$ branes recombine (see *e.g.* [91]). For small λ, the tachyon vacuum corresponds on the field theory side to the local minimum of the one-loop effective potential. It would be interesting to expand the Lagrangian around the minimum and relate this field theory calculation to the bulk phenomenon of brane recombination.

Chapter 4

The Veneziano Limit of $\mathcal{N} = 2$ Superconformal QCD

4.1 Motivation

How general is the gauge/string correspondence? 't Hooft's topological argument [13] suggests that any large N gauge theory should be dual to a closed string theory. However, the four-dimensional gauge theories for which an independent definition of the dual string theory is presently available are rather special. Even among conformal field theories, which are the best understood, an explicit dual string description is known only for a sparse subset of models. In some sense all examples are close relatives of the original paradigm of $\mathcal{N} = 4$ super Yang-Mills [1–3] and are found by considering stacks of branes at local singularities in critical string theory, or variations of this setup, *e.g.* [39, 40, 49, 105–108].[1] Conformal field theories in this class can have lower or no supersymmetry, but are far from being "generic". Some of their special features are:

(i) The a and c conformal anomaly coefficients are equal at large N [111].

(ii) The fields are in the adjoint or in bifundamental representations of the gauge group. (Except possibly for a small number of fundamental flavors – "small" in the large N limit – as in [89]).

(iii) The dual geometry is ten dimensional.

[1] We should perhaps emphasize from the outset that our focus is on *string* duals of gauge theories. There are strongly coupled field theories that admit *gravity* duals with no perturbative string limit, see *e.g.* [109, 110].

(iv) The conformal field theory has an exactly marginal coupling λ, which corresponds to a geometric modulus on the dual string side. For large λ the string sigma model is weakly coupled and the supergravity approximation is valid.[2]

The situation certainly does not improve if one breaks conformal invariance – the field theories for which we can directly describe the string dual remain a very special set, which does not include some of the most relevant cases, such as pure Yang-Mills theory. Many more field theories, including pure Yang-Mills, can be described indirectly, as low-energy limits of deformations of $\mathcal{N} = 4$ SYM (as *e.g.* in [112] for $\mathcal{N} = 1$ SYM) or of other UV fixed points, not necessarily four-dimensional (as in [113] for $\mathcal{N} = 0$ YM or [114, 115] for $\mathcal{N} = 1$ SYM). These constructions count as physical "existence proofs" of the string duals, but if one wishes to focus just on the low-energy dynamics, one invariably encounters strong coupling on the dual string side. In the limit where the unwanted UV degrees of freedom decouple, the dual appears to be described (in the most favorable duality frame) by a closed-string sigma model with strongly curved target. This may well be only a technical problem, which would be overcome by an analytic or even a numerical solution of the worldsheet CFT. The more fundamental problem is that we lack a precise recipe to write, let alone solve, the limiting sigma model that describes only the low-energy degrees of freedom.

To break this impasse and enlarge the list of dual pairs outside the $\mathcal{N} = 4$ SYM universality class, we can try to attack the "next simplest case". A natural candidate for this role is $\mathcal{N} = 2$ SYM with gauge group $SU(N_c)$ and $N_f = 2N_c$ flavor hypermultiplets in the fundamental representation of $SU(N_c)$. The number of flavors is tuned to obtain a vanishing beta function. We refer to this model as $\mathcal{N} = 2$ superconformal QCD (SCQCD). The theory violates properties (i) and (ii) but it still has a large amount of symmetry (half the maximal superconformal symmetry) and it shares with $\mathcal{N} = 4$ SYM the crucial simplifying feature of a tunable, exactly marginal gauge coupling g_{YM}. (The theory also exhibits S-duality [116–118], though this will not be important for our considerations, since we will work in the large N limit, which does not commute with S-duality.)

The large N expansion of $\mathcal{N} = 2$ SCQCD is the one defined by Veneziano [119]: the number of colors N_c and the number of fundamental flavors N_f are both sent to infinity, keeping fixed their ratio ($N_f/N_c \equiv 2$ in our case) and the combination $\lambda \equiv g_{YM}^2 N_c$. Which, if any, is the dual string theory? And what

[2] In some cases, as in $\mathcal{N} = 4$ SYM, the opposite limit of small λ corresponds to a weakly coupled Lagrangian description on the field theory side. In other cases, like the Klebanov-Witten theory [106], the Lagrangian description is never weakly coupled.

happens to it for large λ?

4.2 The Veneziano Limit and Dual Strings

4.2.1 A general conjecture

To understand in which sense we should expect a dual string description of a gauge theory in the Veneziano limit, we start by reviewing general elementary facts about large N counting, Feynman-diagrams topology, and operator mixing. At this stage we have in mind a generic field theory that contains both adjoint fields, which we collectively denote by ϕ^a_b, with $a, b = 1, \ldots, N_c$, and fundamental fields, denoted by q^a_i, with $i = 1, \ldots, N_f$. We can consider the theory both in the 't Hooft limit of large N_c with N_f fixed, and in the Veneziano limit of large $N_c \sim N_f$.

$\mathbf{N_c \to \infty}$, $\mathbf{N_f}$ fixed

Let us first recall the familiar analysis in the 't Hooft limit [13], where the number of colors N_c is sent to infinity, with $\lambda = g^2_{YM} N_c$ *and* the number of flavors N_f kept fixed. In this limit it is useful to represent propagators for adjoint fields with *double* lines, and propagators for fundamental fields with *single* lines – the lines keep track of the flow of the a type (color) indices. Vacuum Feynman diagrams admit a topological classification as Riemann surfaces with boundaries: each flavor loop is interpreted as a boundary. The N dependence is $N_c^{2-2h-b} N_f^b$, for h the genus and b the number of boundaries.

The natural dual interpretation is then in terms of a string theory with coupling $g_s \sim 1/N_c$, containing both a closed and an open sector – the latter arising from the presence of N_f explicit "flavor" branes where open strings can end. Indeed this is the familiar way to introduce a small number of flavors in the AdS/CFT correspondence [71]: by adding explicit flavor branes to the bulk geometry (the simplest examples is adding D7 branes to the $AdS_5 \times S^5$ background). Since $N_f \ll N_c$, the backreaction of the flavor branes can be neglected (probe approximation).

According to the standard AdS/CFT dictionary, single-trace "glueball" composite operators, of the schematic form $\mathrm{Tr}\, \phi^\ell$ (where Tr is a color trace) are dual to closed string states, while "mesonic" composite operators, of the schematic form $\bar{q}^i \phi^\ell q_j$, are dual to open string states. At large N_c, these two classes of operators play a special role since they can be regarded as "elementary" building blocks: all other gauge-invariant composite operators of finite dimension can be built by taking products of the elementary (single-trace and mesonic) operators, and their correlation functions factorize into the

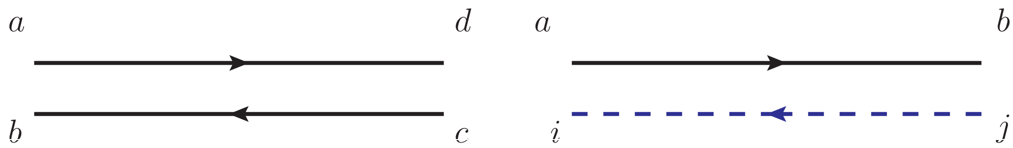


Figure 4.1: Double line propagators. The adjoint propagator $\langle \phi^a_{\ b}\ \phi^c_{\ d} \rangle$ on the left, represented by two color lines, and the fundamental propagator $\langle q^a_{\ i}\ \bar{q}^j_{\ b} \rangle$ on the right, represented by a color and a flavor line.

correlation functions of the elementary constituents.[3] This factorization is dual to the fact for $g_s \to 0$ the string Hilbert space becomes the free multiparticle Fock space of open and closed strings.

Flavor-singlet mesons, of the form $\sum_{i=1}^{N_f} \bar{q}^i \phi^\ell q_i$, mix with glueballs in perturbation theory, but the mixing is suppressed by a factor of $N_f/N_c \ll 1$, so the distinction between the two classes of operators is meaningful in the 't Hooft limit. On the dual side, this translates into the statement that the mixing of open and closed strings in subleading since each boundary comes with a suppression factor of $g_s N_f \sim N_f/N_c$.

$\mathbf{N_c \sim N_f \to \infty}$

We can now repeat the analysis in the Veneziano limit of large N_c and large N_f with $\lambda = g_{YM}^2 N_c$ and N_f/N_c fixed. In this limit it is appropriate to use a double-line notation with two distinct types of lines [119]: color lines (joining a indices) and flavor lines (joining i indices). A ϕ propagator decomposes as two color lines with opposite orientations, while a q propagator is made of a color and a flavor line (Figure 4.1). Since $N_f \sim N_c \equiv N$, color and flavor lines are on the same footing in the counting of factors of N. It is natural to regard all vacuum Feynman diagrams as *closed* Riemann surfaces, whose N dependence is N^{2-2h}, for h the genus. At least at this topological level, by the same logic of [13], we should expect a gauge theory in the Veneziano limit to be described by the perturbative expansion of a closed string theory, with coupling $g_s \sim 1/N$. More precisely, there should be a dual *purely closed* string description of the *flavor-singlet* sector of the gauge theory.

This point can be sharpened looking at operator mixing. It is consistent to truncate the theory to flavor-singlets, since they close under operator product expansion. The new feature that arises in the Veneziano limit is the order-one mixing of "glueballs" and flavor-singlet "mesons". For large $N_c \sim N_f$, the

[3]Note that in this discussion we are not considering baryonic operators, since they have infinite dimension in the strict large N_c limit. Baryons are interpreted as solitons of the large N_c theory; as familiar, in AdS/CFT they correspond to non-perturbative (D-brane) states on the string theory side [105].

basic "elementary" operators are what we may call *generalized single-trace* operators, of the form

$$\mathrm{Tr}\left(\phi^{k_1}\mathcal{M}^{\ell_1}\phi^{k_2}\ldots\phi^{k_n}\mathcal{M}^{\ell_n}\right)\,, \qquad \mathcal{M}^a_{\ b} \equiv \sum_{i=1}^{N_f} q^a_{\ i}\,\bar{q}^{\,i}_{\ b}\,. \tag{4.1}$$

Here we have introduced a flavor-contracted combination of a fundamental and an antifundamental field, $\mathcal{M}^a_{\ b}$, which for the purpose of the large N expansion plays the role of just another adjoint field. The usual large N factorization theorems apply: correlators of generalized multi-traces factorize into correlators of generalized single-traces. In the conjectural duality with a closed string theory, generalized single-trace operators are dual to single-string states.

We can imagine to start with a dual closed string description of the field theory with $N_f = 0$, and first introduce a small number of flavors $N_f \ll N_c$ by adding flavor branes in the probe approximation. As we increase N_f to be $\sim N_c$, the probe approximation breaks down: boundaries are not suppressed and for fixed genus we must sum over worldsheets with arbitrarily many boundaries. The result of this resummation – we are saying – is a new closed string background dual to the flavor-singlet sector of the field theory. The large mixing of closed strings and flavor singlet open strings gives rise to *new effective closed-string degrees of freedom*, propagating in a backreacted geometry. This is the string theory interpretation of the generalized single-trace operators (4.1).

In stating the conjectured duality we have been careful to restrict ourselves to the flavor-singlet sector of the field theory. One may entertain the idea that "generalized mesonic operators" of the schematic form $\bar{q}^i\,\phi^{k_1}\,\mathcal{M}^{\ell_1}\,\phi^{k_2}\,q_j$ (with open flavor indices i and j) would map to elementary open string states in the bulk. However this cannot be correct, because generalized mesons and generalized single-trace operators are not independent – already in free field theory they are constrained by algebraic relations – so adding an independent open string sector in the dual theory would amount to overcounting.

4.2.2 Relation to previous work

The idea that sub-critical string theories play a role in the gauge/gravity correspondence is of course not new. Polyakov's conjecture that pure Yang-Mills theory should be dual to a $5d$ string theory, with the Liouville field playing the role of the fifth dimension, predates the AdS/CFT correspondence (see *e.g.* [35, 36, 120]). In fact one of the surprises of AdS/CFT was that some supersymmetric gauge theories are dual to simple backgrounds of *critical* string theory. General studies of AdS solutions of non-critical spacetime effective actions include [121, 122]. Non-critical holography has been mostly considered, starting with [123, 124], in the $\mathcal{N}=1$ supersymmetric case, notably for $\mathcal{N}=1$ super QCD in the Seiberg conformal window, which is argued to be dual to $6d$ non-critical backgrounds of the form $AdS_5 \times S^1$ with string-size curvature. There is an interesting literature on the RNS worldsheet description of these $6d$ non-critical backgrounds and their gauge-theory interpretation, see *e.g.* [125–128]. Non-critical RNS superstrings were formulated in [129, 130] and shown in [131–133, 133–135] to describe subsectors of critical string theory – the degrees of freedom localized near NS5 branes or (in the mirror description) Calabi-Yau singularities. Non-critical superstrings have been also considered in the Green-Schwarz and pure-spinor formalisms, see *e.g.* [136–140].

Our analysis in sections 6 and 7 for $\mathcal{N}=2$ SCQCD will be in the same spirit as the analysis of [125, 128] for $\mathcal{N}=1$ super QCD. We will use the double-scaling limit defined in [134, 135] and further studied in *e.g.* [141–143]. One of our points is that the $\mathcal{N}=2$ supersymmetric case should be the simplest for non-critical gauge/string duality. On the string side, more symmetry does not hurt, but the real advantage is on the field theory side. Little is known about the SCFTs in the Seiberg conformal window, since generically they are strongly coupled, isolated fixed points. By contrast $\mathcal{N}=2$ SCQCD has an exactly marginal coupling λ, which takes arbitrary non-negative values. There is a weakly coupled Lagrangian description for $\lambda \to 0$, and we can bring to bear all the perturbative technology that has been so successful for $\mathcal{N}=4$ SYM, for example in uncovering integrable structures.[4] At the same time we may hope, again in analogy with $\mathcal{N}=4$ SYM, that the string dual will simplify in the strong coupling limit $\lambda \to \infty$.

There are also interesting approaches to holography for gauge theories with a large number of fundamental flavors in *critical* string theory/supergravity, see *e.g.* [144–152]. The critical setup inevitably implies that the boundary gauge theory will have UV completions with extra degrees of freedom (*e.g.* higher supersymmetry and/or higher dimensions).

[4] $\mathcal{N}=1$ SQCD at the Seiberg self-dual point $N_f = 2N_c$ admits an exactly marginal coupling (the coefficient of a quartic superpotential), which however is bounded from below – the theory is never weakly coupled.

4.3 Protected Spectrum of the Interpolating Theory

In the present and in the following section we will study the protected spectrum of $\mathcal{N}=2$ SCQCD at large N, in the flavor singlet sector, and its relation with the protected spectrum of the interpolating SCFT. We have argued that in the large N Veneziano limit, flavor singlets that diagonalize the dilation operator take the "generalized single-trace" form (4.1). We will look for the generalized single-trace operators belonging to short multiplets of the superconformal algebra. These are the operators expected to map to the Kaluza-Klein tower of massless single closed string states, so they are the first place to look in a "bottom-up" search for the string dual.

The determination of the complete list of short multiplets of $\mathcal{N}=2$ SCQCD in this "generalized single-trace" sector turns out to be more subtle than expected. A class of short multiplets is relatively easy to isolate, namely the multiplets based on the following superconformal primaries:

$$\operatorname{Tr}\mathcal{M}_{\mathbf{3}} = (Q^a_i \bar{Q}^i_a)_{\mathbf{3}}\,, \qquad \operatorname{Tr}\phi^{\ell+2}\,, \qquad \operatorname{Tr}[\,T\phi^\ell\,]\,, \qquad \ell \geq 0\,. \tag{4.2}$$

Here $T \equiv \phi\bar{\phi} - \mathcal{M}_{\mathbf{1}}$. We hasten to add that this will turn out to be only a small fraction of the complete set of protected operators. The set (5.44) is the complete list of one-loop protected primaries *in the scalar sector*, as we show in the next chapter 5 by a systematic evaluation of the one-loop anomalous dimension of all operators that are made out of scalars and obey shortening conditions. The operators $\operatorname{Tr}\phi^\ell$ correspond to the vacuum of the spin-chain studied in [9], while the $\operatorname{Tr}T\phi^\ell$ correspond to the $p \to 0$ limit of a gapless magnon $T(p)$ of momentum p.

The operators $\operatorname{Tr}\mathcal{M}_{\mathbf{3}}$ and $\operatorname{Tr}\phi^{\ell+2}$ obey the familiar BPS condition $\Delta = 2R+|r|$, where R is the $SU(2)_R$ spin and r the $U(1)_r$ charge, and they are generators of the chiral ring (with respect to an $\mathcal{N}=1$ subalgebra), see appendix B.[5] By contrast $\operatorname{Tr}[\,T\phi^\ell]$ obey a "semi-shortening" condition and it may be missed in a naive analysis; in these operators there is a large mixing of "glueballs" and "mesons" and the idea of considering "generalized single-traces" is

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[5] Incidentally, the analysis of the chiral ring extends immediately to flavor non-singlets. The only chiral ring generator which is not a flavor singlet is the $SU(2)_R$ triplet bilinear

$$\mathcal{O}^i_{\mathbf{3}\,j} \equiv (\bar{Q}^i{}_a Q^a{}_j)_{\mathbf{3}} = \bar{Q}^i_{a\,\{\mathcal{I}} Q^a_{\mathcal{J}\}\,j}\,, \tag{4.3}$$

in the adjoint of $SU(N_f)$. The conserved currents for the $SU(N_f) \subset U(N_f)$ flavor symmetry belong to the short multiplet with bottom component $\mathcal{O}^i_{\mathbf{3}\,j}$. Similarly the current for the $U(1) \subset U(N_f)$ baryon number belongs to the $\operatorname{Tr}\mathcal{M}_{\mathbf{3}}$ multiplet.

essential. The $\mathrm{Tr}\,T$ multiplet plays a distinguished role since it contains the stress-energy tensor and R-symmetry currents.

Protection of the operators (5.44) can be understood from the viewpoint of the interpolating SCFT connecting $\mathcal{N}=2$ SCQCD with the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM, as follows. The complete spectrum of short multiplets at the orbifold point $g=\check{g}$ is well-known. We will argue, using superconformal representation theory [154], that the protected multiplets found at the orbifold point *cannot* recombine into long multiplets as we vary $\check{g}$, so in particular taking $\check{g}\to 0$ they must evolve into protected multiplets of the theory

$$\{\mathcal{N}=2\ \text{SCQCD}\ \oplus\ \text{decoupled}\ SU(N_{\check{c}})\ \text{vector multiplet}\}\,. \tag{4.4}$$

The list (5.44) is precisely recovered by restricting to $U(N_f)$ singlets. Remarkably however, the superconformal index of $\mathcal{N}=2$ SCQCD, evaluated in the next section, will show the existence of *many more* protected states. The extra protected states arise from the splitting *long* multiplets of the interpolating theory into short multiplets as $\check{g}\to 0$.

We will make extensive use of the the list given by Dolan and Osborn[154] of all possible shortening conditions of the $\mathcal{N}=2$ superconformal algebra. We summarize their results and establish notations in appendix B.1.

4.3.1 Protected Spectrum at the Orbifold Point

At the orbifold point ($g=\check{g}$) the state space of the field theory is the direct sum of an untwisted and a twisted sector, respectively even and odd under the "quantum" $\mathbb{Z}_2$ symmetry (5.23).

Untwisted sector

Operators in the untwisted sector of the orbifold descend from operators of $\mathcal{N}=4$ SYM by projection onto the $\mathbb{Z}_2$ invariant subspace. Their correlators coincide at large N_c with $\mathcal{N}=4$ correlators [41, 45]. In particular the complete list of untwisted protected states is obtained by projection of the protected states of $\mathcal{N}=4$. We will be interested in single-trace operators; as is well-known, the only protected single-trace operators of $\mathcal{N}=4$ belong to the $\frac{1}{2}$ BPS multiplets $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0]}$, built on the chiral primaries $\mathrm{Tr}X^{\{i_1}\dots X^{i_p\}}$, with $p\geq 2$, in the $[0,p,0]$ representation of $SU(4)_R$ (symmetric traceless of $SO(6)$) The decomposition of each $\frac{1}{2}$ BPS multiplet $\mathcal{N}=4$ into $\mathcal{N}=2$ multiplets reads [154]

$$
\begin{aligned}
\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0]} \;\simeq\;& (p+1)\hat{\mathcal{B}}_{\frac{1}{2}p} \oplus \mathcal{E}_{p(0,0)} \oplus \bar{\mathcal{E}}_{-p(0,0)} \\
&\oplus (p-1)\hat{\mathcal{C}}_{\frac{1}{2}p-1(0,0)} \oplus p(\mathcal{D}_{\frac{1}{2}(p-1)(0,0)} \oplus \bar{\mathcal{D}}_{\frac{1}{2}(p-1)(0,0)} \\
&\oplus \bigoplus_{k=1}^{p-2}(k+1)(\mathcal{B}_{\frac{1}{2}k,p-k(0,0)} \oplus \bar{\mathcal{B}}_{\frac{1}{2}k,k-p(0,0)}) \\
&\oplus \bigoplus_{k=0}^{p-3}(k+1)(\mathcal{C}_{\frac{1}{2}k,p-k-2(0,0)} \oplus \bar{\mathcal{C}}_{\frac{1}{2}k,k-p+2(0,0)}) \\
&\oplus \bigoplus_{k=0}^{p-4}\bigoplus_{l=0}^{p-k-4}(k+1)\mathcal{A}^{p}_{\frac{1}{2}k,p-k-4-2l(0,0)}\,, \qquad (4.5)
\end{aligned}
$$

which can be understood by considering all possible ways to substitute $X^i \to \mathcal{Z}, \bar{\mathcal{Z}}, \mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}}$, *i.e.* $\mathbf{6} \to (0,0)_1 \oplus (0,0)_{-1} \oplus (\frac{1}{2},\frac{1}{2})_0$ in the branching $SU(4)_R \to SU(2)_L \times SU(2)_R \times U(1)_r$. The $\mathbb{Z}_2$ orbifold projection is then accomplished by the substitution (5.16); states with an even (odd) number of $\mathcal{X}$s are kept (projected out), or equivalently, states with integer (half-odd) $SU(2)_R$ spin are kept (projected out). Table 4.1 lists all the superconformal primaries of the orbifold theory obtained by this procedure.

Let us explain the notation. The explicit expressions in terms of fields are schematic. The symbol $\sum$ indicates summation over all "symmetric traceless" permutations of the component fields allowed by the index structure. The symbol $\mathcal{T}$ stands for the appropriate combination of two scalar fields, neutral under the R symmetry. In the case of the multiplet $\hat{\mathcal{C}}_{0(0,0)}$, $\mathrm{Tr}\,\mathcal{T} = \mathrm{Tr}\,[T + \check{\phi}\bar{\check{\phi}}]$, the bottom component of the stress tensor multiplet of the orbifold theory. The $SU(2)_R \times U(1)_r$ quantum numbers are manifest as labels of the $\mathcal{N}=2$ multiplets, while the $SU(2)_L$ quantum numbers can be seen from the multiplicity of each multiplet on the right hand side of (4.5) – the $SU(2)_L$ spin always equals the $SU(2)_R$ spin of the multiplet, because $SU(2)_R$ and $SU(2)_L$ indices always come in pairs $(\mathcal{I}\hat{\mathcal{I}})$ and are separately symmetrized.

Twisted sector

In the twisted sector, we claim that the complete list of single-trace superconformal primary operators obeying shortening conditions is

$$
\begin{aligned}
&\mathrm{Tr}[\tau Z^\ell] = \mathrm{Tr}[\phi^\ell - \check{\phi}^\ell] \quad \text{for} \quad \ell \geq 2 \qquad (4.6)\\
\text{and} \qquad &\mathrm{Tr}[\tau \mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}}\mathcal{X}_{\mathcal{J}\hat{\mathcal{J}}}\epsilon^{\mathcal{I}\mathcal{J}}] = -\mathrm{Tr}[Q_{\hat{\mathcal{I}}\{\mathcal{I}}\bar{Q}^{\hat{\mathcal{I}}}_{\mathcal{J}\}}] = -\mathrm{Tr}\mathcal{M}_{\mathbf{3}}\,.
\end{aligned}
$$

Multiplet	Orbifold operator ($R, \ell \geq 0$, $n \geq 2$)
$\hat{\mathcal{B}}_{R+1}$	$\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}]$
$\bar{\mathcal{E}}_{-(\ell+2)(0,0)}$	$\mathrm{Tr}[\phi^{\ell+2} + \check{\phi}^{\ell+2}]$
$\hat{\mathcal{C}}_{R(0,0)}$	$\mathrm{Tr}[\sum \mathcal{T}(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}]$
$\bar{\mathcal{D}}_{R+1(0,0)}$	$\mathrm{Tr}[\sum (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}(\phi + \check{\phi})]$
$\bar{\mathcal{B}}_{R+1,-(\ell+2)(0,0)}$	$\mathrm{Tr}[\sum_i (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}\phi^i\check{\phi}^{\ell+2-i}]$
$\bar{\mathcal{C}}_{R,-(\ell+1)(0,0)}$	$\mathrm{Tr}[\sum_i \mathcal{T}(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}\phi^i\check{\phi}^{\ell+1-i}]$
$\mathcal{A}^{\Delta=2R+\ell+2n}_{R,-\ell(0,0)}$	$\mathrm{Tr}[\sum_i \mathcal{T}^n(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}\phi^i\check{\phi}^{\ell-i}]$

Table 4.1: Superconformal primary operators in the untwisted sector of the orbifold theory. They descend from the $\frac{1}{2}$ BPS primaries of $\mathcal{N} = 4$ SYM. The symbol $\sum$ indicates summation over all "symmetric traceless" permutations of the component fields allowed by the index structure.

Multiplet	Orbifold operator ($\ell \geq 0$)
$\hat{\mathcal{B}}_1$	$\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{-}} - Q^{+\hat{-}}\bar{Q}^{+\hat{+}})] = \mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$
$\bar{\mathcal{E}}_{-\ell-2(0,0)}$	$\mathrm{Tr}[\phi^{\ell+2} - \check{\phi}^{\ell+2}]$

Table 4.2: Superconformal primary operators in the twisted sector of the orbifold theory.

That these operators are protected can be seen by the fact that they are the generators of the $\mathcal{N} = 1$ chiral ring in the twisted sector, as we show in appendix B.2. A priori there could be extra twisted states that do not belong to the chiral ring, as is the case for the untwisted sector. In the next section we will evaluate the superconformal index of the orbifold theory and find that it matches perfectly with the contribution of our claimed list of short multiplets.

The primary $\mathrm{Tr}[\phi^\ell - \check{\phi}^\ell]$ corresponds for each $\ell \geq 2$ to a second copy of the chiral multiplet $\bar{\mathcal{E}}_{-\ell(0,0)}$ – the first copy being the one in the untwisted sector built on $\mathrm{Tr}[\phi^\ell + \check{\phi}^\ell]$. The operator $\mathrm{Tr}[Q_{\hat{\mathcal{I}}\{\mathcal{I}}\bar{Q}^{\hat{\mathcal{I}}}_{\mathcal{J}\}}]$ is an $SU(2)_R$ triplet with vanishing $U(1)_r$ charge and $\Delta = 2$, and must be identified with the primary of a $\hat{\mathcal{B}}_1$ multiplet. This protected multiplet has been overlooked in previous discussions of the orbifold field theory. It is protected only in the theory where the relative $U(1)$ has been correctly subtracted (see section 3.2), as seen both in the chiral ring analysis of appendix B and in an explicit one-loop calculation.

4.3.2 From the orbifold point to $\mathcal{N} = 2$ SCQCD

As we move away from the orbifold point by changing $\check{g}$, the short multiplets that we have just enumerated may *a priori* recombine into long multiplets and acquire a non-zero anomalous dimension. The possible recombinations of short multiplets of the $\mathcal{N} = 2$ superconformal algebra were classified in [154]. For short multiplets with a Lorentz-*scalar* bottom component, the relevant rule is

$$\mathcal{A}^{2R+\ell+2}_{R,-\ell(0,0)} \simeq \bar{\mathcal{C}}_{R,-\ell(0,0)} \oplus \bar{\mathcal{B}}_{R+1,-(\ell+1)(0,0)} \,. \tag{4.7}$$

In the special case $\ell = 0$, the short multiplets on the right hand side further decompose into even shorter multiplets as

$$\mathcal{A}^{2R+2}_{R,0(0,0)} \simeq \hat{\mathcal{C}}_{R(0,0)} \oplus \mathcal{D}_{R+1(0,0)} \oplus \bar{\mathcal{D}}_{R+1(0,0)} \oplus \hat{\mathcal{B}}_{R+2(0,0)} \tag{4.8}$$

. It follows that the short multiplets of the orbifold theory that that could in principle recombine are

$$\mathrm{Tr}[\sum_i T(Q^{++\hat{+}}\bar{Q}^{++\hat{+}})^R\phi^i\check{\phi}^{\ell-i}] \oplus \mathrm{Tr}[\sum_i (Q^{++\hat{+}}\bar{Q}^{++\hat{+}})^{R+1}\phi^i\check{\phi}^{\ell-i}] \longrightarrow \mathcal{A}^{2R+\ell+2}_{R,-\ell(0,0)} \tag{4.9}$$

$$\begin{aligned}&\mathrm{Tr}[\sum T(Q^{++\hat{+}}\bar{Q}^{++\hat{+}})^R] \oplus \mathrm{Tr}[\sum_i (Q^{++\hat{+}}\bar{Q}^{++\hat{+}})^{R+1}\bar{\phi}^i\bar{\check{\phi}}^{1-i}] \oplus \\ &\mathrm{Tr}[\sum_i (Q^{++\hat{+}}\bar{Q}^{++\hat{+}})^{R+1}\phi^i\check{\phi}^{1-i}] \oplus \mathrm{Tr}[\sum (Q^{++\hat{+}}\bar{Q}^{++\hat{+}})^{R+2}] \longrightarrow \mathcal{A}^{2R+2}_{R,0(0,0)} \,.\end{aligned} \tag{4.10}$$

However we see that the proposed recombinations entail short multiplets with *different* $SU(2)_L$ quantum numbers, which is impossible since the supercharges are neutral under $SU(2)_L$. Thus $SU(2)_L$ selection rules forbid the recombination, and the protected multiplets of the orbifold theory remain short for all values of g and $\check{g}$. This conclusion was reached using superconformal representation theory, and it is a rigorous result valid at the full quantum level.[6]

In the limit $\check{g} \to 0$, we must be able to match the protected states of the interpolating SCFT with protected states of $\{\mathcal{N} = 2$ SCQCD $\oplus$ decoupled vector multiplet$\}$. In [9] we follow this evolution in detail using the one-loop spin chain Hamiltonian. The basic features of this evolution can be understood just from group theory. The protected states naturally splits into two sets: $SU(2)_L$ singlets and $SU(2)_L$ non-singlets. It is clear that all the (generalized) single-trace operators of $\mathcal{N} = 2$ SCQCD must arise from the $SU(2)_L$ singlets.

The $SU(2)_L$ singlets are:

(i) One $\hat{\mathcal{B}}_1$ multiplet, corresponding to the primary $\mathrm{Tr}[Q_{\hat{\mathcal{I}}\{\mathcal{I}}\bar{Q}^{\hat{\mathcal{I}}}_{\mathcal{J}\}}] = \mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$. Since this is the only operator with these quantum numbers, it cannot mix with anything and its form is independent of $\check{g}$.

(ii) Two $\bar{\mathcal{E}}_{-\ell(0,0)}$ multiplets for each $\ell \geq 2$, corresponding to the primaries $\mathrm{Tr}\,[\phi^\ell \pm \check{\phi}^\ell]$. For each ℓ, there is a two-dimensional space of protected operators, and we may choose whichever basis is more convenient. For $g = \check{g}$, the natural basis vectors are the untwisted and twisted combinations (respectively even and odd under $\phi \leftrightarrow \check{\phi}$), while for $\check{g} = 0$ the natural basis vectors are $\mathrm{Tr}\,\phi^\ell$ (which is an operator of $\mathcal{N} = 2$ SCQCD) and $\mathrm{Tr}\,\check{\phi}^\ell$ (which belongs to the decoupled sector).

(iii) One $\hat{\mathcal{C}}_{0(0,0)}$ multiplet (the stress-tensor multiplet), corresponding to the primary $\mathrm{Tr}\,\mathcal{T} = \mathrm{Tr}\,[T + \check{\phi}\bar{\check{\phi}}]$. We have checked that this combination is an eigenstate with zero eigenvalue for all $\check{g}$. For $\check{g} = 0$, we may trivially subtract out the decoupled piece $\mathrm{Tr}\,\check{\phi}\bar{\check{\phi}}$ and recover $\mathrm{Tr}\,T$, the stress-tensor multiplet of $\mathcal{N} = 2$ SCQCD.

(iv) One $\bar{\mathcal{C}}_{0,-\ell(0,0)}$ multiplet for each $\ell \geq 1$. In the limit $\check{g} \to 0$, we expect this multiplet to evolve to the $\mathrm{Tr}\,T\phi^\ell$ multiplet of $\mathcal{N} = 2$ SCQCD. We have checked this in detail in [9].

All in all, we see that this list reproduces the list (5.44) of one-loop protected scalar operators of $\mathcal{N} = 2$ SCQCD, *plus* the extra states $\mathrm{Tr}\check{\phi}^\ell$ that decouple for $\check{g} = 0$.

[6] We will rephrase the same result in the next section by computing a refined superconformal index that also keeps track of the $SU(2)_L$ quantum number.

The basic protected primary of $\mathcal{N}=2$ SCQCD which is charged under $SU(2)_L$ is the $SU(2)_L$ triplet contained in the mesonic operator $\mathcal{O}^i_{\mathbf{3_R}\,j} = (\bar{Q}^i_a Q^a_j)_{\mathbf{3_R}}$ (see footnote 5). Indeed writing the $U(N_f = 2N_c)$ flavor indices i as $i = (\check{a}, \hat{\mathcal{I}})$, with $\check{a} = 1, \dots N_f/2 = N_c$ "half" flavor indices and $\mathcal{I} = \hat{\pm}$ $SU(2)_L$ indices, we can decompose

$$\mathcal{O}^i_{\mathbf{3_R}\,j} \to \mathcal{O}^{\check{a}}_{\mathbf{3_R}\mathbf{3_L}\,\check{b}}\,, \quad \mathcal{O}^{\check{a}}_{\mathbf{3_R}\mathbf{1_L}\,\check{b}}\,. \tag{4.11}$$

In particular we may consider the highest weight combination for both $SU(2)_L$ and $SU(2)_R$,

$$(\bar{Q}^{+\hat{+}} Q^{+\hat{+}})^{\check{a}}_{\check{b}}\,. \tag{4.12}$$

States with higher $SU(2)_L$ spin can be built by taking products of $\mathcal{O}_{\mathbf{3_R}\mathbf{3_L}}$ with $SU(2)_L$ and $SU(2)_R$ indices separately symmetrized – and this is the only way to obtain protected states of $\mathcal{N}=2$ SCQCD charged under $SU(2)_L$ which have finite conformal dimension in the Veneziano limit. It is then a priori clear that a protected primary of the interpolating theory with $SU(2)_L$ spin L must evolve as $\check{g} \to 0$ into a product of L copies of $(\bar{Q}^{+\hat{+}} Q^{+\hat{+}})$ and of as many additional decoupled scalars $\check{\phi}$ and $\bar{\check{\phi}}$ as needed to make up for the correct $U(1)_r$ charge and conformal dimension. Examples of this evolution are given in [9].

4.3.3 Summary

In summary all the short multiplets of the interpolating theory remain short as $\check{g} \to 0$, and have a natural interpretation in this limit. The $SU(2)_L$-singlet protected states evolve into the list (5.44) of protected states of SCQCD, plus some extra states made purely from the decoupled vector multiplet. The interpolating theory has also many single-trace protected states with non-trivial $SU(2)_L$ spin, which are flavor non-singlets from the point of view of $\mathcal{N}=2$ SCQCD: we have seen that in the limit $\check{g} \to 0$, a state with $SU(2)_L$ spin L can be interpreted as a "multiparticle state", obtained by linking together L short "open" spin-chains with of SCQCD and decoupled fields $\check{\phi}$. This is also suggestive of a dual string theory interpretation: as $\check{g} \to 0$, single closed string states carrying $SU(2)_L$ quantum numbers disintegrate into multiple open strings.

Thus by embedding $\mathcal{N}=2$ SCQCD into the interpolating SCFT we have confirmed that the operators (5.44) are protected at the full quantum level, since they arise as the limit of operators whose protection can be shown at the orbifold point and is preserved by the exactly marginal deformation. However this argument does *not* guarantee that (5.44) is the *complete* set of protected generalized single-trace primaries of $\mathcal{N} = 2$ SCQCD. Indeed we will exhibit many more such states in the next section: they arise from *long* multiplets of the interpolating theory splitting into short multiplets at $\check{g} = 0$.

4.4 Extra Protected Operators of $\mathcal{N} = 2$ SCQCD from the Index

The superconformal index [18] (see also [155]) computes "cohomological" information about the protected spectrum of a superconformal field theory. It counts (with signs) the multiplets obeying shortening conditions, up to equivalence relations that set to zero all sequences of short multiplets that may in principle recombine into long multiplets. The index is invariant under exactly marginal deformations and can thus be evaluated in the free field limit (if the theory admits a Lagrangian description). It should be kept in mind that the index does not completely fix the protected spectrum. A first issue is a certain ambiguity in the quantum numbers of the protected multiplets detected by the index. Short multiplets can be organized into "equivalence classes", such that each short multiplet in a class gives the same contribution to the index. For $\mathcal{N} = 2$ 4d superconformal theories these equivalence classes contain a finite number of short multiplets. This finite ambiguity could in principle be resolved by an explicit one-loop calculation, but in practice this is difficult since the diagonalization of the one-loop dilation operator becomes rapidly complicated as the conformal dimension increases. A second issue is that some sequences of short multiplets that are kinematically allowed to recombine into long multiplets may in fact remain protected for dynamical reasons. This dynamical protection is known to occur at large N_c in $\mathcal{N} = 4$ SYM for certain multi-trace operators, but not for single-trace operators.

Despite these caveats, the index is a very valuable tool. In this section, after reviewing the definition of the index [18], we explain exactly what kind of information can be extracted from it, by characterizing the "equivalence classes" of short multiplets that give the same contribution to the index. We then proceed to concrete calculations, evaluating the index for the interpolating SCFT and for $\mathcal{N} = 2$ SCQCD. The free field contents of the two theories, and thus their indices, are different: recall that the interpolating SCFT has an extra vector multiplet in the adjoint of $SU(N_{\check{c}})$. The index for the interpolating theory confirms the protected spectrum of single-trace operators discussed in the previous section. By contrast, the index for $\mathcal{N} = 2$ SCQCD reveals the existence of many more generalized single-trace operators obeying shortening conditions: their degeneracy grows exponentially with the conformal dimension. Interestingly, we find protected operators with arbitrarily high spin, though none of them is a higher-spin conserved current. We account for the origin of these extra protected states by identifying long multiplets of the interpolating theory that at $\check{g} = 0$ split into short multiplets: some of the resulting short multiplets belong purely to $\mathcal{N} = 2$ SCQCD (*i.e.* do not contain fields in the decoupled vector multiplet) and comprise the extra states.

4.4.1 Review of the Superconformal Index

The superconformal index [18] is just the Witten index with respect to one of the Poincaré supercharges, call it $\mathcal{Q}$, of the superconformal algebra. Let $\mathcal{S} = \mathcal{Q}^\dagger$ be the conformal supercharge conjugate to $\mathcal{Q}$, and $\delta \equiv 2\{\mathcal{S}, \mathcal{Q}\}$. Every state in a unitary representation of the superconformal algebra has $\delta \geq 0$. The index is defined as

$$\mathcal{I} = \mathrm{Tr}\,(-1)^F e^{-\alpha\delta + M}\,, \tag{4.13}$$

where the trace is over the Hilbert space of the theory on S^3, in the usual radial quantization, and M is any operator that commutes with $\mathcal{Q}$ and . The index receives contributions only from states with $\delta = 0$, which are in one-to-one correspondence with the cohomology classes of $\mathcal{Q}$. It is thus independent of α.

There are in fact two inequivalent possibilities for the choice of $\mathcal{Q}$, leading to a "left" index $\mathcal{I}^{\mathtt{L}}$ and a "right" index $\mathcal{I}^{\mathtt{R}}$. The choice $\mathcal{Q} = \mathcal{Q}^1_-$ leads to the "left" index $\mathcal{I}^{\mathtt{L}}$. In this case

$$\delta^{\mathtt{L}} = \Delta - 2j - 2R - r\,. \tag{4.14}$$

Introducing chemical potentials for all the operators that commute with $\mathcal{Q}$ and , one defines

$$\mathcal{I}^{\mathtt{L}}(t, y, v) \equiv \mathrm{Tr}\,(-1)^F\, t^{2(\Delta+j)}\, y^{2\bar{j}} v^{r-R}\,. \tag{4.15}$$

The choice $\mathcal{Q} = \bar{\mathcal{Q}}_{2+}$ gives instead the "right" index $\mathcal{I}^{\mathtt{R}}$. In this case

$$\begin{aligned} \delta^{\mathtt{R}} &\equiv \Delta - 2\bar{j} - 2R + r \qquad (4.16)\\ \mathcal{I}^{\mathtt{R}}(t, y, v) &= \mathrm{Tr}\,(-1)^F\, t^{2(\Delta+\bar{j})}\, y^{2j} v^{-r-R}. \qquad (4.17) \end{aligned}$$

The relation between the left and right index is simply $j \leftrightarrow \bar{j}$ and $r \leftrightarrow -r$. For an $\mathcal{N} = 2$ theory, which is necessarily non-chiral, the left and right indices are in fact equal as functions of the chemical potentials, $\mathcal{I}^{\mathtt{L}}(t, y, v) = \mathcal{I}^{\mathtt{R}}(t, y, v)$, but it will be useful to have introduced the definitions of both.

4.4.2 Equivalence Classes of Short Multiplets

We have mentioned that there is a certain finite ambiguity in extracting from the index which are the actual multiplets that remain short. Schematically, the issue is the following. Suppose that two short multiplets, S_1 and S_2, can recombine to form a long multiplet L_1,

$$S_1 \oplus S_2 = L_1 \,, \tag{4.18}$$

and similarly that S_2 can recombine with a third short multiplet S_3 to give another long multiplet L_2,

$$S_2 \oplus S_3 = L_2 \,. \tag{4.19}$$

By construction, the index evaluates to zero on long multiplets, so

$$\mathcal{I}(S_1) = -\mathcal{I}(S_2) = \mathcal{I}(S_3) \,. \tag{4.20}$$

We say that the two multiplets S_1 and S_3 belong to the same equivalence class, since their indices are the same. Note that S_2 *can* be distinguished from $S_1 \sim S_3$ by the overall sign of its index.

The recombination rules for $\mathcal{N}=2$ superconformal algebra are [154]

$$\begin{aligned}
\mathcal{A}^{2R+r+2j+2}_{R,r(j,\bar{j})} &\simeq \mathcal{C}_{R,r(j,\bar{j})} \oplus \mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j-\frac{1}{2},\bar{j})} &(4.21)\\
\mathcal{A}^{2R-r+2\bar{j}+2}_{R,r(j,\bar{j})} &\simeq \bar{\mathcal{C}}_{R,r(j,\bar{j})} \oplus \bar{\mathcal{C}}_{R+\frac{1}{2},r-\frac{1}{2}(j,\bar{j}-\frac{1}{2})} &(4.22)\\
\mathcal{A}^{2R+j+\bar{j}+2}_{R,j-\bar{j}(j,\bar{j})} &\simeq \hat{\mathcal{C}}_{R(j,\bar{j})} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j-\frac{1}{2},\bar{j})} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j,\bar{j}-\frac{1}{2})} \oplus \hat{\mathcal{C}}_{R+1(j-\frac{1}{2},\bar{j}-\frac{1}{2})} \,. &(4.23)
\end{aligned}$$

Notations are reviewed in appendix B.1. The $\mathcal{C}$, $\bar{\mathcal{C}}$ and $\hat{\mathcal{C}}$ multiplets obey certain "semi-shortening" conditions, see Table B.1, while $\mathcal{A}$ multiplets are generic long multiplets. A long multiplet whose conformal dimension is exactly at the unitarity threshold can be decomposed into shorter multiplets according to (4.21,4.22,4.23). We can formally regard any multiplet obeying some shortening condition (with the exception of the $\mathcal{E}$ and $\bar{\mathcal{E}}$ types) as a multiplet of type $\mathcal{C}$, $\bar{\mathcal{C}}$ or $\hat{\mathcal{C}}$ by allowing the spins j and $\bar{j}$, whose natural range is over the non-negative half-integers, to take the value $-1/2$ as well. The translation is as follows:

$$\mathcal{C}_{R,r(-\frac{1}{2},\bar{j})} \simeq \mathcal{B}_{R+\frac{1}{2},r+\frac{1}{2}(0,\bar{j})} \,. \tag{4.24}$$

$$\hat{\mathcal{C}}_{R(-\frac{1}{2},\bar{j})} \simeq \mathcal{D}_{R+\frac{1}{2}(0,\bar{j})}, \qquad \hat{\mathcal{C}}_{R(j,-\frac{1}{2})} \simeq \bar{\mathcal{D}}_{R+\frac{1}{2}(j,0)} . \tag{4.25}$$

$$\hat{\mathcal{C}}_{R(-\frac{1}{2},-\frac{1}{2})} \simeq \mathcal{D}_{R+\frac{1}{2}(0,-\frac{1}{2})} \simeq \bar{\mathcal{D}}_{R+\frac{1}{2}(-\frac{1}{2},0)} \simeq \hat{\mathcal{B}}_{R+1} . \tag{4.26}$$

Note how these rules flip statistics: a multiplet with bosonic primary ($j+\bar{j}$ integer) is turned into a multiplet with fermionic primary ($j+\bar{j}$ half-odd), and viceversa. With these conventions, the rules (4.21, 4.22, 4.23) are the most general recombination rules. The $\mathcal{E}$ and $\bar{\mathcal{E}}$ multiplets never recombine.

Let us start by characterizing the equivalent classes for $\mathcal{C}$-type multiplets. The right index vanishes identically on $\mathcal{C}$ multiplets. From (4.21), we have

$$\mathcal{I}^{\mathtt{L}}[\mathcal{C}_{R,r(j,\bar{j})}] + \mathcal{I}^{\mathtt{L}}[\mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j-\frac{1}{2},\bar{j})}] = 0 . \tag{4.27}$$

Clearly $\tilde{R} \equiv R+j$, $\tilde{r} \equiv r+j$ and $\bar{j}$ and the overall sign are the invariant quantum numbers that label an equivalence class. We denote by $[\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{+}$ the equivalence class of $\mathcal{C}$ multiplets with $\mathcal{I}^{\mathtt{L}} = \mathcal{I}^{\mathtt{L}}[\mathcal{C}_{\tilde{R},\tilde{r}(0,\bar{j})}]$, and by $[\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{-}$ the class with $\mathcal{I}^{\mathtt{L}} = -\mathcal{I}^{\mathtt{L}}[\mathcal{C}_{\tilde{R},\tilde{r}(0,\bar{j})}]$,

$$\begin{aligned} [\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{+} &= \{\mathcal{C}_{\tilde{R}-m,\tilde{r}-m\,(m,\bar{j})}\,|\, m = 0,1,2\ldots, m \leq \tilde{R}\} && (4.28)\\ [\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{-} &= \{\mathcal{C}_{\tilde{R}-m,\tilde{r}-m\,(m,\bar{j})}\,|\, m = -\frac{1}{2},\frac{1}{2},\frac{3}{2}\ldots, m \leq \tilde{R}\} . && (4.29) \end{aligned}$$

Explicitly, the left index of the class $[\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{\pm}$ is:

$$\begin{aligned} &\mathcal{I}^{\mathtt{L}}_{[\tilde{R},\tilde{r},\bar{j}]^{\mathtt{L}}_{\pm}} = && (4.30)\\ &\pm(-1)^{2\bar{j}+1} t^{6+4\tilde{R}+2\tilde{r}} v^{-2+\tilde{r}-\tilde{R}} \frac{(1-t^2 v)(t-\frac{v}{y})(t-vy)}{(1-t^3 y)(1-\frac{t^3}{y})} (y^{2\bar{j}}+\ldots+y^{-2\bar{j}}) \end{aligned}$$

We have illustrated the equivalence classes $[1,1,0]^{\mathtt{L}}_{\pm}$ in Figure 4.2 by listing multiplets on the j axis. The allowed values of $\tilde{R}$ and $\bar{j}$ are $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, ...,

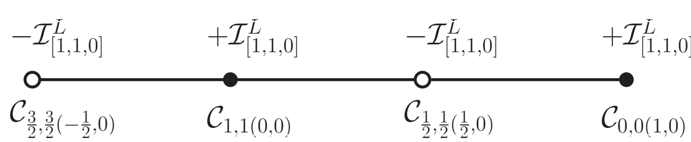


Figure 4.2: The equivalence classes $[1,1,0]^{\mathtt{L}}_{\pm}$. The multiplets belonging to $[1,1,0]^{\mathtt{L}}_{\pm}$ have index $\pm\mathcal{I}^{\mathtt{L}}_{[1,1,0]}$. The sum of the indices of adjacent multiplets is zero, as required by the recombination rule.

with the proviso that $j = -\frac{1}{2}$ or $\bar{j} = -\frac{1}{2}$ must be interpreted according to

(4.24). For the lowest value of $\tilde{R}$, $\tilde{R} = -\frac{1}{2}$, the class $[-\frac{1}{2}, \tilde{r}, \bar{j}]^{\mathtt{L}}_{+}$ is empty while the class $[-\frac{1}{2}, \tilde{r}, \bar{j}]^{\mathtt{L}}_{-} = \mathcal{B}_{\frac{1}{2}, \tilde{r}+1(0,\bar{j})}$ consists of a single multiplet, which can then be determined without any ambiguity. For $\tilde{R} = 0$, $[0, \tilde{r}, \bar{j}]^{\mathtt{L}}_{+} = \mathcal{C}_{0,\tilde{r}(0,\bar{j})}$ and $[0, \tilde{r}, \bar{j}]^{\mathtt{L}}_{-} = \mathcal{B}_{1,\tilde{r}+1(0,\bar{j})}$ both contain a single multiplet and again there is no ambiguity. Finally for $\tilde{R} = \frac{1}{2}$, $[\frac{1}{2}, \tilde{r}, \bar{j}]_{+} = \mathcal{C}_{\frac{1}{2},\tilde{r}(0,\bar{j})}$ contains a single multiplet, but $[\frac{1}{2}, \tilde{r}, \bar{j}]_{-}$ already has two and from the index alone cannot decide which of the two actually remains protected. Clearly the ambiguity grows linearly with $\tilde{R}$.

The analysis for the $\bar{\mathcal{C}}$ multiplets is entirely analogous, and follows from the previous discussion by the substitutions $j \leftrightarrow \bar{j}$, $r \leftrightarrow -r$. One needs to consider $\mathcal{I}^{\mathtt{R}}$, since now it is $\mathcal{I}^{\mathtt{L}}$ that evaluates to zero. The equivalence classes are defined to be the set of all the $\bar{\mathcal{C}}$ multiplets with same $\mathcal{I}^{\mathtt{R}}$ up to sign, and are denoted as $[\bar{\tilde{R}}, \bar{\tilde{r}}, j]^{\mathtt{R}}_{\pm}$, where $\bar{\tilde{R}} \equiv R + \bar{j}$, $\bar{\tilde{r}} \equiv -r + \bar{j}$.

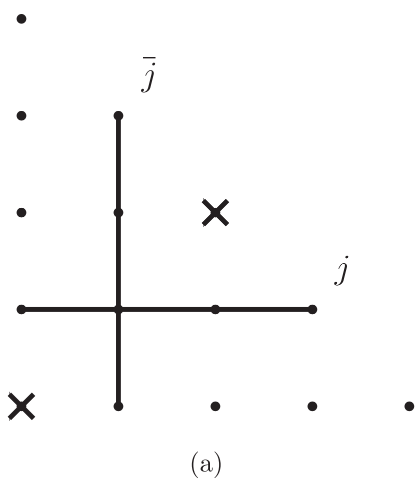


Figure 4.3: Example of two configurations of the $\hat{\mathcal{C}}$. In subfigure (a) $\hat{\mathcal{C}}_{0(\frac{1}{2},\frac{1}{2})}$ and $\hat{\mathcal{C}}_{2(-\frac{1}{2},-\frac{1}{2})} \equiv \hat{\mathcal{B}}_{3(0,0)}$ and in subfigure (b) $\hat{\mathcal{C}}_{1(-\frac{1}{2},\frac{1}{2})} \equiv \mathcal{D}_{\frac{3}{2}(0,\frac{1}{2})}$ and $\hat{\mathcal{C}}_{1(\frac{1}{2},-\frac{1}{2})} \equiv \bar{\mathcal{D}}_{\frac{3}{2}(0,\frac{1}{2})}$

multiplets with $R + j + \bar{j} = 1$ contributing the same to both $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$. The multiplets are denoted by crosses on the $(j, \bar{j})$ grid. The indices are the same for (a) and (b) because the *projections* on the j and $\bar{j}$ (*i.e.* the *sets* of j and $\bar{j}$ values) are the same.

The analysis for the $\hat{\mathcal{C}}$ multiplets is slightly more involved. Unlike $\mathcal{C}$ and $\bar{\mathcal{C}}$ multiplets, $\hat{\mathcal{C}}$ multiplets contribute to both $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$. Moreover the quantum number r is fixed by the additional shortening condition $r = \bar{j} - j$. The left and right equivalence classes of $\hat{\mathcal{C}}_{R(j,\bar{j})}$ are $[R+j, \bar{j}, \bar{j}]^{\mathtt{L}}_{\pm}$ and $[R+\bar{j}, j, j]^{\mathtt{R}}_{\pm}$ respectively. The left index determines $\tilde{R} = R + j$ and the right index $\tilde{\tilde{R}} = R + \bar{j}$, so all in all no two different $\hat{\mathcal{C}}$ multiplets give the same contribution to both $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$. Nevertheless different *direct sums* of $\hat{\mathcal{C}}$ multiplets can have the same $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$. It is convenient to introduce the quantum number $\hat{R} \equiv R + j + \bar{j}$, which is an invariant for both the left and the right equivalence classes, and to label the equivalence classes for $\hat{\mathcal{C}}$ multiplets as $[\hat{R}, \bar{j}]^{\mathtt{L}}_{\pm}$ and $[\hat{R}, j]^{\mathtt{R}}_{\pm}$. This new way to label the classes does not entail any loss of information, and makes it more convenient to analyze both the indices simultaneously. Explicitly, the left and right indices for these equivalence classes are:

$$
\begin{aligned}
\mathcal{I}^{\mathtt{L}}_{[\hat{R},\bar{j}]^{\mathtt{L}}_{\pm}} &= \pm(-1)^{2\bar{j}} \frac{t^{6-2\bar{j}+4\hat{R}} v^{-1+2\bar{j}-\hat{R}}(1-t^2 v)}{(1-t^3 y)(1-t^3/y)} \\
&\quad (t(y^{2\bar{j}+1} + \ldots + y^{-(2\bar{j}+1)}) - v(y^{2\bar{j}} + \ldots + y^{-2\bar{j}})) \qquad (4.31) \\
\mathcal{I}^{\mathtt{R}}_{[\hat{R},j]^{\mathtt{R}}_{\pm}} &= \pm(-1)^{2j} \frac{t^{6-2j+4\hat{R}} v^{-1+2j-\hat{R}}(1-t^2 v)}{(1-t^3 y)(1-t^3/y)} \\
&\quad (t(y^{2j+1} + \ldots + y^{-(2j+1)}) - v(y^{2j} + \ldots + y^{-2j}))\,. \qquad (4.32)
\end{aligned}
$$

Now the point is that given a collection of $\hat{\mathcal{C}}$ multiplets with the same value of $\hat{R}$, the left index determines the *set* of $\bar{j}$ values while the right index determines the set of j values, but in general there is not enough information to fix uniquely all quantum numbers. Figure 4.3 illustrates the ambiguity in a simple example: two different configurations, each consisting of two $\hat{\mathcal{C}}$ multiplets, give the same contribution to both $\mathcal{I}^{\mathtt{L}}$ and $\mathcal{I}^{\mathtt{R}}$.

Multiplet	Equivalence class
$\mathcal{C}$	$[\tilde{R}, \tilde{r}, \bar{j}]^{\mathtt{L}}_{\pm} \equiv [R+j, r+j, \bar{j}]^{\mathtt{L}}_{\pm}$
$\bar{\mathcal{C}}$	$[\tilde{R}, \bar{\tilde{r}}, j]^{\mathtt{R}}_{\pm} \equiv [R+\bar{j}, -r+\bar{j}, j]^{\mathtt{R}}_{\pm}$
$\hat{\mathcal{C}}$	$[\hat{R}, \bar{j}]^{\mathtt{L}}_{\pm} \equiv [R+j+\bar{j}, \bar{j}]^{\mathtt{L}}_{\pm}$ $[\hat{R}, j]^{\mathtt{R}}_{\pm} \equiv [R+j+\bar{j}, j]^{\mathtt{R}}_{\pm}$

Table 4.3: Summary of notation for equivalence classes of short multiplets.

Letters	Δ	j	$\bar{j}$	R	r	$\mathcal{I}^{\mathtt{R}}$
ϕ	1	0	0	0	-1	$t^2 v$
λ^1_+	3/2	1/2	0	1/2	-1/2	$-t^3 y$
λ^1_-	3/2	-1/2	0	1/2	-1/2	$-t^3 y^{-1}$
$\bar{\lambda}_{2+}$	3/2	0	1/2	1/2	1/2	$-t^4 v^{-1}$
$\bar{F}_{++}$	2	0	1	0	0	t^6
∂_{++}	1	1/2	1/2	0	0	$t^3 y$
∂_{-+}	1	-1/2	1/2	0	0	$t^3 y^{-1}$
$\partial_{-+}\lambda^1_+ + \partial_{++}\lambda^1_- = 0$	5/2	0	1/2	1/2	1/2	t^6

Table 4.4: Letters with $\delta^{\mathtt{R}} = 0$ from the $\mathcal{N} = 2$ vector multiplet

4.4.3 The Index of the Interpolating Theory

We now review the calculation of the index for the orbifold theory [18, 156].[7] The index is invariant under exactly marginal deformation and is thus the same for the whole family of interpolating SCFTs. The procedure is well-established. One enumerates the "letters" of the theory with $\delta = 0$ and then counts all possible gauge-invariants words. This is done efficiently by a matrix model, which for large N can be evaluated by saddle point. Tables 4.4 and 4.5 list the $\delta^{\mathtt{R}} = 0$ letters from the $\mathcal{N} = 2$ vector and hyper multiplets.[8] Equations of motion are accounted for by introducing words with "wrong" statistics.

[7] While we agree with the general procedure followed in [156], we disagree with the final result, equ.(3.5) of [156]. The discrepancy can be traced to an incorrect subtraction of the $U(1)$ factors in [156], they are apparently taken to be $\mathcal{N} = 1$ rather than $\mathcal{N} = 2$ vector multiplets (equ.(2.12) of [156]). For the same reason we disagree with the expression ((3.7) of [156]) for the contribution to the index of the $6d$ $(2,0)$ massless tensor multiplet, which we evaluate in appendix B.3.

[8] For definiteness we evaluate $\mathcal{I}^{\mathtt{R}}$, but recall that $\mathcal{I}^{\mathtt{L}}(t, y, v) = \mathcal{I}^{\mathtt{R}}(t, y, v)$. The concrete letters with $\delta^{\mathtt{L}} = 0$ are different but the left and right single-letter indices coincide.

Letters	Δ	j	$\bar{j}$	R	r	$\mathcal{I}^{\mathtt{R}}$
q	1	0	0	1/2	0	$t^2v^{-1/2}$
$\bar{\psi}_+$	3/2	0	1/2	0	-1/2	$-t^4v^{1/2}$
$\tilde{q}$	1	0	0	1/2	0	$t^2v^{-1/2}$
$\bar{\tilde{\psi}}_+$	3/2	0	1/2	0	-1/2	$-t^4v^{1/2}$

Table 4.5: Letters with $\delta^{\mathtt{R}} = 0$ from the hyper multiplet

One finds the single-letter indices for the vector multiplet and the "half" hyper multiplet

$$f_V(t,y,v) = \frac{t^2v - t^3\left(y+y^{-1}\right) - t^4v^{-1} + 2t^6}{\left(1-t^3y\right)\left(1-t^3y^{-1}\right)} \tag{4.33}$$

$$f_H(t,y,v) = \frac{t^2}{v^{1/2}}\frac{\left(1-t^2v\right)}{\left(1-t^3y\right)\left(1-t^3y^{-1}\right)}\,. \tag{4.34}$$

The single-letter index then reads

$$\begin{aligned} i_{orb}(t,y,v;U,\check{U}) &= f_V(t,y,v)(\mathrm{Tr}U\,\mathrm{Tr}U^\dagger - 1) + f_V(t,y,v)(\mathrm{Tr}\check{U}\,\mathrm{Tr}\check{U}^\dagger - 1) \\ &\quad + \left(w + \frac{1}{w}\right) f_H(t,y,v)(\mathrm{Tr}U\,\mathrm{Tr}\check{U}^\dagger + \mathrm{Tr}U^\dagger\,\mathrm{Tr}\check{U})\,. \end{aligned} \tag{4.35}$$

Here U and $\check{U}$ are is an $N_c \times N_c$ unitary matrices out of which we construct the relevant characters of $SU(N_c)$ and $SU(N_{\check{c}})$. We have also introduced a potential w that keeps track of $SU(2)_L$ quantum numbers: $w + \frac{1}{w}$ is the character of the fundamental representation of $SU(2)_L$. The index is obtained by enumerating all gauge-invariant operators in terms of the matrix integral

$$\mathcal{I}_{orb} = \int [dU][d\check{U}]\exp\left(\sum_n \frac{1}{n} i_{orb}(t^n, y^n, v^n; U^n\check{U}^n)\right)\,, \tag{4.36}$$

which for large N_c can be carried out explicitly,

$$\mathcal{I}_{orb} \cong \prod_{n=1}^{\infty} \frac{e^{-\frac{2}{n}f_V(t^n,y^n,v^n)}}{(1-f_V(t^n,y^n,v^n))^2 - (w^{2n}+w^{-2n}+2)f_H^2(t^n,y^n,v^n)} \equiv \mathcal{I}_{orb}^{m.t.} \tag{4.37}$$

This expression contains the contribution from all the gauge-invariant operators of the theory, which at large N_c are multi-traces, hence the superscript in $\mathcal{I}_{orb}^{m.t.}$. To extract the contribution from single-traces we evaluate the plethystic logarithm (see *e.g.* [157])

$$
\begin{aligned}
\mathcal{I}^{s.t.}_{orb} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log[\mathcal{I}^{m.t}_{orb}(t^n, y^n, v^n)] \qquad (4.38)\\
&= -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log[(1 - f_V(t^n, y^n, v^n))^2 - (w^{2n} + w^{-2n} + 2) f_H^2(t^n, y^n, v^n)] \\
&\quad -2 f_V(t, y, v) \\
&= 2\left[\frac{t^2 v}{1 - t^2 v} - \frac{t^3 y}{1 - t^3 y} - \frac{t^3 y^{-1}}{1 - t^3 y^{-1}}\right] + \frac{\frac{t^4 w^2}{v}}{1 - \frac{t^4 w^2}{v}} + \frac{\frac{t^4}{v w^2}}{1 - \frac{t^4}{v w^2}} \\
&\quad -2 f_V(t, y, v)\,.
\end{aligned}
$$

Here $\mu(n)$ is the Moebius function ($\mu(1) \equiv 1$, $\mu(n) \equiv 0$ if n has repeated prime factors, and $\mu(n) = (-1)^k$ if n is the product of k distinct primes), and $\varphi(r)$ is the Euler Phi function, defined as the number of positive integers less than or equal to r that are coprime with respect to r. We have used the properties

$$
\sum_{d|n} d\, \mu\left(\frac{n}{d}\right) = \varphi(n)\,, \qquad \sum_r \frac{\varphi(r)}{r} \log(1 - x^r) = \frac{-x}{1 - x}\,. \qquad (4.39)
$$

The index is of course independent of g and $\check{g}$. At the orbifold point $g = \check{g}$ it makes sense organize the spectrum into a twisted and an untwisted sector. Protected operators in the untwisted sectors are known from inheritance from $\mathcal{N} = 4$ SYM. To evaluate the contribution to the index from the untwisted sector we start with the single-trace index for $SU(N_c)$ $\mathcal{N} = 4$ SYM and project onto the $\mathbb{Z}_2$ invariant subspace. The single-trace index for $\mathcal{N} = 4$ is found by regarding $\mathcal{N} = 4$ as an $\mathcal{N} = 2$ theory with one adjoint vector and one adjoint hyper. A short calculation gives [18][9]

$$
\begin{aligned}
\mathcal{I}_{\mathcal{N}=4} &= \frac{t^2 v}{1 - t^2 v} + \frac{\frac{t^2 w}{\sqrt{v}}}{1 - \frac{t^2 w}{\sqrt{v}}} + \frac{\frac{t^2}{w\sqrt{v}}}{1 - \frac{t^2}{w\sqrt{v}}} - \frac{t^3 y}{1 - t^3 y} - \frac{t^3 y^{-1}}{1 - t^3 y^{-1}} \\
&\quad - f_V(t, y, v) - (w + \frac{1}{w}) f_H(t, y, v)\,. \qquad (4.40)
\end{aligned}
$$

[9]Our notations for the chemical potentials are slightly different from [18].

The $\mathbb{Z}_2$ acts as $w \to -w$ leaving invariant the under potentials, so the index of the untwisted sector of the $\mathbb{Z}_2$ orbifold theory is

$$\begin{aligned}
\mathcal{I}_{untwist} &= \frac{1}{2}(\mathcal{I}_{\mathcal{N}=4}(t,y,v,w) + \mathcal{I}_{\mathcal{N}=4}(t,y,v,-w)) \qquad (4.41)\\
&= \frac{t^2 v}{1-t^2 v} - \frac{t^3 y}{1-t^3 y} - \frac{t^3 y^{-1}}{1-t^3 y^{-1}} + \frac{\frac{t^4 w^2}{v}}{1-\frac{t^4 w^2}{v}} + \frac{\frac{t^4}{vw^2}}{1-\frac{t^4}{vw^2}} - f_V(t,y,v)\,.
\end{aligned}$$

Subtracting the contribution of the untwisted sector from the total index (4.39), we finally find

$$\mathcal{I}_{twist} = \frac{t^2 v}{1-t^2 v} - \frac{t^3 y}{1-t^3 y} - \frac{t^3 y^{-1}}{1-t^3 y^{-1}} - f_V(t,y,v)\,. \qquad (4.42)$$

In appendix B.3 we confirm that this precisely matches with the contribution from the twisted multiplets $\{\mathcal{M}_{\mathbf{3}}, \mathrm{Tr}(\phi^{2+\ell} - \check{\phi}^{2+\ell})\,, \ell \geq 0\}$, which are the generators of the $\mathcal{N}=1$ chiral ring in the twisted sector.

4.4.4 The Index of $\mathcal{N}=2$ SCQCD and the Extra States

The single-letter index for $\mathcal{N}=2$ SCQCD is

$$\begin{aligned}
i_{QCD}(t,y,v;U,V) &= f_V(t,y,v)(\mathrm{Tr}U\,\mathrm{Tr}U^\dagger - 1) \qquad (4.43)\\
&\quad + f_H(t,y,v)(\mathrm{Tr}U\,\mathrm{Tr}V^\dagger + \mathrm{Tr}U^\dagger\,\mathrm{Tr}V)\,,
\end{aligned}$$

where U an $N_c \times N_c$ matrix and V an $N_f \times N_f$ matrix, with $N_f = 2N_c$. We are interested in gauge *and* flavor-singlets, so we integrate over both U and V,

$$\mathcal{I}_{QCD} = \int [dU][dV] \exp\left(\sum_n \frac{1}{n} i_{QCD}(t^n, y^n, v^n; U^n V^n)\right)\,. \qquad (4.44)$$

For large N_c and N_f with N_f/N_c fixed we can again use saddle point,

$$\mathcal{I}_{QCD} \cong \prod_{n=1}^{\infty} \frac{e^{-\frac{1}{n} f_V(t^n,y^n,v^n)}}{(1 - f_V(t^n,y^n,v^n)) - f_H^2(t^n,y^n,v^n)} \equiv \mathcal{I}^{m.t.}_{QCD}\,. \qquad (4.45)$$

The index that enumerates (generalized) single-trace operators is then

$$\mathcal{I}^{s.t.}_{QCD} = -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log[(1 - f_V(t^n,y^n,v^n)) - f_H^2(t^n,y^n,v^n)] - f_V(t,y,v) \qquad (4.46)$$

Unlike the orbifold theory, there is no nice factorization of the single-letter index and we cannot extract the plethystic log explicitly. This is already an indication of a more complicated structure than expected. The naive expectation is that all protected generalized single-trace multiplets of $\mathcal{N}=2$ SCQCD are exhausted by the list $\{\mathcal{M}_{\mathbf{3}}\,, \mathrm{Tr}\,\phi^{2+\ell}\,, \mathrm{Tr}\,T\phi^{\ell}\,, \ell \geq 0\}$, obtained by projecting the protected single-trace spectrum of the interpolating theory onto $U(N_f)$ singlets. We evaluate the corresponding index in appendix B.3,

$$\begin{aligned}\mathcal{I}_{naive} &= \frac{1}{(1-t^3y)(1-\frac{t^3}{y})}\times \\ &\left[-t^6(1-\frac{t}{v}(y+\frac{1}{y}))-\frac{t^{10}}{v}+\frac{t^4v^2(1-\frac{t}{vy})(1-\frac{ty}{v})}{1-t^2v}+\frac{t^4}{v}(1-t^2v)\right],\end{aligned} \tag{4.47}$$

which is different from the correct index (4.46). Expanding in powers of t, the first discrepancy appears at $O(t^{13})$.

To get some insight, let us rewrite the single-trace index of the orbifold theory as

$$\begin{aligned}\mathcal{I}^{s.t.}(h,k) &= -\sum_{n=1}^{\infty}\Big[\frac{\varphi(n)}{n}\log[(1-f_V(t^n,y^n,v^n))(1-hf_V(t^n,y^n,v^n)) \\ &\quad -(k(w^{2n}+1+w^{-2n})+1)f_H^2(t^n,y^n,v^n)\Big] \\ &\quad -f_V(t,y,v)\,.\end{aligned} \tag{4.48}$$

We have introduced a variable h that keeps track of the number of $SU(N_{\check{c}})$ vector multiplets, and a variable k associated with the *triplet* combination of two neighboring $SU(2)_L$ indices. The index (4.46) for $\mathcal{N}=2$ SCQCD is recovered in the limit $(h,k)\to(0,0)$. Indeed setting $(h,k)=(0,0)$: this amounts to omitting the “second” vector multiplet and to project onto $U(N_f)$ singlets, which is equivalent to first projecting onto $SU(N_{\check{c}})$ singlets (automatically done in the interpolating theory) and then contracting all neighboring $SU(2)_L$ indices into the singlet combination. The grading of gauge-invariant words by powers of h (number of letters in the $SU(N_{\check{c}})$ vector multiplet) makes sense only for $\check{g}=0$. Similarly, for $\check{g}\neq 0$ only the overall $SU(2)_L$ spin of a state is a meaningful quantum number, not the specific way neighboring $SU(2)_L$ indices are contracted. (For example it is clearly possible to construct $SU(2)_L$ singlets which are not $U(N_f)$ singlets.) At $\check{g}\neq 0$ words with different h or k grading will generically mix.

The origin of the extra protected states is then clear. As $\check{g}\to 0$, a long multiplets of the interpolating theory, which obviously does not contribute to

$\mathcal{I}_{orb}$, may hit the unitarity bound and decompose into a sum of short multiplets, some of which are $U(N_f)$ singlets and thus belong to $\mathcal{N}=2$ SCQCD, but some of which have instead non-trivial h or k grading. Schematically

$$\lim_{\check{g}\to 0} L = \oplus S_{(h,k)=(0,0)} \;\oplus S_{(h,k)\neq(0,0)}\,. \tag{4.49}$$

The operators $\{S_{(h,k)=(0,0)}\}$ are the extra states. They are protected in $\mathcal{N}=2$ SCQCD because they have no partners to recombine with.

Remarkably the extra protected states are vastly more numerous than the naive list. The asymptotic growth of states in the naive list is clearly linear in the conformal dimension – the number of states with $\Delta < N$ grows as $\sim 2N$, in other terms the density of states $\rho(\Delta)$ is constant. This modest growth is consistent with the fact that the naive single-trace index does not "deconfine", *i.e.* it does not diverge as a function of $t = e^{-1/T}$ for any finite temperature T. The same behavior holds for the orbifold theory or for $\mathcal{N}=4$ SYM. By contrast, the single-trace index of $\mathcal{N}=2$ SCQCD exhibits Hagedorn behavior. Setting for simplicity all other potentials to 1, we encounter a divergence at $t = t_H$ such that

$$1 - f_V(t_H,1,1) - f_H^2(t_H,1,1) = 0 \longrightarrow t_H \cong 0.897769\,. \tag{4.50}$$

This implies an exponential growth in the density of states contributing to the index,

$$\rho(E') \sim e^{\beta_H E'}\,, \quad E' \equiv \Delta + j\,, \quad \beta_H = -\ln t_H \cong 0.107842\,. \tag{4.51}$$

It is interesting to compare this behavior with the density of *generic* generalized single-trace operators of $\mathcal{N}=2$ SCQCD. The density of generic states, unlike the density of protected states, is of course a function of the coupling. For $g=0$, it is obtained by calculating the phase transition temperature of the complete generalized single-trace partition function (with no $(-1)^F$). We find $\sim e^{\beta'_H(\Delta+j)}$ with $\beta'_H = 1.34254$. Not surprisingly, $\beta_H < \beta'_H$. The density of protected states, while exponential, grows at a much slower rate than the density of the generic states, or at least this is the behavior for small g.

4.4.5 Sieve Algorithm

We would like to list the quantum numbers of the extra protected states, up to the finite equivalence class ambiguity intrinsic to the index. There is no closed-form expression for $\mathcal{I}^{s.t.}_{QCD}$ but we can identity the equivalence classes contributing to it in a systematic expansion in powers of t, by implementing a "sieve" algorithm similar in spirit to the one of [158].

The first discrepancy between $\mathcal{I}^{s.t.}_{QCD}$ is the $O(t^{13})$ term

$$\mathcal{I}_{QCD} - \mathcal{I}_{naive} = -\frac{t^{13}}{v}(y + \frac{1}{y}) + \dots \tag{4.52}$$

On the other hand, expanding (4.31) in powers of t, the lowest term is

$$- t^{6+4\tilde{R}+2\tilde{r}} v^{\tilde{r}-\tilde{R}} (y^{2\bar{j}} + \dots + y^{-2\bar{j}}) . \tag{4.53}$$

Matching with (4.52) we determine the equivalence class of the first new protected multiplet to be $[\tilde{R}, \tilde{r}, \bar{j}]^{\mathtt{L}}_{+} = [\frac{3}{2}, \frac{1}{2}, \frac{1}{2}]^{\mathtt{L}}_{+}$. Since $\tilde{r} = \bar{j}$, this is actually a $\hat{\mathcal{C}}$ multiplet so we rewrite its equivalence class as $[\hat{R}, \bar{j}]^{\mathtt{L}} = [2, \frac{1}{2}]^{\mathtt{L}}_{+}$. Subtracting the whole index of the class from the discrepancy we proceed to the next mismatch in the t expansion, and so on. In this way, we can systematically construct the equivalence classes of all the extra protected multiplets of the SCQCD. The results from $\mathcal{I}^{\mathtt{L}}$ for first few multiplets are:

- $\mathcal{C}$ multiplets: $[2, 2, 0]^{\mathtt{L}}_{+}$, $[2, 3, 0]^{\mathtt{L}}_{+}$, $[2, 4, 0]^{\mathtt{L}}_{+}$, $[3, 2, 0]^{\mathtt{L}}_{-}$, $[3, 2, 1]^{\mathtt{L}}_{-}$, ...
- $\hat{\mathcal{C}}$ multiplets: $[2, \frac{1}{2}]^{\mathtt{L}}_{+}$, $[4, 1]^{\mathtt{L}}_{+}$, $[4, \frac{3}{2}]^{\mathtt{L}}_{+}$, ...

From the analysis of $\mathcal{I}^{\mathtt{R}}$ we can write down the *right* equivalence classes of the protected multiplets. Since $\mathcal{I}^{\mathtt{R}} = \mathcal{I}^{\mathtt{L}}$, the list of *right* equivalence classes is obtained immediately from the list of *left* equivalence classes by the substitutions $\mathcal{C} \to \bar{\mathcal{C}}$ and $\mathtt{L} \to \mathtt{R}$.

Protected $\bar{\mathcal{C}}$ multiplets are just conjugates of protected $\mathcal{C}$ multiplets. The $\hat{\mathcal{C}}$ multiplets, however, appear in both left and right classes, and as we discussed this gives more information. For example the $\hat{\mathcal{C}}$ multiplet in $[2, \frac{1}{2}]^{\mathtt{L}}_{+}$ also belongs to $[2, \frac{1}{2}]^{\mathtt{R}}_{+}$ and furthermore it is the only multiplet with $\hat{R} = R + j + \bar{j} = 2$. The left equivalence class determines $\bar{j} = \frac{1}{2}$, the right equivalence class $j = \frac{1}{2}$ and both also imply $R = \hat{R} - j - \bar{j} = 1$. This determines the lowest-lying extra protected $\hat{\mathcal{C}}$ multiplet to be $\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}$. For $\hat{R} = 4$, there are two multiplets with $\bar{j} = 1$, $\frac{3}{2}$ and with same values of j. Two possible $(j, \bar{j})$ Lorentz spins are $(1, 1)$, $(\frac{3}{2}, \frac{3}{2})$ or $(1, \frac{3}{2})$, $(\frac{3}{2}, 1)$ but we also know that it is a bosonic multiplet from the subcript $+$. This picks out the pair $(1, 1)$, $(\frac{3}{2}, \frac{3}{2})$ with $R = 4-1-1 = 2$ and $R = 4 - \frac{3}{2} - \frac{3}{2} = 1$ respectively. This determines the next protected $\hat{\mathcal{C}}$ multiplets to be $\hat{\mathcal{C}}_{1(\frac{3}{2},\frac{3}{2})}$ and $\hat{\mathcal{C}}_{2(1,1)}$. To summarize, the first three protected $\hat{\mathcal{C}}$ multiplets are:

- $\hat{\mathcal{C}}$ multiplets: $\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}$, $\hat{\mathcal{C}}_{1(\frac{3}{2},\frac{3}{2})}$, $\hat{\mathcal{C}}_{2(1,1)}$, ...

A striking feature of the extra protected multiplets is that they contain states with higher spin, in fact we believe that the sieve will produce arbitrarily high spin. To the best of our knowledge this is the first time that higher-spin protected multiplets are found in an *interacting* 4d superconformal field theory. Note that *none* of the protected states we find are higher spin *conserved currents*, which correspond to the multiplets $\hat{\mathcal{C}}_{0(j,\bar{j})}$. This is not surprising: higher spin conserved currents are the hallmark of a free theory, but $\mathcal{N}=2$ SCQCD is most definitely an interacting quantum field theory. As in $\mathcal{N}=4$ SYM [159], higher spin conserved currents exist at strictly zero coupling, but they are anomalous and recombine into long multiplets at non-zero coupling.

4.5 Dual Interpretation of the Protected Spectrum

As we have repeatedly emphasized, $\mathcal{N}=2$ SCQCD can be obtained as the $\check{g}_{YM} \to 0$ limit of a family of $\mathcal{N}=2$ superconformal field theories, which reduces for $g_{YM} = \check{g}_{YM}$ to the $\mathcal{N}=2$ $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. This latter theory has a familiar dual description has IIB string theory on $AdS_5 \times S^5/\mathbb{Z}_2$ [39], so it would seem that to find the dual of $\mathcal{N}=2$ SCQCD we simply need to follow the fate of the bulk string theory under the exactly marginal deformation. Recall that at the orbifold point the NSNS B-field has half-unit period through the blown-down S^2 of the orbifold singularity, $\int_{S^2} B_{NS} = 1/2$ [160]. Taking $\check{g}_{YM} \neq g_{YM}$ is dual to changing the period of B-field, according to the dictionary [40, 161]

$$\frac{1}{g_{YM}^2} + \frac{1}{\check{g}_{YM}^2} = \frac{1}{2\pi g_s} \tag{4.54}$$

$$\frac{\check{g}_{YM}^2}{g_{YM}^2} = \frac{\beta}{1-\beta}\,, \qquad \beta \equiv \int_{S^2} B_{NS}\,. \tag{4.55}$$

The catch is that the limit $\check{g}_{YM} \to 0$ translates on the dual side to the singular limit of vanishing B_{NS} and vanishing string coupling g_s, and the IIB background $AdS_5 \times S^5/\mathbb{Z}_2$ becomes ill-defined. We will study in the next section how to handle this subtle limit. In this section we will try to learn about the string dual of $\mathcal{N}=2$ SCQCD from the "bottom-up", collecting the clues offered by the spectrum of protected operators. We start by reviewing the well-known bulk-boundary dictionary for the protected states of the orbifold theory.

4.5.1 KK interpretation of the orbifold protected specrum

The untwisted spectrum of the orbifold field theory (summarized in Table 4.1), has a transparent dual interpretation as the Kaluza-Klein spectrum of IIB supergravity on $AdS_5 \times S^5/\mathbb{Z}_2$. It is appropriate to write the metric of $S^5/\mathbb{Z}_2$ as [89]

$$ds^2_{S^5/\mathbb{Z}_2} = d\alpha^2 + \sin^2\alpha\, d\varphi^2 + \cos^2\alpha\, ds^2_{S^3/\mathbb{Z}_2}\,,\ 0 \le \varphi \le 2\pi\,,\ 0 \le \alpha \le \frac{\pi}{2}\,. \quad (4.56)$$

Momentum on S^1 corresponds to the $U(1)_r$ charge r. The $SO(4) \cong SU(2)_L \otimes SU(2)_R$ isometry of the 3-sphere is broken to $SO(3)_L \otimes SU(2)_R$ by the $\mathbb{Z}_2$ orbifold, which projects out harmonics with j_L half-odd. Needless to say, $SU(2)_R$ and $SO(3)_L$ are interpreted as the field theory symmetry groups of the same name, so in particular the right spin j_R is identified with the quantum number R. Finally the harmonics on the α interval are parametrized by an integer n, dual to the power of neutral scalar $\mathcal{T}$ (with $\Delta = 2$) in the schematic expressions of the operators in Table 4.1. It is not difficult to carry an explicit KK expansion and confirm that $\Delta = |r| + 2R + 2n$. A nice shortcut is to consider the KK expansion of the ten dimensional dilaton-axion [89], since only *scalar* harmonics on $S^5/\mathbb{Z}_2$ are required. Scalar harmonics on $S^3/\mathbb{Z}_2$ have $(j_L, j_R) = (2R, 2R)$ with $2R$ a non-negative integer. One finds $\Delta = |r| + 2R + 2n + 4$ [89], as expected from the fact that the KK modes of the dilaton-axion are dual to the descendants obtained by acting with $\mathcal{Q}^4\bar{\mathcal{Q}}^4$ on the superconformal primaries of Table 4.1.

The twisted states of the orbifold field theory (shown in Table 5.5), must map on the dual side to twisted closed string states localized at the fixed locus of the orbifold, which is $AdS_5 \times S^1$, corresponding to $\alpha = \pi/2$ in the parametrization (4.56). The massless twisted states of IIB on the A_1 singularity comprise one massless six-dimensional tensor multiplet, so the KK reduction of the tensor multiplet on $AdS_5 \times S^1$ must reproduce the protected twisted states of the orbifold field theory. It does, as we review in appendix B.4 following the analysis of [162], to which we add a detailed treatment of the zero modes. We find that the zero modes of the tensor multiplet correspond to the multiplet build on the "exceptional state" $\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$.

4.5.2 Interpretation for $\mathcal{N} = 2$ SCQCD?

The protected spectrum of $\mathcal{N} = 2$ SCQCD (restricting as usual to flavor singlets, and in the large N Veneziano limit) consists of two sectors: the "naive" list of protected primaries (5.44) easily found by a one-loop calculation in the scalar sector [9]; and the many more extra "exotic" states found in the analysis of the superconformal index.

The "naive" spectrum arises from a truncation of the protected spectrum of the interpolating theory (as $\check{g} \to 0$) to $U(N_f)$ singlets. We have discussed in section 2 the reason to focus on the flavor-singlet sector: flavor-singlet operators, which necessarily are of "generalized single-trace type" in the Veneziano limit, are expected to map to single closed string states. The restriction to $U(N_f)$ singlets has an interesting geometric interpretation: flavor singlets are in particular $SU(2)_L$ singlets, and thus they are dual to supergravity states with *no* angular momentum on $S^3/\mathbb{Z}_2$ in the parametrization (4.56). So in performing this restriction we are "losing" three spatial dimensions. As explained around (4.12), the protected primaries of the interpolating theory that are *not* flavor-singlets can be decomposed in the limit $\check{g} \to 0$ as products of "mesonic" operators $(\bar{Q}^{++\hat{+}} Q^{++\hat{+}})^{\check{a}}_{\check{b}}$ and decoupled scalars of the "second" vector multiplet. The dual interpretation in the bulk is that as $\check{g} \to 0$ KK modes on $S^3/\mathbb{Z}_2$ become multi-particle states of open strings. The flavor singlet sector of $\mathcal{N} = 2$ SCQCD does not "see" the $S^3/\mathbb{Z}_2$ portion of the geometry. We regard the "loss" of $S^3/\mathbb{Z}_2$ as a first hint that the string dual to the singlet sector of $\mathcal{N} = 2$ SCQCD should be a *sub-critical* string background. The S^1 factor on the other hand is preserved.

We may also ignore the relation of $\mathcal{N} = 2$ SCQCD with the orbifold theory, and consider the protected states (5.44) at face value: they are immediately suggestive of Kaluza-Klein reduction on a circle. The dual geometry must contain an AdS_5 factor to implement the conformal symmetry, and an S^1 factor to generate the two KK towers dual to $\{\mathrm{Tr}\, T\, \phi^\ell\}$ and $\{\mathrm{Tr}\, \phi^{\ell+2}\}$. Moreover the radii of the AdS_5 and S^1 factor must be equal. Indeed Kaluza-Klein reduction on S^1 gives a mass spectrum $m^2 \sim \ell^2/R^2_{S^1}$ (for ℓ large), and correspondingly a conformal dimension $\Delta \cong m R_{AdS} \cong \ell \frac{R_{AdS}}{R_{S^1}}$. Inspection of (5.44) gives $R_{AdS} = R_{S^1}$. The isometry of S^1 is interpreted as the $U(1)_r$ R-symmetry. On the other hand, there is no hint in the protected spectrum (5.44) of a "geometrically" realized $SU(2)_R$. The relation with the interpolating theory makes it clear that indeed the geometric factor $S^3/\mathbb{Z}_2$, with isometry $SU(2)_R \otimes SO(3)_L$, is lost in the limit $\check{g} \to 0$.

We can further split the "naive" spectrum (5.44) into the primaries $\{\mathrm{Tr}\, \mathcal{M}_{\mathbf{3}}, \mathrm{Tr}\phi^\ell\}$ and the primaries $\{\mathrm{Tr} T \phi^\ell\}$. The first set, of course, is isomorphic to the twisted states of the orbifold, and can be precisely matched with the KK reduction on $AdS_5 \times S^1$ of one tensor multiplet of $(2,0)$ chiral supergravity. A first guess is that the primaries $\{\mathrm{Tr}\, T \phi^\ell\}$ correspond to the KK reduction of the $6d$ $(2,0)$ *gravity* multiplet on $AdS_5 \times S^1$, but this is incorrect. The zero modes of the $6d$ gravity multiplet correctly match the stress-energy tensor multiplet (whose bottom component is the primary $\mathrm{Tr}\, T$), but there are not enough states in the higher KK modes to match the states in the $\mathrm{Tr}\, T\phi^\ell$ for $\ell > 0$. This could have been anticipated by tracing the origin of the states $\{\mathrm{Tr}\, T\phi^\ell\}$ in the orbifold theory: the dual supergravity states have no angular momentum on $S^3/\mathbb{Z}_2$ in the parametrization (4.56), but they are extended in the remaining *seven* dimensions. So a better guess is that the states $\{\mathrm{Tr}\, T\phi^\ell\}$ should have an interpretation in seven-dimensional supergravity.

In summary, with some hindsight, the "naive" spectrum appears to indicate a sub-critical string background, with seven "geometric" dimensions, and containing both an AdS_5 and an S^1 factor, with $R_{AdS} = R_{S^1}$.

The extra exotic protected states teach another important lesson. They arise in the limit $\check{g} \to 0$ from long multiplets on the interpolating theory that hit the unitarity bound and split into short multiplets. In the dual string theory, this means that a fraction of the massive closed string states become massless in the limit $\check{g} \to 0$. It is a substantial enough fraction to give rise to a Hagedorn degeneracy, as we saw in section 4.4.4. This has the crucial implication that *the dual description of $\mathcal{N} = 2$ SCQCD is never in terms of supergravity*, since even in the limit $\lambda \equiv g_{YM}^2 N_c \to \infty$ there is an infinite tower of "light" closed string states, with a mass of the order of the AdS scale. However it seems plausible to conjecture that there is also a second sector of "heavy" string states that decouple for $\lambda \to \infty$.

The picture that we have in mind is the following. There are really two 't Hooft couplings in the interpolating theory, $\lambda \equiv g_{YM}^2 N_c$ and $\check{\lambda} \equiv \check{g}_{YM}^2 N_c$, and correspondingly *two* effective string tensions $T_s \sim 1/l_s^2$ and $\check{T}_s \sim 1/\check{l}_s^2$. The idea of two effective string tensions is intuitive from the spin chain viewpoint, since the bifundamental fields separate different regions of the chain, occupied by adjoint fields of the two different groups $SU(N_c)$ and $SU(N_{\check{c}})$ and thus governed by the two different gauge couplings. At the orbifold point, of course, $\lambda = \check{\lambda}$. In the limit in which the unique 't Hooft coupling of the orbifold theory is sent to infinity the string length goes to zero in AdS units according to the usual AdS/CFT dictionary $R_{AdS_5}/l_s \sim \lambda^{1/4}$, leading to the decoupling of all massive string states. To approach $\mathcal{N} = 2$ SCQCD we are interested in what happens as λ is kept large, but $\check{\lambda}$ is sent to zero. At present we do not know how to modify the AdS/CFT dictionary in this limit. The most naive extrapolation would suggest a hierarchy between two different scales: there should be one sector of closed string states governed by $l_s \sim \lambda^{-1/4} R_{AdS}$ and thus very massive, and another governed by $\check{l}_s \sim R_{AdS}$ and thus light. The latter would correspond to the exotic protected states revealed by the index.

4.6 Brane Constructions and Non-Critical Strings

The interpolating SCFT has a dual description as IIB on $AdS_5 \times S^5/\mathbb{Z}_2$, but this description breaks down in the $\check{g} \to 0$ limit that we wish to study. We must describe the theory in a different duality frame. We will argue that the correct description is in terms of a *non-critical* superstring background. In this section we reconsider the IIB brane setup leading to the interpolating SCFT, and review how it can be T-dualized to a IIA Hanany-Witten setup (see *e.g.* [163] for a review). The T-dual frame allows for a more transparent understanding of the limit $\check{g} \to 0$, as a double-scaling limit in which two brane NS5 collide while the string coupling is sent to zero. In this limit the near-horizon dynamics is described a non-critical string background, which (before the backreaction of the D-branes) admits an exact worldsheet description as $\mathbb{R}^{5,1}$ times $SL(2)_2/U(1)$, the supersymmetric cigar CFT. We are led to identify the near-horizon backreacted background, where D-branes are replaced by flux, with the dual of $\mathcal{N} = 2$ SCQCD.

4.6.1 Brane Constructions

The interpolating SCFT arises at the low-energy limit on N_c D3 branes sitting at the orbifold singularity $\mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$. The blow-up modes of the orbifold are set to zero, since they correspond to massive deformations of the $4d$ field theory. The NSNS period β is related to g_{YM} and $\check{g}_{YM}$ by the dictionary (4.54). As $\beta \to 0$ the D-strings obtained by wrapping $D3$ branes on the blow-down cycle of the orbifold become tensionless and string perturbation theory breaks down. It is useful to T-dualize to a IIA Hanany-Witten description, where the deformation β can be pictured more easily. To perform the T-duality we should first replace the A_1 singularity $\mathbb{R}^4/\mathbb{Z}_2$ with its S^1 compactification, a two-center Taub-NUT space of radius $\tilde{R}$. The local singularity is recovered for $\tilde{R} \to \infty$.

Recall, more generally, that the S^1 compactification of the resolved A_{k-1} singularity is a k-center Taub-NUT, a hyperkäler manifold which can be concretely described as an S^1 fibration of $\mathbb{R}^3$. Let $\tilde{\tau}$ be the coordinate of the S^1 fiber and $\boldsymbol{y}$ the coordinates of the $\mathbb{R}^3$ base. The S^1 fiber degenerates to zero size at k points on the base, $\boldsymbol{y} = \boldsymbol{y}^{(a)}$, $a = 1, \dots k$, and goes to a finite radius $\tilde{R}$ at the infinity of $\mathbb{R}^3$. (Topologically the S^1 is non-trivially fibered over the S^2 boundary of $\mathbb{R}^3$, with monopole charge k.) Rotations of the $\boldsymbol{y}$ coordinates are interpreted as the $SU(2)$ symmetry that rotates the complex structures. From the viewpoint of the worldvolume theory of D3 branes probing the singularity, this is the $SU(2)_R$ R-symmetry. The geometry has also an extra $U(1)_L$ symmetry acting as angular rotation in the S^1 fiber.[10] (Finally the $U(1)_r$ of the $4d$ gauge theory corresponds to an isometry outside the Taub-NUT, namely rotations in the $\mathbb{R}^2$ factor of $\mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$.)

The metric of a k-center Taub-NUT space has $3(k-1)$ non-trivial hyperkähler moduli (after setting say $\boldsymbol{y}^{(1)} \equiv 0$ by an overall translation), which correspond to the blow-up modes of the $(k-1)$ cycles – one $SU(2)_R$ triplet for each cycle. In the string sigma model one needs to further specify the periods of B_{NSNS} and B_{RR} on each cycle,which gives two extra real moduli for each cycle, singlets under $SU(2)_R$. Altogether the $5 = 3+1+1$ moduli for each cycle are the scalar components of a tensor multiplet living in the six transverse directions to the Taub-NUT (or ALE) space. T-duality along the $\tilde{\tau}$ direction yields a string background with non-zero NSNS H flux and non-trivial dilaton, which is interpreted as the background produced by k NS5 branes [131, 164]. The NS5 branes sit at $\boldsymbol{y}^a$ in the $\mathbb{R}^3$ directions, and are localized on the dual circle.[11] The NSNS periods map to the relative angles of the NS5 branes on the dual circle.

Let us apply these rules to our case. We start on the IIB side with the configuration

IIB	x_0	x_1	x_2	x_3	x_4	x_5	$\tilde{\tau}$	y_1	y_2	y_3
TN_2							$\times$	$\times$	$\times$	$\times$
D3	$\times$	$\times$	$\times$	$\times$						

The two-center Taub-NUT TN_2 has radius $\tilde{R}$, vanishing blow-up modes ($\boldsymbol{y}^{(1)} = \boldsymbol{y}^{(2)} = 0$) and $\int_{S^2} B_{NSNS} = \beta$. T-duality gives the IIA configuration

IIA	x_0	x_1	x_2	x_3	x_4	x_5	τ	y_1	y_2	y_3
2 NS5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
D4	$\times$	$\times$	$\times$	$\times$			$\times$			

The two NS5 branes, at the origin of $\mathbb{R}^3$ are localized on the dual circle of radius $R = \alpha'/\tilde{R}$ and at an angle $2\pi\beta$ from each other. The string couplings are related as

$$g_s^A = \frac{R}{l_s} g_s^B = \frac{l_s}{\tilde{R}} g_s^B \,. \tag{4.57}$$

[10]The A_1 singularity ($k=2$, $\boldsymbol{y}_a = 0$, $\tilde{R} = \infty$) has a symmetry enhancement $U(1)_L \to SO(3)_L$, whose field theory manifestation is the $SO(3)_L$ global symmetry of the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM, discussed in section 5.2.2. The symmetry is broken to $U(1)_L$ for finite $\tilde{R}$; the full $SO(3)_L$ is recovered in the infrared.

[11]Naive application of the T-duality rules gives NS5 branes smeared on the dual circle. The localized solution arises after taking into account worldsheet instanton corrections [165].

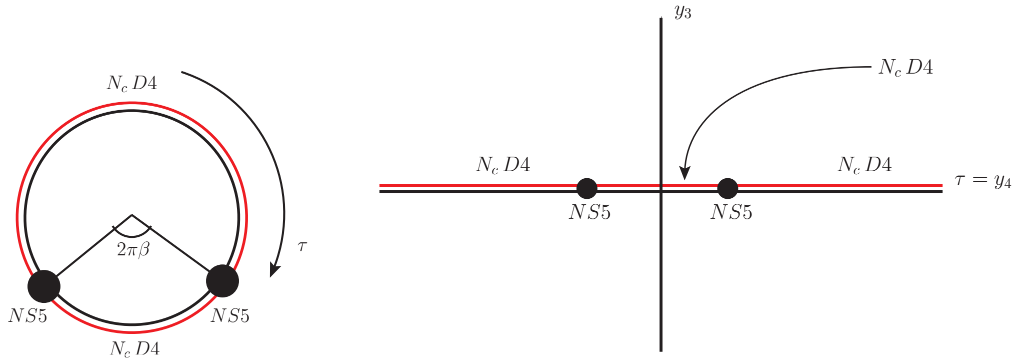


Figure 4.4: Hanany-Witten setup for the interpolating SCFT (on the left) and for $\mathcal{N} = 2$ SCQCD (on the right).

T-duality maps the N_c $D3$ branes on the IIB side (which can also be thought as *two stacks* of fractional branes [166]) to two stacks of N_c D4 branes on the IIA side, each stack ending on the two NS5 branes and extended along either arc segment of the τ circle (see Figure 4.4). This is the familiar Hanany-Witten setup for the $\mathbb{Z}_2$ orbifold field theory. The four-dimensional field theory living on the non-compact directions 0123 decouples from the higher dimensional and stringy degrees of freedom in the limit

$$\begin{aligned} & g_s^A \to 0 \quad l_s \to 0\,, \quad R \to 0\,, \\ & \text{with } \frac{\beta R}{2\pi g_s^A l_s} \equiv \frac{1}{g_{YM}^2} \text{ and } \frac{(1-\beta)R}{2\pi g_s^A l_s} \equiv \frac{1}{\check{g}_{YM}^2} \text{ fixed}\,. \end{aligned} \tag{4.58}$$

At this stage we are still keeping both gauge couplings g_{YM} and $\check{g}_{YM}$ finite. If L is the 4d length scale above which the field theory is a good description, we have the hierarchy of scales

$$L \gg l_s \gg R \cong g_s^A l_s\,. \tag{4.59}$$

Again, rotations in the y_i directions correspond to the $SU(2)_R$ R-symmetry of the $\mathcal{N} = 2$ $4d$ field theory, while rotations in the 45 plane correspond to the $4d$ $U(1)_r$ symmetry. Finally the $U(1)_L$ symmetry, which was related to momentum conservation along the S^1 fiber in the IIB setup, is T-dualized to *winding* symmetry in the Hanany-Witten IIA setup. It gets enhanced in the infrared to the $SO(3)_L$ symmetry of the $4d$ field theory.

4.6.2 From Hanany-Witten to a Non-Critical Background

The limit $\breve{g}_{YM} \to 0$ (with g_{YM} fixed) can now be understood more geometrically: it corresponds to $\beta \to 0$, the limit of coincident NS5 branes. In this limit we can ignore the periodicity of the τ direction and think of two NS5 branes located in $\mathbb{R}^4$ at a distance $\tau_0 \equiv 2\pi\beta R$ from each other, with $\tau_0 \to 0$. There is a stack of N_c D4 branes suspended between the two NS5s and two stacks of N_c semi-infinite D4s, ending on either NS5 brane. As is well-known, $k \geq 2$ *coincident* NS5 branes generate a string frame background with a strongly coupled near horizon region – the string coupling blows up down the infinite throat towards the location of the branes. The throat region is the CHS background [167]

$$\mathbb{R}^{5,1} \times SU(2)_k \times \mathbb{R}_\rho \,, \quad \text{with dilaton } \Phi = -\frac{\rho}{\sqrt{2k}} \,, \tag{4.60}$$

where ρ is the radial direction (the NS5 branes are located at $\rho = -\infty$). The supersymmetric $SU(2)_k$ WZW model describes the angular S^3; it arises by combining the bosonic $SU(2)_{k-2}$ and three free fermions ψ_i, $i = 1, 2, 3$, which make up an $SU(2)_2$. This description breaks down for large negative ρ where the string coupling e^Φ is large. In Type IIA (our case), we must uplift to M-theory to obtain the correct description of the near horizon region strictly coincident NS5 branes. However, what we are really interested in is bringing the branes together in a controlled fashion, simultaneously turning off the string coupling g_s^A. We can break the limit (4.58) into two steps:

(i) We first take the double scaling limit [134, 135]

$$\tau_o \to 0 \,, \quad g_s^A \to 0 \,, \quad \frac{\tau_0}{l_s g_s^A} \equiv \frac{1}{g_{eff}} \sim \frac{1}{g_{YM}^2} \text{ fixed} \,, \quad l_s \text{ fixed.} \tag{4.61}$$

(ii) We then send $l_s \to 0$.

Let us first consider the purely closed background without the D4 branes. The double-scaling limit (i) has been studied in detail in [134, 135], precisely with the motivation of avoiding strong coupling. In this limit the region near the location of the NS5 branes decouples from the rest of the geometry and is described by a perfectly regular background of *non-critical* superstring theory [134, 135]. To describe the background as a worldsheet CFT it is useful to perform a further T-duality, in an angular direction around the branes. If $\tau \equiv y_4$ is the direction along which the branes are separated, we pick say the

$y_3 y_4$ plane and perform a T-duality around $\chi = \arctan y_3/y_4$. The result is the exact IIB background

$$\mathbb{R}^{5,1} \times SL(2)_2/U(1)/\mathbb{Z}_2 \,. \tag{4.62}$$

The $\mathbb{Z}_2$ orbifold implements the GSO projection. The Kazama-Susuki coset $SL(2)_2/U(1)$ is the supersymmetric Euclidean 2d black hole, or supersymmetric cigar, at level $k = 2$. The corresponding sigma-model background is

$$ds^2 = d\rho^2 + \tanh^2(\frac{Q\rho}{2})d\theta^2 + dX^\mu dX_\mu \qquad \theta \sim \theta + \frac{4\pi}{Q} \tag{4.63}$$

$$\Phi = -\ln\cosh(\frac{Q\rho}{2}), \qquad B_{ab} = 0 \,. \tag{4.64}$$

In appendix E we review several properties of this background. An equivalent (mirror) description of $SL(2)/U(1)$ is as the $\mathcal{N} = 2$ superLiouville theory [168]. The two descriptions are manifestly equal in the asymptotic region $\rho \to \infty$, where they reduce to ($S^1\times$ linear dilaton). At large ρ, the leading perturbation away from the linear dilaton takes a different form in the semiclassical cigar and Liouville descriptions, but in the complete quantum description both the cigar and Liouville perturbations are present. The cigar description is more appropriate for $k \to \infty$, since in this limit the cigar perturbation dominates at large ρ over the Liouville perturbation, while the Liouville description is more appropriate for $k \to 0$, where the opposite is true. For $k = 2$ both descriptions are precisely on the same footing – the cigar and Liouville perturbations are present with equal strength and are in fact rotated into each another by the $SU(2)_R$ symmetry [143]. For $k = 2$ the asymptotic radius of the cigar is $\sqrt{2\alpha'}$, which is the free fermion radius, implying that for large ρ the angular coordinate θ and its superpartner ψ_θ can then be replaced by three free fermions ψ_i, or equivalently by $SU(2)_2$. The cigar background is thus a smoothed out version of the CHS background (4.60) – the negative ρ region of CHS has been cut-off and the string coupling is now bounded from above by its value g_{eff} at the tip of the cigar.[12]

[12] As an aside, it is worth recalling the generalization of this discussion to k NS5 branes, equally spaced on a contractible circle in the $y_3 y_4$ plane. T-duality around the angular coordinate χ produces the background [134]

$$\mathbb{R}^{5,1} \times (SL(2)_k/U(1) \times SU(2)_k/U(1))/\mathbb{Z}_k \,. \tag{4.65}$$

The central charges are of the Kazama-Susuki cosets are

$$c(SL(2)_k/U(1)) = 3 + \frac{6}{k}\,, \quad c(SU(2)_k/U(1)) = 3 - \frac{6}{k}\,. \tag{4.66}$$

To summarize, we started from a IIA configuration of two separated NS5 branes in flat space, and took the double-scaling limit (4.61). In this limit the near-horizon region decouples from the asymptotic flat space region, and is described by the exact non-critical IIB background (4.62). (The switch from IIA and IIB is due to the angular T-duality along χ.) The reduction of degrees of freedom from critical to non-critical strings happens because we are focusing on a subsector of the full theory, namely the degrees of freedom near the singularity produced by the colliding NS5 branes. The transverse direction ρ can be thought of as a worldsheet RG scale, with the asymptotically flat region at large ρ playing the role of the UV and the cigar geometry playing the role of the IR – in focusing to the near horizon region we lose the asymptotic flat space degrees of freedom. In particular, what remains of the transverse S^3 is just the "stringy" $SU(2)_2$ associated with the free fermions ψ_i, $i = 1, 2, 3$.

We can easily follow the fate of the D-branes through the double scaling limit and T_χ-duality: the D4 branes suspended between the two NS5s become D3 branes localized at the tip of the cigar, while the semi-infinite D4 branes become D5 branes extended on the cigar. This at least is the intuitive geometric picture. Since the cigar background has string-size curvature near the tip, a more appropriate description of the D-branes is in terms of the exact boundary states. Boundary states for the Kazama-Susuki coset $SL(2)/U(1)$ (equivalently, for the superLiouville CFT) have been studied in several papers [169–173], following the construction of boundary states in bosonic Liouville theory, and used in $\mathcal{N} = 1$ non-critical holography in [125, 126, 128]. There are indeed natural candidates for the two types of cigar D-branes that we need. The branes localized near the tip of the cigar are the analog of Liouville ZZ [174] branes, while the branes extended along the cigar are the analog of the Liouville FZZT [175, 176] branes. The non-critical string setup can be summarized by the following diagram:

IIB	x_0	x_1	x_2	x_3	x_4	x_5	ρ	θ
D3	$\times$	$\times$	$\times$	$\times$				
D5	$\times$	$\times$	$\times$	$\times$			$\times$	$\times$

We could have taken this as our starting point. The theory on the worldvolume of the N_c D3 branes (the "color" branes) reduces for energies much smaller than the string scale to $\mathcal{N} = 2$ $SU(N_c)$ SYM, coupled to $N_f = 2N_c$ hypermultiplets arising from the open strings stretched between the D3s and the "flavor" D5s. This is true by construction, since we obtained this non-critical setup as a

The CFT (4.65) In the semiclassical limit $k \to \infty$ we have a weakly curved "geometric" $10d$ background, while in the opposite limit $k = 2$ the curvature is string scale, the $SU(2)/U(1)$ piece disappears and we have the "non-critical" string background (4.62).

limit of a well-known brane realization of the same field theory, and it could also be checked directly, by examining the open string spectrum and preserved supersymmetries.

To decouple the field theory we need to take $l_s \to 0$ (step (ii) in our previous discussion of the field theory limit). This amounts on the gravity side to the near-horizon limit of the geometry produced by the D-branes. By the usual arguments [1], we are led to conjecture that the resulting non-critical string background is dual to $\mathcal{N} = 2$ SCQCD.

4.7 Towards the String Dual of $\mathcal{N} = 2$ SCQCD

The explicit construction of the background after the backreaction of the D-branes is left for future work. In this section we outline a line of attack, based on a $7d$ "effective action" which we identify as maximal supergravity with $SO(4)$ gauging. In fact several features of the background can be determined from symmetry considerations alone, and just assuming that a solution exists we will find a nice qualitative agreement with the bottom-up field theory analysis, notably in the protected spectrum of operators.

4.7.1 Symmetries

Let us start by recapitulating the symmetries. The obvious bosonic symmetries of the closed string background (4.62) (the background before introducing D-branes, henceforth the "cigar background") are the Poincaré group in $\mathbb{R}^{5,1}$ and the $U(1)$ isometry of the θ circle. In fact since as $\rho \to \infty$ the θ circle is at the free fermion radius, there is an asymptotic "stringy" enhancement of the $U(1)$ symmetry to $SU(2)_{\psi_i} \times SU(2)_{\tilde{\psi}_i} \cong SO(4)$. At finite ρ the cigar and super-Liouville interactions break this symmetry to the *diagonal* $SU(2)$. This has a clear geometric interpretation in the HW picture (before the angular T_χ-duality) of the two colliding NS5 branes: the $SO(4)$ symmetry is the isometry of the transverse four directions to two *coincident* NS5 branes; separating the branes along one direction ($\tau = y_4$ in the picture on the right of Figure 4) breaks the symmetry to $SO(3) \cong SU(2)$ (rotations of y_i, $i = 1, 2, 3$). This surviving diagonal $SU(2)$ is interpreted as the $SU(2)_R$ R-symmetry of the $\mathcal{N} = 2$ $4d$ gauge theory. Adding the color D3 branes and the flavor D5 branes breaks the $6d$ Poincaré symmetry to $4d$ Poincaré symmetry in the directions x_m, $m = 0, 1, 2, 3$, times the rotational symmetry in the 45 plane. The latter is interpreted as the $U(1)_r$ R-symmetry of the gauge theory. Note that the branes preserve the same (diagonal) $SU(2)$ as the cigar and super-Liouliville interactions. This is again transparent in the picture of colliding NS5 branes, since both the "compact" D4 branes and the "non-compact" D4 branes, which become respectively the color D3s and the flavor D5s after T_χ-duality, are oriented along the same $\tau = y_4$ direction in which the two NS5s are separated. Finally we should mention the fermionic symmetries. As we review in appendix E, the background (4.62) has 16 real supercharges, corresponding to the $(2,0)$ Poincaré superalgebra in $\mathbb{R}^{5,1}$. Adding the D-branes breaks the supersymmetry in half, so that 8 Poincaré supercharges survive (that D3s and D5s break the *same* half is again obvious in the T-dual frame where they are both (parallel) D4 branes). Taking the near-horizon geometry is expected to give the usual supersymmetry enhancement, restoring a total of 16 supercharges that form the $\mathcal{N} = 4$ AdS_5 superalgebra (isomorphic to the $\mathcal{N} = 2$ $4d$ superconformal algebra).

4.7.2 The cigar background and $7d$ maximal $SO(4)$-gauged supergravity

The cigar background (4.62) is analyzed in some detail in appendix E, which the reader is invited to read at this point. Let us summarize some of the relevant points. The physical spectrum of the cigar background consists of: (i) normalizable states localized at the tip of the cigar $\rho \sim 0$, living in $\mathbb{R}^{5,1}$: they fill a tensor multiplet of $(2,0)$ $6d$ supersymmetry; (ii) delta-function normalizable states, corresponding to plane waves in the radial ρ direction; (iii) non-normalizable vertex operators, supported in the large ρ region.

We are only interested in the cigar background as an intermediate step towards the background dual to $\mathcal{N} = 2$ SCQCD, obtained in the near-horizon limit of the D3/D5 brane configuration. A possible strategy is to use the cigar background, which admits an exact CFT description, to derive a spacetime "effective action". The spacetime action is expected to be background independent and should admit as classical solutions *both* the cigar background *and* the background dual to $\mathcal{N} = 2$ SCQCD. (In this respect, the cigar background is analogous to the $10d$ flat background of IIB string theory, which is described at low energies by $10d$ IIB supergravity; *another* solution of IIB supergravity is the $AdS_5 \times S^5$ background dual to $\mathcal{N} = 4$ SYM.) For the purpose of deriving an "effective action" the relevant part of the spectrum is (ii), the continuum of plane-wave states. Performing a KK reduction on the θ circle, the plane-wave states are naturally organized in a tower of increasing $7d$ mass (which gets contribution both from the θ momentum and from string oscillators). There is is no real separation of scales between the lowest mass level and the higher ones, because the linear dilaton has string-size gradient. Nevertheless the states belonging to lowest level are special: although they obey "massive"

$7d$ wave-equations, this is an artifact of the linear dilaton; the counting of degrees of freedom is that of massless $7d$ states because of gauge invariances.

Remarkably, we find that for large ρ the lowest-mass level of the continuum spectrum is described by seven dimensional *maximally supersymmetric* supergravity (32 supercharges), but with a non-standard gauging: only an $SO(4)$ of the full $SO(5)$ R-symmetry is gauged. This supergravity has been constructed only quite recently [177, 178]. The maximal supersymmetry (which, as we shall see momentarily, is spontaneously broken to half-maximal, consistently with our previous counting) can be understood as follows. After fermionizing the angular coordinate θ, we have a total of ten left-moving fermions, ψ_μ, $\mu = 0 \ldots 5$ along $\mathbb{R}^{5,1}$, ψ_ρ and ψ_i, $i = 1, 2, 3$ (the last three corresponding to $\partial\theta$, ψ_θ), and similarly ten right-moving fermions. So the construction of the lowest-level physical states of our sub-critical theory is entirely isomorphic to the construction of the massless states of the standard critical IIB string theory, except of course that the momenta are now seven dimensional. The $SO(4)$ that is being gauged is the asymptotic $SU(2)_{\psi_i} \times SU(2)_{\tilde{\psi}_i} \cong SO(4)$ that we have mentioned. It turns out that unlike the standard $SO(5)$-gauged $7d$ sugra, which admits the maximally supersymmetric AdS_7 vacuum, the $SO(4)$-gauged theory breaks half of the supersymmetry spontaneously. The scalar potential of the $SO(4)$-gauged theory does not admit a stationary solution but only a domain wall solution [177, 178], which is nothing but the linear dilaton background, with 16 unbroken supercharges – the $6d$ $(2, 0)$ super-Poincaré invariance discussed earlier.

Incidentally, we believe that this is a general phenomenon: non-critical superstrings in various dimensions must admit (non-standard) gauged supergravities as their spacetime "effective actions", in the sense that we have discussed. It may be worth to explore this connection systematically.

4.7.3 An Ansatz

We expect the $SO(4)$-gauged $7d$ sugra that describes the "massless" fields to be a useful tool, though not a perfect one because we know that the higher levels are not truly decoupled. The next step is to look for a solution of this supergravity with all the expected symmetries. In the seven dimensional theory the $SU(2)_R$ symmetry is not realized geometrically – its last remnant was the (string-size) θ circle, over which we have KK reduced to get down to $7d$. On the other hand, the $U(1)_r$ symmetry *is* geometric, and conformal symmetry is expected to arise in the near-horizon geometry, which must then contain both an S^1 and an AdS_5 factor. The most general ansatz for the $7d$

metric with the expected isometries is

$$ds^2 = f(y)ds^2_{AdS_5} + g(y)d\varphi^2 + C(y)dy^2\,. \qquad (4.67)$$

Here φ is the angular coordinate of the S^1 associated to $U(1)_r$ isometry, while the y has range in a finite interval, say $y \in [0,1]$. Restoring the θ coordinate, the non-critical background would have the form

$$ds^2 = f(y)ds^2_{AdS_5} + g(y)d\varphi^2 + h(y)d\theta^2 + C(y)dy^2\,. \qquad (4.68)$$

Comparing with the brane setup, which is again

IIB	x_0	x_1	x_2	x_3	x_4	x_5	ρ	θ
D3	$\times$	$\times$	$\times$	$\times$				
D5	$\times$	$\times$	$\times$	$\times$			$\times$	$\times$

we identify φ is angular coordinate in the 45 plane, while y could be taken to be a relative angle between the radial distance in the 45 plane and the radial distance ρ along the cigar, $y = \frac{2}{\pi}\arctan(\rho/\sqrt{x_4^2+x_5^2})$. The D5 branes sit at $y = 1$.

The program is then to look for a solution (4.67) of the $SO(4)$-gauged $7d$ supergravity, possibly allowing for singular behavior at the original location $y = 1$ of the flavor branes. For fixed N_c and $N_f(= 2N_c)$, we expect a one-parameter family of solutions, because the 't Hooft coupling λ is exactly marginal – the AdS scale should be a modulus, as in the familiar $AdS_5 \times S^5$ case. The color (D3) branes are magnetically charged under the RR one-form $C^{(2,2)}_{\hat{\mu}}$ (see Table 18) and the flavor branes (which are actually D4 branes from the viewpoint in the $7d$ theory) are magnetically charged under the RR zero-form $C^{(2,2)}$. The corresponding fluxes will be turned on in the solution. As usual the color branes will be completely replaced by flux. Our analysis of the large N Veneziano limit suggests that new effective closed string degrees of freedom, dual to "generalized single-trace" operators, arise from the resummation of open string perturbation theory. This favors the scenario in which also the flavor branes are completely replaced by flux. This fundamental issue would be illuminated by an explicit solution.

The program of finding a supergravity background for $\mathcal{N} = 2$ SCQCD was also discussed in critical IIB supergravity [179] and in $11d$ supergravity [110], but no explicit solutions are yet known. It would be interesting to understand the relation of these approaches with our sub-critical setup. In particular a somewhat singular limit of solutions found in [110] should correspond to $\mathcal{N} = 2$ SCQCD, and it would be nice to understand this in detail.

4.7.4 Spectrum

Already at this stage we can recognize that the top-down (string theory) and bottom-up (field theory) analyses are in qualitative agreement. Both suggest that the string dual of $\mathcal{N} = 2$ SCQCD is a sub-critical background with an AdS_5 and an S^1 factor. In the field theory protected spectrum we found a sharp difference between the $U(1)_r$ and $SU(2)_R$ factors of the R-symmetry group: there are towers of states with increasing $U(1)_r$, but no analogous towers for $SU(2)_R$. The brane construction confirms the natural interpretation of this fact: while the $U(1)_r$ is realized geometrically as the isometry of a "large" S^1_φ, with its towers of KK modes, the $SU(2)_R$ is associated to the string-sized S^1_θ of the cigar (and in fact the very enhancement from the θ isometry $U(1) \subset SU(2)_R$ to the full $SU(2)_R$ is a stringy phenomenon). The "naive" part of the protected spectrum nicely matches:

(i) The multiplets built on the primaries $\{\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}\,, \mathrm{Tr}\,\phi^{2+\ell}\}$ correspond to the KK modes on S^1_φ of the $6d$ tensor multiplet (see appendix D): these are the truly normalizable states of the cigar background, localized at the tip of the cigar ($y = 0$ in the parametrization (4.67)).

(ii) The multiplets built on $\{\mathrm{Tr}\,T\phi^\ell\}$ correspond to the KK modes on S^1_φ of the bulk $7d$ $SO(4)$-gauged supergravity: this is the lowest level of the plane-wave spectrum of the cigar background. While we have not performed a detailed KK reduction, for which the precise geometry is required, it is clear that the bulk graviton maps to the stress tensor, which is part of the $\mathrm{Tr}\,T$ multiplet, and that the ℓ-th KK mode of the graviton maps to the unique spin 2 state in the $\mathrm{Tr}\,T\phi^\ell$ multiplet. Supersymmetry should do the rest.

The "extra" protected states of the field theory must correspond to light string states in the bulk, with mass of order of the AdS scale, but we do not know how to establish a more precise dictionary at this point. We have suggested in section 6 that the string theory dual to $\mathcal{N} = 2$ SCQCD may contain two sectors of string states, in correspondence with the two effective string scales l_s and $\check{l}_s$ of the interpolating theory: a light sector, controlled by $\check{l}_s \sim R_{AdS}$ for all λ, and a heavy sector, controlled by $l_s \ll R_{AdS}$ for $\lambda \gg 1$. The string length of the cigar background should be identified with l_s, so the massive string states of the cigar background would correspond to the heavy sector and decouple for large λ. The light sector is more mysterious. A tantalizing speculation is that the light states correspond to cohomology classes with non-normalizable $\mathcal{N} = 2$ Liouville dressing, *i.e.* supported at large ρ (operators of type (iii) in the list of section E.4). It is clearly possible to tune the ρ-momentum to achieve "massless" six-dimensional states, at the expense of making them non-normalizable in the ρ direction. Perhaps the extra protected states of $\mathcal{N} = 2$ SCQCD are somewhat analogous to the discrete states of the $c = 1$ matrix model, which are indeed dual to vertex operators with non-normalizable Liouville dressing.[13]

If indeed $l_s \ll R_{AdS}$ for large λ, the $7d$ supergravity, while not capturing the whole theory even in this limit (as we know from the existence of the extra protected states), may still offer a useful description of a subsector.

4.8 Discussion

We may now look back to section 1, at the list of special features shared by all $4d$ CFTs for which an explicit string dual is presently known. We have studied in some detail perhaps the most symmetric theory that violates property (i) (since $a \neq c$ at large N) and property (ii) (since it has a large number of fields in the fundamental representation), while still satisfying the nice simplifying feature (iv) of an exactly marginal coupling λ. We have argued that the dual string theory is not ten dimensional, thus violating (iii), and proposed a sub-critical string dual in eight dimensions (including the string-size θ). The theory emerges as a limit of a family of superconformal field theories that have $a = c$ and admit ten dimensional string duals. In this singular limit some fields decouple on the field theory side, leading to $a \neq c$, while on the string side two dimensions are lost (counting θ as a dimension). It is tempting to link the two phenomena. The natural speculation is that the $4d$ gauge theories in the "$\mathcal{N} = 4$ universality class" (which among other things are characterized by $a = c$) have $10d$ string dual, while theories with "genuinely" fewer supersymmetries have sub-critical duals. A plausible pattern for (susy, dimension) is ($\mathcal{N}$, d) = (4, 10), (2,8), (1,6), (0,5). We have given evidence for the $\mathcal{N} = 2 \leftrightarrow d = 8$ connection, while [124, 125, 128] focused on $\mathcal{N} = 1 \leftrightarrow d = 6$.

Our example is in harmony with the no-go theorem that $a = c$ for all field theories with an AdS_5 gravity dual, since we argued that even for large λ the supergravity approximation to the dual of $\mathcal{N} = 2$ SCQCD cannot be entirely valid. The imbalance between a and c must arise from higher-curvature terms in the AdS_5 gravity theory [180]. We believe that the stringy origin of these higher curvature terms is the Wess-Zumino action of the flavor branes, as in the example studied in [26, 181]: the flavor Wess-Zumino terms were shown

[13] Alternatively, our idea of two effective string scales may be wrong, and the unique scale l_s may be of the order of R_{AdS} for all λ. In this case all anomalous dimensions would remain small for large λ. The extra protected states would be special only in that their anomalous dimension is exactly zero for all λ. This is certainly a logical possibility.

to generate $\mathcal{R}^2$ corrections to the $5d$ Einstein-Hillbert action, contributing at order $O(N_f/N_c)$ to $a-c$. In the example of [26, 181] $N_f \ll N_c$, while in our case $N_f \sim N_c$ and $a-c=O(1)$, but the mechanism must be the same. It is important to keep in mind that the higher-curvature terms from the WZ action are topological in nature and are on a different footing from the higher-curvature corrections due to the closed string sigma-model loops, which are instead suppressed by powers of l_s/R_{AdS}. So there is no contradiction in principle between our suggestion that for large λ the non-critical background has a string length $l_s \ll R_{AdS}$, and the fact that $a-c=O(1)$, since $a-c$ arises from the higher-curvature terms coming from the WZ action, since they are not suppressed.

It is worth pointing out a simple relation between our $\mathcal{N}=2$ story and the $\mathcal{N}=1$ story of [124, 125, 128], if we specialize their setup to $\mathcal{N}=1$ super QCD with $N_f=2N_c$, the Seiberg self-dual theory. This theory can be viewed as the $\check{g}\to 0$ limit of a family of $\mathcal{N}=1$ SCFTs with product gauge-group $SU(N_c)\times SU(N_{\check{c}})$; when the couplings are equal the family reduces to the Klebanov-Witten theory [106], which is dual to $AdS_5\times T^{1,1}$. This is entirely analogous to the relation between $\mathcal{N}=2$ SCQCD and the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM, and of course this is not a coincidence: the two-parameter family of $\mathcal{N}=1$ theories is obtained from the two-parameter family of $\mathcal{N}=2$ theories flowing in the IR by a *relevant* deformation. For $g=\check{g}$, this is the well-known RG flow from the $\mathbb{Z}_2$ orbifold to the KW theory triggered by $\mathrm{Tr}(\phi^2-\check{\phi}^2)$ [106]. Unlike the $\mathcal{N}=2$ family, for $\mathcal{N}=1$ the couplings are bounded from below and the family of $\mathcal{N}=1$ SCFTs is never weakly coupled. The exactly marginal coupling of the self-dual $\mathcal{N}=1$ super QCD is the coefficient of a quartic superpotential – it cannot be taken arbitrarily small but it can be taken arbitrarily large. Our analysis of appendix E should easily generalize to this case, to find the gauged supergravity describing the lightest modes of the continuum spectrum. Only an isolated supergravity solution exists [124] (for arbitrary $N_f\sim N_c$), but in the special case $N_f=2N_c$ a one-parameter family of solutions is expected. This is also confirmed by the vanishing of the dilaton tadpole when $N_f=2N_c$ [128]. It would be nice to understand this point better.

Clearly there are many open questions. The bottom-up analysis would be greatly enhanced if we could determine the large λ behavior of generic non-protected operators. This may eventually be possible if $\mathcal{N}=2$ SCQCD exhibits an all-loop integrable structure. In chapter 5 we find a preliminary hint of one-loop integrability. In the top-down approach, work is in progress to verify whether the ansatz (4.67) is indeed a solution of the $SO(4)$-gauged supergravity. It will be interesting to understand its physical implications, especially the role of the warping factors and their possible singularity at $y = 1$.

Chapter 5

The One-Loop Spin Chain of $\mathcal{N} = 2$ Superconformal QCD

5.1 Introduction

In this chapter we take the next step of the bottom-up (field theory) analysis, by evaluating the one-loop dilation operator in the scalar sector of $\mathcal{N} = 2$ SCQCD.

Perturbative calculations of anomalous dimensions have given important clues into the nature of $\mathcal{N} = 4$ SYM. They gave the first hint for integrability of the planar theory: the one-loop dilation operator in the scalar sector is the Hamiltonian of the integrable $SO(6)$ spin-chain – a result later generalized to the full theory and to higher loops, using the formalism of the asymptotic Bethe ansatz. Remarkably, the asymptotic S-matrix of magnon excitations in the field theory spin chain can be exactly matched with the analogous S-matrix for the dual string sigma model. Thus perturbative calculations open a window into the structure of the dual string theory.

It is natural to attempt the same strategy for $\mathcal{N} = 2$ SCQCD. The theory admits a large N expansion à la Veneziano: the number of colors N_c and the number of fundamental flavors N_f are both sent to infinity keeping fixed their ratio ($N_f/N_c \equiv 2$) in our case and the combination $\lambda = g_{YM}^2 N_c$. We focus on the flavor-singlet sector of the theory, which is a consistent truncation since flavor singlets close under operator product expansion. The usual large N factorization theorems apply: correlators of generalized multi-traces factorize into correlators of generalized single-traces. In particular, acting with the dilation operator on a generalized single-trace operator yields (at leading order in N) another generalized single-trace operator, so we may consistently diagonalize the dilation operator in the space of generalized single-traces. The dilation operator acting on generalized single-traces can then be interpreted, in the usual fashion, as the Hamiltonian of a closed spin chain. At one-loop, the spin chain is local (nearest neighbor interactions): the well-known relation between planarity and locality of the spin chain continues to hold in the Veneziano limit.

More insight is gained by viewing $\mathcal{N} = 2$ SCQCD as part of an "interpolating" $\mathcal{N} = 2$ superconformal field theory (SCFT) that has a product gauge group $SU(N_c) \times SU(N_{\check{c}})$, with $N_{\check{c}} \equiv N_c$, and correspondingly two exactly marginal couplings g and $\check{g}$. For $\check{g} \to 0$ one recovers $\mathcal{N} = 2$ SCQCD *plus* a decoupled free vector multiplet, while for $\check{g} = g$ one finds the familiar $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. We have evaluated the one-loop dilation operator for the whole interpolating theory, in the sector of operators made out of scalar fields. The magnon excitations of the spin chain and their bound states undergo an interesting evolution as a function of $\kappa = \check{g}/g$. For $\kappa = 0$ (that is, for $\mathcal{N} = 2$ SCQCD itself), the basic asymptotic excitations of the spin chain are linear combinations of the the adjoint impurity $\bar{\phi}$ and of "dimer" impurities $\mathcal{M}^a{}_b$ (we refer to them as dimers since they occupy two sites of the chain). From the point of view of the interpolating theory with $\kappa > 0$, these dimeric asymptotic states of $\mathcal{N} = 2$ SCQCD are s are *bound states* of two elementary magnons; the bound-state wavefunction localizes in the limit $\kappa \to 0$, giving an impurity that occupies two sites.

Armed with the one-loop Hamiltonian in the scalar sector, we can easily determine the complete spectrum of one-loop protected composite operators made of scalar fields. It is instructive to follow the evolution of the protected eigenstates as a function of κ, from the orbifold point to $\mathcal{N} = 2$ SCQCD. Some of these results were quoted with no derivation in the previous chapter 4, where they served as input to the analysis of the full protected spectrum, carried out with the help of the superconformal index.

An important question is whether the one-loop spin chain of $\mathcal{N} = 2$ SCQCD is integrable. The spin chain for the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM (which by definition has $\check{g} = g$) is known to be integrable. We find that as we move away from the orbifold point integrability is broken, indeed for general $\kappa = \check{g}/g$ the Yang-Baxter equation for the two-magnon S-matrix does not hold. Remarkably however the Yang-Baxter equation is satisfied again in the $\mathcal{N} = 2$ SCQCD limit $\kappa \to 0$. Ordinarily a check of the Yang-Baxter equation is very strong evidence in favor of integrability. In our case we should trivializes – what is really relevant is the S-matrix of their dimeric bound states. But since for infinitesimal κ the Yang-Baxter equation for the S-matrix the elementary magnons fails only infinitesimally, we can hope that the Yang-Baxter equation for the S-matrix of their dimeric bound states will also fail infinitesimally, and that the multi-particle S-matrix really factorizes $\kappa \to 0$. While this intuition is reason for optimism, it is no substitute for a careful analysis, which we leave for future work.

5.2 Lagrangian and Symmetries

In this section we briefly review the structure and symmetries of $\mathcal{N} = 2$ SCQCD, and its relation to the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. Much insight is gained by viewing $\mathcal{N} = 2$ SCQCD, which has *one* exactly marginal parameter (the $SU(N_c)$ gauge coupling g_{YM}), as the limit of a *two*-parameter family of $\mathcal{N} = 2$ superconformal field theories. This is the family of $\mathcal{N} = 2$ theories with product gauge group[1] $SU(N_c) \times SU(N_{\check{c}})$ and two bifundamental hypermultiplets; its exactly marginal parameters are the two gauge-couplings g_{YM} and $\check{g}_{YM}$. For $\check{g}_{YM} \to 0$ one recovers $\mathcal{N} = 2$ SCQCD *plus* a decoupled free vector multiplet in the adjoint of $SU(N_{\check{c}})$. At $\check{g}_{YM} = 0$, the second gauge group is interpreted as a subgroup of the global flavor symmetry, $SU(N_{\check{c}}) \subset U(N_f = 2N_c)$. For $\check{g}_{YM} = g_{YM}$, we have instead the familiar $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. Thus by tuning $\check{g}_{YM}$ we interpolate continuously between $\mathcal{N} = 2$ SCQCD and the $\mathcal{N} = 4$ universality class.

The a and c anomalies are constant, and equal to each other, along this exactly marginal line: at the end point $\check{g}_{YM} = 0$, the $SU(N_{\check{c}})$ vector multiplets decouples, accounting for the "missing" $a - c$ in $\mathcal{N} = 2$ SCQCD.

5.2.1 $\mathcal{N} = 2$ SCQCD

Our main interest is $\mathcal{N} = 2$ SYM theory with gauge group $SU(N_c)$ and $N_f = 2N_c$ fundamental hypermultiplets. We refer to this theory as $\mathcal{N} = 2$ SCQCD. Its global symmetry group is $U(N_f) \times SU(2)_R \times U(1)_r$, where $SU(2)_R \times U(1)_r$ is the R-symmetry subgroup of the superconformal group. We use indices $\mathcal{I}, \mathcal{J} = \pm$ for $SU(2)_R$, $i, j = 1, \ldots N_f$ for the flavor group $U(N_f)$ and $a, b = 1, \ldots N_c$ for the color group $SU(N_c)$.

Table 5.1 summarizes the field content and quantum numbers of the model: The Poincar é supercharges $\mathcal{Q}^{\mathcal{I}}_{\alpha}$, $\bar{\mathcal{Q}}_{\mathcal{I}\,\dot{\alpha}}$ and the conformal supercharges $\mathcal{S}_{\mathcal{I}\,\alpha}$, $\bar{\mathcal{S}}^{\mathcal{I}}_{\dot{\alpha}}$ are $SU(2)_R$ doublets with charges $\pm 1/2$ under $U(1)_r$. The $\mathcal{N} = 2$ vector multiplet consists of a gauge field A_μ, two Weyl spinors $\lambda^{\mathcal{I}}_{\alpha}$, $\mathcal{I} = \pm$, which form a doublet under $SU(2)_R$, and one complex scalar ϕ, all in the adjoint representation of $SU(N_c)$. Each $\mathcal{N} = 2$ hypermultiplet consists of an $SU(2)_R$

[1]The ranks of the two groups coincide, $N_c \equiv N_{\check{c}}$, but it will be useful to always distinguish graphically with a "check" all quantities pertaining to the second group $SU(N_{\check{c}})$.

	$SU(N_c)$	$U(N_f)$	$SU(2)_R$	$U(1)_r$
$\mathcal{Q}^{\mathcal{I}}_{\alpha}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$+1/2$
${}_{\mathcal{I}\,\alpha}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
A_μ	Adj	$\mathbf{1}$	$\mathbf{1}$	0
ϕ	Adj	$\mathbf{1}$	$\mathbf{1}$	-1
$\lambda^{\mathcal{I}}_{\alpha}$	Adj	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
$Q_{\mathcal{I}}$	$\square$	$\square$	$\mathbf{2}$	0
ψ_α	$\square$	$\square$	$\mathbf{1}$	$+1/2$
$\tilde{\psi}_\alpha$	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$	$+1/2$
$\mathcal{M}_{\mathbf{1}}$	Adj + $\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0
$\mathcal{M}_{\mathbf{3}}$	Adj + $\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0

Table 5.1: Symmetries of $\mathcal{N} = 2$ SCQCD. We show the quantum numbers of the supercharges $\mathcal{Q}^{\mathcal{I}}$, $\mathcal{S}_{\mathcal{I}}$, of the elementary components fields and of the mesonic operators $\mathcal{M}$. Complex conjugate objects (such as $\bar{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}$ and $\bar{\phi}$) are not written explicitly.

doublet $Q_{\mathcal{I}}$ of complex scalars and of two Weyl spinors ψ_α and $\tilde{\psi}_\alpha$, $SU(2)_R$ singlets. It is convenient to define the flavor contracted mesonic operators

$$\mathcal{M}_{\mathcal{J}\ b}^{\mathcal{I}a} \equiv \frac{1}{\sqrt{2}} Q_{\mathcal{J}\ i}^{\ a}\, \bar{Q}^{\mathcal{I}\ i}_{\ b}\,, \tag{5.1}$$

which may be decomposed into into the $SU(2)_R$ singlet and triplet combinations

$$\mathcal{M}_{\mathbf{1}} \equiv \mathcal{M}_{\mathcal{I}}^{\mathcal{I}} \quad \text{and} \quad {\mathcal{M}_{\mathbf{3}}}_{\mathcal{J}}^{\ \mathcal{I}} \equiv \mathcal{M}_{\mathcal{J}}^{\ \mathcal{I}} - \frac{1}{2}\mathcal{M}_{\mathcal{K}}^{\mathcal{K}}\,\delta_{\mathcal{J}}^{\mathcal{I}}\,. \tag{5.2}$$

The operators $\mathcal{M}$ decompose into adjoint plus singlet representations of the color group $SU(N_c)$; the singlet piece is however subleading in the large N_c limit.

The lagrangian of this theory is

$$\mathcal{L} = \mathcal{L}_V + \mathcal{L}_H \tag{5.3}$$

where, $\mathcal{L}_V$ stands for the lagrangian of $N = 2$ vector multiplet and $\mathcal{L}_H$, for the lagrangian of $N = 2$ hypermultiplet. Written explicitely,

$$\begin{aligned}
\mathcal{L}_V &= -\mathrm{Tr}\Big[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\,\bar{\lambda}_{\mathcal{I}}\bar{\sigma}^{\mu}D_{\mu}\lambda^{\mathcal{I}} + (D^{\mu}\phi)(D_{\mu}\phi)^{\dagger} \\
&\quad + i\sqrt{2}\,(g_{YM}\,\epsilon_{\mathcal{I}\mathcal{J}}\lambda^{\mathcal{I}}\lambda^{\mathcal{J}}\phi^{\dagger} - g_{YM}\,\epsilon^{\mathcal{I}\mathcal{J}}\bar{\lambda}_{\mathcal{I}}\bar{\lambda}_{\mathcal{J}}\phi) + \frac{g_{YM}^2}{2}\big[\phi\,,\,\phi^{\dagger}\big]^2\Big] \qquad (5.4)
\end{aligned}$$

in the above we follow conventions where $D_{\mu} = \partial_{\mu} + i\,g_{YM}\,A_{\mu}$ and $\epsilon_{\mathcal{I}\mathcal{J}}\,\epsilon^{\mathcal{J}\mathcal{K}} = \delta_{\mathcal{I}}^{\mathcal{K}}$.

$$\begin{aligned}
\mathcal{L}_H &= -\Big[(D^{\mu}\bar{Q}^{\mathcal{I}})(D_{\mu}Q_{\mathcal{I}}) + i\,\bar{\psi}\bar{\sigma}^{\mu}D_{\mu}\psi + i\,\tilde{\psi}\bar{\sigma}^{\mu}D_{\mu}\bar{\tilde{\psi}} \\
&\quad + i\sqrt{2}\,(g_{YM}\,\epsilon^{\mathcal{I}\mathcal{J}}\bar{\psi}\bar{\lambda}_{\mathcal{I}}Q_{\mathcal{J}} -, g_{YM}\,\epsilon_{\mathcal{I}\mathcal{J}}\bar{Q}^{\mathcal{I}}\lambda^{\mathcal{J}}\psi \qquad (5.5)\\
&\quad + g_{YM}\,\tilde{\psi}\lambda^{\mathcal{I}}Q_{\mathcal{I}} - g_{YM}\,\bar{Q}^{\mathcal{I}}\bar{\lambda}_{\mathcal{I}}\bar{\tilde{\psi}} \\
&\quad + g_{YM}\,\tilde{\psi}\phi\psi - g_{YM}\,\bar{\psi}\bar{\phi}\bar{\tilde{\psi}}) \qquad (5.6)\\
&\quad + g_{YM}^2\bar{Q}_{\mathcal{I}}(\phi^{\dagger}\phi + \phi\phi^{\dagger})Q^{\mathcal{I}} + g_{YM}^2\mathcal{V}\Big] \qquad (5.7)
\end{aligned}$$

where the quartic potential for the squarks is,

$$\begin{aligned}
\mathcal{V} &= (\bar{Q}^{\mathcal{I}}{}_a{}^i Q_{\mathcal{I}}{}^a{}_j)(\bar{Q}^{\mathcal{J}}{}_b{}^j Q_{\mathcal{J}}{}^b{}_i) - \frac{1}{2}(\bar{Q}^{\mathcal{I}}{}_a{}^i Q_{\mathcal{J}}{}^a{}_j)(\bar{Q}^{\mathcal{J}}{}_b{}^j Q_{\mathcal{I}}{}^b{}_i) \\
&\quad + \frac{1}{N_c}(\frac{1}{2}(\bar{Q}^{\mathcal{I}}{}_a{}^i Q_{\mathcal{I}}{}^a{}_i)(\bar{Q}^{\mathcal{J}}{}_b{}^j Q_{\mathcal{J}}{}^b{}_j) - (\bar{Q}^{\mathcal{I}}{}_a{}^i Q_{\mathcal{J}}{}^a{}_i)(\bar{Q}^{\mathcal{J}}{}_b{}^j Q_{\mathcal{I}}{}^b{}_j)) \qquad (5.8)
\end{aligned}$$

Using the flavor contracted mesonic operator (5.1), $\mathcal{V}$ can be written in an instructive and compact form.

$$\begin{aligned}
\mathcal{V} &= \mathrm{Tr}[\mathcal{M}^{\mathcal{J}}{}_{\mathcal{I}}\mathcal{M}^{\mathcal{I}}{}_{\mathcal{J}}] - \frac{1}{2}\mathrm{Tr}[\mathcal{M}^{\mathcal{I}}{}_{\mathcal{I}}\mathcal{M}^{\mathcal{J}}{}_{\mathcal{J}}] \\
&\quad - \frac{1}{N_c}\mathrm{Tr}[\mathcal{M}^{\mathcal{J}}{}_{\mathcal{I}}]\mathrm{Tr}[\mathcal{M}^{\mathcal{I}}{}_{\mathcal{J}}] + \frac{1}{2}\frac{1}{N_c}\mathrm{Tr}[\mathcal{M}^{\mathcal{I}}{}_{\mathcal{I}}]\mathrm{Tr}[\mathcal{M}^{\mathcal{J}}{}_{\mathcal{J}}] \\
&= \mathrm{Tr}[\mathcal{M}_{\mathbf{3}}\mathcal{M}_{\mathbf{3}}] - \frac{1}{N_c}\mathrm{Tr}[\mathcal{M}_{\mathbf{3}}]\mathrm{Tr}[\mathcal{M}_{\mathbf{3}}]
\end{aligned}$$

5.2.2 $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ and interpolating family of SCFTs

$\mathcal{N} = 2$ SCQCD can be viewed as a limit of a family of superconformal theories; in the opposite limit the family reduces to a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. In this subsection we first describe the orbifold theory and then its connection to $\mathcal{N} = 2$ SCQCD.

As familiar, the field content of $\mathcal{N} = 4$ SYM comprises the gauge field A_m, four Weyl fermions λ^A_α and six real scalars X_{AB}, where $A, B = 1, \dots 4$ are indices of the $SU(4)_R$ R-symmetry group. Under $SU(4)_R$, the fermions are in the **4** representation, while the scalars are in **6** (antisymmetric self-dual) and obey the reality condition[2]

$$X^\dagger_{AB} = \frac{1}{2}\epsilon^{ABCD} X_{CD}\,. \tag{5.9}$$

We may parametrize X_{AB} in terms of six real scalars X_k, $k = 4, \dots 9$,

$$X_{AB} = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 0 & X_4 + iX_5 & X_7 + iX_6 & X_8 + iX_9 \\ -X_4 - iX_5 & 0 & X_8 - iX_9 & -X_7 + iX_6 \\ \hline -X_7 - iX_6 & -X_8 + iX_9 & 0 & X_4 - iX_5 \\ -X_8 - iX_9 & X_7 - iX_6 & -X_4 + iX_5 & 0 \end{array} \right) \tag{5.10}$$

Next, we pick an $SU(2)_L \times SU(2)_R \times U(1)_r$ subgroup of $SU(4)_R$,

$$\begin{array}{cc} 1 & + \\ 2 & - \\ 3 & \hat{+} \\ 4 & \hat{-} \end{array} \left(\begin{array}{c|c} SU(2)_R \times U(1)_r & \\ & \\ \hline & \\ & SU(2)_L \times U(1)^*_r \end{array} \right)\,. \tag{5.11}$$

We use indices $\mathcal{I}, \mathcal{J} = \pm$ for $SU(2)_R$ (corresponding to $A, B = 1, 2$) and indices $\hat{\mathcal{I}}, \hat{\mathcal{J}} = \hat{\pm}$ for $SU(2)_L$ (corresponding to $A, B = 3, 4$). To make more manifest their transformation properties, the scalars are rewritten as the $SU(2)_L \times SU(2)_R$ singlet Z (with charge -1 under $U(1)_r$) and as the bifundamental $\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}}$ (neutral under $U(1)_r$),

$$\mathcal{Z} \equiv \frac{X_4 + iX_5}{\sqrt{2}}\,, \qquad \mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} X_7 + iX_6 & X_8 + iX_9 \\ X_8 - iX_9 & -X_7 + iX_6 \end{pmatrix}\,. \tag{5.12}$$

Note the reality condition $\mathcal{X}^\dagger_{\mathcal{I}\hat{\mathcal{I}}} = -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\mathcal{X}_{\mathcal{J}\hat{\mathcal{J}}}$. Geometrically, $SU(2)_L \times SU(2)_R \cong SO(4)$ is the group of 6789 rotations and $U(1)_R \cong SO(2)$ the group of 45 rotations. Diagonal $SU(2)$ transformations $\mathcal{X} \to U\mathcal{X}U^{-1}$ ($U_R = U, U_L = U^*$) preserve the trace, $\mathrm{Tr}[\mathcal{X}] = 2iX_6$, and thus correspond to 789 rotations.

We are now ready to discuss the orbifold projection. In R-symmetry space,

[2] The $\dagger$ indicates hermitian conjugation of the matrix in color space. We choose hermitian generators for the color group.

the orbifold group is chosen to be $\mathbb{Z}_2 \subset SU(2)_L$ with elements $\pm\mathbb{I}_{2\times 2}$. This is the well-known quiver theory [153] obtained by placing N_c D3 branes at the A_1 singularity $\mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$, with $(X_6, X_7, X_8, X_9) \to \pm(X_6, X_7, X_8, X_9)$ and X_4 and X_5 invariant. Supersymmetry is broken to $\mathcal{N} = 2$, since the supercharges with $SU(2)_L$ indices are projected out. The $SU(4)_R$ symmetry is broken to $SU(2)_L \times SU(2)_R \times U(1)_r$, or more precisely to $SO(3)_L \times SU(2)_R \times U(1)_r$ since only objects with integer $SU(2)_L$ spin survive. The $SU(2)_R \times U(1)_r$ factors are the R-symmetry of the unbroken $\mathcal{N} = 2$ superconformal group, while $SO(3)_L$ is an extra global symmetry under which the unbroken supercharges are neutral.

In color space, we start with gauge group $SU(2N_c)$, and declare the non-trivial element of the orbifold to be

$$\tau \equiv \begin{pmatrix} \mathbb{I}_{N_c\times N_c} & 0 \\ 0 & -\mathbb{I}_{N_c\times N_c} \end{pmatrix} . \tag{5.13}$$

All in all the $\mathbb{Z}_2$ action on the $\mathcal{N} = 4$ fields is

$$A_m \to \tau A_m \tau\,, \quad Z_{\mathcal{I}\mathcal{J}} \to \tau Z_{\mathcal{I}\mathcal{J}}\tau\,, \quad \lambda_{\mathcal{I}} \to \tau\lambda_{\mathcal{I}}\tau\,, \tag{5.14}$$

$$\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} \to -\tau\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}}\tau\,, \quad \lambda_{\hat{\mathcal{I}}} \to -\tau\lambda_{\hat{\mathcal{I}}}\tau \qquad . \tag{5.15}$$

The components that survive the projection are

$$A_m = \begin{pmatrix} A^a_{\mu b} & 0 \\ 0 & \check{A}^{\check{a}}_{\mu\check{b}} \end{pmatrix} \quad Z = \begin{pmatrix} \phi^a{}_b & 0 \\ 0 & \check{\phi}^{\check{a}}{}_{\check{b}} \end{pmatrix} \tag{5.16}$$

$$\lambda_{\mathcal{I}} = \begin{pmatrix} \lambda^a_{\mathcal{I}b} & 0 \\ 0 & \check{\lambda}^{\check{a}}_{\mathcal{I}\check{b}} \end{pmatrix} \quad \lambda_{\hat{\mathcal{I}}} = \begin{pmatrix} 0 & \psi^a_{\hat{\mathcal{I}}\check{a}} \\ \tilde{\psi}^{\check{b}}_{\hat{\mathcal{I}}b} & 0 \end{pmatrix} \tag{5.17}$$

$$\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} = \begin{pmatrix} 0 & Q_{\mathcal{I}\hat{\mathcal{I}}\check{a}}{}^{a} \\ -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\bar{Q}^{\check{b}\hat{\mathcal{J}}\mathcal{J}}{}_b & 0 \end{pmatrix} . \tag{5.18}$$

The gauge group is broken to $SU(N_c) \times SU(N_{\check{c}}) \times U(1)$, where the $U(1)$ factor is the *relative*[3] $U(1)$ generated by τ (equ.(5.13)): it must be removed by hand, since its beta function is non-vanishing. The process of removing the relative $U(1)$ modifies the scalar potential by double-trace terms, which arise from the fact that the auxiliary fields (in $\mathcal{N} = 1$ superspace) are now missing the $U(1)$ component. Equivalently we can evaluate the beta function for the double-trace couplings, and tune them to their fixed point [44].

[3]Had we started with $U(2N_c)$ group, we would also have an extra *diagonal* $U(1)$, which would completely decouple since no fields are charged under it.

	$SU(N_c)_1$	$SU(N_c)_2$	$SU(2)_R$	$SU(2)_L$	$U(1)_R$
$\mathcal{Q}^{\mathcal{I}}_{\alpha}$	**1**	**1**	**2**	**1**	+1/2
${}_{\mathcal{I}\,\alpha}$	**1**	**1**	**2**	**1**	–1/2
A_m	Adj	**1**	**1**	**1**	0
$\check{A}_m$	**1**	Adj	**1**	**1**	0
ϕ	Adj	**1**	**1**	**1**	–1
$\check{\phi}$	**1**	Adj	**1**	**1**	–1
$\lambda^{\mathcal{I}}$	Adj	**1**	**2**	**1**	–1/2
$\check{\lambda}^{\mathcal{I}}$	**1**	Adj	**2**	**1**	–1/2
$Q_{\mathcal{I}\hat{\mathcal{I}}}$	$\square$	$\overline{\square}$	**2**	**2**	0
$\psi_{\hat{\mathcal{I}}}$	$\square$	$\overline{\square}$	**1**	**2**	+1/2
$\tilde{\psi}_{\hat{\mathcal{I}}}$	$\overline{\square}$	$\square$	**1**	**2**	+1/2

Table 5.2: Symmetries of the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM and of the interpolating family of $\mathcal{N}=2$ SCFTs.

Supersymmetry organizes the component fields into the $\mathcal{N}=2$ vector multiplets of each factor of the gauge group, $(\phi, \lambda_{\mathcal{I}}, A_m)$ and $(\check{\phi}, \check{\lambda}_{\mathcal{I}}, \check{A}_m)$, and into two bifundamental hypermultiplets, $(Q_{\mathcal{I},\hat{+}}, \psi_{\hat{+}}, \tilde{\psi}_{\hat{+}})$ and $(Q_{\mathcal{I},\hat{-}}, \psi_{\hat{-}}, \tilde{\psi}_{\hat{-}})$. Table 2 summarizes the field content and quantum numbers of the orbifold theory.

The two gauge-couplings g_{YM} and $\check{g}_{YM}$ can be independently varied while preserving $\mathcal{N}=2$ superconformal invariance, thus defining a two-parameter family of $\mathcal{N}=2$ SCFTs. Some care is needed in adjusting the Yukawa and scalar potential terms so that $\mathcal{N}=2$ supersymmetry is preserved. We find

$$\begin{aligned}\mathcal{L}_{Yuk}(g_{YM}, \check{g}_{YM}) = \;& i\sqrt{2}\mathrm{Tr}\big[- g_{YM}\epsilon^{\mathcal{I}\mathcal{J}}\bar{\lambda}_{\mathcal{I}}\bar{\lambda}_{\mathcal{J}}\phi - \check{g}_{YM}\epsilon^{\mathcal{I}\mathcal{J}}\bar{\check{\lambda}}_{\mathcal{I}}\bar{\check{\lambda}}_{\mathcal{J}}\check{\phi} \\ & + g_{YM}\epsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}}\tilde{\psi}_{\hat{\mathcal{I}}}\phi\psi_{\hat{\mathcal{J}}} + \check{g}_{YM}\epsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}}\psi_{\hat{\mathcal{J}}}\check{\phi}\tilde{\psi}_{\hat{\mathcal{I}}} \\ & + g_{YM}\epsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}}\tilde{\psi}_{\hat{\mathcal{J}}}\lambda^{\mathcal{I}}Q_{\mathcal{I}\hat{\mathcal{I}}} + \check{g}_{YM}\epsilon^{\hat{\mathcal{I}}\hat{\mathcal{J}}}Q_{\mathcal{I}\hat{\mathcal{I}}}\check{\lambda}^{\mathcal{I}}\tilde{\psi}_{\hat{\mathcal{J}}} \\ & - g_{YM}\epsilon_{\mathcal{I}\mathcal{J}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{I}}\lambda^{\mathcal{J}}\psi_{\hat{\mathcal{J}}} - \check{g}_{YM}\epsilon_{\mathcal{I}\mathcal{J}}\psi_{\hat{\mathcal{J}}}\check{\lambda}^{\mathcal{I}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}\big] \\ & + h.c. \end{aligned} \tag{5.19}$$

$$
\begin{aligned}
&\mathcal{V}(g_{YM}, \check{g}_{YM}) = \\
&g^2_{YM}\mathrm{Tr}\big[\frac{1}{2}[\bar{\phi},\phi]^2 + \mathcal{M}_{\mathcal{I}}{}^{\mathcal{I}}(\phi\bar{\phi}+\bar{\phi}\phi) + \mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}\mathcal{M}_{\mathcal{J}}{}^{\mathcal{I}} - \frac{1}{2}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{I}}\mathcal{M}_{\mathcal{J}}{}^{\mathcal{J}}\big] \\
&+\check{g}^2_{YM}\mathrm{Tr}\big[\frac{1}{2}[\bar{\check{\phi}},\check{\phi}]^2 + \check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{I}}(\check{\phi}\bar{\check{\phi}}+\bar{\check{\phi}}\check{\phi}) + \check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{J}}\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{I}} - \frac{1}{2}\check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{I}}\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{J}}\big] \\
&+g_{YM}\check{g}_{YM}\mathrm{Tr}\big[-2Q_{\mathcal{I}\hat{\mathcal{I}}}\check{\phi}\bar{Q}^{\hat{\mathcal{I}}\mathcal{I}}\bar{\phi} + h.c.\big] - \frac{1}{N_c}\mathcal{V}_{d.t.}\,,
\end{aligned}
\tag{5.20}
$$

where the mesonic operators $\mathcal{M}$ are defined as[4]

$$
\mathcal{M}_{\mathcal{J}}{}^{\mathcal{I}a}{}_{b} \equiv \frac{1}{\sqrt{2}}Q^{a}_{\mathcal{J}\hat{\mathcal{J}}\check{a}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{I}\check{a}}{}_{b}\,, \qquad \check{\mathcal{M}}^{\mathcal{I}\check{a}}_{\mathcal{J}\check{b}} \equiv \frac{1}{\sqrt{2}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{I}\check{a}}{}_{a}Q^{a}_{\mathcal{J}\hat{\mathcal{J}}\check{b}}\,, \tag{5.21}
$$

and the double-trace terms in the potential are

$$
\begin{aligned}
\mathcal{V}_{d.t.} &= g^2_{YM}\big(\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}]\mathrm{Tr}[\mathcal{M}_{\mathcal{J}}{}^{\mathcal{I}}] - \frac{1}{2}\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}{}^{\mathcal{I}}]\mathrm{Tr}[\mathcal{M}_{\mathcal{J}}{}^{\mathcal{J}}]\big) \qquad (5.22)\\
&\quad +\check{g}^2_{YM}\big(\mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{J}}]\mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{I}}] - \frac{1}{2}\mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{I}}{}_{\mathcal{I}}]\mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{J}}]\big) \\
&= \big(g^2_{YM}+\check{g}^2_{YM}\big)\big(\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}]\mathrm{Tr}[\mathcal{M}_{\mathcal{J}}{}^{\mathcal{I}}] - \frac{1}{2}\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}{}^{\mathcal{I}}]\mathrm{Tr}[\mathcal{M}_{\mathcal{J}}{}^{\mathcal{J}}]\big)\,.
\end{aligned}
$$

The $SU(2)_L$ symmetry is present for all values of the couplings (and so is the $SU(2)_R \times U(1)_r$ R-symmetry, of course). At the orbifold point $g_{YM} = \check{g}_{YM}$ there is an extra $\mathbb{Z}_2$ symmetry (the quantum symmetry of the orbifold) acting as

$$
\phi \leftrightarrow \check{\phi}\,, \quad \lambda_{\mathcal{I}} \leftrightarrow \check{\lambda}_{\mathcal{I}}\,, \quad A_m \leftrightarrow \check{A}_m\,, \quad \psi_{\hat{\mathcal{I}}} \leftrightarrow \tilde{\psi}_{\hat{\mathcal{I}}}\,, \quad Q_{\mathcal{I}\hat{\mathcal{I}}} \leftrightarrow -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\bar{Q}^{\mathcal{J}\hat{\mathcal{J}}}\,. \tag{5.23}
$$

Setting $\check{g}_{YM} = 0$, the second vector multiplet $(\check{\phi}, \check{\lambda}_{\mathcal{I}}, \check{A}_m)$ becomes free and completely decouples from the rest of theory, which happens to coincide with $\mathcal{N} = 2$ SCQCD (indeed the field content is the same and $\mathcal{N} = 2$ susy does the rest). The $SU(N_{\check{c}})$ symmetry can now be interpreted as a global flavor symmetry. In fact there is a symmetry enhancement $SU(N_{\check{c}}) \times SU(2)_L \to U(N_f = 2N_c)$: one sees in (5.19, 5.20) that for $\check{g}_{YM} = 0$ the $SU(N_{\check{c}})$ index $\check{a}$ and the $SU(2)_L$ index $\hat{\mathcal{I}}$ can be combined into a single flavor index $i \equiv (\check{a}, \hat{I}) = 1, \ldots 2N_c$.

In the rest of the paper, unless otherwise stated, we will work in the large $N_c \equiv N_{\check{c}}$ limit, keeping fixed the 't Hooft couplings

$$
\lambda \equiv g^2_{YM}N_c \equiv 8\pi^2 g^2\,, \qquad \check{\lambda} \equiv \check{g}^2_{YM}N_{\check{c}} \equiv 8\pi^2\check{g}^2\,. \tag{5.24}
$$

[4]Note that $\mathrm{Tr}[\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}] = \mathrm{Tr}[\check{\mathcal{M}}^{\mathcal{J}}{}_{\mathcal{I}}]$.

We will refer to the theory with arbitrary g and $\check{g}$ as the "interpolating SCFT", thinking of keeping g fixed as we vary $\check{g}$ from $\check{g} = g$ (orbifold theory) to $\check{g} = 0$ ($\mathcal{N} = 2$ SCQCD $\oplus$ extra $N_{\check{c}}^2 - 1$ free vector multiplets).

5.3 One-loop dilation operator in the scalar sector

At large $N_c \sim N_f$ the natural gauge-invariant operators of $\mathcal{N} = 2$ SCQCD are of the generalized single-trace form (4.1). Motivated by the success of the analogous calculation in $\mathcal{N} = 4$ SYM [11], we have evaluated the one-loop dilation operator on generalized single-trace operators made out of scalar fields. An example of such an operator is

$$\mathrm{Tr}[\bar{\phi}\phi\phi Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}\bar{\phi}] = \bar{\phi}^a{}_b \phi^b{}_c \phi^c{}_d Q^d_{\mathcal{I}\,i} \bar{Q}^{\mathcal{J}i}{}_e \bar{\phi}^e{}_a\,,\ a,b,c,d,e = 1,\dots N_c\,,\ i = 1,\dots N_f\,. \tag{5.25}$$

Since the color or flavor indices of consecutive elementary fields are contracted, we can assign each field to a definite "lattice site"[5] and think of a generalized single-trace operator as a state in a periodic spin-chain. In the scalar sector, the state space V_l at each lattice site is six-dimensional, spanned by $\{\phi, \bar{\phi}, Q_{\mathcal{I}}, \bar{Q}^{\mathcal{J}}\}$. However the index structure of the fields imposes restrictions on the total space $\otimes_{l=1}^{L} V_l$: not all states in the tensor product are allowed. Indeed a Q at site l must always be followed by a $\bar{Q}$ at site $l+1$, and viceversa a $\bar{Q}$ must always be preceded by a Q. Equivalently, as in appendix C.2, we may use instead the color-adjoint objects ϕ, $\bar{\phi}$, $\mathcal{M}_{\mathbf{1}}$ and $\mathcal{M}_{\mathbf{3}}$ (recall the definitions (A.17), where the $\mathcal{M}$'s are viewed as "dimers" occupying two sites of the chain.

As usual, we may interpret the perturbative dilation operator as the hamiltonian of the spin chain. It is convenient to factor out the overall coupling from the definition of the hamiltonian H,

$$\Gamma^{(1)} \equiv g^2 H\,, \qquad g^2 \equiv \frac{\lambda}{8\pi^2}\,, \quad \lambda \equiv g_{YM}^2 N_c\,, \tag{5.26}$$

where $\Gamma^{(1)}$ is the one-loop anomalous dimension matrix. By an immediate extension of the usual arguments, the Veneziano double-line notation (Figure 1) makes it clear that for large $N_c \times N_f$ (with λ fixed) the perturbative dilation operator acts *locally* on the spin-chain. The one-loop hamiltonian is of nearest-neighbor type, $H = \sum_{l=1}^{L} H_{kk+1}$ (with $k \equiv k+L$), where $H_{k,k+1} : V_k \otimes V_{k+1} \to V_k \otimes V_{k+1}$. The two loop correction is next-to-nearest-neighbor and so on.

The matrix elements of the dilation operator are evaluated by computing

[5] Up to cyclic re-ordering of course, under which the trace is invariant.

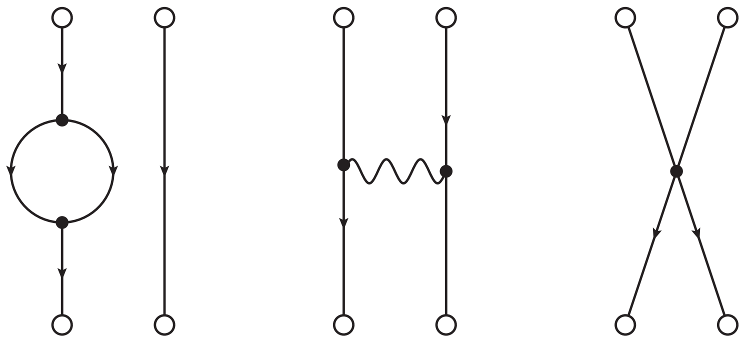


Figure 5.1: Various types of Feynman diagrams that contribute, at one loop, to anomalous dimension. The first diagram is the self energy contribution. The second diagram represents the gluon exchange contribution whereas the third one stands for the quartic interaction between the fields. The first and the second diagrams are proportional to the identity in the R symmetry space while the third one carries a nontrivial R symmetry index structure.

the correlator of the two composite operators. Working in the large N_c limit, we focus on the planar Feynman diagram. These diagrams can be classified as self energy diagrams, gluon interaction diagrams and quartic vertex diagrams as shown in figure 5.1. The results of the one-loop anomalous dimension are presented in section 5.3.1. In appendix C.1, we present an alternate derivation of the one loop dilation operator without explicit one loop computation. In stead, that method uses the knowledge of energies and R charges of some of the composite operators. It serves as a check for the explicit one loop computation of the dilation operator.

In section 5.3.2 we study the magnon excitations in the spin chain "vacuum" for the SCQCD theory. We repeat the exercise for the interpolating family of SCFTs in section 5.3.3 and 5.3.4.

5.3.1 Hamiltonian for SCQCD

We now simply quote our result for H_{kk+1} in $\mathcal{N}=2$ SYM with gauge group $SU(N_c)$ and N_f fundamental hypermultiplets [6],

[6]The spin-chain with this nearest-neighbor hamiltonian reproduces the one-loop anomalous dimension of all operators with $L>2$, where L is the number of sites. The $L=2$ case is special: the double-trace terms in the scalar potential, which give subleading contributions (at large N) for $L>2$, become important for $L=2$ and must be added separately. This special case plays a role in the protection of $\mathrm{Tr}\mathcal{M}_{\mathbf{3}}$, see section 5.4.

$$H_{k,k+1} =$$

$$\begin{array}{c} \\ \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'} \\ \bar{Q}^{\mathcal{I}'}Q_{\mathcal{J}'} \\ \\ Q_{\mathcal{K}'}\bar{Q}^{\mathcal{L}'} \\ \\ Q_{\mathcal{I}'}\phi_{\mathfrak{p}'} \end{array} \begin{array}{c} \begin{array}{cccc} \phi^{\mathfrak{p}}\phi^{\mathfrak{q}} & Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}} & \bar{Q}^{\mathcal{K}}Q_{\mathcal{L}} & \bar{Q}^{\mathcal{I}}\phi^{\mathfrak{p}} \end{array} \\ \begin{pmatrix} 2\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'} & \sqrt{\frac{N_f}{N_c}}g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}} & 0 & 0 \\ \sqrt{\frac{N_f}{N_c}}g^{\mathfrak{p}\mathfrak{q}}\delta^{\mathcal{I}'}_{\mathcal{J}'} & \begin{array}{c}(2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'})\frac{N_f}{N_c} \\ +\frac{1}{2}(1+\xi)\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'}\end{array} & 0 & 0 \\ 0 & 0 & \begin{array}{c}2\delta^{\mathcal{K}}_{\mathcal{L}}\delta^{\mathcal{L}'}_{\mathcal{K}'} \\ -\frac{1}{2}(1+\xi)\delta^{\mathcal{K}}_{\mathcal{K}'}\delta^{\mathcal{L}'}_{\mathcal{L}}\end{array} & 0 \\ 0 & 0 & 0 & \frac{1}{4}(7-\xi)\delta^{\mathcal{I}}_{\mathcal{I}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'} \end{pmatrix} \end{array}$$

Here the indices $\mathfrak{p}, \mathfrak{q} = \pm$ label the $U(1)_r$ charges of ϕ and $\bar{\phi}$, in other terms we have defined $\phi^- \equiv \phi$, $\phi^+ \equiv \bar{\phi}$, and $g_{\mathfrak{p}\mathfrak{q}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The result is valid for arbitrary large $N_f \sim N_c$. (For $N_f \neq 2N_c$ the one-loop beta function is non-vanishing, but, as seen from the Callan-Symanzik equation, this does not affect the calculation of the one-loop dilation operator.) Here, ξ is the standard gauge parameter that appears in the gluon propagator as $\frac{1}{k^2}(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2})$. Notice that although the form of nearest neighbor hamiltonian depends on gauge choice ξ, the anomalous dimension of gauge invariant flavor singlet operators doesn't. We will set $\xi = -1$ as this choice correctly produces the anomalous dimension of flavor non-singlets as well.

We introduce the symbols $\mathbb{I}, \mathbb{P}$ and $\mathbb{K}$ for identity, permutation and trace operators respectively. Their position in the matrix specifies the space in which they act. For example, the operator $\mathbb{P}$ that appears in the matrix element of $\langle \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'} | \phi^{\mathfrak{p}}\phi^{\mathfrak{q}} \rangle$ is $\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'}$, the operator $\mathbb{K}$ that appears in the matrix element $\langle \phi_{\mathfrak{p}}\phi_{\mathfrak{q}} | Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}} \rangle$ stands for the operator $g_{\mathfrak{p}}g_{\mathfrak{q}}\delta^{\mathcal{J}}_{\mathcal{I}}$ and so on. With this notation, and setting $\xi = -1$, we may re-write H_{kk+1} more concisely,

$$H_{k,k+1} = \begin{array}{c} \\ \phi\phi \\ \bar{Q}Q \\ Q\bar{Q} \\ Q\phi \end{array} \begin{array}{c} \begin{array}{cccc} \phi\phi & Q\bar{Q} & \bar{Q}Q & \bar{Q}\phi \end{array} \\ \begin{pmatrix} 2\mathbb{I} + \mathbb{K} - 2\mathbb{P} & \sqrt{\frac{N_f}{N}}\mathbb{K} & 0 & 0 \\ \sqrt{\frac{N_f}{N}}\mathbb{K} & (2\mathbb{I} - \mathbb{K})\frac{N_f}{N_c} & 0 & 0 \\ 0 & 0 & 2\mathbb{K} & 0 \\ 0 & 0 & 0 & 2\mathbb{I} \end{pmatrix} \end{array} \tag{5.27}$$

We introduce σ matrices to write this Hamiltonian manifestly as spin-spin interaction of the nearest neighbour spins.

$$
\begin{array}{cccccc}
 & \phi\phi & Q\bar{Q} & \bar{Q}Q & \bar{Q}\phi & \\
H_{k,k+1} = & \multicolumn{4}{c}{\begin{pmatrix} -\frac{1}{2}(\sigma_x^2+\sigma_y^2+3(\sigma_z^2-1)) & \sqrt{\frac{N_f}{N}}\frac{1}{2}(1+\sigma_x^2+\sigma_y^2-\sigma_z^2) & 0 & 0 \\ \sqrt{\frac{N_f}{N}}\frac{1}{2}(1+\sigma_x^2+\sigma_y^2-\sigma_z^2) & -\frac{N_f}{2N_c}(-3+\sigma_x^2+\sigma_y^2-\sigma_z^2) & 0 & 0 \\ 0 & 0 & 1+\sigma_x^2+\sigma_y^2-\sigma_z^2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}} & \quad (5.28)
\end{array}
$$

We have used the shorthand σ^2 for $\sigma_l \otimes \sigma_{l+1}$.

Artificially restricting the hamiltonian to the space of ϕ (and $\bar{\phi}$) gives $2\mathbb{I}_{\phi\phi} + \mathbb{K}_{\phi\phi} - 2\mathbb{P}_{\phi\phi}$, which is hamiltonian of the XXZ spin chain, confirming the result found in [**?**] for pure $\mathcal{N} = 2$ SYM. The ϕ sector is not closed in our case due to the leading order glueball-meson mixing. The mixing element that is responsible for $\phi\phi \to QQ$ is proportional to $\mathbb{K}$ in ϕ space. Both of these elements vanish when the neighbouring ϕ fields have the same $\pm$ index. This implies that the operator $\mathrm{Tr}[\phi^k]$ is protected, and we can think of it as the ferromagnetic ground state of the spin chain (all spins are pointing down). The impurities that can be excited on this ground state are $\bar{\phi}$, $\mathcal{M}_{\mathbf{1}}$ and $\mathcal{M}_{\mathbf{3}}$, where the last two are "dimeric" impurities which occupy two sites. All of these impurities are double excitations, a single excitation (a single "Q" excitation) is impossible as it is not allowed by the color index structure. It is this fact that makes it hard to study the scattering of two impurities. As we will see, the fundamental excitations on the spin chain vacuum of the interpolating theory are single excitations and their two body scattering can be easily studied.

Nonetheless, the "one body" problem i.e. the dispersion relation of these doubly excited impurities is studied in the next subsection.

5.3.2 Magnon excitations in the SCQCD spin chain

In the map from the composite operators to spin chain, the cyclicity of the trace gives periodic boundary conditions on the spin-chain, along with the constraint that the total momentum of all the impurities in the spin be zero. As usual, it is convenient to first consider the chain to be infinite, and impose later the zero-momentum constraint on multi-impurity states. The action of the Hamiltonian on single-impurities in position space is

$$
\begin{aligned}
H[\bar{\phi}(x)] &= 6\bar{\phi}(x) - \bar{\phi}(x+1) - \bar{\phi}(x-1) && (5.29)\\
&\quad + \sqrt{\frac{2N_f}{N_c}}\mathcal{M}_{\mathbf{1}}(x) + \sqrt{\frac{2N_f}{N_c}}\mathcal{M}_{\mathbf{1}}(x-1) && (5.30)\\
H[\mathcal{M}_{\mathbf{1}}(x)] &= 4\mathcal{M}_{\mathbf{1}}(x) + \sqrt{\frac{2N_f}{N_c}}\bar{\phi}(x) + \sqrt{\frac{2N_f}{N_c}}\bar{\phi}(x+1) \\
H[\mathcal{M}_{\mathbf{3}}(x)] &= 8\mathcal{M}_{\mathbf{3}}(x)\,, && (5.31)
\end{aligned}
$$

where the coordinate x denotes the position (site) of the impurity on the chain; for the dimeric impurities $\mathcal{M}_{\mathbf{1}}$ and $\mathcal{M}_{\mathbf{3}}$ we use the coordinate of the first site. To diagonalize the hamiltonian on the $\bar{\phi}/\ \mathcal{M}_{\mathbf{1}}$ sector, we go to momentum space,

$$
\bar{\phi}(p) \equiv \sum_x \bar{\phi}(x)e^{ipx}\,, \quad \mathcal{M}_{\mathbf{1}}(p) \equiv \sum_x \mathcal{M}_{\mathbf{1}}(x)e^{ipx} \tag{5.32}
$$

$$
H\begin{pmatrix} \bar{\phi}(p) \\ \mathcal{M}_{\mathbf{1}} \end{pmatrix} = \begin{pmatrix} 6 - e^{ip} - e^{-ip} & (1+e^{-ip})\sqrt{\frac{2N_f}{N_c}} \\ (1+e^{ip})\sqrt{\frac{2N_f}{N_c}} & 4 \end{pmatrix} \begin{pmatrix} \bar{\phi}(p) \\ \mathcal{M}_{\mathbf{1}} \end{pmatrix} \tag{5.33}
$$

Eigenvalues and the form of eigenstates is not very illuminating for generic values of the ratio N_f/N_c. For the case of $N_f = 2N_c$, however, they simplify. In that case, the eigenstates are

$$
\begin{aligned}
T(p) &\equiv -\frac{1}{2}(1+e^{-ip})\bar{\phi}(p) + \mathcal{M}_{\mathbf{1}}(p) \\
&= \sum_x e^{ipx}[-\frac{1}{2}(\bar{\phi}(x) + \bar{\phi}(x+1)) + \mathcal{M}_{\mathbf{1}}(x)] && (5.34)\\
\widetilde{T}(p) &\equiv \bar{\phi}(p) + \frac{1}{2}(1+e^{ip})\mathcal{M}_{\mathbf{1}}(p) \\
&= \sum_x e^{ipx}[\bar{\phi}(x) + \frac{1}{2}(\mathcal{M}_{\mathbf{1}}(x) + \mathcal{M}_{\mathbf{1}}(x-1))]\,, && (5.35)
\end{aligned}
$$

with eigenvalues

$$
\begin{aligned}
HT(p) &= 4\sin^2(\frac{p}{2})\,T(p) && (5.36)\\
H\widetilde{T}(p) &= 8\,\widetilde{T}(p)\,. && (5.37)
\end{aligned}
$$

For $\mathcal{N} = 2$ SCQCD, $N_f = 2N_c$ and the magnon excitation $T(p)$ becomes gapless. From now on we will only consider the superconformal case and set

$N_f \equiv 2N_c$.

5.3.3 Hamiltonian for the interpolating SCFTs

We have generalized the calculation of the one-loop dilation operator to the full interpolating family of $\mathcal{N}=2$ SCFTs [8],

$$
\begin{array}{ccc}
 & & \begin{array}{cc} \phi^{\mathfrak{p}}\phi^{\mathfrak{q}} & Q_{\mathcal{I}\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}} \end{array} \\
H \;=\; & \begin{array}{c} \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'} \\ \bar{Q}^{\hat{\mathcal{I}}'\mathcal{I}'}Q_{\mathcal{J}'\hat{\mathcal{J}}'} \end{array} & \begin{pmatrix} (2\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'}) & \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}g_{\mathfrak{p}'\mathfrak{q}'} \\ \delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{J}}'}g^{\mathfrak{p}\mathfrak{q}} & (2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'})\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{J}}'} + 2\kappa^2\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'} \end{pmatrix} \\
 & & \begin{array}{cc} \check{\phi}^{\mathfrak{p}}\check{\phi}^{\mathfrak{q}} & \bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}Q_{\mathcal{I}\hat{\mathcal{I}}} \end{array} \\
\oplus & \begin{array}{c} \check{\phi}_{\mathfrak{p}'}\check{\phi}_{\mathfrak{q}'} \\ Q_{\mathcal{J}'\hat{\mathcal{J}}'}\bar{Q}^{\hat{\mathcal{I}}'\mathcal{I}'} \end{array} & \begin{pmatrix} \kappa^2(2\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'}) & \kappa^2\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}g_{\mathfrak{p}'\mathfrak{q}'} \\ \kappa^2\delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{J}}'}g^{\mathfrak{p}\mathfrak{q}} & \kappa^2(2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'})\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{J}}'} + 2\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'} \end{pmatrix} \\
 & & \begin{array}{cc} \phi^{\mathfrak{p}}Q_{\mathcal{I}\hat{\mathcal{I}}} & Q_{\mathcal{I}\hat{\mathcal{I}}}\check{\phi}^{\mathfrak{p}} \end{array} \\
\oplus & \begin{array}{c} \phi_{\mathfrak{p}'}\bar{Q}^{\hat{\mathcal{I}}'\mathcal{I}'} \\ \bar{Q}^{\hat{\mathcal{I}}'\mathcal{I}'}\check{\phi}_{\mathfrak{p}'} \end{array} & \begin{pmatrix} 2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\mathfrak{p}}_{\mathfrak{p}'} & -2\kappa\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\mathfrak{p}}_{\mathfrak{p}'} \\ -2\kappa\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\mathfrak{p}}_{\mathfrak{p}'} & 2\kappa^2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\hat{\mathcal{I}}'}_{\hat{\mathcal{I}}}\delta^{\mathfrak{p}}_{\mathfrak{p}'} \end{pmatrix} \\
 & & \begin{array}{cc} \check{\phi}^{\mathfrak{p}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}} & \bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}\phi^{\mathfrak{p}} \end{array} \\
\oplus & \begin{array}{c} \check{\phi}_{\mathfrak{p}'}Q_{\mathcal{J}'\hat{\mathcal{J}}'} \\ Q_{\mathcal{J}'\hat{\mathcal{J}}'}\phi_{\mathfrak{p}'} \end{array} & \begin{pmatrix} 2\kappa^2\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'} & -2\kappa\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'} \\ -2\kappa\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'} & 2\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{J}}'}\delta^{\mathfrak{p}}_{\mathfrak{p}'} \end{pmatrix}
\end{array}
\tag{5.38}
$$

In concise form, [7]

$$
H_{k,k+1} =
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} \phi\phi & Q\bar{Q} & \check{\phi}\check{\phi} & \bar{Q}Q & \phi Q & Q\check{\phi} & \check{\phi}\bar{Q} & \bar{Q}\phi \end{array} \\
\begin{array}{c} \phi\phi \\ Q\bar{Q} \\ \check{\phi}\check{\phi} \\ \bar{Q}Q \\ \phi Q \\ Q\check{\phi} \\ \check{\phi}\bar{Q} \\ \bar{Q}\phi \end{array} &
\begin{pmatrix}
(2+\mathbb{K}-2\mathbb{P}) & \mathbb{K} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{K} & (2-\mathbb{K})\hat{\mathbb{K}}+2\kappa^2\mathbb{K} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \kappa^2(2+\mathbb{K}-2\mathbb{P}) & \kappa^2\mathbb{K} & 0 & 0 & 0 & 0 \\
0 & 0 & \kappa^2\mathbb{K} & \kappa^2(2-\mathbb{K})\hat{\mathbb{K}}+2\mathbb{K} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2\kappa & 0 & 0 \\
0 & 0 & 0 & 0 & -2\kappa & 2\kappa^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2\kappa^2 & -2\kappa \\
0 & 0 & 0 & 0 & 0 & 0 & -2\kappa & 2
\end{pmatrix}
\end{array}
$$

[7] The meaning of the different operators can be read off by comparing with the explicit form above. Note in particular that to avoid cluttering we have dropped the identity symbol $\mathbb{I}$ (terms proportional to unity are proportional to the identity in the respective spaces). Also in the subspaces $Q\bar{Q}$, $\bar{Q}Q$ we use the notation $\mathbb{K}$ for the trace operator acting on $SU(2)_R$ indices and $\hat{\mathbb{K}}$ that acts on the $SU(2)_L$ indices.

where $\kappa \equiv \frac{g}{\check{g}}$ and $g^2 \equiv \frac{g_{YM}^2 N}{8\pi^2}$, $\check{g}^2 \equiv \frac{\check{g}_{YM}^2 N}{8\pi^2}$.

The hamiltonian could be written in more compact fashion directly in terms of the $\mathbb{Z}_2$ projected fields of the orbifold (eq.(5.16)) Z and $\mathcal{X}$. We rewrite their definitions here for reader's convenience.

$$Z = \begin{pmatrix} \phi & 0 \\ 0 & \check{\phi} \end{pmatrix}, \qquad \mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} = \begin{pmatrix} 0 & Q_{\mathcal{I}\hat{\mathcal{I}}} \\ -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}} & 0 \end{pmatrix} \tag{5.39}$$

The hamiltonian for a generic value of κ can now be manifestly written as interpolating between the hamiltonian of $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM and that of SCQCD.

$$g^2 H = \begin{array}{c} \begin{array}{cccc} ZZ & \mathcal{X}\mathcal{X} & Z\mathcal{X} & \mathcal{X}Z \end{array} \\ \begin{pmatrix} (g_+ + \gamma g_-)^2(2 + \mathbb{K} - 2\mathbb{P}) & (g_+ + \gamma g_-)^2\mathbb{K}\hat{\mathbb{K}} & 0 & 0 \\ (g_+ + \gamma g_-)^2\mathbb{K}\hat{\mathbb{K}} & \begin{array}{c}(g_+ + \gamma g_-)^2(2\hat{\mathbb{K}} - \mathbb{K}\hat{\mathbb{K}}) \\ +2(g_+ - \gamma g_-)^2\mathbb{K}\end{array} & 0 & 0 \\ 0 & 0 & 2(g_+ + \gamma g_-)^2 & -2({g_+}^2 - {g_-}^2) \\ 0 & 0 & -2({g_+}^2 - {g_-}^2) & 2(g_+ - \gamma g_-)^2 \end{pmatrix} \end{array} \tag{5.40}$$

Here we have defined, $g \equiv g_+ + g_-$, $\check{g} = g_+ - g_-$. The matrix $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is called the twist.

In the limit, $\check{g} \to 0$, we see that this hamiltonian reduces to that of SCQCD spin chain as it should. In appendix C.1, we have also derived this hamiltonian using the similar arguments as for the SCQCD case. In the next subsection, we study the dispersion relation of the fundamental excitations. As we will see they are single excitations, in contrast with the SCQCD case where they are double excitations.

5.3.4 Magnon excitations of the interpolating SCFT spin chain

As before, we think of a composite operator as a spin chain where the sites are occupied by the fields. But one needs to be careful in construction of such a spin chain as the index structure already imposes constraints on the sequence in which various impurities can appear in the spin chain. These constraints are the following

1. ϕ can only be followed by Q

2. Q can be followed either by $\bar{Q}$ or $\check{\phi}$.

3. $\check{\phi}$ can only be followed by $\bar{Q}$.

4. $\bar{Q}$ can be followed either by Q or ϕ.

With these constraints if we start with the sea of ϕ as a vacuum (the other choice would be to start with the sea of $\check{\phi}$), inserting one Q impurity will amount to changing the ϕs following the impurity into $\check{\phi}$s. This is the only way one could consistently insert impurities. Let us study the dynamics of a single impurity Q. The position space state where Q is localised to a position x in the spin chain is denoted by $Q(x)$.

Using the Hamiltonian above, we write the evolution

$$g^2 H Q_{\mathcal{I}\hat{\mathcal{I}}}(x) = 2(g^2 + \check{g}^2) Q_{\mathcal{I}\hat{\mathcal{I}}}(x) - 2g\check{g}[Q_{\mathcal{I}\hat{\mathcal{I}}}(x-1) + Q_{\mathcal{I}\hat{\mathcal{I}}}(x+1)] \tag{5.41}$$

Fourier transforming as, $Q(p) = \sum_x e^{ipx} Q(x)$ we get,

$$\begin{aligned} g^2 H Q_{\mathcal{I}\hat{\mathcal{I}}}(p) &= 2(g^2 + \check{g}^2 - 2g\check{g}\cos p) Q_{\mathcal{I}\hat{\mathcal{I}}}(p) \\ &= [2(g-\check{g})^2 + 4g\check{g}(1-\cos p)] Q_{\mathcal{I}\hat{\mathcal{I}}}(p) \\ &= [2(g-\check{g})^2 + 8g\check{g}\sin^2(\frac{p}{2})] Q_{\mathcal{I}\hat{\mathcal{I}}}(p) \end{aligned} \tag{5.42}$$

Hence the dispersion relation for $Q_{\mathcal{I}\hat{\mathcal{I}}}(p)$ is,

$$g^2 E(p) = \Delta + 8g\check{g}\sin^2(\frac{p}{2}) \tag{5.43}$$

As we can see, we get a $\sin^2(\frac{p}{2})$ dispersion relation for Q with a gap $\Delta = 2(g-\check{g})^2$, the gap is zero only when $g = \check{g}$ as expected. The similar analysis holds for $\bar{Q}$ impurity and we end up with the same dispersion relation.

5.4 The protected spectrum in detail

In the previous chapter 4, we analyzed the protected spectrum of $\mathcal{N}=2$ SCQCD with the help of superconformal index [18]. In this section we determine all the generalized single trace operators in the scalar sector of SCQCD having vanishing one-loop anomalous dimension. We find the complete list of such operators to be:

$$\mathrm{Tr}\,\phi^{k+2}, \qquad \mathrm{Tr}[T\phi^k], \qquad \mathrm{Tr}\mathcal{M}_{\mathbf{3}}. \tag{5.44}$$

Here, $T \equiv \phi\bar{\phi} - \mathcal{M}_{\mathbf{1}}$ and $k \geq 0$. We are first led to (5.44) by an educated guess. In section B.1 we list all operators in the scalar sector that obey any of the the shortening or semi-shortening conditions of the $\mathcal{N}=2$ superconformal algebra, which have been completely classified by Dolan and Osborn [154]. Using the spin-chain hamiltonian, we compute the one-loop anomalous dimension of these candidate protected states, and find that only (5.44) have zero anomalous dimension. We see that the result agrees with the first class of protected operators, i.e. the operators in the scalar sector, obtained in [8]. The spin chain hamiltonian of section 5.3.1 is insufficient to reproduce the second class of protected operators as they contain fields with higher spins. In section 5.4.2, we list the protected operators of the orbifold theory that could be obtained by variety of methods and follow the evolution of these operators along the exactly marginal line $\kappa \to 0$ towards SCQCD.

Even though we concern ourselves with one loop analysis in this section, (5.44) can be seen to be protected at full quantum level using superconformal index [8].

5.4.1 Protected states in $\mathcal{N}=2$ SCQCD

A generic long multiplet $\mathcal{A}^{\Delta}_{R,r(j,\bar{j})}$ of the $\mathcal{N}=2$ superconformal algebra is generated by the action of the 8 Poincaré supercharges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ on a superconformal primary, which by definition is annihilated by all conformal supercharges . If some combination of the $\mathcal{Q}$'s also annihilates the primary, the corresponding multiplet is shorter and the conformal dimensions of all its members are protected against quantum corrections. A comprehensive list of the possible shortening conditions for the $\mathcal{N}=2$ superconformal algebra was given in [154] . Their findings are summarized in Table B.1. [8] We refer to [154] for more details.

[8]We follow the conventions of [154], except that we have introduced the labels $\mathcal{D}$, $\mathcal{F}$, $\hat{\mathcal{F}}$ and $\mathcal{G}$ to denote some shortening conditions that were left nameless in [154].

Scalar Multiplets	SCQCD operators	Protected
$\bar{\mathcal{B}}_{R,-\ell(0,0)}$	$\mathrm{Tr}[\phi^\ell \mathcal{M}_3^R]$	
$\bar{\mathcal{E}}_{-\ell(0,0)}$	$\mathrm{Tr}[\phi^\ell]$	✓
$\hat{\mathcal{B}}_R$	$\mathrm{Tr}[\mathcal{M}_3^R]$	✓ for $R=1$
$\bar{\mathcal{C}}_{R,-\ell(0,0)}$	$\mathrm{Tr}[T\mathcal{M}_3^R\phi^\ell]$	
$\bar{\mathcal{C}}_{0,-\ell(0,0)}$	$\mathrm{Tr}[T\phi^\ell]$	✓
$\hat{\mathcal{C}}_{R(0,0)}$	$\mathrm{Tr}[T\mathcal{M}_3^R]$	
$\hat{\mathcal{C}}_{0(0,0)}$	$\mathrm{Tr}[T]$	✓
$\bar{\mathcal{D}}_{R(0,0)}$	$\mathrm{Tr}[\mathcal{M}_3^R\phi]$	

Table 5.3: $\mathcal{N}=2$ SCQCD protected operators at one loop

In Table 5.3 we list all the generalized single-trace operators of $\mathcal{N}=2$ SCQCD made out of scalar fields, which obey any of the possible shortening conditions. Using the spin chain Hamiltonian of section 5.3.1, we find that the only operators with zero anomalous dimension are the one listed in (5.44)[9]. The operators $\mathrm{Tr}\,\phi^\ell$ correspond to the vacuum of the spin-chain, while the operators $\mathrm{Tr}\,T\,\phi^\ell$ correspond to the zero-momentum limit of the gapless excitation $T(p)$, eq. (5.36) . There is one more protected operator, which is "exceptional" in not belonging to an infinite sequence: $\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$. Its anomalous dimension is zero for gauge group $SU(N_c)$ but not for gauge group $U(N_c)$: the double-trace terms in the Lagrangian that arise from the removal of the $U(1)$ are crucial for the protection of this operator.

5.4.2 Protected spectrum of the interpolating family of $\mathcal{N}=2$ SCFTs

As we have reviewed in section 2.2, $\mathcal{N}=2$ SCQCD can be obtained as the $\check{g}_{YM}\to 0$ limit of a family of $\mathcal{N}=2$ superconformal field theories, which reduces for $g_{YM}=\check{g}_{YM}$ to the $\mathcal{N}=2$ $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. In this section we find the protected spectrum of single-trace operators of the interpolating family. We start at the orbifold point, where the protected states are easy to determine, and follow their fate along the exactly marginal line towards $\mathcal{N}=2$ SCQCD.

[9] Together of course with their conjugates. Note that since in our conventions ϕ has $r=-1$, the multiplet $\bar{\mathcal{E}}_{-\ell(0,0)}$, $\ell>0$, is represented by $\mathrm{Tr}\phi^\ell$. The conjugate multiplet $\mathcal{E}_{\ell(0,0)}$ is represented by $\mathrm{Tr}\bar{\phi}^\ell$ and is of course also protected.

At the orbifold point, operators fall into two classes: untwisted and twisted. In the untwisted sector, the protected states are well-known, since they are inherited from $\mathcal{N}=4$ SYM. The protected operators in the twisted sector are chiral with respect to $\mathcal{N}=1$ subalgebra and could be obtained by analyzing the chiral ring [162]. [10] Both the classes of operators can be rigorously checked to be protected by computing the superconformal index.[11] Using the index one can also argue that the protected multiplets found at the orbifold point *cannot* recombine into long multiplets as we vary $\check{g}$ [8], so in particular taking $\check{g} \to 0$ they must evolve into the protected multiplets of the theory

$$\{\mathcal{N}=2 \text{ SCQCD } \oplus \text{ decoupled } SU(N_{\check{c}}) \text{ vector multiplet}\} . \tag{5.45}$$

In section 5.4.3 we follow this evolution in detail. We find that the $SU(2)_L$-singlet protected states of the interpolating theory evolve into the list (5.44) of protected states of SCQCD, plus some extra states made purely from the decoupled vector multiplet. On the other hand, the interpolating theory has also many single-trace protected states with non-trivial $SU(2)_L$ spin, which are of course absent from the list (5.44): we see that in the limit $\check{g} \to 0$, a state with $SU(2)_L$ spin L can be interpreted as a "multiparticle state", obtained by linking together L short "open" spin-chains with of SCQCD and decoupled fields $\check{\phi}$. By this route we confirm that (5.44) is the correct and complete list of protected single-traces in the scalar sector for $\mathcal{N}=2$ SCQCD. The results are also suggestive of a dual string theory interpretation: as $\check{g} \to 0$, single closed string states carrying $SU(2)_L$ quantum numbers disintegrate into multiple open strings. The above argument, however, doesn't imply that all the protected operators of SCQCD are obtained as degenrations of protected operators of the interpolating theory. Indeed, they aren't. In [8], we discuss an alternative mechanism that brings about more protected SCQCD operators from the decomposition of long multiplets of the interpolating theory as $\check{g} \to 0$.

In summary, the degeneracy of protected states is independent of the exactly marginal deformation that changes $\check{g}_{YM}$ and is thus the same for the orbifold theory and for the theory (5.45). At $\check{g}_{YM}=0$ there is a symmetry enhancement, $SU(2)_L \times SU(N_{\check{c}}) \to U(N_f = 2N_c)$, and we can consistently truncate the spectrum of generalized single trace operators to singlets of the flavor group $U(N_f)$ – which in particular do not contain any of the decoupled states $\check{\phi}$. This is the flavor singlet spectrum of $\mathcal{N}=2$ SCQCD that we have analyzed in the previous section.

[10] We confirm the spectrum in [8] up to one operator that was missed in the analysis of [162].

[11] The calculation for the orbifold was carried out already in [156], which we confirm up to a minor emendation in [8].

Multiplet	Orbifold operator $(R, \ell \geq 0,\, n \geq 2)$
$\hat{\mathcal{B}}_{R+1}$	$\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}]$
$\bar{\mathcal{E}}_{-(\ell+2)(0,0)}$	$\mathrm{Tr}[\phi^{\ell+2} + \check{\phi}^{\ell+2}]$
$\hat{\mathcal{C}}_{R(0,0)}$	$\mathrm{Tr}[\sum \mathcal{T}(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}]$
$\bar{\mathcal{D}}_{R+1(0,0)}$	$\mathrm{Tr}[\sum (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}(\phi^{+}\check{\phi})]$
$\bar{\mathcal{B}}_{R+1,-(\ell+2)(0,0)}$	$\mathrm{Tr}[\sum_i (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R+1}\phi^i\check{\phi}^{\ell+2-i}]$
$\bar{\mathcal{C}}_{R,-(\ell+1)(0,0)}$	$\mathrm{Tr}[\sum_i \mathcal{T}(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}\phi^i\check{\phi}^{\ell+1-i}]$
$\mathcal{A}^{\Delta=2R+\ell+2n}_{R,-\ell(0,0)}$	$\mathrm{Tr}[\sum_i \mathcal{T}^n(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^{R}\phi^i\check{\phi}^{\ell-i}]$

Table 5.4: Superconformal primary operators in the untwisted sector of the orbifold theory that descent from the $\frac{1}{2}$ BPS primary of $\mathcal{N} = 4$. The symbol $\sum$ indicates summation over all "symmetric traceless" permutations of the component fields allowed by the index structure.

Multiplet	Orbifold operator $(\ell \geq 0)$
$\hat{\mathcal{B}}_1$	$\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{-}} - Q^{+\hat{-}}\bar{Q}^{+\hat{+}})] = \mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$
$\bar{\mathcal{E}}_{-(\ell+2)(0,0)}$	$\mathrm{Tr}[\phi^{\ell+2} - \check{\phi}^{\ell+2}]$

Table 5.5: Superconformal primary operators in the twisted section of the orbifold theory.

5.4.3 Away from the orbifold point: matching with $\mathcal{N} = 2$ SCQCD

In the limit $\check{g} \to 0$, we must be able to match the protected states of the interpolating family with protected states of $\{\mathcal{N} = 2 \text{ SCQCD} \oplus \text{decoupled vector multiplet}\}$. For the purpose of this discussion, the protected states naturally splits into two sets: $SU(2)_L$ singlets and $SU(2)_L$ non-singlets. It is clear that all the (generalized) single-trace operators of $\mathcal{N} = 2$ SCQCD must arise from the $SU(2)_L$ singlets.

$\mathbf{SU(2)_L}$ singlets

They are:

(i) One $\hat{\mathcal{B}}_1$ multiplet, corresponding to the primary $\mathrm{Tr}[Q_{\hat{\mathcal{I}}\{\mathcal{I}}\bar{Q}^{\hat{\mathcal{I}}}_{\mathcal{J}\}}] = \mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$. Since this is the only operator with these quantum numbers, it cannot mix with anything and its form is independent of $\check{g}$.

(ii) Two $\bar{\mathcal{E}}_{-\ell(0,0)}$ multiplets for each $\ell \geq 2$, corresponding to the primaries $\mathrm{Tr}\,[\phi^\ell \pm \check{\phi}^\ell]$. For each ℓ, there is a two-dimensional space of protected operators, and we may choose whichever basis is more convenient. For $g = \check{g}$, the natural basis vectors are the untwisted and twisted combinations (respectively even and odd under $\phi \leftrightarrow \check{\phi}$), while for $\check{g} = 0$ the natural basis vectors are $\mathrm{Tr}\,\phi^\ell$ (which is an operator of $\mathcal{N} = 2$ SCQCD) and $\mathrm{Tr}\,\check{\phi}^\ell$ (which belongs to the decoupled sector).

(iii) One $\hat{\mathcal{C}}_{0(0,0)}$ multiplet (the stress-tensor multiplet), corresponding to the primary $\mathrm{Tr}\,\mathcal{T} = \mathrm{Tr}\,[T + \check{\phi}\bar{\check{\phi}}]$. We have checked that this combination is an eigenstate with zero eigenvalue for all $\check{g}$. For $\check{g} = 0$, we may trivially subtract out the decoupled piece $\mathrm{Tr}\,\check{\phi}\bar{\check{\phi}}$ and recover $\mathrm{Tr}\,T$, the stress-tensor multiplet of $\mathcal{N} = 2$ SCQCD.

(iv) One $\bar{\mathcal{C}}_{0,-\ell(0,0)}$ multiplet for each $\ell \geq 1$. In the limit $\check{g} \to 0$, we expect this multiplet to evolve to the $\mathrm{Tr}\,T\phi^\ell$ multiplet of $\mathcal{N} = 2$ SCQCD. Let us check this in detail.

The primary of $\bar{\mathcal{C}}_{0,-\ell(0,0)}$ has $R = 0$, $r = -\ell$ and $\Delta = \ell + 2$. The space of operators which classically have these quantum numbers is spanned by

$$|a\rangle = \mathrm{Tr}[\check{\phi}^{\ell+1}\bar{\check{\phi}}], \quad |b_i\rangle \equiv \tfrac{1}{2}\mathrm{Tr}[\phi^i Q_{\mathcal{I}\hat{\mathcal{I}}}\check{\phi}^{\ell-i}\bar{Q}^{\hat{\mathcal{I}}\mathcal{I}}] \quad \text{for} \quad 0 \leq i \leq \ell$$
$$\text{and} \quad |c_\ell\rangle \equiv \mathrm{Tr}[\phi^{\ell+1}\bar{\phi}] \tag{5.46}$$

Diagonalizing the Hamiltonian in Fourier space, we find the protected operator to be

$$|\bar{\mathcal{C}}_{0,-\ell(0,0)}\rangle_\kappa = \kappa^\ell|a\rangle - \sum_{i=0}^{\ell}\kappa^{\ell-i}|b_i\rangle + |c_\ell\rangle \tag{5.47}$$

where $\kappa \equiv \check{g}/g$. In the limit $\kappa \to 0$,

$$|\bar{\mathcal{C}}_{0,-\ell(0,0)}\rangle_{\kappa\to 0} = \mathrm{Tr}[(\phi\bar{\phi} - \frac{1}{2}Q_{\mathcal{I}\hat{\mathcal{I}}}\bar{Q}^{\mathcal{I}\hat{\mathcal{I}}})\phi^\ell] = \mathrm{Tr}[T\phi^\ell]\,, \tag{5.48}$$

as claimed.

All in all, we see that this list reproduces the list (5.44) of one-loop protected scalar operators of $\mathcal{N}=2$ SCQCD, *plus* the extra states $\mathrm{Tr}\check{\phi}^\ell$ which decouple for $\check{g}=0$. This concludes the argument that that the operators (5.44) are protected at the full quantum level, and that they are the *complete* set of protected generalized single-trace primaries of $\mathcal{N}=2$ SCQCD.

$\mathbf{SU(2)_L}$ non-singlets

The basic protected primary of $\mathcal{N}=2$ SCQCD which is charged under $SU(2)_L$ is the $SU(2)_L$ triplet contained in the mesonic operator $\mathcal{O}^i_{\mathbf{3_R}\,j} = (\bar{Q}^i_a Q^a_j)_{\mathbf{3_R}}$. Indeed writing the $U(N_f=2N_c)$ flavor indices i as $i=(\check{a},\hat{\mathcal{I}})$, with $\check{a} = 1,\ldots N_f/2 = N_c$ "half" flavor indices and $\mathcal{I}=\hat{\pm}$ $SU(2)_L$ indices, we can decompose

$$\mathcal{O}^i_{\mathbf{3_R}\,j} \to \mathcal{O}^{\check{a}}_{\mathbf{3_R}\mathbf{3_L}\,\check{b}}\,, \quad \mathcal{O}^{\check{a}}_{\mathbf{3_R}\mathbf{1_L}\,\check{b}}\,. \tag{5.49}$$

In particular we may consider the highest weight combination for both $SU(2)_L$ and $SU(2)_R$,

$$(\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^{\check{a}}_{\check{b}}\,. \tag{5.50}$$

States with higher $SU(2)_L$ spin can be built by taking products of $\mathcal{O}_{\mathbf{3_R}\mathbf{3_L}}$ with $SU(2)_L$ and $SU(2)_R$ indices separately symmetrized – and this is the only way to obtain protected states of $\mathcal{N}=2$ SCQCD charged under $SU(2)_L$ which have finite conformal dimension in the Veneziano limit. It is then a priori clear that a protected primary of the interpolating theory with $SU(2)_L$ spin L must evolve as $\check{g}\to 0$ into a product of L copies of $(\bar{Q}^{+\hat{+}}Q^{+\hat{+}})$ and of as many additional decoupled scalars $\check{\phi}$ and $\bar{\check{\phi}}$ as needed to make up for the correct $U(1)_r$ charge and conformal dimension. It is amusing to follow in more detail this evolution for the various multiplets:

(i) $\hat{\mathcal{B}}_R$ multiplet.

This is a trivial case, since for each R there is only one operator with the correct quantum numbers, namely

$$|\hat{\mathcal{B}}_R\rangle_\kappa \equiv \mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^R]\,, \tag{5.51}$$

for all g and $\check{g}$. We have checked that it is indeed an eigenstate of zero eigenvalue for all couplings.

(ii) $\bar{\mathcal{D}}_{R(0,0)}$ multiplet.

The primary of $\bar{\mathcal{D}}_{R(0,0)}$ has $SU(2)_R$ spin equal R, $U(1)_r$ charge $r=-1$ and $\Delta = 2R+1$. The space of operators which classically have these quantum numbers is two-dimensional, spanned by $\mathrm{Tr}[(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^R\phi]$ and $\mathrm{Tr}[(\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^R\check{\phi}]$. The spin-chain Hamiltonian in this subspace reads

$$g^2 H_{\bar{\mathcal{D}}} = \begin{pmatrix} 4g^2 & -4g\check{g} \\ -4g\check{g} & 4\check{g}^2 \end{pmatrix} \tag{5.52}$$

The protected operator (eigenvector with zero eigenvalue) is

$$|\bar{\mathcal{D}}_{R(0,0)}\rangle_\kappa \equiv \mathrm{Tr}[\kappa(Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^R\phi + (\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^R\check{\phi}]\,. \tag{5.53}$$

For $\kappa = 0$, the protected operator is interpreted as a "multi-particle state" of R open chains of SCQCD and one decoupled scalar $\check{\phi}$. For example for $R=2$, the operator will be broken into the following gauge-invariant pieces,

$$(\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^{\check{a}}{}_{\check{b}}\,, \quad (\bar{Q}^{+\hat{+}}Q^{+\hat{+}})^{\check{b}}{}_{\check{c}} \quad \text{and} \quad \check{\phi}^{\check{c}}{}_{\check{a}}\,. \tag{5.54}$$

In the limit $\check{g}\to 0$, the "closed chain" of the interpolating theory effectively breaks into "open chains" of $\{\mathcal{N}=2$ SCQCD $\oplus$ decoupled multiplet$\}$, with the rupture points at the contractions of the "half-flavor" indices $\check{a}$, $\check{b}$, $\check{c}$.

(iii) $\bar{\mathcal{B}}_{R,r(0,0)}$ multiplet.

Finding the protected multiplet for arbitrary coupling amounts to diagonalizing the spin-chain Hamiltonian of the interpolating theory in the space of operators with quantum numbers R, r and $\Delta = 2R - r$. The dimension of this space increases rapidly with R and r. Let us focus on two simple cases.

case 1*:* $R = 1$, $r \equiv -\ell < 0$

In this case, the space is $\ell + 1$ dimensional, spanned by

$$|\psi_i\rangle \equiv \mathrm{Tr}[\phi^i Q^{+\hat{+}} \check{\phi}^{\ell-i} \bar{Q}^{+\hat{+}}]\,, \quad i = 0, \ldots \ell\,. \tag{5.55}$$

The protected operator is found to be

$$|\bar{\mathcal{B}}_{1,-\ell(0,0)}\rangle_\kappa \equiv \sum_{i=0}^{\ell} \kappa^i |\psi_i\rangle \tag{5.56}$$

In our schematic notation of $\sum$, introduced earlier, the same operator would read

$$|\bar{\mathcal{B}}_{1,-\ell(0,0)}\rangle_\kappa = \mathrm{Tr}[\sum_i \kappa^i (Q^{+\hat{+}} \bar{Q}^{+\hat{+}}) \phi^i \check{\phi}^{\ell-i}]\,. \tag{5.57}$$

Note that at $\kappa = 0$, the $U(1)_r$ charge of the operator is all carried by the decoupled scalars $\check{\phi}$ – there are no ϕ. This is again consistent with the picture of the closed chain disintegrating into open pieces.

case 2: $r = -2$, $R = 2$

The relevant vector space is spanned by the operators

$$\begin{array}{rclcrcl}
|0\rangle &=& \mathrm{Tr}[\phi\phi Q^{+\hat{+}} \bar{Q}^{+\hat{+}} Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] & \quad & |\check{0}\rangle &=& \mathrm{Tr}[Q^{+\hat{+}} \check{\phi}\check{\phi} \bar{Q}^{+\hat{+}} Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] \\
|1\rangle &=& \mathrm{Tr}[\phi Q^{+\hat{+}} \check{\phi} \bar{Q}^{+\hat{+}} Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] & & |\check{1}\rangle &=& \mathrm{Tr}[Q^{+\hat{+}} \check{\phi} \bar{Q}^{+\hat{+}} \phi Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] \\
|2\rangle &=& \mathrm{Tr}[\phi Q^{+\hat{+}} \bar{Q}^{+\hat{+}} \phi Q^{+\hat{+}} \bar{Q}^{+\hat{+}}] & & |\check{2}\rangle &=& \mathrm{Tr}[Q^{+\hat{+}} \check{\phi} \bar{Q}^{+\hat{+}} Q^{+\hat{+}} \check{\phi} \bar{Q}^{+\hat{+}}]
\end{array} \tag{5.58}$$

The Hamiltonian in this subspace is (the basis vectors are read in the sequence $|0\rangle$, $|\check{0}\rangle$, $|1\rangle$, $\ldots$)

$$g^2 H_{\bar{\mathcal{B}}_{2,-2(0,0)}} = \begin{pmatrix}
4g^2 & 0 & -2g\check{g} & -2g\check{g} & 0 & 0 \\
0 & 4\check{g}^2 & -2g\check{g} & -2g\check{g} & 0 & 0 \\
-2g\check{g} & -2g\check{g} & 4g^2 + 4\check{g}^2 & 0 & -2g\check{g} & -2g\check{g} \\
-2g\check{g} & -2g\check{g} & 0 & 4g^2 + 4\check{g}^2 & -2g\check{g} & -2g\check{g} \\
0 & 0 & -2g\check{g} & -2g\check{g} & 4g^2 & 0 \\
0 & 0 & -2g\check{g} & -2g\check{g} & 0 & 4\check{g}^2
\end{pmatrix} \tag{5.59}$$

There is an eigenvector with zero eigenvalue for all κ, namely

$$\begin{aligned} |\bar{\mathcal{B}}_{2,-2(0,0)}\rangle_\kappa &\equiv \kappa^2|0\rangle + |\check{0}\rangle + \kappa|1\rangle + \kappa|\check{1}\rangle + \kappa^2|2\rangle + |\check{2}\rangle \\ &= \mathrm{Tr}[\sum_i \kappa^i (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^2 \phi^i \check{\phi}^{2-i}] \end{aligned}$$

As expected, for $\kappa = 0$ the operator contains $\check{\phi}$ and no ϕ.

Extrapolating from these cases, we make an educated guess for the form for general protected operator,

$$|\bar{\mathcal{B}}_{R,-\ell(0,0)}\rangle_\kappa = \mathrm{Tr}[\sum_i \kappa^i (Q^{+\hat{+}}\bar{Q}^{+\hat{+}})^R \phi^i \check{\phi}^{\ell-i}]\,. \tag{5.60}$$

In the limit $\kappa \to 0$, this operator breaks into R mesons $(\bar{Q}Q)^{\check{a}}_{\check{b}}$ of $\mathcal{N}=2$ SCQCD and ℓ decoupled scalars $\check{\phi}^{\check{a}}_{\check{b}}$.

(iv) $\hat{\mathcal{C}}_{R(0,0)}$ and $\bar{\mathcal{C}}_{R,-\ell(0,0)}$ multiplets.

We have not studied these cases in detail since they are technically quite involved. It is clear however that for $\check{g} \to 0$ the protected primaries must evolve into states of the schematic form

$$\mathrm{Tr}\left[\mathcal{O}^R_{\mathbf{3_R 3_L}} \check{\phi}^{\ell+n} \bar{\check{\phi}}^n\right]\,, \tag{5.61}$$

with $\ell = 0$, $n = 1$ for $\hat{\mathcal{C}}_{R(0,0)}$ and $n = 1$ for $\bar{\mathcal{C}}_{R,-\ell(0,0)}$.

Finally, let us consider the the long multiplets ($\mathcal{A}$-type multiplets) that appear on the right hand side of the decomposition (4.21). Their schematic expression at the orbifold point is given in the last row of Table 5. At the orbifold point, the planar inheritance theorem [41, 45] guarantees that the multiplets, even if long, have protected dimension at large N_c, since they descend from 1/2 BPS chiral primary operators of $\mathcal{N}=4$ SYM. In fact AdS/CFT predicts that they remain protected (at large N_c) even away from the orbifold point, since they correspond to supergravity states which are unaffected by a change in the value B_{NS}. It would be interesting to check this explicitly using our one-loop hamiltonian: for all $\check{g}$ there must be an eigenvector with zero eigenvalue with the quantum numbers of each of the long multiplets on on the right hand side of (4.21). As $\tilde{g} \to 0$, these operators must evolve into states of the form (5.61), with the appropriate quantum numbers.

5.5 Two-body scattering

In this section we study the scattering of two magnons in the spin chain for the interpolating SCFT. We take the chain to be infinite. Because of the index structure of the impurities, one of the asymptotic magnons must be a Q and the other a $\bar{Q}$, and their ordering is fixed – we can have a Q impurity always to the left of a $\bar{Q}$ impurity, or viceversa. The scattering is thus pure reflection. For the case of Q to the *left* of $\bar{Q}$, and suppressing momentarily the $SU(2)_L \times SU(2)_R$ quantum numbers, the asymptotic form of the eigenstates of the Hamiltonian is

$$\sum_{x_1 \ll x_2} \left(e^{ip_1 x_1 + ip_2 x_2} + S(p_2, p_1) e^{ip_2 x_1 + ip_1 x_2} \right) | \ldots \phi Q(x_1) \check{\phi} \ldots \check{\phi} \bar{Q}(x_2) \phi \ldots \rangle \,. \tag{5.62}$$

This is the definition of the two-body S-matrix. In fact, thanks to the nearest-neighbor nature of the spin chain, if the impurities are not adjacent we are already in the "asymptotic" region, so $x_1 \ll x_2$ should be interpreted as $x_1 < x_2 - 1$. Similarly, for the case where Q to the *right* of $\bar{Q}$ the asymptotic form of the two-magnon state is

$$\sum_{x_1 \ll x_2} \left(e^{ip_1 x_1 + ip_2 x_2} + \check{S}(p_2, p_1) e^{ip_2 x_1 + ip_1 x_2} \right) | \ldots \check{\phi} \bar{Q}(x_1) \phi \ldots \phi Q(x_2) \check{\phi} \ldots \rangle \,, \tag{5.63}$$

which defines $\check{S}$. The two-body S-matrices S and $\check{S}$ are related by exchanging $g \leftrightarrow \check{g}$,

$$S(p_1, p_2; g, \check{g}) = \check{S}(p_1, p_2; \check{g}, g) \,. \tag{5.64}$$

The total energy of a two-magnon state is just the sum of the energy of the individual magnons,

$$E(p_1, p_2; \kappa) = \left(2(1-\kappa)^2 + 8\kappa (\sin^2 \frac{p_1}{2}) \right) + \left(2(1-\kappa)^2 + 8\kappa (\sin^2 \frac{p_2}{2}) \right) . \tag{5.65}$$

Besides the continuum of states with real momenta p_1 and p_2, there can be bound and "anti-bound" states for special complex values of the momenta. A bound state occurs when

$$S(p_1, p_2) = \infty \,, \quad \text{with} \quad p_1 = \frac{P}{2} - iq \,, \quad p_2 = \frac{P}{2} + iq \,, \quad q > 0 \,. \tag{5.66}$$

Since $S(p_2,p_1) = 1/S(p_1,p_2) \to 0$, the asymptotic wave-function is

$$e^{iP\frac{x_1+x_2}{2} - q(x_2-x_1)}\,,, \tag{5.67}$$

which is indeed normalizable (since $x_2 > x_1$ in our conventions). A bound state has smaller energy than any state in the two-particle continuum with the same total momentum P. An anti-bound state occurs when

$$S(p_1,p_2) = \infty\,, \quad \text{with} \quad p_1 = \frac{P}{2} - iq + \pi\,, \quad p_2 = \frac{P}{2} + iq - \pi\,, \quad q > 0 \tag{5.68}$$

The asymptotic wave-function is now

$$(-1)^{x_2-x_1} e^{iP\frac{x_1+x_2}{2} - q(x_2-x_1)}\,. \tag{5.69}$$

The energy of an anti-bound state is strictly bigger than the two-particle continuum. It is easy to see that (5.66) and (5.68) are the only allowed possibilities for complex p_1 and p_2 such that the total momentum and the total energy are real.

The analysis of two-body scattering proceeds independently in four different sectors, corresponding the choice of the triplet or singlet combinations for $SU(2)_L$ and $SU(2)_R$. In each sector, we will compute the S-matrix and look for the (anti)bound states associated to its poles.

5.5.1 $3_L \otimes 3_R$ Sector

In the $3_L \otimes 3_R$ sector, we write the general two-impurity state with Q to the left of $\bar{Q}$ as as

$$|\Psi_{3\otimes 3}\rangle = \sum_{x_1<x_2} \Psi_{3\otimes 3}(x_1,x_2)|\ldots\phi Q(x_1)\check{\phi}\ldots\check{\phi}\bar{Q}(x_2)\phi\ldots\rangle_{3\otimes 3}\,. \tag{5.70}$$

There is no mixing with states containing $\bar{\phi}$ and $\bar{\check{\phi}}$ since they have different $SU(2)_L \times SU(2)_R \times U(1)_r$ quantum numbers. Acting with the Hamiltonian, one finds:

- For $x_2 > x_1 + 1$,

$$\begin{aligned} g^2 H \cdot \Psi_{3\otimes 3}(x_1,x_2) \;\; = \;\; & 4(g^2+\check{g}^2)\Psi_{3\otimes 3}(x_1,x_2) \\ & -2g\check{g}\Psi_{3\otimes 3}(x_1+1,x_2) - 2g\check{g}\Psi_{3\otimes 3}(x_1-1,x_2) \\ & -2g\check{g}\Psi_{3\otimes 3}(x_1,x_2+1) - 2g\check{g}\Psi_{3\otimes 3}(x_1,x_2-1) \end{aligned} \tag{5.71}$$

- For $x_2 = x_1 + 1$,

$$\begin{aligned} g^2 H \cdot \Psi_{3\otimes 3}(x_1, x_2) &= 4g^2 \Psi_{3\otimes 3}(x_1, x_2) \\ &\quad -2g\check{g}\Psi_{3\otimes 3}(x_1 - 1, x_2) - 2g\check{g}\Psi_{3\otimes 3}(x_1, x_2 + 1)\,. \end{aligned} \tag{5.72}$$

The plane wave states $e^{i(p_1 x_1 + p_2 x_2)}$ and $e^{i(p_1 x_2 + p_2 x_1)}$ are separately eigenstates for the "bulk" action of the Hamiltonian (5.71), with eigenvalue (5.65). The action of the Hamiltonian on the state with adjacent impurities, equ.(5.72), provides the boundary condition that fixes the exact eigenstate of asymptotic momenta p_1, p_2,

$$\Psi_{3\otimes 3}(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S_{3\otimes 3}(p_1, p_2) e^{i(p_1 x_2 + p_2 x_1)}\,, \tag{5.73}$$

where

$$S_{3\otimes 3}(p_1, p_2) = -\frac{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_1}}{1 + e^{ip_1 + ip_2} - 2\kappa e^{ip_2}}\,, \qquad \kappa \equiv \frac{\check{g}}{g}\,. \tag{5.74}$$

In this sector, the S-matrix coincides with the familiar S-matrix of the XXZ chain, with the identification $\Delta_{XXZ} = \kappa$. The pole of the S-matrix,

$$e^{ip_2} = \frac{1 + e^{i(p_1 + p_2)}}{2\kappa}\,, \tag{5.75}$$

is associated to a bound state. Writing $p_1 = P/2 - iq$, $p_2 = P/2 + iq$, we have

$$e^{-q} = \frac{\cos(\frac{P}{2})}{\kappa}\,. \tag{5.76}$$

The wave-function is normalizable provided $q > 0$, which implies $2 \arccos \kappa < |P| < \pi$. Substituting p_1 and p_2 into the expression for the total energy (5.65), we find that the dispersion relation of the bound state is simply

$$\left[Q\bar{Q}\right]^{bound}_{3_L\, 3_R}: \qquad E = 4\sin^2(\frac{P}{2})\,, \quad 2\arccos\kappa < |P| < \pi\,. \tag{5.77}$$

This dispersion relation is plotted as the dashed orange curve in the left column of Figure 2.2. When the total momentum P is smaller than $2 \arccos \kappa$ the bound state dissolves into the two-particle continuum. The bound state exists for the full range of P at the orbifold point $\kappa = 1$, but the allowed range of P shrinks as κ is decreased, and the bound state disappears in the SCQCD limit $\kappa \to 0$.

The S-matrix in the $3_L \otimes 3_R$ sector with Q to the *right* of $\bar{Q}$ is obtained by switching $g \leftrightarrow \check{g}$,

$$\check{S}_{3\otimes 3}(p_1, p_2; \kappa) = S_{3\otimes 3}(p_1, p_2; 1/\kappa) - \frac{1 + e^{ip_1 + ip_2} - \frac{2}{\kappa} e^{ip_1}}{1 + e^{ip_1 + ip_2} - \frac{2}{\kappa} e^{ip_2}} \,. \tag{5.78}$$

Now the pole of the S-matrix is associated to a bound state with

$$e^{-q} = \kappa \cos(\frac{P}{2}) \,. \tag{5.79}$$

The bound state exists for all P in the whole range of $\kappa \in (0, 1]$. Its dispersion relation is

$$\left[\bar{Q}Q\right]^{bound}_{3_L\, 3_R} : \qquad E = 4\kappa^2 \sin^2(\frac{P}{2}) \,, \tag{5.80}$$

plotted as the dashed orange curve in the right column of Figure 2.2. The existence of this bound state is consistent with our analysis of the protected spectrum.

5.5.2 $1_L \otimes 3_R$ Sector

The general two-body state with Q to the left of $\bar{Q}$ is

$$|\Psi_{1\otimes 3}\rangle = \sum_{x_1 < x_2} \Psi_{1\otimes 3}(x_1, x_2) | \ldots \phi Q(x_1) \check{\phi} \ldots \check{\phi} \bar{Q}(x_2) \phi \ldots \rangle_{1\otimes 3} \tag{5.81}$$

The action of the Hamiltonian for $x_2 = x_1 + 1$ is now

$$g^2 H \cdot \Psi_{1\otimes 3}(x, x{+}1) = 8g^2 \Psi_{1\otimes 3}(x, x{+}1) - 2g\check{g}\Psi_{1\otimes 3}(x{-}1, x{+}1) - 2g\check{g}\Psi_{1\otimes 3}(x, x{+}2) \,. \tag{5.82}$$

Writing

$$\Psi_{1\otimes 3}(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S_{1\otimes 3}(p_2, p_1) e^{i(p_1 x_2 + p_2 x_1)} \,, \tag{5.83}$$

we find

$$S_{1\otimes 3}(p_1, p_2; \kappa) = -\frac{1 + e^{ip_1 + ip_2} - 2(\kappa - \frac{1}{\kappa}) e^{ip_1}}{1 + e^{ip_1 + ip_2} - 2(\kappa - \frac{1}{\kappa}) e^{ip_2}} \,, \tag{5.84}$$

which is again the S-matrix of the XXZ chain, now with $\Delta = \kappa - \frac{1}{\kappa}$. The S-matrix blows up for

$$e^{ip_2} = \frac{1 + e^{i(p_1 + p_2)}}{2(\kappa - \frac{1}{\kappa})} \,. \tag{5.85}$$

This pole is associated to an *anti*-bound state. Parametrizing $p_1 = P/2 - iq + \pi$, $p_2 = P/2 - iq - \pi$, the location of the pole is given by

$$e^{-q} = \frac{\cos(\frac{P}{2})}{\kappa - \frac{1}{\kappa}}\,. \tag{5.86}$$

Normalizability of the wave-function requires $q > 0$, which occurs for a restricted range of P for $\kappa_* < \kappa < 1$, and for the full range of P for $\kappa < k_*$,

$$\begin{aligned} 2\arccos(\frac{1}{\kappa} - \kappa) < |P| < \pi \quad &\text{for } \frac{\sqrt{5}-1}{2} < \kappa < 1 \\ 0 < |P| < \pi \quad &\text{for } 0 < \kappa < \frac{\sqrt{5}-1}{2}\,. \end{aligned} \tag{5.87}$$

Substituting in $E(p_1, p_2; \kappa)$ we find the dispersion relation for the anti-bound state,

$$\big[Q\bar{Q}\big]^{antibound}_{1_L\,3_R}: \qquad E = \frac{4(2-\kappa^2)}{1-\kappa^2} - \frac{4\kappa^2}{1-\kappa^2}\sin^2\frac{P}{2}\,, \tag{5.88}$$

which is plotted as the red curve in the left column of Figure 2.2. The anti-bound state is absent at the orbifold point $\kappa = 1$. For $\kappa \to 0$, $q \to +\infty$, so that the wave-function (5.69) localizes to two neighboring sites and in fact coincides with the dimeric excitation $\mathcal{M}_{\mathbf{3}} = (Q\bar{Q})_{\mathbf{3}}$ of $\mathcal{N} = 2$ SCQCD; in the limit we smoothly recover the $\mathcal{M}_{\mathbf{3}}$ dispersion relation $E(P) = 8$.

For $\bar{Q}Q$ scattering, we have

$$\check{S}_{1\otimes 3}(p_1, p_2; \kappa) = S_{1\otimes 3}(p_1, p_2; 1/\kappa) = -\frac{1 + e^{ip_1 + ip_2} - 2(\frac{1}{\kappa} - \kappa)e^{ip_1}}{1 + e^{ip_1 + ip_2} - 2(\frac{1}{\kappa} - \kappa)e^{ip_2}}\,. \tag{5.89}$$

Now the pole corresponds to a bound state, indeed it occurs for $p_1 = P/2 - iq$, $p_2 = P/2 + iq$ with q and P related as in (5.86). Clearly the allowed range of P is as in (5.87). We find the dispersion relation

$$\big[Q\bar{Q}\big]^{bound}_{1_L\,3_R}: \qquad E = \frac{4\kappa^2}{(1-\kappa^2)}(1 - 2\kappa^2 + \sin^2\frac{P}{2})\,, \tag{5.90}$$

which is plotted as the red curve in the right column of Figure 2.2.

5.5.3 $3_L \otimes 1_R$ Sector

The scattering problem in the $3_L \otimes 1_R$ sector is solved by the same technique. We find

$$S_{3\otimes 1}(p_1, p_2) = \check{S}_{3\otimes 1}(p_1, p_2) - 1\,, \tag{5.91}$$

which coincides with the scattering matrix of free fermions, or with the $\Delta_{XXZ} \to \infty$ limit of the S-matrix for the XXZ chain. Clearly there are no (anti-)bound states.

5.5.4 $1_L \otimes 1_R$ Sector

The analysis for the $1_L \otimes 1_R$ sector is slightly more involved because a two-impurity state is not closed under the action of Hamiltonian: when Q and $\bar{Q}$ are next to each other they can transform into $\phi\bar{\phi}$. The general state for $Q\bar{Q}$ scattering in the singlet sector is

$$\begin{aligned} |\Psi_{1\otimes 1}\rangle &= \sum_{x_1<x_2} \Psi_{1\otimes 1}(x_1, x_2)|\ldots\phi Q(x_1)\check{\phi}\ldots\check{\phi}\bar{Q}(x_2)\phi\ldots\rangle_{1\otimes 1} \\ &\quad + \sum_{x} \Psi_{\bar{\phi}}(x)|\ldots\phi\bar{\phi}(x)\phi\ldots\rangle \end{aligned}$$

The first term is an eigenstate for "bulk" action of the Hamiltonian ($x_2 > x_1 + 1$) with the usual eigenvalue $E(p_1, p_2; \kappa)$ of equ.(5.65). The action of the Hamiltonian for $x_2 = x_1 + 1$ is

$$\begin{aligned} g^2 H \cdot \Psi_{1\otimes 1}(x, x+1) &= 4(g^2 + \check{g}^2)\Psi_{1\otimes 1}(x, x+1) \\ &\quad -2g\check{g}\Psi_{1\otimes 1}(x-1, x+1) - 2g\check{g}\Psi_{1\otimes 1}(x, x+2) \\ &\quad +2g^2\Psi_{\bar{\phi}}(x) + 2g^2\Psi_{\bar{\phi}}(x+1)\,. \end{aligned}$$

Furthermore,

$$\begin{aligned} g^2 H \cdot \Psi_{\bar{\phi}}(x) &= 6g^2\Psi_{\bar{\phi}}(x) - g^2\Psi_{\bar{\phi}}(x+1) - g^2\Psi_{\bar{\phi}}(x-1) \\ &\quad +2g^2\Psi_{1\otimes 1}(x, x+1) + 2g^2\Psi_{1\otimes 1}(x-1, x)\,. \end{aligned}$$

We take the ansatz

$$\begin{aligned} \Psi_{1\otimes 1}(x_1, x_2) &= e^{i(p_1 x_1 + p_2 x_2)} + S_{1\otimes 1}(p_2, p_1)e^{i(p_1 x_2 + p_2 x_1)} \\ \Psi_{\bar{\phi}}(x) &= S_{\bar{\phi}}(p_2, p_1)e^{i(p_1+p_2)x} \end{aligned}$$

Note that $S_{1\otimes 1}(p_1, p_2)$ still has the interpretation of the scattering matrix of the magnons Q and $\bar{Q}$, which are the asymptotic excitations, while $\bar{\phi}$ is an "unstable" excitations created during the collision of Q and $\bar{Q}$. We find

$$\begin{aligned} S_{1\otimes 1}(p_1,p_2) &= -\left(\frac{1+e^{ip_1+ip_2}-2(\kappa-\frac{1}{\kappa})e^{ip_1}}{1+e^{ip_1+ip_2}-2(\kappa-\frac{1}{\kappa})e^{ip_2}}\right)\left(\frac{1+e^{ip_1+ip_2}-2\kappa e^{ip_1}}{1+e^{ip_1+ip_2}-2\kappa e^{ip_2}}\right)^{-1} \\ S_{\bar{\phi}}(p_1,p_2) &= \frac{4ie^{i(p_1+p_2)}(\sin p_1-\sin p_2)}{(1+e^{ip_1+ip_2}-2\kappa e^{ip_1})(1+e^{ip_1+ip_2}-2(\kappa-\frac{1}{\kappa})e^{ip_2})}\,. \end{aligned}$$

$S_{1\otimes 1}$ is the product of two factors, and it admits two poles. The first factor coincides with $S_{1\otimes 3}$, so its pole is associated to an anti-bound state entirely analogous to the anti-bound state in the $1_L \otimes 3_R$ sector. The pole is located at $p_1 = P/2 - iq + \pi$, $p_2 = P/2 + iq - \pi$, with

$$e^{-q} = \frac{\cos(P/2)}{\frac{1}{\kappa}-\kappa}\,. \tag{5.92}$$

The dispersion relation is again

$$\left[Q\bar{Q}\right]^{antibound}_{1_L\,1_R}: \qquad E = \frac{4(2-\kappa^2)}{1-\kappa^2} - \frac{4\kappa^2}{1-\kappa^2}\sin^2\frac{P}{2}\,, \tag{5.93}$$

and the range of P for which the wave-function is normalizable is as in (5.87) – see the red curve in the left column of Figure 2.2. It is interesting to analyze the explicit form of the wave-function in the $\kappa \to 0$ limit. The $Q\bar{Q}$ piece has the form

$$\Psi_{1\otimes 1}(x_1,x_2) = (-1)^{x_2-x_1} e^{iP(\frac{x_1+x_2}{2})} e^{-q(x_2-x_1)}\,, \quad q\to\infty \tag{5.94}$$

so only the $x_2 = x_1 + 1$ term survives in the limit, and we recover the dimeric impurity $\mathcal{M}_{\mathbf{1}}$ of SCQCD. A short calculation gives

$$\frac{\Psi_{\bar{\phi}}(x)}{\Psi(x,x+1)}|_{\kappa\to 0} = \frac{2}{(1+e^{iP})}\,. \tag{5.95}$$

Comparison with (**??**) shows that that in the $\kappa \to 0$ limit the antibound state in the $Q\bar{Q}$ singlet sector becomes precisely the dimeric excitation $\tilde{T}$ of SCQCD.

The pole in the second factor of $S_{1\otimes 1}$ o corresponds instead to a bound state, with

$$e^{q} = \frac{\cos(P/2)}{\kappa}\,. \tag{5.96}$$

The dispersion relation and range of existence are

$$[Q\bar{Q}]^{bound}_{1_L\,1_R}: \qquad E = 4\sin^2\frac{q}{2}\,, \quad 0 < |P| < 2\,\mathrm{arccos}\kappa\,, \tag{5.97}$$

which are shown as the green curve on the left column of Figure 2.2. This bound state is absent at the orbifold point and comes into full existence (for any P) in the SCQCD limit $\kappa \to 0$. The natural guess is that in this limit it reduces to the gapless $T(p)$ magnon of SCQCD, and it does:

$$\frac{\Psi_{\bar{\phi}}(x)}{\Psi(x,x+1)}|_{\kappa\to 0} = -\frac{1+e^{-iP}}{2}\,, \tag{5.98}$$

in agreement with (5.34).

5.5.5 Summary

We have seen that the two-body scattering problem in the spin-chain of the interpolating SCFT admits a rich spectrum of bound and anti-bound states. The results are summarized in Table 5.6 and Figure 5.7. The $Q\bar{Q}$ scattering channel (that is, the channel with Q to the left of $\bar{Q}$, and the ϕ vacuum on the outside) is the one relevant to make contact with $\mathcal{N}=2$ SCQCD, which is obtained in the $\kappa \to 0$ limit. Remarkably, the magnon excitations of SCQCD are recovered as the smooth limits of the $Q\bar{Q}$ (anti)bound states: as $\kappa \to 0$ the wavefunctions of the (anti)bound states localize to two neighboring sites and reproduce the "dimeric" magnons $T(p)$, $\tilde{T}(p)$ and $\mathcal{M}_{\mathbf{3}}(p)$ of SCQCD.

5.5.6 Let/right factorization of the two-body S-matrix

As is well-known, the magnon excitations of the $\mathcal{N}=4$ SYM spin-chain transform in the fundamental representation of $SU(2|2)\times SU(2|2)$, and their two-body S-matrix factorizes into the product of the S-matrices for the "left" and "right" $SU(2|2)$. The $\mathbb{Z}_2$ orbifold preserves this factorization. Remarkably, this left/right factorization persists even away from the orbifold point, for the full interpolation SCFT – or at least this is what happens at one-loop in the scalar sector. Our results for the S-matrix in the $Q\bar{Q}$ channel in the four different $SU(2)_L\times SU(2)_R$ sectors are summarized in Table 5.8, where we have defined

$$\mathcal{S}(p_1,p_2,\kappa) \equiv -\frac{1-2\kappa e^{ip_1}+e^{i(p_1+p_2)}}{1-2\kappa e^{ip_2}+e^{i(p_1+p_2)}} \tag{5.99}$$

	Pole of S-matrix	Range of existence	Dispersion relation
$\mathcal{M}_{33}$	$e^{-q} = \frac{1}{\kappa}\cos(\frac{P}{2})$	$2\arccos\kappa < \lvert P\rvert < \pi$	$4\sin^2(\frac{P}{2})$
T	$e^{q} = \frac{1}{\kappa}\cos(\frac{P}{2})$	$0 < \lvert P\rvert < 2\arccos\kappa$	$4\sin^2(\frac{P}{2})$
$\tilde{T}$	$e^{-q} = \frac{\cos(\frac{P}{2})}{(\kappa-\frac{1}{\kappa})}$	See equ.(5.87)	$\frac{4\kappa^2}{(1-\kappa^2)}(\frac{2}{\kappa^2} - 1 - \sin^2\frac{P}{2})$
$\mathcal{M}_3$	$e^{-q} = \frac{\cos(\frac{P}{2})}{(\kappa-\frac{1}{\kappa})}$	See equ.(5.87)	$\frac{4\kappa^2}{(1-\kappa^2)}(\frac{2}{\kappa^2} - 1 - \sin^2\frac{P}{2})$
$\check{\mathcal{M}}_{33}$	$e^{-q} = \kappa\cos(\frac{P}{2})$	$0 < \lvert P\rvert < \pi$	$4\kappa^2\sin^2(\frac{P}{2})$
$\check{T}$	$e^{q} = \kappa\cos(\frac{P}{2})$	No solution	
$\check{\tilde{T}}$	$e^{-q} = \frac{\cos(\frac{P}{2})}{(\kappa-\frac{1}{\kappa})}$	$2\arccos(\frac{1}{\kappa} - \kappa) < \lvert P\rvert < \pi$	$\frac{4\kappa^2}{(1-\kappa^2)}(1 - 2\kappa^2 + \sin^2\frac{P}{2})$
$\check{\mathcal{M}}_3$	$e^{-q} = \frac{\cos(\frac{P}{2})}{(\kappa-\frac{1}{\kappa})}$	$2\arccos(\frac{1}{\kappa} - \kappa) < \lvert P\rvert < \pi$	$\frac{4\kappa^2}{(1-\kappa^2)}(1 - 2\kappa^2 + \sin^2\frac{P}{2})$

Table 5.6: Dispersion relations and range of existence of the various (anti)bound states in two-body scattering. The first three entries correspond to the $Q\bar{Q}$ channel and the last three entries to the $\bar{Q}Q$ channel. The color-coding of the third entry is a reminder that these are *anti*-bound states with energy above the two-particle continuum.

i.e. the standard S-matrix of the XXZ chain, with $\Delta_{XXZ} = \kappa$.

$$\mathcal{S}(p_1, p_2, 1) = \mathcal{S}_{SU(2)}(p_1, p_2) \tag{5.100}$$

$$\mathcal{S}(p_1, p_2, 0) = -1 \text{ (Free fermion)} \tag{5.101}$$

$$\mathcal{S}(p_1, p_2, \pm\infty) = -e^{i(p_1-p_2)} \text{ (Free Boson?)} \tag{5.102}$$

We see that we can write

$$S(p_1, p_2; \kappa) = \frac{S_L(p_1, p_2; \kappa) S_R(p_1, p_2; \kappa)}{S_{3\otimes 3}(p_1, p_2; \kappa} \tag{5.103}$$

where S_L and S_R are defined in Table 5.9.

In the analysis of the Yang-Baxter equation, it will be useful to write the S-matrices in both the $SU(2)_L$ and $SU(2)_R$ sectors using the identity ($\mathbb{I}$) and trace ($\mathbb{K}$) tensorial structures,

$$S_L(p_1, p_2; \kappa) = A_L(p_1, p_2; \kappa)\,\mathbb{I} + B_L(p_1, p_2; \kappa)\,\mathbb{K} \tag{5.104}$$

$$S_R(p_1, p_2; \kappa) = A_R(p_1, p_2; \kappa)\,\mathbb{I} + B_R(p_1, p_2; \kappa)\,\mathbb{K}\,. \tag{5.105}$$

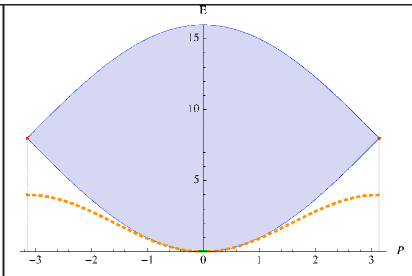
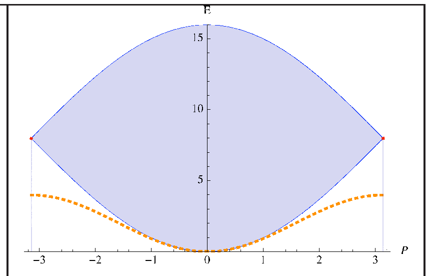
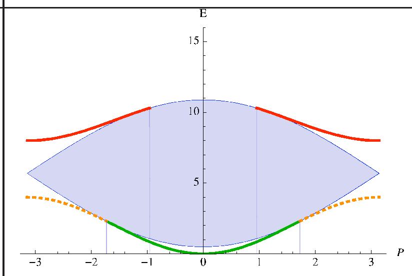
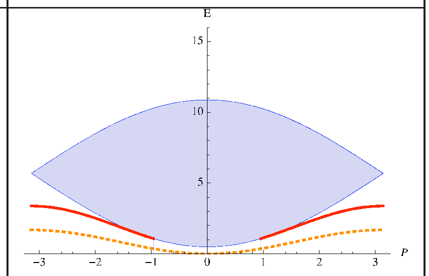
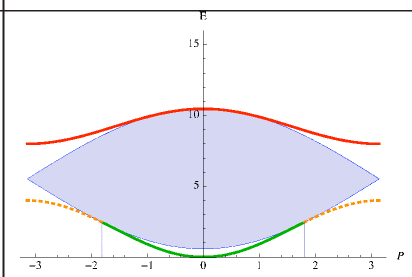
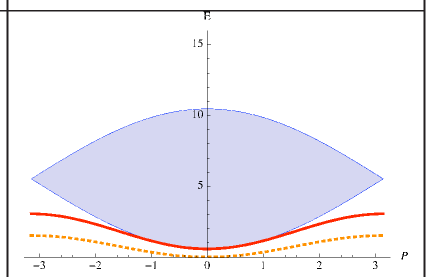
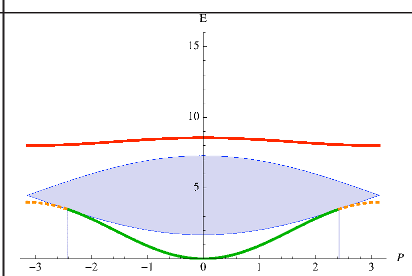
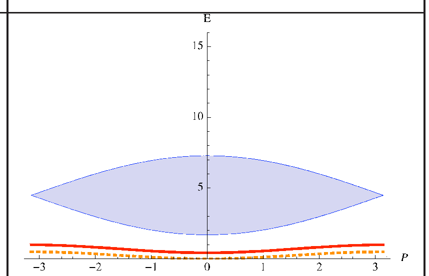
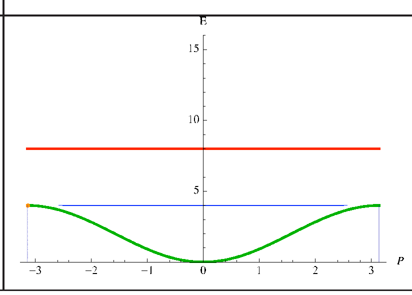
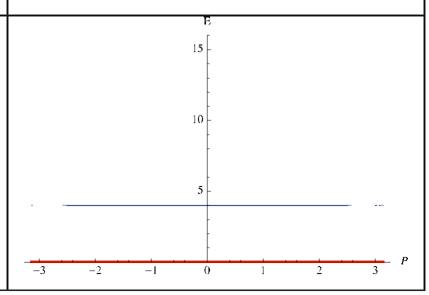
	$Q\bar{Q}$ scattering channel	$\bar{Q}Q$ scattering channel
$\kappa = 0.999$		
$\kappa = 0.65$		
$\kappa = \frac{-1+\sqrt{5}}{2}$		
$\kappa = 0.35$		
$\kappa = 0.001$		

Table 5.7: Plots of the dispersion relations of the (anti)bound states for different values of κ. The shaded region represents the two-particle continuum.

$L \otimes R$	$S(p_1, p_2, \kappa)$
$1 \otimes 1$	$-\mathcal{S}(p_1, p_2, \kappa - \frac{1}{\kappa})\mathcal{S}^{-1}(p_1, p_2, \kappa)$
$1 \otimes 3$	$\mathcal{S}(p_1, p_2, \kappa - \frac{1}{\kappa})$
$3 \otimes 1$	-1
$3 \otimes 3$	$\mathcal{S}(p_1, p_2, \kappa)$

Table 5.8: The S-matrix in the $Q\bar{Q}$ scattering channel.

$SU(2)_L$	$S_L(p_1, p_2; \kappa)$	$SU(2)_R$	$S_R(p_1, p_2; \kappa)$
$\mathbf{1}$	$\mathcal{S}(p_1, p_2; \kappa - \frac{1}{\kappa})$	$\mathbf{1}$	-1
$\mathbf{3}$	$\mathcal{S}(p_1, p_2; \kappa)$	$\mathbf{3}$	$\mathcal{S}(p_1, p_2; \kappa)$

Table 5.9: Definitions of the $SU(2)_L$ and $SU(2)_R$ S-matrices.

Writing the indices explicitly,

$$(S_R)_{\mathcal{I}\mathcal{J}}^{\mathcal{M}\mathcal{N}} = A_R\, \delta_{\mathcal{I}}^{\mathcal{M}}\delta_{\mathcal{J}}^{\mathcal{N}} + B_R\, g_{\mathcal{I}\mathcal{J}}g^{\mathcal{M}\mathcal{N}}\,, \tag{5.106}$$

Recalling that eigenvalue of $\mathbb{K}$ on the triplet is zero while it is two on the singlet, we see that

$$A = S_{\mathbf{3}} \tag{5.107}$$

$$B = \frac{1}{2}(S_{\mathbf{1}} - S_{\mathbf{3}})\,, \tag{5.108}$$

where the values of $S_{\mathbf{1}}$ and $S_{\mathbf{3}}$ in both the $SU(2)_L$ and $SU(2)_R$ sectors can be read off from Table 5.9.

In complete analogy, in the $\bar{Q}Q$ channel we have the factorization

$$\check{S}(p_1, p_2; \kappa) = \frac{\check{S}_L(p_1, p_2; \kappa)\check{S}_R(p_1, p_2; \kappa)}{\check{S}_{3\otimes 3}(p_1, p_2; \kappa}\,, \tag{5.109}$$

and we can write

$$\check{S}_L = \check{A}_L\,\mathbb{I} + \check{B}_L\,\mathbb{K} \tag{5.110}$$

$$\check{S}_R = \check{A}_R\,\mathbb{I} + \check{B}_R\,\mathbb{K}\,. \tag{5.111}$$

As always, each "checked" quantity is obtained from the corresponding unchecked one by sending $\kappa \to 1/\kappa$.

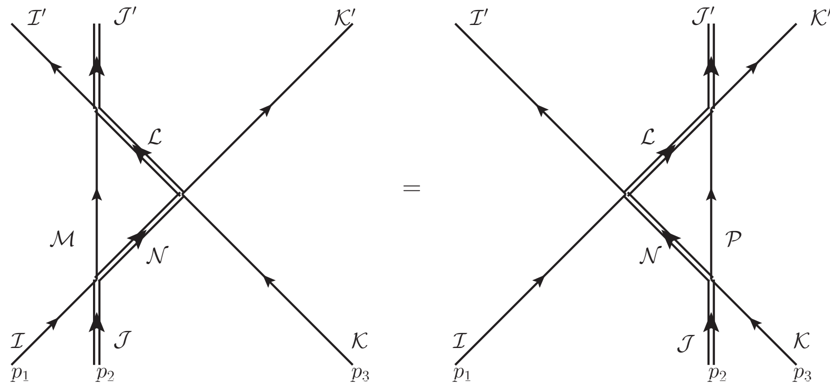


Figure 5.2: Yang Baxter equation for Orbifold theory

5.6 Yang-Baxter Equation

Integrability of the spin chain is the statement of existence of infinitely many conserved charges or equivalently it amounts to factorization of many body scattering matrix into two body scattering matrices. The exact check of this statement could be quite hard for a generic spin chain.

The factorization of many body scattering matrix in particular implies that the three body scattering matrix should factor. For integrable systems the three body scattering process, diagrammatically, can be denoted as product of two body scatterings in two ways as in figure 5.2. Integrability implies that the two body scatterings should "commute" and hence both the processes should be equivalent. This is the statement of "coordinate" Yang Baxter equation. This should necessarily hold if the system is integrable but of course it is not a sufficient condition.

We check this condition as it is the simplest step towards the exploring the integrability of the system. Failure to satisfy this condition would necessarily mean that the system is not integrable. For the system that we are studying, the Yang Baxter equation factorizes into two equations on account of factorization of the full scalar two body scattering matrix into $SU(2)_L$ and $SU(2)_R$ sectors. So the problem of checking the Yang Baxter equation for $SU(2)_L \otimes SU(2)_R$ spin chain reduces to checking it for $SU(2)_L$ and $SU(2)_R$ sectors separately. It will be satisfied for the full spin chain if and only if it is satisfied for both the $SU(2)$ sectors.

Let us start with a generic $SU(2)$ spin chain which can be either $SU(2)_L$ or $SU(2)_R$ spin chains in that it has two types of excitations Q and $\bar{Q}$ and transmission coefficient is zero.

The figure 5.2 represents the three body Yang Baxter equation to be verified. Written explicitly in indices, this equation is:

$$S^{\mathcal{MN}}_{\mathcal{IJ}}(p_1,p_2)\check{S}^{\mathcal{LK}'}_{\mathcal{NK}}(p_1,p_3)S^{\mathcal{I}'\mathcal{J}'}_{\mathcal{ML}}(p_2,p_3) = \check{S}^{\mathcal{J}'\mathcal{K}'}_{\mathcal{LP}}(p_1,p_2)S^{\mathcal{I}'\mathcal{L}}_{\mathcal{IN}}(p_1,p_3)\check{S}^{\mathcal{NP}}_{\mathcal{JK}}(p_2,p_3) \tag{5.112}$$

We simplify the equation by decomposing the two body scattering matrices into identity and trace piece as described in the previous section. While writing the following equation, we suppress the momentum arguments with the convention that the first symbol in each term is a function of (p_1,p_2), the second is function of (p_1,p_3) and the third (p_2,p_3). We do not change the order of symbols while simplifying the expression.

$$\begin{aligned}
& S^{\mathcal{MN}}_{\mathcal{IJ}}(p_1,p_2)\check{S}^{\mathcal{LK}'}_{\mathcal{NK}}(p_1,p_3)S^{\mathcal{I}'\mathcal{J}'}_{\mathcal{ML}}(p_2,p_3) \\
= \; & A\check{A}A\delta^{\mathcal{K}'}_{\mathcal{K}}\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}'}_{\mathcal{J}} + \check{B}Bg_{\mathcal{JK}}\delta^{\mathcal{K}'}_{\mathcal{I}}g^{\mathcal{I}'\mathcal{J}'} + B\check{B}Ag_{\mathcal{IJ}}\delta^{\mathcal{I}'}_{\mathcal{K}}g^{\mathcal{J}'\mathcal{K}'} \\
+ \; & (A\check{A}B + B\check{A}A + 2B\check{A}B + B\check{B}B)\delta^{\mathcal{K}'}_{\mathcal{K}}g_{\mathcal{IJ}}g^{\mathcal{I}'\mathcal{J}'} + A\check{B}Ag_{\mathcal{JK}}g^{\mathcal{J}'\mathcal{K}'}\delta^{\mathcal{I}'}_{\mathcal{I}}
\end{aligned}$$

The right hand side can be simplified in the same way,

$$\begin{aligned}
& \check{S}^{\mathcal{J}'\mathcal{K}'}_{\mathcal{LP}}(p_1,p_2)S^{\mathcal{I}'\mathcal{L}}_{\mathcal{IN}}(p_1,p_3)\check{S}^{\mathcal{NP}}_{\mathcal{JK}}(p_2,p_3) \\
= \; & \check{A}A\check{A}\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}'}_{\mathcal{J}}\delta^{\mathcal{K}'}_{\mathcal{K}} + \check{A}B\check{B}g^{\mathcal{I}'\mathcal{J}'}g_{\mathcal{JK}}\delta^{\mathcal{K}'}_{\mathcal{I}} + \check{B}B\check{A}g^{\mathcal{J}'\mathcal{K}'}g_{\mathcal{IJ}}\delta^{\mathcal{I}'}_{\mathcal{K}} \\
+ \; & \check{A}B\check{A}g_{\mathcal{IJ}}g^{\mathcal{I}'\mathcal{J}'}\delta^{\mathcal{K}'}_{\mathcal{K}} + (\check{A}A\check{B} + \check{B}A\check{A} + 2\check{B}A\check{B} + \check{B}B\check{B})g^{\mathcal{J}'\mathcal{K}'}\delta^{\mathcal{I}'}_{\mathcal{I}}g_{\mathcal{JK}}
\end{aligned}$$

Collecting the terms with various index structures, the Yang Baxter equation reduces to the following five equations.

$$A\check{A}A = \check{A}A\check{A} \tag{5.113}$$

$$A\check{B}B = \check{A}B\check{B} \tag{5.114}$$

$$B\check{B}A = \check{B}B\check{A} \tag{5.115}$$

$$(A\check{A}B + B\check{A}A + 2B\check{A}B + B\check{B}B) = \check{A}B\check{A} \tag{5.116}$$

$$A\check{B}A = (\check{A}A\check{B} + \check{B}A\check{A} + 2\check{B}A\check{B} + \check{B}B\check{B}) \tag{5.117}$$

The full Yang Baxter equation can be reduced to the check of above set of equations for both $SU(2)_L$ and $SU(2)_R$ sectors. In $SU(2)_L$ sector,

$$A_L(p_1,p_2,\kappa) = \check{A}_L(p_1,p_2,\frac{1}{\kappa}) = \mathcal{S}(p_1,p_2,\kappa) \tag{5.118}$$

$$B_L(p_1,p_2,\kappa) = \check{B}_L(p_1,p_2,\frac{1}{\kappa}) = \frac{1}{2}(\mathcal{S}(p_1,p_2,\kappa-\frac{1}{\kappa}) - \mathcal{S}(p_1,p_2,\kappa)) \tag{5.119}$$

while in $SU(2)_R$ sector,

$$A_R(p_1,p_2,\kappa) = \check{A}_R(p_1,p_2,\frac{1}{\kappa}) = \mathcal{S}^{-1}(p_1,p_2,\kappa) \tag{5.120}$$

$$B_R(p_1,p_2,\kappa) = \check{B}_R(p_1,p_2,\frac{1}{\kappa}) = -\frac{1}{2}(1+\mathcal{S}^{-1}(p_1,p_2,\kappa))\,. \tag{5.121}$$

In the case of $\mathbb{Z}_2$ orbifold at $g=\check{g}$ hence $S=\check{S}$ and the nontrivial check of Yang Baxter equation reduces to (5.116). It can be verified that (5.116) is indeed satisfied at $g=\check{g}$. Away from the orbifold point, the Yang Baxter equation fails to be satisfied proving that the spin chain doesn't remain integrable under this marginal deformation. At $\kappa=0$ i.e. for $\mathcal{N}=2$ SCQCD, Yang Baxter equation is again satisfied giving us a hint towards the integrability of SCQCD. There is a subtlety here. Strictly at $\kappa=0$, the two body scattering matrices in both $SU(2)_L$ and $SU(2)_R$ sectors become trivial and the information about the bound states is lost. This can be seen from

In $\mathcal{N}=2$ SCQCD the nontrivial information that the spin chain contains is about the dynamics of "bound states". If we loose the information about the bound state themselves at $\kappa=0$ it would not amount to an indication of the integrability of the SCQCD spin chain. Nonetheless, at infinitesimally small κ, one could derive the S matrix of the bound states from the S matrix of the fundamental excitations assuming integrability. We can check that the failure to satisfy the Yang Baxter equation (i.e. the difference between the left and right hand side of the set of equations) goes continuously to zero as $\kappa\to 0$. At infinitesimal κ the failure to satisfy the Yang Baxter equation is infinitesimal. This could imply that the Yang Baxter equation for the bound states as well is almost satisfied at small κ. In the limiting process of sending κ to zero one could preserve the information about the bound state and show that the Yang Baxter equation is satisfied.

If we assume that the failure to satisfy the Yang Baxter equation is infinitesimal for the bound state scattering matrix if it is infinitesimal for the scattering matrix of the fundamental excitations then this would imply that the Yang-Baxter equation is satisfied in the SCQCD. A concrete check would amount to computing the scattering matrix of the bound states from the fundamental scattering matrix assuming the integrability of the spin chain and then explicitly showing that at $\kappa=0$, the bound state Yang Baxter equation is satisfied.

In the case with $g=\check{g}$, $S=\check{S}$. Hence, in that case the check of the Yang Baxter just reduces to (5.116). These equations are only satisfied at $\kappa=1$ and $\kappa=0$. This means that for $\kappa\neq 0,1$ the spin chain is not integrable. We know that the spin chain of the $\mathbb{Z}_2$ orbifold theory is integrable. This is consistent with the fact that Yang-Baxter equation is satisfied at $\kappa = 1$. For $\kappa = 0$, however, we can see that this primitive check doesn't rule out the integrability.

Chapter 6

Conclusions

In chapter 2 we considered a generic large N, non-supersymmetric field theory that is "conformal in its single-trace sector". In such a theory, quantum effects induce double-trace couplings $f \int d^4x \, \mathcal{O}\bar{\mathcal{O}}$. If β_f has no real zeros, conformal invariance is broken. The symmetric vacuum $\langle\phi\rangle = 0$ is stable if and only if β_f has a fixed point. Conversely, if β_f has no real zeros, dynamical symmetry breaking occurs. The field theory instability associated with double-trace renormalization is the boundary counterpart of the instability associated with a closed string tachyon in the AdS bulk. We showed that at large N the double-trace beta function is quadratic in the coupling f, to all orders in perturbation theory, and $\Delta_{\mathcal{O}}$ is linear. If the discriminant of $\beta_f = 0$ is negative, there are no physical (real) values of f for which the theory is conformal. But if we insist on formally preserving conformal invariance by tuning f to one of its two complex fixed points, then the operator dimension also becomes complex, $\Delta_{\mathcal{O}} = 2 \pm i\, b(\lambda)$. Using the usual AdS/CFT dictionary we found that the AdS_5 scalar dual to $\mathcal{O}$ has mass $m^2(\lambda)R^2 = -4 - b(\lambda)^2 < m^2_{BF}$, *i.e.* it is a true tachyon, and the bulk theory is unstable. Following research should address the questions outlined below:

- Using this general framework we can study specific cases of non supersymmetric field theories that are "conformal in their single-trace sector". In our original paper [6], we presented two examples of non-supersymmetric orbifolds while in our subsequent work [7] we studied an open string example with intersecting $D7$ flavor branes. Further examples, including gamma deformations of $\mathcal{N} = 4$ SYM, orientifolds of $\mathcal{N} = 4$ SYM[1] and non-supersymmetric defect conformal field theory, should be studied.

[1]Pedro Liendo is currently working on this case.

- In [6] we showed that the double-trace beta function is quadratic in the coupling to all orders in planar perturbation theory. Using integrability techniques to calculate the coefficients $v(\lambda)$, $\gamma(\lambda)$ and $a(\lambda)$ of the beta function to all orders in λ and consequently the mass of the tachyon, would provide a novel non-perturbative understanding of tachyons in string theory.

- Finally, our formal work might have applications to real world systems.

  For a second order phase transition the characteristic energy scale m (inverse correlation length) near the ctitical point goes to zero as $m \sim |\alpha - \alpha^*|^\nu$, where α is a parameter that can vary continuously and $\alpha = \alpha^*$ is the location of the critical point. For an other wide class of phase transitions the correlation length vanishes exponentially on one side of the phase transition, while being strictly zero on the other side

$$m \sim \Lambda_{UV}\, \Theta\left(\alpha - \alpha^*\right)\, e^{-\frac{c}{\sqrt{\alpha-\alpha^*}}}\,. \tag{6.1}$$

  This behavior has been observed before in the Berezinskii-Kosterlitz-Thouless (BKT) phase transition in two dimensions. For non-supersymmetric field theories that are "conformal in their single-trace sector" the characteristic energy scale m near the ctitical point always go to zero as in (6.1). This is a consequence of the fact that the beta function is quadratic in the coupling. As the discriminant $D(\lambda)$ decreases, the two fixed points approach each other until they merge at $f_\pm = f_*$ for $D(\lambda) = 0$. For $D(\lambda) < 0$ the solutions to $\beta = 0$ are complex, and the theory no longer has a conformal phase. To see that the fixed point merger generically gives rise to BKT scaling, we simply use our results from section 2 and compute the ratio

$$\frac{\mu_{IR}}{\mu_{UV}} = e^{-\frac{\pi}{b(\lambda)}}\,, \tag{6.2}$$

  which is identical to (6.1) after the identification $D(\lambda) = \alpha - \alpha^*$.

  One example of such an application was presented in Rey's talk at Strings 2007. He discussed the problem of graphene using a non-supersymmetric defect conformal field theory. More examples with this behavior are considered in [189, 190].

In chapter 3 we used the dictionary developed in chapter 2 to study the specific example of the $AdS_5 \times S^5$ background with two probe $D7$ branes intersecting at general angles. In this case, supersymmetry is completely broken and an open string tachyon emerges between the branes. On the field theory side we compute the one-loop Coleman-Weinberg effective potential for the fundamental scalars and discover an instability. We identify the triplet mesonic operator as the dual of the open string tachyon, calculate the tachyon mass for small λ and find that it violates the AdS stability bound.

- The main unresolved puzzle from chapter 3 is to completely fix the functions $f(\theta_1, \theta_2)$ and $d(\theta_1, \theta_2)$ in the quadratic potential.
- Combining the knowledge we gained from chapters 3 and 5 we could study the integrability of the $D3/D7$ brane configuration with intersecting $D7$ branes. The supersymmetric open spin chain with fundamental matter has been studied [98] and found to be integrable. It would be very interesting to check integrability for the non-supersymmetric case. Demanding integrability would provide extra conditions for $f(\theta_1, \theta_2)$ and $d(\theta_1, \theta_2)$ that might correctly fix them.
- An other followup of [7] would be to expand the Lagrangian around the local minimum of the effective potential and recover the shape of the recombined intersecting $D7$ branes after tachyon condensation with a field theory calculation [91].

In chapter 4 we presented work on the direction of finding string theory duals for gauge theories with genuinely less supersymmetry and unquenched flavor. We searched for the gravity dual for $\mathcal{N} = 2$ $SU(N_c)$ SYM coupled to $N_f = 2N_c$ fundamental hypermultiplets using the Veneziano expansion of large N_c and large N_f with the ratio $N_f/N_c = 2$ and $\lambda = g^2 N_c$ kept fixed. In order to advance this research program, additional steps are needed. Some of these are described below. Integrability techniques have been proven an extremely powerful tool in studying planar $\mathcal{N} = 4$ SYM. The same integrability structures arise both on the field theory and on its dual string theory side, allowing a very precise check of the AdS/CFT correspondence. In chapter 5 we began the " bottom-up" study of the field theory dilation operator, by calculating it at one-loop for the scalar sector. Following this line of research there are many open questions we should assess:

- An important open question is whether the one-loop spin chain for $\mathcal{N} = 2$ SCQCD is integrable. The spin chain for the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM (which by definition has $\check{g} = g$) is known to be integrable [63]. In chapter 5 we find that as we move away from the orbifold point integrability is broken. Indeed for general $\kappa = \check{g}/g$ the Yang-Baxter equation for the two-magnon S-matrix does not hold. Remarkably however the Yang-Baxter equation is satisfied again in the $\mathcal{N} = 2$ SCQCD limit $\kappa \to 0$. Ordinarily a check of the Yang-Baxter equation is very strong evidence in favor of integrability. In our case we should be more cautious. Strictly at $\kappa = 0$ the Yang-Baxter equation for the elementary magnons trivializes – what is really relevant is the S-matrix of their dimeric bound states. But since for infinitesimal κ the Yang-Baxter equation for the S-matrix of the elementary magnons fails only infinitesimally, we can hope that the the S-matrix of their dimeric bound states will also fail infinitesimally, and that the theory really becomes integrable for $\kappa \to 0$. While this intuitive argument is a reason for optimism, it is no substitute for a careful calculation of the bound-state S-matrix, which should be carried out.

- The next step is to compute the complete one-loop Hamiltonian of $\mathcal{N} = 2$ SCQCD (and for the deformed orbifold theory) for all possible sectors, including fermions and derivatives. In [9] we computed the one-loop Hamiltonian for the scalar sector that turns out to be the hardest. The other sectors look simpler and shall provide a complete understanding of the problem [12, 191, 192]. What is more, we should proceed to higher loops [193], especially using algebraic techniques as in [194] and check the integrability of $\mathcal{N} = 2$ SCQCD to all orders in perturbation theory.

- On the dual side, the most important question is to find the precise supergravity solution. Studying the massless spectrum of the non-critical theory, it seems that there is a low energy supergravity description, that is an $SO(4)$ (non-maximally) gauged supergravity in seven dimensions (after reduction on the cigar circle) since it has the same spectrum. The conjecture is that this should be the appropriate supergravity dual for the strong coupling of $\mathcal{N} = 2$ SCQCD. This supergravity was recently written by Martin Weidner in his PhD thesis "Gauged Supergravities in Various Spacetime Dimensions" [178]. This type of observation might be more general, and one may be able to find non-maximally gauged supergravities for several noncritical string theories in various dimensions [188]. In [178] a very comprehensive review of several non-maximally gauged supergravities is presented.

- Ultimately an accurate description of the string dual will require the full non-critical sigma-model in RR background. It would be very interesting to start with the sigma-model for type IIB $AdS_5 \times S^5/\mathbb{Z}_2$, which can be quantized either in the generalized light-cone gauge or in the pure-spinor formalism, and understand the transition to a non-critical sigma-model in the limit $\kappa \to 0$. This may well be the simplest instance of such a transition. We should learn the rules of the game in this highly symmetric example.

- The deformed orbifold theory is not integrable (as we show in [9]), yet it would be highly nontrivial to compute the scattering matrix from the sting theory side and match it (qualitatively; the two calculations are done in different regimes) with the field theory calculation following the $\mathcal{N} = 4$ matching [195, 196]. To achieve that, one would first have to have the worldsheet theory of the type IIB $AdS_5 \times S^5/\mathbb{Z}_2$ deformed orbifold with the non-zero B-field in either Green-Schwarz or pure spinor formalism.

- The Wilson loop calculation, following the methods developed by Vasily Pestun [197], will immediately reveal whether the AdS scale is small or can be large. In [198] this calculation was performed. We should use it to make a statement about the size of AdS.

- The c and a conformal anomaly coefficients are equal at large N for all the known field theories with AdS duals. For $\mathcal{N} = 2$ SCQCD it is well known from the field theory side that $c \neq a$. We should compute the c and a anomalies from the gravity side and match the field theory result. As discussed in chapter 4, the imbalance between c and a must arise from higher-curvature terms in the AdS_5 gravity theory. The stringy origin of these higher curvature terms is the Wess-Zumino action of the flavor branes.

- A more ambitious, but very interesting step would be to study deviations from conformality $|N_f - 2N_c| = \epsilon$. There are some ten dimensional examples that might help, like [112, 114] and $\mathcal{N} = 1^*$ mass deformations of $\mathcal{N} = 4$ [199].

- We should try to carry on a similar analysis for $\mathcal{N} = 1$ $SU(N_c)$ SYM coupled to N_f fundamental chiral multiplets in the conformal window.

As we move to more realistic theories, complete integrability is expected to be lost, but many ideas and techniques developed in the context of $\mathcal{N} = 4$ SYM will continue to be useful. One long-term research direction, would be to understand more systematically the criteria for a theory to be integrable, and to investigate how we can use part of the integrability machinery for non-integrable theories. For example, given an exact AdS/CFT duality, one may consider the S-matrix of magnon excitations of the field theory spin chain

[196], and compare it with the analogous S-matrix of the dual sigma model on the AdS side. In the presence of integrability, the n-body S-matrix is completely determined in terms of the 2-body S-matrix, greatly facilitating the quantitative analysis. However, the AdS/CFT correspondence is logically independent of integrability, and in principle one should always be able to match the field theory and string theory S-matrices. While a precise extrapolation from weak to strong coupling will be difficult for non-integrable theories, qualitative checks should always be possible.

Bibliography

[1] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386, [hep-th/9905111].

[5] I. R. Klebanov, *TASI lectures: Introduction to the AdS/CFT correspondence*, hep-th/0009139.

[6] E. Pomoni and L. Rastelli, *Large N Field Theory and AdS Tachyons*, arXiv:0805.2261.

[7] E. Pomoni and L. Rastelli, *Intersecting Flavor Branes*, arXiv:1002.0006.

[8] A. Gadde, E. Pomoni, and L. Rastelli, *The Veneziano Limit of $\mathcal{N} = 2$ Superconformal QCD: Towards the String Dual of $\mathcal{N} = 2$ $SU(N_c)$ SYM with $N_f = 2N_c$*, arXiv:0912.4918.

[9] A. Gadde, E. Pomoni, and L. Rastelli, *The One-Loop Spin Chain of $\mathcal{N} = 2$ Superconformal QCD*, to appear.

[10] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang Mills*, *JHEP* **04** (2002) 013, [hep-th/0202021].

[11] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013, [hep-th/0212208].

[12] N. Beisert, *The complete one-loop dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B676** (2004) 3–42, [hep-th/0307015].

[13] G. 't Hooft, *A PLANAR DIAGRAM THEORY FOR STRONG INTERACTIONS*, *Nucl. Phys.* **B72** (1974) 461.

[14] G. 't Hooft, *Dimensional reduction in quantum gravity*, gr-qc/9310026.

[15] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377–6396, [hep-th/9409089].

[16] J. D. Bekenstein, *Entropy bounds and black hole remnants*, *Phys. Rev.* **D49** (1994) 1912–1921, [gr-qc/9307035].

[17] E. D'Hoker, S. D. Mathur, A. Matusis, and L. Rastelli, *The operator product expansion of $N = 4$ SYM and the 4- point functions of supergravity*, *Nucl. Phys.* **B589** (2000) 38–74, [hep-th/9911222].

[18] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209–254, [hep-th/0510251].

[19] N. Beisert, V. Dippel, and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang- Mills*, *JHEP* **07** (2004) 075, [hep-th/0405001].

[20] N. Beisert, *The dilatation operator of $N = 4$ super Yang-Mills theory and integrability*, *Phys. Rept.* **405** (2005) 1–202, [hep-th/0407277].

[21] N. Beisert, *The su(2—2) dynamic S-matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945, [hep-th/0511082].

[22] N. Gromov, V. Kazakov, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **103** (2009) 131601, [arXiv:0901.3753].

[23] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar $N = 4$ Supersymmetric Yang-Mills Theory: TBA and excited states*, *Lett. Math. Phys.* **91** (2010) 265–287, [arXiv:0902.4458].

[24] N. Gromov, V. Kazakov, and P. Vieira, *Exact AdS/CFT spectrum: Konishi dimension at any coupling*, `arXiv:0906.4240`.

[25] D. Anselmi and A. Kehagias, *Subleading corrections and central charges in the AdS/CFT correspondence*, *Phys. Lett.* **B455** (1999) 155–163, [`hep-th/9812092`].

[26] M. Blau, K. S. Narain, and E. Gava, *On subleading contributions to the AdS/CFT trace anomaly*, *JHEP* **09** (1999) 018, [`hep-th/9904179`].

[27] S. G. Naculich, H. J. Schnitzer, and N. Wyllard, *1/N corrections to anomalies and the AdS/CFT correspondence for orientifolded $N = 2$ orbifold models and $N = 1$ conifold models*, *Int. J. Mod. Phys.* **A17** (2002) 2567–2594, [`hep-th/0106020`].

[28] O. Aharony and E. Witten, *Anti-de Sitter space and the center of the gauge group*, *JHEP* **11** (1998) 018, [`hep-th/9807205`].

[29] J. Bagger and N. Lambert, *Comments On Multiple M2-branes*, *JHEP* **02** (2008) 105, [`arXiv:0712.3738`].

[30] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091, [`arXiv:0806.1218`].

[31] M. Yamazaki, *Brane Tilings and Their Applications*, *Fortsch. Phys.* **56** (2008) 555–686, [`arXiv:0803.4474`].

[32] T. Sakai and S. Sugimoto, *Low energy hadron physics in holographic QCD*, *Prog. Theor. Phys.* **113** (2005) 843–882, [`hep-th/0412141`].

[33] T. Sakai and S. Sugimoto, *More on a holographic dual of QCD*, *Prog. Theor. Phys.* **114** (2005) 1083–1118, [`hep-th/0507073`].

[34] M. Van Raamsdonk and K. Whyte, *Baryons from embedding topology and a continuous meson spectrum in a new holographic gauge theory*, `arXiv:0912.0752`.

[35] A. M. Polyakov, *String theory and quark confinement*, *Nucl. Phys. Proc. Suppl.* **68** (1998) 1–8, [`hep-th/9711002`].

[36] A. M. Polyakov, *The wall of the cave*, *Int. J. Mod. Phys.* **A14** (1999) 645–658, [`hep-th/9809057`].

[37] A. Parkes and P. C. West, *Finiteness in Rigid Supersymmetric Theories*, *Phys. Lett.* **B138** (1984) 99. ; P. C. West, *The Yukawa beta Function in N=1 Rigid Supersymmetric Theories*, *Phys. Lett.* **B137** (1984) 371. ; D. R. T. Jones and L. Mezincescu, *The Chiral Anomaly and a Class of Two Loop Finite Supersymmetric Gauge Theories*, *Phys. Lett.* **B138** (1984) 293. ; A. V. Ermushev, D. I. Kazakov, and O. V. Tarasov, *FINITE N=1 SUPERSYMMETRIC GRAND UNIFIED THEORIES*, *Nucl. Phys.* **B281** (1987) 72–84. ; D. I. Kazakov, *FINITE N=1 SUSY FIELD THEORIES AND DIMENSIONAL REGULARIZATION*, *Phys. Lett.* **B179** (1986) 352–354. ; D. R. T. Jones, *COUPLING CONSTANT REPARAMETRIZATION AND FINITE FIELD THEORIES*, *Nucl. Phys.* **B277** (1986) 153. ; R. Oehme, *REDUCTION AND REPARAMETRIZATION OF QUANTUM FIELD THEORIES*, *Prog. Theor. Phys. Suppl.* **86** (1986) 215. ; C. Lucchesi, O. Piguet, and K. Sibold, *NECESSARY AND SUFFICIENT CONDITIONS FOR ALL ORDER VANISHING BETA FUNCTIONS IN SUPERSYMMETRIC YANG-MILLS THEORIES*, *Phys. Lett.* **B201** (1988) 241. ; X.-d. Jiang and X.-j. Zhou, *A CRITERION FOR EXISTENCE OF FINITE TO ALL ORDERS N=1 SYM THEORIES*, *Phys. Rev.* **D42** (1990) 2109–2114.

[38] T. Banks and A. Zaks, *On the Phase Structure of Vector-Like Gauge Theories with Massless Fermions*, *Nucl. Phys.* **B196** (1982) 189.

[39] S. Kachru and E. Silverstein, *4d conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855–4858, [hep-th/9802183].

[40] A. E. Lawrence, N. Nekrasov, and C. Vafa, *On conformal field theories in four dimensions*, *Nucl. Phys.* **B533** (1998) 199–209, [hep-th/9803015].

[41] M. Bershadsky, Z. Kakushadze, and C. Vafa, *String expansion as large N expansion of gauge theories*, *Nucl. Phys.* **B523** (1998) 59–72, [hep-th/9803076].

[42] A. Adams and E. Silverstein, *Closed string tachyons, AdS/CFT, and large N QCD*, *Phys. Rev.* **D64** (2001) 086001, [hep-th/0103220].

[43] A. Dymarsky, I. R. Klebanov, and R. Roiban, *Perturbative search for fixed lines in large N gauge theories*, *JHEP* **08** (2005) 011, [hep-th/0505099].

[44] A. Dymarsky, I. R. Klebanov, and R. Roiban, *Perturbative gauge theory and closed string tachyons*, *JHEP* **11** (2005) 038, [hep-th/0509132].

[45] M. Bershadsky and A. Johansen, *Large N limit of orbifold field theories*, *Nucl. Phys.* **B536** (1998) 141–148, [hep-th/9803249].

[46] A. A. Tseytlin and K. Zarembo, *Effective potential in non-supersymmetric SU(N) x SU(N) gauge theory and interactions of type 0 D3-branes*, *Phys. Lett.* **B457** (1999) 77–86, [hep-th/9902095].

[47] C. Csaki, W. Skiba, and J. Terning, *Beta functions of orbifold theories and the hierarchy problem*, *Phys. Rev.* **D61** (2000) 025019, [hep-th/9906057].

[48] P. Breitenlohner and D. Z. Freedman, *Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity*, *Phys. Lett.* **B115** (1982) 197.

[49] O. Lunin and J. M. Maldacena, *Deforming field theories with U(1) x U(1) global symmetry and their gravity duals*, *JHEP* **05** (2005) 033, [hep-th/0502086].

[50] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069, [hep-th/0503201].

[51] S. Ananth, S. Kovacs, and H. Shimada, *Proof of ultra-violet finiteness for a planar non- supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B783** (2007) 227–237, [hep-th/0702020].

[52] G. T. Horowitz, J. Orgera, and J. Polchinski, *Nonperturbative Instability of AdS_5xS^5/Z_k*, *Phys. Rev.* **D77** (2008) 024004, [0709.4262].

[53] K. Copsey and R. B. Mann, *States of Negative Energy and $AdS_5 \times S_5/Z_k$*, *JHEP* **05** (2008) 069, [0803.3801].

[54] O. Aharony, M. Berkooz, and E. Silverstein, *Multiple-trace operators and non-local string theories*, *JHEP* **08** (2001) 006, [hep-th/0105309].

[55] O. Aharony, M. Berkooz, and E. Silverstein, *Non-local string theories on AdS(3) x S**3 and stable non-supersymmetric backgrounds*, *Phys. Rev.* **D65** (2002) 106007, [hep-th/0112178].

[56] E. Witten, *Multi-trace operators, boundary conditions, and AdS/CFT correspondence*, hep-th/0112258.

[57] M. Berkooz, A. Sever, and A. Shomer, *Double-trace deformations, boundary conditions and spacetime singularities*, *JHEP* **05** (2002) 034, [hep-th/0112264].

[58] P. Minces, *Multi-trace operators and the generalized AdS/CFT prescription*, *Phys. Rev.* **D68** (2003) 024027, [hep-th/0201172].

[59] D. Z. Freedman, K. Johnson, and J. I. Latorre, *Differential regularization and renormalization: A New method of calculation in quantum field theory*, *Nucl. Phys.* **B371** (1992) 353–414.

[60] D. Z. Freedman, K. Johnson, R. Munoz-Tapia, and X. Vilasis-Cardona, *A Cutoff procedure and counterterms for differential renormalization*, *Nucl. Phys.* **B395** (1993) 454–496, [hep-th/9206028].

[61] J. I. Latorre, C. Manuel, and X. Vilasis-Cardona, *Systematic differential renormalization to all orders*, *Ann. Phys.* **231** (1994) 149–173, [hep-th/9303044].

[62] S. R. Coleman and E. J. Weinberg, *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*, *Phys. Rev.* **D7** (1973) 1888–1910.

[63] N. Beisert and R. Roiban, *The Bethe ansatz for Z(S) orbifolds of $N = 4$ super Yang- Mills theory*, *JHEP* **11** (2005) 037, [hep-th/0510209].

[64] A. Dymarsky, S. Franco, I. R. Klebanov, and R. Roiban: Work in progress.

[65] A. Armoni, E. Lopez, and A. M. Uranga, *Closed strings tachyons and non-commutative instabilities*, *JHEP* **02** (2003) 020, [hep-th/0301099].

[66] P. H. Frampton and C. Vafa, *Conformal approach to particle phenomenology*, hep-th/9903226.

[67] P. H. Frampton and T. W. Kephart, *Quiver Gauge Theory and Conformality at the Large Hadron Collider*, *Phys. Rept.* **454** (2008) 203–269, [0706.4259].

[68] Z. Kakushadze, *Tachyon-Free Non-Supersymmetric Strings on Orbifolds*, *Int. J. Mod. Phys.* **A23** (2008) 4371–4386, [0711.4108].

[69] A. Armoni, *Non-Perturbative Planar Equivalence and the Absence of Closed String Tachyons*, *JHEP* **04** (2007) 046, [hep-th/0703229].

[70] A. Sen, *Tachyon dynamics in open string theory*, *Int. J. Mod. Phys.* **A20** (2005) 5513–5656, [hep-th/0410103].

[71] A. Karch and E. Katz, *Adding flavor to AdS/CFT*, *JHEP* **06** (2002) 043, [hep-th/0205236].

[72] T. Sakai and J. Sonnenschein, *Probing flavored mesons of confining gauge theories by supergravity*, *JHEP* **09** (2003) 047, [hep-th/0305049].

[73] S. Kuperstein, *Meson spectroscopy from holomorphic probes on the warped deformed conifold*, *JHEP* **03** (2005) 014, [hep-th/0411097].

[74] D. Arean, D. E. Crooks, and A. V. Ramallo, *Supersymmetric probes on the conifold*, *JHEP* **11** (2004) 035, [hep-th/0408210].

[75] P. Ouyang, *Holomorphic D7-branes and flavored $N = 1$ gauge theories*, *Nucl. Phys.* **B699** (2004) 207–225, [hep-th/0311084].

[76] F. Canoura, J. D. Edelstein, and A. V. Ramallo, *D-brane probes on L(a,b,c) superconformal field theories*, *JHEP* **09** (2006) 038, [hep-th/0605260].

[77] R. Apreda, J. Erdmenger, D. Lust, and C. Sieg, *Adding flavour to the Polchinski-Strassler background*, *JHEP* **01** (2007) 079, [hep-th/0610276].

[78] C. Sieg, *Holographic flavour in the $N = 1$ Polchinski-Strassler background*, *JHEP* **08** (2007) 031, [0704.3544].

[79] S. Penati, M. Pirrone, and C. Ratti, *Mesons in marginally deformed AdS/CFT*, *JHEP* **04** (2008) 037, [0710.4292].

[80] X.-J. Wang and S. Hu, *Intersecting branes and adding flavors to the Maldacena- Nunez background*, *JHEP* **09** (2003) 017, [hep-th/0307218].

[81] C. Nunez, A. Paredes, and A. V. Ramallo, *Flavoring the gravity dual of $N = 1$ Yang-Mills with probes*, *JHEP* **12** (2003) 024, [hep-th/0311201].

[82] A. Karch and L. Randall, *Open and closed string interpretation of SUSY CFT's on branes with boundaries*, *JHEP* **06** (2001) 063, [hep-th/0105132].

[83] O. DeWolfe, D. Z. Freedman, and H. Ooguri, *Holography and defect conformal field theories*, *Phys. Rev.* **D66** (2002) 025009, [hep-th/0111135].

[84] J. Erdmenger, Z. Guralnik, and I. Kirsch, *Four-Dimensional Superconformal Theories with Interacting Boundaries or Defects*, *Phys. Rev.* **D66** (2002) 025020, [hep-th/0203020].

[85] K. Skenderis and M. Taylor, *Branes in AdS and pp-wave spacetimes*, *JHEP* **06** (2002) 025, [hep-th/0204054].

[86] N. R. Constable, J. Erdmenger, Z. Guralnik, and I. Kirsch, *Intersecting D3-branes and holography*, *Phys. Rev.* **D68** (2003) 106007, [hep-th/0211222].

[87] N. Marcus, A. Sagnotti, and W. Siegel, *TEN-DIMENSIONAL SUPERSYMMETRIC YANG-MILLS THEORY IN TERMS OF FOUR-DIMENSIONAL SUPERFIELDS*, *Nucl. Phys.* **B224** (1983) 159.

[88] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1–548, [hep-th/0108200].

[89] O. Aharony, A. Fayyazuddin, and J. M. Maldacena, *The large N limit of N = 2,1 field theories from three- branes in F-theory*, *JHEP* **07** (1998) 013, [hep-th/9806159].

[90] M. Kruczenski, D. Mateos, R. C. Myers, and D. J. Winters, *Meson spectroscopy in AdS/CFT with flavour*, *JHEP* **07** (2003) 049, [hep-th/0304032].

[91] F. T. J. Epple and D. Lust, *Tachyon condensation for intersecting branes at small and large angles*, *Fortsch. Phys.* **52** (2004) 367–387, [hep-th/0311182].

[92] N. Berkovits, *A Ten-dimensional superYang-Mills action with off-shell supersymmetry*, *Phys. Lett.* **B318** (1993) 104–106, [hep-th/9308128].

[93] L. Baulieu, N. J. Berkovits, G. Bossard, and A. Martin, *Ten-dimensional super-Yang-Mills with nine off-shell supersymmetries*, *Phys. Lett.* **B658** (2008) 249–254, [0705.2002].

[94] L. J. Dixon, D. Friedan, E. J. Martinec, and S. H. Shenker, *The Conformal Field Theory of Orbifolds*, *Nucl. Phys.* **B282** (1987) 13–73.

[95] M. Cvetic and I. Papadimitriou, *Conformal field theory couplings for intersecting D-branes on orientifolds*, *Phys. Rev.* **D68** (2003) 046001, [hep-th/0303083].

[96] S. A. Abel and A. W. Owen, *Interactions in intersecting brane models*, *Nucl. Phys.* **B663** (2003) 197–214, [hep-th/0303124].

[97] I. Antoniadis, K. Benakli, and A. Laugier, *Contact interactions in D-brane models*, *JHEP* **05** (2001) 044, [hep-th/0011281].

[98] T. Erler and N. Mann, *Integrable open spin chains and the doubling trick in N = 2 SYM with fundamental matter*, *JHEP* **01** (2006) 131, [hep-th/0508064].

[99] N. Mann and S. E. Vazquez, *Classical open string integrability*, *JHEP* **04** (2007) 065, [hep-th/0612038].

[100] D. H. Correa and C. A. S. Young, *Reflecting magnons from D7 and D5 branes*, *J. Phys.* **A41** (2008) 455401, [0808.0452].

[101] D. H. Correa and C. A. S. Young, *Asymptotic Bethe equations for open boundaries in planar AdS/CFT*, 0912.0627.

[102] E. D'Hoker, D. Z. Freedman, and W. Skiba, *Field theory tests for correlators in the AdS/CFT correspondence*, *Phys. Rev.* **D59** (1999) 045008, [hep-th/9807098].

[103] N. R. Constable *et al.*, *PP-wave string interactions from perturbative Yang-Mills theory*, *JHEP* **07** (2002) 017, [hep-th/0205089].

[104] E. D'Hoker and A. V. Ryzhov, *Three-point functions of quarter BPS operators in N = 4 SYM*, *JHEP* **02** (2002) 047, [hep-th/0109065].

[105] E. Witten, *Baryons and branes in anti de Sitter space*, *JHEP* **07** (1998) 006, [hep-th/9805112].

[106] I. R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, *Nucl. Phys.* **B536** (1998) 199–218, [hep-th/9807080].

[107] D. R. Morrison and M. R. Plesser, *Non-spherical horizons. I*, *Adv. Theor. Math. Phys.* **3** (1999) 1–81, [hep-th/9810201].

[108] S. Benvenuti, S. Franco, A. Hanany, D. Martelli, and J. Sparks, *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, *JHEP* **06** (2005) 064, [hep-th/0411264].

[109] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, *Supersymmetric AdS(5) solutions of M-theory*, *Class. Quant. Grav.* **21** (2004) 4335–4366, [hep-th/0402153].

[110] D. Gaiotto and J. Maldacena, *The gravity duals of N=2 superconformal field theories*, arXiv:0904.4466.

[111] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023, [hep-th/9806087].

[112] J. Polchinski and M. J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[113] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505–532, [hep-th/9803131].

[114] I. R. Klebanov and M. J. Strassler, *Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities*, *JHEP* **08** (2000) 052, [hep-th/0007191].

[115] J. M. Maldacena and C. Nunez, *Towards the large N limit of pure N = 1 super Yang Mills*, *Phys. Rev. Lett.* **86** (2001) 588–591, [hep-th/0008001].

[116] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, *Nucl. Phys.* **B431** (1994) 484–550, [hep-th/9408099].

[117] P. C. Argyres and N. Seiberg, *S-duality in N=2 supersymmetric gauge theories*, *JHEP* **12** (2007) 088, [arXiv:0711.0054].

[118] D. Gaiotto, *N=2 dualities*, arXiv:0904.2715.

[119] G. Veneziano, *Some Aspects of a Unified Approach to Gauge, Dual and Gribov Theories*, *Nucl. Phys.* **B117** (1976) 519–545.

[120] A. M. Polyakov, *String theory as a universal language*, *Phys. Atom. Nucl.* **64** (2001) 540–547, [hep-th/0006132].

[121] S. Kuperstein and J. Sonnenschein, *Non-critical supergravity (d ¿ 1) and holography*, *JHEP* **07** (2004) 049, [hep-th/0403254].

[122] M. Alishahiha, A. Ghodsi, and A. E. Mosaffa, *On isolated conformal fixed points and noncritical string theory*, *JHEP* **01** (2005) 017, [hep-th/0411087].

[123] A. M. Polyakov, *Conformal fixed points of unidentified gauge theories*, *Mod. Phys. Lett.* **A19** (2004) 1649–1660, [hep-th/0405106].

[124] I. R. Klebanov and J. M. Maldacena, *Superconformal gauge theories and non-critical superstrings*, *Int. J. Mod. Phys.* **A19** (2004) 5003–5016, [hep-th/0409133].

[125] A. Fotopoulos, V. Niarchos, and N. Prezas, *D-branes and SQCD in non-critical superstring theory*, *JHEP* **10** (2005) 081, [hep-th/0504010].

[126] S. K. Ashok, S. Murthy, and J. Troost, *D-branes in non-critical superstrings and minimal super Yang-Mills in various dimensions*, *Nucl. Phys.* **B749** (2006) 172–205, [hep-th/0504079].

[127] F. Bigazzi, R. Casero, A. L. Cotrone, E. Kiritsis, and A. Paredes, *Non-critical holography and four-dimensional CFT's with fundamentals*, *JHEP* **10** (2005) 012, [hep-th/0505140].

[128] S. Murthy and J. Troost, *D-branes in non-critical superstrings and duality in $N = 1$ gauge theories with flavor*, *JHEP* **10** (2006) 019, [hep-th/0606203].

[129] D. Kutasov and N. Seiberg, *Noncritical superstrings*, *Phys. Lett.* **B251** (1990) 67–72.

[130] D. Kutasov, *Some properties of (non)critical strings*, hep-th/9110041.

[131] H. Ooguri and C. Vafa, *Two-Dimensional Black Hole and Singularities of CY Manifolds*, *Nucl. Phys.* **B463** (1996) 55–72, [hep-th/9511164].

[132] O. Aharony, M. Berkooz, D. Kutasov, and N. Seiberg, *Linear dilatons, NS5-branes and holography*, *JHEP* **10** (1998) 004, [hep-th/9808149].

[133] A. Giveon, D. Kutasov, and O. Pelc, *Holography for non-critical superstrings*, *JHEP* **10** (1999) 035, [hep-th/9907178].

[134] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *JHEP* **10** (1999) 034, [hep-th/9909110].

[135] A. Giveon and D. Kutasov, *Comments on double scaled little string theory*, *JHEP* **01** (2000) 023, [hep-th/9911039].

[136] W. Siegel, *Subcritical superstrings*, *Phys. Rev.* **D52** (1995) 3563–3566, [hep-th/9503173].

[137] P. A. Grassi and Y. Oz, *Non-critical covariant superstrings*, hep-th/0507168.

[138] N. Wyllard, *Pure-spinor superstrings in $d = 2, 4, 6$*, *JHEP* **11** (2005) 009, [hep-th/0509165].

[139] I. Adam, P. A. Grassi, L. Mazzucato, Y. Oz, and S. Yankielowicz, *Non-critical pure spinor superstrings*, *JHEP* **03** (2007) 091, [hep-th/0605118].

[140] I. Adam, A. Dekel, L. Mazzucato, and Y. Oz, *Integrability of type II superstrings on Ramond-Ramond backgrounds in various dimensions*, *JHEP* **06** (2007) 085, [hep-th/0702083].

[141] O. Aharony, B. Fiol, D. Kutasov, and D. A. Sahakyan, *Little string theory and heterotic/type II duality*, *Nucl. Phys.* **B679** (2004) 3–65, [hep-th/0310197].

[142] O. Aharony, A. Giveon, and D. Kutasov, *LSZ in LST*, *Nucl. Phys.* **B691** (2004) 3–78, [hep-th/0404016].

[143] S. Murthy, *Notes on non-critical superstrings in various dimensions*, *JHEP* **11** (2003) 056, [hep-th/0305197].

[144] B. A. Burrington, J. T. Liu, L. A. Pando Zayas, and D. Vaman, *Holographic duals of flavored $N = 1$ super Yang-Mills: Beyond the probe approximation*, *JHEP* **02** (2005) 022, [hep-th/0406207].

[145] R. Casero, C. Nunez, and A. Paredes, *Towards the string dual of $N = 1$ SQCD-like theories*, *Phys. Rev.* **D73** (2006) 086005, [hep-th/0602027].

[146] A. Paredes, *On unquenched $N = 2$ holographic flavor*, *JHEP* **12** (2006) 032, [hep-th/0610270].

[147] F. Benini, F. Canoura, S. Cremonesi, C. Nunez, and A. V. Ramallo, *Unquenched flavors in the Klebanov-Witten model*, *JHEP* **02** (2007) 090, [hep-th/0612118].

[148] F. Benini, F. Canoura, S. Cremonesi, C. Nunez, and A. V. Ramallo, *Backreacting Flavors in the Klebanov-Strassler Background*, *JHEP* **09** (2007) 109, [arXiv:0706.1238].

[149] R. Casero, C. Nunez, and A. Paredes, *Elaborations on the String Dual to N=1 SQCD*, *Phys. Rev.* **D77** (2008) 046003, [arXiv:0709.3421].

[150] E. Caceres, R. Flauger, M. Ihl, and T. Wrase, *New Supergravity Backgrounds Dual to N=1 SQCD-like Theories with $N_f = 2N_c$*, *JHEP* **03** (2008) 020, [arXiv:0711.4878].

[151] C. Hoyos-Badajoz, C. Nunez, and I. Papadimitriou, *Comments on the String dual to N=1 SQCD*, *Phys. Rev.* **D78** (2008) 086005, [arXiv:0807.3039].

[152] F. Bigazzi, A. L. Cotrone, A. Paredes, and A. V. Ramallo, *The Klebanov-Strassler model with massive dynamical flavors*, *JHEP* **03** (2009) 153, [arXiv:0812.3399].

[153] M. R. Douglas and G. W. Moore, *D-branes, Quivers, and ALE Instantons*, hep-th/9603167.

[154] F. A. Dolan and H. Osborn, *On short and semi-short representations for four dimensional superconformal symmetry*, *Ann. Phys.* **307** (2003) 41–89, [hep-th/0209056].

[155] C. Romelsberger, *Counting chiral primaries in N = 1, d=4 superconformal field theories*, *Nucl. Phys.* **B747** (2006) 329–353, [hep-th/0510060].

[156] Y. Nakayama, *Index for orbifold quiver gauge theories*, *Phys. Lett.* **B636** (2006) 132–136, [hep-th/0512280].

[157] B. Feng, A. Hanany, and Y.-H. He, *Counting Gauge Invariants: the Plethystic Program*, *JHEP* **03** (2007) 090, [hep-th/0701063].

[158] N. Beisert, M. Bianchi, J. F. Morales, and H. Samtleben, *On the spectrum of AdS/CFT beyond supergravity*, *JHEP* **02** (2004) 001, [hep-th/0310292].

[159] N. Beisert, M. Bianchi, J. F. Morales, and H. Samtleben, *Higher spin symmetry and N = 4 SYM*, *JHEP* **07** (2004) 058, [hep-th/0405057].

[160] P. S. Aspinwall, *Enhanced gauge symmetries and K3 surfaces*, *Phys. Lett.* **B357** (1995) 329–334, [hep-th/9507012].

[161] I. R. Klebanov and N. A. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, *Nucl. Phys.* **B574** (2000) 263–274, [hep-th/9911096].

[162] S. Gukov, *Comments on N = 2 AdS orbifolds*, *Phys. Lett.* **B439** (1998) 23–28, [hep-th/9806180].

[163] A. Giveon and D. Kutasov, *Brane dynamics and gauge theory*, *Rev. Mod. Phys.* **71** (1999) 983–1084, [hep-th/9802067].

[164] R. Gregory, J. A. Harvey, and G. W. Moore, *Unwinding strings and T-duality of Kaluza-Klein and H- monopoles*, *Adv. Theor. Math. Phys.* **1** (1997) 283–297, [hep-th/9708086].

[165] D. Tong, *NS5-branes, T-duality and worldsheet instantons*, *JHEP* **07** (2002) 013, [hep-th/0204186].

[166] A. Karch, D. Lust, and D. J. Smith, *Equivalence of geometric engineering and Hanany-Witten via fractional branes*, *Nucl. Phys.* **B533** (1998) 348–372, [hep-th/9803232].

[167] C. G. Callan, Jr., J. A. Harvey, and A. Strominger, *Supersymmetric string solitons*, `hep-th/9112030`.

[168] K. Hori and A. Kapustin, *Duality of the fermionic 2d black hole and N = 2 Liouville theory as mirror symmetry*, *JHEP* **08** (2001) 045, [`hep-th/0104202`].

[169] D. Israel, A. Pakman, and J. Troost, *D-branes in N = 2 Liouville theory and its mirror*, *Nucl. Phys.* **B710** (2005) 529–576, [`hep-th/0405259`].

[170] A. Fotopoulos, V. Niarchos, and N. Prezas, *D-branes and extended characters in SL(2,R)/U(1)*, *Nucl. Phys.* **B710** (2005) 309–370, [`hep-th/0406017`].

[171] K. Hosomichi, *N=2 Liouville Theory with Boundary*, *JHEP* **12** (2006) 061, [`hep-th/0408172`].

[172] T. Eguchi, *Modular bootstrap of boundary N = 2 Liouville theory*, *Comptes Rendus Physique* **6** (2005) 209–217, [`hep-th/0409266`].

[173] D. Israel, A. Pakman, and J. Troost, *D-branes in little string theory*, *Nucl. Phys.* **B722** (2005) 3–64, [`hep-th/0502073`].

[174] A. B. Zamolodchikov and A. B. Zamolodchikov, *Liouville field theory on a pseudosphere*, `hep-th/0101152`.

[175] V. Fateev, A. B. Zamolodchikov, and A. B. Zamolodchikov, *Boundary Liouville field theory. I: Boundary state and boundary two-point function*, `hep-th/0001012`.

[176] J. Teschner, *Remarks on Liouville theory with boundary*, `hep-th/0009138`.

[177] H. Samtleben and M. Weidner, *The maximal D = 7 supergravities*, *Nucl. Phys.* **B725** (2005) 383–419, [`hep-th/0506237`].

[178] M. Weidner, *Gauged Supergravities in Various Spacetime Dimensions*, *Fortsch. Phys.* **55** (2007) 843–945, [`hep-th/0702084`].

[179] M. Grana and J. Polchinski, *Gauge / gravity duals with holomorphic dilaton*, *Phys. Rev.* **D65** (2002) 126005, [`hep-th/0106014`].

[180] S. Nojiri and S. D. Odintsov, *On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence*, *Int. J. Mod. Phys.* **A15** (2000) 413–428, [`hep-th/9903033`].

[181] O. Aharony, J. Pawelczyk, S. Theisen, and S. Yankielowicz, *A note on anomalies in the AdS/CFT correspondence*, *Phys. Rev.* **D60** (1999) 066001, [hep-th/9901134].

[182] F. Cachazo, M. R. Douglas, N. Seiberg, and E. Witten, *Chiral Rings and Anomalies in Supersymmetric Gauge Theory*, *JHEP* **12** (2002) 071, [hep-th/0211170].

[183] D. Berenstein and R. G. Leigh, *Discrete torsion, AdS/CFT and duality*, *JHEP* **01** (2000) 038, [hep-th/0001055].

[184] I. R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B556** (1999) 89–114, [hep-th/9905104].

[185] S. Mizoguchi, *Localized Modes in Type II and Heterotic Singular Calabi-Yau Conformal Field Theories*, *JHEP* **11** (2008) 022, [arXiv:0808.2857].

[186] A. Konechny, A. Parnachev, and D. A. Sahakyan, *The ground ring of $N = 2$ minimal string theory*, *Nucl. Phys.* **B729** (2005) 419–440, [hep-th/0507002].

[187] L. Rastelli and M. Wijnholt, *Minimal AdS(3)*, hep-th/0507037.

[188] H. J. Boonstra, K. Skenderis, and P. K. Townsend, *The domain wall/QFT correspondence*, *JHEP* **01** (1999) 003, [hep-th/9807137].

[189] D. B. Kaplan, J.-W. Lee, D. T. Son, and M. A. Stephanov, *Conformality Lost*, *Phys. Rev.* **D80** (2009) 125005, [arXiv:0905.4752].

[190] L. Vecchi, *The Conformal Window of deformed CFT's in the planar limit*, arXiv:1004.2063.

[191] N. Beisert, C. Kristjansen, and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B664** (2003) 131–184, [hep-th/0303060].

[192] N. Beisert and M. Staudacher, *The N=4 SYM Integrable Super Spin Chain*, *Nucl. Phys.* **B670** (2003) 439–463, [hep-th/0307042].

[193] N. Beisert and M. Staudacher, *Long-range PSU(2,2—4) Bethe ansaetze for gauge theory and strings*, *Nucl. Phys.* **B727** (2005) 1–62, [hep-th/0504190].

[194] N. Beisert, *The su(2—3) dynamic spin chain*, *Nucl. Phys.* **B682** (2004) 487–520, [hep-th/0310252].

[195] G. Arutyunov and S. Frolov, *Foundations of the AdS_5xS^5 Superstring. Part I*, *J. Phys.* **A42** (2009) 254003, [arXiv:0901.4937].

[196] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054, [hep-th/0412188].

[197] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, arXiv:0712.2824.

[198] S.-J. Rey and T. Suyama, *Exact Results and Holography of Wilson Loops in N=2 Superconformal (Quiver) Gauge Theories*, arXiv:1001.0016.

[199] S. G. Naculich, H. J. Schnitzer, and N. Wyllard, *Vacuum states of N = 1* mass deformations of N = 4 and N = 2 conformal gauge theories and their brane interpretations*, *Nucl. Phys.* **B609** (2001) 283–312, [hep-th/0103047].

Appendix A: Intersecting $D7$ Flavor Branes

A.1 The Supersymmetric Field Theory

In this appendix we spell out our conventions and write the $\mathcal{N}=2$ supersymmetric action for the usual $D3/D7$ system [71]. We first present the Lagrangian in $\mathcal{N}=1$ superspace and then in components. Unless otherwise stated, we follow the superspace notations of [88].

As familiar, the $\mathcal{N}=4$ vector multiplet decomposes into an $\mathcal{N}=1$ vector multiplet,

$$V=\bar{\theta}\sigma^{\mu}\theta A_{\mu}+i\theta^{2}\bar{\theta}\bar{\lambda}-i\bar{\theta}^{2}\theta\lambda+\theta^{2}\bar{\theta}^{2}D \quad \text{(Wess-Zumino gauge)} \tag{A.1}$$

and three chiral multiplets

$$\Phi^{a}=\phi^{a}+\theta\chi^{a}-\theta^{2}F^{a}\,, \quad a=1,2,3\,, \tag{A.2}$$

all in the adjoint representation of the $SU(N)$ gauge group. For zero theta angle, the superspace Lagrangian reads

$$\mathcal{L}_{\mathcal{N}=4} \;=\; \mathrm{Tr}\left[\int d^{4}\theta\, e^{-gV}\,\bar{\Phi}_{a}\, e^{gV}\,\Phi^{a}+\int d^{2}\theta\, W^{2}+\left(\frac{i\,g}{3!}\int d^{2}\theta\,\epsilon_{abc}\,\Phi^{a}\left[\Phi^{b},\,\Phi^{c}\right]+h.c.\right)\right], \tag{A.3}$$

where $W_{\alpha}\equiv i\bar{D}^{2}D_{\alpha}V$ is the usual field strength chiral superfield. In this $\mathcal{N}=1$ language, only an $SU(3)_{R}\times U(1)_{r}$ subgroup of the $SU(4)_{R}$ R-symmetry is visible. The $SU(3)_{R}$ rotates the three chiral superfields leaving V invariant, while the $U(1)_{r}$ is the usual $\mathcal{N}=1$ R-symmetry, with the chiral superfields having charge 2/3. [1]

[1] Note that we are making a graphical distinction between this $U(1)_{r}$ symmetry and the $U(1)_{R}$ symmetry defined in (B.23). See the footnote in Appendix A.2.

In components[2],

$$\mathcal{L}_{\mathcal{N}=4} = \mathrm{Tr}\Bigg[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\bar{\lambda}_A\bar{\sigma}^\mu D_\mu \lambda^A - \frac{1}{2}D^\mu \bar{X}_{AB} D_\mu X^{AB} \tag{A.4}$$

$$+ i\sqrt{2}\, g\, X^{AB}\,\bar{\lambda}_A\bar{\lambda}_B - i\sqrt{2}\, g\, \bar{X}_{AB}\lambda^A\lambda^B - \frac{g^2}{4}[X^{AB}, X^{CD}]\,[\bar{X}_{CD}, \bar{X}_{AB}]\Bigg], \tag{A.5}$$

where $A, B = 1, \ldots, 4$. The scalars X^{AB} are related to the three complex scalars ϕ^a as

$$X^{AB} = \left(\begin{array}{cc|cc} 0 & \phi^3 & \phi^2 & \phi^1 \\ -\phi^3 & 0 & \phi_1^* & -\phi_2^* \\ \hline -\phi^2 & -\phi_1^* & 0 & \phi_3^* \\ -\phi^1 & \phi_2^* & -\phi_3^* & 0 \end{array}\right) \tag{A.6}$$

and obey the self-duality constraint (3.1).

We can also think the $\mathcal{N} = 4$ vector multiplet as an $\mathcal{N} = 2$ vector multiplet (comprising V and Φ^3) and an $\mathcal{N} = 2$ hyper multiplet (comprising Φ^1 and Φ^2). We wish to couple the $\mathcal{N} = 2$ vector multiplet (V, Φ^3) to N_f "flavor" hyper multiplets in the fundamental representation of the gauge group. Each $\mathcal{N} = 2$ flavor hyper multiplet decomposes into two $\mathcal{N} = 1$ chiral multiplets

$$Q = q + \theta\psi - \theta^2 f \quad \text{and} \quad \tilde{Q} = \tilde{q} + \theta\tilde{\psi} - \theta^2\tilde{f}\,,$$

where Q is in the fundamental representation of $SU(N)$ and $\tilde{Q}$ in the antifundamental representation. In $\mathcal{N} = 1$ superspace, the flavor part of the Lagrangian reads[3]

$$\mathcal{L}_{hyper} = \int d^4\theta\, \bar{Q}_i\, e^{gV}\, Q^i + \int d^4\theta\, \tilde{Q}_i\, e^{-gV}\, \bar{\tilde{Q}}^i + \left(g\int d^2\theta\, \tilde{Q}_i\, \Phi^3\, Q^i + h.c. \right) \tag{A.7}$$

where $i = 1, \ldots, N_f$ is a flavor index. In components,

$$\mathcal{L}_{hyper} = -D^\mu \bar{Q}_{\mathcal{I}\,i} D_\mu Q^{\mathcal{I}\,i} - i\bar{\psi}_i\bar{\sigma}^\mu D_\mu \psi^i - i\tilde{\psi}_i\sigma^\mu D_\mu \bar{\tilde{\psi}}^{\,i} \tag{A.8}$$

[2]In going from superspace to components, we redefine the coupling, $g_{superspace} = \sqrt{2}g_{components}$, to recover the usual normalization.

[3]Strictly speaking, this is the Lagrangian for gauge group $U(N)$. For $SU(N)$ there is a $O(1/N)$ correction to the Q^4 potential, which we neglect since we are interested in the large N limit.

$$-\sqrt{2}\,i\,g\,\tilde{\psi}_i X^{12}\psi^i + \sqrt{2}\,i\,g\,\bar{\psi}_i \bar{X}_{12}\bar{\tilde{\psi}}^{\,i} \tag{A.9}$$

$$+ig\sqrt{2}\bar{Q}_{\mathcal{I}\,i}\bar{\Lambda}^{\mathcal{I}}\bar{\tilde{\psi}}^{\,i} - ig\sqrt{2}\tilde{\psi}_i\Lambda_{\mathcal{I}}Q^{\mathcal{I}\,i} + ig\sqrt{2}\bar{Q}_{\mathcal{I}\,i}\epsilon^{\mathcal{I}\mathcal{J}}\Lambda_{\mathcal{J}}\psi^i - ig\sqrt{2}\bar{\psi}_i\bar{\Lambda}^{\mathcal{I}}\epsilon_{\mathcal{I}\mathcal{J}}Q^{\mathcal{J}\,i} \tag{A.10}$$

$$-\frac{1}{2}g^2\bar{Q}_{\mathcal{I}\,i}\bar{X}_{AB}X^{AB}Q^{\mathcal{I}\,i} - g^2\bar{Q}_{\mathcal{J}\,i}\mathcal{X}_{\mathcal{I}\hat{\mathcal{K}}}\mathcal{X}^{\mathcal{J}\hat{\mathcal{K}}}Q^{\mathcal{I}\,i}\,. \tag{A.11}$$

$$-\frac{g^2}{2}(\bar{Q}_{\mathcal{I}\,i}\cdot Q^{\mathcal{J}\,j})(\bar{Q}_{\mathcal{J}\,j}\cdot Q^{\mathcal{I}\,i}) - g^2\epsilon_{\mathcal{I}\mathcal{K}}\epsilon^{\mathcal{L}\mathcal{J}}(\bar{Q}_{\mathcal{L}\,i}\cdot Q^{\mathcal{K}\,j})(\bar{Q}_{\mathcal{J}\,j}\cdot Q^{\mathcal{I}\,i})\,. \tag{A.12}$$

Following [98], we have introduced the $SU(2)_R$ doublets

$$Q^{\mathcal{I}} \equiv \begin{pmatrix} q \\ \tilde{q}^* \end{pmatrix}\,, \quad \Lambda_{\mathcal{I}} \equiv \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -\chi_3 \end{pmatrix}\,, \quad \mathcal{I} = 1, 2\,. \tag{A.13}$$

The other two Weyl spinors can be assembled into an $SU(2)_L$ doublet,

$$\hat{\Lambda}_{\hat{\mathcal{I}}} \equiv \begin{pmatrix} \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -\chi_2 \\ -\chi_1 \end{pmatrix}\,, \tag{A.14}$$

which does not couple to the flavor hyper multiplets. Note that to avoid cluttering we keep color indices implicit. Color contractions are almost always obvious. When ambiguity may arise, we indicate the contraction with a dot. For example in the term

$$(\bar{Q}_{\mathcal{I}\,i}\cdot Q^{\mathcal{J}\,j})(\bar{Q}_{\mathcal{J}\,j}\cdot Q^{\mathcal{I}\,i}) \tag{A.15}$$

the first pair is color contracted, and so is the second pair.

The Q^4 term in the Lagrangian can be written more compactly by introducing flavor-contracted composite operators, in the adjoint of the gauge group,

$$\mathcal{M}_{\mathcal{J}}{}^{\mathcal{I}\mathfrak{a}}{}_{\mathfrak{b}} \equiv \frac{1}{\sqrt{2}}Q_{\mathcal{J}}{}^{\mathfrak{a}}{}_i\,\bar{Q}^{\mathcal{I}}{}_{\mathfrak{b}}{}^i\,, \tag{A.16}$$

which may be decomposed into the $SU(2)_R$ singlet and triplet combinations

$$\mathcal{M}_{\mathbf{1}} \equiv \mathcal{M}_{\mathcal{I}}^{\mathcal{I}} \quad \text{and} \quad \mathcal{M}_{\mathbf{3}\mathcal{J}}{}^{\mathcal{I}} \equiv \mathcal{M}_{\mathcal{J}}^{\mathcal{I}} - \frac{1}{2}\mathcal{M}_{\mathcal{K}}^{\mathcal{K}}\,\delta_{\mathcal{J}}^{\mathcal{I}}\,. \tag{A.17}$$

In terms of the component fields q and $\tilde{q}$,

$$\begin{aligned}
\mathcal{M}_{\mathbf{1}} &= \frac{1}{\sqrt{2}}\,(q\,\bar{q} + \bar{\tilde{q}}\,\tilde{q}) \\
\mathcal{M}_{\mathbf{3}^+} &= q\,\tilde{q} \\
\mathcal{M}_{\mathbf{3^0}} &= \frac{1}{\sqrt{2}}\,(q\,\bar{q} - \bar{\tilde{q}}\,\tilde{q}) \\
\mathcal{M}_{\mathbf{3}^-} &= \tilde{q}^*\,q^*\,,
\end{aligned} \tag{A.18}$$

where the superscripts refer to the eigenvalues under the Cartan generator of $SU(2)_R$. The F and $\bar{F}$ auxiliary fields couple to $\mathcal{M}_{\mathbf{3}^\pm}$, while the D auxiliary field couples to $\mathcal{M}_{\mathbf{3}^0}$. Thus the Q^4 scalar potential is the square of the triplet composite,

$$\mathcal{L}_{Q^4} = -g^2\,\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}\,\mathcal{M}_{\mathbf{3}} \equiv -g^2\,\mathrm{Tr}\,[2\mathcal{M}_{\mathbf{3}^+}\mathcal{M}_{\mathbf{3}^-} + \mathcal{M}_{\mathbf{3}^0}\mathcal{M}_{\mathbf{3}^0}]\ . \tag{A.19}$$

Finally, let us write the Q^4 potential using the gauge-invariant mesonic operators. The explicit expressions of the mesons in components are

$$\begin{aligned}
\mathcal{O}^{ij}_{\mathbf{1}} &= \frac{1}{\sqrt{2}}\left(q^{i\,\mathfrak{a}}\,\bar{q}_{j\,\mathfrak{a}} + \bar{\tilde{q}}^{i\,\mathfrak{a}}\,\tilde{q}_{j\,\mathfrak{a}}\right) \\
\mathcal{O}^{ij}_{\mathbf{3}^+} &= q^{i\,\mathfrak{a}}\,\tilde{q}_{j\,\mathfrak{a}} \\
\mathcal{O}^{ij}_{\mathbf{3^0}} &= \frac{1}{\sqrt{2}}\left(q^{i\,\mathfrak{a}}\,\bar{q}_{j\,\mathfrak{a}} - \bar{\tilde{q}}^{i\,\mathfrak{a}}\,\tilde{q}_{j\,\mathfrak{a}}\right) \\
\mathcal{O}^{ij}_{\mathbf{3}^-} &= \tilde{q}^{*\,i\,\mathfrak{a}}\,q^*_{j\,\mathfrak{a}}\,.
\end{aligned} \tag{A.20}$$

Withe these definitions,

$$\mathcal{L}_{Q^4} = -\frac{g^2}{2}\,\mathrm{Tr}\left(3\,\mathcal{O}^{ij}_{\mathbf{1}}\,\mathcal{O}^{ij}_{\mathbf{1}} - \mathcal{O}^{ij}_{\mathbf{3}}\,\mathcal{O}^{ij}_{\mathbf{3}}\right)\ . \tag{A.21}$$

A.2 The Field Theory for General Angles

In this appendix we derive the Lagrangian dual to the system with two flavor branes at general angles, up to an ambiguity in the Q^4 terms of the scalar potential, for which we give a general parametrization. As explained in the text, we need to rotate the $\mathcal{N} = 4$ fields (including in principle the auxiliary fields) in the terms of the Lagrangian where they are coupled to the second hyper multiplet.

A.2.1 R-symmetry rotations of the $\mathcal{N} = 4$ fields

Rotation of the X_m scalars in the 49 plane (with angle θ_1) and in the 85 plane (with angle θ_2) is performed by the matrix

$$\mathcal{R}^{(\mathbf{6})}(\theta_1, \theta_2) = \begin{pmatrix} \cos\theta_1 & 0 & 0 & 0 & 0 & -\sin\theta_1 \\ 0 & \cos\theta_2 & 0 & 0 & \sin\theta_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin\theta_2 & 0 & 0 & \cos\theta_2 & 0 \\ \sin\theta_1 & 0 & 0 & 0 & 0 & \cos\theta_1 \end{pmatrix} \tag{A.22}$$

A short calculation using the the Clebsh-Gordon coefficients (5.10) gives the corresponding $SU(4)_R$ transformation for the fermions λ_A,

$$\mathcal{R}^{(\mathbf{4})}(\theta_1, \theta_2) = \begin{pmatrix} \cos\left(\frac{\theta_1-\theta_2}{2}\right) & 0 & i\sin\left(\frac{\theta_1-\theta_2}{2}\right) & 0 \\ 0 & \cos\left(\frac{\theta_1+\theta_2}{2}\right) & 0 & i\sin\left(\frac{\theta_1+\theta_2}{2}\right) \\ i\sin\left(\frac{\theta_1-\theta_2}{2}\right) & 0 & \cos\left(\frac{\theta_1-\theta_2}{2}\right) & 0 \\ 0 & i\sin\left(\frac{\theta_1+\theta_2}{2}\right) & 0 & \cos\left(\frac{\theta_1+\theta_2}{2}\right) \end{pmatrix} \tag{A.23}$$

For completeness, we also list the $SU(4)_R$ transformations corresponding to various $U(1)$ subgroups. The subgroup $U(1)_R$ (see (B.23)) corresponds to an 89 rotation:

$$\left(\begin{array}{cc|cc} e^{i\theta_{89}/2} & & & \\ & e^{i\theta_{89}/2} & & \\ \hline & & e^{-i\theta_{89}/2} & \\ & & & e^{-i\theta_{89}/2} \end{array}\right). \tag{A.24}$$

Rotations in the 45 and 67 planes are given respectively by

$$
\begin{gathered}
\left(\begin{array}{cc|cc}
e^{i\theta_{45}/2} & & & \\
 & e^{-i\theta_{45}/2} & & \\
\hline
 & & e^{-i\theta_{45}/2} & \\
 & & & e^{i\theta_{45}/2}
\end{array}\right) \\
\left(\begin{array}{cc|cc}
e^{i\theta_{67}/2} & & & \\
 & e^{-i\theta_{67}/2} & & \\
\hline
 & & e^{i\theta_{67}/2} & \\
 & & & e^{-i\theta_{67}/2}
\end{array}\right) .
\end{gathered}
\tag{A.25}
$$

Finally the $U(1)_r$ symmetry of $\mathcal{N}=1$ superspace is

$$
r = \left(\begin{array}{c|ccc}
e^{-ir} & & & \\
\hline
 & e^{+ir/3} & & \\
 & & e^{+ir/3} & \\
 & & & e^{+ir/3}
\end{array}\right) . \tag{A.26}
$$

Ideally, at this point we would provide the corresponding $SU(4)_R$ transformation of the F and D auxiliary fields. An unsuccessful attempt to find such transformation rules using the formalism of [87, 88] is described in Appendix A.3.

Clearly, the $\mathcal{N}=4$ part does not depend on the angles, since $SU(4)_R$ is an exact symmetry,

$$
\mathcal{L}_{total}(\theta) = \mathcal{L}_{\mathcal{N}=4} + \mathcal{L}_{hyper}(\theta) . \tag{A.27}
$$

We write

$$
\begin{aligned}
\mathcal{L}_{hyper}(\theta) = \quad & \mathcal{L}_{kin} + \mathcal{L}^{(1)}_{Yukawa} + \mathcal{L}^{(1)}_{\bar{Q}X^2Q} \\
& + \mathcal{L}^{(2)}_{Yukawa}(\theta) + \mathcal{L}^{(2)}_{\bar{Q}X^2Q}(\theta) + \mathcal{L}_{Q^4}(\theta) ,
\end{aligned} \tag{A.28}
$$

where the superscripts (1) and (2) refer to the first and second flavor, respectively. We have indicated which terms are θ dependent. The terms $\mathcal{L}^{(2)}_{Yukawa}(\theta)$ and $\mathcal{L}^{(2)}_{\bar{Q}X^2Q}(\theta)$ are fixed unambiguously by the transformations (A.22, A.23). By contrast to determine $\mathcal{L}_{Q^4}(\theta)$ we would need an off-shell superspace formulation for the $\mathcal{N}=4$ multiplet, which is not available at present.

The terms that we *can* fix are:

$$\mathcal{L}_{kin} = -D^{\mu}\bar{q}_i D_{\mu} q^i - D^{\mu}\tilde{q}_i D_{\mu}\bar{\tilde{q}}^i - i\bar{\psi}_i\bar{\sigma}^{\mu}D_{\mu}\psi^i - i\tilde{\psi}_i\sigma^{\mu}D_{\mu}\bar{\tilde{\psi}}^{\,i} \tag{A.29}$$

$$\begin{aligned}\mathcal{L}^{(1)}_{Yukawa} = \; & -\sqrt{2}ig\,\tilde{\psi}_1\,\lambda_1\,q^1 - \sqrt{2}ig\,\tilde{q}_1\,\lambda_1\,\psi^1 - \sqrt{2}ig\,\bar{\psi}_1\,\bar{\lambda}^2\,q^1 - \sqrt{2}ig\,\tilde{\psi}_1\,\lambda_2\,\tilde{q}^{*\,1} \\ & -i\,g\,\tilde{\psi}_1\,(X_8 + iX_9)\,\psi^1 \,+\, h.c.\end{aligned} \tag{A.30}$$

$$\begin{aligned}\mathcal{L}^{(1)}_{\bar{Q}X^2Q} \;=\; & -i\,g^2\;q_1^*\Big(\,[\,X_4\,,\,X_6\,] + i\,[\,X_4\,,\,X_7\,] + i\,[\,X_5\,,\,X_6\,] - [\,X_5\,,\,X_7\,]\,\Big)\,\tilde{q}^{*\,1} \\ & -i\,g^2\,\tilde{q}_1\Big(\,[\,X_4\,,\,X_6\,] - i\,[\,X_4\,,\,X_7\,] - i\,[\,X_5\,,\,X_6\,] - [\,X_5\,,\,X_7\,]\,\Big)\,q^1 \\ & +i\,g^2\,q_1^*\Big(\,[\,X_4\,,\,X_5\,] + \,[\,X_6\,,\,X_7\,]\,\Big)\,q^1 \\ & -i\,g^2\,\tilde{q}_1\Big(\,[\,X_4\,,\,X_5\,] + \,[\,X_6\,,\,X_7\,]\,\Big)\,\tilde{q}^{*\,1} \\ & -g^2\,q_1^*\Big(X_8^2 + X_9^2\Big)\,q^1 - \;g^2\,\tilde{q}_1\Big(X_8^2 + X_9^2\Big)\,\tilde{q}^{*\,1}\end{aligned} \tag{A.31}$$

$$\begin{aligned}\mathcal{L}^{(2)}_{Yukawa}(\theta) \;=\; & -\sqrt{2}ig\,\cos\left(\frac{\theta_1-\theta_2}{2}\right)\tilde{\psi}_2\,\lambda_1\,q^2 + \sqrt{2}g\,\sin\left(\frac{\theta_1-\theta_2}{2}\right)\tilde{\psi}_2\,\lambda_3\,q^2 \\ & - \;\sqrt{2}ig\,\tilde{q}_2\,\cos\left(\frac{\theta_1-\theta_2}{2}\right)\lambda_1\,\psi^2 + \sqrt{2}g\,\tilde{q}_2\,\sin\left(\frac{\theta_1-\theta_2}{2}\right)\lambda_3\,\psi^2 \\ & - \;\sqrt{2}ig\,\cos\left(\frac{\theta_1+\theta_2}{2}\right)\bar{\psi}_2\,\bar{\lambda}^2\,q^2 - \sqrt{2}g\,\sin\left(\frac{\theta_1+\theta_2}{2}\right)\bar{\psi}_2\,\bar{\lambda}^4\,q^2 \\ & - \;\sqrt{2}ig\,\cos\left(\frac{\theta_1+\theta_2}{2}\right)\tilde{\psi}_2\,\lambda_2\,\tilde{q}^{*\,2} + \sqrt{2}g\,\sin\left(\frac{\theta_1+\theta_2}{2}\right)\tilde{\psi}_2\,\lambda_4\,\tilde{q}^{*\,2} \\ & - \;i\,g\,\tilde{\psi}_2\,(\cos\theta_2\,X_8 + \sin\theta_2 X_5 + i\cos\theta_1\,X_9 - i\sin\theta_1 X_4)\,\psi^2 + h.c.\end{aligned} \tag{A.32}$$

$$
\begin{aligned}
&\mathcal{L}^{(2)}_{\bar{Q}X^2Q}(\theta) = \\
&- i\, g^2\, q_2^* \Big(\cos\theta_1 \left([\,X_4\,,\,X_6\,] + i\,[\,X_4\,,\,X_7\,]\right) + \cos\theta_2 \left(i\,[\,X_5\,,\,X_6\,] - [\,X_5\,,\,X_7\,]\right) \Big)\, \tilde{q}^{*\,2} \\
&- i\, g^2\, \tilde{q}_2 \Big(\cos\theta_1 \left([\,X_4\,,\,X_6\,] - i\,[\,X_4\,,\,X_7\,]\right) + \cos\theta_2 \left(-i\,[\,X_5\,,\,X_6\,] - [\,X_5\,,\,X_7\,]\right) \Big)\, q^2 \\
&- i\, g^2\, q_2^* \Big(-\sin\theta_1 \left([\,X_9\,,\,X_6\,] + i\,[\,X_9\,,\,X_7\,]\right) + \sin\theta_2 \left(i\,[\,X_8\,,\,X_6\,] - [\,X_8\,,\,X_7\,]\right) \Big)\, \tilde{q}^{*\,2} \\
&- i\, g^2\, \tilde{q}_2 \Big(-\sin\theta_1 \left([\,X_9\,,\,X_6\,] - i\,[\,X_9\,,\,X_7\,]\right) + \sin\theta_2 \left(-i\,[\,X_8\,,\,X_6\,] - [\,X_8\,,\,X_7\,]\right) \Big)\, q^2 \\
&+ i\, g^2\, q_2^* \Big(\cos\theta_1 \cos\theta_2\, [\,X_4\,,\,X_5\,] + [\,X_6\,,\,X_7\,] - \sin\theta_1 \sin\theta_2\, [\,X_8\,,\,X_9\,] \Big)\, q^2 \\
&+ i\, g^2\, q_2^* \Big(\cos\theta_1 \sin\theta_2\, [\,X_4\,,\,X_8\,] - \sin\theta_1 \cos\theta_2\, [\,X_5\,,\,X_9\,] \Big)\, q^2 \\
&- i\, g^2\, \tilde{q}_2 \Big(\cos\theta_1 \cos\theta_2\, [\,X_4\,,\,X_5\,] + [\,X_6\,,\,X_7\,] - \sin\theta_1 \sin\theta_2\, [\,X_8\,,\,X_9\,] \Big)\, \tilde{q}^{*\,2} \\
&- i\, g^2\, \tilde{q}_2 \Big(\cos\theta_1 \sin\theta_2\, [\,X_4\,,\,X_8\,] - \sin\theta_1 \cos\theta_2\, [\,X_5\,,\,X_9\,] \Big)\, \tilde{q}^{*\,2} \\
&- g^2\, q_2^* \left(\Big| \cos\theta_2\, X_8 + \sin\theta_2 X_5 \Big|^2 + \Big| \cos\theta_1\, X_9 - \sin\theta_1 X_4 \Big|^2 \right)\, q^2 \\
&- g^2\, \tilde{q}_2 \left(\Big| \cos\theta_2\, X_8 + \sin\theta_2 X_5 \Big|^2 + \Big| (\cos\theta_1\, X_9 - \sin\theta_1 X_4) \Big|^2 \right)\, \tilde{q}^{*\,2}\ .
\end{aligned}
$$

A.2.2 Parametrizing the Q^4 potential

Writing

$$
-\,\mathcal{L}_{Q^4}(\theta_1,\theta_2) = V_{Q_1^4} + V_{Q_2^4} + V_{Q_1^2 Q_2^2}(\theta_1,\theta_2)\,, \tag{A.33}
$$

it is clear that the terms $V_{Q_1^4}$ and $V_{Q_2^4}$ (involving respectively only the scalars of the first and second hyper multiplet) are unaffected by the rotation. Indeed if we set to zero one of the two hyper multiplets we must recover the standard supersymmetric Lagrangian (possibly after a change of variables: if we set to zero the *first* hyper multiplet in (A.28) we must rotate back the X and λ fields to restore $\bar{Q}X^2Q$ and Yukawa terms to the standard form).

Recall (Appendix A.1) that the potential for a single flavor Q_1 is

$$
V_{Q_1^4} = \mathrm{Tr}\,\left[2\,\mathcal{M}^{11}_{\mathbf{3}^+}\mathcal{M}^{11}_{\mathbf{3}^-} + \mathcal{M}^{11}_{\mathbf{3}^0}\mathcal{M}^{11}_{\mathbf{3}^0}\right]\,, \tag{A.34}
$$

where $\mathcal{M}^{11}_{\mathbf{3}} \sim (Q_{i=1}\bar{Q}^{i=1})_{\mathbf{3}}$ is the color–adjoint composite in the triplet of $SU(2)_R$ containing only scalars in the first flavor $i=1$ (compare with (A.17)), and similarly of course for $V_{Q_2^4}$, with $\mathcal{M}^{11}_{\mathbf{3}} \to \mathcal{M}^{22}_{\mathbf{3}} \sim (Q_{i=2}\bar{Q}^{i=2})_{\mathbf{3}}$. The mixed terms can be parametrized by two unknown functions of the angles,

$$V_{Q_1^2 Q_2^2} \;=\; 2\, g^2\, \mathrm{Tr}\left[f\left(\theta_1, \theta_2\right) \left(\mathcal{M}^{11}_{\mathbf{3}^+} \mathcal{M}^{22}_{\mathbf{3}^-} + \mathcal{M}^{11}_{\mathbf{3}^-} \mathcal{M}^{22}_{\mathbf{3}^+} \right) + d\left(\theta_1, \theta_2\right)\, \mathcal{M}^{11}_{\mathbf{3}^0} \mathcal{M}^{22}_{\mathbf{3}^0} \right]. \tag{A.35}$$

We have imposed neutrality of the potential under a Cartan generator $U(1) \subset SU(2)_R$. This $U(1)$ is preserved for general θ_1, θ_2 and corresponds geometrically to rotations in the 67 plane. When one of the two angles is zero, say $\theta_2 = 0$, an $SU(2)$ symmetry is preserved, corresponding geometrically to rotations in the 567 directions, which is a certain diagonal combination of $SU(2)_L$ and $SU(2)_R$. The hyper multiplets are neutral under $SU(2)_L$, so under a 567 rotation they just undergo just an $SU(2)_R$. It follows that for $\theta_2 = 0$ the Q^4 must be $SU(2)_R$ invariant and the functions f and d are related as

$$f\left(\theta, 0\right) = d\left(\theta, 0\right) \,. \tag{A.36}$$

The only assumption we have made in writing (A.35) is that the rotation does not introduce any terms containing the $SU(2)_R$ singlet composites $\mathcal{M}^{11}_{\mathbf{1}}$ and $\mathcal{M}^{22}_{\mathbf{1}}$. This is generally the case if the rotated Lagrangian can be obtained from *some* off-shell superspace formulation of $\mathcal{N} = 4$ SYM with manifest $SU(4)_R$ symmetry. Indeed we know that for zero angles the $\mathcal{N} = 4$ auxiliary fields only couple to $\mathcal{M}_{\mathbf{3}}$, and rotating the auxiliary fields can never generate $\mathcal{M}_{\mathbf{1}}$.

When $\theta_1 = \theta_2 = \theta$ $\mathcal{N} = 1$ supersymmetry is preserved, and the $SU(4)$ R-symmetry transformation corresponds to a matrix $\mathcal{R}^{(\mathbf{4})} \subset SU(3)$, which acts on F^a leaving D invariant. This is a manifest symmetry of the $\mathcal{N} = 1$ superspace formulation. As reviewed in Appendix A.3, in this special case one can unambiguously find

$$\begin{aligned} f\left(\theta, \theta\right) &= \cos\theta \\ d\left(\theta, \theta\right) &= 1 \,. \end{aligned} \tag{A.37}$$

Further constraints follow from discrete symmetries. The 89 reflection $X_8, X_9 \to -X_8, -X_9$ corresponds

$$\theta_1\,, \theta_2 \to -\theta_1\,, -\theta_2\,. \tag{A.38}$$

Invariance of the Q^4 potential under this parity symmetry implies

$$f\left(\theta_1, \theta_2\right) = f\left(-\theta_1, -\theta_2\right) \tag{A.39}$$

$$d\left(\theta_1, \theta_2\right) = d\left(-\theta_1, -\theta_2\right) \,. \tag{A.40}$$

Similarly, invariance under the discrete symmetry $X_4 \leftrightarrow X_5$ and $X_8 \leftrightarrow X_9$, or

$$\theta_1 \leftrightarrow \theta_2\,, \tag{A.41}$$

implies

$$f\,(\theta_1,\theta_2) = f\,(\theta_2,\theta_1) \quad \text{and} \quad d\,(\theta_1,\theta_2) = d\,(\theta_2,\theta_1)\ . \tag{A.42}$$

Unfortunately this set of relations is not sufficient to fix the functions f and d uniquely.

A crucial assumption we shall make is *positivity* of the classical Q^4 potential. It would follow from the existence of a superspace formulation with manifest $SU(4)_R$ symmetry: the scalar potential would be proportional to the square of some auxiliary fields, and it would thus be positive even the susy-breaking $SU(4)_R$ rotation in the terms that couple the auxiliary fields to the second hyper multiplet. Assuming positivity, we have

$$V_{Q^4} \geq 0 \quad \Rightarrow \quad f\,(\theta_1,\theta_2) \leq 1 \quad \text{and} \quad d\,(\theta_1,\theta_2) \leq 1 \quad \forall\,\theta_1,\theta_2\,. \tag{A.43}$$

For small angles, taking into account the discrete symmetries, we can expand

$$\begin{aligned} f\,(\theta_1,\theta_2) &= 1-\alpha(\theta_1^2+\theta_2^2)-\beta\theta_1\theta_2+O(\theta^3) & \text{(A.44)}\\ d\,(\theta_1,\theta_2) &= 1-\tilde{\alpha}\left(\theta_1^2+\theta_2^2\right)-\tilde{\beta}\theta_1\theta_2+O(\theta^3) & \text{(A.45)} \end{aligned}$$

for some coefficients $\alpha,\beta,\tilde{\alpha},\tilde{\beta}$. Imposing (A.37) gives $2\alpha+\beta=\frac{1}{2}$ and $2\tilde{\alpha}+\tilde{\beta}=0$, while (A.36) gives $\alpha=\tilde{\alpha}$. This leaves us with with a single unknown coefficient,

$$\begin{aligned} f\,(\theta_1,\theta_2) &= 1-\alpha(\theta_1-\theta_2)^2-\frac{1}{2}\theta_1\theta_2+O(\theta^3) & \text{(A.46)}\\ d\,(\theta_1,\theta_2) &= 1-\alpha\,(\theta_1-\theta_2)^2+O(\theta^3)\,. & \text{(A.47)} \end{aligned}$$

Positivity of the Q^4 potential implies that $\alpha>0$.

Finally we record the explicit expressions of the Q^4 terms, both in terms of component fields,[4]

$$
\begin{aligned}
\mathcal{L}_{Q^4}(\theta_1,\theta_2) \;=\;& -2\,g^2\left(\tilde{q}_1\cdot\tilde{q}^{*\,1}\right)\left(q_1^*\cdot q^1\right)-2\,g^2\left(\tilde{q}_2\cdot\tilde{q}^{*\,2}\right)\left(q_2^*\cdot q^2\right) \\
& -2\,g^2 f\left(\theta_1,\theta_2\right)\Big[\left(\tilde{q}_1\cdot\tilde{q}^{*\,2}\right)\left(q_2^*\cdot q^1\right)+\left(\tilde{q}_2\cdot\tilde{q}^{*\,1}\right)\left(q_1^*\cdot q^2\right)\Big] \\
& -\frac{g^2}{2}\left(q^1\,q_1^*-\tilde{q}^{*\,1}\,\tilde{q}_1\right)^2-\frac{g^2}{2}\left(q^2\,q_2^*-\tilde{q}^{*\,2}\,\tilde{q}_2\right)^2 \\
& -g^2 d\left(\theta_1,\theta_2\right)\left(q^1\,q_1^*-\tilde{q}^{*\,1}\,\tilde{q}_1\right)\cdot\left(q^2\,q_2^*-\tilde{q}^{*\,2}\,\tilde{q}_2\right)\,,
\end{aligned}
\tag{A.49}
$$

and in terms of the gauge invariant mesonic operators,

$$
\begin{aligned}
\mathcal{L}_{Q^4}(\theta_1,\theta_2) \;=\;& -\frac{g^2}{2}\,\mathrm{Tr}\Big[3\,\mathcal{O}^{11}_{\mathbf{1}}\,\mathcal{O}^{11}_{\mathbf{1}}-\mathcal{O}^{11}_{\mathbf{3}}\,\mathcal{O}^{11}_{\mathbf{3}}\Big]-\frac{g^2}{2}\,\mathrm{Tr}\Big[3\,\mathcal{O}^{22}_{\mathbf{1}}\,\mathcal{O}^{22}_{\mathbf{1}}-\mathcal{O}^{22}_{\mathbf{3}}\,\mathcal{O}^{22}_{\mathbf{3}}\Big] \\
& -g^2\left(d\left(\theta_1,\theta_2\right)+2\,f\left(\theta_1,\theta_2\right)\right)\,\mathrm{Tr}\Big[\mathcal{O}^{12}_{\mathbf{1}}\,\mathcal{O}^{21}_{\mathbf{1}}\Big] \\
& -g^2\left(d\left(\theta_1,\theta_2\right)-2\,f\left(\theta_1,\theta_2\right)\right)\,\mathrm{Tr}\Big[\mathcal{O}^{12}_{\mathbf{3}^0}\,\mathcal{O}^{21}_{\mathbf{3}^0}\Big] \\
& +g^2 d\left(\theta_1,\theta_2\right)\mathrm{Tr}\Big[\mathcal{O}^{12}_{\mathbf{3}^+}\,\mathcal{O}^{21}_{\mathbf{3}^-}+\mathcal{O}^{12}_{\mathbf{3}^-}\,\mathcal{O}^{21}_{\mathbf{3}^+}\Big]\,.
\end{aligned}
\tag{A.50}
$$

A.3 R-symmetry in $\mathcal{N}=1$ Superspace

In this appendix we describe an attempt to derive the Q^4 potential for general angles, using a formalism developed in [87, 88] to describe the general global transformations of $\mathcal{N}=4$ SYM in $\mathcal{N}=1$ superspace language. The attempt fails, for reasons that could have been anticipated: while the formalism prescribes how auxiliary fields must transform under general R-symmetry transformations so that the action is invariant, the transformations do not close off-shell. Nevertheless we believe that the exercise contains some relevant lessons in the search of a more complete superspace formulation of $\mathcal{N}=4$ SYM and we reproduce it here for the benefit of the technically inclined reader. We also present an application of the formalism to the analogous problem for $\mathcal{N}=2$ SYM coupled to $\mathcal{N}=1$ chiral matter: how to break supersymmetry by inequivalent embeddings of two $\mathcal{N}=1$ subalgebras into $\mathcal{N}=2$. In this case the formalism works, because the algebra of global transformations closes off-shell. The simplified problem is interesting in its own right and provides a

[4] We use dots as shorthand notation for color contractions. Using $\mathfrak{a},\mathfrak{b}$ for the color indices, and suppressing all other indices, we set

$$
(q\,q^*)\cdot(q\,q^*)\equiv\left(q^{\mathfrak{a}}\,q^*_{\mathfrak{b}}\right)\cdot\left(q^{\mathfrak{b}}\,q^*_{\mathfrak{a}}\right)\,,\quad (q\,q^*)^2\equiv\left(q^{\mathfrak{a}}\,q^*_{\mathfrak{b}}\right)\cdot\left(q^{\mathfrak{b}}\,q^*_{\mathfrak{a}}\right)\,. \tag{A.48}
$$

model for how things should work in the yet-to-be-found improved superspace formulation of $\mathcal{N}=4$.

In $\mathcal{N}=1$ superspace, $\mathcal{N}=2$ SYM has a manifest $U(1)_r \times U(1)_u$ subgroup of the $SU(2)_R \times U(1)_R$ R-symmetry, and $\mathcal{N}=4$ SYM a manifest $SU(3)_R \times U(1)_r$ subgroup of the $SU(4)_R$ R-symmetry. Nevertheless, the remaining R-symmetry transformations, while realized non-linearly, are legitimate off-shell symmetries of the superspace action. They close off-shell for $\mathcal{N}=2$ but not for $\mathcal{N}=4$. The explicit transformations rules were originally given in [87]. We follow the presentation of [88]. Here we review the superspace formalism of [87, 88], translate it into components and apply it to our problem.

A.3.1 Global symmetries in $\mathcal{N}=1$ superspace

$\mathcal{N}=1$ supersymmetric theories are invariant under translations, supersymmetry transformations and (under certain conditions) R-symmetry transformations. The parameters of these transformations can be assembled into a single x-independent real superfield ζ, subject to the gauge-invariance

$$\delta\zeta = i(\bar{\xi} - \xi)\,, \tag{A.51}$$

where ξ is an x-independent chiral superfield. The physical components of ζ (in Wess-Zumino gauge) are

$$\zeta_{\alpha\dot{\alpha}} = \frac{1}{2}\left[\bar{D}_{\dot{\alpha}}, D_\alpha\right]\zeta|\,, \quad \epsilon_\alpha = i\bar{D}^2 D_\alpha \zeta|\,, \quad r = \frac{1}{2} D^\alpha \bar{D}^2 D_\alpha\, \zeta|\,. \tag{A.52}$$

The vector $\zeta_{\alpha\dot{\alpha}}$ parametrizes the translations, the spinor ϵ_α the supersymmetry transformations and the scalar r the $U(1)_r$ symmetry.

Let us next consider the $\mathcal{N}=2$ SYM theory. In $\mathcal{N}=1$ superspace, the field content consists of a vector superfield V and a chiral superfield Φ. It was shown [88] that the $\mathcal{N}=2$ SYM action is invariant under the global transformations

$$\begin{aligned} \delta\Phi &= -W^\alpha \nabla_\alpha \eta - i\,\bar{\nabla}^2\left(\nabla^\alpha\zeta\right)\nabla_\alpha\Phi \\ e^{-V}\delta e^V &= i\left(\bar{\eta}\,\Phi - \eta\,\tilde{\bar{\Phi}}\right) + \left(W^\alpha\nabla_\alpha + \bar{W}^{\dot{\alpha}}\bar{\nabla}_{\dot{\alpha}}\right)\zeta \quad . \end{aligned} \tag{A.53}$$

Moreover, the algebra of these transformations closes off-shell. The parameters η and ζ are x-independent, color neutral superfields. The symbol ∇_α denotes the gauge-covariant derivative, $\nabla_\alpha \equiv e^{-V} D_\alpha e^V$, and $\tilde{\bar{\Phi}} \equiv e^{-V}\bar{\Phi}e^V$. Note that $\nabla_\alpha\eta = D_\alpha\eta$ and $\nabla_\alpha\zeta = D_\alpha\zeta$. Another obvious global symmetry of the action is a phase rotation of Φ,

$$\delta\Phi = iu\Phi\,, \quad \delta V = 0\,. \tag{A.54}$$

The parameter ζ is the real superfield of $\mathcal{N}=1$ transformations, as above. Its vectorial component $\zeta_{\alpha\dot{\alpha}}$ parametrizes translations, its spinorial component ϵ^1_α parametrizes the manifest $\mathcal{N}=1$ supersymmetry, and its auxiliary component r parametrizes the manifest $U(1)_r$ symmetry

$$\Phi(y,\theta) \to e^{-2ir}\Phi(y, e^{ir}\theta)\,, \quad V(x,\theta,\bar{\theta}) \to V(x, e^{ir}\theta, e^{-ir}\bar{\theta})\,. \tag{A.55}$$

The parameter η is a chiral superfield that mixes V and Φ. Its components are

$$z = \eta|\,, \quad \epsilon^2_\alpha = D_\alpha\eta|\,, \quad \mu = D^2\eta|\,. \tag{A.56}$$

The complex scalar z corresponds to the central charge, the spinor ϵ^2_α parametrizes the second (non-manifest in $\mathcal{N}=1$ superspace) supersymmetry transformation, and finally the complex scalar μ parametrizes the non-manifest internal symmetries $U(2)/(U(1)_r \times U(1)_u)$. The complete internal symmetry group of the classical action is $U(2)$, parametrized by u, r and μ. (Note that together the parameters ζ and η form an $\mathcal{N}=2$ vector multiplet, mimicking the field content of the theory.)

From (A.53, A.54), we find the following global symmetry transformation rules on the component fields and auxiliary fields:

$$\begin{aligned}
\delta\phi &= -2\,i\,r\,\phi + i\,u\,\phi && \text{(A.57)}\\
\delta\chi &= -\mu\lambda - i\,r\,\chi + i\,u\,\chi && \text{(A.58)}\\
\delta\lambda &= \bar{\mu}\chi - i\,r\,\lambda &&\\
\delta F &= 2\,i\,\mu D + i\,u\,F && \text{(A.59)}\\
\delta D &= i\left(\bar{\mu}F - \mu\bar{F}\right)\,. &&
\end{aligned}$$

The two U(1)s in these transformations are the r-symmetry of $\mathcal{N}=1$ superspace $U(1)_r$ and the global phase rotation $U(1)_u$: these are natural symmetries from an $\mathcal{N}=1$ superspace point of view. The $\mathcal{N}=2$ SYM theory has a $SU(2)_R \times U(1)_R$ R-symmetry where the $U(1)_R$ and the diagonal $T_3 \subset SU(2)_R$

are related to r and u as $R = r - \frac{u}{2}$, $\mu_3 = u$, so that we can equivalently write

$$\begin{aligned}
\delta\phi &= -2\,i\,R\,\phi && \text{(A.60)}\\
\delta\chi &= -\mu\lambda - i\,R\,\chi + i\,\frac{\mu_3}{2}\,\chi && \text{(A.61)}\\
\delta\lambda &= \bar{\mu}\chi - i\,R\,\lambda - i\,\frac{\mu_3}{2}\,\lambda\\
\delta F &= 2\,i\,\mu D - i\,\mu_3\,F && \text{(A.62)}\\
\delta D &= i\left(\bar{\mu}F - \mu\bar{F}\right)\,.
\end{aligned}$$

Note that the auxiliary fields transform as a triplet under $SU(2)_R$.

We finally come to $\mathcal{N}=4$ SYM. The $\mathcal{N}=4$ action is invariant under the global transformations [88]

$$\begin{aligned}
\delta\Phi^a &= -\left(W^\alpha\nabla_\alpha\eta^a + \epsilon^{abc}\bar{\nabla}^2\bar{\eta}_b\tilde{\Phi}_c\right) && \text{(A.63)}\\
&\quad -i\left[\bar{\nabla}^2\left(\nabla^\alpha\zeta\right)\nabla_\alpha\Phi^a + \frac{2}{3}\bar{\nabla}^2\left(\nabla^2\zeta\right)\Phi^a\right]
\end{aligned}$$

$$e^{-V}\delta e^V = i\left(\bar{\eta}_a\Phi^a - \eta^a\tilde{\Phi}_a\right) + \left(W^\alpha\nabla_\alpha + \bar{W}^{\dot{\alpha}}\bar{\nabla}_{\dot{\alpha}}\right)\zeta\,. \qquad \text{(A.64)}$$

Unlike the $\mathcal{N}=2$ case, the algebra does not close off-shell. The parameters ζ and η^a, $a=1,2,3$ have the same interpretation as before. The real superfield ζ contains the parameters of the manifest symmetries, while the chiral superfields η^a contain the parameters of the non-manifest symmetries. In particular their auxiliary auxiliary components of μ^a are the parameters of the $SU(4)/(SU(3)\times U(1)_r)$ R-symmetries. From (A.63), after some algebra we find the following $SU(4)/SU(3)$ transformation rules on the component fields,

$$\begin{aligned}
\delta\phi^a &= -\epsilon^{abc}\bar{\mu}_b\bar{\phi}_c - i\frac{2}{3}r\,\phi^a && \text{(A.65)}\\
\delta\chi^a &= -\mu^a\lambda + \frac{1}{3}i\,r\,\chi^a && \text{(A.66)}\\
\delta\lambda &= \bar{\mu}_a\chi^a - i\,r\,\lambda\\
\delta F^a &= 2\,i\,\mu^a D + i\frac{4}{3}r\,F^a && \text{(A.67)}\\
\delta D &= i\left(\bar{\mu}_a F^a - \mu^a\bar{F}_a\right)\,.
\end{aligned}$$

A.3.2 Application to $\mathcal{N}=2$

Let us consider $\mathcal{N}=2$ SYM coupled to N_f $\mathcal{N}=1$ chiral multiplets Q^i, $i=1,\dots N_f$. The only term in the $\mathcal{N}=1$ superspace Lagrangian that couples the different flavors is the Kähler term for the chiral multiplets,

$$\int d^4\theta\, \bar{Q}_i\, e^{gV}\, Q^i\,. \tag{A.68}$$

(There is no superpotential term that preserves gauge invariance and the $U(N_f)$ global flavor symmetry.) In component language, and before integrating out the auxiliary fields, the relevant terms in the Lagrangian are

$$\mathcal{L}=\cdots+D\left(q_i^* q^i+\left[\phi\,,\,\bar{\phi}\right]\right)+D^2+\bar{F}F+\dots \tag{A.69}$$

We then perform an off diagonal $SU(2)_R$ transformation (A.62) with $\mu=i\,\theta/2$ to the $\mathcal{N}=2$ SYM component fields that couple to the second chiral multiplet.

$$\mathcal{L}=\cdots+D\left(q_1^* q^1+q_2^* q^2\cos\theta+\left[\phi\,,\,\bar{\phi}\right]\right)+\mathrm{Re}(F)q_2^* q^2\sin\theta+D^2+\bar{F}F+. \tag{A.70}$$

Integrating out the auxiliary fields, we find the scalar potential

$$V_{q^4}=|q_1|^4+|q_2|^4+2\,\cos\theta\,|q_1^* q^2|^2\,, \tag{A.71}$$

which is positive definite for any θ since it is proportional to $D^2+\bar{F}F$.

A.3.3 Application to $\mathcal{N}=4$

From (A.22), we see that for infinitesimal θ_1, θ_2,

$$\begin{aligned}\delta\phi_1 &= i\left(\frac{\theta_1+\theta_2}{2}\right)\phi_3-i\left(\frac{\theta_1-\theta_2}{2}\right)\bar{\phi}_3\\ \delta\phi_3 &= i\left(\frac{\theta_1+\theta_2}{2}\right)\phi_1+i\left(\frac{\theta_1-\theta_2}{2}\right)\bar{\phi}_1\\ \delta\phi_2 &= 0\,.\end{aligned} \tag{A.72}$$

The holomorphic part of the variation is an infinitesimal $SU(3)$ rotation $\delta\phi^a=i(\theta_1+\theta_2)(\hat{T}_6\,\phi)^a$ generated by the Lie algebra element

$$\hat{T}_6=\frac{1}{2}\,\hat{\lambda}_6=\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\,. \tag{A.73}$$

Comparison with (A.65) shows that the antiholomorphic part of the variation is an $SU(4)/SU(3)$ transformation with parameters

$$r = \mu_1 = \mu_3 = 0\,, \quad \mu_2 = \pm i\,\frac{\theta_2 - \theta_1}{2}\,. \tag{A.74}$$

Recalling that F^a transform in the **3** of $SU(3)$, and using (A.67) for the transformation rules under $SU(4)/SU(3)$, we find the corresponding variations of the auxiliary fields,

$$\begin{aligned} \delta F^1 &= i\left(\frac{\theta_1+\theta_2}{2}\right)F^3 \\ \delta F^3 &= i\left(\frac{\theta_1+\theta_2}{2}\right)F^1 \end{aligned} \tag{A.75}$$

$$\begin{aligned} \delta D &= -\left(\theta_1-\theta_2\right)\mathrm{Re}\left(F_2\right) \\ \delta \mathrm{Re}\left(F_2\right) &= \left(\theta_1-\theta_2\right)D\,. \end{aligned} \tag{A.76}$$

Their naive exponentiation gives

$$\left(\begin{array}{c} D \\ \mathrm{Re}\left(F_2\right) \end{array}\right)_{rot} = \left(\begin{array}{cc} \cos\left(\theta_1-\theta_2\right) & -\sin\left(\theta_1-\theta_2\right) \\ \sin\left(\theta_1-\theta_2\right) & \cos\left(\theta_1-\theta_2\right) \end{array}\right)\left(\begin{array}{c} D \\ \mathrm{Re}\left(F_2\right) \end{array}\right) \tag{A.77}$$

$$\left(\begin{array}{c} F_1 \\ F_3 \end{array}\right)_{rot} = \left(\begin{array}{cc} \cos\left(\frac{\theta_1+\theta_2}{2}\right) & i\,\sin\left(\frac{\theta_1+\theta_2}{2}\right) \\ i\,\sin\left(\frac{\theta_1+\theta_2}{2}\right) & \cos\left(\frac{\theta_1+\theta_2}{2}\right) \end{array}\right)\left(\begin{array}{c} F_1 \\ F_3 \end{array}\right)\,. \tag{A.78}$$

Given these explicit transformations for the auxiliary fields we can proceed to derive the form of the Q^4 potential after rotation. The prescription is to transform the auxiliary fields that couple to the second hyper multiplet, leaving untouched the auxiliary fields that couple to the first hyper multiplet. This method predicts

$$\begin{aligned} f\left(\theta_1,\theta_2\right) &= \cos\left(\frac{\theta_1+\theta_2}{2}\right) \\ d\left(\theta_1,\theta_2\right) &= \cos\left(\theta_1-\theta_2\right) \end{aligned} \tag{A.79}$$

for the parameter functions introduced in (A.35). This result is clearly incorrect. It does not satisfy condition (A.36). Moreover $f(\theta,0)$ has the wrong periodicity – the potential should come back to itself after a 2π rotation. Since the transformation rules do not close off-shell, it was not permissible to simply exponentiate the infinitesimal variations. It appears that this is a fundamental flaw of this formalism and that what is required is a different superspace formulation where the $SU(4)_R$ closes off-shell.

A.4 Anomalous Dimensions

In this appendix we describe the computation of the one-loop anomalous dimensions of the mesonic operators $\mathcal{O}^{\mathcal{I}}{}_{\mathcal{J}}$. Following [11], we view a single-trace composite operator as a closed spin chain whose sites correspond to the elementary fields. In the large N limit and at the one-loop level only nearest neighbor interactions are present. The nearest neighbor interaction is conveniently expressed in terms of three elementary operators acting on the vector space of two successive sites. These three operators represent the three independent ways to map two $SU(2)_R$ symmetry indices of an "incoming" operator $\mathcal{O}^{\mathcal{I}}{}_{\mathcal{J}}$ to the indices of an "outgoing" operator $\bar{\mathcal{O}}_{\mathcal{L}}{}^{\mathcal{K}}$. They are the trace operator $\mathbb{K}$, the permutation operator $\mathbb{P}$ and the identity operator $\mathbb{I}$:

$$\mathbb{K}^{\mathcal{J}\mathcal{L}}_{\mathcal{I}\mathcal{K}} \equiv \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{L}}_{\mathcal{K}}\,, \qquad \mathbb{P}^{\mathcal{J}\mathcal{L}}_{\mathcal{I}\mathcal{K}} \equiv \delta_{\mathcal{I}\mathcal{K}}\delta^{\mathcal{J}\mathcal{L}}\,, \qquad \mathbb{I}^{\mathcal{J}\mathcal{L}}_{\mathcal{I}\mathcal{K}} \equiv \delta^{\mathcal{L}}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{K}}\,. \tag{A.80}$$

The anomalous dimension of the mesonic operators receives contributions from the Feynman diagrams shown schematically in Figure 3.2. Since the gauge boson exchange is R-symmetry blind, the Feynman diagram shown in figure 3.2(a) is proportional to the identity operator,

$$Z_A - 1 = (1-\xi)\,\frac{g^2\,N}{8\pi^2}\,\mathbb{I}\,\ln\Lambda\ . \tag{A.81}$$

Here ξ is the gauge fixing parameter in the propagator of the gauge boson, which is $\frac{g_{\mu\nu}-\xi\frac{k_\mu k_\nu}{k^2}}{k^2}$ in our conventions. The $SU(2)_R$ structure of the quartic interaction (figure 3.2(b)) is more interesting. The scalar vertex has index structure

$$\frac{1}{2}\,\delta^{\mathcal{L}}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{K}} - \,\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{L}}_{\mathcal{K}} = \frac{1}{2}\,\mathbb{I} - \mathbb{K}\ , \tag{A.82}$$

with trace part arising from F-terms and the identity from the D-terms. The contribution of the quartic interaction to the renormalization of the mesonic operators is then

$$Z_{Q^4} - 1 = \frac{g^2\,N}{8\pi^2}\,(\mathbb{I} - \,2\,\mathbb{K})\,\ln\Lambda\ . \tag{A.83}$$

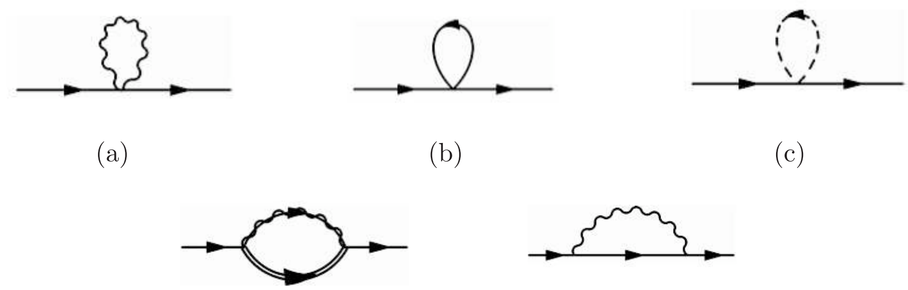


Figure A.1: One-loop Feynman diagrams contributing to γ_Q.

Finally, we need to consider the squark self-energy corrections (figure 3.2(c), shown in more detail in figure A.1). The contribution of the squark self-energy to the meson renormalization is

$$Z_Q - 1 = -\,(2-\xi)\,\frac{\lambda}{8\pi^2}\,\mathbb{I}\,\ln\Lambda\,. \tag{A.84}$$

We also record for future use the anomalous dimension of the squark,

$$\gamma_Q = \frac{\lambda}{8\pi^2}\,\delta_{\mathcal{I}}^{\mathcal{J}}\,\left(2-\xi\right)\,. \tag{A.85}$$

Adding the diagrams, we find

$$Z = 1 + \frac{\lambda}{4\pi^2}\,\mathbb{K}\,\ln\Lambda \tag{A.86}$$

and we read off the matrix of anomalous dimensions,

$$\Gamma^{(1)} \equiv \frac{dZ}{d\ln\Lambda}Z^{-1} = \frac{\lambda}{4\pi^2}\,\mathbb{K}\,. \tag{A.87}$$

This answer is due entirely to the F-terms, since all other contributions (D-terms, gluon exchange and self-energy diagram) add up to zero. This is an example of a general property of theories with extended supersymmetry [102–104].

The trace operator is a 4×4 matrix that acts on the four dimensional $\mathbf{2}\times\bar{\mathbf{2}}$ vector space $Q^{\mathcal{I}}\bar{Q}_{\mathcal{J}}$. Its eigenstates are the singlet and the triplet states of $SU(2)_R$, with eigenvalues 2 and 0 respectively. In this basis the anomalous dimension matrix $\Gamma^{(1)}$ is diagonal, with eigenvalues:

$$\gamma_{\mathbf{1}} = \frac{\lambda}{2\pi^2} \quad \text{and} \quad \gamma_{\mathbf{3}} = 0 \tag{A.88}$$

This result is expected because $\mathcal{O}_{\mathbf{3}}$ is an $\mathcal{N}=2$ chiral primary that obeys the shortening condition $\Delta = 2R$, while $\mathcal{O}_{\mathbf{1}}$ belongs to a long multiplet and is not protected.

A.5 Coleman-Weinberg Potential

The calculation of the one-loop effective potential is straightforward but somewhat lengthy. Here we provide some intermediate steps for the sake of the reader who would like to reproduce our result. Following the original paper by Coleman and Weinberg [62], the bosonic and fermionic contributions to the one-loop effective potential are

$$\mathcal{V}_{bose} = \frac{1}{64\pi^2} \mathrm{Tr}\, \mathcal{M}_b^4 \ln\left(\mathcal{M}_b^2\right) \tag{A.89}$$

$$\mathcal{V}_{fermi} = -\frac{1}{64\pi^2} \mathrm{Tr}\left(\mathcal{M}_f \mathcal{M}_f^\dagger\right)^2 \ln\left(\mathcal{M}_f \mathcal{M}_f^\dagger\right) , \tag{A.90}$$

where the mass matrices read off by expanding the Lagrangian (A.28) around the classical background. We choose the background (3.33)

$$Q_1 = \begin{pmatrix} q \\ 0 \end{pmatrix} , \quad Q_2 = \begin{pmatrix} 0 \\ -q \end{pmatrix} , \qquad q \in \mathbb{C} \, . \tag{A.91}$$

Setting to zero the extra "double-trace" couplings, $f \equiv 0$, we find the following partial contributions (we write $V \equiv v \, \frac{N\, g^4\, |q|^4}{16\pi^2} \ln |q|^2$):

$$\begin{aligned}
v_Q &= \left(\frac{5 + d^2(\theta_1, \theta_2)}{2} \right) , \\
v_{fermi}\,(\theta) &= \left(-8 + 2 \sin^2\left(\frac{\theta_1+\theta_2}{2}\right) + 2 \sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\right) , \\
v_A &= 3 \, , \\
v_X &= \left(2 - 2 \sin^2\left(\frac{\theta_1-\theta_2}{2}\right) - 2 \sin^2\left(\frac{\theta_1+\theta_2}{2}\right) + 4 \sin^2\left(\frac{\theta_1-\theta_2}{2}\right) \sin^2\left(\frac{\theta_1+\theta_2}{2}\right)\right) .
\end{aligned} \tag{A.92}$$

The contribution of the gauge fields is calculated in the Landau gauge[5]. Adding these partial contributions,

$$\mathcal{V}_{1-loop}\,(\theta_1,\theta_2;f=0) \equiv \frac{\lambda^2}{N}\,v_{1-loop}\,|q|^4\ln|q| \tag{A.93}$$

with

$$v_{1-loop} = \frac{1}{8\pi^2}\left[4\,\sin^2\left(\frac{\theta_1+\theta_2}{2}\right)\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)+\frac{d^2(\theta_1,\theta_2)-1}{2}\right]\,. \tag{A.94}$$

For $\theta_1 = \theta_2 = \theta$, using $d(\theta,\theta) = 1$ we find

$$\mathcal{V}_{1-loop}\,(\theta,\theta;f=0) = 0\,, \tag{A.95}$$

as expected since this configuration preserves $\mathcal{N} = 1$ susy.

Along this classical background, the tree-level potential $\mathcal{V}_{tree} \equiv -(\mathcal{L}_{Q^4} + \delta\mathcal{L}_{fund})$ (see (A.49) and (3.26)) evaluates to

$$\mathcal{V}_{tree}(q) = \frac{\lambda}{N}\,|q|^4\Big(1-d(\theta_1,\theta_2)\Big)+\frac{f_{\mathbf{3}^+}}{N}\,|q|^4 \equiv \frac{\lambda}{N}\,\mathcal{C}_\lambda+\frac{f_{\mathbf{3}^+}}{N}\,\mathcal{C}_f\,. \tag{A.96}$$

The classical background was chosen precisely to ensure that the only double-trace coupling contributing at tree level is $f_{\mathbf{3}^+}$. The Callan-Symanzik equation for the effective potential reads

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_f\frac{\partial}{\partial f}+\beta_\lambda\frac{\partial}{\partial\lambda}-\gamma_q^{(1)}\,q\frac{\partial}{\partial q}\right]\mathcal{V}(q,\mu,f,\lambda) = 0\,. \tag{A.97}$$

We can drop the β_λ term since it is subleading for large N. Writing $\mathcal{V}(f,\lambda) = \mathcal{V}_{tree} + \mathcal{V}_{1-loop}(f=0) + O(f) + O(\lambda^3)$, the CZ equation allows to extract the

[5]This is convenient for the following reason. The formula (A.89) arises from the resummation of polygonal one-loop diagrams with the background fields at the external legs which have zero momenta. When gauge fields are inside the loop there are more diagrams than just the gauge polygons (the polygons are made purely out of gauge fields). But in the Landau gauge the only diagrams that are non-zero are gauge polygons. Then their contribution is simply (A.89) multiplied by 3. The extra factor of 3 stems from the trace of the numerator of the Landau gauge propagator.

f-independent one-loop coefficient of β_f, $\beta_f(f=0) = a(\lambda) = a^{(1)}\lambda^2 + O(\lambda^3)$,

$$\begin{aligned} a^{(1)} &= 4\,\gamma_q^{(1)}\,(\mathcal{C}_\lambda/\mathcal{C}_f) + v_{1-loop} \\ &= \frac{1}{16\pi^2}\left[\Big(1-d(\theta_1,\theta_2)\Big) + \frac{1}{2}\Big(1-d(\theta_1,\theta_2)\Big)^2 \right. \\ &+ \left. 4\,\sin^2\left(\frac{\theta_1+\theta_2}{2}\right)\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\right]. \end{aligned} \tag{A.98}$$

To study the stability of the CW potential, we must first perform the standard RG improvement [62]. A detailed discussion for the problem at hand can be found in section 3 of [6]. One finds that the “perturbative vacuum” $q = 0$ is stable if and only if β_f admits real zeros. In our case β_f has *imaginary* zeros and symmetry breaking does occur. This is one of the manifestations of the tachyonic instability on the field theory side.

Appendix B: Holography in the Veneziano Limit

B.1 Shortening Conditions of the N =2 Superconformal Algebra

A generic long multiplet $\mathcal{A}^{\Delta}_{R,r(j,\bar{j})}$ of the $\mathcal{N}=2$ superconformal algebra is generated by the action of the 8 Poincaré supercharges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ on a superconformal primary, which by definition is annihilated by all conformal supercharges . If some combination of the $\mathcal{Q}$'s also annihilates the primary, the corresponding multiplet is shorter and the conformal dimensions of all its members are protected against quantum corrections. A comprehensive list of the possible shortening conditions for the $\mathcal{N}=2$ superconformal algebra was given in [154] . Their findings are summarized in Table B.1. We take a moment to explain the notation.[1] The state $|R,r\rangle^{h.w.}_{(j,\bar{j})}$ is the highest weight state with $SU(2)_R$ spin $R>0$, $U(1)_r$ charge r, which can have either sign, and Lorentz quantum numbers $(j,\bar{j})$. The multiplet built on this state is denoted as $\mathcal{X}_{R,r(j,\bar{j})}$, where the letter $\mathcal{X}$ characterizes the shortening condition. The left column of Table B.1 labels the condition. A superscript on the label corresponds to the index $\mathcal{I}=1,2$ of the supercharge that kills the primary: or example $\mathcal{B}^1$ refers to $\mathcal{Q}^1_{\alpha}$. Similarly a "bar" on the label refers to the conjugate condition: for example $\bar{\mathcal{B}}^2$ corresponds to $\bar{Q}_{2\,\dot{\alpha}}$ annihilating the state; this would result in the short anti-chiral multiplet $\bar{\mathcal{B}}_{R,r(j,0)}$, obeying $\Delta=2R-r$. Note that conjugation reverses the signs of r, j and $\bar{j}$ in the expression of the conformal dimension. We refer to [154] for more details.

[1] We follow the conventions of [154], except that we have introduced the labels $\mathcal{D}$, $\mathcal{F}$, $\hat{\mathcal{F}}$ and $\mathcal{G}$ to denote some shortening conditions that were left nameless in [154].

Shortening Conditions				Multiplet
$\mathcal{B}_1$	$\mathcal{Q}^1_\alpha \lvert R,r\rangle^{h.w.} = 0$	$j=0$	$\Delta = 2R+r$	$\mathcal{B}_{R,r(0,\bar{j})}$
$\bar{\mathcal{B}}_2$	$\bar{\mathcal{Q}}_{2\dot{\alpha}} \lvert R,r\rangle^{h.w.} = 0$	$\bar{j}=0$	$\Delta = 2R-r$	$\bar{\mathcal{B}}_{R,r(j,0)}$
$\mathcal{E}$	$\mathcal{B}_1 \cap \mathcal{B}_2$	$R=0$	$\Delta = r$	$\mathcal{E}_{r(0,\bar{j})}$
$\bar{\mathcal{E}}$	$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$R=0$	$\Delta = -r$	$\bar{\mathcal{E}}_{r(j,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B}_1 \cap \bar{\mathcal{B}}_2$	$r=0,\ j,\bar{j}=0$	$\Delta = 2R$	$\hat{\mathcal{B}}_R$
$\mathcal{C}_1$	$\epsilon^{\alpha\beta}\mathcal{Q}^1_\beta \lvert R,r\rangle^{h.w.}_\alpha = 0$ $(\mathcal{Q}^1)^2 \lvert R,r\rangle^{h.w.} = 0$ for $j=0$		$\Delta = 2+2j+2R+r$ $\Delta = 2+2R+r$	$\mathcal{C}_{R,r(j,\bar{j})}$ $\mathcal{C}_{R,r(0,\bar{j})}$
$\bar{\mathcal{C}}_2$	$\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\mathcal{Q}}_{2\dot{\beta}} \lvert R,r\rangle^{h.w.}_{\dot{\alpha}} = 0$ $(\bar{\mathcal{Q}}_2)^2 \lvert R,r\rangle^{h.w.} = 0$ for $\bar{j}=0$		$\Delta = 2+2\bar{j}+2R-r$ $\Delta = 2+2R-r$	$\bar{\mathcal{C}}_{R,r(j,\bar{j})}$ $\bar{\mathcal{C}}_{R,r(j,0)}$
$\mathcal{F}$	$\mathcal{C}_1 \cap \mathcal{C}_2$	$R=0$	$\Delta = 2+2j+r$	$\mathcal{C}_{0,r(j,\bar{j})}$
$\bar{\mathcal{F}}$	$\bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$R=0$	$\Delta = 2+2\bar{j}-r$	$\bar{\mathcal{C}}_{0,r(j,\bar{j})}$
$\hat{\mathcal{C}}$	$\mathcal{C}_1 \cap \bar{\mathcal{C}}_2$	$r=\bar{j}-j$	$\Delta = 2+2R+j+\bar{j}$	$\hat{\mathcal{C}}_{R(j,\bar{j})}$
$\hat{\mathcal{F}}$	$\mathcal{C}_1 \cap \mathcal{C}_2 \cap \bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$R=0, r=\bar{j}-j$	$\Delta = 2+j+\bar{j}$	$\hat{\mathcal{C}}_{0(j,\bar{j})}$
$\mathcal{D}$	$\mathcal{B}_1 \cap \bar{\mathcal{C}}_2$	$r=\bar{j}+1$	$\Delta = 1+2R+\bar{j}$	$\mathcal{D}_{R(0,\bar{j})}$
$\bar{\mathcal{D}}$	$\bar{\mathcal{B}}_2 \cap \mathcal{C}_1$	$-r=j+1$	$\Delta = 1+2R+j$	$\bar{\mathcal{D}}_{R(j,0)}$
$\mathcal{G}$	$\mathcal{E} \cap \bar{\mathcal{C}}_2$	$r=\bar{j}+1, R=0$	$\Delta = r = 1+\bar{j}$	$\mathcal{D}_{0(0,\bar{j})}$
$\bar{\mathcal{G}}$	$\bar{\mathcal{E}} \cap \mathcal{C}_1$	$-r=j+1, R=0$	$\Delta = -r = 1+j$	$\bar{\mathcal{D}}_{0(j,0)}$

Table B.1: Shortening conditions and short multiplets for the $\mathcal{N}=2$ superconformal algebra [154].

B.2 $\mathcal{N}=1$ Chiral Ring

An important subset of the protected operators of a supersymmetry theory are the operators in the chiral ring. Chiral operators, by definition, are annihilated by the supercharge of one chirality, $\bar{\mathcal{Q}}^{\dot{\alpha}}$, and thus obey a $\mathcal{B}$-type shortening condition. (If the theory has extended supersymmetry we focus on an $\mathcal{N}=1$ subalgebra.) The product of two chiral operators is again chiral. Chiral operators are normally considered modulo $\bar{\mathcal{Q}}^{\dot{\alpha}}$-exact operators. The chiral cohomology classes can be specified by a set of generators and relations, which are easy to determine at weak (infinitesimal but non-zero) coupling. At higher orders the relations may get corrected, but the basic counting of chiral states is not expected to change [18, 182].

Let us first consider the case of pure $\mathcal{N}=2$ SYM with gauge group $SU(N_c)$. Under an $\mathcal{N}=1$ subalgebra the field content is decomposed as a chiral superfield Φ and a vector superfield W_α, both in the adjoint representation of the gauge group.. A generic chiral operator of the theory in the adjoint representation of the gauge group obeys

$$[W_\alpha, \mathcal{O}\} = \left[\bar{\mathcal{Q}}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}\mathcal{O}\right\} \ . \tag{B.1}$$

Substituting $\mathcal{O}=\Phi$ and $\mathcal{O}=W_\beta$ we see that, modulo $\bar{\mathcal{Q}}$ exact terms, W_α

(anti-)commutes with Φ and W_β respectively. Using these relations we can narrow down the single-trace chiral operators to

$$\mathrm{Tr}\,\Phi^{k+2}, \qquad \mathrm{Tr}\,\Phi^{k+1}W_\alpha, \qquad \mathrm{Tr}\,\Phi^k \epsilon^{\alpha\beta}W_\alpha W_\beta\,, \qquad \text{for } k \geq 0\,. \tag{B.2}$$

We have listed one representative from each cohomology class. For finite N_c the operators are further related by trace relations. In the large N_c limit of $N = 2$ supersymmetric Yang Mills, (B.2) is the complete and unconstrained list of single-trace chiral operators. Taking products we generate the whole chiral ring. In $\mathcal{N} = 2$ language the chiral operators are assembled in a single supermultiplet for each k, the multiplet with primary $\mathrm{Tr}\,\phi^{k+2}$.

To obtain $\mathcal{N} = 2$ SCQCD we add N_f fundamental hypermultiplets, equivalent to N_f fundamental chiral multiplets $\mathfrak{Q}$ and N_f antifundamental chiral multiplets $\tilde{\mathfrak{Q}}$, with the $\mathcal{N} = 2$ invariant superpotential $\tilde{\mathfrak{Q}}\Phi\mathfrak{Q}$. There are no chiral operators containing both W_α and $\mathfrak{Q}$ because $W_\alpha\mathfrak{Q}$ is $\bar{\mathcal{Q}}$ exact. Generally, in a theory with superpotential, further relations are imposed by the equations of motion

$$\partial_A W(A_i) = \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}A \quad \Rightarrow \quad \partial_A W(A_i)_{c.r.} = 0\,, \tag{B.3}$$

where $\{A_i\}$ is the set of chiral superfields. The subscript $c.r.$ denotes that the relation is valid in the chiral ring. In our case this implies that operators containing both Φ and $\mathfrak{Q}$ are constrained by the equations of motion

$$\Phi\mathfrak{Q} = 0, \quad \tilde{\mathfrak{Q}}\Phi = 0 \quad \text{and} \quad \mathfrak{Q}^a{}_i\tilde{\mathfrak{Q}}^i{}_b - \frac{1}{N_c}\delta^a_b\mathfrak{Q}^c{}_i\tilde{\mathfrak{Q}}^i{}_c = 0\,. \tag{B.4}$$

These relations set to zero all generalized single-trace operators[2] containing $\mathfrak{Q}$, except for $\mathrm{Tr}\,\mathfrak{Q}\tilde{\mathfrak{Q}}$. When expressed in $SU(2)_R$ covariant fashion, this operator corresponds to the $\mathcal{N} = 2$ superconformal primary $\mathrm{Tr}\mathcal{M}_{\mathbf{3}}$. Note that for gauge group $U(N_c)$ instead of $SU(N_c)$ the third relation gets modified to $\mathfrak{Q}^a{}_i\tilde{\mathfrak{Q}}^i{}_b = 0$ implying that even $\mathrm{Tr}\,\mathfrak{Q}\tilde{\mathfrak{Q}}$ is absent from the chiral ring. (For $U(N_c)$ we would have to also *add* the operator $\mathrm{Tr}\,\Phi$ to the list (B.2)). All in all, consideration of the chiral ring for $\mathcal{N} = 2$ SCQCD has led to identify the following protected $\mathcal{N} = 2$ superconformal primaries:

$$\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}\,, \quad \mathrm{Tr}\,\phi^{\ell+2}\,, \quad \ell \geq 0\,. \tag{B.5}$$

Note that the multiplets $\{\mathrm{Tr}\,T\phi^\ell\}$, as well as the extra exotic protected states discussed in section 4.4.4, are not part of the chiral ring.

It is straightforward to repeat this exercise for the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$

[2] In the flavor non-singlet sector they also allow for $\mathfrak{Q}^a_i\tilde{\mathfrak{Q}}^j_a$.

SYM. In $\mathcal{N}=1$ language the field content of the orbifold theory consists of vector multiplets (Φ, W_α) and $(\check{\Phi}, \check{W}_\alpha)$, in the adjoint representation of $SU(N_c)$ and $SU(N_{\check{c}})$ respectively. They are coupled to bifundamental chiral multiplets $(\mathfrak{Q}_{\hat{\mathcal{I}}}, \tilde{\mathfrak{Q}}^{\hat{\mathcal{J}}})$ through the superpotential $\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}}\Phi\mathfrak{Q}_{\hat{\mathcal{I}}} + \mathfrak{Q}_{\hat{\mathcal{I}}}\check{\Phi}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}}$. Here $\hat{\mathcal{I}}, \hat{\mathcal{J}}$ are $SU(2)_L$ indices. At large N_c, the chiral ring of the orbifold is generated by the operators (B.2), by a second copy of (B.2) with $\Phi, W_\alpha \to \check{\Phi}, \check{W}_\alpha$ corresponding to the two vector multiplets, and by single-trace operators involving the fields from hypermultiplets. The latter obey following constraints due to the superpotential:

$$\begin{aligned} &\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}}\Phi = -\check{\Phi}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}}, && \Phi\mathfrak{Q}_{\hat{\mathcal{I}}} = -\mathfrak{Q}_{\hat{\mathcal{I}}}\check{\Phi} \\ &\mathfrak{Q}^{a}_{\hat{\mathcal{I}}\,\check{a}}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}\check{a}}{}_{b} - \frac{1}{N_c}\delta^a_b\mathfrak{Q}^{c}_{\hat{\mathcal{I}}\,\check{a}}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}\check{a}}{}_{c} = 0, && \tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}\check{a}}{}_{a}\mathfrak{Q}^{a}_{\hat{\mathcal{I}}\,\check{b}} - \frac{1}{N_{\check{c}}}\delta^{\check{a}}_{\check{b}}\tilde{\mathfrak{Q}}^{\hat{\mathcal{I}}\check{c}}{}_{a}\mathfrak{Q}^{a}_{\hat{\mathcal{I}}\,\check{c}} = 0 \end{aligned} \tag{B.6}$$

Using the first two equivalence relations we could always choose a class representative that doesn't contain any $\check{\Phi}$. Then the relations in the second line allow for highest $SU(2)_L$ spin chiral operators of schematic form $\mathrm{Tr}\,(\mathfrak{Q}\tilde{\mathfrak{Q}})^{\ell+1}_{\mathbf{3_L}}\Phi^k$. This operator is in the untwisted sector as it is invariant under quantum $\mathbb{Z}_2$ symmetry of the orbifold upto $\bar{\mathcal{Q}}^{\dot{\alpha}}$ exact terms. As before, the chiral ring of the $SU(N_c)$ theory (as opposed to $U(N_c)$), also contains the "exceptional" operator $\mathrm{Tr}\,(\mathfrak{Q}\tilde{\mathfrak{Q}})_{\mathbf{1_L}}$, which belongs to the twisted sector. Assembling these $\mathcal{N}=1$ chiral multiplets into full $\mathcal{N}=2$ multiplets, we find the following list of $\mathcal{N}=2$ superconformal primaries:

$$\mathrm{Tr}\,(\phi^{k+2} + \check{\phi}^{k+2})\,, \qquad \mathrm{Tr}\,(\mathcal{M}^{\ell+1}_{\mathbf{3_R}\mathbf{3_L}}\,\phi^k)\,, \tag{B.7}$$

$$\mathrm{Tr}\,(\phi^{k+2} - \check{\phi}^{k+2})\,, \qquad \mathrm{Tr}\,\mathcal{M}_{\mathbf{3_R}\mathbf{1_L}}\,, \qquad \text{for } k \geq 0,\ \ell \geq 0\,. \tag{B.8}$$

The primaries in the first line belong to the untwisted sector and the primaries in the second line belong to the twisted sector. We know from inheritance from $\mathcal{N}=4$ SYM that in the untwisted sector there are additional protected operators (see section 4.3.1). On the other hand, in the twisted sector this is plausibly the complete list, as confirmed by the calculation of the superconformal index in appendix B.3.

As we move away from the orbifold point by taking $\check{g} \neq g$, the calculation of the chiral ring is almost unchanged, we only need to perform the substitutions $\check{\Phi}, \check{W}_\alpha \to \kappa\check{\Phi}, \kappa\check{W}_\alpha$, with $\kappa \equiv \check{g}/g$ that take into account the deformation of the superpotential. The quantum numbers of the chiral operators remain unchanged.

B.3 The Index of Some Short multiplets

In this appendix we calculate the index of various short multiplets. A first goal is to determine the index of the set $\{\ \hat{\mathcal{B}}_1,\ \mathcal{E}_{\ell(0,0)},\ \ell \geq 2\ \}$ (the multiplets found by the analysis of the chiral ring in the twisted sector of the orbifold), and show that it agrees with (4.42). A second goal is to calculate $\mathcal{I}_{naive}$, the index of the "naive" protected spectrum (5.44) of $\mathcal{N} = 2$ SCQCD.

B.3.1 $\mathcal{E}_{\ell(0,0)}$ multiplet

The chiral multiplet $\mathcal{E}_{\ell(0,0)}$ [154] is defined to be the multiplet that descends from the operator with $R = 0$, that is annihilated by both $\mathcal{Q}^1$ and $\mathcal{Q}^2$. The shortening condition is $\Delta = \ell$. We have arranged the operator content of the multiplet in the array below. We represent the action of the supercharge $\mathcal{Q}$ to the left and $\bar{\mathcal{Q}}$ to the right. As $\mathcal{E}_{\ell(0,0)}$ is annihilated by $\mathcal{Q}$s, it only extends to the right.

$$
\begin{array}{c|ccccc}
\Delta & & & & & \\
\ell & \underline{0_{(0,0)}} & & & & \\
\ell+\frac{1}{2} & & \underline{\tfrac{1}{2}_{(0,\frac{1}{2})}} & & & \\
\ell+1 & & & 0_{(0,1)},\, \underline{1_{(0,0)}} & & \\
\ell+\frac{3}{2} & & & & \tfrac{1}{2}_{(0,\frac{1}{2})} & \\
\ell+2 & & & & & 0_{(0,0)} \\
\hline
r & \ell & \ell-\frac{1}{2} & \ell-1 & \ell-\frac{3}{2} & \ell-2
\end{array}
\tag{B.9}
$$

This multiplet contributes only to the left index $\mathcal{I}^{\mathrm{L}}$. The operators with $\delta^{\mathrm{L}} = 0$ are underlined and their contribution to the index is listed in table B.2.

Δ	$R_{(j,\bar{j})}$	$\mathcal{I}^{\mathrm{L}}(t,y,v)$
ℓ	$0_{(0,0)}$	$t^{2\ell}v^{\ell}$
$\ell+\frac{1}{2}$	$\frac{1}{2}_{(0,\frac{1}{2})}$	$-t^{2\ell+1}v^{\ell-1}\left(y+\frac{1}{y}\right)$
$\ell+1$	$1_{(0,0)}$	$t^{2\ell+2}v^{\ell-2}$

Table B.2: Operators with $\delta^{\mathrm{L}} = 0$ in $\mathcal{E}_{\ell(0,0)}$

For $\ell > 1$, we sum the contribution of the operators from the above table and divide it by the contribution $(1-t^3y)\,(1-t^3y^{-1})$ from the derivatives,

$$
\begin{aligned}
\sum_{\ell=2}^{\infty}\mathcal{I}^{\mathtt{L}}_{\mathcal{E}_{\ell(0,0)}} &= \frac{1}{(1-t^3y)\,(1-t^3y^{-1})}\sum_{\ell=2}^{\infty}t^{2\ell}v^{\ell}(1-t^1v^{-1}(y+y^{-1})+t^2v^{-2}) \\
&= \frac{t^4v^2(1-\frac{t}{vy})(1-\frac{ty}{v})}{(1-t^2v)\,(1-t^3y)\,(1-t^3y^{-1})}
\end{aligned}
$$

The conjugate multiplet $\bar{\mathcal{E}}_{-\ell(0,0)}$ contributes exactly the same but to $\mathcal{I}^{\mathtt{R}}$.

B.3.2 $\hat{\mathcal{B}}_1$ multiplet

Next we consider the nonchiral multiplet $\hat{\mathcal{B}}_1$ [154], with the shortenning condition that the highest weight state is anihilated by $\mathcal{Q}^2, \bar{\mathcal{Q}}_1$. This shortening condition requires $r=0$, $j=\bar{j}=0$ and $\Delta=2$ for the highest weight state.

$$
\begin{array}{c|ccccc}
\Delta & & & & & \\
2 & & & \underline{1_{(0,0)}} & & \\
\frac{5}{2} & & \underline{\frac{1}{2}_{\left(\frac{1}{2},0\right)}} & & \frac{1}{2}_{\left(0,\frac{1}{2}\right)} & \\
3 & 0_{(0,0)} & & 0_{\left(\frac{1}{2},\frac{1}{2}\right)} & & 0_{(0,0)} \\
\frac{7}{2} & & & & & \\
4 & & & -0_{(0,0)} & & \\
\hline
r & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1
\end{array}
\tag{B.10}
$$

The operator $-0_{(0,0)}$ at $\Delta=4$ stands for an equation of motion – the negative sign in front of it means that its contribution to the index (partition function in general) has to be subtracted. We have underlined the operators with $\delta^{\mathtt{L}}=0$ and their contribution to $\mathcal{I}^{\mathtt{L}}$ is listed in table B.3.

Δ	$R_{(j,\bar{j})}$	$\mathcal{I}^{\mathtt{L}}(t,y,v)$
2	$1_{(0,0)}$	$\frac{t^4}{v}$
$\frac{5}{2}$	$\frac{1}{2}_{\left(\frac{1}{2},0\right)}$	$-t^6$

Table B.3: Operators with $\delta^{\mathtt{L}}=0$ in $\mathcal{B}_1$

Summing the individual contributions and dividing with the contribution from the derivatives, we get the index for this multiplet as,

$$\mathcal{I}^{\mathtt{L}}_{\mathcal{B}_1} = \frac{t^4\,(1-t^2 v)}{v\,(1-t^3 y)\,(1-t^3 y^{-1})}\,. \tag{B.11}$$

B.3.3 $\hat{\mathcal{C}}_{0(0,0)}$ multiplet

The stress tensor, supercurrents and R-symmetry currents of the $\mathcal{N} = 2$ theory are part of this multiplet. Its shortening condition $\hat{\mathcal{C}}$ is explained in table B.1. The operator content of this multiplet is displayed in the array below.

$$
\begin{array}{c|ccccc}
\Delta & & & & & \\
2 & & & 0_{(0,0)} & & \\
\frac{5}{2} & & \underline{\tfrac{1}{2}_{(\frac{1}{2},0)}} & & \tfrac{1}{2}_{(0,\frac{1}{2})} & \\
3 & \underline{0_{(1,0)}} & & \underline{1_{(\frac{1}{2},\frac{1}{2})}},\ 0_{(\frac{1}{2},\frac{1}{2})} & & 0_{(0,1)} \\
\frac{7}{2} & & \underline{\tfrac{1}{2}_{(1,\frac{1}{2})}} & & \tfrac{1}{2}_{(\frac{1}{2},1)} & \\
4 & & & \begin{array}{c} 0_{(1,1)} \\ -0_{(0,0)},\ -1_{(0,0)} \end{array} & & \\
\frac{9}{2} & & -\tfrac{1}{2}_{(\frac{1}{2},0)} & & -\tfrac{1}{2}_{(0,\frac{1}{2})} & \\
10 & & & -0_{(\frac{1}{2},\frac{1}{2})} & & \\
\hline
r & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1
\end{array}
\tag{B.12}
$$

The operators with negative signs stand for equations of motion as before. We have underlined the operators with $\delta^{\mathrm{L}} = 0$ and their contribution is listed in the table below. Summing the contributions, we get the left index of this multiplet to be

$$
\mathcal{I}^{\mathrm{L}}_{\hat{\mathcal{C}}_{(0,0)}} = -t^6(1 - vt^2)(1 - \frac{t}{v}(y + \frac{1}{y}))\,. \tag{B.13}
$$

Being a nonchiral multipet, it contributes the same to the right index as well.

Δ	$R_{(j,\bar{j})}$	$\mathcal{I}^{\mathtt{L}}(t,y,v)$
$\frac{5}{2}$	$\frac{1}{2}_{(\frac{1}{2},0)}$	$-t^6$
3	$0_{(1,0)}$	$t^8 v$
3	$1_{(\frac{1}{2},\frac{1}{2})}$	$\frac{t^7}{v}(y+\frac{1}{y})$
$\frac{7}{2}$	$\frac{1}{2}_{(1,\frac{1}{2})}$	$-t^9(y+\frac{1}{y})$

Table B.4: Operators with $\delta^{\mathtt{L}}=0$ in $\hat{\mathcal{C}}_{0(0,0)}$

B.3.4 $\mathcal{C}_{\ell(0,0)}$ multiplet, $\ell \geq 1$

This multiplet obeys the shortening condition $\mathcal{F} = \mathcal{C}_1 \cap \mathcal{C}_2$. The operator content of $\mathcal{C}_{\ell(0,0)}$ is displayed below.

Δ \ r	$\ell+1$	$\ell+\frac{1}{2}$	ℓ	$\ell-\frac{1}{2}$	$\ell-1$	$\ell-\frac{3}{2}$	$\ell-2$
$\ell+2$			$0_{(0,0)}$				
$\ell+\frac{5}{2}$		$\underline{\frac{1}{2}_{(\frac{1}{2},0)}}$		$\frac{1}{2}_{(0,\frac{1}{2})}$			
$\ell+3$	$\underline{0_{(1,0)}}$		$\underline{1_{(\frac{1}{2},\frac{1}{2})}}$, $0_{(\frac{1}{2},\frac{1}{2})}$		$0_{(0,1)}$, $1_{(0,0)}$		
$\ell+\frac{7}{2}$		$\underline{\frac{1}{2}_{(1,\frac{1}{2})}}$		$\frac{1}{2}_{(\frac{1}{2},1)}$, $\frac{1}{2}_{(\frac{1}{2},0)}$, $\underline{\frac{3}{2}_{(\frac{1}{2},0)}}$		$\frac{1}{2}_{(0,\frac{1}{2})}$	
$\ell+4$			$0_{(1,1)}$, $\underline{1_{(1,0)}}$		$0_{(\frac{1}{2},\frac{1}{2})}$, $1_{(\frac{1}{2},\frac{1}{2})}$		$0_{(0,0)}$
$\ell+\frac{9}{2}$				$\frac{1}{2}_{(1,\frac{1}{2})}$		$\frac{1}{2}_{(\frac{1}{2},0)}$	
$\ell+5$					$0_{(1,0)}$		

The operators with $\delta^{\mathtt{L}}=0$ are underlined as usual. Table B.5 lists their contribution to $\mathcal{I}^{\mathtt{L}}$. Summing the contribution to the left index from $\mathcal{C}_{\ell(0,0)}$ with $\ell \geq 1$ we get,

$$\sum_{\ell=1}^{\infty} \mathcal{I}^{\mathtt{L}}_{\mathcal{C}_{\ell(0,0)}} = -t^8 v(1-vt^2)(1-\frac{t}{v}(y+\frac{1}{y})) - \frac{t^{10}}{v}\,. \tag{B.14}$$

B.3.5 The $\mathcal{I}_{twist}$ of the orbifold and $\mathcal{I}_{naive}$ of SCQCD

The protected operators in the twisted sector of the orbifold are listed in Table 5.5. The conjugates, which contribute to $\mathcal{I}^{\mathtt{L}}$, are of the type:

$$\hat{\mathcal{B}}_1, \qquad \mathcal{E}_{\ell(0,0)} \qquad\qquad \text{for} \quad \ell \geq 2\,. \tag{B.15}$$

Δ	$R_{(j,\bar{j})}$	$\mathcal{I}^{\mathtt{L}}(t,y,v)$
$\ell+\frac{5}{2}$	$\frac{1}{2}_{(\frac{1}{2},0)}$	$-t^{6+2\ell}v^{\ell}$
$\ell+3$	$0_{(1,0)}$	$t^{8+2\ell}v^{\ell+1}$
$\ell+3$	$1_{(\frac{1}{2},\frac{1}{2})}$	$t^{7+2\ell}v^{\ell-1}(y+\frac{1}{y})$
$\ell+\frac{7}{2}$	$\frac{1}{2}_{(1,\frac{1}{2})}$	$-t^{9+2\ell}v^{\ell}(y+\frac{1}{y})$
$\ell+\frac{7}{2}$	$\frac{3}{2}_{(\frac{1}{2},0)}$	$-t^{8+2\ell}v^{\ell-2}$
$\ell+4$	$1_{(1,0)}$	$t^{10+2\ell}v^{\ell-1}$

Table B.5: Operators with $\delta^{\mathtt{L}}=0$ in $\mathcal{C}_{\ell(0,0)}$

So we get,

$$\begin{aligned}
\mathcal{I}_{twist} &= \mathcal{I}_{\hat{\mathcal{B}}_1}+\sum_{\ell=2}^{\infty}\mathcal{I}_{\mathcal{E}_{\ell(0,0)}} && \text{(B.16)}\\
&= \frac{t^4\,(1-t^2v)}{v\,(1-t^3y)\,(1-t^3y^{-1})}+\frac{t^4v^2(1-\frac{t}{vy})(1-\frac{ty}{v})}{(1-t^2v)\,(1-t^3y)\,(1-t^3y^{-1})} && \text{(B.17)}\\
&= \frac{t^2v}{1-t^2v}-\frac{t^3y}{1-t^3y}-\frac{t^3y^{-1}}{1-t^3y^{-1}}-f_V(t,y,v)\,. && \text{(B.18)}
\end{aligned}$$

This precisely matches with (4.42), confirming the protected operators in the twisted sector of the orbifold. Let us now compute the $\mathcal{I}_{naive}$ of SCQCD that follows from the preliminary list 5.44 of protected operators. Their conjugates, which contribute to $\mathcal{I}^{\mathtt{L}}$, are of the type:

$$\hat{\mathcal{B}}_1,\qquad \mathcal{E}_{\ell+2(0,0)},\qquad \hat{\mathcal{C}}_{0,0},\qquad \mathcal{C}_{\ell+1(0,0)}\qquad\qquad \text{for}\quad \ell\geq 0\,. \qquad \text{(B.19)}$$

The $\mathcal{I}_{naive}$ then is

$$\begin{aligned}
\mathcal{I}_{naive} &= \mathcal{I}_{\hat{\mathcal{B}}_1}+\sum_{\ell=2}^{\infty}\mathcal{I}_{\mathcal{E}_{\ell(0,0)}}+\mathcal{I}_{\hat{\mathcal{C}}_{0,0}}+\sum_{\ell=1}^{\infty}\mathcal{I}_{\mathcal{C}_{\ell(0,0)}} && \text{(B.20)}\\
&= \frac{-t^6(1-\frac{t}{v}(y+\frac{1}{y}))-\frac{t^{10}}{v}+\frac{t^4v^2(1-\frac{t}{vy})(1-\frac{ty}{v})}{1-t^2v}+\frac{t^4}{v}(1-t^2v)}{(1-t^3y)(1-\frac{t^3}{y})} && \text{(B.21)}
\end{aligned}$$

B.4 KK Reduction of the $6d$ Tensor Multiplet on $AdS_5 \times S^1$

In this appendix we discuss the Kaluza-Klein reduction of the $6d$ tensor multiplet on $AdS_5 \times S^1$, and its matching with the twisted spectrum of the orbifold theory.

The tensor multiplet of maximal chiral supersymmetry in six dimensions (we will refer to it as (2,0) susy) has the following field content

$$B^-_{\mu\nu}\,, \quad \lambda^{\mathfrak{I}}_\alpha\,, \quad \Phi^{[\mathfrak{I}\mathfrak{K}]}\,. \tag{B.22}$$

The indices $\mathfrak{I}, \mathfrak{K}$ are the $USp(4)$ indices which is the R-symmetry group of the chiral supergravity. The spinors $\lambda^{\mathfrak{I}}_\alpha$ are in the $\mathbf{4}$ (complex) representation of $USp(4)$ and the scalars $\Phi^{[\mathfrak{I}\mathfrak{K}]}$ in the $\mathbf{5}$ (real) representation. The $\lambda^{\mathfrak{I}}_\alpha$ are Weyl, symplectic Majorana spinors. The symplectic Majorana condition is a psuedo-reality condition, $\bar{\lambda}_{\mathfrak{I}} = \Omega_{\mathfrak{I}\mathfrak{K}}\lambda^{\mathfrak{K}}$, where Ω is the symplectic form.

Consider now the background $AdS_5 \times S^1$. The natural embedding of the $SU(2)_R \times U(1)_r$ R-symmetry of the $\mathcal{N} = 4$ AdS_5 superalgebra (or equivalently of the $\mathcal{N} = 2$ $4d$ superconformal algebra) into $USp(4)$ is

$$\left(\begin{array}{c|c} SU(2)_R \times U(1)_r & \\ \hline & SU(2)_R \times U(1)^*_r \end{array}\right) \tag{B.23}$$

The five scalars decompose as

$$\begin{array}{rcl} \Phi^{[\mathfrak{I}\mathfrak{K}]} & \longrightarrow & \Phi^i + \Phi \;+\; \bar{\Phi} \\ \mathbf{5} & \longrightarrow & \mathbf{3}_0 + \mathbf{1}_{-\mathbf{1}} + \mathbf{1}_{+\mathbf{1}}\,, \end{array} \tag{B.24}$$

where the subscripts denote $U(1)_r$ charges. The spinors decompose as two (conjugate) $SU(2)_R$ doublets, with opposite $U(1)_r$ charges $r = \pm\frac{1}{2}$.

We are interested in the Kaluza-Klein reduction of the tensor multiplet on the S^1. We borrow the results of [162] (see also [183]), where all the KK modes with non-zero momentum were matched with the multiplets $\{\bar{\mathcal{E}}_{2+\ell(0,0)}\; \ell \geq 0\}$, corresponding to the twisted primaries $\{\mathrm{Tr}\phi^{2+\ell} - \mathrm{Tr}\check{\phi}^{\ell+2}\}$ of the orbifold theory. We will add the zero modes to the analysis of [162].

Let us indeed start with the zero modes on S^1. The bosonic zero modes comprise the following AdS_5 fields [162]: a complex scalar Φ, with $m^2 = -3$ (in

AdS units)[3]; a triplet of scalars Φ^i, with $m^2 = -4$; a massless two form $B_{\hat{m}\hat{n}}$, *or* equivalently a massless gauge field $A_{\hat{m}}$. The massless two-form $B_{\hat{m}\hat{n}}$ arises from the $6d$ anti-selfdual two-form $B^-_{\mu\nu}$ when both indices are taken to be along AdS_5, while the gauge field $A_{\hat{m}}$ arises from $B^-_{\mu\nu}$ when one index is taken to be along AdS_5 and the other along S^1. Because of the anti-selfduality of $B^-_{\mu\nu}$, the two possibilities are not independent: $B_{\hat{m}\hat{n}}$ and $A_{\hat{m}}$ are dual to each other as $5d$ fields, and we must pick one *or* the other. This ambiguity translates into two alternative ways to fit the zero modes into supermultiplets of the $\mathcal{N} = 2$ $4d$ superconformal algebra. Let us look at them in turn:

- Choosing $B_{\hat{m}\hat{n}}$.

  The massless two-form $B_{\hat{m}\hat{n}}$ is dual to a boundary two-form operator F'_{mn} of dimension $\Delta = 2$. We claim that the full supermultiplet of boundary operators is $\{\phi' \, , \lambda'^{\mathcal{I}}_{\alpha} \, , F'_{mn} \, D'_i\}$, which is the the familiar off-shell $\mathcal{N} = 2$ vector multiplet (or $\mathcal{N} = 2$ "supersingleton" multiplet). Here ϕ' is a complex scalar with $r = \pm 1$ and $\Delta = 1$, dual to the bulk scalar Φ of $m^2 = -3$. The mass of Φ is in the range that allows both the Δ_+ and the Δ_- quantization schemes [48, 184], and supersymmetry forces the choice of $\Delta_- = 2 - \sqrt{m^2 + 4} = 1$. Since ϕ' saturates the unitarity bound, it must be a free scalar field. We recognize F'_{mn} as the Maxwell field strength and D'_i, $i = 1, 2, 3$, which form $SU(2)_R$ triplet with $\Delta = 2$ and are dual to the bulk fields Φ^i, as the auxiliary fields. Finally $\lambda'^{\mathcal{I}}_{\alpha}$ are the free fermionic fields with $\Delta = \frac{3}{2}$. The AdS/CFT relation for spin $\frac{1}{2}$ fields is usually quoted as $\Delta = 2 + |m|$, but this is evidently a case where we must pick instead $\Delta_- = 2 - |m|$, with $m = \frac{1}{2}$. We are not aware of an explicit discussion of the $\Delta_\pm$ quantization ambiguity for spinors, but it must be there because of supersymmetry. (Incidentally, similar issues arise in the familiar IIB on $AdS_5 \times S^5$ background if one looks at the zero modes, which can be organized in the $\mathcal{N} = 4$ supersingleton multiplet. Again both the scalars in the $\mathbf{6}$ of $SU(4)$ *and* the spinors in the $\mathbf{4}$ must be quantized in the Δ_- scheme.)

- Choosing $A_{\hat{\mu}}$.

  The boundary dual to $A_{\hat{m}}$ is a conserved current J_m ($\Delta = 3$). In this case we claim that supersymmetry forces the usual Δ_+ quantization scheme for Φ and $\lambda^{\mathfrak{I}}_{\alpha}$. It is easy to check that the zero modes can be precisely organized into the $\hat{\mathcal{B}}_1$ multiplet (summarized in (B.10)).

[3] The complex scalar Φ corresponds to the $k = -1$ real scalar in Family 2 and the $k = 1$ real scalar in Family 3 of [162]. We have just relabeled them as $n = 0$ modes.

Field Theory			Gravity	
Operator	$U(1)_r$	Δ	Mass	Field
$\mathrm{Tr}[\bar{\phi}^{n+1}] - \mathrm{Tr}[\check{\bar{\phi}}^{n+1}]$	$n+1$	$n+1$	$(n+1)(n-3)$	$\bar{\Phi}$
$\mathrm{Tr}[F\bar{\phi}^{n}] - \mathrm{Tr}[\check{F}\check{\bar{\phi}}^{n}]$	n	$n+2$	n^2	$B_{\hat{m}\hat{n}}$
$\mathrm{Tr}[\lambda\lambda\bar{\phi}^{n-1}] - \mathrm{Tr}[\check{\lambda}\check{\lambda}\check{\bar{\phi}}^{n-1}]$	n	$n+2$	n^2-4	Φ^i
$\mathrm{Tr}[F^2\bar{\phi}^{n-1}] - \mathrm{Tr}[\check{F}^2\check{\bar{\phi}}^{n-1}]$	$n-1$	$n+3$	$(n-1)(n+3)$	Φ

Table B.6: Matching of the positive KK modes ($n \geq 1$) [162]. The negative KK modes ($n \leq -1$) correspond to the conjugate operators.

The two possibilities have a nice physical interpretation. The first alternative corresponds to keeping the $U(1)$ degree of freedom in the twisted sector (this is the "relative" $U(1)$ in the product gauge recall the discussion after equ.(5.16)) – in other terms we should identify $\phi' = \mathrm{Tr}(\phi - \hat{\phi})$. The second possibility corresponds instead to *removing* the relative $U(1)$. Then clearly the multiplet built on $\mathrm{Tr}(\phi - \hat{\phi})$ is lost, but as we have emphasized in section 4.3.1 and appendix B, an *additional* protected multiplet appears, the $\hat{\mathcal{B}}_1$ multiplet built on the primary $\mathrm{Tr}\,\mathcal{M}_{\mathbf{3}}$. The AdS/CFT dictionary handles this subtle ambiguity in a very elegant way. For our purposes, the second alternative is the relevant one, since we must remove the relative $U(1)$ in order to have a truly conformal field theory.

The matching of the higher Kaluza-Klein modes was discussed in [162], we summarize the results in Table B.6.

B.5 The Cigar Background and 7d Gauged Sugra

This appendix collects some facts about the non-critical string theory obtained in the double-scaling limit of two colliding NS branes [134, 135], namely IIB on $\mathbb{R}^{5,1} \times SL(2)_2/U(1)$. We start by reviewing well-known results, see *e.g.* [132, 133, 133–135, 141–143], and then make a new claim about a space-time "effective action" description. We are going to argue that the "lighest" delta-function normalizable modes in the continuum are described by a $7d$ maximally supersymmetric supergravity with non-standard gauging, recently constructed in [177, 178].

B.5.1 Preliminaries and Worldsheet Symmetries

A class of "non-critical" supersymmetric string backgrounds can defined in the RNS formalism by taking the tensor product of $\mathbb{R}^{d-1,1}$ with the Kazama Suzuki supercoset $SL_2(\mathbb{R})_k/U(1)$. The $\mathbb{R}^{d-1,1}$ part is described as usual by d free bosons X^μ and d free fermions ψ^μ. The coset $SL_2(\mathbb{R})_k/U(1)$ has a sigma-model description with target space the "cigar" background (setting $\alpha' = 2$)

$$ds^2 = d\rho^2 + \tanh^2(\frac{Q\rho}{2})d\theta^2 \qquad \rho \geq 0 \quad \theta \sim \theta + \frac{4\pi}{Q} \tag{B.25}$$

with vanishing B field and dilaton varying as

$$\Phi = -\ln\cosh(\frac{Q\rho}{2})\,. \tag{B.26}$$

The level k of the coset is related to the parameter Q as $k = 2/Q^2$. The central charge is

$$c_{cig} = 3 + \frac{6}{k} = 3 + 3Q^2\,. \tag{B.27}$$

Adding the usual superconformal ghost system $\{b\,,c\,,\beta\,,\gamma\}$ of central charge -15 and requiring cancellation of the total conformal anomaly, one finds $Q = \sqrt{\frac{1}{2}(8-d)}$. In the asymptotic region $\rho \to \infty$ the cigar becomes a cylinder of radius $\frac{2}{Q}$, with the dilaton varying linearly with ρ, and the theory is thus a free CFT. We will soon restrict to the $d = 6$ case, implying $c_{cig} = 6$, $Q = 1$ and $k = 2$.

For generic level k the Kazama-Susuki coset $SL(2)_k/U(1)$ has $(2,2)$ supersymmetry. In the asymptotic linear-dilaton region the holomorphic currents of $\mathcal{N} = 2$ susy take the form

$$\begin{aligned}
T_{\rm cig} &= -\frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\theta)^2 - \frac{1}{2}(\psi_\rho\partial\psi_\rho + \psi_\theta\partial\psi_\theta) - \frac{1}{2}Q\partial^2\rho && \text{(B.28)}\\
J_{\rm cig} &= -i\psi_\rho\psi_\theta + iQ\partial\theta \equiv i\partial H + iQ\partial\theta \equiv i\partial\phi && \text{(B.29)}\\
G^\pm_{\rm cig} &= \frac{i}{2}(\psi_\rho \pm i\psi_\theta)\partial(\rho \mp i\theta) + \frac{i}{2}Q\partial(\psi_\rho \pm i\psi_\theta)\,, && \text{(B.30)}
\end{aligned}$$

with analogous expressions for the anti-holomorphic currents. For $k = 2$, which is the case of interest for us, worldsheet supersymmetry is enhanced to $(4,4)$. This is the generic enhancement of worldsheet susy from $\mathcal{N} = 2$ to $\mathcal{N} = 4$ that takes place when $c = 6$. Indeed for this value of the central charge the currents $J^i_{\rm cig} = \{e^{\pm\int J_{\rm cig}}\,, J_{\rm cig}\}$, $i = \pm, 3$, generate a left-moving $SU(2)$ current algebra, the R subalgebra of the left-moving $\mathcal{N}=4$ worldsheet superconformal algebra. The two extra odd currents $\hat{G}^{\pm}_{\rm cig}$ are generated in the OPE of $G^{\pm}_{\rm cig}$ with $J^i_{\rm cig}$. Similarly for the right-movers. In the full cigar background the wordsheet superconformal currents have more complicated expressions but the theory still has exact $(2,2)$ susy, enhanced to $(4,4)$ for $k=2$.

In the free linear dilaton theory, $i\partial\theta$ and $i\partial H$ defined in (B.29) are separately holomorphic, but only their linear combination $J_{\rm cig}$ is holomorphic in the full cigar background. This reflects the non-conservation of winding around the cigar (strings can unwrap at the tip). Momentum P^θ around the cigar is still conserved, and there is a corresponding Noether current with both holomorphic and anti-holomorphic components, which asymptotically takes the form $\frac{1}{Q}(i\partial\theta\,, i\bar{\partial}\theta)$. For $k=2$, the field θ is asymptotically at the free fermion radius. Thus in the linear dilaton theory the left-moving susy $U(1)$ generated by $(i\partial\theta\,,\psi_\theta)$ is enhanced to a left-moving $SU(2)_2$ current algebra, which can be represented by three free fermions ψ_i, with $\psi_3\equiv\psi_\theta$ and $\psi_\pm\equiv e^{\pm i\theta}$. To avoid confusions with other $SU(2)$ symmetries will refer to this algebra as $SU(2)_{\psi_i}$. Similarly in the right-moving sector we have the analogous $SU(2)_{\tilde{\psi}_i}$. In the full cigar background the $SU(2)_{\psi_i}$ and $SU(2)_{\tilde{\psi}_i}$ current algebras are *not* symmetries, and only a *global* diagonal $SU(2)$ survives, whose Cartan generator is the momentum P^θ. This is interpreted as the $SU(2)_R$ *spacetime* R-symmetry.

B.5.2 Cigar Vertex Operators

To characterize the primary vertex operators of the cigar it is sufficient to give their asymptotic form in the linear-dilaton region. While the exact expressions are more complicated, their quantum numbers (including conformal dimensions) remain the same and can thus be evaluated in the asymptotic region. Splitting the vertex operators in left-moving and right-moving parts, we have the asymptotic left-moving expressions

$$\begin{aligned} V^{NS}_{j,m} &= e^{iQm\theta}e^{Qj\rho} \\ V^{R}_{j,m} &= e^{\pm\frac{i}{2}\phi}e^{iQm\theta}e^{Qj\rho} \end{aligned} \tag{B.31}$$

and the asymptotic anti-holomorphic expressions

$$\begin{aligned} \tilde{V}^{NS}_{j,\tilde{m}} &= e^{-iQ\tilde{m}\bar{\theta}}e^{Qj\bar{\rho}} \\ \tilde{V}^{R}_{j,\tilde{m}} &= e^{\pm\frac{i}{2}\tilde{\phi}}e^{-iQ\tilde{m}\bar{\theta}}e^{Qj\bar{\rho}}\,. \end{aligned} \tag{B.32}$$

Left-moving and right-moving terms can be glued together provided they have the same value of the quantum number j. We will sometimes re-express j in terms of p, the momentum in the radial direction, as

$$j = -\frac{Q}{2} + ip\,. \tag{B.33}$$

The quantum numbers m and $\tilde{m}$ are related to the integer winding w and the integer momentum n in the angular direction of the cylinder as

$$m = \frac{1}{2}(n + wk) \qquad \tilde{m} = -\frac{1}{2}(n - wk)\,. \tag{B.34}$$

Recall however that winding is not a conserved quantum number in the cigar background. Conformal dimensions of the primary operators (B.31,B.32) are

$$\begin{aligned}
\Delta^{NS}_{j,m} &= \frac{m^2 - j(j+1)}{k} \\
\bar{\Delta}^{NS}_{j,\tilde{m}} &= \frac{\tilde{m}^2 - j(j+1)}{k} \\
\Delta^{R\pm}_{j,m} &= \frac{1}{8} + \frac{(m \pm \frac{1}{2})^2 - j(j+1)}{k} \\
\bar{\Delta}^{R\pm}_{j,\tilde{m}} &= \frac{1}{8} + \frac{(\tilde{m} \mp \frac{1}{2})^2 - j(j+1)}{k}
\end{aligned}$$

.

B.5.3 Spacetime Supersymmetry

From now on we restrict to the case of interest, $d = 6$. The RNS vertex operators for $\mathbb{R}^{5,1}$ are familiar. To describe the Ramond sector, we bosonize the fermions in the usual fashion,

$$\begin{aligned}
\pm\psi_0 + \psi_1 &= e^{\pm\phi_0} \\
\psi_2 \pm i\psi_3 &= e^{\pm i\phi_1} \\
\psi_4 \pm i\psi_5 &= e^{\pm i\phi_2}
\end{aligned}$$

Spinors of $\mathbb{R}^{5,1}$ are then written

$$V_\alpha = e^{\frac{1}{2}(\epsilon_0\phi_0 + i\epsilon_1\phi_1 + i\epsilon_2\phi_2)} \tag{B.35}$$

with $\epsilon_a = \pm 1$. With these notations at hand, the BRST invariant vertex operators for the spacetime supercharges for the IIB theory read

$$\begin{aligned} S_\alpha &= e^{-\varphi/2} e^{+\frac{i}{2}\phi} V_\alpha^+ \qquad \bar{S}_\alpha = e^{-\varphi/2} e^{-\frac{i}{2}\phi} V_\alpha^+ \\ \tilde{S}_\alpha &= e^{-\tilde{\varphi}/2} e^{+\frac{i}{2}\tilde{\phi}} \tilde{V}_\alpha^+ \qquad \bar{\tilde{S}}_\alpha = e^{-\tilde{\varphi}/2} e^{-\frac{i}{2}\tilde{\phi}} \tilde{V}_\alpha^+ \end{aligned}$$

where φ is the usual chiral boson arising in the bosonization of the $\beta\gamma$ system. We use a bar to denote conjugation, and a tilde to distinguish the right-movers. By V_α^+ we mean the positive chirality spinor, *i.e.* we impose $\epsilon_0 \epsilon_1 \epsilon_2 = 1$. Choosing the same chirality in the left and right-moving sectors is the statement of the type IIB GSO projection. The supercharges obey the supersymmetry algebra

$$\{S_\alpha \,, \bar{S}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu \qquad \{\tilde{S}_\alpha \,, \bar{\tilde{S}}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu \,, \tag{B.36}$$

where P_μ is the momentum in $\mathbb{R}^{5,1}$. Thus the theory has $(2,0)$ supersymmetry in the six Minkowski directions. Note that

$$[P^\theta, S_\alpha\,(\tilde{S}_\alpha)] = \frac{1}{2} S_\alpha\,(\tilde{S}_\alpha)\,, \quad [P^\theta, \bar{S}_\alpha\,(\bar{\tilde{S}}_\alpha)] = -\frac{1}{2} \bar{S}_\alpha\,(\bar{\tilde{S}}_\alpha)\,, \tag{B.37}$$

confirming the interpretation of P^θ as a spacetime R-symmetry.

Physical vertex operators are constrained to be local with the spacetime supercharges. Locality implies the GSO condition

$$\begin{aligned} m + F_L &\in 2\mathbb{Z} + 1 \qquad (NS) \\ m + F_L &\in 2\mathbb{Z} \qquad\qquad (R) \end{aligned}$$

where F_L is the left-moving worldsheet fermion number. The analogous condition holds for the right-movers. In the asymptotic region we may fermionize the field θ into $\psi^\pm$. Then the quantum number m, instead of denoting left-moving momentum in the θ direction, gets re-interpreted as $\psi^\pm$ fermion number. Denoting by $F_L' = F_L + m$ the new total left-moving fermion number, the GSO projection becomes simply

$$\begin{aligned} F_L' &\in 2\mathbb{Z} + 1 \qquad (NS) \\ F_L' &\in 2\mathbb{Z} \qquad\qquad (R) \end{aligned}$$

and analogously for the right-movers.

B.5.4 Spectrum: generalities

The physical spectrum of the theory comprises:

(i) A discrete set of truly normalizable states, localized at the tip of the cigar.
$(j < -Q/2)$

(ii) A continuum of delta-function normalizable states, corresponding to incoming and outgoing waves in the ρ direction.
$(j = -Q/2 + i\mathbb{R}$, *i.e.* $p \in \mathbb{R})$

(iii) Non-normalizable vertex operators, supported in the asymptotic large ρ region.
$(j > -Q/2)$

States of type (i) live in $\mathbb{R}^{5,1}$ at $\rho \sim 0$ and they fill in a massless tensor multiplet of the 6d $(2, 0)$ supersymmetry. More precisely they are:

NSNS: four scalars, in the $\mathbf{3} + \mathbf{1}$ of $SU(2)_R$;

RR: one scalar and one anti-selfdual antisymmetric tensor, both $SU(2)_R$ singlets;

RNS: one left-handed Weyl spinor, which can be thought of an $SU(2)_R$ doublet of left-handed Majorana-Weyl spinors;

NSR: same as RNS.

See [185] for a detailed analysis.

In the rest of this appendix we will focus on the states of type (ii). These are the states relevant for the determination of a spacetime "effective action" for the non-critical string. Recall that our philosophy is to use the $\mathbb{R}^{5,1}\times$ cigar background as an intermediate step towards the AdS background dual to $\mathcal{N} = 2$ SCQCD. Both backgrounds should arise as solutions of the same non-critical string field theory. We would like to use the cigar background, for which we have a solvable worldsheet CFT, to derive an "effective action" description. The "effective action" is expected to be background independent and should admit both the cigar background and the AdS background as different classical solutions. We will restrict to the lowest level in a "Kaluza-Klein expansion" on the cigar circle (to be defined more precisely below). The states will then propagate in seven dimensions, $\mathbb{R}^{5,1}$ times the radial direction ρ. Because of the linear dilaton, they obey massive field equations in 7d, but they are in another sense "massless" – they are closely related to the massless states of the critical IIB 10d theory and possess the gauge invariances expected for massless 7d fields. We should emphasize from the outset that the linear dilaton varies with a string-scale gradient, so there is no real separation of scales between the "massless" level that we are keeping and the higher levels. This is why we are using "effective action" in quotation marks. Nevertheless the distinction between the lowest level obeying massless gauge-invariances and the higher genuinely massive levels is a meaningful one, and we still expect such an "effective action" to contain useful information. Remarkably, we will see that it is a $7d$ gauged supergravity with non-standard gauging.

Finally we should mention the operators of type (iii). They have an interesting holographic interpretation as "off-shell" observables of little string theory, which "lives" on the $\mathbb{R}^{5,1}$ boundary at $\rho = \infty$. However we are not interested in the cigar background per se and we are after a different incarnation of holography, so it is not immediately clear what the significance of these operators is for our story. In analogy with $c = 1$ non-critical string, our non-critical superstring background is expected to possess a rich spectrum of "discrete states", with Liouville dressing of type (iii). A closely related phenomenon is the existence of a chiral ring, which has been demonstrated in [186] (see also [187]). This infinite tower of discrete states may be related to the exotic extra protected states of $\mathcal{N} = 2$ SCQCD.

B.5.5 Delta-function normalizable states: the lowest mass level

We are now going to exhibit in detail the physical states of type (ii) at the lowest mass level. We first organize the states according their symmetries in the asymptotic linear dilaton region, and later discuss the symmetry breaking induced by the cigar interaction. The asymptotic cylinder is at free-fermion radius, and we wish to work covariantly in the enhanced $SU(2)_{\psi_i} \times SU(2)_{\tilde{\psi}_i}$ symmetry.

After fermionizing θ into $\psi^{\pm}$, we have in total ten worldsheet fermions: ψ_μ, $\mu = 0, \ldots 5$ associated with $\mathbb{R}^{5,1}$, ψ_ρ associated to the radial direction and ψ_i, $i = 3, \pm$ associated to the stringy circle. It is then clear from outset that the lowest mass level of our theory will be formally similar to the massless spectrum of 10d *critical* IIB string theory, but of course the states will propagate only in the seven dimensions $x_{\hat{\mu}} = (x_\mu, \rho)$.

NS sector

In the left-moving NS sector the lowest states are the three 7d scalars

$$V_i^{\mathrm{NS}} = \psi_i e^{-\varphi} e^{j\rho} e^{ik\cdot X}\,, \tag{B.38}$$

in a triplet of $SU(2)_{\psi_i}$, and the 7d vector

$$V_{\hat\mu}^{\mathrm{NS}} = \psi_{\hat\mu} e^{-\varphi} e^{j\rho} e^{ik\cdot X}\,, \tag{B.39}$$

where $\hat\mu = \mu\,, \rho$. The mass-shell condition $L_0 = 1$ gives, for both the scalar and the vector,

$$\frac{1}{2}k^2 - \frac{1}{2}j(j+1) = 0\,, \tag{B.40}$$

which using $j = -1/2 + ip$ we may write as

$$-k^2 - p^2 = k_0^2 - \boldsymbol{k}^2 - p^2 = \frac{1}{4}\,. \tag{B.41}$$

Because of the linear dilaton, the wave equations appear to be "massive" with $m^2 = \frac{1}{4}$. Introducing a polarization vector $e^{\hat\mu} = (e_\mu\, e_\rho)$, the superconformal invariance condition $G_{\frac{1}{2}} e^{\hat\mu} V_{\hat\mu}^{\mathrm{NS}} = 0$ gives a modified transversality equation for the vector[4]

$$k\cdot e - \sqrt{-1}(j+1)e_\rho = 0\,. \tag{B.42}$$

A short calculation shows that the polarization

$$e = k \quad \text{and} \quad e_\rho = -\sqrt{-1}\,j \tag{B.43}$$

corresponds to a null state. Thus despite the mass term in the wave equation, $V_{\hat\mu}^{\mathrm{NS}}$ the 7-2 = 5 physical degrees of freedom of a massless 7d vector.

The theory is super-Poincaré invariant in $\mathbb{R}^{5,1}$, and we may label the states in terms of 6d quantum numbers. In assigning 6d Lorentz quantum numbers, we may focus for convenience on the states with radial momentum $p = \frac{1}{2}$, which obey a massless 6d wave-equation (see B.41). We can then label them according to the 6d little group $SO(4) = SU(2)\times SU(2)$. It must kept in mind that this is just a notational device, since the states are really part of a 7d continuum with arbitrary real p. We use the notation $|j_1\,, j_2\rangle^{2I+1}$ for a state with spins (j_1, j_2) under the 6d little group, and in the $2I+1$-dimensional representation of $SU(2)_{\psi_i}$. All in all, in this 6d notation we may summarize

[4] Apologies for the $\sqrt{-1}$, but here the symbol i would look confusing next to the momentum j.

the lowest NS states as

$$|\frac{1}{2},\frac{1}{2}\rangle^1 \oplus |0,0\rangle^1 \oplus |0,0\rangle^3\,. \tag{B.44}$$

R sector

The construction of vertex operators in the Ramond sector proceeds just as in to the familiar critical (10d) case, except of course that momenta are only seven-dimensional,

$$V^R = e^{-\varphi/2} e^{\frac{i}{2}(\epsilon_0\phi_0+\epsilon_1\phi_1+\epsilon_2\phi_2\epsilon_\theta\theta+\epsilon_H H)} e^{j\rho} e^{ip\cdot X}\,, \quad \epsilon_0\epsilon_1\epsilon_2\epsilon_\theta\epsilon_H = 1\,, \tag{B.45}$$

which we may write as

$$\Psi_\alpha(p_\mu) e^{\pm\frac{i}{2}(\theta+H)}\, e^{j\rho}\,, \qquad \Psi_{\dot\alpha}(p_\mu) e^{\pm\frac{i}{2}(\theta-H)}\, e^{j\rho}\,. \tag{B.46}$$

Here Ψ_α and $\Psi_{\dot\alpha}$ are 6d pseudo-real (Majorana-Weyl) spinors, respectively left-handed and right-handed. Choosing the 7d momentum as $p=\frac{1}{2}$ the spinors obey a massless 6d wave equation, but as above we should keep in mind that they are really part of 7d continuum. For each chirality we have an $SU(2)$ doublet of 6d Majorana-Weyl spinors (equivalently, one complex Weyl spinor) so in "massless 6d notation" we write the spectrum as

$$|\frac{1}{2},0\rangle^2 \oplus |0,\frac{1}{2}\rangle^2\,. \tag{B.47}$$

In 7d the wave-equation looks "massive", but the counting of degrees of freedom is again the one for massless states.

Gluing

Table B.7–B.10 show the result of gluing the left- and right-moving sectors. In the first column of each table we list the $(m,\tilde m)$ quantum numbers, recall (B.34). In the second and third columns the Lorentz quantum numbers are specified in the the 6d "massless" notation, that is we label states by their spins (j_1,j_2) of the little group $SO(4)=SU(2)_1\times SU(2)_2$. The superscripts $2I+1$ and $2\tilde I+1$ in the second column denote the dimensions of the representations under $SU(2)_{\psi_i}$ and $SU(2)_{\tilde\psi_i}$, respectively (the superscript is omitted for singlets). Finally the superscript $2R+1$ in the third column denotes the dimension of the $SU(2)_R$ representation, with $SU(2)_R$ defined as the *diagonal* combination of $SU(2)_{\psi_i}$ and $SU(2)_{\tilde\psi_i}$ which is preserved by the cigar interaction.

$(\{m\},\{\tilde m\})$	$\vert j_1,j_2\rangle^{2I+1}\otimes\vert j_1,j_2\rangle^{2\tilde I+1}$	Decomposition: $\vert j_1,j_2\rangle^{2R+1}$	$6d$ Fields
$(\{0\},\{0\})$	$\vert\frac12,\frac12\rangle\otimes\vert\frac12,\frac12\rangle$	$\vert 1,1\rangle\oplus\vert 1,0\rangle\oplus\vert 0,1\rangle\oplus\vert 0,0\rangle$	$G_{\mu\nu}, B_{\mu\nu}, \phi$
	$\vert\frac12,\frac12\rangle\otimes\vert 0,0\rangle$	$\vert\frac12,\frac12\rangle$	V_μ
	$\vert 0,0\rangle\otimes\vert\frac12,\frac12\rangle$	$\vert\frac12,\frac12\rangle$	$\tilde V_\mu$
	$\vert 0,0\rangle\otimes\vert 0,0\rangle$	$\vert 0,0\rangle$	ρ
$(\{\pm1,0\},\{0\})$	$\vert 0,0\rangle^3\otimes\vert\frac12,\frac12\rangle$	$\vert\frac12,\frac12\rangle^3$	$\tilde V^3_\mu$
	$\vert 0,0\rangle^3\otimes\vert 0,0\rangle$	$\vert 0,0\rangle^3$	ρ^3
$(\{0\},\{\pm1,0\})$	$\vert\frac12,\frac12\rangle\otimes\vert 0,0\rangle^3$	$\vert\frac12,\frac12\rangle^3$	V^3_μ
	$\vert 0,0\rangle\otimes\vert 0,0\rangle^3$	$\vert 0,0\rangle^3$	$\tilde\rho^3$
$(\{\pm1,0\},\{\pm1,0\})$	$\vert 0,0\rangle^3\otimes\vert 0,0\rangle^3$	$\vert 0,0\rangle^5\oplus\vert 0,0\rangle^3\oplus\vert 0,0\rangle$	T^5, T^3, T

Table B.7: Field Content in NSNS sector.

$(\{m\},\{\tilde m\})$	$\vert j_1,j_2\rangle^{2I+1}\otimes\vert j_1,j_2\rangle^{2\tilde I+1}$	Decomposition: $\vert j_1,j_2\rangle^{2R+1}$	$6d$ Fields
$(\{0\},\{0\})$	$\vert\frac12,0\rangle^2\otimes\vert\frac12,0\rangle^2$	$\vert 1,0\rangle^3\oplus\vert 1,0\rangle\oplus\vert 0,0\rangle^3\oplus\vert 0,0\rangle$	$A^{3+}_{\mu\nu}, A^{+}_{\mu\nu}, A^3, A$
$(\{\pm1\},\{0\})$	$\vert 0,\frac12\rangle^2\otimes\vert\frac12,0\rangle^2$	$\vert\frac12,\frac12\rangle^3\oplus\vert\frac12,\frac12\rangle$	A^3_μ, A_μ
$(\{0\},\{\pm1\})$	$\vert\frac12,0\rangle^2\otimes\vert 0,\frac12\rangle^2$	$\vert\frac12,\frac12\rangle^3\oplus\vert\frac12,\frac12\rangle$	$\tilde A^3_\mu, \tilde A_\mu$
$(\{\pm1\},\{\pm1\})$	$\vert 0,\frac12\rangle^2\otimes\vert 0,\frac12\rangle^2$	$\vert 0,1\rangle^3\oplus\vert 0,1\rangle\oplus\vert 0,0\rangle^3\oplus\vert 0,0\rangle$	$A^{3-}_{\mu\nu}, A^{-}_{\mu\nu}, A'^3, A'$

Table B.8: Field Content in RR sector

$(\{m\},\{\tilde m\})$	$\vert j_1,j_2\rangle^{2I+1}\otimes\vert j_1,j_2\rangle^{2\tilde I+1}$	Decomposition: $\vert j_1,j_2\rangle^{2R+1}$	$6d$ Fields
$(\{0\},\{0\})$	$\vert\frac12,\frac12\rangle\otimes\vert\frac12,0\rangle^2$	$\vert 1,\frac12\rangle^2\oplus\vert 0,\frac12\rangle^2$	$\Psi^2_{\mu\dot\alpha}, \Psi^2_{\dot\alpha}$
	$\vert 0,0\rangle\otimes\vert\frac12,0\rangle^2$	$\vert\frac12,0\rangle^2$	Ψ^2_α
$(\{\pm1,0\},\{0\})$	$\vert 0,0\rangle^3\otimes\vert\frac12,0\rangle^2$	$\vert\frac12,0\rangle^4\oplus\vert\frac12,0\rangle^2$	$\Psi^4_\alpha, \Psi^2_\alpha$
$(\{0\},\{\pm1\})$	$\vert\frac12,\frac12\rangle\otimes\vert 0,\frac12\rangle^2$	$\vert\frac12,1\rangle^2\oplus\vert\frac12,0\rangle^2$	$\Psi^2_{\mu\alpha}, \Psi^2_\alpha$
	$\vert 0,0\rangle\otimes\vert 0,\frac12\rangle^2$	$\vert 0,\frac12\rangle^2$	$\Psi^2_{\dot\alpha}$
$(\{\pm1,0\},\{\pm1\})$	$\vert 0,0\rangle^3\otimes\vert 0,\frac12\rangle^2$	$\vert 0,\frac12\rangle^4\oplus\vert 0,\frac12\rangle^2$	$\Psi^4_{\dot\alpha}, \Psi^2_{\dot\alpha}$

Table B.9: Field Content in NSR sector

$(\{m\},\{\tilde m\})$	$\vert j_1,j_2\rangle^{2I+1}\otimes\vert j_1,j_2\rangle^{2\tilde I+1}$	Decomposition: $\vert j_1,j_2\rangle^{2R+1}$	$6d$ Fields
$(\{0\},\{0\})$	$\vert\frac12,0\rangle^2\otimes\vert\frac12,\frac12\rangle$	$\vert 1,\frac12\rangle^2\oplus\vert 0,\frac12\rangle^2$	$\Psi^2_{\mu\dot\alpha}, \Psi^2_{\dot\alpha}$
	$\vert\frac12,0\rangle^2\otimes\vert 0,0\rangle$	$\vert\frac12,0\rangle^2$	Ψ^2_α
$(\{\pm1\},\{0\})$	$\vert 0,\frac12\rangle^2\otimes\vert\frac12,\frac12\rangle$	$\vert\frac12,1\rangle^2\oplus\vert\frac12,0\rangle^2$	$\Psi^2_{\mu\alpha}, \Psi^2_\alpha$
$(\{0\},\{\pm1,0\})$	$\vert 0,\frac12\rangle^2\otimes\vert 0,0\rangle$	$\vert 0,\frac12\rangle^2$	$\Psi^2_{\dot\alpha}$
	$\vert\frac12,0\rangle^2\otimes\vert 0,0\rangle^3$	$\vert\frac12,0\rangle^4\oplus\vert\frac12,0\rangle^2$	$\Psi^4_\alpha, \Psi^2_\alpha$
$(\{\pm1\},\{\pm1,0\})$	$\vert 0,\frac12\rangle^2\otimes\vert 0,0\rangle^3$	$\vert 0,\frac12\rangle^4\oplus\vert 0,\frac12\rangle^2$	$\Psi^4_{\dot\alpha}, \Psi^2_{\dot\alpha}$

Table B.10: Field Content in RNS sector

It is interesting to organize the spectrum according to massless supermultiplets of $6d$ supersymmetry (again, we may pretend that the states are massless in $6d$ by focussing on the value $p = \frac{1}{2}$ of the momentum along ρ). Massless supermultiplets are constructed by taking the direct product of a primary $|j_1, j_2\rangle^{2R+1}$ with a set $\mathcal{R}$ of raising operators. For $(2,0)$ susy in six dimensions,

$$\mathcal{R} = (1,0) + 2(\frac{1}{2},0)^2 + (0,0)^3 + 2(0,0) \tag{B.48}$$

For example the graviton multiplet is obtained acting with $\mathcal{R}$ on the primary $|0,1\rangle$, while the tensor multiplet is obtained starting with the primary $|0,0\rangle$. The complete field content of (the lowest level of) the cigar theory is obtained by action of $\mathcal{R}$ on the set of primaries,

$$|0,1\rangle + 2|0,\frac{1}{2}\rangle^2 + |0,0\rangle^3 + 2|0,0\rangle \tag{B.49}$$

Comparison with (B.48) suggests us that there are two other hidden supercharges at work, of opposite chirality, namely $(0,2)$, which relate the primaries of all the $(2,0)$ supermultiplets. In other words, we might conclude that we have obtained the maximally supersymmetric non-chiral $(2,2)$ supergravity in six dimensions. This is correct as the counting of states with $7d$ momentum $p = \frac{1}{2}$ goes, but the right-handed supersymmetries are broken by interactions. Nevertheless this is a useful hint: we should regard the effective theory for the lowest level as a spontaneously broken version of a maximally supersymmetric theory. And since the $7d$ momentum can be arbitrary, the candidate theory before symmetry breaking is maximally supersymmetry *seven*-dimensional supergravity.

B.5.6 Maximal 7d Supergravity with $SO(4)$ Gauging

To pursue this hint, in Table B.11 we have organized the lowest level of the linear-dilaton theory (before turning on the cigar interaction) according to $7d$ quantum numbers. The little group in $7d$ is $SO(5) \cong USp(4)$ and we label $USp(4)$ representations by their dimension. In the linear dilaton theory the full $SU(2)_{\psi_i} \otimes SU(2)_{\tilde{\psi}_i} \cong SO(4)$ is unbroken and we label states with superscripts $(2I+1, 2\tilde{I}+1)$ indicating the representation dimensions of the two $SU(2)$s. Remarkably, the resulting spectrum is precisely the field content of maximal $7d$ supergravity with $SO(4)$ gauging, a theory that has been fully constructed only quite recently [177, 178]. The massless vector $V_{\hat{\mu}}^{(3,1)+(1,3)}$ are the $SO(4)$ gauge fields. On the other hand the vectors $C_{\hat{\mu}}^{4}$ are eaten by the two forms $C_{\hat{\mu}\hat{\nu}}^{4}$, which become massive through a vectorial Higgs mechanism [177, 178].

Sector	$\lvert USp(4)\rangle^{2I+1} \otimes \lvert USp(4)\rangle^{2\tilde{I}+1}$	$\lvert USp(4)\rangle^{(2I+1,2\tilde{I}+1)}$	$7d$ Fields
NSNS	$\lvert 5\rangle \otimes \lvert 5\rangle$	$\lvert 14\rangle \oplus \lvert 10\rangle \oplus \lvert 1\rangle$	$G_{\hat{\mu}\hat{\nu}}$, $B_{\hat{\mu}\hat{\nu}}$, ϕ
	$\lvert 5\rangle \otimes \lvert 1\rangle^3$	$\lvert 5\rangle^{(3,1)\oplus(1,3)}$	$V_{\hat{\mu}}^{(3,1)\oplus(1,3)}$
	$\lvert 1\rangle^3 \otimes \lvert 5\rangle$		
	$\lvert 1\rangle^3 \otimes \lvert 1\rangle^3$	$\lvert 1\rangle^{(3,3)}$	$T^{(3,3)}$
RR	$\lvert 4\rangle^2 \otimes \lvert 4\rangle^2$	$\lvert 10\rangle^{(2,2)} \oplus \lvert 5\rangle^{(2,2)} \oplus \lvert 1\rangle^{(2,2)}$	$C_{\hat{\mu}\hat{\nu}}^{(2,2)}$, $C_{\hat{\mu}}^{(2,2)}$, $C^{(2,2)}$
RNS	$\lvert 4\rangle^2 \otimes \lvert 5\rangle$	$\lvert 16\rangle^{(2,1)\oplus(1,2)} \oplus \lvert 4\rangle^{(2,1)\oplus(1,2)}$	$\Psi_{\hat{\mu}}^{(2,1)\oplus(1,2)}$, $\Psi^{(2,1)\oplus(1,2)}$
NSR	$\lvert 5\rangle \otimes \lvert 4\rangle^2$		
	$\lvert 4\rangle^2 \otimes \lvert 1\rangle^3$	$\lvert 4\rangle^{(2,3)\oplus(3,2)}$	$\Psi^{(2,3)\oplus(3,2)}$
	$\lvert 1\rangle^3 \otimes \lvert 4\rangle^2$		

Table B.11: Seven-dimensional labeling of the spectrum of the linear-dilaton theory

Recall that the standard gauging of maximal $7d$ sugra is of the full $SO(5)$ R-symmetry – this is the famous supergravity that arises by consistent truncation of $11d$ supergravity compactified on S^4 and that admits a maximally supersymmetric AdS_7 vacuum. By contrast, the scalar potential of the $SO(4)$ theory does not allow for a stationary solution, but only for a domain wall solution [177, 178], that is, our linear-dilaton background. A closely related interpretation of the $SO(4)$ gauged supergravity was given in [188] (before its explicit construction!) as the effective $7d$ supergravity arising from a "warped compactification" of IIB supergravity on the near-horizon NS5 brane background $\mathbb{R}^{5,1}\times$ linear dilaton $\times S^3$.

The cigar background is obtained by further turning on a "tachyon" perturbation, a profile for the NSNS scalar fields $T^{(3,3)}$ that decays for large ρ and acts as a wall for $\rho \sim 0$. Note that the scalars are in the symmetric traceless tensor of $SO(4)$, and choosing a vev for them breaks $SO(4) \to SO(3) \cong SU(2)_R$, the diagonal combination of $SU(2)_{\psi_i} \times SU(2)_{\tilde{\psi}_i}$, as expected. In the IIA set-up of colliding NS5 branes, this breaking corresponds to choosing an angular direction in the transverse S^3 to the coincident NS5 brane – the direction along which the branes are separated (we called it τ in Figure 4.4). Under the preserved diagonal $SU(2)_R$, the nine NSNS scalars $T^{(3,3)}$ decompose as $\mathbf{5}+\mathbf{3}+\mathbf{1}$. The $\mathbf{1}$ and the $\mathbf{3}$ are associated to moduli, corresponding respectively (in the T-dual picture) to the radial and angular separations of the two NS5 branes; together with an extra $SU(2)_R$-singlet scalar from the RR sector they comprise the five scalars of the $6d$ tensor multiplet localized at the tip of the cigar.

In the application of the $SO(4)$-gauged $7d$ supergravity to our problem of finding the dual $\mathcal{N}=2$ SCQCD, we are not interested in turning on a background for the NSNS scalars, but rather for the RR fields corresponding to N_c D3 branes and N_f D5 branes. D3 branes are magnetically charged unde the RR one-form $C_\mu^{(2,2)}$ and D5 branes are magnetically charged under the RR zero-form $C^{(2,2)}$. As the superscripts indicate both of the RR one-form and zero-form transform as vectors of $SO(4)$. It is possible to choose a common direction in $SO(4)$ space for both forms, so that again we break $SO(4) \to SO(3) \cong SU(2)_R$. This is again consistent with the IIA Hanany-Witten picture. Separating the NS5 branes in breaks $SO(4)$ to $SO(3)$, and it is clear that both the compact and the non-compact $D4$-branes are extended in the same direction along which the NS5 branes are separated, so that their fluxes are oriented coherently in $SO(4)$ space. The surviving $SO(3) \cong SU(2)_R$ is interpreted as the $SU(2)_R$ R-symmetry of the $\mathcal{N} = 2$ gauge theory.

Appendix C: Spin chain

C.1 Computation one-loop dilation operator

In this appendix, we describe a shortcut to determine the one loop spin chain hamiltonian from the knowledge of the scaling dimensions and R charges of some of the local operators. As this method uses some of the eigenvalues from the spectrum of dilation operator, it doesn't substitute the explicit one loop computation of the dilation operator but serves as a quick check.

As usual, the composite operators are thought of as spin chains and the one-loop dilation operator as the nearest neighbor type hamiltonian acting on the spin chain. If the kth site in the spin chain has the vector space V_k associated with it, then $H = \sum_{k=1}^{L} H_{k,k+1}$ (with $k \equiv k+L$) acts as $H_{k,k+1} : V_k \otimes V_{k+1} \to V_k \otimes V_{k+1}$. The interactions contributing to $H_{k,k+1}$ at one loop are listed schematically in figure 5.1. The first and second interactions (self energy and gluon interaction) in figure 5.1 are proportional to identity in $V_k \otimes V_{k+1}$ while the non-trivial element is contributed only by the third interaction (quartic interaction). As a general strategy in this section, we first determine the contribution of the quartic vertex to the hamiltonian and then use anomalous dimensions of some operators to fix the coefficient of the remaining identity piece.

C.1.1 SCQCD

A brief pause to introduce some notation. Let the indices $\mathfrak{p}, \mathfrak{q} = \pm$ label the $U(1)_r$ charges of ϕ and $\bar{\phi}$, in other terms we define $\phi^- \equiv \phi$, $\phi^+ \equiv \bar{\phi}$, and $g_{\mathfrak{p}\mathfrak{q}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $|e^i e^j\rangle$ denotes an element of $V_k \otimes V_{k+1}$ (i.e. $e^i \in V_k$) then let $\langle e_{i'} e_{j'}|H|e^i e^j\rangle_{L_4}$ denote the contribution of the quartic interaction L_4 to the said matrix element.

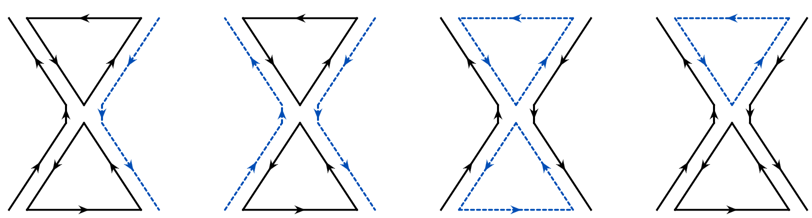
The nonzero elements of the hamiltonian due to quartic vertices are listed below:

$$\begin{aligned}
\langle \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'}|H|\phi^{\mathfrak{p}}\phi^{\mathfrak{q}}\rangle_{\phi^4} &= \delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'} && \text{(C.1)}\\
\langle \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'}|H|Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}\rangle_{Q^2\phi^2} &= \sqrt{\frac{N_f}{N_c}}g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}} && \text{(C.2)}\\
\langle \bar{Q}^{\mathcal{I}'}Q_{\mathcal{J}'}|H|Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}\rangle_{Q^4} &= \frac{N_f}{N_c}(2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'}) && \text{(C.3)}\\
\langle Q_{\mathcal{J}'}\bar{Q}^{\mathcal{I}'}|H|\bar{Q}^{\mathcal{J}}Q_{\mathcal{I}}\rangle_{Q^4} &= 2\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'} - \delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} && \text{(C.4)}
\end{aligned}$$

The factors of $\frac{N_f}{N_c}$ are explained in figure C.1. Figures C.1(a),C.1(b),C.1(c),C.1(d) correspond to equations (C.1,C.2,C.3,C.4) respectively. This fixes the hamiltonian up to identity terms.

$$H_{k,k+1} =$$

$$\begin{array}{c c}
 & \begin{array}{cccc} \phi^{\mathfrak{p}}\phi^{\mathfrak{q}} & Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}} & \bar{Q}^{\mathcal{K}}Q_{\mathcal{L}} & Q_{\mathcal{I}}\phi^{\mathfrak{p}} \end{array} \\
\begin{array}{c} \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'} \\ \bar{Q}^{\mathcal{I}'}Q_{\mathcal{J}'} \\ Q_{\mathcal{K}'}\bar{Q}^{\mathcal{L}'} \\ \bar{Q}^{\mathcal{I}'}\phi_{\mathfrak{p}'} \end{array} &
\begin{pmatrix}
\alpha\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'} & \sqrt{\frac{N_f}{N_c}}g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}} & 0 & 0 \\
\sqrt{\frac{N_f}{N_c}}g^{\mathfrak{p}\mathfrak{q}}\delta^{\mathcal{I}'}_{\mathcal{J}'} & \beta\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} - \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'}\frac{N_f}{N_c} & 0 & 0 \\
0 & 0 & \gamma\delta^{\mathcal{K}}_{\mathcal{K}'}\delta^{\mathcal{L}'}_{\mathcal{L}} + 2\delta^{\mathcal{K}}_{\mathcal{L}}\delta^{\mathcal{L}'}_{\mathcal{K}'} & 0 \\
0 & 0 & 0 & \eta\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathfrak{p}}_{\mathfrak{p}'}
\end{pmatrix}
\end{array}$$



(a) (b) (c) (d)

Figure C.1: The color/flavor structure of the quartic vertex. The solid black line represents the flow of the color index while the dotted blue line show the flow of the flavor index. (a) shows the ϕ^4 interaction vertex, whose contribution is proportional to N_c as compared to the tree level. In (b) the $Q^2\phi^2$ interaction vertex has a factor of N_f/N_c compared to (a) because of the presence of one flavor loop. The Q^4 vertex in (c) has an additional factor of $(N_f/N_c)^2$ compared to (a) due to the presence of two flavor loops. The figure (d), however, does not carry any additional N_f/N_c factors.

Next task is to determine the coefficients α, β, γ and η. From the fact that operator $\mathrm{Tr}\phi^k$ is chiral and hence protected, we determine $\alpha = 2$. Another protected multiplet is the multiplet of stress energy tensor and R symmetry currents (and supercurrents). Its superconformal primary, called $\mathrm{Tr}\, T$, has $R, r = 0$ and $\Delta = 2$. Hence, it is a linear combination of $\mathrm{Tr}[Q_{\mathcal{I}}\bar{Q}^{\mathcal{I}}]$ and $\mathrm{Tr}[\phi\bar{\phi}]$. Reduction of the hamiltonian over these operators is

$$H = \begin{array}{c c} & \begin{array}{cc} \mathrm{Tr}[\phi\bar{\phi}] & \mathrm{Tr}[Q_{\mathcal{I}}\bar{Q}^{\mathcal{I}}] \end{array} \\ \begin{array}{c} \mathrm{Tr}[\phi\bar{\phi}] \\ \mathrm{Tr}[Q_{\mathcal{I}}\bar{Q}^{\mathcal{I}}] \end{array} & \begin{pmatrix} 4 & 4\sqrt{\frac{N_f}{N}} \\ 4\sqrt{\frac{N_f}{N}} & 2(\beta+\gamma) - 4(\frac{N_f}{N_c} - 2) \end{pmatrix} \end{array} \tag{C.5}$$

We require a zero eigenvalue of this matrix at the superconformal point $N_f = 2N_c$, yielding $\beta + \gamma = 4$. Another fact that $\mathrm{Tr}\, T\phi$ is also a protected operator imposes a relation $\beta + 2\eta = 8$. Also, one loop anomalous dimension of flavor nonsinglet operator $\bar{Q}^i\phi \ldots \phi Q_i$ is calculated to be 4. The contribution comes only due to the "boundary" i.e. due to $\langle \bar{Q}\phi | \phi Q\rangle$ matrix element, as the "bulk" contribution from pairs of adjacent ϕs is zero. This determines $\eta = 2$. Summarizing we get,

$$\alpha = 2, \qquad \beta = 4, \qquad \gamma = 0, \qquad \eta = 2. \tag{C.6}$$

C.1.2 Interpolating SCFT

We repeat the same exercise as before. The quartic vertices contribute the following to the hamiltonian.

$$\langle \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'} | \phi^{\mathfrak{p}}\phi^{\mathfrak{q}}\rangle_{\phi^4} = \delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'} \tag{C.7}$$

$$\langle \check{\phi}_{\mathfrak{p}'}\check{\phi}_{\mathfrak{q}'} | \check{\phi}^{\mathfrak{p}}\check{\phi}^{\mathfrak{q}}\rangle_{\check{\phi}^4} = \kappa^2(\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathfrak{q}}_{\mathfrak{q}'} + g^{\mathfrak{p}\mathfrak{q}}g_{\mathfrak{p}'\mathfrak{q}'} - 2\delta^{\mathfrak{p}}_{\mathfrak{q}'}\delta^{\mathfrak{q}}_{\mathfrak{p}'}) \tag{C.8}$$

$$\begin{aligned} \langle \bar{Q}^{\hat{\mathcal{L}}\mathcal{L}}Q_{\mathcal{K}\hat{\mathcal{K}}} | Q_{\mathcal{I}\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}\rangle_{Q^4} &= 2\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{J}}_{\mathcal{K}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{K}}}\delta^{\mathcal{L}}_{\mathcal{I}} - \delta^{\mathcal{J}}_{\mathcal{I}}\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\,\delta^{\mathcal{L}}_{\mathcal{K}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{K}}} \\ &+ \kappa^2(2\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{K}}}\delta^{\mathcal{J}}_{\mathcal{I}}\,\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{L}}_{\mathcal{K}} - \delta^{\mathcal{L}}_{\mathcal{I}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{J}}_{\mathcal{K}}\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{K}}}) \end{aligned} \tag{C.9}$$

$$\begin{aligned} \langle Q_{\mathcal{I}\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}} | \bar{Q}^{\hat{\mathcal{L}}\mathcal{L}}Q_{\mathcal{K}\hat{\mathcal{K}}}\rangle_{Q^4} &= 2\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{J}}_{\mathcal{K}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{K}}}\delta^{\mathcal{L}}_{\mathcal{I}} - \delta^{\mathcal{J}}_{\mathcal{I}}\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\,\delta^{\mathcal{L}}_{\mathcal{K}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{K}}} \\ &+ \kappa^2(2\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{K}}}\delta^{\mathcal{J}}_{\mathcal{I}}\,\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{L}}_{\mathcal{K}} - \delta^{\mathcal{L}}_{\mathcal{I}}\delta^{\hat{\mathcal{L}}}_{\hat{\mathcal{I}}}\delta^{\mathcal{J}}_{\mathcal{K}}\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{K}}}) \end{aligned} \tag{C.10}$$

$$\langle \phi_{\mathfrak{p}'}\phi_{\mathfrak{q}'} | Q_{\mathcal{I}\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}\rangle_{Q^2\phi^2} = g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}}\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}} \tag{C.11}$$

$$\langle \check{\phi}_{\mathfrak{p}'}\check{\phi}_{\mathfrak{q}'} | \bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}Q_{\mathcal{I}\hat{\mathcal{I}}}\rangle_{Q^2\check{\phi}^2} = \kappa^2 g_{\mathfrak{p}'\mathfrak{q}'}\delta^{\mathcal{J}}_{\mathcal{I}}\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}} \tag{C.12}$$

$$\langle \bar{Q}^{\hat{\mathcal{J}}\mathcal{J}}\check{\phi}_{\mathfrak{q}} | \phi^{\mathfrak{p}}Q_{\mathcal{I}\hat{\mathcal{I}}}\rangle_{\phi Q\check{\phi}\bar{Q}} = -2\kappa\delta^{\mathfrak{p}}_{\mathfrak{q}}\delta^{\mathcal{J}}_{\mathcal{I}}\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}} \tag{C.13}$$

$$\langle \phi^{\mathfrak{p}}\bar{Q}^{\hat{\mathcal{J}}\mathcal{J}} | Q_{\mathcal{I}\hat{\mathcal{I}}}\check{\phi}_{\mathfrak{q}}\rangle_{\phi Q\check{\phi}\bar{Q}} = -2\kappa\delta^{\mathfrak{p}}_{\mathfrak{q}}\delta^{\mathcal{J}}_{\mathcal{I}}\,\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}} \tag{C.14}$$

The first four elements could have an additional identity pieces. They are easily determined by imposing the symmetry under $g \leftrightarrow \check{g}$, $Q \leftrightarrow \bar{Q}$ and

$\phi \leftrightarrow \check{\phi}$ and by requiring the hamiltonian to reduce to that of SCQCD in the limit $\kappa \to 0$. The one loop hamiltonian (5.38) is precisely reproduced by this method.

C.2 Composite Impurities

To implement the aforementioned constraint that $Q_{\mathcal{I}}$ can only occur as a combination $Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}$, we regard the combined object $Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}} \equiv \mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}$ as a basic impurity object in the spin chain. For future convenience, we define the singlet combination $\mathcal{M} = \frac{1}{\sqrt{2}}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}\delta^{\mathcal{I}}_{\mathcal{J}}$ and the triplet $\mathcal{M}^i = \frac{1}{\sqrt{2}}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}(\sigma^i)^{\mathcal{I}}_{\mathcal{J}}$, where σ^i are three Pauli matrices . These can be rewritten in an $SO(4)$ notation as $\mathcal{M}^m = \frac{1}{\sqrt{2}}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}(\sigma^m)^{\mathcal{I}}_{\mathcal{J}}$, where m takes the values $0, \ldots, 4$ with $\sigma^0 \equiv \mathbb{I}_{2\times 2}$. With these basic objects at hand, we would like to write down the way they interact in the spin chain.

C.2.1 Neighbouring $\mathcal{M}^m$ and $\bar{\phi}$

We now write down the nearest neighbour Hamiltonian in this picture, with $\mathcal{M}^m$ and $\bar{\phi}$ as the basic impurity objects. Let us concentrate on the matrix element that acts on neighbouring ϕ and $\mathcal{M}$. Consider the following sequence in the spin chain, $\ldots\phi^{\mathfrak{p}}Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}\phi^{\mathfrak{q}}\ldots$.

$$
\begin{array}{ccccccc}
\phi^{\mathfrak{p}} & & Q_{\mathcal{I}} & & \bar{Q}^{\mathcal{J}} & & \phi^{\mathfrak{q}} \\
 & \frac{1}{2}(3+\frac{\xi}{2})\mathbb{I}_{\phi Q} & & (5-\frac{\xi}{2})\mathbb{I}_{QQ} - 2\mathbb{K}_{QQ} & & \frac{1}{2}(3+\frac{\xi}{2})\mathbb{I}_{\phi Q} & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\phi_{\mathfrak{p}'} & & \bar{Q}^{\mathcal{I}'} & & Q_{\mathcal{J}'} & & \phi_{\mathfrak{q}'}
\end{array}
\tag{C.15}
$$

In this new picture, where we see $\mathcal{M}$ as the basic impurity, we need to include the middle term $(\frac{\xi}{2}-1)\mathbb{I}_{QQ} + 2\mathbb{K}_{QQ}$ of the Hamiltonian i.e. "self energy" of $\mathcal{M}$, in both the nearest neighbour Hamiltonian that acts on consecutive ϕ and

$\mathcal{M}$s. So we write,

$$\begin{aligned}
& g^{-2}\langle \ldots \phi_{\mathfrak{p}'}\bar{\mathcal{M}}^{\mathcal{I}'}{}_{\mathcal{J}'}\ldots | \ldots \phi^{\mathfrak{p}}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}\ldots\rangle \\
= & [\frac{1}{2}(3+\frac{\xi}{2})+\frac{1}{2}(5-\frac{\xi}{2})]\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'} \\
& -\delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'} \\
= & (4\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'}-\delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'})\delta^{\mathfrak{p}}_{\mathfrak{p}'} \\
& g^{-2}\langle \ldots \phi_{\mathfrak{p}'}\bar{\mathcal{M}}^{m'}\ldots | \ldots \phi^{\mathfrak{p}}\mathcal{M}^{m}\ldots\rangle \\
= & \frac{1}{2}(\sigma^{m'})^{\mathcal{J}'}_{\mathcal{I}'}(\sigma^{m})^{\mathcal{I}}_{\mathcal{J}}\langle \ldots \phi_{\mathfrak{p}'}\bar{\mathcal{M}}^{\mathcal{I}'}{}_{\mathcal{J}'}\ldots | \ldots \phi^{\mathfrak{p}}\mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}\ldots\rangle \\
= & \delta^{\mathfrak{p}}_{\mathfrak{p}'}\delta^{mm'}(4-2\delta^{m0}) \\
= & 4\mathbb{I}_{\phi}\otimes\mathbb{I}_{\mathcal{M}}-2\mathbb{I}_{\phi}\otimes\mathcal{P}^{0}
\end{aligned}$$

Here, we have introduced a new operator $\mathcal{P}^0$. This operator is a projection operator which projects the four dimensional space spanned by $\mathcal{M}^n$ onto $\mathcal{M}^0$.

C.2.2 Neighbouring $\mathcal{M}^m$ and $\mathcal{M}^n$

It is easy to repeat the exercise to get the nearest neighbour matrix element acting between two $\mathcal{M}$s. For this purpose we consider the sequence $\ldots Q_{\mathcal{I}}\bar{Q}^{\mathcal{J}}Q_{\mathcal{K}}\bar{Q}^{\mathcal{L}}\ldots$.

$$\begin{array}{ccccccc}
Q_{\mathcal{I}} & & \bar{Q}^{\mathcal{J}} & & Q_{\mathcal{K}} & & \bar{Q}^{\mathcal{L}} \\
& (5-\frac{\xi}{2})\mathbb{I}_{QQ}-2\mathbb{K}_{QQ} & & (\frac{\xi}{2}-1)\mathbb{I}_{QQ}+2\mathbb{K}_{QQ} & & (5-\frac{\xi}{2})\mathbb{I}_{QQ}-2\mathbb{K}_{QQ} & \\
& \downarrow & & \downarrow & & \downarrow & \\
\bar{Q}^{\mathcal{I}'} & & Q_{\mathcal{J}'} & & \bar{Q}^{\mathcal{K}'} & & Q_{\mathcal{L}'}
\end{array} \tag{C.16}$$

The downward arrow shows the action of Hamiltonian in various neighbouring sites. This simply says that

$$g^{-2}\langle \ldots \bar{\mathcal{M}}^{\mathcal{I}'}{}_{\mathcal{J}'}\bar{\mathcal{M}}^{\mathcal{K}'}_{\mathcal{L}'}\ldots | \ldots \mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}}\mathcal{M}_{\mathcal{K}}{}^{\mathcal{L}}\ldots\rangle = \tag{C.17}$$

$$\begin{array}{rcl}
\frac{1}{2}(5-\frac{\xi}{2})\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\mathcal{K}'}_{\mathcal{K}}\delta^{\mathcal{L}}_{\mathcal{L}'} & - & \delta^{\mathcal{J}}_{\mathcal{I}}\delta^{\mathcal{I}'}_{\mathcal{J}'}\delta^{\mathcal{K}'}_{\mathcal{K}}\delta^{\mathcal{L}}_{\mathcal{L}'} \\
(\frac{\xi}{2}-1)\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\mathcal{K}'}_{\mathcal{K}}\delta^{\mathcal{L}}_{\mathcal{L}'} & + & 2\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{K}}\delta^{\mathcal{K}'}_{\mathcal{J}'}\delta^{\mathcal{L}}_{\mathcal{L}'} \\
\frac{1}{2}(5-\frac{\xi}{2})\delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\mathcal{K}'}_{\mathcal{K}}\delta^{\mathcal{L}}_{\mathcal{L}'} & - & \delta^{\mathcal{I}'}_{\mathcal{I}}\delta^{\mathcal{J}}_{\mathcal{J}'}\delta^{\mathcal{L}}_{\mathcal{K}}\delta^{\mathcal{K}'}_{\mathcal{L}'}
\end{array}$$

$$
\begin{aligned}
g^{-2}\langle\ldots\bar{\mathcal{M}}^{m'}\bar{\mathcal{M}}^{n'}\ldots|\ldots\mathcal{M}^{m}\mathcal{M}^{n}\ldots\rangle &= \delta^{mm'}\delta^{nn'}(13-4\delta^{m0}-4\delta^{n0}) \\
&\quad +\delta^{mn}\delta^{m'n'}-\delta^{mn'}\delta^{nm'}+i\epsilon^{mnn'm'}
\end{aligned}
$$

Following Minahan and Zarembo, we introduce the $SO(4)$ spin operators,

$$
(\Sigma^{ij})_{mn} \equiv \delta^i_m\delta^j_n - \delta^i_n\delta^j_m \tag{C.18}
$$

Now we can rewrite this piece of Hamiltonian when the spin-spin interactions are manifest.

$$
\begin{aligned}
g^{-2}\langle\ldots\bar{\mathcal{M}}^{m'}\bar{\mathcal{M}}^{n'}\ldots|\ldots\mathcal{M}^{m}\mathcal{M}^{n}\ldots\rangle &= 13\mathbb{I}_{\mathcal{M}}\otimes\mathbb{I}_{\mathcal{M}}-4(\mathcal{P}^0\otimes\mathbb{I}_{\mathcal{M}}+\mathbb{I}_{\mathcal{M}}\otimes\mathcal{P}^0) \\
&\quad +\frac{1}{2}\Sigma^{ij}\otimes\Sigma^{ij}-\frac{i}{4}\epsilon_{ijkl}\Sigma^{ij}\otimes\Sigma^{kl}
\end{aligned}
$$

C.3 Spin Chain

C.3.1 The Undynamic Spin Chain

In this appendix, we construct the Hamiltonian for a spin chain which has $\mathcal{M}$ and $\bar{\phi}$ to be the elementary excitations.

We run into difficulty if we continue writing the nearest neighbour Hamiltonian in this picture where $\mathcal{M}$ is the basic impurity and not Q and $\bar{Q}$. This is because while writing the Hamiltonian in this fashion we implicitly assume that the number of sites is preserved by the Hamiltonian, which is clearly not true. There is an element of the Hamiltonian which is responsible for the "decay" of the impurity $\mathcal{M}$ into ϕ and $\bar{\phi}$ and vice versa i.e. for the transition $\mathcal{M}^0 \leftrightarrow \phi\bar{\phi} + \bar{\phi}\phi$. Clearly, this transition does not preserve the number of sites in the spin chain. To remedy this situation we follow Beisert's treatment called the undynamic spin chain. Let us introduce an integer subscript l which counts the number of ϕs following an impurity.

$$\bar{\phi}_l \equiv \bar{\phi}\phi^l \qquad \mathcal{M}^m_l \equiv \mathcal{M}^m\phi^l \tag{C.19}$$

with these redefinitions of the impurities, each site in this chain hosts a vector space $V\otimes\mathbb{Z}$, where V is spanned by $\bar{\phi}$ and $\mathcal{M}^n$s, for example, $\bar{\phi}_l$ can be written as $\bar{\phi}\otimes|l\rangle$. We introduce the raising and lowering operators acting on $\mathbb{Z}$, with $a^\dagger|l\rangle = |l+1\rangle$ and $a|l\rangle = |l-1\rangle$. The hardcore constraint is implemented by defining $|0\rangle$ to be the vacuum, hence annihilated by a. Let l_i be the number of ϕs following the i th impurity. a_i and $a_i^\dagger$ are the raising and lowering operators which act on i th impurity.

$$\begin{aligned} H[\bar{\phi}_{l_i}\bar{\phi}_{l_{i+1}}] &= 6\bar{\phi}_{l_i}\bar{\phi}_{l_{i+1}} - \bar{\phi}_{l_i+1}\bar{\phi}_{l_{i+1}-1} - \bar{\phi}_{l_i-1}\bar{\phi}_{l_{i+1}+1} \\ &\quad +2\bar{\phi}_{l_i}\mathcal{M}^0_{l_{i+1}-1} + 2\bar{\phi}_{l_i-1}\mathcal{M}^0_{l_{i+1}} - \delta_{l_1 0}6\bar{\phi}_{l_1}\bar{\phi}_{l_2} \end{aligned}$$

$\delta_{l_i,0}$ can be rewritten as $(1-a_i^\dagger a_i)$,

$$\begin{aligned} H[\bar{\phi}_{l_i}\bar{\phi}_{l_{i+1}}] &= (6a_i^\dagger a_i - a_i^\dagger a_{i+1} - a_{i+1}^\dagger a_i)\bar{\phi}_{l_i}\bar{\phi}_{l_{i+1}} \\ &\quad +2(a_i + a_{i+1})\bar{\phi}_{l_i}\mathcal{M}^0_{l_{i+1}}) \end{aligned}$$

Doing the same exercise for all the matrix elements,

$$
\begin{array}{c}
\begin{array}{cccc}
\bar{\phi}_{l_i}\bar{\phi}_{l_{i+1}} & \bar{\phi}_{l_i}\mathcal{M}^m_{l_{i+1}} & \mathcal{M}^m_{l_i}\bar{\phi}_{l_{i+1}} & \mathcal{M}^m_{l_i}\mathcal{M}^n_{l_{i+1}}
\end{array} \\
\begin{pmatrix}
6a^\dagger_i a_i - a^\dagger_i a_{i+1} - a^\dagger_{i+1} a_i & 2(a^\dagger_i + a^\dagger_{i+1})\mathcal{P}^0_{i+1} & 0 & 0 \\
2(a_i + a_{i+1})\mathcal{P}^0_{i+1} & 4 + 3a^\dagger_i a_i - 2\mathcal{P}^0_{i+1} & 0 & 0 \\
0 & 0 & \begin{array}{c} 4 + 3a^\dagger_i a_i - a^\dagger_i a_{i+1} \\ -a^\dagger_{i+1} a_i - 2\mathcal{P}^0_i \end{array} & 2(a^\dagger_i + a^\dagger_{i+1})\mathcal{P}^0_{i+1} \\
0 & 0 & 2(a_i + a_{i+1})\mathcal{P}^0_{i+1} & \begin{array}{c} a^\dagger_i a_i(8 - 2(\mathcal{P}^0_i + \mathcal{P}^0_{i+1})) - (1 - a^\dagger_i a_i) \\ (13 - 4(\mathcal{P}^0_i + \mathcal{P}^0_{i+1}) \\ +\frac{1}{2}\Sigma_i \cdot \Sigma_{i+1} - \frac{i}{4}\Sigma_i \times \Sigma_{i+1}) \end{array}
\end{pmatrix}
\end{array}
\tag{C.20}
$$

Here, $\mathcal{P}^0_i$ is a projection operator acting on i th $\mathcal{M}^n$ impurity, and $\Sigma \times \Sigma$ stands for $\epsilon_{ijkl}\Sigma^{ij}\Sigma^{kl}$. In the matrix elements of Hamiltonian $H_{i,i+1}$ we have included the half of the self energy of the i th and $i+1$ th nearest neighbours, and nontrivial movements of $i+1$ th impurity. The block diagonal structure is due to this feature of construction. Please note that we can not include the movement of i th impurity in $H_{i,i+1}$ as it would change the subscript of $i-1$ th impurity as well. Our treatment is consistent because of the cyclicity of the spin chain, which makes the above Hamiltonian act on all the impurities.

Consider a two impurity sector in a periodic spin chain of length l. This sector will be spanned by the operators $\bar{\phi}_{l_1}\bar{\phi}_{l_2}$, $\bar{\phi}_{l_1}\mathcal{M}^m_{l_2-1}$ and $\mathcal{M}^m_{l_1-1}\mathcal{M}^n_{l_2-1}$, with the constraint $l_1 + l_2 = l - 2$. Let us write down the Hamiltonian in this basis,

$$
\begin{array}{c}
\begin{array}{cccc}
\bar{\phi}_{l_1}\bar{\phi}_{l_2} & \bar{\phi}_{l_1}\mathcal{M}^m_{l_2} & \mathcal{M}^m_{l_1}\bar{\phi}_{l_2} & \mathcal{M}^m_{l_1}\mathcal{M}^n_{l_2}
\end{array} \\
\begin{pmatrix}
\begin{array}{c} 2(3(a^\dagger_1 a_1 + a^\dagger_2 a_2) \\ -a^\dagger_1 a_2 - a^\dagger_2 a_1) \end{array} & 2(a^\dagger_1 + a^\dagger_2)\mathcal{P}^0_2 & 2(a^\dagger_1 + a^\dagger_2)\mathcal{P}^0_1 & 0 \\
2(a_1 + a_2)\mathcal{P}^0_2 & \begin{array}{c} 3(a^\dagger_1 a_1 + a^\dagger_2 a_2) \\ -a^\dagger_2 a_1 - a^\dagger_1 a_2 + 8 - 4\mathcal{P}^0_2 \end{array} & 0 & 2(a^\dagger_1 + a^\dagger_2)\mathcal{P}^0_1 \\
2(a_1 + a_2)\mathcal{P}^0_1 & 0 & \begin{array}{c} 3(a^\dagger_1 a_1 + a^\dagger_2 a_2) \\ -a^\dagger_2 a_1 - a^\dagger_1 a_2 + 8 - 4\mathcal{P}^0_1 \end{array} & 2(a^\dagger_1 + a^\dagger_2)\mathcal{P}^0_2 \\
0 & 2(a_1 + a_2)\mathcal{P}^0_1 & 2(a_1 + a_2)\mathcal{P}^0_2 & \begin{array}{c} 26 - 5(a^\dagger_1 a_1 + a^\dagger_1 a_1) \\ -(8 - 2(a^\dagger_1 a_1 + a^\dagger_2 a_2))(\mathcal{P}^0_1 + \mathcal{P}^0_2) \\ (2 - a^\dagger_1 a_1 - a^\dagger_2 a_2) \\ \times(\frac{1}{2}\Sigma_1 \cdot \Sigma_2 - \frac{i}{4}\Sigma_1 \times \Sigma_2) \end{array}
\end{pmatrix}
\end{array}
\tag{C.21}
$$

C.4 Superspace for $\mathcal{N} = 2$ orbifold theory

In this appendix, we explain the superspace construction of the most general $\mathcal{N} = 2$ superconformal Lagrangian with the field content displayed in Table 5.2. This corresponds to the theory of the "deformed" orbifold of $\mathcal{N} = 4$ SYM.

The component Lagrangian of this theory is presented in Section 5.2.1.

The component fields of the Table 5.2, can be organized into two $\mathcal{N}=2$ vector multiplets and a hypermultiplet. The $\mathcal{N}=2$ vector multiplet in $\mathcal{N}=1$ superspace language is made out of a vector multiplet and a chiral multiplet.

In this theory of two $SU(N_c)$ gauge groups, the color indices of the $\mathcal{N}=2$ vector superfields under both the gauge groups are $V^a{}_b, \Phi^a{}_b, \check{V}^{\check{a}}{}_{\check{b}}$ and $\check{\Phi}^{\check{a}}{}_{\check{b}}$. Their component expansion is as follows,

$$
\begin{aligned}
V &= \bar{\theta}\sigma^\mu\theta A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}^2 D && \text{(C.22)}\\
\Phi &= \phi + \theta\chi - \theta^2 F && \text{(C.23)}\\
\check{V} &= \bar{\theta}\sigma^\mu\theta \check{A}_\mu + i\theta^2\bar{\theta}\bar{\check{\lambda}} - i\bar{\theta}^2\theta\check{\lambda} + \theta^2\bar{\theta}^2 \check{D} && \text{(C.24)}\\
\check{\Phi} &= \check{\phi} + \theta\check{\chi} - \theta^2 \check{F} && \text{(C.25)}
\end{aligned}
$$

The $\mathcal{N}=2$ hypermultiplets, however, carry the bi-fundamental index structure with respect to the two gauge groups as $Q^a_{\hat{\mathcal{I}}\check{a}}$ and $\tilde{Q}^{\hat{\mathcal{I}}\check{a}}{}_a$. They are decomposed in the components as,

$$
\begin{aligned}
Q_{\hat{\mathcal{I}}} &= q_{\hat{\mathcal{I}}} + \theta\psi_{\hat{\mathcal{I}}} - \theta^2 f_{\hat{\mathcal{I}}} && \text{(C.26)}\\
\tilde{Q}^{\hat{\mathcal{I}}} &= \tilde{q}^{\hat{\mathcal{I}}} + \theta\tilde{\psi}^{\hat{\mathcal{I}}} - \theta^2 \tilde{f}^{\hat{\mathcal{I}}} && \text{(C.27)}
\end{aligned}
$$

The unique $\mathcal{N}=2$ superconformal Lagrangian with above multiplets is,

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_V + \mathcal{L}_H && \text{(C.28)}\\
\mathcal{L}_V &= \int d^4\theta e^{-g_{YM}V}\bar{\Phi}e^{g_{YM}V} + (\int d^2\theta W^\alpha W_\alpha + h.c.)\\
&\quad + \int d^4\theta e^{-\check{g}_{YM}\check{V}}\bar{\check{\Phi}}e^{\check{g}_{YM}\check{V}} + (\int d^2\theta \check{W}^\alpha \check{W}_\alpha + h.c.) && \text{(C.29)}\\
\mathcal{L}_H &= \int d^4\theta \bar{Q}e^{g_{YM}V}Q + \int d^4\theta Q e^{-\check{g}_{YM}\check{V}}\bar{Q}\\
&\quad + \int d^4\theta \bar{\tilde{Q}}e^{-g_{YM}V}\tilde{Q} + \int d^4\theta \tilde{Q}e^{\check{g}_{YM}\check{V}}\bar{\tilde{Q}}\\
&\quad + (\int d^2\theta \tilde{Q}\Phi Q + h.c.) + (\int d^2\theta Q\check{\Phi}\tilde{Q} + h.c.) && \text{(C.30)}
\end{aligned}
$$

One could add the term $m\int d^2\theta\tilde{Q}Q$ to the Lagrangian which would preserve the $\mathcal{N}=2$ supersymmetry but this would break the conformal invariance.