is $\sigma[\mathrm{J}/\psi+\eta_c]\times B^{\eta_c}[\geqslant 2]=(17.6\pm 2.8^{+1.5}_{-2.1})$ fb by BABAR [12]. Here $B^{\eta_c}[\geqslant 2]$ denotes the branching fraction for the η_c decaying into at least two charged tracks. Meanwhile, predictions at the LO in α_s and v, given by Braaten and Lee [13], Liu, He and Chao [14], and Hagiwara, Kou and Qiao [15] are about 2.3–5.5 fb. This is almost an order of magnitude smaller than the experimental results. Such a large discrepancy becomes a challenge to the current understanding of charmonium production based on NRQCD. Many studies have been performed in order to resolve this problem. For example, Ma and Si [16] treated the process by using the light-cone method, which was also performed by Bondar and Chernyak [17], and Bodwin, Kang and Lee [18]. A possible contribution from intermediate meson rescatterings was considered by Zhang, Zhao, and Qiao [19]. It was also studied in the Bethe-Salpeter formalism by Guo, Ke, Li, and Wu in Ref. [20]. Based on NRQCD, Braaten and Lee [13] have shown that the relativistic corrections would increase the cross section by a factor of about 2. The NLO QCD correction of the process has been studied by Zhang, Gao and Chao [21], and Gong and Wang [22], which can enhance the cross section with a K factor (the ratio of the NLO contribution to the LO one) of about 2. The corresponding relativistic corrections have been studied by Bodwin, Kang, Kim, Lee and Yu [23] and by He, Fan and Chao [4], which is also significant. More detailed treatment, through the resummation of a class of relativistic correction, was achieved by Bodwin and Lee and Yu [24]. Meanwhile, Bodwin, Lee and Braaten [25] showed that the cross section of the process $e^+e^-\to \mathrm{J}/\psi+\mathrm{J}/\psi$ may be larger than that of $\mathrm{J}/\psi+\eta_c$ by a factor of 1.8, in spite of a suppression factor α^2/α_s^2 associated with the QED and QCD coupling constants. They suggested that a significant part of the discrepancy of $\mathrm{J}/\psi+\eta_c$ production may be explained by comparing with this process. Hagiwara, Kou and Qiao [15] also calculated and discussed this process. In 2004, a new analysis of double charmonium production in e^+e^- annihilation was performed by Belle [26], based on a three times larger data set, and no evidence for the process $e^+e^-\to \mathrm{J}/\psi+\mathrm{J}/\psi$ was found. Since both the NLO QCD corrections and relativistic corrections to $e^+e^-\to\mathrm{J}/\psi+\eta_c$ give a large K factor of about 2, it is reasonable that these two types of corrections to $e^+e^-\to\mathrm{J}/\psi+\mathrm{J}/\psi$ should be studied to explain the experimental results. In fact, they have been studied by Bodwin, Lee and Braaten for the dominant photon-fragmentation contribution diagrams [27]. Their results show that the cross section is decreased by K factors of 0.39 and 0.78 for the NLO QCD and relativistic corrections, respectively. A more reliable estimate, 1.69±0.35 fb, was given by Bodwin, Lee, Braaten and Yu in Ref. [28]. The light-cone method is used in Ref. [29] by V. V. Braguta. Gong and Wang performed a complete NLO QCD calculation on $e^+e^-\to\mathrm{J}/\psi+\mathrm{J}/\psi$ [30], and the results show that the cross section would be much smaller than the rough estimate in Ref. [27]. This explains why there was no evidence for the process $e^+e^-\to\mathrm{J}/\psi+\mathrm{J}/\psi$ at B-factories.

The results show that both the QCD correction (α_s) and relativistic correction (v^2) are very large for $e^+e^-\to\mathrm{J}/\psi+\eta_c$ at B-factory energy, and the experimental measurement can be explained with these corrections. Therefore, it is natural to ask about the situation for the higher order corrections beyond α_s and v^2 correction. An α_s^2 correction is very difficult to do, but recent progress makes the order $\alpha_s v^2$ correction available. It is very interesting to see that the $\alpha_s v^2$ correction given in a recent work [31] is small. This convinces us that, in some sense (with α_s^2 correction absent), the double expansions in NRQCD converges quite well on this problem. Since the calculation is quite complicated and plays an important role on the convergence of the theoretical prediction, which can explain the experimental data, in this paper we have performed an independent calculation of it. The calculation is done by using the package Feynman Diagram Calculation (FDC) [32] which employs a built-in method to calculate a relativistic correction.

The remainder of this paper is organized as follows. Based on the NRQCD frame, we briefly introduce theoretical formalism for the calculation of heavy quarkonium production and give the corresponding results in perturbative NRQCD in Section 2. The details in perturbative QCD are summarized in Section 3. We give the numerical results of $\alpha_s v^2$ corrections and some discussion in Section 4. Finally, in Section 5, we present a brief summary.

2 NRQCD factorization formula up to v^2 order

According to NRQCD effective theory, charmonium production is factorized into two parts, the short-distance part and the long-distance part. The long-distance part is related to the four fermion operators, characterized by the velocity v of the charm quark in the meson rest frame. The long-distance matrix elements can be estimated by lattice calculations, phenomenological models, or determined by fitting experimental data. The production cross section up to v^2 order is expressed as

$$\sigma(e^+e^-\to\mathrm{J}/\psi+\eta_c) = (c_{00}+c_{10}\langle v^2\rangle_{\mathrm{J}/\psi} +c_{01}\langle v^2\rangle_{\eta_c})\langle\mathcal{O}_1\rangle_{\eta_c}\langle\mathcal{O}_1\rangle_{\mathrm{J}/\psi}, \quad (1)$$

where the long-distance matrix elements are defined by using related operators as

$$\langle v^2\rangle_{\mathrm{J}/\psi} = \frac{\langle\mathcal{P}_1\rangle_{\mathrm{J}/\psi}}{m_c^2\langle\mathcal{O}_1\rangle_{\mathrm{J}/\psi}},$$

$$\begin{aligned}\langle\mathcal{O}_1\rangle_{\mathrm{J}/\psi} &= \langle 0|\chi^\dagger\sigma^{\mathrm{i}}\psi(a^\dagger_{\mathrm{J}/\psi}a_{\mathrm{J}/\psi})\psi^\dagger\sigma^{\mathrm{i}}\chi|0\rangle,\\ \langle\mathcal{P}_1\rangle_{\mathrm{J}/\psi} &= \langle 0|\frac{1}{2}\left[\chi^\dagger\sigma^{\mathrm{i}}\psi(a^\dagger_{\mathrm{J}/\psi}a_{\mathrm{J}/\psi})\psi^\dagger\sigma^{\mathrm{i}}\left(-\frac{\mathrm{i}}{2}\overleftrightarrow{\boldsymbol{D}}\right)^2\chi\right.\\ &\left.+\chi^\dagger\sigma^{\mathrm{i}}\left(-\frac{\mathrm{i}}{2}\overleftrightarrow{\boldsymbol{D}}\right)^2\psi(a^\dagger_{\mathrm{J}/\psi}a_{\mathrm{J}/\psi})\psi^\dagger\sigma^{\mathrm{i}}\chi\right]|0\rangle, \quad (2)\end{aligned}$$

for J/ψ and

$$\begin{aligned}\langle v^2\rangle_{\eta_c} &= \frac{\langle\mathcal{P}_1\rangle_{\eta_c}}{m_c^2\langle\mathcal{O}_1\rangle_{\eta_c}},\\ \langle\mathcal{O}_1\rangle_{\eta_c} &= \langle 0|\chi^\dagger\psi(a^\dagger_{\eta_c}a_{\eta_c})\psi^\dagger\chi|0\rangle,\\ \langle\mathcal{P}_1\rangle_{\eta_c} &= \langle 0|\frac{1}{2}\left[\chi^\dagger\psi(a^\dagger_{\eta_c}a_{\eta_c})\psi^\dagger\left(-\frac{\mathrm{i}}{2}\overleftrightarrow{\boldsymbol{D}}\right)^2\chi\right.\\ &\left.+\chi^\dagger\left(-\frac{\mathrm{i}}{2}\overleftrightarrow{\boldsymbol{D}}\right)^2\psi(a^\dagger_{\eta_c}a_{\eta_c})\psi^\dagger\chi\right]|0\rangle, \quad (3)\end{aligned}$$

for η_c. m_c is the charm quark mass. This is the basic point that the NRQCD factorization for hadron related process will also hold when the hadron state are replaced by $Q\bar{Q}$ states with exactly the same quantum numbers as the corresponding hadron state. In this way, the short-distance coefficients c_{00}, c_{01} and c_{10} can be obtained in perturbative calculation through the matching condition, and they are calculated up to QCD next-to-leading (NLO) order. In order to obtain the short-distant coefficients, the matrix elements of the operators for quantum states need to be calculated perturbatively, and there are

$$\langle\mathcal{O}_1\rangle_{^1S_0}=2N_c(2E_{q_1})^2,\ \ \langle\mathcal{O}_1\rangle_{^3S_1}=6N_c(2E_{q_2})^2, \quad (4)$$

where there are $N_c=3$ for $SU(3)$ group and $E_q=\sqrt{m_c^2+\boldsymbol{q}^2}$. From the NRQCD effective Lagrangian, we could easily get the Feynman rules. Therefore, we have calculated order $\alpha_s v^2$ corrections to the leading order $\langle\mathcal{O}_1\rangle_{^{2s+1}S_s}$ in perturbative NRQCD with the dimensional regularization and defined the renormalization constants $Z_O^{\overline{\mathrm{MS}}}$ of the operator by using the $\overline{\mathrm{MS}}$ scheme [2, 33].

$$\delta Z_O^{\overline{\mathrm{MS}}}=-\frac{4\alpha_s C_F}{3\pi}\left(\frac{\mu_r^2}{\mu_\Lambda^2}\right)^\epsilon\left(\frac{1}{\epsilon_{\mathrm{UV}}}+\ln 4\pi-\gamma_E\right)\frac{\boldsymbol{q}^2}{m_c^2}, \quad (5)$$

$$\begin{aligned}\langle\mathcal{O}_1\rangle^{\mathrm{R}}_{^{2s+1}S_s} &= \left[1+\frac{4\alpha_s C_F}{3\pi}\left(\frac{\mu_r^2}{\mu_\Lambda^2}\right)^\epsilon\left(\frac{1}{\epsilon}+\ln 4\pi-\gamma_E\right)\frac{\boldsymbol{q}^2}{m_c^2}\right]\\ &\times\langle\mathcal{O}_1\rangle_{^{2s+1}S_s}. \quad (6)\end{aligned}$$

$$\langle\mathcal{P}_1\rangle_{^{2s+1}S_s}=\boldsymbol{q}^2\langle\mathcal{O}_1\rangle_{^{2s+1}S_s}. \quad (7)$$

At last we can easily give the perturbative NRQCD results.

$$\begin{aligned}&\sigma(e^+e^-\to Q\bar{Q}(^3S_1^1)+Q\bar{Q}(^1S_0^1))\Big|_{\mathrm{pertNRQCD}}\\ &=\left\{c_{00}+\frac{\boldsymbol{q}_1^2}{m_c^2}\left[c_{10}+\frac{4\alpha_s C_F}{3\pi}\left(\frac{\mu_r^2}{\mu_\Lambda^2}\right)^\epsilon\left(\frac{1}{\epsilon}+\ln 4\pi-\gamma_E\right)c^0_{00}\right]\right.\\ &\left.+\frac{\boldsymbol{q}_2^2}{m_c^2}\left[c_{01}+\frac{4\alpha_s C_F}{3\pi}\left(\frac{\mu_r^2}{\mu_\Lambda^2}\right)^\epsilon\left(\frac{1}{\epsilon}+\ln 4\pi-\gamma_E\right)c^0_{00}\right]\right\}\\ &\times 192(N_c E_{q_1}E_{q_2})^2. \quad (8)\end{aligned}$$

3 Details of perturbative QCD calculation

For a $Q(p)\bar{Q}(\bar{p})$ quantum state, we denote P as the total momentum and q as the relative momentum between Q and $\bar{Q}$ pair. Therefore, there are

$$\begin{aligned}p &= \frac{1}{2}P+q,\ \ \bar{p}=\frac{1}{2}P-q.\\ p^2 &= \bar{p}^2=m_Q^2,\ \ P^2=4E_q^2,\ \ E_q=\sqrt{m_Q^2+\boldsymbol{q}^2}, \quad (9)\end{aligned}$$

where m_Q is the mass of the heavy quark Q, and the Q and $\bar{Q}$ are on their mass shells.

To do the perturbative calculation in related process for the quantum states, we should obtain the projectors for each quantum states. The spin-singlet and spin-triplet components of each $Q\bar{Q}$ state can be projected out by making use of the spin projectors. After multiplying corresponding Clebsch-Gordan coefficients to the spin component of the outer product of the spinors for each $Q\bar{Q}$ pair, we give the expressions of $\bar{\Pi}_1$ and $\bar{\Pi}_3$, respectively, which are the spin-singlet and spin-triplet projectors of the $Q\bar{Q}$ production. The spin projectors that are valid to all orders in the relative momentum can be found in Ref. [34].

$$\begin{aligned}\Pi_1 &= \frac{1}{4\sqrt{2}E(E+m_Q)}(\not{\bar{p}}-m_Q)\gamma_5(\not{P}+2E)(\not{p}+m_Q),\\ \Pi_3 &= \frac{1}{4\sqrt{2}E(E+m_Q)}(\not{\bar{p}}-m_Q)\not{\epsilon}^*(\lambda)(\not{P}+2E)(\not{p}+m_Q),\end{aligned} \quad (10)$$

where $\epsilon^*(\lambda)$ is the polarization vector of the spin-triplet state. For process

$$\begin{aligned}&e^+(p_1)e^-(p_2)\to Q\left(\frac{p_3}{2}-q_1\right)\bar{Q}\left(\frac{p_3}{2}+q_1\right)(^3S_1^1)\\ &+Q\left(\frac{p_4}{2}-q_2\right)\bar{Q}\left(\frac{p_4}{2}-q_2\right)(^1S_0^1),\end{aligned}$$

the Feyman amplitude of the production is expressed as

$$\mathcal{M}\left(\mathrm{e^+e^-}\to \mathrm{Q\bar{Q}}(^3S_1^1)+\mathrm{Q\bar{Q}}(^1S_0^1)\right)=\epsilon_\mu(S_z)A^\mu(q_1,q_2)$$

$$= \epsilon_\mu(S_z)\left(A^\mu\Big|_{q_1=0,q_2=0}+\frac{\boldsymbol{q}_1^2}{2(D-1)}I^{\alpha\beta}\frac{\mathrm{d}^2A^\mu}{\mathrm{d}q_1^\alpha \mathrm{d}q_1^\beta}\Big|_{q_1=0,q_2=0}\right.$$

$$\left.+\frac{\boldsymbol{q}_2^2}{2(D-1)}I^{\alpha\beta}\frac{\mathrm{d}^2A^\mu}{\mathrm{d}q_2^\alpha \mathrm{d}q_2^\beta}\Big|_{q_1=0,q_2=0}\right)+\mathcal{O}(q_1^4,q_2^4), \qquad (11)$$

where we have used the following relation

$$\int\frac{\mathrm{d}\Omega}{4\pi}q^\mu=0,\int\frac{\mathrm{d}\Omega}{4\pi}q^\mu q^\nu=\frac{\boldsymbol{q}^2}{D-1}I^{\mu\nu},I^{\alpha\beta}=-g^{\alpha\beta}+\frac{P^\alpha P^\beta}{P^2}.$$

As for the expansion of q, we should consider the effect that the external momentum and polarization vector may be the implicit function of q. From the momentum conservation and on-shell conditions, $p_3^2=4E_{q_1}^2$, $p_4^2=4E_{q_2}^2$, we know that p_3, p_4 are implicit functions of q_1, q_2, respectively. However, it is obvious that the short-distance coefficients, to be obtained in the perturbative calculation, are functions of the independent variables, which are the invariant mass s of the e^+ and e^- system and $\cos\theta$. θ is the angle between J/ψ and the electron. That is to say, s and $\cos\theta$ are independent of the relative momentum q. So the relation between the external momentum and q can be given.

Since the final results are Lorentz invariance and irrelevant to the reference frame, we choose to do the calculation in the center-of-mass of this system, where $p_1+p_2=p_3+p_4=(\sqrt{s},\ 0,\ 0,\ 0)$ is the explicit expression of the momentum conservation. Therefore, the following results are obtained:

$$\frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}\cdot p_3=2,\ \ \frac{\mathrm{d}p_4}{\mathrm{d}\boldsymbol{q}_1^2}\cdot p_4=0,\ \ \frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}+\frac{\mathrm{d}p_4}{\mathrm{d}\boldsymbol{q}_1^2}=0,\ \ \frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}\cdot p_4=0. \quad (12)$$

We choose two vectors $r_1=(0,\overrightarrow{r_1})$ and $r_2=(0,\overrightarrow{r_2})$ with $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ being unit vectors, while $\overrightarrow{r_1}$, $\overrightarrow{r_2}$ and $\overrightarrow{p_3}$ are perpendicular to each other; that is, $r_1r_2=0,p_3r_1=0,p_3r_2=0$. Then, vector $\frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}$ can be expressed as linear combination of four independent vectors as $\frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}=a_1p_3+a_2p_4+a_3r_1+a_4r_2$. From the following conditions

$$\frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}\cdot r_1=0,\ \ \frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}\cdot r_2=0 \qquad (13)$$

together with previous conditions in Eq. (12), we can easily obtain the solution

$$\frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}=\frac{-2p_4^2}{(p_3\cdot p_4)^2-p_3^2p_4^2}p_3+\frac{2p_3\cdot p_4}{(p_3\cdot p_4)^2-p_3^2p_4^2}p_4. \qquad (14)$$

For the $\epsilon^*(\lambda)$, the polarization four-vector of the $|\mathrm{Q\bar{Q}}(^3S_1)\rangle$ with helicity λ, there are the relation $\frac{\mathrm{d}\epsilon^*(\pm1)}{\mathrm{d}\boldsymbol{q}_1^2}=0$ since θ is independent of the relative momentum q. It is easy to obtain

$$\frac{\mathrm{d}\epsilon^*(0)}{\mathrm{d}\boldsymbol{q}_1^2}\cdot p_3 = -\frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}\cdot\epsilon^*(0),\ \ \frac{\mathrm{d}\epsilon^*(0)}{\mathrm{d}\boldsymbol{q}_1^2}\cdot\epsilon^*(0)=0,$$

$$\frac{\mathrm{d}\epsilon^*(0)}{\mathrm{d}\boldsymbol{q}_1^2}\cdot\epsilon^*(1) = 0,\ \ \frac{\mathrm{d}\epsilon^*(0)}{\mathrm{d}\boldsymbol{q}_1^2}\cdot\epsilon^*(-1)=0. \qquad (15)$$

Therefore, we obtain the relation between the polarization four-vector and q as

$$\frac{\mathrm{d}\epsilon^*(\lambda)}{\mathrm{d}\boldsymbol{q}_1^2}=\frac{-\frac{\mathrm{d}p_3}{\mathrm{d}\boldsymbol{q}_1^2}\cdot\epsilon^*(\lambda)p_3}{p_3^2}=\frac{-2p_3\cdot p_4p_4\cdot\epsilon^*(\lambda)p_3}{((p_3\cdot p_4)^2-p_3^2p_4^2)p_3^2}. \qquad (16)$$

The treatment of q_2 is similar to these.

We should also expand the two body phase space.

$$\mathrm{d}\varGamma=\int d\cos\theta\frac{2|\overrightarrow{p'}|}{16\pi\sqrt{s}},$$

where $|\overrightarrow{p'}|=\frac{\lambda^{1/2}(s,4E_{q_1}^2,4E_{q_2}^2)}{2\sqrt{s}}$, $\lambda(x,y,z)=x^2+y^2+z^2-2(xy+yz+xz)$. Only $|\overrightarrow{p'}|$ needs to be expanded since $\cos\theta$ and s are independent of q_1 and q_2. Then, there is

$$\mathrm{d}\varGamma=\int d\cos\theta\frac{2|\overrightarrow{p}|}{16\pi\sqrt{s}}\left(1-\frac{1}{|\overrightarrow{p}|^2}(\boldsymbol{q}_1^2+\boldsymbol{q}_2^2)\right),$$

where $|\overrightarrow{p}|=\frac{\lambda^{1/2}(s,4m_\mathrm{c}^2,4m_\mathrm{c}^2)}{2\sqrt{s}}$. After we square the amplitude, integrate over the phase space, and expand in powers of q, the desired perturbative result to v^2 is obtained.

$$\sigma(\mathrm{e^+e^-}\to\mathrm{Q\bar{Q}}(^3S_1^1)+\mathrm{Q\bar{Q}}(^1S_0^1))\Big|_{\mathrm{pertQCD}}=\int\mathrm{d}\varGamma\sum_{s_z}\Big|\mathcal{M}\Big|^2. \qquad (17)$$

Most of the steps in this section are realized in a small program in the FDC package, and the final Fortran source for numerical calculation is prepared by using the FDC package together with a small program for $\boldsymbol{q}^2$ expansion.

Since there is no $\mathcal{O}(\alpha_\mathrm{s})$ real process in NLO, we only need to calculate virtual corrections. Dimensional regularization has been adopted for isolating the ultraviolet (UV) and infrared (IR) singularities. UV divergences are cancelled upon the renormalization of the QCD gauge coupling constant, the charm quark mass and field, and the gluon field. A similar renormalization scheme is chosen as the Ref. [35], except that both light quarks and charm quark are included in the quark loop to obtain the renormalization constants. The renormalization constants of the charm quark mass Z_m and field Z_2, and the gluon field Z_3 are defined in the on-mass-shell (OS) scheme while that of the QCD gauge coupling Z_g is defined in the modified-minimal-subtraction ($\overline{\rm MS}$) scheme:

$$\delta Z_{\rm m}^{\rm OS} = -3C_F\frac{\alpha_{\rm s}}{4\pi}\left[\frac{1}{\epsilon_{UV}}-\gamma_{\rm E}+\ln\frac{4\pi\mu^2}{m_{\rm c}^2}+\frac{4}{3}+\mathcal{O}(\epsilon)\right],$$

$$\delta Z_2^{\rm OS} = -C_{\rm F}\frac{\alpha_{\rm s}}{4\pi}\times\left[\frac{1}{\epsilon_{\rm UV}}+\frac{2}{\epsilon_{\rm IR}}-3\gamma_{\rm E}+3\ln\frac{4\pi\mu^2}{m_{\rm c}^2}+4+\mathcal{O}(\epsilon)\right],$$

$$\delta Z_3^{\rm OS} = \frac{\alpha_{\rm s}}{4\pi}\left[(\beta_0'-2C_A)\left(\frac{1}{\epsilon_{\rm UV}}-\frac{1}{\epsilon_{\rm IR}}\right)-\frac{4}{3}T_{\rm F}\left(\frac{1}{\epsilon_{\rm UV}}-\gamma_{\rm E}+\ln\frac{4\pi\mu^2}{m_{\rm c}^2}\right)+\mathcal{O}(\epsilon)\right],$$

$$\delta Z_{\rm g}^{\overline{\rm MS}} = -\frac{\beta_0}{2}\frac{\alpha_{\rm s}}{4\pi}\left[\frac{1}{\epsilon_{\rm UV}}-\gamma_{\rm E}+\ln(4\pi)+\mathcal{O}(\epsilon)\right], \qquad (18)$$

where $\gamma_{\rm E}$ is Euler's constant, $\beta_0=\frac{11}{3}C_{\rm A}-\frac{4}{3}T_{\rm F}n_{\rm f}$ is the one-loop coefficient of the QCD beta function and $n_{\rm f}$ is the number of active quark flavors. There are three massless light quarks u, d, s, and one heavy quark c, so $n_{\rm f}=4$. In $SU(3)_{\rm c}$, color factors are given by $T_{\rm F}=\frac{1}{2}$, $C_{\rm F}=\frac{4}{3}$, $C_{\rm A}=3$. And $\beta_0'\equiv\beta_0+(4/3)T_{\rm F}=(11/3)C_{\rm A}-(4/3)T_{\rm F}n_{\rm lf}$ where $n_{\rm lf}\equiv n_{\rm f}-1=3$ is the number of light quarks flavors. Actually, in the NLO total amplitude level, the terms proportion to $\delta Z_3^{\rm OS}$ cancel each other, thus the result is independent of the renormalization scheme of the gluon field.

4 Results

The final results are obtained by using the matching method with the UV and IR divergences being cancelled.

$$\sigma=\sigma_{\rm LO}+\sigma_{{\rm NLO}(\alpha_{\rm s})}+\sigma_{{\rm NLO}(v^2)}+\sigma_{{\rm NLO}(\alpha_{\rm s}v^2)}, \qquad (19)$$

$\sigma_{\rm LO}$, $\sigma_{{\rm NLO}(\alpha_{\rm s})}$, $\sigma_{{\rm NLO}(v^2)}$, $\sigma_{{\rm NLO}(\alpha_{\rm s}v^2)}$ are the contributions from the leading order, the next leading order in $\alpha_{\rm s}$, the next leading in v^2 and the next leading in $\alpha_{\rm s}v^2$. Then, the production cross section up to $\mathcal{O}(\alpha_{\rm s}v^2)$ order is expressed as

$$\sigma = \frac{8192\pi^3C_{\rm F}^2e_{\rm c}^2\alpha_{\rm s}^2(u_r)\alpha^2(1-4r)^{3/2}}{9N_{\rm c}^2s^4}\langle\mathcal{O}_1\rangle_{\eta_{\rm c}}\langle\mathcal{O}_1\rangle_{{\rm J}/\psi}$$
$$\times\Bigg\{1+v_{{\rm J}/\psi}^2f_1(r)+v_{\eta_{\rm c}}^2f_2(r)+\frac{\alpha_{\rm s}(\mu_r)}{\pi}\left[\beta_0\ln\frac{\mu_r}{2m_{\rm c}}+f_3(r)\right]+\frac{\alpha_{\rm s}(\mu_r)}{\pi}v_{{\rm J}/\psi}^2\left[\beta_0\ln\frac{\mu_r}{2m_{\rm c}}f_1(r)+\frac{32}{9}\ln\frac{\mu_\Lambda}{2m_{\rm c}}+f_4(r)\right]+\frac{\alpha_{\rm s}(\mu_r)}{\pi}v_{\eta_{\rm c}}^2$$
$$\times\left[\beta_0\ln\frac{\mu_r}{2m_{\rm c}}f_2(r)+\frac{32}{9}\ln\frac{\mu_\Lambda}{2m_{\rm c}}+f_5(r)\right]\Bigg\}, \qquad (20)$$

where there are $e_{\rm c}=\frac{2}{3}$, $r=\frac{4m_{\rm c}^2}{s}$, $f_1(r)=\frac{9-74r+80r^2}{6(1-4r)}$, $f_2(r)=\frac{11-82r+80r^2}{6(1-4r)}$ and μ_r is the renormalization scale. Therefore, the obtained analytic expressions of the v^2 correction are in agreement with that in the Ref. [31]. At the same time, the analytic expression of $f_3(r)$ in the results of the $\alpha_{\rm s}$ correction is also in agreement with that in Ref. [22]. Since the analytic expressions of $f_4(r)$ and $f_5(r)$ in the $\mathcal{O}(\alpha_{\rm s}v^2)$ correction are so lengthy, we just give their numerical results. In the numerical calculation, there are

$$f_1 = 0.97466,\ f_2=1.3080,\ f_3=12.358,\ f_4=3.8382,$$
$$f_5 = 3.2537,\ \text{for } r=\frac{4\times1.4^2}{10.58^2};$$
$$f_1 = 0.87643,\ f_2=1.2098,\ f_3=11.806,\ f_5=2.0543,$$
$$f_4 = 2.6668,\ \text{for } r=\frac{4\times1.5^2}{10.58^2}.$$

We take $\sqrt{s}=10.58$ GeV and $\mu_\Lambda=m_{\rm c}$. The running strong coupling constant is evaluated by using the two-loop formula with $\Lambda_{\overline{\rm MS}}^{(4)}=0.338$ GeV, as used in Ref. [22]. Our results, presented in Table 1 with parameters given in table caption, are in agreement with that in Ref. [31], in which the contribution from the $\mathcal{O}(\alpha_{\rm s}v^2)$ order is small. In the Table 2, we also present results that are a little bit different by using the long-distance matrices and QED coupling constant chosen in Ref [4], and we find that the correction at $\mathcal{O}(\alpha_{\rm s}v^2)$ order is also small. We also give the renormalization scale μ_r dependence of the cross sections in Fig. 1. There is about 10 percent difference in the total cross sections between

Table 1. With the follow parameters: $\alpha(\sqrt{s})=1/130.9$, $\langle\mathcal{O}_1\rangle_{{\rm J}/\psi}=1.161$ GeV^3, $\langle\mathcal{O}_1\rangle_{\eta_{\rm c}}=0.387$ GeV^3, $m_{\rm c}=1.4$ GeV, $\langle v^2\rangle_{{\rm J}/\psi}=0.223$, $\langle v^2\rangle_{\eta_{\rm c}}=0.133$, we give the cross sections with different renormalization scale μ and $\sqrt{s}$=10.58. Their units are fb.

$\alpha_{\rm s}(\mu_r)$	$\sigma_{\rm LO}$	$\sigma_{{\rm NLO}(\alpha_{\rm s})}$	$\sigma_{{\rm NLO}(v^2)}$	$\sigma_{{\rm NLO}(\alpha_{\rm s}v^2)}$	σ
$\alpha_{\rm s}\left(\frac{\sqrt{s}}{2}\right)=0.211$	4.381	5.196	1.714	0.731	12.023
$\alpha_{\rm s}(2m_{\rm c})=0.267$	7.0156	7.368	2.745	0.245	17.374

Table 2. In the follow parameters: $\alpha(\sqrt{s})=1/137$, $\langle\mathcal{O}_1\rangle_{J/\psi}=1.719\ \mathrm{GeV}^3$, $\langle\mathcal{O}_1\rangle_{\eta_c}=0.432\ \mathrm{GeV}^3$, $\langle v^2\rangle_{J/\psi}=0.090$, $\langle v^2\rangle_{\eta_c}$=0.119. We give the cross sections with different m and renormalization scale μ_r and $\sqrt{s}$=10.6 GeV.

m	$\alpha_s(\mu_r)$	σ_{LO}	$\sigma_{\mathrm{NLO}(\alpha_s)}$	$\sigma_{\mathrm{NLO}(v^2)}$	$\sigma_{\mathrm{NLO}(\alpha v^2)}$	σ
1.5	$\alpha_s\left(\frac{\sqrt{s}}{2}\right)=0.211$	5.973	6.645	1.335	0.416	14.369
1.5	$\alpha_s(2m_c)=0.259$	9.000	8.771	2.011	−0.017	19.766
1.4	$\alpha_s\left(\frac{\sqrt{s}}{2}\right)=0.211$	6.526	7.754	1.591	0.667	16.538
1.4	$\alpha_s(2m_c)=0.267$	10.450	10.989	2.548	0.1989	24.185

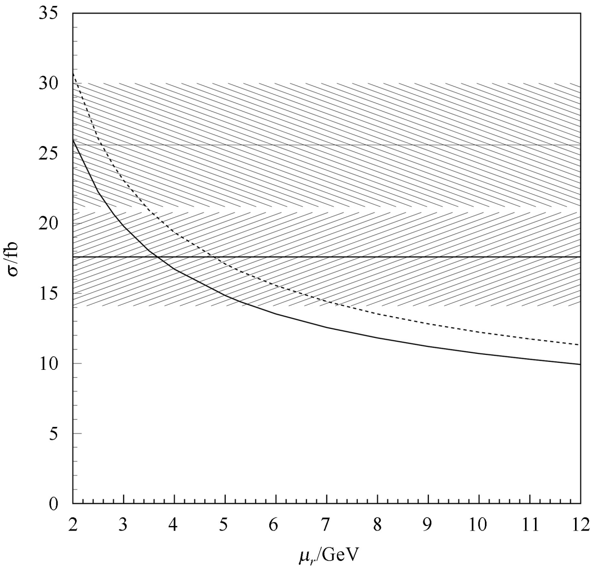

Fig. 1. The cross section as a function of the μ_r at $\sqrt{s}$=10.58 GeV. The dashed and solid curves are the cross sections in the m_c=1.4 and m_c=1.5 respectively. The above and under reseau bands represent the measured cross sections by the Belle and BABAR experiments, with respective systematic and statistical errors.

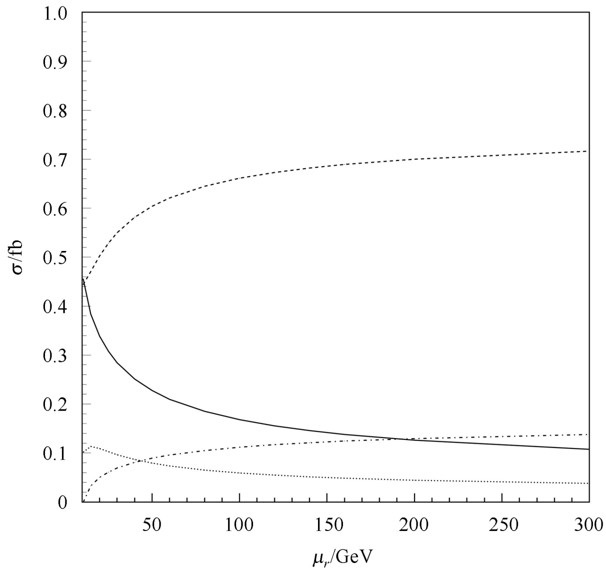

Fig. 2. The ratios of the different parts as a function of the $\sqrt{s}$. The solid,dashed,dotted,dot-dashed lines are the fraction of the leading order, α_s order, v^2 order, $\alpha_s v^2$ order contribution, respectively.

m_c=1.5 and m_c=1.4, which shows that the uncertainty of the total cross sections from m_c is not small. If we choose $\mu_r=2m_c$ and $m_c=1.5$, we present the e^+e^- center-mass-energy $\sqrt{s}$ dependence of the fraction of different parts in total cross section in Fig. 2. We find that the contributions from $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s v^2)$ become important and that the one from LO becomes small when $\sqrt{s}$ is large, and the maximum fraction from $\mathcal{O}(\alpha_s v^2)$ is about 14 percent. At the same time, the asymptotic behavior of the Fig. 2 in the limit $\sqrt{s}\gg m_c$ is agreement with the analytic expressions in Ref. [31].

5 Summary

In this work we have calculated the $\mathcal{O}(\alpha_s v^2)$ correction in detail for the processes $e^+e^-\to J/\psi+\eta_c$ within the frame of NRQCD. The result at $\mathcal{O}(\alpha_s v^2)$ order gives about 6 percent contribution to the total theoretical prediction, while the $\mathcal{O}(\alpha_s)$ correction and $\mathcal{O}(v^2)$ are about 40 percent and 14 percent, respectively. This indicates that the convergence in the double perturbative expansions in QCD α_s and relativistic v^2 are very good for the theoretical calculation on the production rate of the process $e^+e^-\to J/\psi+\eta_c$. Up to $O(\alpha_s v^2)$ order, the theoretical prediction with reasonable variations of the charm quark mass and renormalization scale can describe the experimental measurement.

References

1 Brambilla N, Eidelman S, Heltsley B et al. Eur. Phys. J. C, 2011, **71**: 1534
2 Bodwin G T, Braaten E, Lepage G P. Phys. Rev. D, 1995, **51**: 1125
3 Campbell J, Maltoni F, Tramontano F. Phys. Rev. Lett., 2007, **98**: 252002; GONG Bin, WANG Jian-Xiong. Phys. Rev. Lett., 2008, **100**: 232001; GONG Bin, LI Xue-Qian, WANG Jian-Xiong. Phys. Lett. B, 2009, **673**: 197; MA Yan-Qing, WANG Kai, CHAO Kuang-Ta. Phys. Rev. Lett., 2011, **106**: 042002; Butenschoen M, Kniehl B A. Phys. Rev. Lett., 2011, **106**: 022003; KANG Zhong-Bo, QIU Jian-Wei, Sterman G. Phys. Rev. Lett., 2012, **108**: 102002; Butenschoen M, Kniehl B A. Phys. Rev. Lett., 2012, **108**: 172002; CHAO Kuang-Ta, MA Yan-Qing, SHAO Hua-Sheng et al. Phys. Rev. Lett., 2012, **108**: 242004; GONG Bin, WAN Lu-Ping, WANG Jian-Xiong et al. Phys. Rev. Lett., 2012, **110**: 042002
4 HE Zhi-Guo, FAN Ying, CHAO Kuang-Ta. Phys. Rev. D, 2007, **75**: 074011
5 ZHANG Yu-Jie, CHAO Kuang-Ta. Phys. Rev. Lett., 2007, **98**: 092003; MA Yan-Qing, ZHANG Yu-Jie, CHAO Kuang-Ta. Phys. Rev. Lett., 2009, **102**: 162002; GONG Bin, WANG Jian-Xiong. Phys. Rev. Lett., 2009, **102**: 162003; GONG Bin, WANG Jian-Xiong. Phys. Rev. D, 2009, **80**: 054015; HE Zhi-Guo, FAN Ying, CHAO Kuang-Ta. Phys. Rev. D, 2010, **81**: 054036; JIA Yu. Phys. Rev. D, 2010, **82**: 034017
6 ZHANG Yu-Jie, MA Yan-Qing, WANG Kai et al. Phys. Rev. D, 2010, **81**: 034015
7 Luke M E, Savage M J. Phys. Rev. D, 1998, **57**: 413
8 GUO Huai-Ke, MA Yian-Qing, CHAO Kuang-Ta. Phys. Rev. D, 2011, **83**: 114038
9 YU JIA, YANG Xiu-Ting, SANG Wen-Long et al. JHEP, 2011, **1106**: 097
10 Abe K et al. (Belle collaboration). Phys. Rev. Lett., 2002, **89**: 142001
11 Pakhlov P (Belle). arXiv:hep-ex/0412041
12 Aubert B et al. (BABAR collaboration). Phys. Rev. D, 2005, **72**: 031101
13 Braaten E, Lee J. Phys. Rev. D, 2003, **67**: 054007
14 LIU Kui-Yong, HE Zhi-Guo, CHAO Kuang-Ta. Phys. Lett. B, 2003, **557**: 45
15 Hagiwara K, Kou E, QIAO Cong-Feng. Phys. Lett. B, 2003, **570**: 39
16 MA Jian-Ping. Phys. Rev. D, 2004, **70**: 074007
17 Bondar A E, Chernyad V L. Phys. Lett. B, 2005, **612**: 215
18 Bodwin G T, Kang D, Lee J. Phys. Rev. D, 2006, **74**: 114028
19 ZHANG Yuan-Jiang, ZHAO Qiang, QIAO Cong-Feng. Phys. Rev. D, 2008, **78**: 054014
20 GUO Xin-Heng, KE Hong-Wei, LI Xue-Qian et al. arXiv:0804.0949 [hep-ph]
21 ZHANG Yu-Jie, GAO Ying-jia, CHAO Kuang-Ta. Phys. Rev. Lett, 2006, **96**: 092001
22 GONG Bin, WANG Jian-Xiong. Phys. Rev. D, 2008, **77**: 054028
23 Bodwin G T, Kang D, Kim T et al. AIP Conf. Proc., 2007, **892**: 315
24 Bodwin G T, Lee J, Yu C. Phys. Rev. D, 2008, **77**: 094018
25 Bodwin G T, Lee J, Braaten E. Phys. Rev. Lett., 2003, **90**: 162001
26 Abe K et al. (Belle collaboration). Phys. Rev. D, 2004, **70**: 071102
27 Bodwin G T, Lee J, Braaten E. Phys. Rev. D, 2003, **67**: 054023
28 Bodwin G T, Braaten E, Lee J et al. Phys. Rev. D, 2006, **74**: 074014
29 Braguta V V. arXiv:0712.1475 [hep-ph]
30 GONG Bin, WANG Jian-Xiong. Phys. Rev. Lett., 2008, **100**: 181803
31 DONG Hai-Rong, FENG Feng, JIA Yu. Phys. Rev. D, 2012, **85**: 114018
32 WANG Jian-Xiong. Nucl. Instrum. Methods A, 2004, **534**: 241
33 Bodwin G T, CHEN Yun-Qi. Phys. Rev. D, 1999, **60**: 054008
34 Bodwin G T, Petrelli A. Phys. Rev. D, 2002, **66**: 094011
35 Klasen M, Kniehl B, Mihaila L, Steinhauser M. Nucl. Phys. B, 2005, **713**: 487