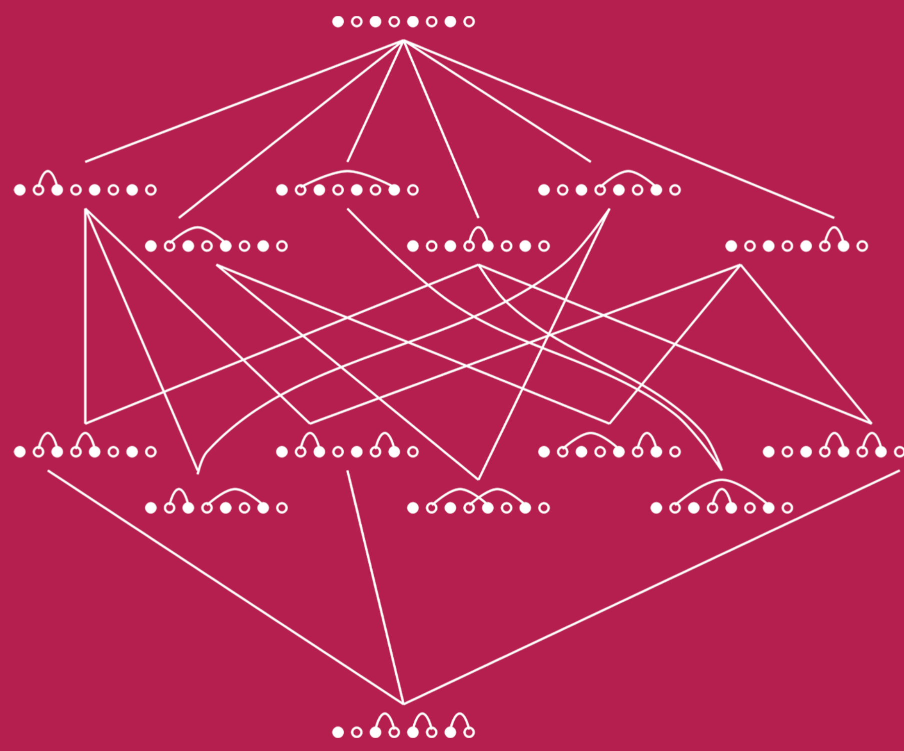
DISCRETE MATHEMATICS AND ITS APPLICATIONS

Commutation Relations, Normal Ordering, and Stirling Numbers



$$(\hat{a}^\dagger \hat{a})^4 = S(4,4)(\hat{a}^\dagger)^4 \hat{a}^4 + S(4,3)(\hat{a}^\dagger)^3 \hat{a}^3 + S(4,2)(\hat{a}^\dagger)^2 \hat{a}^2 + S(4,1)\hat{a}^\dagger \hat{a}$$

Toufik Mansour • Matthias Schork

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Preface

Very early in my study of physics, Weyl became one of my gods. I use the word "god" rather than, say, "outstanding teacher" for the ways of gods are mysterious, inscrutable, and beyond the comprehension of ordinary mortals.

Julian Schwinger

This book gives an introduction to combinatorial aspects of normal ordering in the Weyl algebra and some of its close relatives. For our considerations, the *Weyl algebra* is the complex algebra generated by two letters U and V (with unit I) satisfying the *commutation relation*

$$UV - VU = I.$$

A concrete representation is given by the operators $D = \frac{d}{dx}$ and X, where $(Xf)(x) = xf(x)$ for any function f. In this representation, the noncommutative nature of D and X was recognized by the pioneers of calculus. *Normal ordering* a word in D and X means to bring it, using the commutation relation, into a form where all operators D stand to the right. For example, $(XD)^2 = X^2D^2 + XD$. Presumably, Scherk in 1823 was the first to explicitly normal order $(XD)^n$ (and a few other words). The coefficients which appear upon normal ordering are the *Stirling numbers of the second kind*. However, Scherk did not recognize the coefficients he determined as the numbers Stirling had considered in a different context. In the middle of the 19th century, many – mostly formal – results were derived in the *operational* or *symbolical calculus*, often in connection with special polynomials (this line of research was revived in the 1970s, in particular after Rota's work on *finite operator calculus*, a modern incarnation of *umbral calculus*). Later, noncommutative structures – first in the form of Lie algebras and Lie groups as well as in the emerging abstract algebra – rose to a central place in mathematics, where they have stayed ever since.

In the physical discourse of this time, noncommutative structures per se played no role. This changed suddenly when Heisenberg postulated in 1925 the fundamental commutation relation

$$\mathsf{pq} - \mathsf{qp} = -i\hbar\mathsf{1}$$

for the physical observables representing momentum and location, and where $\hbar$ denotes Planck's constant. (In fact, the postulate in this form was written in a follow-up publication by Born, Heisenberg, and Jordan.) Thus, the basic structure of this "matrix version" of *quantum mechanics* is the Weyl algebra. Since then, noncommutative structures pervade theoretical physics. One particularly important toy model is the *harmonic oscillator*. To describe it, one makes use of the *creation operator* $\hat{a}^\dagger$ and the *annihilation operator* $\hat{a}$. These two operators also satisfy the commutation relation of the Weyl algebra. Since the harmonic oscillator describes the first-order deviation from equilibrium it is an important model, and its properties can be applied in many different situations. From the practical point of view, where one has to determine expectation values of operator functions in $\hat{a}^\dagger$ and $\hat{a}$, it is advantageous to write them in normal ordered form where all operators $\hat{a}$ stand to the right. One way to achieve this is to use *Wick's theorem*, which expresses the normal ordered form of an operator function as a sum over all its possible "contractions". In this context, Katriel discovered in 1974 that by normal ordering powers of the *number operator* $\hat{n} = \hat{a}^\dagger \hat{a}$, the Stirling numbers appear as normal ordering coefficients, thereby rediscovering Scherk's result in a physical context. Until this seminal work, individual research papers on various aspects of normal ordering appeared, but there was no focused research interest. This changed when several groups of authors developed new research directions. They studied normal and antinormal ordering and its connections to combinatorics, for example, set partitions, lattice paths, and rooks. Other focuses were on the coefficients that appear in normal ordered forms and on their applications. More generally, the new academic discipline "combinatorial physics" (even "physical combinatorics" is used) has emerged, devoted to the interplay of combinatorics and physics. One particular aspect has been the study of "q-deformed" structures, which began in the mid 1990s. Roughly speaking, a structure gets q-deformed by introducing a parameter q into its defining relation (such that for $q \to 1$ the defining relation of the undeformed structure is recovered). For example, the relation

$$UV - qVU = I$$

defines the *q-deformed Weyl algebra*. In the physical context, the creation and annihilation operator of a *q-boson* satisfy this commutation relation, and normal ordering these operators is beneficial in diverse physical applications. However, the extension of normal ordering results to the q-deformed situation is not always straightforward.

In this book we give an introduction to the topics mentioned above. The Stirling numbers, some closely related generalizations, and their role in normal and antinormal ordering are discussed. We also consider several variants of the Weyl algebra, all of which are special cases of the algebra generated by letters U and V satisfying the commutation relation

$$UV - qVU = hV^s.$$

We describe combinatorial aspects of these algebras and of normal ordering words in the letters U and V. In addition to the combinatorial aspects, we describe the relation to operational calculus. Also, the physical motivation as well as some physical applications are sketched. To give a comprehensive account of this field of research and some of its ramifications, many additional topics are treated in remarks (or problems). Even if the subject looks rather focused, many connections to different mathematical objects are mentioned. A similar study of algebras generated by three generators would be much more ambitious.

Although it is impossible to give an exhaustive or complete bibliography, we strive to provide a comprehensive bibliography with many references to original publications (but, alas, neither of us is a historian). We also indicate some of the early historical development of Stirling and Bell numbers.

The later chapters of this book are based on our own research and on that of our collaborators and other researchers in the field. We present these results with consistent notation and we have modified some proofs to relate them to other results in the book. As a general rule, results listed without specific references either are well-known and presented in standard references mentioned, or give results from articles by the authors and their collaborators, while results from other authors are given with specific references.

Audience

The book is intended for advanced undergraduate and graduate students in discrete mathematics as well as for graduate students or researchers in physics interested in combinatorial aspects of normal ordering operators. Additionally, the book serves as a one-stop reference for a bibliography of research activities on the subject, known results, and research directions for any researcher who is interested in studying this topic.

Outline

In *Chapter 1* we present a historical perspective of the research on normal ordering and Stirling numbers and give an overview of the major themes of the book: Stirling and Bell numbers as well as generalizations thereof; the Weyl algebra, quantum theory, and normal ordering; the q-deformed Weyl algebra and the meromorphic Weyl algebra; the q-deformed generalized Weyl algebra.

In *Chapter 2* we introduce techniques to solve recurrence relations, which arise naturally when dealing with normal ordering and Stirling numbers, and illustrate them with several examples. We also provide definitions and combinatorial techniques that are used later on, such as lattice paths, partitions, Ferrers boards, rooks, Riordan arrays, and Sheffer sequences.

In *Chapter 3* we recall the definition and basic properties of the classical Stirling and Bell numbers. Furthermore, we discuss the Dobiński formula as well as Spivey's Bell number relation. Also, a q-deformation of Stirling and Bell numbers is introduced and several of its properties are discussed.

In *Chapter 4* we consider several generalizations of Stirling and Bell numbers. The starting point for generalizations are the operational interpretation of Stirling numbers and their interpretation as connection coefficients. We survey many properties of these generalizations. Connections between different versions of generalized Stirling numbers are mentioned.

In *Chapter 5* we define the Weyl algebra and mention some of its early history. The main focus of the chapter is on elementary quantum theory and some of its consequences. We show why the Weyl algebra is of interest to physicists and discuss the operator ordering problem of "quantization". The harmonic oscillator is discussed and the creation and annihilation operators are introduced. Several examples for normal ordering are presented.

In *Chapter 6* we continue the study of normal ordering in the Weyl algebra and collect many results. In addition, we discuss Viskov's identity, the connection of normal ordering to rook numbers, an identity of Bender, Mead, and Pinsky, and Wick's theorem. Connections between normal ordering and further combinatorial structures are mentioned and a survey of other operator ordering schemes is given.

In *Chapter 7* we consider normal ordering in three variants of the Weyl algebra: the q-deformed Weyl algebra ($UV-qVU = h$), the meromorphic Weyl algebra ($UV-VU = hV^2$), and the q-deformed meromorphic Weyl algebra ($UV - qVU = hV^2$). To warm up, we begin with a brief discussion of the quantum plane ($UV = qVU$).

In *Chapter 8* we introduce a generalization of the Weyl algebra where one has $UV-VU = hV^s$. After discussing some general aspects of normal ordering, we introduce generalized

Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ as normal ordering coefficients of $(VU)^n$. The properties of these numbers – and of the corresponding generalized Bell numbers $\mathfrak{B}_{s;h}(n)$ – are investigated in detail.

In *Chapter 9* we extend the results of the previous chapter to variables U and V satisfying $UV - qVU = hV^s$. We discuss the binomial formula for $(U+V)^n$, and we describe other "noncommutative binomial formulas" and "noncommutative Bell polynomials". Also, we define associated q-deformed generalized Stirling numbers $\mathfrak{S}_{s;h|q}(n,k)$ as normal ordering coefficients of $(VU)^n$ and present several properties of these numbers.

In *Chapter 10* we study a generalization of the Touchard polynomials which is motivated by its connection to normal ordering and the generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ and Bell numbers $\mathfrak{B}_{s;h}(n)$.

The *Appendices* provide basic background from different areas of mathematics, namely, q-calculus, symmetric functions, graph theory, Lie algebras, and Hilbert spaces.

Most chapters start with a section describing the history of the particular topic and its relation to previous chapters. New methods and definitions are illustrated with examples. At the end of each chapter we present some exercises and research problems.

Acknowledgment

Every book has a story of how it came into being and the people who supported the author(s) along the way. This book is no exception. The origin of this project might be located several years ago when Simone Severini suggested Toufik and Matthias join forces and study combinatorial aspects of normal ordering as a team, bringing, in particular, Toufik and Matthias into contact with each other. Matthias had been working in mathematical physics and was turning to questions of normal ordering, while Simone and Toufik had been cooperating on projects having a more combinatorial flavor and were turning to concrete applications in normal ordering. A fruitful collaboration resulted, and the outcomes can be found in the present book. To pursue the idea of introducing generalized Stirling numbers via normal ordering, Toufik and Matthias had the good luck to win Mark Shattuck as a collaborator. This collaboration also proved to be very fertile, and Mark's influence can be felt in many places in this book. These developments prepared the ground, and when Nenad Cakić encouraged Toufik to write a book on Stirling numbers and applications, the idea for the present book was born.

In the first place, we thank our collaborators Simone Severini and Mark Shattuck. Large parts of the material covered in this book are based on joint work with them. A special thanks to Nenad Cakić for giving us the essential nudge to write this book. We wish to express our heartfelt gratitude to Christian Nassau for reading a previous version of this book and his several remarks and suggestions. Arnold Knopfmacher, Armend Sh. Shabani, and David G.L. Wang also read parts of a previous version of this book – thank you! Many thanks to Simone Severini and Jonathan L. Gross for providing us with copies of essential references.

And then there are the people in our lives who supported us on a daily basis by giving us the time and the space to write this book. Their moral support has been very important. Toufik thanks his wife Ronit and his daughters Itar, Atil, and Hadel for their support and understanding when the work on the book took him away from spending time with them. Matthias thanks his entire family for their constant support and understanding.

About the Authors

Toufik Mansour obtained his PhD degree in mathematics from the University of Haifa in 2001. He spent one year as a postdoctoral researcher at the University of Bordeaux (France) supported by a Bourse Chateaubriand scholarship, and a second year at the Chalmers Institute in Gothenburg (Sweden) supported by a European Research Training Network grant. Between 2003–2006, he received a prestigious MAOF grant from the Israeli Council for Education. Toufik has been a permanent faculty member at the University of Haifa since 2003 and was promoted to associate professor in 2008, and to full professor in 2014. He spends his summers as a visitor at institutions around the globe, for example, at the Center for Combinatorics at Nankai University (China) where he was a faculty member from 2004 to 2007, and at The John Knopfmacher Center for Applicable Analysis and Number Theory, University of the Witwatersrand (South Africa). Toufik's area of specialty is enumerative combinatorics and more generally, discrete mathematics and its applications. Originally focusing on pattern avoidance in permutations, he has extended his interest to colored permutations, set partitions, words, compositions, and normal ordering. Toufik has authored or co-authored more than 250 papers in this area, many of them concerning the enumeration of normal ordering. He has given talks at national and international conferences and is very active as a reviewer for several journals.

Matthias Schork obtained his PhD degree in mathematics from the Johann Wolfgang Goethe University of Frankfurt (Germany) in 2001 for work done in mathematical physics. He joined the IT department of Deutsche Bahn – the largest German railway company – in 2002 and still works there. In his spare time he studies recent developments in mathematical physics as well as discrete mathematics and its applications to physics. Originally focusing on topics motivated directly by physical application, he has extended his interest to include more conventional mathematical topics, for example, special differential equations and q-calculus. Matthias has authored or coauthored more than 40 papers in this area, many of them together with Toufik concerning Stirling numbers, normal ordering, and its ramifications. Matthias is also active as a reviewer for several journals.

Chapter 1

Introduction

In this chapter we introduce the main objects of study and describe their early history as well as some later developments. In Section 1.1 the most classical of these objects – set partitions – are introduced and first properties of the corresponding Stirling and Bell numbers are discussed. Several further results are mentioned which will be discussed in later chapters in detail (and from different angles). In Section 1.2 the early history of the formal theory of operational or symbolical calculus is described and several results mentioned. Furthermore, the connection to the physical theory of quantum mechanics is elucidated, thereby motivating the same structure from a physical point of view. In Section 1.3 the "abstract" Weyl algebra and some close relatives are introduced and some of the more recent developments mentioned. Finally, in Section 1.4, the content of the book is described in more detail.

1.1 Set Partitions, Stirling, and Bell Numbers

The first known application of set partitions arose in the context of tea ceremonies and incense games in Japanese upper-class society around 1500. Guests at a Kado ceremony would be smelling cups with burned incense with the goal to either identify the incense or to identify which cups contained identical incense. There are many variations of the game, even today. One particular game is named *genji-ko*, and it is the one that originated the interest in n-set partitions. Five different incense sticks were cut into five pieces, each piece put into a separate bag, and then five of these bags were chosen to be burned. Guests had to identify which of the five were the same. The Kado ceremony masters developed symbols for the different possibilities, so-called *genji-mon*. Each such symbol consists of vertical bars, some of which are connected by horizontal bars. For example, the symbol  indicates that incense 1, 2, and 3 are the same, while incense 4 and 5 are different from the first three and also from each other (recall that the Japanese write from right to left). Fifty-two symbols were created, and for easier memorization, each symbol was identified with one of the chapters of the famous *Tale of Genji* by Lady Murasaki. Figure 1.1 shows the diagrams[1] used in the tea ceremony game. In time, these genji-mon and two additional symbols started to be displayed at the beginning of each chapter of the *Tale of Genji* and in turn became part of numerous Japanese paintings. They continued to be popular symbols for family crests and Japanese kimono patterns in the early 20th century, and can be found on T-shirts sold today.

How does the tea ceremony game relate to set partitions? Before making the connection, let us define what we mean by a set partition in general.

[1] www.viewingjapaneseprints.net/texts/topictexts/artist_varia_topics/genjimon7.html

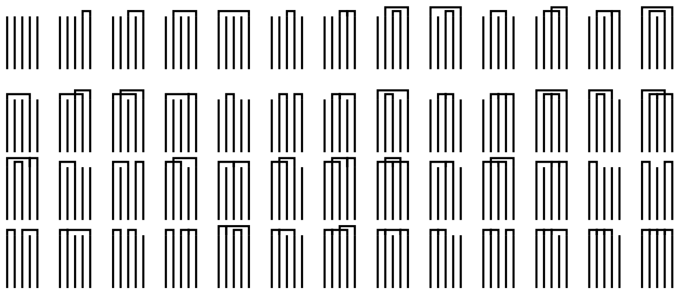


FIGURE 1.1: Diagrams used to represent set partitions in 16th century Japan.

1.1.1 Definition of Stirling and Bell Numbers

In the following S will be a set of natural numbers where 0 is included, that is, $S \subseteq \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For the particular set of the first n natural numbers we use the convenient notation

$$[n] = \{1, 2, 3, \ldots, n\}.$$

Definition 1.1 *A* set partition π *of a set* S *is a collection* $B_1, B_2, \ldots, B_k$ *of nonempty disjoint subsets of* S *such that* $\cup_{i=1}^k B_i = S$. *The elements of a set partition are called* blocks, *and the* size *of a block* B *is given by* $|B|$, *the number of elements in* B. *We assume that* $B_1, B_2, \ldots, B_k$ *are listed in increasing order of their minimal elements, that is,* $\min B_1 < \min B_2 < \cdots < \min B_k$. *The set of all set partitions of* S *is denoted by* $\Pi(S)$.

Note that an equivalent way of representing a set partition is to order the blocks by their maximal element, that is, $\max B_1 < \max B_2 < \cdots < \max B_k$. Unless otherwise noted, we will use the ordering according to the minimal element of the blocks.

Example 1.2 *The set partitions of the set* $\{1, 3, 5\}$ *are given by*

$$\{1, 3, 5\};\ \{1, 3\}, \{5\};\ \{1, 5\}, \{3\};\ \{1\}, \{3, 5\} \text{ and } \{1\}, \{3\}, \{5\}.$$

Definition 1.3 *The set of all set partitions of* $[n]$ *is denoted by* $\Pi_n = \Pi([n])$, *and the number of all set partitions of* $[n]$ *by* $\varpi_n = |\Pi_n|$, *with* $\varpi_0 = 1$ *(as there is only one set partition of the empty set).*

Example 1.4 *For* $[1]$, *there exists exactly one set partition. Thus,* $\varpi_1 = 1$. *For* $[2]$, *the set partitions are* $\{1\}, \{2\}$ *and* $\{1, 2\}$, *implying* $\varpi_2 = 2$. *The set partitions of* $[3]$ *are given by* $\{1, 2, 3\}; \{1, 2\}, \{3\}; \{1, 3\}, \{2\}; \{1\}, \{2, 3\}$ *and* $\{1\}, \{2\}, \{3\}$, *giving* $\varpi_3 = 5$. *In the same way one determines* $\varpi_4 = 15$ *as well as* $\varpi_5 = 52$. *Thus, the sequence of* ϖ_n *starts with* $1, 1, 2, 5, 15, 52, \ldots$.

Definition 1.5 *Let* π *be any set partition of* $[n]$. *We represent* π *in either* sequential *or* canonical form. *In the sequential form, each block is represented as sequence of increasing numbers and different blocks are separated by the symbol* /. *In the canonical representation, we indicate for each integer the block in which it occurs, that is,* $\pi = \pi_1 \pi_2 \cdots \pi_n$ *such that* $j \in B_{\pi_j}$, $1 \leq j \leq n$.

Example 1.6 *The set partitions of* [3] *in sequential form are* 123, 12/3, 13/2, 1/23, *and* 1/2/3, *while the set partitions of* [3] *in canonical representation are* 111, 112, 121, 122, *and* 123, *respectively.*

Example 1.7 *The set partition* $14/257/3/6$ *has canonical form* 1231242.

The two representations can be distinguished easily due to the symbol /, except in the single case when all elements of $[n]$ are in a single block. In this case, $\pi = 12345\cdots n$, and its corresponding canonical form is $11\cdots 1$. On the other hand, the set partition $12345\cdots n$ in canonical form represents the set partition $1/2/\cdots/n$ in sequential form. The canonical representations can be formulated in terms of words satisfying certain conditions. At first, we explain what we mean by the concept of a word, and then we characterize which kind of words correspond to a canonical representation of a set partition.

Definition 1.8 *Let a finite set* $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ *of objects be given. We call each* a_k *(for* $k = 1, \ldots, n$*) a* letter *and* $\mathcal{A}$ *the* alphabet*. An element of* $\mathcal{A}^N$ *will be called a* word *in the alphabet* $\mathcal{A}$ *(of* length N*). A word* $\omega = (a_{i_1}, a_{i_2}, \ldots, a_{i_N})$ *will be written in the form* $\omega = a_{i_1}a_{i_2}\cdots a_{i_N}$*, that is, as concatenation of its letters. For convenience, we also introduce the* empty word $\emptyset \in \mathcal{A}^0$. *If* ω *is a word, we denote the concatenation* $\omega\omega\cdots\omega$ *(*k *times) briefly by* ω^k. *In the case* $\mathcal{A} = [k]$*, an element of* $\mathcal{A}^n$ *is called* k-ary word of size n. *Words with letters from the set* $\{0, 1\}$ *are called* binary words *or* binary strings*, and words with letters from the set* $\{0, 1, 2\}$ *are called* ternary words *or* ternary strings.

Example 1.9 *The* 2*-ary words of size three are* 111*,* 112*,* 121*,* 122*,* 211*,* 212*,* 221*, and* 222*, the binary strings of size two are given by* 00*,* 01*,* 10*, and* 11*, while the ternary strings of size two are given by* 00*,* 01*,* 02*,* 10*,* 11*,* 12*,* 20*,* 21*, and* 22.

Example 1.10 *Let* $\mathcal{A} = \{a, b\}$ *be an alphabet with two letters. Then* $\omega_1 = abba$, $\omega_2 = baba$ *and* $\omega_3 = aabb$ *are words of length 4 which in general are not related. Note that we can write briefly* $\omega_1 = ab^2a$, $\omega_2 = (ba)^2$ *and* $\omega_3 = a^2b^2$.

In the following we are interested in expressions which are sums of words. Two words can be added if they are equal and we then write $\omega + \omega = 2\omega$ (since in our applications the letters are not numbers, no confusion can arise).

After having clarified what we mean by a word, we can characterize which words arise as the canonical representation of a set partition of $[n]$.

Fact 1.11 *A (canonical representation of a) set partition* $\pi = \pi_1\pi_2\cdots\pi_n$ *of* $[n]$ *is a word* π *such that* $\pi_1 = 1$*, and the first occurrence of the letter* $i \geq 1$ *precedes that of* j *if* $i < j$.

Now we draw the connection between genji-ko and set partitions: each of the possible incense selections corresponds to a set partition of [5], where the partition is according to flavor of the incense. Thus,  can be written as the set partition 123/4/5 of [5]. As $\varpi_5 = 52$, there are 52 genji-mon, as mentioned at the beginning of Section 1.1 and drawn in Figure 1.1. According to Knuth [675], a systematic investigation to find the number of set partitions of $[n]$ for any n, was first undertaken by Takakazu Seki and his students in the early 1700s. One of his pupils, Yoshisuke Matsunaga, found a recurrence relation for the number of set partitions of $[n]$, as well as a formula for the number of set partitions of $[n]$ with exactly k blocks of sizes $n_1, n_2, \ldots, n_k$ with $n_1 + \cdots + n_k = n$.

Theorem 1.12 (Matsunaga) *Let* ϖ_n *be the number of set partitions of* $[n]$. *Then* ϖ_n *satisfies the recurrence relation*

$$\varpi_n = \sum_{j=0}^{n-1} \binom{n-1}{j} \varpi_j \tag{1.1}$$

with initial condition $\varpi_0 = 1$.

Proof Assume that the first block contains $j+1$ elements from the set $[n]$, where $0 \le j \le n-1$. Since the first block contains the minimal element of the set, namely 1, we need to choose j elements from the set $\{2,3,\ldots,n\}$ to complete the first block. Thus, the number of set partitions of $[n]$ with exactly $j+1$ elements in the first block is given by $\binom{n-1}{j}\varpi_{n-1-j}$. Summing over all possible values of j, we obtain that

$$\varpi_n = \sum_{j=0}^{n-1}\binom{n-1}{j}\varpi_{n-1-j} = \sum_{j=0}^{n-1}\binom{n-1}{n-1-j}\varpi_{n-1-j} = \sum_{j=0}^{n-1}\binom{n-1}{j}\varpi_j,$$

with $\varpi_0 = 1$. ■

Theorem 1.13 (Matsunaga) *The number of set partitions of $[n]$ with exactly k blocks of sizes $n_1,\ldots,n_k$ with $n_1+\cdots+n_k = n$ is given by*

$$\prod_{j=1}^{k}\binom{n-1-n_1-\cdots-n_{j-1}}{n_j-1}.$$

Proof The proof is similar to the one for Theorem 1.12. For the first block, we choose n_1-1 elements from the set $\{2,3,\ldots,n\}$. From the $n-n_1$ available elements, we place the minimal element into the second block and then choose n_2-1 elements from the $n-n_1-1$ remaining elements, and so on, until we have placed all elements. Thus, the number of set partitions of $[n]$ with exactly k blocks of sizes $n_1,n_2,\ldots,n_k$ with $n_1+n_2+\cdots+n_k = n$ is given by

$$\binom{n-1}{n_1-1}\binom{n-1-n_1}{n_2-1}\cdots\binom{n-1-n_1-\cdots-n_{s-1}}{n_s-1},$$

which completes the proof. ■

A more general formula for the number of set partitions of $[n]$ into k_j blocks of sizes n_j with $k_1n_1+\cdots+k_mn_m = n$ can be obtained directly from Theorem 1.13. These results were not published by Matsunaga himself, but were mentioned (with proper credit given) in Yoriyuki Arima's book *Shūki Sanpō*, which was published in 1769. One of the questions posed in this text was to find the value of n for which the number of set partitions of $[n]$ is equal to 678.570 (the answer is $n = 11$). Additional results were derived by Masanobu Saka in 1782 in his work *Sanpō-Gakkai*. Saka established a recurrence for the number of set partitions of $[n]$ into k subsets, and using this recurrence, he computed the values for $n \le 11$.

Definition 1.14 *The set of all set partitions of $[n]$ with exactly k blocks is denoted by $\Pi_{n,k}$. The number $|\Pi_{n,k}|$ of set partitions of $[n]$ into k blocks is denoted by $S(n,k)$ and is called* Stirling number of the second kind *(Sequence A008277 in [1019]).*

Example 1.15 *From Example 1.4 one reads off that the set $[3]$ has exactly one partition with one block (123), three partitions into two blocks ($1/23$, $12/3$ and $13/2$), and one partition into three blocks ($1/2/3$). Thus, $S(3,1) = 1, S(3,2) = 3$ and $S(3,3) = 1$. In particular, $\varpi_3 = S(3,1)+S(3,2)+S(3,3)$.*

Remark 1.16 *Note that, by definition,*

$$\varpi_n = \sum_{k=0}^{n} S(n,k). \tag{1.2}$$

The numbers ϖ_n are also known as Bell numbers *(in honor of Eric Temple Bell) and denoted by B_n (Sequence A000110 in [1019]).*

Theorem 1.17 (Saka) *The number $S(n,k)$ of set partitions of $[n]$ into exactly k blocks satisfies the recurrence relation*

$$S(n+1,k) = S(n,k-1) + kS(n,k),$$

with $S(1,1)=1, S(n,0)=0$ for $n \geq 1$, and $S(n,k)=0$ for $n<k$.

Proof For any partition of $[n+1]$ into k blocks, there are two possibilities: either $n+1$ forms a single block, or the block containing $n+1$ has more than one element. In the first case, there are $S(n,k-1)$ such set partitions, while in the second case, the element $n+1$ can be placed into one of the k blocks of a partition of $[n]$ into k blocks, that is, there are $kS(n,k)$ such partitions. ■

Saka was not the first one to discover the numbers $S(n,k)$. James Stirling, on the other side of the globe in England, had found these numbers in a purely algebraic setting in his book *Methodus Differentialis* [1040] in 1730. Stirling's interest was in speeding up convergence of series, and the $S(n,k)$ arise as *connection coefficients* between monomials and falling polynomials.

Definition 1.18 *Polynomials of the form $z(z-1)\cdots(z-n+1)$ are called* falling polynomials *and are denoted by $(z)_n$.*

Example 1.19 *The first three monomials can be expressed in terms of falling polynomials as*

$$\begin{aligned}
z^1 &= z = (z)_1,\\
z^2 &= z + z(z-1) = (z)_1 + (z)_2,\\
z^3 &= z + 3z(z-1) + z(z-1)(z-2) = (z)_1 + 3(z)_2 + (z)_3.
\end{aligned}$$

The values of the coefficients in the falling polynomials were given in the introduction of *Methodus Differentialis*, reproduced as Figure 1.2, where columns correspond to n, and rows correspond to k. For example, $S(7,3)=301$. The relation (1.2) shows that ϖ_n is given as the sum of the entries in the nth column of Figure 1.2. Thus, the sequence ϖ_n of Bell numbers starts with $1,1,2,5,15,52,203,877,4.140,21.146,\ldots$.

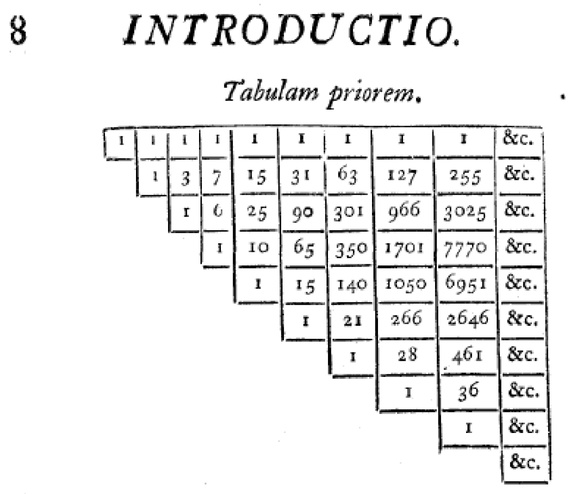
The description given by Stirling on how to compute these values makes it clear that he did not use the recurrence given by Saka (Theorem 1.17). To read more about how Stirling used the falling polynomials for series convergence, see the English translation of *Methodus Differentialis* with annotations by Tweddle [1091] (or [1090]). Despite Stirling's earlier discovery of the numbers $S(n,k)$, Saka receives credit for being the first one to associate a combinatorial meaning to these numbers, which are now named after James Stirling.

Theorem 1.20 (Stirling) *For all $n \geq 1$, one has that*

$$z^n = \sum_{k=1}^{n} S(n,k)(z)_k. \tag{1.3}$$

Proof We proceed the proof by induction on n. The first few cases can be checked by comparing Example 1.19 and Figure 1.2. Assume that the claim holds for n and let us prove it for $n+1$. By the induction hypothesis, we have that $z^{n+1} = \sum_{k=1}^{n} S(n,k)(z)_k(z-k+k)$. Using that $(z)_k(z-k) = (z)_{k+1}$ and shifting the index from k to $k-1$, this yields

$$z^{n+1} = \sum_{k=1}^{n+1} S(n,k-1)(z)_k + \sum_{k=1}^{n+1} kS(n,k)(z)_k.$$



8 *INTRODUCTIO.*

Tabulam priorem.

1	1	1	1	1	1	1	1	1	&c.
	1	3	7	15	31	63	127	255	&c.
		1	6	25	90	301	966	3025	&c.
			1	10	65	350	1701	7770	&c.
				1	15	140	1050	6951	&c.
					1	21	266	2646	&c.
						1	28	461	&c.
							1	36	&c.
								1	&c.
									&c.

FIGURE 1.2: Stirling numbers of the second kind from Stirling's *Methodus Differentialis.*

Writing this in one sum and using the recurrence given in Theorem 1.17, one obtains that $z^{n+1} = \sum_{k=1}^{n+1} S(n+1,k)(z)_k$, as was to be shown. ■

We introduce *Stirling numbers of the first kind* in analogy to (1.3) as connection coefficients.

Definition 1.21 *The* Stirling numbers of the first kind $s(n,k)$ *are defined as connection coefficients between falling polynomials and monomials,*

$$(z)_n = \sum_{k=1}^{n} s(n,k) z^k. \tag{1.4}$$

Combining (1.3) and (1.4), this gives the *orthogonality relations*

$$\sum_{k=1}^{n} s(n,k)S(k,l) = \sum_{k=1}^{n} S(n,k)s(k,l) = \delta_{n,l}, \tag{1.5}$$

where $\delta_{n,l}$ is the *Kronecker symbol* ($\delta_{n,l} = 1$ if $n = l$ and $\delta_{n,l} = 0$ if $n \neq l$).

Let us mention that another notation is also used for Stirling numbers, see, for example, [508] and the discussion in [674]. One writes

$$\left\{ {n \atop k} \right\} = S(n,k), \qquad \left[{n \atop k} \right] = (-1)^{n-k} s(n,k).$$

1.1.2 Early History of Stirling and Bell Numbers

While set partitions were studied by several Japanese authors and Toshiaki Honda devised algorithms to generate a list of all set partitions of $[n]$, the problem did not receive equal interest in Europe. There were isolated incidences of research, but no systematic study. The first known occurrence of set partitions in Europe also occurred outside of mathematics, in the context of the structure of poetry. In the second book of *The Arte of English*

Poesie [921], George Puttenham in 1589 compared the metrical form of verses to arithmetical, geometrical, and musical patterns. Several diagrams, which are in essence the same as the genji-mon, were given in [921].

The first mathematical investigation of set partitions was conducted by Gottfried Wilhelm Leibniz in the late 1600s (the manuscript was written probably in 1676). The unpublished manuscript shows that he tried to enumerate the number of ways to write a^n as a product of k factors, which is equivalent to the question of partitioning a set of n elements into k blocks. He enumerated the cases for $n \leq 5$, and, unfortunately, double-counted the case for $n = 4$ into two blocks of size 2 and the case for $n = 5$ into three blocks of sizes one, two, and two. These two mistakes prevented him from discovering that $S(n, 2) = 2^{n-1} - 1$ and also the recurrence given in Theorem 1.17. Further details can be found in the commentary by Knobloch [668, Pages 229–233], [669] and the reprint of Leibniz's original manuscript [670, Pages 316–321].

The second investigation was made by John Wallis, who asked a more general question in the third chapter of his *Discourse of Combinations, Alternations, and Aliquot Parts* in 1685 [1125]. (For example, see Jordan [610], Riordan [935], Goldberg et al. [484], or Knuth [672].) He was interested in questions relating to proper divisors (=*aliquot parts*) of numbers in general and integers in particular. The question of finding all the ways to factor an integer is equivalent to finding all partitions of the multiset consisting of the prime factors of the integer (with multiplicities). He devised an algorithm to list all factorizations of a given integer, but did not investigate special cases.

Back in Japan, a modification of Theorem 1.12 was given by Saka in 1782, when he showed that the number of set partitions of $[n]$ with exactly k blocks is given by $S(n, k)$, the Stirling number of the second kind. After 1782, the Bell numbers ϖ_n received more attention. It seems that the first occurrence in print of the Bell numbers has never been traced, but these numbers have been attributed to Euler (see Bell [73], but there is no reference for this statement). Following Bell [73,74], they are also called *exponential numbers*. Touchard [1077,1079] used the notation a_n to celebrate the birth of his daughter Anne, and later Becker and Riordan [67] used the notation B_n in honor of Bell. Throughout this book, we will use the notation B_n or ϖ_n.

The first appearance of the numbers B_n seems to be in a paper by Christian Kramp [686] from 1796, who considered an expansion of the function e^{e^x-1} (which we now know is the exponential generating function of the B_n). Tate [1058] gave in 1845 formula (1.26), which is equivalent to the *Dobiński formula* (1.25). This formula was discussed by Dobiński [358] in 1877 and he gave an explicit formula for the nth Bell number. One year later, in 1878, Ligowski [729] gave a more general formula involving the exponential generating function e^{e^x-1}. These results were preceded by the work of Grunert [521], who in 1843 had considered expressions which contain the Dobiński formula. The Dobiński formula also appeared as a problem in *Mathematicheskii Sbornik* in 1868 with solution provided in the following year [1, 2]. Whitworth [1139] discussed in the classical book *Choice and Chance* from 1870 problems of set partitions and derived explicit formulas for the Stirling and Bell numbers using the generating function e^{e^x-1}. In 1880, Peirce [900] gave explicit expressions for the Bell numbers. In the context of difference equations, Cesàro [214] also considered the Bell numbers and rederived the Dobiński formula in 1885. D'Ocagne [360] studied in 1887 the generating function for the sequence $\{\varpi_n\}_{n\geq 0}$. In 1901, Anderegg [32] showed that

$$2e = \sum_{k\geq 1} \frac{k^2}{k!}, \quad 5e = \sum_{k\geq 1} \frac{k^3}{k!}, \quad 15e = \sum_{k\geq 1} \frac{k^4}{k!},$$

and also obtained the general Dobiński's formula. In the 1920s, Ramanujan studied the Bell and Stirling numbers in his unpublished notebooks. His work is presented and discussed in

[93]. In the 1930s, Becker and Riordan [67] studied several arithmetic properties of Bell and Stirling numbers, and Bell [73, 74] recovered the Bell numbers. Later, Epstein [399] studied the exponential generating function for the Bell numbers (see also Williams [1146] and Touchard [1077,1079]). Rota [946] presented in 1964 a modern approach to Bell numbers and set partitions. Concerning Bell numbers, we refer the reader to the bibliography compiled by Gould [502], which contains over 200 entries.

The Stirling numbers of the first and second kind were considered in different contexts, for example, as connection coefficients, in the calculus of finite differences, in the theory of factorials, in connection with Bernoulli and Euler numbers, and evaluation of particular series (and, of course, in connection with the Bell numbers). Apart from Stirling's work mentioned above, they were considered – explicitly or implicitly – by many famous mathematicians, for instance, Euler (1755 [404]), Emerson (1763 [397]), Kramp (1799 [687]), Lacroix (1800 [702]), Ivory (1806 [580]), Brinkley (1807 [154]), Laplace (1812 [714]), Herschel (1816 [552], 1820 [553]), Scherk (1823 [959], 1834 [960]), Ettingshausen (1826 [403]), Grunert (1827 [520], 1843 [521]), Gudermann (1830 [523]), Oettinger (1831 [880]), Schlömilch (1846 [966–968], 1852 [969], 1858 [970], 1859 [971]), Schläfli (1852 [964], 1867 [965]), Catalan (1856 [204]), Jeffery (1861 [600]), Blissard (1867 [119], 1868 [120]), Whitworth (1870 [1139]), Worpitzky (1883 [1158]), and Cayley (1888, [208]). The Stirling numbers were so named by Nielsen [872–874] in 1904 in honor of James Stirling. From the beginning of the 20th century we single out Tweedie (1918 [1092]), Ramanujan (1920s, see [93]), Ginsburg (1928 [475]), Carlitz (1930 [183], 1932 [184]), Aitken (1933 [15]), Jordan (1933 [609]), Touchard (1933 [1077]), Becker and Riordan (1934 [67]), Bell (1934 [73, 74]), Goldstein (1934 [487]), Toscano (1936 [1068]), Epstein (1939 [399]) and Williams (1945 [1146]).

The Stirling numbers of the first kind were also discussed by Stirling [1040] in 1730. In fact, in roughly the same context Thomas Harriot had come across these numbers already in 1618 in his unpublished manuscript *Magisteria Magna* [531] (reprinted and annotated in [68]). Some remarks concerning the history of Stirling numbers can be found in [140, 230, 232, 609, 610, 674, 675].

1.2 Commutation Relations and Operator Ordering

A *commutation relation* describes the discrepancy between different orders of operation of two operations U and V. To describe it, we use the *commutator* $[U, V] \equiv UV - VU$. If U and V commute, then the commutator vanishes. Nowadays, many examples for noncommuting structures are well-known, for example, matrices, Grassmann algebras, quaternions, Lie algebras, but the formal recognition of the algebraic properties like commutativity or associativity emerged rather slowly and at first in concrete examples. How far a given structure deviates from the commutative case is described by the right-hand side of the commutation relation. For example, in a complex Lie algebra $\mathfrak{g}$ one has a set of generators $\{X_\alpha\}_{\alpha\in I}$ with the *Lie bracket* $[X_\alpha X_\beta] = \sum_{\gamma\in I} f^\gamma_{\alpha\beta} X_\gamma$, where the coefficients $f^\gamma_{\alpha\beta} \in \mathbb{C}$ are called *structure constants*. The associated *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is an associative algebra generated by $\{X_\alpha\}_{\alpha\in I}$, and the above bracket becomes the commutation relation

$$[X_\alpha, X_\beta] = \sum_{\gamma\in I} f^\gamma_{\alpha\beta} X_\gamma.$$

One of the earliest instances of a noncommutative structure was recognized in the context of *operational calculus* (also called *symbolical calculus*). Recall that one of the basic properties of calculus is the product rule, which implies that $D(x \cdot f(x)) = D(x) \cdot f(x) + x \cdot Df(x)$. Interpreting the multiplication with the variable as an application of the multiplication operator X, this can be written in the form $(D \circ X - X \circ D)f = f$, or, suppressing "$\circ$" and the operand f, as commutation relation between the operators X and D,

$$DX - XD = I. \tag{1.6}$$

1.2.1 Operational (or Symbolical) Calculus

In this section we present some of the early development of operational calculus, following mainly the account given by Koppelman [681] (and, in addition, the remarks given in [331, Chapter 1]). In both accounts many references to the original literature can be found. Furthermore, the classical book [201] of Carmichael from 1855 and [127] of Boole from 1859 are recommended.

The first steps in the formal theory of linear operators can be traced back to a letter from Leibniz to Johann Bernoulli in 1695; a published account appeared in 1710 [718]. In it Leibniz discussed the formula for higher derivatives of a product of functions (what we call today the *Leibniz rule*) and stressed the analogy to the binomial formula. Furthermore, he discussed a beautiful combinatorial argument for the coefficients appearing. If we denote the derivative with respect to x by D and let $D^m f \equiv f^{(m)}$, then Leibniz showed that

$$D^n(\psi u)(x) = \sum_{k=0}^{n} \binom{n}{k} \psi^{(n-k)}(x) u^{(k)}(x). \tag{1.7}$$

In 1772 Lagrange [703] discussed many operational formulas which would later be interpreted as the first steps in the calculus of finite differences. Let us introduce in addition to D the *shift operator*

$$Eu(x) = u(x+1) \tag{1.8}$$

and the *operator of finite difference*

$$\Delta u(x) = u(x+1) - u(x). \tag{1.9}$$

Clearly, one has $Eu(x) = (1+\Delta)u(x)$. In this notation, *Taylor's theorem* can be formally denoted by $f(x+h) = e^{hD} f(x)$, where the right-hand side has to be expanded using the conventional exponential series. Thus, $Eu(x) = e^D u(x)$. Introducing a constant ξ and denoting $\Delta_\xi u(x) = u(x+\xi) - u(x)$, Lagrange derived the operational relation

$$\Delta_\xi^\lambda u = \left(e^{\xi \frac{du}{dx}} - 1 \right)^\lambda. \tag{1.10}$$

A proof of (1.10) was given by Laplace [713] in 1776. The next big step was taken by Arbogast in his book *Du Calcul Des Dérivations* [40] from 1800 (following ideas of Lorgna). His idea was to separate the "symbols" (that is, operators) from the subject on which they act and to consider the rules the symbols satisfy algebraically. For example, he wrote (1.10) for $\lambda = 1$ as

$$1 + \Delta_\xi = e^{\xi D},$$

that is, as equation between the symbols itself. By considering the symbols apart from the subjects on which they act and manipulating them as if they were algebraic quantities, he was clearly working in the realm of operational calculus. In 1814, Servois published two notable papers [987, 988], in which he showed that the reason for the analogy between operational and algebraical symbols was that both types of symbols satisfy the distributive, associative and commutative law. Servois introduced the names "distributive" and "commutative", but the name "associative" seems to be due to Hamilton. Cauchy [205] discussed operational calculus in 1827 (mentioning, in particular, the work of Brisson) and showed, among many other results, that

$$F(D)[e^{rx} f(x)] = e^{rx} F(r + D) f(x), \tag{1.11}$$

where F is a polynomial. Cauchy used these results to solve particular differential equations, and he inquired into the convergence of the series obtained by formal processes and considered methods for establishing the validity of results of operational methods.

However, the operational methods did not become popular on the continent. The acceptance of the methods and notation of the continentals was surprisingly quick in England, and it was here that the calculus of operations was extended in scope and its applications. The first mathematicians in England who were responsible for this development were Babbage, Herschel, Peacock, and Woodhouse; see [681] for a discussion. In the next 30–40 years, from the late 1830s to the 1870s, many important results were achieved. In 1837 Murphy published a paper [855] in which a very clear and general account of the theory of linear operations was given, and in which he also noticed explicitly the difference between commuting and noncommuting operations. The next mathematician whom we single out is Gregory, who in the late 1830s and early 1840s published several papers in which the operational calculus was applied to differential and difference equations. He also discussed more general questions concerning operational calculus and its algebraic contents, see, for example, [516, 517]. Some information about Gregory, who died at the early age of 30, can be found in [29, 342, 681]. The work of Murphy and Gregory influenced Boole and his most important work concerned with operational calculus appeared in 1844 [126] (and can also be found in his book [127]). For example, in [126] he considered symbols π and ρ which are assumed to be associative and distributive and which satisfy for any function f, which can be developed into a power series in x, that $\rho f(\pi) = \lambda f(\pi)\rho$, where λ acts on π so that $\lambda f(\pi) = f(\phi(\pi))$. He showed that one can write

$$f(\pi + \rho) = \sum_{m \geq 0} f_m(\pi) \rho^m,$$

where $f_0(\pi) = f(\pi)$ and $f_m(\pi) = \frac{\lambda - 1}{(\lambda^m - 1)\pi} f_{m-1}(\pi)$. Furthermore, he showed that $f(\pi)\rho^m u = \rho^m f(\pi + m)u$. Choosing $\pi = \frac{d}{d\theta} = D$ and $x = \rho = e^\theta$, this implied that

$$f(D) e^{m\theta} u = e^{m\theta} f(D + m) u, \tag{1.12}$$

reproducing (1.11). As a second application, Boole derived for $D = \frac{d}{dx}$ that

$$xD(xD - 1)(xD - 2) \cdots (xD - n + 1)u = x^n D^n u, \tag{1.13}$$

which he called "known relation". In the late 1840s and early 1850s many attempts to extend and generalize Boole's results appeared. One of the most prolific adherents was the Reverend Bronwin, who devoted several papers to the symbolic method; see, for example, [157, 158]. Another follower was Hargreave, whose most important contribution appeared in 1848 [530]. His generalization of the Leibniz rule (1.7) can be written as

$$\phi(D)[\psi(x) \cdot u(x)] = \psi(x)\phi(D)u(x) + \phi'(x)\psi'(D)u(x) + \frac{1}{2!}\psi''(x)\phi''(D)u(x) + \cdots,$$

where ϕ and ψ were assumed to be functions which can be developed in ascending or descending integral powers of the variable. Shortly after that, the Reverend Graves [510] discussed the symbolical content of Hargreaves's results and introduced for that purpose symbols π and ρ satisfying

$$\pi\rho - \rho\pi = \alpha, \tag{1.14}$$

where α was assumed to commute with π and ρ. Graves discussed that a particular representation for this commutation relation is given (for $\alpha = 1$) by $\pi \mapsto D = \frac{d}{dx}$ and $\rho \mapsto X$, where X denotes again the operator of multiplication with the independent variable x. He also showed that this commutation relation implies an abstract version of Hargreaves' results. In fact, a few years earlier, in 1850, Donkin [362] had considered a more general situation in which symbols $\omega, \rho_1, \ldots, \rho_{n+1}$ are involved which satisfy

$$\begin{aligned} \omega\rho - \rho\omega &= \rho_1 \\ \omega\rho_1 - \rho_1\omega &= \rho_2 \\ &\vdots \\ \omega\rho_n - \rho_n\omega &= \rho_{n+1}. \end{aligned}$$

Clearly, if $\rho_k = 0$ for $k \geq 2$, this reduces to the situation considered by Graves. Assuming that $f(x)$ can be expanded in integral powers of x, Donkin showed, for example, that $f(\omega)\rho = \rho f(\omega) + \rho_1 f'(\omega) + \frac{\rho_2}{2!} f''(\omega) + \cdots$, and applied this to several questions in differential and difference calculus. In another interesting paper [509], Graves considered the action of $e^{g(x)D}$ on functions $u(x)$. He discovered that if $e^{g(x)D}u(x) = f(x)$, then f can be described as

$$f(x) = u\{G^{-1}[G(x) + 1]\}, \tag{1.15}$$

where $G(x) = \int^x \frac{dt}{g(t)}$ and G^{-1} is the inverse function of G. For example, if $g(x) = x^m$ with $m \in \mathbb{N} \setminus \{1\}$, then

$$e^{\lambda x^m D}u(x) = u\left\{\frac{x}{\sqrt[m-1]{1 - (m-1)\lambda x^{m-1}}}\right\}. \tag{1.16}$$

In the particular case $m = 1$, one obtains for the exponential of the *Euler operator* xD due to $G(x) = \ln(x)$ for $\lambda \in \mathbb{R}$ the well-known result

$$e^{\lambda xD}u(x) = u(e^{\lambda}x). \tag{1.17}$$

In the early 1860s a series of papers of Russel [951–953] appeared in which he considered noncommutative symbols along the lines of Boole (but satisfying slightly different commutation relations), and in 1882 Cazzaniga [209, 210] gave a systematic exposition of symbolical calculus. Around this time, Crofton [308–311] and Glaisher [476–479] published several interesting papers. From 1881 on, Heaviside worked out his operational calculus (the so-called *Heaviside calculus*) in a long series of publications; see the discussion in [331, 903]. The importance of this work was recognized in 1910–1920, and several mathematicians tried to give it a rigorous foundation (for a well-known early work; see Wiener [1141] and the references therein). Let us also mention the work of Carmichael, Cockle, Greatheed, DeMorgan, Roberts, and Spottiswoode (see the discussion in [681]). As Davis [331] remarked, the period of formal development of operational methods may be regarded as having ended by 1900. At this time, the theory of integral equations began fascinating mathematicians, and from these beginnings the modern theory of functional analysis emerged. Since the 1970s, Gian-Carlo Rota and collaborators revived many of these classical topics in *finite operator calculus* – or also under the classical name *umbral calculus*; see, for example, [939–941, 947].

1.2.2 Early Results for Normal Ordering Operators

The problem of bringing operators into a convenient order arose with the appearance of noncommutative objects ("symbols"). Clearly, if the operators under consideration commute, one can write them in any order one wishes. Recall the terminology considering words introduced in Definition 1.8. Let us turn to the concrete situation where the alphabet consists of the two operators X and D. An arbitrary word ω in these letters can be written as

$$\omega = X^{r_n} D^{s_n} \cdots X^{r_2} D^{s_2} X^{r_1} D^{s_1} \tag{1.18}$$

for some $r_k, s_k \in \mathbb{N}_0$. In our context (1.6) holds true, that is, two adjacent letters X and D in a word can be interchanged according to this relation. Each time one uses it in a word ω, two new words result. If we write the original word as $\omega = \omega_1 DX \omega_2$ (where each ω_k can be the empty word), then applying (1.6) yields that $\omega = \omega_1 XD\omega_2 + \omega_1\omega_2$.

Example 1.22 *The simplest example results when $\omega_1 = \omega_2 = \emptyset$, and $\omega = DX$ can be written as $DX = XD + 1$. The more complex word D^2XD can be written as $DDXD = DXDD + DD = DXD^2 + D^2$.*

Using successively (1.6), one can transform each word in X and D into a sum of words, where each of these words has all the powers of D to the right.

Definition 1.23 *A word ω in the letters X and D is in* normal ordered form *if $\omega = a_{r,s}X^rD^s$ for $r, s \in \mathbb{N}_0$ (and arbitrary coefficients $a_{r,s} \in \mathbb{C}$). An expression consisting of a sum of words is called normal ordered if each of the summands is normal ordered. The process of bringing a word (or a sum of words) into its normal ordered form is called* normal ordering. *Writing the word ω in its normal ordered form,*

$$\omega = \sum_{r,s\in\mathbb{N}_0} A_{r,s}(\omega) X^r D^s,$$

the – uniquely determined – coefficients $A_{r,s}(\omega)$ are called normal ordering coefficients of ω *(the sum is only finite). In a similar fashion, $\omega = b_{r,s}D^rX^s$ is called* antinormal ordered.

As shown above, the normal ordered form of DX is $XD + 1$. As the next example, we show that it is possible to interpret the Leibniz rule (1.7) as a formula concerning normal ordering. Indeed, if we consider the left-hand side $D^n(\psi u)(x)$ as the successive application of the multiplication operator $\psi(X)$ followed by D^n on u, we can write this relation as

$$(D^n \circ \psi(X))\, u(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\psi^{(n-k)}(X) \circ D^k\right) u(x),$$

which we interpret as the following normal ordering relation

$$D^n\psi(X) = \sum_{k=0}^{n} \binom{n}{k} \psi^{(n-k)}(X) D^k.$$

Choosing $\psi(X) = X$, one gets back (1.6) for $n = 1$. Choosing $\psi(X) = X^m$ with $m \geq n$, one can use that $D^l(x^m) = l!\binom{m}{l}x^{m-l}$ to find

$$D^nX^m = \sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} k! X^{m-k} D^{n-k}. \tag{1.19}$$

Let us point out that this interpretation of the Leibniz rule (1.7) is an unhistorical one. Maybe the first explicit results concerning normal ordering were derived by Scherk [959] in his dissertation from 1823.

Theorem 1.24 (Scherk) *Let X and D satisfy* (1.6).

1. *The powers $(XD)^n$ have for $n \in \mathbb{N}$ the normal ordered form*

$$(XD)^n = \sum_{k=1}^{n} a_n^k X^k D^k, \tag{1.20}$$

where the coefficients a_n^k satisfy the recurrence relation

$$a_n^k = a_{n-1}^{k-1} + k a_{n-1}^k. \tag{1.21}$$

2. *The powers $(e^X D)^n$ have for $n \in \mathbb{N}$ the normal ordered form*

$$(e^X D)^n = e^{nX} \sum_{k=1}^{n} c_n^k D^k, \tag{1.22}$$

where the coefficients c_n^k satisfy the recurrence relation

$$c_n^k = c_{n-1}^{k-1} + (n-1) c_{n-1}^k.$$

Note that e^X is an infinite series and is treated formally. Scherk also gave combinatorial interpretations and explicit expressions for the expansion coefficients a_n^k and c_n^k. He considered in his dissertation also briefly the expansion of $(X^p D)^n$ with $p \in \mathbb{N}$ and wrote

$$(X^p D)^n = X^{np-n} \sum_{k=1}^{n} b_n^k X^k D^k, \tag{1.23}$$

where the coefficients b_n^k are described combinatorially as a sum over certain partitions. Scherk [960] mentioned in 1834 the following recurrence for them,

$$b_n^k = b_{n-1}^{k-1} + ((n-1)p - n + k + 1)\, b_{n-1}^k.$$

Murphy derived in the already mentioned paper [855] from 1837 several remarkable formulas. If v denotes an arbitrary function, he found the expansion

$$(vD)^n = v^n D^n + \binom{n}{2} v' v^{n-1} D^{n-1} + \binom{n}{3} \left\{ \frac{3n-5}{4} (v')^2 + v'' v \right\} v^{n-2} D^{n-2} + \cdots, \tag{1.24}$$

but gave no explicit expression for the general term; in fact, Scherk [959] had also considered this expansion. Relation (1.13) mentioned by Boole [126] was used frequently as a starting point to obtain generalizations. Grunert [521] considered in 1843 the expansion (1.20) and found the recurrence (1.21). Cesàro [214] considered in 1885 (1.20) and derived for the coefficients the expression $a_n^k = \frac{\Delta^k 0^n}{k!}$, where a symbolic notation of the calculus of finite differences is used. Applying (1.20) to e^x and using on the left-hand side $e^x = \sum_{k \geq 0} \frac{x^k}{k!}$ as well as $(XD)^n x^k = k^n x^k$, the left-hand side gives $\sum_{k \geq 1} \frac{k^n x^k}{k!}$. On the right-hand side, one obtains $\sum_{k=1}^{n} a_n^k x^k e^x$. Comparing both sides for $x = 1$, one obtains that (here we use that $a_n^k = S(n,k)$)

$$\frac{1}{e} \sum_{k \geq 1} \frac{k^n}{k!} = \varpi_n, \tag{1.25}$$

where ϖ_n are the Bell numbers (Cesàro did not recognize these numbers as partition numbers). Thus, Cesàro derived the *Dobiński formula* (1.25), and he also derived the exponential generating function e^{e^x-1} of the numbers ϖ_n. Note that we can write (1.25) also as

$$\frac{1^n}{1!}+\frac{2^n}{2!}+\frac{3^n}{3!}+\cdots=e\left\{\frac{\Delta 0^n}{1!}+\frac{\Delta^2 0^n}{2!}+\frac{\Delta^3 0^n}{3!}+\cdots\right\}. \tag{1.26}$$

In this form, (1.26) was already shown by Tate [1058] in 1845, and he considered the case $n=3$ explicitly (where $\varpi_3=5$). In the beautiful paper [360] from 1887 d'Ocagne obtained several results for the "remarkable numbers" K_n^k, which he defined by (1.21), that is, $K_n^k=K_{n-1}^{k-1}+kK_{n-1}^k$ (since the initial values coincide, one has $K_n^k=a_n^k$). D'Ocagne derived (1.20) and also

$$(DX)^n=\sum_{k=0}^{n}K_{n+1}^{k+1}X^kD^k.$$

In addition, denoting $\phi_{m+1}(x)=\sum_{k=0}^{m}K_{m+1}^{k+1}x^k$, he derived the expression

$$(X+DX)^n=\sum_{k=0}^{n}\frac{\phi_{m+1}^{(k)}(X)}{k!}X^kD^k.$$

Several other expansions were treated, in particular in connection with higher derivatives of "functions of functions". In this context, we should mention the work of Meyer [812,813], Schlömilch [966–971], and Schläfli [964, 965]. These authors noticed the appearance of interesting coefficients and studied their properties. Nielsen [872, 873] introduced in 1904 the name "Stirling numbers" for the coefficients a_n^k, and Tweedie [1092] wrote a first comprehensive paper in 1918. Shortly after that, Schwatt [984] noticed in 1924 that the coefficients in (1.20) are given by the Stirling numbers (of the second kind), that is, we can write

$$(XD)^n=\sum_{k=1}^{n}S(n,k)X^kD^k. \tag{1.27}$$

In the early 1930s, Carlitz [183, 184], seemingly unaware of the work of Scherk, *defined* in analogy to (1.20) and (1.23) *generalized Stirling numbers* $S_{r,s}(n,k)$ as normal ordering coefficients for $r\geq s$ by

$$(X^rD^s)^n=X^{n(r-s)}\sum_{k\geq 0}S_{r,s}(n,k)X^kD^k. \tag{1.28}$$

Clearly, $S_{1,1}(n,k)=S(n,k)=a_n^k$ and $S_{p,1}(n,k)=b_n^k$. Independently, Toscano followed the same idea and, beginning in 1935, treated the generalized Stirling numbers in a long series of papers [1067–1075]. McCoy [791] considered in 1934 arbitrary words (1.18) in X and D,

$$X^{r_n}D^{s_n}\cdots X^{r_2}D^{s_2}X^{r_1}D^{s_1}=X^{|\mathbf{r}|-|\mathbf{s}|}\sum_{k\geq 0}S_{\mathbf{r},\mathbf{s}}(k)X^kD^k, \tag{1.29}$$

where $\mathbf{r}=(r_1,\dots,r_n)$ and $|\mathbf{r}|=r_1+\cdots+r_n$ (and, similarly, for $\mathbf{s}$). The coefficients $S_{\mathbf{r},\mathbf{s}}(k)$ generalize the Stirling numbers of Carlitz: If $r_k=r$ and $s_k=s$ for $k=1,\dots,n$, then $S_{\mathbf{r},\mathbf{s}}(k)=S_{r,s}(n,k)$ and (1.29) reduces to (1.28). Since many classical polynomials – for example, the Bell, Bessel, Hermite, and Laguerre polynomials – allow an operational treatment, many other researchers followed this line of research and discovered many interesting relations involving Stirling numbers or their generalizations; see, for example, Cakić [175–178, 392, 393, 826], Chak [216–219], Comtet [279], Gould [496–499, 501, 503], Lang [710–712], Mitrinović [834–836], Al-Salam [17, 18], and Al-Salam [19–21].

Let us point out another way to generalize Stirling numbers. Introducing the *generalized factorial* $(z|\gamma)_n = z(z-\gamma)\cdots(z-(n-1)\gamma)$, Hsu and Shiue [568] defined generalized Stirling numbers $S(n,k;\alpha,\beta,r)$ as connection coefficients,

$$(z|\alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)(z-r|\beta)_k. \tag{1.30}$$

Clearly, (1.3) and (1.4) are particular instances of (1.30), and many previous generalizations of Stirling numbers are special cases of $S(n,k;\alpha,\beta,r)$.

1.2.3 Operator Ordering in Quantum Theory

Recall from the preceding section that Graves discovered in the 1850s that the main property to derive many of the algebraic consequences of operational calculus is the commutation relation (1.14) (with $\alpha \in \mathbb{C}$), which is an abstract version of (1.6). Unfortunately, he was roughly 70 years ahead of his time. In 1925, Werner Heisenberg [545] discovered that to understand the physics of the atom one should depart from classical notions, implying in particular that the mathematical objects representing physical properties need not commute. The relations he postulated for the momentum and location were recognized immediately by Born and Jordan [131] as the commutation relation

$$\mathsf{pq} - \mathsf{qp} = -i\hbar 1 \tag{1.31}$$

for the infinite matrices p (resp. q) which represent the momentum (resp. location) and where $\hbar = h/2\pi$ denotes Planck's constant. Independently, Dirac [349–351] considered abstract *q-numbers* satisfying (1.31) and developed a *quantum algebra* for them. Thus, this noncommutative structure – coinciding with (1.14) considered by Graves – lies at the heart of quantum theory. Very shortly after the discovery of this *matrix mechanics*, a different version of quantum theory was found by Erwin Schrödinger in the form of *wave mechanics* – the famous Schrödinger equation. However, it was soon established that both versions of the theory are equivalent.

Born and Jordan [131] recognized that one needs to consider particularly ordered forms of expressions in the noncommuting objects p and q. Calling an expression in these two variables *normal ordered* (resp. *antinormal ordered*), if all powers of p stand to the right (resp. left) of the powers of q, they gave the following normal ordering formula

$$\mathsf{p}^n\mathsf{q} = \mathsf{q}\mathsf{p}^n + n(-i\hbar)\mathsf{p}^{n-1},$$

as well as the analogous antinormal ordering formula

$$\mathsf{q}^n\mathsf{p} = \mathsf{p}\mathsf{q}^n - n(-i\hbar)\mathsf{q}^{n-1}.$$

More generally, they also mentioned that

$$\mathsf{p}^n\mathsf{q}^m = \sum_{k=0}^{\min(n,m)} k!\binom{n}{k}\binom{m}{k}(-i\hbar)^k\mathsf{q}^{m-k}\mathsf{p}^{n-k}, \tag{1.32}$$

and gave the analogous antinormal ordering formula. Note that (1.32) has the same structure as (1.19) due to the common algebraic structure. In the subsequent paper [130] together with Heisenberg , they derived for any function $f(\mathsf{q},\mathsf{p})$, which can be formally expressed as power series in p and q, the rule

$$\mathsf{p}f - f\mathsf{p} = (-i\hbar)\frac{\partial f}{\partial \mathsf{q}},$$

as well as

$$\mathsf{q}f - f\mathsf{q} = (-i\hbar)\frac{\partial f}{\partial \mathsf{p}}. \tag{1.33}$$

Dirac, who independently found (1.31) in [349], considered in his subsequent work [351] algebraic consequences of (1.31) and also derived (1.33). Furthermore, Dirac showed that

$$f(\mathsf{q},\mathsf{p})e^{i\alpha\mathsf{q}} = e^{i\alpha\mathsf{q}}f(\mathsf{q},\mathsf{p}+\alpha\hbar),$$

which one recognizes as (1.12) already discussed by Boole – or, even earlier, by Cauchy (1.11). Dirac [350] considered many other interesting consequences of (1.31). Coutinho [304] gave a beautiful account of the early history of the underlying *Weyl algebra*. From a more mathematical point of view, several consequences of relation (1.31) were discussed in the early 1930s by McCoy [786–791] as well as Kermack and McCrea [649,650,792]. An instructive discussion of their work from a modern perspective can be found in [306]. Motivated by the example of "quantum algebra", Littlewood [733] started in 1933 a thorough examination of this algebra.

Let us turn back to quantum theory. In its applications, it is often convenient to switch to *Fock space* and consider two adjoint operators in it satisfying the *bosonic commutation relation*

$$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1. \tag{1.34}$$

Note that this is again an instance of (1.14)! The *creation operator* $\hat{a}^\dagger$ (resp. *annihilation operator* $\hat{a}$) has the interpretation of creating (resp. annihilating) one *quantum* in the system considered (for example, a photon). In the simplest example a physical *state* just denotes the number of quanta present in the system, and a state representing n quanta is denoted by $|n\rangle$. Fock space $\mathscr{F}$ is the linear span $\{|1\rangle, |2\rangle, \ldots, |n\rangle, \ldots\}$ of these states, and one has that

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle.$$

Destroying the last quantum, only the *vacuum* remains, that is, $\hat{a}|1\rangle = 0$. The *number operator* $\hat{n} = \hat{a}^\dagger\hat{a}$ has the property $\hat{n}|n\rangle = n|n\rangle$, hence its name. To calculate expectation values of interesting operators in $\hat{a}$ and $\hat{a}^\dagger$, it is advantageous to write them in normal ordered form, meaning that the powers of $\hat{a}^\dagger$ stand to the left of the powers of $\hat{a}$. The reason for this is that destroying more quanta than are present, the vacuum results, that is, $(\hat{a}^\dagger)^m\hat{a}^k|n\rangle = 0$ if $k > n$.

For the states one has that $\langle n||m\rangle = \delta_{n,m}$. A simple calculation gives $\langle n|\hat{a}^\dagger|m\rangle = \sqrt{m+1}\langle n||m+1\rangle = \sqrt{m+1}\delta_{n,m+1}$. One easily finds that $\langle m|\hat{n}^k|m\rangle = m^k$ for any $k \in \mathbb{N}$. As an example, consider $k = 2$, where $\hat{n}^2 = \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}$. Using (1.34), one obtains that $\hat{n}^2 = (\hat{a}^\dagger)^2\hat{a}^2 + \hat{a}^\dagger\hat{a}$, hence,

$$\langle m|\hat{n}^2|m\rangle = \langle m|(\hat{a}^\dagger)^2\hat{a}^2|m\rangle + \langle m|\hat{a}^\dagger\hat{a}|m\rangle = m(m-1) + m = m^2,$$

as it should. Higher powers of the number operator can be written as

$$\hat{n}^n = \sum_{k=1}^{n} T_{n,k}\,(\hat{a}^\dagger)^k\hat{a}^k \tag{1.35}$$

for some coefficients $T_{n,k}$. Normal ordered expressions for powers of $\hat{n}$ were derived by

Agarwal and Wolf [8] in 1970. In the same context, similar relations had been discussed a few years earlier by Schwinger [985,986], Mandel [755], Louisell and Walker [741], Marburger [775,776], Wilcox [1102,1142,1143]), Peřina [904,905]), and Cahill and Glauber [170]. Gluck [482] considered in 1972 closely related operators. For our considerations, two important papers appeared in the mid 1970s: Navon [861] considered in 1973 the *anticommutation relation*

$$\hat{f}\hat{f}^{\dagger} + \hat{f}^{\dagger}\hat{f} = 1 \tag{1.36}$$

for *fermionic creation and annihilation operators* – compare with (1.34) – and showed that the normal ordering coefficients for arbitrary words in the multi-mode case can be expressed as *rook numbers*. Katriel [634] recognized in 1974 that the coefficients in (1.35) are Stirling numbers of the second kind, that is,

$$\hat{n}^n = \sum_{k=1}^{n} S(n,k)(\hat{a}^{\dagger})^k \hat{a}^k. \tag{1.37}$$

The work of Katriel was generalized from the 1980s up to the present, beginning by himself [635–638] and Mikhaĭlov [822,823], to more general expressions. Katriel [637] discovered in 2000 (see also [638]) that the Bell numbers appear as expectation values of $\hat{n}^n$ with respect to *coherent states*. Since normal ordered expressions are useful in applications, this more combinatorial approach gained speed after 2000 and more and more authors contributed to an understanding of normal ordered expressions. By considering instead of $\hat{n}^n = (\hat{a}^{\dagger}\hat{a})^n$ the expressions $((\hat{a}^{\dagger})^r\hat{a}^s)^n$, generalized Stirling numbers $\mathcal{S}_{r,s}(n,k)$ were introduced by Blasiak, Penson, and Solomon [114–116] in 2003 for $r \geq s$ by

$$((\hat{a}^{\dagger})^r\hat{a}^s)^n = (\hat{a}^{\dagger})^{n(r-s)} \sum_{k=0}^{n} \mathcal{S}_{r,s}(n,k)(\hat{a}^{\dagger})^k a^k, \tag{1.38}$$

and many of their properties were studied. Since $\hat{a} \mapsto D$ and $\hat{a}^{\dagger} \mapsto X$ (hence, $\hat{n} = \hat{a}^{\dagger}\hat{a} \mapsto XD$) furnishes a representation of the commutation relation, the generalized Stirling numbers $\mathcal{S}_{r,s}(n,k)$ from (1.38) equal the generalized Stirling numbers $S_{r,s}(n,k)$ introduced by Carlitz (1.28).

In the above physical situation, Arik and Coon [41] considered a q-analog of (1.34), that is, they introduced the *q-deformed commutation relation*

$$\hat{a}_q\hat{a}_q^{\dagger} - q\hat{a}_q^{\dagger}\hat{a}_q = 1 \tag{1.39}$$

of a *q-boson* (where $q \in \mathbb{C}$). Considering $q \to 1$ gives the bosonic commutation relation (1.34), while considering $q \to -1$ gives the fermionic commutation relation (1.36). Here the same problems as in the undeformed case appear and normal ordering powers of the corresponding number operator involves the q-deformed Stirling numbers of the second kind, as was shown in 1992 by Katriel and Kibler [642]. Many properties of this algebra have been considered, and an extensive bibliography up to 2000 can be found in [549].

Since Katriel's seminal work [634], the combinatorial aspects of boson normal ordering have received a lot of attention; see, for example, [101, 113, 114, 116, 117, 181, 356, 357, 417, 419, 453, 494, 637, 639, 711, 764, 768–770, 807, 822, 974, 976, 989, 991, 1099, 1100, 1149] (more references are given in later chapters). *Wick's theorem* is the physicist's way to determine the normal ordered form of an arbitrary operator function in $\hat{a}$ and $\hat{a}^{\dagger}$. A closer look reveals that the *contractions* used in it can be described in terms of set partitions, providing a conceptual reason for the appearance of $S(n,k)$ in (1.37).

Concerning introductions to normal ordering, we recommend the beautiful survey of Blasiak and Flajolet [106], where many combinatorial aspects are discussed. An older reference is [740], while [113] provides an elementary first introduction. Also, [761] contains a discussion on normal ordering.

1.3 Normal Ordering in the Weyl Algebra and Relatives

For us, the *Weyl algebra* A_h (where $h \in \mathbb{C}$) is the complex algebra generated by the letters U and V satisfying $UV - VU = hI$, where the identity I on the right-hand side will usually be suppressed. This relation is exactly (1.14) considered by Graves, and a concrete representation is given for $h = 1$ by $V \mapsto X$ and $U \mapsto D$, see (1.6) (or, $V \mapsto \hat{a}^\dagger$ and $U \mapsto \hat{a}$, see (1.34)). The normal ordering results mentioned above only depend on the commutation relation between the "symbols", so also hold in A_1. For example, normal ordering $(VU)^n$ gives rise to the $S(n,k)$ as normal ordering coefficients. In 2005, Varvak [1100] showed that the normal ordering coefficients of an arbitrary word in U and V can be expressed as rook numbers (Fomin [448] had shown the same in a different context in 1994). More precisely, to a word ω in U and V one can associate a *Ferrers board* B_ω, and it is then possible to write for a word ω having m appearances of V (resp. n of U) the normal ordered expression

$$\omega = \sum_{k=0}^{\min(m,n)} r_k(B_\omega) V^{m-k} U^{n-k}, \tag{1.40}$$

where $r_k(B_\omega)$ denotes the kth rook number of the board B_ω. For example, if $\omega = (VU)^n$, then the corresponding Ferrers board is given by the *staircase board* $J_{n,1}$, for which one knows $r_{n-k}(J_{n,1}) = S(n,k)$. Thus, (1.40) gives

$$(VU)^n = \sum_{k=0}^{n} r_{n-k}(J_{n,1}) V^k U^k = \sum_{k=0}^{n} S(n,k) V^k U^k, \tag{1.41}$$

that is, the well-known result (1.27).

The *q-deformed Weyl algebra* $\mathsf{A}_{h|q}$ is defined – in analogy to A_h – to be the complex algebra generated by the letters U and V satisfying

$$UV - qVU = h, \tag{1.42}$$

where $q \in \mathbb{C}$ is assumed to be generic. A physical representation is given for $h = 1$ by $U \mapsto \hat{a}_q$ and $V \mapsto \hat{a}_q^\dagger$; see (1.39). An operational representation of (1.42) is given for $h = 1$ by $V \mapsto X$ and $U \mapsto D_q$, where D_q denotes the *Jackson derivative*. The action of the Jackson derivative on a function f is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

Diaz and Pariguan [345] considered in 2005 the *meromorphic Weyl algebra* which results by considering X^{-1} and D (instead of X and D, as in the Weyl algebra). One finds that $DX^{-1} - X^{-1}D = -(X^{-1})^2$, that is, abstractly,

$$UV - VU = -V^2. \tag{1.43}$$

One can consider different combinatorial aspects in this algebra, for example define associated Stirling numbers as normal ordering coefficients of $(VU)^n$. In the context of algebraic geometry this algebra is known as *Jordan plane* and appeared occasionally in the literature. In more recent times, Shirikov [1005–1008] studied it thoroughly; see also [581]. From a different point of view, Benaoum [77] had considered in 1998 the binomial formula for variables U and V satisfying (1.43) in the form $UV - VU = hV^2$. For these variables, he introduced *h-binomial coefficients* and derived a normal ordered expansion

$$(U+V)^n = \sum_{k=0}^{n} \binom{n}{k}_h V^k U^{n-k},$$

in close analogy to the conventional case (recovered for $h = 0$). In 1999 Benaoum [78] considered a q-deformation of this situation, where

$$UV - qVU = hV^2, \tag{1.44}$$

and introduced *(q,h)-binomial coefficients*, which reduce for $q = 1$ to the h-binomial coefficients. Note in particular that the degenerate case $h = 0$ of (1.44) leads to *q-commuting variables*, that is, $UV = qVU$, and the corresponding binomial formula is the classical *q-binomial theorem* [917, 983],

$$(U+V)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q V^k U^{n-k}, \tag{1.45}$$

where q-binomial coefficients are used. Variables U and V satisfying (1.44) have been considered also by other authors, for example, [255, 346, 544, 945, 1184]. In a completely different context, Burde [162] considered in 2005 finite dimensional matrices U and V satisfying the commutation relation

$$UV - VU = V^p \tag{1.46}$$

for $p \in \mathbb{N}$, and also considered the coefficients resulting from normal ordering $(UV)^n$. In the same year, Varvak [1100] suggested to consider normal ordering expressions in variables U and V satisfying (1.46) and drew a connection to *p-rook numbers* introduced by Goldman and Haglund [485] in 2000. Comparing the different algebras considered above, a common generalization emerges.

Definition 1.25 *The* q-deformed generalized Weyl algebra $\mathsf{A}_{s;h|q}$ *is defined for* $s \in \mathbb{N}_0, h \in \mathbb{C} \setminus \{0\}$ *and* $q \in \mathbb{C}$ *as the complex algebra generated by* U *and* V *satisfying*

$$UV - qVU = hV^s. \tag{1.47}$$

Relation (1.47) can be specialized in different ways, thereby reducing to relations considered above. For example, the Weyl algebra A_h corresponds to $\mathsf{A}_{0;h|1}$, and $\mathsf{A}_{2;-1|1}$ is the meromorphic Weyl algebra; see (1.43). Recall from (1.41) that the Stirling numbers $S(n,k)$ can be defined as normal ordering coefficients of $(VU)^n$ in A_h. This motivates the following definition [765].

Definition 1.26 *The* q-deformed generalized Stirling numbers $\mathfrak{S}_{s;h|q}(n,k)$ *are defined as normal ordering coefficients of* $(VU)^n$ *in* $\mathsf{A}_{s;h|q}$, *that is,*

$$(VU)^n = \sum_{k=1}^{n} \mathfrak{S}_{s;h|q}(n,k) V^{s(n-k)+k} U^k. \tag{1.48}$$

The generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k) = \mathfrak{S}_{s;h|q=1}(n,k)$ are a subfamiliy of the generalized Stirling numbers $S(n,k;\alpha,\beta,r)$ from (1.30), and one has that $\mathfrak{S}_{0;1}(n,k) = S(n,k)$. Particularly interesting is the case $s = 2$ (corresponding to the meromorphic Weyl algebra), where the generalized Stirling numbers are given by Bessel numbers. These generalized Stirling numbers were studied in several papers [289, 290, 763, 765–767, 771–773].

Now that we have defined the chief characters of the book, we can succintly describe its focus as follows: We discuss different aspects of normal ordering in $\mathsf{A}_{s;h|q}$ and in several interesting specializations, like $\mathsf{A}_{2;-1|1}$. Apart from general results, we are particularly interested in the word $(VU)^n$, giving rise to the generalized Stirling and Bell numbers, and in $(U+V)^n$. Along the way we also present rewarding ramifications.

1.4 Content of the Book

In *Chapter 2* we introduce techniques to solve recurrence relations which occur naturally when enumerating set partitions. This chapter also contains many examples of important integer sequences, such as the Fibonacci and Catalan numbers, to illustrate the techniques of setting up and of solving recurrence relations. Methods for solving recurrence relations include guess and check, iteration, characteristic polynomial, and generating function. Lattice paths and trees as basic combinatorial structures are treated, including Dyck and Motzkin paths, rooted trees and k-ary trees. We also discuss other combinatorial objects (rooks, Sheffer sequences, etc.) in this chapter for easy reference in later chapters.

In *Chapter 3* we discuss the classical Stirling and Bell numbers. After presenting some basic properties, such as recurrence relations and generating functions, several combinatorial interpretations are given. We then treat Touchard (or exponential) polynomials and discuss some more specialized topics which will be generalized in later chapters, for example, a differential equation for the generating function of the Bell numbers, the Dobiński formula, and Spivey's Bell number relation. Also, a q-deformation as well as a (p,q)-deformation of the Stirling and Bell numbers are reviewed.

In *Chapter 4* several generalizations of the Stirling and Bell numbers are considered. The first starting point for generalization is the operational interpretation of Stirling numbers; see (1.27). Considering instead of $(XD)^n$ other words in X and D gives rise to different generalizations of Stirling numbers; see, for example, (1.28) and (1.29). We present Comtet's result about normal ordering $\left(v(x)\frac{d}{dx}\right)^n$, and give an explicit expression for the general term in (1.24). The second starting point for generalization is the interpretation of the Stirling numbers as connection coefficients; see (1.3). We present the generalization (1.30) due to Hsu and Shiue, which unified many of the previous generalizations of the Stirling numbers. After surveying many of their properties, a q-deformation and a (p,q)-deformation are treated. At the end of the chapter we briefly mention a selection of further recent generalizations of the Stirling numbers.

In *Chapter 5* we focus on the Weyl algebra, which is the complex algebra generated by U and V satisfying $UV - VU = h$ for some $h \in \mathbb{C}$. After presenting some elementary properties and a few remarks on its history, we give an introduction to elementary aspects of quantum mechanics (stressing its connection to the Weyl algebra). The "operator ordering problem" in quantization is discussed and several approaches to handle it are mentioned. As a particularly important toy example the harmonic oscillator is treated in detail, and the creation and annihilation operators are introduced. Several examples for normal ordering words in these operators are considered, and the connection to (generalized) Stirling and Bell numbers is elucidated.

In *Chapter 6* we continue the study of normal ordering in the Weyl algebra. We discuss some special relations, for example, Viskov's identity and the identity of Bender, Mead, and Pinsky, and also the connection to rook numbers. Also, Wick's theorem is discussed from a combinatorial as well as a physical point of view. Considering the normal ordering of particular expressions gives connections to a variety of combinatorial problems, for example, counting trees with particular properties. In addition to the operator ordering schemes discussed in more detail (normal ordering, antinormal ordering, Weyl ordering), we mention a collection of other such schemes. At the end of the chapter we briefly discuss a few aspects of the multi-mode case and provide some literature.

In *Chapter 7* normal ordering in several variants of the Weyl algebra is treated. We begin with a brief discussion of the quantum plane, where the generating variables satisfy $UV = qVU$, and derive the q-binomial formula (1.45). Then we turn to the q-deformed Weyl algebra characterized by (1.42) and show how the q-deformed Stirling and Bell numbers arise upon normal ordering. Several examples are treated and the q-deformed Wick's theorem derived. A connection to rooks is presented and a binomial formula given. Then, we consider normal ordering in the meromorphic Weyl algebra characterized by (1.43) and derive a binomial formula. The associated Stirling and Bell numbers are defined as normal ordering coefficients and some of their properties are studied. Most of these results are then extended to the q-meromorphic Weyl algebra.

In *Chapter 8* the generalized Weyl algebra $\mathsf{A}_{s;h} = \mathsf{A}_{s;h|1}$ is introduced; see Definition 1.25. We first survey the literature and point out close relatives of this algebra. Since it is an example of an Ore extension, we mention a few properties of Ore extensions and also describe some elementary normal ordering results for them. Then we discuss basic properties of $\mathsf{A}_{s;h}$ and also derive normal ordering results. In the main part of the chapter we introduce generalized Stirling numbers as in Definition 1.26 and study their properties (and those of the associated Bell numbers) in detail. Since it turns out that they are a particular subfamily of the generalized Stirling numbers of Hsu and Shiue, many properties follow from those reviewed in Chapter 4. We single out the particularly nice case $s = 2$, where the generalized Stirling numbers are given by Bessel numbers.

In *Chapter 9* we treat the algebra $\mathsf{A}_{s;h|q}$, see Definition 1.25. After deriving some basic normal ordering results, we turn to the binomial formula for $(U + V)^n$ and give operational interpretations for several special cases. We present "noncommutative Bell polynomials" and a "noncommutative binomial formula" in two different versions. Then we introduce the q-deformed generalized Stirling numbers as in Definition 1.26 and study their properties. An interpretation in terms of rook numbers is given and special cases are related to other q-deformed numbers.

In *Chapter 10* we introduce a generalization of Touchard polynomials related to normal ordering $\left(x^m \frac{d}{dx}\right)^n$. By definition, there exists a close connection to the generalized Stirling and Bell numbers considered in Chapter 8. Due to the operational treatment one can obtain binomial formulas for particular values of parameters, giving new examples for the results of Chapter 9. Generalizing from operators of the form $\left(x^m \frac{d}{dx}\right)^n$ to $\left(v(x) \frac{d}{dx}\right)^n$, one can use Comtet's result discussed in Chapter 4 to introduce and study so-called Comtet–Touchard functions. Finally, a q-deformation of the generalized Touchard polynomials is introduced and several properties are studied, in particular, a Spivey-like relation.

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