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# SO(9, 1) Group and Examples of Analytic Functions

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## Abstract.

Octonionic representation of  $O(9, 1)$  with special emphasis placed on the analytic properties of octonionic functions is investigated.

## 1. Introduction

There exist four normed division algebras, which are the real numbers ( $\mathbb{R}$ ), the complex numbers ( $\mathbb{C}$ ), the quaternions ( $\mathbb{H}$ ), and the octonions ( $\mathbb{O}$ ). Octonions provide a tenable path to a grand unified field theory. An octonion is a linear combination of eight basis vectors, or, unit octonions. It is well known that the exceptional groups of  $E_6$  and  $E_8$  are part and parcel to unification. Exceptional groups consist of associated Lie algebras, which are intimately connected to Jordan algebras of  $3 \times 3$  Hermitian matrices consisting of octonionic entries. In prior work, we detailed crucial properties of octonions, unit octonions, the construction of split octonions and a relevant algebra, as well as the construction of an octonionic Hilbert space from which we were able to represent elements of  $E_8$  [1,2]. All the exceptional groups,  $G_2, F_4, E_6, E_7, E_8$ , have octonions as their building blocks. Of these groups,  $E_6$  internally contains  $SO(10)$  or  $SU(5)$  as subgroups, which facilitate the existence of grand unified field theories [3], [4].

The Standard Model provides a gauge field theory of electromagnetic, weak and strong interactions. These forces are conjoined in grand unified theories (GUTs). Supersymmetry (SUSY) is required to stabilize GUTs, and we thereby obtain supersymmetric GUTs. Unification with gravitation is achieved via supergravity (SUGRA). Supersymmetric GUTs and SUGRA must themselves be combined so as to



reach an ultimate, final theory of everything (TOE). There are two leading candidates for a TOE. One is a 10 - dimensional extended supergravity theory incorporating super Yang-Mills matter, and the other is superstring theory. These theories possess Yang-Mills and gravitational anomalies. However, these anomalies cancel if the Yang-Mills theory is based on  $E_8 \times E_8$  or the SO(32) group. It is also well understood that  $E_8 \times E_8$  Yang-Mills theory gives rise to  $E_6 \times E_8$  theory in  $d = 4$ .  $E_6$ , as a GUT, is known to be broken into SO(10) or SU(5) GUTs. In addition, heterotic superstring theory is known to give in  $d = 10$ , SUGRA coupled to  $E_8 \times E_8$  Yang-Mills theory.

These observations demonstrate that unification of fields is possible only if the Yang-Mills sector is one of the aforementioned groups, and the Einstein sector is the theory of gravitation in  $(9 + 1)$  dimensions. Underlying both sectors is an octonionic structure. The gravitational part involves local Lorentz invariance in  $(9 + 1)$  dimensions. For example,  $O(9 + 1)$  has non-linear representations for massless particles, which can be expressed in terms of fractional linear transformations of octonions. Moreover, the internal charge space, that is,  $E_8 \times E_8$  or  $O(32)$ , corresponds to the dimensions of spinors in  $d = 10$ . In fact, requiring both Weyl and Majorana conditions be satisfied on  $d = 10$  spinors leads to a chiral Majorana spinor with 16 components. Such a spinor can be suitably represented by a pair of octonions. Each octonion separately represents the root space of  $E_8$  as shown by Coxeter [5]. A vector and a spinor together represent a point in a superspace of dimension 26. As a consequence, the chargeless superstring and the internal charge space of a charged superstring separately exhibit octonionic structures. As such, there is motivation into reformulating string and superstring theories, as well as their local supergravity limits, in terms of the octonion algebra.

The group  $O(8)$ , being the helicity group of the Poincare group in  $d = 10$  for a massless particle, serves a very important role. Its irreducible representations are imperative for classifying massless particle excitations in  $d = 10$ . We also note that if the octonions are replaced by quaternions, the relevant Lorentz group becomes  $O(5, 1)$  which operates in  $d = 6$  Minkowski space. Spinors in such a space have dimension 8, and so the superspace has dimension 14. Replacing quaternions by complex numbers generates  $O(3, 1)$ , the Lorentz group in which spinors have dimension 4. Therefore the superspace has dimension 8. In the case of real numbers, the corresponding Lorentz group is  $O(2, 1)$ , and spinors have dimension 2 so that the superspace has dimension 5. We remark that  $d = 3, 4, 6$  and  $10$  are the only dimensions in which one can write classical supersymmetric Yang-Mills theories. Quantized superstrings only exist for  $d = 10$ , hence in the octonionic case.

## 2. Octonionic Analyticity

Since octonions are neither commutative nor associative it was thought that a viable theory of functions of an octonionic variable may differ vastly from that of a complex and that of quaternionic variable. Theory of functions of a quaternionic variable was developed by Fueter [7], and later by Dentori and Sce [8], and it was shown that Fueter's analysis can be extended to the theory of functions of an octonionic variable. First, let us review complex analyticity before placing it into a form in which a natural generalization to octonions can be made.

Let  $z = x + iy$  be a complex variable and  $\omega(z) = u(x, y) + iv(x, y)$  be a complex-valued function of  $z$ . Then,  $\omega$  is analytic at  $z_0$  iff  $u_x - v_y = 0$ ,  $u_y + v_x = 0$  at the point  $z_0$  provided partial derivatives exist. These are known as the Cauchy-Riemann equations. Defining the differential operator  $D = \partial_x + i\partial_y$ , we have

$$D\omega = (\partial_x + i\partial_y)\omega = (\partial_x u - \partial_y v) + i(\partial_y u + \partial_x v) \quad (1)$$

Thus  $\omega(z)$  is analytic at  $z_0$  iff  $D\omega = 0$  at  $z_0$ .

Now, similar to the complex case we define an octonionic differential operator  $D$  by  $D = e_n \frac{\partial}{\partial x_n} = e_n \partial_n$ . Then the functions  $\ell(x) = \ell_n(x)e_n$ ,  $r(x) = r_n(x)e_n$ , and  $g(x) = g_n(x)e_n$  are said to be left-analytic, right-analytic and left-right-analytic respectively if they satisfy  $D\ell(x) = 0$ ,  $r(x)\underline{D} = 0$ ,

$Dg(x) = g(x)\underline{D} = 0$ . Note,  $\underline{D}$  operates to the left only. Now we have

$$\begin{aligned} D\ell(x) &= (\partial_0 + e_\mu \partial_\mu)(\ell_0 + \ell_\nu e_\nu) \\ &= (\partial_0 \ell_0 - \partial_\mu \partial_\mu) + e_\alpha (\partial_0 \ell_\alpha + \partial_\alpha \ell_0 + \phi_{\alpha\mu\nu} \partial_\mu \ell_\nu) = 0 \end{aligned} \quad (2)$$

$$\begin{aligned} r(x)\underline{D} &= (r_0 + r_\nu e_\nu)(\underline{\partial}_0 + e_\mu \underline{\partial}_\mu) \\ &= (\partial_0 r_0 - \partial_\mu r_\mu) + e_\alpha (\partial_0 r_\alpha + \partial_\alpha r_0 - \phi_{\alpha\mu\nu} \partial_\mu r_\nu) = 0 \end{aligned} \quad (3)$$

$$Dg(x) = (\partial_0 g_0 - \partial_\mu g_\mu) + e_\alpha (\partial_0 g_\alpha + \partial_\alpha g_0 + \phi_{\alpha\mu\nu} \partial_\mu g_\nu) = 0 \quad (4)$$

$$g(x)\underline{D} = (\partial_0 g_0 - \partial_\mu g_\mu) + e_\alpha (\partial_0 g_\alpha + \partial_\alpha g_0 - \phi_{\alpha\mu\nu} \partial_\mu g_\nu) = 0 \quad (5)$$

Therefore we have for the left analyticity  $\partial_\nu \ell_0 - \partial_\mu \ell_\mu = 0$  and  $\partial_0 \ell_\alpha + \partial_\alpha \ell_0 + \phi_{\alpha\mu\nu} \partial_\mu \ell_\nu = 0$ ; and for the right analyticity  $\partial_0 r_0 - \partial_\mu r_\mu = 0$  and  $\partial_0 r_\alpha + \partial_\alpha r_0 - \phi_{\alpha\mu\nu} \partial_\mu r_\nu = 0$ ; and for the left-right analyticity three equations:  $\partial_0 g_0 - \partial_\mu g_\mu = 0$ ,  $\partial_0 g_\alpha + \partial_\alpha g_0 = 0$ , and  $\phi_{\alpha\mu\nu} \partial_\mu g_\nu = 0$ .

We can now produce octonionic Gauss' theorem as follows: Let  $\Omega$  be an 8-dimensional volume with  $\partial\Omega$ . Then Gauss' theorem states

$$\int_{\Omega} (\partial_n \ell_m) d^8 x = \int_{\partial\Omega} \ell_m \partial \Sigma_n \quad (6)$$

where  $\partial \Sigma_n$  is a component of the surface element  $d\Sigma = d\Sigma_n e_n$ . Then the expression  $(d\Sigma)\ell$  reads

$$\begin{aligned} (d\Sigma)\ell &= (d\Sigma_0 + d\Sigma_\mu e_\mu)(\ell_0 + \ell_\nu e_\nu) \\ &= (d\Sigma_0)\ell_0 - (d\Sigma_\mu)\ell_\mu + e_\alpha ((d\Sigma_0)\ell_\alpha + (d\Sigma_\alpha)\ell_0 + \phi_{\alpha\mu\nu} (d\Sigma_\mu)\ell_\nu) \end{aligned} \quad (7)$$

Comparing the last expression with  $D\ell$  and applying the Gauss' theorem on each component we arrive at octonionic Gauss' theorem

$$\int_{\Omega} (D\ell) d^8 x = \int_{\partial\Omega} (d\Sigma)\ell \quad (8)$$

Similarly, we have

$$\int_{\Omega} (r \underline{D}) d^8 x = \int_{\partial\Omega} r d\Sigma. \quad (9)$$

Now if  $\ell(x)$  is left analytic in  $\Omega$  and on  $\partial\Omega$ , then  $D\ell = 0$  and we get

$$\int_{\partial\Omega} (d\Sigma)\ell = \int_{\Omega} (D\ell) d^8 x = 0 \quad (10)$$

and if  $r(x)$  is right-analytic in  $\Omega$  and on  $\partial\Omega$ , the  $r \underline{D} = 0$  and we get

$$\int_{\partial\Omega} r d\Sigma = \int_{\Omega} (r \underline{D}) d^8 x = 0. \quad (11)$$

Also, if  $g(x)$  is left-right analytic in  $\Omega$  and  $\partial\Omega$ , then

$$\int_{\partial\Omega} (d\Sigma)g = \int_{\Omega} g d\Sigma = 0. \quad (12)$$

These are the integral forms of analyticity condition for  $\ell(x)$ ,  $r(x)$  and  $g(x)$ . If we assume that instead of being analytic,  $\ell(x)$  satisfies  $D\ell(x) = \delta^8(x - A)$  where the point  $A$  is in  $\Omega$ , then we have

$$\int_{\partial\Omega} (d\Sigma)\ell = \int_{\Omega} (D\ell) d^8 x = 1. \quad (13)$$

A decomposition of a real octonion is useful in studying analytic functions. If we let  $r = x_\mu x_\mu$  and  $\eta = x_\mu e_\mu / r \in S^6$ ,  $\eta^2 = -1$ , then defining the projection operators  $E_\pm$  as

$$E_\pm = \frac{1}{2}(1 \pm i\eta) \quad (14)$$

which satisfy

$$E_+^2 = E_+, \quad E_-^2 = E_-, \quad E_+E_- = E_-E_+ = 0, \quad E_+ + E_- = 1 \quad (15)$$

If now  $z$  be a complex variable defined as  $z = x_0 + ir$ ,  $z^* = x_0 - ir$  then one can write  $x$  as

$$x = zE_- + z^*E_+. \quad (16)$$

We then get for any power of  $x$

$$x^n = z^n E_- + (z^*)^n E_+. \quad (17)$$

If now we let  $F(x)$  be expandable in a power series of  $x$  with real coefficients

$$F(x) = \sum_n x^n c_n, \quad c_n \in R \quad (18)$$

then using the above decomposition we get

$$F(x) = \left( \sum_n c_n z^n \right) E_- + \left( \sum_n c_n z^{*n} \right) E_+ = f(z)E_- + f(z)^*E_+, \quad f(z) = \sum_n c_n z^n \quad (19)$$

Since  $f(z)$  is expandable as a power series, it is complex analytic in the upper half plane:

$$f(z) = u(x_0, r) + iv(x_0, r) \quad (20)$$

$$\frac{\partial u}{\partial x_0} = \frac{\partial v}{\partial r}, \quad \frac{\partial u}{\partial r} = -\frac{\partial v}{\partial x_0} \quad (21)$$

We find for  $F(x)$ :

$$F(x) = (u + iv)\frac{1}{2}(1 - i\eta) + (u - iv)\frac{1}{2}(1 + i\eta) = u + \eta v \quad (22)$$

Thus  $F(z)$  can be obtained from a complex analytic function  $f(z) = u + iv$  by a mere replacement of the complex unit  $i$  by an element of the six-sphere,  $\eta$ . For this reason,  $f(z)$  are called the "stem functions".

Although  $F(x)$  as obtained as shown above is not analytic itself, another function  $G(x)$  obtained from  $F(x)$  as

$$G(x) = \square^3 F(x) \quad (23)$$

is left-right analytic, that is  $DG = \underline{GD} = 0$ .

In 4-dimensional Euclidean space the conformal group is  $O(5, 1)$  or  $Spin(5, 1) \sim SL(2, \mathbb{H})$  with the quaternionic representation

$$y = (ax + b)(cx + d)^{-1} \quad (24)$$

which is also written as

$$y = M\left(\frac{\lambda}{x - A} + \bar{C}\right)^{-1}N \quad (25)$$

where  $M, N, A, C \in \mathbb{H}$ ,  $\lambda \in R$ ,  $M\bar{M} = N\bar{N} = 1$ . This transformation has  $3 + 3 + 1 + 4 + 4 = 15$  parameters. The transformation  $y = Mx'\bar{N}$  is related to  $Spin4 \sim SU(2) \times SU(2)$  with 6 parameters and

$$x' = \left(\frac{\lambda}{x - A} + \bar{C}\right)^{-1} \quad (26)$$

is related to  $\frac{Spin(5,1)}{Spin4}$  with 9 parameters

### 3. Conformal group $SO(9, 1)$ in Euclidean 8-dimensions:

In Euclidean 8-dimensions, the 45 parameter conformal group is  $SO(9, 1)$ , which is also the Lorentz group in 9+1 dimensions. It is obtained by combining  $SL(2, \mathbb{O})$ , the set of fractional linear transformations with octonionic coefficients having  $4 \times 8 - 1 = 31$  parameters, with the automorphism group  $G_2$  having 14 parameters.

Let the octonion  $x$  be the spacetime position in  $R^8$ , then an octonionic representation of a  $SO(9, 1)$  transformation can be written as

$$y = T_{O(8)}X(x)^{-1} = T_{O(8)}x'^{-1} \quad (27)$$

where  $T_{O(8)}$  is a 28 parameter  $SO(8)$  transformation and

$$x' = X(x) = \frac{\lambda}{x - A} + \bar{C} \quad (28)$$

is a 17 parameter transformation related to  $\frac{SO(9,1)}{SO(8)}$ .  $O(8)$  is the helicity group for a state with lightlike momentum. For such a state the little group leaves the norm of the transverse components invariant. Its maximum compact group is  $O(8)$  which is the helicity group of  $m=0$  states in  $d=10$ . Any Poincare' covariant massless state in  $D=10$  must be classified with the unitary representation of  $O(8)$ , true for SUGRA, Yang-Mills theories and superstring theories in 10 dimensions. We can now decompose  $T_{o(8)}$  into  $G_2$ , as

$$\frac{Spin(8)}{G_2} = \frac{Spin(8)}{Spin(7)} \times \frac{Spin(7)}{G_2}. \quad (29)$$

We have

$$\frac{Spin(7)}{G_2} : x'' = Lx'\bar{L} \text{ with } L \in \Omega, |L| = 1 \quad (30)$$

$$\frac{Spin(8)}{Spin(7)} : x''' = Kx''\bar{K} \text{ with } K \in \Omega, |K| = 1 \quad (31)$$

$$G_2 : y = (UV)^{-1}[V(Ux'''U^{-1})V^{-1}](UV) \quad (32)$$

with  $U, V \in \Omega$  and  $|U| = |V| = 1$ . Thus the conformal group in  $R^8$  admits the Mobius representation

$$y = (UV)^{-1}\{V(U[K(L[\frac{\lambda}{x-A} - \bar{C}]^{-1}\bar{L})K]U^{-1})V^{-1}\}(UV) \quad (33)$$

Using fundamental Moufang identities one can derive useful equations for  $X(x)$ . Writing  $\square = D\bar{D} = \bar{D}D$  the eight dimensional Laplace operator, where  $D = e_\mu \frac{\partial}{\partial x_\mu}$  and  $\bar{D} = \bar{e}_\mu \frac{\partial}{\partial x_\mu}$ , we have

$$DX = 6\lambda|x - A|^{-2} = \rho_0, \quad Vec \rho_0 = 0 \quad (34)$$

$$(x - A)^{-1} = -\frac{1}{2}\bar{D}(\ln \rho_0) \quad (35)$$

$$\square^3 D(x - A)^{-1} = -\frac{1}{2}\square^4 \ln \rho_0 = 0 \quad (36)$$

$$\square X = \bar{D}\rho_0 = -2\rho_0(x - A)^{-1} \quad (37)$$

$$\square^2 X = \frac{16}{3\lambda}\rho_0^2(x - A)^{-1} \quad (38)$$

$$\square^3 X = -\frac{32}{3\lambda^2}\frac{\rho_0^3}{x - A} \quad (39)$$

$$\square^3 DX = 0 \quad (40)$$

$$\square^4 X = 0 \quad (41)$$

We see that  $\square^3 X$  is left-right analytic and  $X$  is quadri-harmonic.

We can also apply an inversion to  $y$  as

$$Y(y) = y^{-1} = T_{O(8)} x'^{-1} = T_{O(8)} \left( \frac{\lambda}{x - A} + \bar{C} \right) \quad (42)$$

hence the transformation  $X(x)$  represents the coset space  $\frac{Spin(9,1)}{Spin(8)}$ , having  $(45-28=17)$  parameters, and consequently

$$\square^4 T_{o(8)} X = \square^4 Y = 0. \quad (43)$$

Defining a function  $B(x)$  as the finite sum of the basic transformations  $X_i(x)$ , namely

$$B(x) = \sum \left( \frac{\lambda_i}{x - c_i} + \mu_i \right) \quad (44)$$

with  $DB = \sum \rho_i = \rho$ , then by setting

$$\bar{a} = (\square^3 B)(\square^2 DB)^{-1} \quad (45)$$

we obtain

$$\bar{a} = -6 \left[ \sum \rho_i^3 \lambda_i^2 (x - c_i)^{-1} \right] \left( \sum \rho_i^3 \lambda_i^{-2} \right)^{-1} = \bar{D} \ln(\square^2 \rho) \quad (46)$$

and verify

$$D\bar{a} = (\square^3 DB)(\square^2 \rho)^{-1} - (\square^3 B)(\square^2 \rho)^{-2} (D\square^2 \rho) = -(\square^2 D\rho)(\square^2 \rho)^{-1} \bar{a} = -a\bar{a}, \quad (47)$$

or

$$D\bar{a} + a\bar{a} = 0. \quad (48)$$

The scalar and vector parts of this equation give

$$\partial_n a_n + a_n a_n = 0, \quad \text{and} \quad e'_{nm} \partial_n a_m = 0. \quad (49)$$

As to the non-trivial topology of  $a$ , its explicit form yields

$$a \sim x^{-1} \sim \bar{U} D U \quad (50)$$

as  $|x| \rightarrow \infty$ ,  $U = x|x|^{-1}$  being the unit octonion parametrizing the seven-sphere. Thus this asymptotic behavior identifies  $a$  as non-trivial  $S^7 \rightarrow S^7$  mappings at infinity in  $R^8$ .

We now look into computing the associated winding number. We take an octonionic function  $r(x)$  of an octonion  $x$  on  $S^8$  such that

$$Dr = \sum R_i \delta^8(x - C_i). \quad (51)$$

Then by Stokes' formula

$$\int_{\Omega} DR d^8 x = \oint_{\partial\Omega=S^7} r d\Sigma = \sum_{i=0}^n R_i \quad (52)$$

provided the poles  $C_i$  are located within a domain  $\Omega$  of  $R^8$ . Taking  $r = \square^3 B$  and  $a$  given by  $D\bar{a} + a\bar{a} = 0$ , near the poles  $c_i$ ,

$$\bar{a}(C_i + \epsilon) = -\frac{6}{\epsilon} \quad (53)$$

so that

$$\bar{a} = -\frac{6}{x - C_i} + \text{a regular function in } (x - C_i) \quad (54)$$

and

$$D\Box^3\bar{a} = -6V\delta^8(x - C_i), \quad (55)$$

$V$  being the  $S^7$  volume, since an application of  $\Box^3 D$  on a regular function gives zero. By surrounding each pole  $x = C_i$  by a small 7-sphere  $S_i^7$ , we see that  $\Sigma \int d\Sigma \Box^3 \bar{a}$  simply counts the number of poles in  $a$ . As equation  $\partial_n a_n + a_n a_n = 0$  is clearly conformally invariant in the 8-space  $S^8$ , one of the terms in  $\bar{a}$ , for example

$$(x - C_0)^{-1} \quad (56)$$

can be transformed to the regular function  $x$  by the conformal transformation such as a coordinate inversion.

Consequently out of  $(n + 1)$  poles in  $a$  only  $n$  are significant. We then get the winding number

$$C_4 = \frac{1}{48\pi^4} \oint_{S^7} d\Sigma \Box^3 \bar{a} = n. \quad (57)$$

Again, by Stoke's theorem it can be cast into a 8-dimensional integral

$$C_4 = -\frac{1}{48\pi^4} \int_{S^8} d^8 x \Box^4 \ln(\Box^2 \rho). \quad (58)$$

Just as in the complex  $d = 2$  and quaternionic  $d = 4$  analytic cases, octonion analyticity implies an infinite number of continuity equations. Due to the ring structure of the left-right holomorphic  $B(x)$  functions and power associativity of the octonions, not just  $B(x)$  but any of its powers

$$[B(x)]^n, \quad n = 2, 3, 4, \dots \quad (59)$$

also solves for the case of Cauchy-Riemann equations

$$DJ^{(n)}(x) = D(\Box^3 [B(x)]^n) = 0. \quad (60)$$

This last expression implies an infinite number of octonionic continuity equations. Using arguments paralleling the quaternionic case the octonionic Cauchy integral theorem for octonion analyticity tells us that the Euclidean charges

$$q^n(\tau) = q_\mu^{(n)} e_\mu = \oint_{S^7} \xi_1^{(n)} d\Sigma \xi_2^{(n)} \quad (61)$$

where

$$\xi_1^{(n)} = \Box^3 B_1^{(n)} \quad \text{and} \quad \xi_2^{(n)} = \Box^3 B_2^{(n)} \quad (62)$$

are independent of  $\tau$ , parametrizing the family of  $D = 7$  hypersurfaces  $S_\tau^7$  in  $S^8$ . The  $q^{(n)}$  are therefore conserved:

$$\frac{dq^n}{d\tau} = 0. \quad (63)$$

#### 4. Conclusion

In conclusion, much work needs to be done in the theory of octonionic functions as well as in the topic of exceptional non-associative geometries. A deeper understanding of eight dimensional space and octonion analyticity might be useful, as in the spirit of Kaluza-Klein compactification, for the non-perturbative string theories, and generalized electric/magnetic dualities for extended objects.

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