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Special Issue

Symmetry in Researches of Neutron Stars and Black Holes

Edited by
Dr. Mihaela Parvu and Prof. Dr. Ionel Lazanu

MDPI

https://doi.org/10.3390/sym17010054

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Noncommutative Reissner–Nordström Black Hole from Noncommutative Charged Scalar Field

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Abstract: Within the framework of noncommutative (NC) deformation of gauge field theory by the angular twist, we first rederive the NC scalar and gauge field model from our previous papers, and then generalize it to the second order in the Seiberg–Witten (SW) map. It turns out that SW expansion is finite and that it ceases at the second order in the deformation parameter, ultimately giving rise to the equation of motion for the scalar field in the Reissner–Nordström (RN) metric that is nonperturbative and exact at the same order. As a further step, we show that the effective metric put forth and constructed in our previous work satisfies the equations of Einstein–Maxwell gravity, but only within the first order of deformation and when the gauge field is fixed by the Coulomb potential of the charged black hole. Thus, the obtained NC deformation of the Reissner–Nordström (RN) metric appears to have an additional off-diagonal element which scales linearly with a deformation parameter. We analyze various properties of this metric.

Keywords: noncommutative spaces; angular twist; black hole and effective metric

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Academic Editor: Maxim Y. Khlopov

Received: 1 November 2024
Revised: 16 December 2024
Accepted: 20 December 2024
Published: 31 December 2024

Citation: Dimitrijević Ćirić, M.; Konjik, N.; Jurić, T.; Samsarov, A.; Smolić, I. Noncommutative Reissner–Nordström Black Hole from Noncommutative Charged Scalar Field. *Symmetry* **2025**, *17*, 54. https://doi.org/10.3390/sym17010054

1. Introduction

So far, general relativity (GR) has been shown to be a highly successful theory of gravity, manifested in its remarkable ability to explain a series of observations [1–4] ranging from the early-days examinations of the perihelion precession of Mercury, the bending of light, and the gravitational redshift of radiation from distant stars, to modern day experimental achievements in detecting gravitational waves and imaging of black holes. What was in the not-so-recent past only a mere theoretical conception, following the appearance of advanced ground-based and space-based missions [5–9] like the LIGO and the Event Horizon Telescope, soon became a factual physical reality. While the LIGO experiment set the ground for the first ever detection of gravitational waves from colliding black holes and neutron stars, the Event Horizon Telescope provided an image of the black hole M87* (actually an image of the gas orbiting around the black hole at the center of the supergiant elliptical galaxy Messier 87), thus further adding to GR's enviable predictive power [10–16].

However, in order that a premise of general relativity as the correct theory of gravity be sustained, it was necessary to introduce into consideration a few exotic ingredients, such as dark matter and dark energy [17–19], to explain the galactic rotation curves and the accelerated expansion of the universe. In addition, the conceptual problems with black holes and the Big Bang singularity [20,21] point to the fact that the ultraviolet character of gravity still lacks a complete understanding. With all these issues, any attempt to modify general relativity or to consider alternative gravity models appears to come as a quite natural endeavor.

In this paper, we use the methods of noncommutative geometry and noncommutative gravity [22–30], first to recapitulate a construction of the NC scalar and gauge field model from reference [31], and then to generalize it to the second order in the Seiberg–Witten map [23]. It is shown that the second order in the expansion is at the same time an ultimate order, and consequently the model obtained is nonperturbative and exact. Using certain duality symmetries that are present at the first order in SW expansion, we recap a construction from [31,32] that gives rise to a particular noncommutative deformation of the RN metric. While this construction turns out to be possible at the first order in SW, it fails at the second order. This is due to the fact that duality symmetry breaks at the second order in SW expansion.

As a further step, we show that the metric put forth and constructed in [32] appears to be a deformation of the Reissner–Nordström (RN) black hole that acquires an additional off-diagonal element, linear in the deformation parameter, and satisfying the Einstein–Maxwell equations at the first order of deformation. The construction in reference [32] was carried out by utilizing the methods of noncommutative (NC) gauge field theory [22–30] coupled to an NC spinor field and to a classical geometry of the RN type. The methods of NC gauge theory and gravity offer yet another convenient way to modify general relativity in order to capture effects that are expected to appear close to the Planck scale. The ultimate hope is that the NC modifications of gravity will unravel something of its quantum character. In the rest of the paper, we go on to explore the physical properties of this NC-deformed metric and try to understand its origin and meaning.

It is noteworthy that the construction considered in [32] is not the only attempt in the literature to deform the RN metric (within the framework of noncommutative physics). Indeed, in recent years there have been several investigations concerning the noncommutative versions of Reissner–Nordström (RN) black holes [33–38]. Most of the research in the literature however has dealt with the so-called Moyal-type noncommutativity $[\hat{x}^\mu, \hat{x}^\nu] = \theta^{\mu\nu}$. For example, in [33,34,36] the authors used this type of noncommutativity and implemented it using smeared δ-functions for the mass and charge distributions. The main feature of such systems is the change in Hawking temperature and entropy. An alternative approach was presented in [35], where Moyal noncommutativity was introduced using deformed embedding of RN into deformed Riemannian geometry. Using the framework of the NC gauge theory of gravity, the authors of [37] were able to construct corrections to the RN solution and showed that this could lead to a removal of singularities.

The structure of the paper is as follows: In the following section, we review our model of the NC charged scalar field in a curved background coupled to the NC $U(1)$ gauge field. In Section 3, we extend our results to the second-order expansion in the deformation parameter and show that the equation of motion for the NC scalar field does not contain higher-order terms. This defines the exact (in the NC parameter expansion) model of an NC charged scalar field coupled to the curved (spherically symmetric) background. In Sections 4 and 5, we discuss the properties of the NC charged black hole, obtained from the effective/dual metric in the equation of motion for the NC scalar filed. Section 6 contains further discussion and some conclusions. In particular, we comment on the possibility of constructing the effective metric up to the second order in the deformation parameter by introducing the nonmetricity tensor.

2. NC Scalar Field in the Reissner–Nordström Background

Consider a system consisting of a charged scalar and $U(1)$ gauge field, as well as the classical gravitational field. We want to deform this system in order to ultimately generate one type of deformation of the classical solution to Einstein's gravity. In particular, it is a noncommutative deformation of the RN metric that we focus our attention on. The required steps may be carried out by following [32], where noncommutativity was introduced at the level of the scalar field that is probing the underlying RN background. After a careful derivation of the corresponding equation of motion, which we briefly repeat here, one comes to the conclusion that this system of an RN black hole coupled to an NC scalar and gauge fields is equivalent/dual to a system of a commutative scalar field and a new effective background, which up to the first order in the deformation parameter absorbs all NC effects. We refer to this effective background as the noncommutative Reissner–Nordström (NCRN) black hole.

Let us start with writing the action functional for the NC $U(1)_\star$ gauge theory of a massless charged scalar field $\hat{\phi}$ in an arbitrary background (that has Killing vectors ∂_t and ∂_φ) [31]:

$$\begin{aligned} \mathcal{S}[\hat{\phi}, \hat{A}] &= \int (d\hat{\phi} - i\hat{A} \star \hat{\phi})^\dagger \wedge_\star *(d\hat{\phi} - i\hat{A} \star \hat{\phi}) - \frac{1}{4}\int \hat{F} \wedge_\star *\hat{F} \\ &= \int d^4x \sqrt{|g|} \star g^{\mu\nu} \star D_\mu\hat{\phi}^\dagger \star D_\nu\hat{\phi} - \frac{1}{4}\int d^4x \sqrt{|g|} \star g^{\alpha\beta} \star g^{\mu\nu} \star \hat{F}_{\alpha\mu} \star \hat{F}_{\beta\nu} \end{aligned} \tag{1}$$

where D_μ is the covariant derivative defined by

$$D_\mu\hat{\phi} = \partial_\mu\hat{\phi} - iq\hat{A} \star \hat{\phi} \tag{2}$$

and $\hat{F} = \hat{F}_{\mu\nu} \star dx^\mu \wedge_\star dx^\nu$ is the field strength, defined by

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i[\hat{A}_\mu \stackrel{\star}{,} \hat{A}_\nu]. \tag{3}$$

Action (1) is written in spherical coordinates as $x^\mu = (t, r, \theta, \varphi)$ and the Hodge dual is denoted by $*$.

The $\star$-product is given by the Abelian twist

$$\mathcal{F} = e^{-\frac{ia}{2}(\partial_t \otimes \partial_\varphi - \partial_\varphi \otimes \partial_t)} = e^{-\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha \otimes \partial_\beta} \tag{4}$$

via (m is the multiplication map $m(a \otimes b) = ab$)

$$\begin{aligned} f \star g &= m\left(\mathcal{F}^{-1} \triangleright f \otimes g\right) \\ &= fg + \frac{ia}{2}\left(\frac{\partial f}{\partial t}\frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \varphi}\frac{\partial g}{\partial t}\right) + \mathcal{O}(a^2), \end{aligned} \tag{5}$$

where $f, g \in \mathcal{C}^\infty$ and $\theta^{\alpha\beta}$ are components of the NC deformation, with only $\theta^{t\varphi}$ and $\theta^{\varphi t}$ being different from zero: $\theta^{t\varphi} = -\theta^{\varphi t} = a$. Note that this twist leads to the only nonvanishing commutator $[t \stackrel{\star}{,} e^{i\varphi}] = -ae^{i\varphi}$. The twist (4) may be seen as a special case of the general class of twists related to the Lie-algebraic deformation of Minkowski space [39]. Note that since the twist $\mathcal{F}$ acts trivially on the metric, the $\star$-product in $\sqrt{|g|} \star g^{\alpha\beta} \star g^{\mu\nu}$ can be omitted. Now, it is straightforward to check that action (1) is invariant under the infinitesimal $U(1)_\star$ gauge transformations defined by

$$\delta_\star\hat{\phi} = i\hat{\Lambda} \star \hat{\phi}, \quad \delta_\star\hat{A}_\mu = \partial_\mu\hat{\Lambda} + i[\hat{\Lambda} \stackrel{\star}{,} \hat{A}_\mu], \quad \delta_\star\hat{F}_{\mu\nu} = i[\hat{\Lambda} \stackrel{\star}{,} \hat{F}_{\mu\nu}] \tag{6}$$

where $\hat{\Lambda}$ is the NC gauge parameter.

Using the Seiberg–Witten(SW) map [23], one can express the NC fields as functions of the corresponding commutative fields, which can then be expanded as a series in the deformation parameter a. Using the twist (4), one obtains the following recursion relations [40]:

$$\begin{aligned}\hat{\phi}^{(n+1)} &= -\frac{1}{4(n+1)}\theta^{\rho\sigma}\Big(\hat{A}_\rho \star (\partial_\sigma\hat{\phi} + D_\sigma\hat{\phi})\Big)^{(n)},\\ \hat{A}_\mu^{(n+1)} &= -\frac{1}{4(n+1)}\theta^{\rho\sigma}\Big(\{\hat{A}_\rho \stackrel{\star}{,} (\partial_\sigma\hat{A}_\mu + \hat{F}_{\sigma\mu})\}\Big)^{(n)},\\ \hat{F}_{\mu\nu} &= -\frac{1}{4(n+1)}\theta^{\rho\sigma}\Big(\{\hat{A}_\rho \stackrel{\star}{,} \partial_\sigma\hat{F}_{\mu\nu} + D_\sigma\hat{F}_{\mu\nu}^{(n+1)}\}\Big)^{(n)} + \frac{1}{2(n+1)}\theta^{\rho\sigma}\Big(\{\hat{F}_{\mu\rho}\stackrel{\star}{,}\hat{F}_{\nu\sigma}\}\Big)^{(n)}.\end{aligned} \tag{7}$$

Using the first-order results of (7) and the $\star$-product (5), we expand action (1) up to the first order in the deformation parameter a as follows:

$$\begin{aligned}\mathcal{S} = \int d^4x\sqrt{|g|}\Big(& D_\mu\phi^\dagger D^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}\theta^{\alpha\beta}(F_{\alpha\beta}F_{\mu\nu}F_{\rho\sigma} - 4F_{\mu\alpha}F_{\nu\beta}F_{\rho\sigma})\\ &+ \frac{1}{2}\theta^{\alpha\beta}g^{\mu\nu}(-\frac{1}{2}F_{\alpha\beta}D_\mu\phi^\dagger D_\nu\phi + F_{\alpha\nu}D_\mu\phi^\dagger D_\beta\phi + F_{\alpha\mu}D_\beta\phi^\dagger D_\nu\phi)\Big) + \mathcal{O}(a^2),\end{aligned} \tag{8}$$

where D_μ is the usual $U(1)$ covariant derivative $D_\mu\phi = \partial_\mu\phi - iqA_\mu\phi$. If we add the classical EH action to (8), the resulting functional may be viewed as a deformation of Einstein–Maxwell gravity, leading to an effective theory of gravity akin to some effective models of gravity obtained in the low-energy limit of a string theory action containing the gravitational, gauge, and dilaton or axion fields [41,42].

By varying action (8) with respect to $\phi^\dagger$, one obtains an equation of motion for ϕ:

$$\begin{aligned}& g^{\mu\nu}\Big[D_\mu D_\nu\phi - \Gamma^\lambda_{\mu\nu}D_\lambda\phi\\ &- \frac{1}{4}\theta^{\alpha\beta}\big(D_\mu(F_{\alpha\beta}D_\nu\phi) - \Gamma^\lambda_{\mu\nu}F_{\alpha\beta}D_\lambda\phi - 2D_\mu(F_{\alpha\nu}D_\beta\phi) + 2\Gamma^\lambda_{\mu\nu}F_{\alpha\lambda}D_\beta\phi - 2D_\beta(F_{\alpha\mu}D_\nu\phi)\big)\Big] = 0.\end{aligned} \tag{9}$$

Varying the action with respect to A_λ, one can obtain the NC Maxwell's equations [31].

The gravitational background is defined by the Reissner–Nordström spacetime, with metric

$$g_{\mu\nu} = \begin{pmatrix} -f(r) & 0 & 0 & 0\\ 0 & \frac{1}{f(r)} & 0 & 0\\ 0 & 0 & r^2 & 0\\ 0 & 0 & 0 & r^2\sin^2\theta\end{pmatrix}, \qquad f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{10}$$

where M is the mass and Q the charge of the RN black hole, and the $U(1)$ gauge field is

$$A_\mu = (-\frac{Q}{r}, \vec{0}\,). \tag{11}$$

The corresponding field strength $F_{\mu\nu}$ has the only nonvanishing components:

$$F_{tr} = -F_{rt} = -\frac{Q}{r^2}. \tag{12}$$

Furthermore, since the only nonvanishing components of the NC deformation $\theta^{\alpha\beta}$ are $\theta^{t\varphi} = -\theta^{\varphi t} = a$, by inserting (10), (11), and (12) into (9), we finally obtain

$$\Big(\frac{1}{f}\partial_t^2 - \Delta + (1-f)\partial_r^2 + \frac{2M}{r^2}\partial_r + \frac{2iqQ}{rf}\partial_t - \frac{q^2Q^2}{r^2 f}\Big)\phi + \frac{aqQ}{r^3}\Big(\Big(\frac{M}{r} - \frac{Q^2}{r^2}\Big)\partial_\varphi + rf\partial_r\partial_\varphi\Big)\phi = 0. \tag{13}$$

Equation (13) is the equation of motion of an NC scalar field in the background of the RN black hole. This equation, its quasinormal-mode solutions, and the Bekenstein–Hawking entropy were extensively studied in [31,43,44]. Note that in the limit $a \to 0$ one obtains the usual equation of motion of a commutative scalar field in the RN background.

3. Exact Equation in the Second-Order SW Map

In this section, we extend the previous analysis to the second order in the SW expansion. Remarkably, the SW expansion terminates at this order, and consequently the resulting equations of motion is exact. In order to find the second-order NC corrections, we use recurrent relations for the SW map (7) and follow steps similar to those in Sections 3 and 4 in [40]. Similarly to (7), the recursion relations for action (1) allow us to express corrections in order $n+1$ from the corrections in order n by substituting all pointwise products with the $\star$-products and commutative fields with the corresponding NC fields. By closer inspection and taking into account that the only nonvanishing components of the $\theta^{\mu\nu}$ and $F_{\mu\nu}$ are $\theta^{t\varphi}$ and F_{tr}, we see that the terms from the first-order expansion which give nonzero corrections in the second order are the following (the superscript in $(\ldots)^{(i)}$ denotes that only the i-th order in the deformation parameter a is retained):

$$S^{(2)} = \sqrt{|g|}\frac{\theta^{\alpha\beta}g^{\mu\nu}}{2}(D_\mu\hat{\phi}^\dagger \star \hat{F}_{\alpha\nu} \star D_\beta\hat{\phi} + D_\beta\hat{\phi}^\dagger \star \hat{F}_{\alpha\mu} \star D_\nu\hat{\phi})^{(1)}. \tag{14}$$

Inserting the SW map solutions (7) and expanding the $\star$-products, with the help of the useful method for obtaining manifestly covariant results given in Appendix B in [45], (14) becomes

$$\begin{aligned} S^{(2)} = \quad & \sqrt{|g|}\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}g^{\mu\nu}(-2A_\gamma\partial_\delta(D_\mu\phi^\dagger F_{\alpha\nu}D_\beta\phi) + iD_\gamma(D_\mu\phi^\dagger D_\beta\phi)D_\delta F_{\alpha\nu} \\ & +iF_{\alpha\nu}(D_\gamma D_\mu\phi^\dagger)(D_\delta D_\beta\phi) + D_\mu\phi^\dagger F_{\alpha\nu}F_{\gamma\beta}D_\delta\phi \\ & +D_\delta\phi^\dagger F_{\alpha\nu}F_{\gamma\mu}D_\beta\phi + 2D_\mu\phi^\dagger F_{\gamma\alpha}F_{\delta\nu}D_\beta\phi) \\ & +\sqrt{|g|}\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}g^{\mu\nu}(-2A_\gamma\partial_\delta(D_\beta\phi^\dagger F_{\alpha\mu}D_\nu\phi) + iD_\gamma(D_\beta\phi^\dagger D_\nu\phi)D_\delta F_{\alpha\mu} \\ & +iF_{\alpha\mu}(D_\gamma D_\beta\phi^\dagger)(D_\delta D_\nu\phi) + D_\beta\phi^\dagger F_{\alpha\mu}F_{\gamma\nu}D_\delta\phi \\ & +D_\delta\phi^\dagger F_{\alpha\mu}F_{\gamma\nu}D_\nu\phi + 2D_\beta\phi^\dagger F_{\gamma\alpha}F_{\delta\mu}D_\nu\phi). \end{aligned} \tag{15}$$

After subsequent partial integrations and the use of the identity $i[D_\alpha, D_\beta]\phi = F_{\alpha\beta}\phi$, as well as the fact that derivatives which are contracted with the NC deformation parameter matrix $\theta^{\alpha\beta}$ do not act on the field strength tensor $F_{\mu\nu}$, we obtain

$$\begin{aligned} S^{(2)} = \quad & \sqrt{|g|}\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}g^{\mu\nu}(-F_{\gamma\delta}(D_\mu\phi^\dagger F_{\alpha\nu}D_\beta\phi) + iD_\gamma(D_\mu\phi^\dagger D_\beta\phi)D_\delta F_{\alpha\nu} \\ & -F_{\alpha\nu}(D_\mu\phi^\dagger)F_{\gamma\delta}(D_\beta\phi) + D_\mu\phi^\dagger F_{\alpha\nu}F_{\gamma\beta}D_\delta\phi \\ & +D_\delta\phi^\dagger F_{\alpha\nu}F_{\gamma\mu}D_\beta\phi + 2D_\mu\phi^\dagger F_{\gamma\alpha}F_{\delta\nu}D_\beta\phi) \\ & +\frac{1}{4}\sqrt{|g|}\theta^{\alpha\beta}\theta^{\gamma\delta}g^{\mu\nu}(-F_{\gamma\delta}(D_\beta\phi^\dagger F_{\alpha\mu}D_\nu\phi) + iD_\gamma(D_\beta\phi^\dagger D_\nu\phi)D_\delta F_{\alpha\mu} \\ & +F_{\alpha\mu}(D_\beta\phi^\dagger)F_{\gamma\delta}(D_\nu\phi) + D_\beta\phi^\dagger F_{\alpha\mu}F_{\gamma\nu}D_\delta\phi \\ & +D_\delta\phi^\dagger F_{\alpha\mu}F_{\gamma\beta}D_\nu\phi + 2D_\beta\phi^\dagger F_{\gamma\alpha}F_{\delta\mu}D_\nu\phi). \end{aligned} \tag{16}$$

Since $F_{t\varphi} = 0$, some of the above terms vanish, while the others add to one term given by

$$S^{(2)} = \sqrt{|g|}\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}g^{\mu\nu}(D_\beta\phi^\dagger F_{\alpha\mu}F_{\gamma\nu}D_\delta\phi + D_\beta\phi^\dagger F_{\alpha\mu}F_{\gamma\nu}D_\delta\phi) = \sqrt{|g|}\frac{1}{2}\theta^{\alpha\beta}\theta^{\gamma\delta}g^{\mu\nu}(D_\beta\phi^\dagger F_{\alpha\mu}F_{\gamma\nu}D_\delta\phi). \tag{17}$$

Variation of these terms with respect to $\phi^\dagger$ gives rise to additional terms in the equation of motion. It turns out that only one new term appears, which is of the form

$$\frac{1}{2}\theta^{\alpha\beta}\theta^{\gamma\delta}g^{\mu\nu}F_{\alpha\mu}F_{\gamma\nu}D_\beta D_\delta\phi.$$

More explicitly, we obtain

$$\frac{1}{2}\theta^{t\varphi}\theta^{t\varphi}g^{rr}F_{tr}F_{tr}\partial_\varphi^2\phi = \frac{1}{2}a^2(-f)\frac{q^2Q^2}{r^4}\partial_\varphi^2\phi = -\frac{a^2q^2Q^2}{2r^4}f\partial_\varphi^2\phi.$$

Finally, the resulting equation of motion is

$$\Big(\frac{1}{f}\partial_t^2 - \Delta + (1-f)\partial_r^2 + \frac{2MG}{r^2}\partial_r + 2iqQ\frac{1}{rf}\partial_t - \frac{q^2Q^2}{r^2f}\Big)\phi$$
$$+\frac{aqQ}{r^3}\Big(\big(\frac{MG}{r} - \frac{GQ^2}{r^2}\big)\partial_\varphi + rf\partial_r\partial_\varphi\Big)\phi - \frac{a^2q^2Q^2}{2r^4}f\partial_\varphi^2\phi = 0. \tag{18}$$

As already noted, the equation of motion (18) is not just a perturbative result valid up to the second order in deformation. It is an exact result and may be attributed to the SW map terminating at that same order. As an advantageous outcome, one finds that all analysis that is ever going to follow from this equation requires no perturbative protocols anymore. All results following from (18) are exact automatically. There is one more way to justify why Equation (18) is exact and no higher-order corrections appear. Namely, the SW map is linear in matter fields, while action (1) is quadratic in the matter field ϕ. The only nonzero components of the deformation parameter θ are $\theta^{t\varphi}$, so each new order of expansion contributes one additional set of φ and t indices. Note that the index φ can only appear contracted to $D_\varphi\phi$, since all $F_{\mu\varphi} = 0$. Since action (1) is quadratic in the field ϕ and we can always partially integrate multiple covariant derivatives on ϕ to obtain $F_{\rho\sigma}$, we conclude that the maximal number of D_φ in the expanded action is two, and therefore the expansion of the action has to terminate at the second order.

4. Noncommutative Reissner–Nordström Black Hole

In this section, we focus on the first order in the SW expansion, that is, the equation of motion (13), and identify a duality symmetry that exists at that order. This symmetry will allow us to absorb the noncommutative contributions into a single d'Alembertian operator and ultimately to identify the effective metric related to this problem, which will turn out to be a deformation of the Reissner–Nordström metric. We later discuss possible extensions of the duality symmetry to higher orders.

The equation of motion for the NC scalar field minimally coupled to the RN background can be written in the following form [32]:

$$\frac{1}{\sqrt{|g|}}D_\mu(\sqrt{|g|}g^{\mu\nu}D_\nu\phi) + \Box_a\phi = 0, \tag{19}$$

where $\Box_a$ is the part of (13) which contains only NC contributions. Now, we try to rearrange (19) so that the NC operator $\Box_a$ is absorbed into some effective metric $g'_{\mu\nu}$. Namely, we write

$$\frac{1}{\sqrt{|g'|}}D_\mu(\sqrt{|g'|}g'^{\mu\nu}D_\nu\phi) = \frac{1}{\sqrt{|g|}}D_\mu(\sqrt{|g|}g^{\mu\nu}D_\nu\phi) + \Box_a\phi. \tag{20}$$

We can write an ansatz for $g'_{\mu\nu}$, and after carefully comparing the left- and right-hand sides of (20), we can extract the components of the effective metric $g'_{\mu\nu}$ to obtain

$$g'_{\mu\nu} = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & \frac{1}{f} & 0 & \frac{aqQ}{2}\sin^2\theta \\ 0 & 0 & r^2 & 0 \\ 0 & \frac{aqQ}{2}\sin^2\theta & 0 & r^2\sin^2\theta \end{pmatrix} + \mathcal{O}(a^2). \quad (21)$$

Since we have an effective metric $g'_{\mu\nu}$, we can notice an equivalence between the equation of motion of an NC scalar field in the RN background (9) and the equation of motion guiding a commutative scalar field on some effective background endowed with the effective metric $g'_{\mu\nu}$. The similar property has already been observed for the NC scalar field on the BTZ background [46–49] in the context of κ-deformation. In particular, in [46–49] it was shown that noncommutativity may give rise to black hole spin and that it essentially mimics its advent. It is interesting to note that a similar type of feature, where the noncommutativity is assigned the role of a mimicker of some specific physical property, is quite usual in the literature; see, for example, reference [50]. As the effective metric $g'_{\mu\nu}$ appears to absorb all NC effects, we name this new effective space as NCRN, and in what follows we investigate its physical properties. This effective metric provides a dual picture to the same physical system, comprising the NC scalar field with the charge q and the background metric generated by the black hole with mass M and charge Q. Note that from now on we deal with only one metric, that pertaining to NCRN, and for simplicity we switch the notation accordingly, i.e., $g' \to g$.

Thus, the metric of NCRN is given by

$$g_{\mu\nu} = \begin{pmatrix} -f(r) & 0 & 0 & 0 \\ 0 & \frac{1}{f(r)} & 0 & A\sin^2\theta \\ 0 & 0 & r^2 & 0 \\ 0 & A\sin^2\theta & 0 & r^2\sin^2\theta \end{pmatrix}, \quad (22)$$

written with the abbreviations

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \qquad \text{and} \qquad A = \frac{aqQ}{2}. \quad (23)$$

As $A \to 0$, we recover the commutative limit. Interestingly, when the same procedure is carried out for the spin $1/2$ field up to the first order in deformation, the same effective metric (22) arises [32]. The situation with the extension of this analysis to the vector field is, however, a little bit different. Namely, for the electromagnetic spin 1 field there are no corrections to the equation of motion in the first order, while in higher orders in Θ, due to the SW map (7) the NC Maxwell equation becomes nonlinear in A_μ, rendering any possibility of constructing a dual picture with an effective metric impossible. On the other hand, in order to extend this construction to the second order in the deformation parameter, we need to allow a more general connection. We comment on this in the concluding section.

From now on, we drop the scalar field from any subsequent discussion and the only subject of our interest is a system consisting of the gauge field and the gravitational field (metric tensor).

Now, the main question is what geometry and physics lie behind the NCRN metric (22). Let us evaluate the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\,R g_{\mu\nu} = \begin{pmatrix} \frac{Q^2 f}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{Q^2}{r^4 f} & 0 & A\,\frac{Q^2 \sin^2\theta}{r^4} \\ 0 & 0 & \frac{Q^2}{r^2} & 0 \\ 0 & A\,\frac{Q^2 \sin^2\theta}{r^4} & 0 & \frac{Q^2 \sin^2\theta}{r^2} \end{pmatrix} + \mathcal{O}(A^2)\,. \tag{24}$$

As can be seen, the Einstein tensor is nonzero, so the NCRN metric, as expected, is not a vacuum solution to the Einstein equation.

Thus, the NC effect may be encrypted within some effective matter source, appearing on the right-hand side of the Einstein field equation. The interesting feature is that for the metric (22) this effective matter source may be fixed by Maxwell's energy–momentum tensor:

$$T^M_{\mu\nu} = \frac{1}{4\pi}\Big(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\Big). \tag{25}$$

Indeed, it may be shown that up to first order in the deformation A the metric (22) satisfies the Einstein–Maxwell field equation:

$$G_{\mu\nu} = 8\pi T^M_{\mu\nu}. \tag{26}$$

We first note that the zeroth order in A in (24), i.e., the Einstein tensor for the RN metric, is proportional to the Maxwell energy–momentum tensor (25), where the only nonvanishing component of the electromagnetic tensor $F_{\mu\nu}$ is $F_{rt} = -F_{tr} = Q/r^2$. In order to see what happens in higher orders, in particular the first order in A, we absorb the NC corrections appearing in (24) into the energy–momentum tensor T^M_{ab}, and simultaneously allow the modifications in the electromagnetic tensor F_{ab}. In this way, we propose the following ansatz:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\frac{Q}{r^2} - AF_0 & AF_1 & AF_2 \\ \frac{Q}{r^2} + AF_0 & 0 & AF_3 & AF_4 \\ -AF_1 & -AF_3 & 0 & AF_5 \\ -AF_2 & -AF_4 & -AF_5 & 0 \end{pmatrix} \tag{27}$$

where $F_i = F_i(t, r, \theta, \varphi)$ are yet-unknown functions. Now, using the Einstein tensor (24) calculated for the metric (22) and the energy–momentum tensor (25), evaluated for the ansatz (27), we can calculate the difference tensor:

$$G_{\mu\nu} - 8\pi T^M_{\mu\nu} = \begin{pmatrix} -\frac{AQf}{r^2}F_0 & 0 & -\frac{2AQf}{r^2}F_3 & -\frac{2AQf}{r^2}F_4 \\ 0 & \frac{2AQ}{r^2 f}F_0 & -\frac{2AQ}{r^2 f}F_1 & -\frac{2AQ}{r^2 f}F_2 \\ -\frac{2AQf}{r^2}F_3 & -\frac{2AQ}{r^2 f}F_1 & -2AQF_0 & 0 \\ -\frac{2AQf}{r^2}F_4 & -\frac{2AQ}{r^2 f}F_2 & 0 & -2AQF_0\sin^2\theta \end{pmatrix} + \mathcal{O}(A^2). \tag{28}$$

The only way that the above difference tensor vanishes is if

$$F_0 = F_1 = F_2 = F_3 = F_4 = 0, \tag{29}$$

leaving the function $F_5(t, r, \theta, \varphi)$ still arbitrary. Thus, we see that up to first order in A, the metric (22) satisfies the Einstein–Maxwell field Equation (26).

An alternative perspective on this situation might be that the nonvanishing $G_{\mu\nu}$ in (24) results from a modification of Einstein's gravitational field equation. In that case, we are interpreting all corrections as coming from the (NC) geometry part [40,45,51–60]; i.e., as corrections to the left-hand side of the Einstein equation. In this way, one would obviously fix the energy–momentum part and modify the Einstein tensor $G_{\mu\nu} \longrightarrow \hat{G}_{\mu\nu}$ according to

$$\hat{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \mathcal{O}(A).$$

5. Physical Properties of NCRN

In the following we make a review of some general properties of the metric (22). Later on, we shall see that many of these properties may be easily understood through the lenses of a transition to another coordinate system.

5.1. Various Aspects of NCRN

Primarily, it is easy to see that this metric is static since its stationary Killing vector field $k = \partial/\partial t$ satisfies $k \wedge \underset{\cdot}{k} = 0$ and the metric is written explicitly in block-diagonal form. Furthermore, by the Vishveshwara–Carter theorem we know that ergosurfaces, consisting of points where the Killing vector field k^a becomes null, coincide with the Killing horizon $H[k]$ generated by k^a.

On the other hand, the horizon can be quickly found by looking at the zeros of the metric function $f(r)$, which are formally identical, as in the commutative Reissner–Nordström black hole. However, as the original coordinate system in which the metric is written is *not* regular at the black hole horizon, we have to use some of the light-like coordinates, such as $v = t + r_*$ with the tortoise coordinate r_*, introduced via the $\underset{\cdot}{r}_* = \underset{\cdot}{r}/f(r)$: spacetime metric in the coordinate system $\{v, r, \theta, \varphi\}$, that takes the form

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + 2A\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\varphi + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{30}$$

Nevertheless, here we have $k = \partial/\partial v$ and again $k^2 = g_{vv} = -f(r)$. Let us denote the zeros of $f(r)$ with r_+ and r_-, so that

$$r_\pm = M \pm \sqrt{M^2 - Q^2},$$

as in the case of an RN black hole.

Another interesting point is the temperature of the NCRN black hole. It appears that the temperature of the Reissner–Nordström black hole remains unaltered by the noncommutative corrections. This may be seen from the following line of arguments, starting from the well-known expression for surface gravity:

$$\kappa^2 = -\lim_{H} \frac{(k^b \nabla_b k^a)(k^c \nabla_c k_a)}{k^a k_a}. \tag{31}$$

The evaluation of the expression in the numerator gives

$$k^\alpha \nabla_\alpha k^\mu = k^\alpha \partial_\alpha k^\mu + k^\alpha \Gamma^\mu_{\alpha\beta} k^\beta = 0 + \Gamma^\mu_{tt} = \frac{1}{2} g^{\mu r} \partial_r f(r)$$

and consequently

$$(k^\alpha \nabla_\alpha k^\mu)(k^\beta \nabla_\beta k_\mu) = \frac{1}{4} g_{\mu\nu} g^{\mu r} g^{\nu r} (\partial_r f)^2 = \frac{1}{4} g^{rr} (\partial_r f)^2\,, \quad k^2 = g_{tt} = -f. \tag{32}$$

Since the only component of the metric that we need is g^{rr} and it is given by

$$g^{rr} = \frac{1}{\frac{1}{f} - \frac{A^2 \sin^2\theta}{r^2}},$$

one finally obtains

$$\kappa^2 = \lim_{r\to r_+} \frac{1}{4} \frac{(f')^2}{1 - \frac{A^2\sin^2\theta}{r^2} f} = \lim_{r\to r_+} \frac{(f')^2}{4}.$$

Formally, as above, this has to be checked in a regular coordinate system, such as $\{v, r, \theta, \varphi\}$. Here, we have

$$k^\alpha \nabla_\alpha k^\mu = \Gamma^\mu_{vv} = \frac{1}{2} g^{\mu r} \partial_r f(r)$$

and consequently

$$(k^\alpha \nabla_\alpha k^\mu)(k^\beta \nabla_\beta k_\mu) = \frac{1}{4} g^{rr} (\partial_r f)^2 = \frac{1}{4} \frac{(f')^2}{\frac{1}{f} - \frac{A^2\sin^2\theta}{r^2}}.$$

Again, the conclusion remains unaltered, $\kappa = f'(r_+)/2$. Expectedly, this result is in accordance with that obtained in [31] when calculating the emission rate of the scalar particles using the Parikh–Wilczek tunneling formalism. Moreover, the conclusion that the lowest nonvanishing NC correction to the horizon temperature is beyond the linear one seems to be in agreement with other approaches in the literature [61–63].

5.2. The Newtonian Limit

The Newtonian limit is defined by the three following premises [64]:

1. The particle is moving slowly with respect to the speed of light;
2. The gravitational field is weak and can be considered as perturbation of a flat space;
3. The gravitational field is static.

The mathematical description of premise 1 is given by the requirement

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}, \tag{33}$$

which simplifies the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{tt}\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{34}$$

Moreover, since the gravitational field is static, we have

$$\Gamma^\mu_{tt} = -\frac{1}{2} g^{\mu r} \partial_r g_{tt}.$$

In the subsequent analysis, we will need the inverse of the metric (22), which is given by

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{f} & 0 & 0 & 0 \\ 0 & f & 0 & -A\frac{f}{r^2} \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & -A\frac{f}{r^2} & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} + \mathcal{O}[A]^2. \tag{35}$$

Now, if we examine Equation (34) for $\mu = t$, we obtain

$$\frac{d^2 t}{d\tau^2} = 0 \quad \Longrightarrow \quad \frac{dt}{d\tau} = \text{const.} \tag{36}$$

which enables us to rewrite (34) in terms of the coordinate time t only:

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{tt} = 0\,. \tag{37}$$

At this point, we use premise 2, which tells us that the gravitational field is weak and that it can be treated as a perturbation of the flat metric. In fact, one is dealing here with two types of perturbations: gravitational and noncommutative. Therefore, the inverse of the metric can be written as

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + Ak^{\mu\nu} + \mathcal{O}(A \cdot h, h^2, A^2), \tag{38}$$

where only the lowest order in h and A is kept. Let us calculate the Christoffel symbol in this approximation:

$$\Gamma^\mu_{tt} = -\frac{1}{2} g^{\mu r}\partial_r g_{tt} = -\frac{1}{2}(\eta^{\mu r} - h^{\mu r} + Ak^{\mu r})\partial_r g_{tt} = -\frac{1}{2}(\eta^{\mu r} + Ak^{\mu r})\partial_r g_{tt} + \mathcal{O}[h^2, A^2]. \tag{39}$$

In the last equality, we used the fact that in the lowest order $-g_{tt} = f(r) = 1 + \mathcal{O}[h]$, i.e., $\partial g \cong \mathcal{O}[h]$. Thus, the Christoffel symbols are

$$\Gamma^t_{tt} = \Gamma^\theta_{tt} = 0, \quad \Gamma^r_{tt} = \frac{1}{2}\frac{\partial f}{\partial r}, \quad \Gamma^\varphi_{tt} = -\frac{A}{2r^2}\frac{\partial f}{\partial r} \tag{40}$$

so that (34) in the Newtonian limit reduces to

$$\ddot{r} = -\frac{1}{2}\frac{\partial f}{\partial r}, \quad \ddot{\varphi} = \frac{A}{2r^2}\frac{\partial f}{\partial r}, \quad \ddot{\theta} = 0. \tag{41}$$

While the noncommutativity does not affect the radial equation, it affects the equation for the polar coordinate. The equations of motion (41) can be written in a unified way as

$$\ddot{x}^i = -\tilde{\partial}_i V(r), \tag{42}$$

where $V(r) = \frac{1}{2} f$ is the generalized Newtonian potential (for all practical purposes it is really the Newtonian potential since $f \approx 1 - \frac{2M}{r}$ for $\frac{2M}{r} \gg \frac{Q^2}{r^2}$) and $\tilde{\partial}_i \equiv \partial_i + \tilde{\Theta}_i{}^j \partial_j$ is the generalized Laplacian, with

$$\tilde{\Theta}_i{}^j = \begin{pmatrix} 0 & 0 & \frac{A}{2r^2} \\ 0 & 0 & 0 \\ -\frac{A}{2r^2} & 0 & 0 \end{pmatrix}. \tag{43}$$

Equation (42) represents the noncommutative version of the Newton equation.

5.3. Geodesics in NCRN

Let us investigate the geodesics for the classical, electrically neutral particle moving in the background of NCRN (22). For the sake of simplicity let us examine geodesics in the $\theta = \pi/2$ plane. The 4-velocity is $u^\mu = (\dot{t}, \dot{r}, 0, \dot{\varphi})$, where the dot denotes the derivative with respect to proper time (in case of time-like geodesics), or with respect to some affine parameter (in case of null geodesics). The kinematics is encapsulated in the square of the 4-velocity,

$$-\kappa = u_\mu u^\mu = -f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2 \sin^2\theta\,\dot{\varphi}^2 + 2A\sin^2\theta\,\dot{r}\dot{\varphi}\,, \tag{44}$$

written with the parameter

$$\kappa = \begin{cases} 1\,, & \text{timelike} \\ 0\,, & \text{null} \end{cases} \tag{45}$$

On the other hand, due to the Killing vectors $k = \partial/\partial t$ and $m = \partial/\partial\varphi$, there are two conserved quantities: energy e and angular momentum ℓ,

$$e = -g_{\mu\nu}u^\mu k^\nu = f(r)\dot{t} \quad \Rightarrow \quad \dot{t} = \frac{e}{f(r)},$$

$$\ell = g_{\mu\nu}u^\mu m^\nu = A\sin^2\theta\,\dot{r} + r^2\sin^2\theta\,\dot{\varphi} \quad \Rightarrow \quad \dot{\varphi} = \frac{\ell}{r^2\sin^2\theta} - \frac{A}{r^2}\,\dot{r}.$$

Thus, taking into account that $\theta = \pi/2$ and noting that terms linear in A cancel, we have

$$-\kappa = \left(\frac{1}{f(r)} - \frac{A^2}{r^2}\right)\dot{r}^2 - \frac{e^2}{f(r)} + \frac{\ell^2}{r^2}.$$

Formally, we can put this into standard form with an effective potential via auxiliary function $R(\tau)$:

$$\frac{\dot{R}^2}{2} + V(r) = \frac{e^2}{2}, \tag{46}$$

where

$$V(r) = \frac{f(r)}{r^2}(l^2 + \kappa r^2), \quad \frac{\dot{R}^2}{f(r)} = \left(\frac{1}{f(r)} - \frac{A^2}{r^2}\right)\dot{r}^2 \quad \text{i.e.,} \quad \underset{\cdot}{\mathrm{R}} = \underset{\cdot}{\mathrm{r}}\sqrt{1 - \frac{A^2}{r^2}\,f(r)}\,.$$

However, it is difficult to write this relation explicitly.

Interestingly, in this analysis the circular trajectories ($\dot{r} = 0$) are completely unaffected by noncommutativity. However, a particle that would be released from rest (i.e., with $\ell = 0$) at great distance from the black hole would nevertheless gain some nonvanishing shift in the angle due to the NC term $-A\dot{r}/r^2$. This implies that the total time of the free fall for the photon would display a difference when calculated and compared between commutative and noncommutative cases. Indeed, if $l = 0$ we see that the radial motion is unchanged (up to A^2 it is the same situation as in the commutative case) and unfolds according to

$$r(\tau) = R_0 + e\tau, \tag{47}$$

where R_0 is the initial radius $r(0) = R_0$. The polar coordinate should then acquire the NC correction

$$\dot{\varphi} = -\frac{A}{r^2}\dot{r} \implies \varphi(\tau) = \varphi_0 - Ae\left(\frac{1}{R_0 e - e^2\tau} - \frac{1}{R_0 e}\right). \tag{48}$$

However, one should be careful about proper interpretation of these results, in particular about observational claims. As far as it goes, in our analysis we are relying on a specific coordinate system, and in this particular case it is not as intuitive as one would initially expect, this being due to the presence of the $g_{r\varphi}$ component in the metric. For example, we could say that the experiment is performed by "static observers", that is observers with 4-velocities u^a-tangent to the orbits of the stationary Killing vector field $k^a = \partial_t^a$; more concretely, $u^\mu = (1/\sqrt{-g_{tt}}, 0, 0, 0)$, in which case the conclusions drawn might be somewhat different. These issues are addressed in more detail in the final section, where we take on the task of finding a genuine physical interpretation of the NCRN metric (22) and specifically of its only nonvanishing off-diagonal component $g_{r\varphi}$.

6. Concluding Remarks

This work has provided a study of the noncommutative $U(1)$ gauge field gravity model coupled to a scalar field all up to the second order in the Seiberg–Witten map. If classical Einstein–Hilbert action is added to this model, the resulting setup may be viewed as a deformation of the system consisting of the gauge field, gravitational field, and dilaton field that one usually encounters in some models of quantum gravity after the low-energy limit is taken. The approach that we use provides yet another procedure to modify GR in order to make it more compatible with physics that is expected to occur at the Planck scale.

Using duality symmetry that is present at the first order in SW, we have rederived the effective metric from the reference [32] (see Equation (22)), which turns out to be a noncommutative deformation of the Reissner–Nordström metric, with the only nonvanishing off-diagonal component sitting at the entry (r, φ) and scaling linearly with the deformation parameter a. This metric has been shown to satisfy the equations of Einstein–Maxwell gravity when the gauge field is fixed to be the Coulomb potential with its origin in a black hole charge, albeit only within the first order of deformation. On the contrary, as we demonstrate in the Appendix A, the construction of the effective metric fails at the second order in SW expansion due to duality symmetry being broken there. However, it is worthy to note that if we extend the definition of the connection and in addition to the ordinary Christoffels take it to also involve the contorsion and nonmetricity, then the construction of the effective metric can be pushed through up to the second order and beyond. More precisely, it can be shown that the inverse of the effective metric that in such an extended framework is able to produce the exact (nonperturbative) Equation (18) by means of the general equation of motion (20) appears to pick up an additional term in the component $g^{\varphi\varphi}$:

$$g^{\varphi\varphi(2)} = -\frac{a^2 q^2 Q^2}{2r^4} f. \tag{49}$$

Consequently, the effective metric itself in this more extended framework acquires the corrections

$$g^{(2)}_{\varphi\varphi} = \frac{a^2 q^2 Q^2}{4} f \sin^4\theta, \qquad g^{(2)}_{rr} = -\frac{a^2 q^2 Q^2}{4r^2} \sin^2\theta. \tag{50}$$

Note that the inverse of the effective metric is given by the exact result, while its inverse, i.e., the effective metric itself, has been expanded up to second order in deformation. For details we refer the reader to the Appendix A. Here, we only make a note that such a construction is nonunique.

In Section 4, we have touched upon an important question dealing with an actual interpretation and better understanding of the metric (22), which we here come back to. Specifically, we are interested in the interpretation of the $g_{r\varphi}$ metric component, as well as the meaning of the coordinates in which the metric is expressed and in which the calculations were carried out previously (especially in the preceding section). A call for caution has already been given before, as there might be a possibility that the predictions obtained in the previous section might not be fully trustworthy, due to possible misinterpretation of the coordinates. Indeed, we should not jump to the conclusion that the obtained results are completely reliable just because the coordinates we work with are denoted as standard spherical coordinates. In other words, just because some coordinates are denoted by "r" or "φ" does not automatically mean that they are "usual spherical coordinates" (e.g., it might be that $r \in \langle -\infty, \infty \rangle$ or even $\varphi \in \langle -\infty, \infty \rangle$). For this purpose, here we investigate this point in some more detail.

Using the new coordinates $(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\varphi}) = (t, r, \theta, \varphi - Ar^{-1})$ the metric turns into

$$g_{\tilde{\mu}\tilde{\nu}} = \begin{pmatrix} -f(r) & & & \\ & h(r,\theta) & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}, \quad h(r,\theta) = \frac{1}{f(r)} - \frac{(A \sin\theta)^2}{r^2}. \tag{51}$$

Therefore, we see that upon making a coordinate transformation, the NCRN metric (22) is transformed into a new, more familiar, format, which to a first order of deformation appears to be no different to the RN metric. Moreover, in this coordinate system it is manifestly clear that the metric is asymptotically flat up to the first order in the NC deformation parameter. In addition, it can be easily checked that the same change in coordinates may be used to transform away the deflection of a photon in its free fall toward the center of the black hole studied in the previous section; i.e., to erase the only seemingly nontrivial effect of the NCRN presented in this work. This would consequently mean that the NC corrections present in the metric (22) are trivial and that they do not have any physical meaning. In light of these findings, it does not come as a surprise that the NCRN metric (22) appears to have the properties that we have so far encountered, in particular that all nontrivial changes appear at orders that are not lower than the second.

However, we want to stress that the above reasoning, as well as the conclusions drawn from it, do not present the whole picture, but only a portion of it. As such, this reasoning alone is insufficient to provide any reliable or far-reaching conclusion and in many aspects is misleading. It is indeed true that the metric tensor in the new coordinates at the first order in the NC parameter a seems to be the same as the ordinary RN metric. Nonetheless, the problem with such a stance is that it completely ignores the context which brought about the metric (22) and in which it was derived. Regarding the context in this concrete example, imagine that we have two spacetimes, the background (M, g_{ab}) with RN metric g_{ab}, where we place the NC scalar field, and "effective" spacetime (M, g'_{ab}) with the effective metric g'_{ab}. Unfortunately, as the whole setting (background spacetime and NC action) is prepared in a specific coordinate system, we cannot easily transform components of the effective metric g'_{ab} without going back to the origin of this construction. At best, coordinate changes such as the one above may be trusted at infinity.

An additional point in this case is that the coordinates are noncommutative and the partial derivatives are also noncommutative. In particular,

$$\partial_{\tilde{r}} = \partial_r - \frac{A}{r^2} \partial_\varphi. \tag{52}$$

From these reasons, it is clear that the new geometry will have nontrivial NC effects up to the first order in the NC deformation parameter a, contrary to the argument made around (51). Not a bit less important is that the coupling of the NCRN metric with other fields makes a huge difference in comparison with a situation when this metric is taken alone and studied as an isolated entity. This is where the importance of the duality symmetry and a validity of the corresponding requirement (20) comes into play. Namely, after the coordinate transformation leading to (51), the duality does not hold anymore.

The latter argument is readily confirmed in references [31,32,43], where the metric (22) was coupled to the spin $1/2$ field and scalar field, respectively. We point out that, already at the linear order in the deformation parameter, these couplings lead to QNM spectra that differ from the corresponding QNM spectra when the same fields are coupled to the ordinary RN metric. In this way, the assertion that at the first order in deformation the NCRN metric is essentially the same as the RN metric directly contradicts with the findings in [31,43], where it was explicitly shown that scalar perturbations of RN and NCRN give rise to different QNM spectra already at linear order in the deformation parameter. Also, this assertion is in contradiction with the findings in [32], which show that the spin 1/2 field perturbations of RN and NCRN are governed by different equations of motion.

In summary, when talking about the physical properties of the NCRN metric (22), we may conclude by saying that, observed only by itself, outside of the context in which it was derived, it appears to be just the RN metric in different coordinates. However, what brings something new to this metric and its consequences for physics is when it couples to other types of fields, for example, the scalar, spinor, and gauge fields.

Author Contributions: Conceptualization, M.D.Ć., N.K., T.J., A.S. and I.S.; Methodology, M.D.Ć., N.K., T.J. and A.S.; Formal analysis, M.D.Ć., N.K., T.J., A.S. and I.S.; Investigation, M.D.Ć., N.K., T.J. and A.S.; Data curation, N.K.; Writing—original draft, M.D.Ć., T.J. and A.S.; Writing—review and editing, M.D.Ć., N.K., T.J., A.S. and I.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the Croatian Science Foundation Project IP-2020-02-9614 Search for Quantum spacetime in Black Hole QNM spectrum and Gamma Ray Bursts. The work of M.D.C. and N.K. is supported by project 451-03-9/2021-14/200162 of the Serbian Ministry of Education and Science. This work is partially supported by ICTP-SEENET-MTP Project NT-03 "Cosmology-Classical and Quantum Challenges" in frame of the Southeastern European Network in Theoretical and Mathematical Physics and the COST action CA23130 Bridging high and low energies in search of quantum gravity (BridgeQG).

Data Availability Statement: Data are contained within the article.

Acknowledgments: We would like to thank Kumar Gupta for fruitful discussion and useful comments.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Second-Order Corrections to the Effective Metric

In this appendix, we describe the challenges one encounters when trying to deduce the form of the effective metric in higher orders of deformation. In order to make a construction in higher orders possible, one has to extend the existing framework and redefine the coefficients of affine connection $\Gamma^{\rho}_{\mu\nu}$ in such a way that, in addition to the ordinary Christoffels $\{^{\rho}_{\mu\nu}\}$, they also include the contorsion $K^{\rho}_{\mu\nu}$ and the nonmetricity $C^{\rho}_{\mu\nu}$:

$$\Gamma^{\rho}_{\mu\nu} = \{^{\rho}_{\mu\nu}\} + \frac{1}{2}C^{\rho}_{\mu\nu} + K^{\rho}_{\mu\nu}.$$

Nonmetricity and contorsion are the symmetric and antisymmetric parts of the connection, respectively.

It is readily seen that the second-order effective metric obtained by adding the contribution (49) to the first-order metric (21) can easily account for the terms with second derivatives in Equation (18). However, the issue with the first-order derivatives becomes more involved. It appears that the only way to account for these redundant first derivative terms is to extend the connection as described above, so that nonmetricity and contorsion may absorb these terms. More precisely, from (20) it can be seen that the first derivative terms have the form

$$-g^{\mu\nu}\Gamma^{\rho}_{\mu\nu}\partial_\rho\Phi = -g^{\mu\nu(0)}\Gamma^{\rho(2)}_{\mu\nu}\partial_\rho\Phi - g^{\mu\nu 1}\Gamma^{\rho(1)}_{\mu\nu}\partial_\rho\Phi - g^{\mu\nu(2)}\Gamma^{\rho(0)}_{\mu\nu}\partial_\rho\Phi \quad \text{(A1)}$$

Here, all necessary nonzero Christoffels may be calculated from (49) and (50) to give

$$\left\{{}^{r}_{rr}\right\}^{(2)} = -(aqQ)^2 \frac{f\sin^2\theta}{4r^3}, \quad \left\{{}^{r}_{\varphi\varphi}\right\}^{(2)} = (aqQ)^2 \frac{ff'\sin^4\theta}{8},$$
$$\left\{{}^{\theta}_{rr}\right\}^{(2)} = -(aqQ)^2 \frac{\sin\theta\cos\theta}{4r^4}, \quad \left\{{}^{\theta}_{\varphi\varphi}\right\}^{(2)} = -(aqQ)^2 \frac{f\cos\theta\sin^3\theta}{2r^2},$$
$$\left\{{}^{\theta}_{r\varphi}\right\}^{(1)} = -(aqQ) \frac{\sin\theta\cos\theta}{2r^2}, \quad \left\{{}^{\theta}_{\varphi\varphi}\right\}^{(0)} = -\sin\theta\cos\theta,$$
$$\left\{{}^{r}_{\varphi\varphi}\right\}^{(0)} = -rf\sin^2\theta.$$

Interestingly, as the contorsion needs to be antisymmetric in the last two indices, all its contributions to (A1) will vanish, as they need to be contracted with the inverse metric tensor $g^{\mu\nu}$, which is symmetric. This means that the only terms that may annihilate the first derivative corrections in (A1),

$$(aqQ)^2 \frac{f\sin\theta\cos\theta}{4r^4}\partial_\theta\Phi + (aqQ)^2 (f' - \frac{6f}{r}) \frac{f\sin^2\theta}{8r^2}\partial_r\Phi, \tag{A2}$$

are those that involve components of the nonmetricity tensor.

There are several ways one can remove unwanted first derivative terms:

- Nonmetricity may be introduced as a first-order deformation, so that we may demand

$$-g^{r\varphi(1)}C^{\theta(1)}_{r\varphi} = (aqQ)^2 \frac{f\sin\theta\cos\theta}{4r^4} \Rightarrow C^{\theta(1)}_{r\varphi} = (aqQ)\frac{\sin\theta\cos\theta}{4r^2}$$

$$-g^{r\varphi(1)}C^{r(1)}_{r\varphi} = (aqQ)^2 (f' - \frac{6f}{r}) \frac{f\sin^2\theta}{8r^2} \Rightarrow C^{r(1)}_{r\varphi} = (aqQ)(f' - \frac{6f}{r})\frac{\sin^2\theta}{8}.$$

  From there, the components of nonmetricity immediately follow.
- Nonmetricity may be introduced as a second-order deformation, so that we may demand

$$-\frac{1}{2}g^{rr(0)}C^{\theta(2)}_{rr} = (aqQ)^2 \frac{f\sin\theta\cos\theta}{4r^4} \Rightarrow C^{\theta(2)}_{rr} = -2(aqQ)^2\frac{\sin\theta\cos\theta}{4r^4},$$

$$-\frac{1}{2}g^{rr(0)}C^{r(2)}_{rr} = (aqQ)^2 (f' - \frac{6f}{r}) \frac{f\sin^2\theta}{8r^2} \Rightarrow C^{r(2)}_{rr} = -2(aqQ)^2(f' - \frac{6f}{r})\frac{\sin^2\theta}{8r^2}.$$

  The other possibility is

$$-\frac{1}{2}g^{\varphi\varphi(0)}C^{\theta(2)}_{\varphi\varphi} = (aqQ)^2 \frac{f\sin\theta\cos\theta}{4r^4} \Rightarrow C^{\theta(2)}_{\varphi\varphi} = -(aqQ)^2\frac{f\sin^3\theta\cos\theta}{2r^2},$$

$$-\frac{1}{2}g^{\varphi\varphi(0)}C^{r(2)}_{\varphi\varphi} = (aqQ)^2 (f' - \frac{6f}{r}) \frac{f\sin^2\theta}{8r^2} \Rightarrow C^{r(2)}_{\varphi\varphi} = -(aqQ)^2(f' - \frac{6f}{r})\frac{f\sin^4\theta}{4}.$$

  We may conclude that the way of implementing nonmetricity, which the framework we work in allows, is certainly not unique. It would be interesting to understand more deeply the physical consequences of the effective nonmetric geometry. Also, we point out that while the equation of motion for the NC scalar field is exact, all other results, such as the components of the metric and nonmetricity tensors, are not exact. They are instead perturbative and given only up to a second order in the NC parameter a (i.e., they have higher-order corrections).

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