Kundt geometries in higher-derivative gravity

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Quadratic Gravity is one of the modified theories of gravity; its action contains additional terms quadratic in Riemann tensor and its contractions. We study Kundt metrics, defined geometrically by admitting non-expanding, non-twisting and non-shearing null geodesic congruence, in the framework of Quadratic Gravity. We examine the special cases, namely the *pp*-waves and VSI spacetimes. Equations of geodesic deviation are calculated to provide means of physical interpretation.

Keywords: Quadratic Gravity; Kundt spacetimes; Geodesic deviation.

1. Introduction

Despite many successes of general relativity, including the prediction of gravitational waves that were directly detected nearly four years ago, even such brilliant theory has its limits. Attempts to solve problems of general relativity (e.g., quantization, inflation, dark matter and energy) involve modifying the action.[1] Our focus is on the so-called Quadratic Gravity (QG), where the action contains additional terms that are quadratic combinations of Riemann tensor and its contractions

$$S = \int \left(\tfrac{1}{\kappa}(R - 2\Lambda) + \alpha\, R^2 + \beta\, R_{ab}^2 + \gamma(R_{abcd}^2 - 4R_{ab}^2 + R^2) \right) \sqrt{-g}\, \mathrm{d}^D x\,. \qquad (1)$$

The vacuum field equations derived from this action are then

$$\begin{aligned}
&\tfrac{1}{\kappa}\left(R_{ab} - \tfrac{1}{2}R\, g_{ab} + \Lambda\, g_{ab}\right) + 2\alpha\, R \left(R_{ab} - \tfrac{1}{4}R\, g_{ab}\right) + (2\alpha+\beta)(g_{ab}\Box - \nabla_a\nabla_b)R \\
&+2\gamma\left(R\, R_{ab} - 2R_{abcd}\, R^{cd} + R_{acde}\, R_b^{\ cde} - 2R_{ac}R_b^{\ c} - \tfrac{1}{4}g_{ab}(R_{cdef}^2 - 4R_{cd}^2 + R^2)\right) \\
&+\beta\Box\left(R_{ab} - \tfrac{1}{2}R\, g_{ab}\right) + 2\beta\left(R_{abcd} - \tfrac{1}{4}g_{ab}R_{cd}\right)R^{cd} = 0\,. \qquad (2)
\end{aligned}$$

We can easily see that for $\alpha = \beta = \gamma = 0$, we get famous Einstein's general relativity. In general, Eq. (2) contains second order derivatives of the Ricci scalar and tensor, which implies the fourth order derivatives of the metric. A special case of QG is Gauss–Bonnet theory ($\alpha = 0 = \beta$), that belongs to so called Lovelock's gravity.[2] Our aim is to study spacetimes known in general relativity, which are defined by purely geometrical properties independent of the specific field equations. Metric describing such spacetimes can then be considered as an ansatz and inserted into the QG field equations to observe restrictions on the metric functions and differences between QG and classic general relativity.

Specific representative of such spacetimes is the Kundt family whose line element can be written as[3–5]

$$\mathrm{d}s^2 = g_{pq}(u,x)\,\mathrm{d}x^p\mathrm{d}x^q + 2g_{up}(r,u,x)\,\mathrm{d}u\,\mathrm{d}x^p - 2\mathrm{d}u\,\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,. \qquad (3)$$

Its *geometric definition* is that it admits non-expanding, non-twisting, and non-shearing null geodesic congruence. The optically privileged vector field generating the congruence is $\boldsymbol{k} = \partial_r$, where r is an affine parameter along the congruence. The spacetime is foliated by the null surfaces $u =$ const., to which $\boldsymbol{k}$ is orthogonal (and tangential). The transverse $(D-2)$-space: $u =$ const. and $r =$ const. is described by the metric $g_{pq}(u,x)$, where $x \equiv x^k$, $k = 2, \dots D-1$ are coordinates covering the transverse space. For simplicity, we will study Kundt metrics with $g_{up} = 0$, i.e., without so-called gyratons, that describe spinning null fluid.[6,7]

To algebraically classify these spacetimes, the Weyl tensor projected on a null tetrad $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_i$ can be employed. These (invariant) projections are called Weyl scalars. Spacetime can be classified based on the number of Weyl scalars, sorted by their boost weights, that can be set to zero. Details on the algebraic classification can be found, e.g., in Refs. 8, 9. In Table 1, we list conditions for algebraic types of the Kundt metric.[10]

Table 1. Conditions for algebraic types of the Kundt metric (with $g_{up} = 0$)

Type	Necessary and sufficient conditions[a]
II(ad)	$g_{uu} = -\frac{{}^S R(u,x)}{(D-2)(D-3)}\, r^2 + c(u,x)\, r + d(u,x)$
II(bd)	${}^S R_{pq} = \frac{1}{D-2}\, g_{pq}\, {}^S R$
II(cd)	${}^S C_{mpnq} = 0$
III	II(abcd)
III(a)	${}^S R_{,p} = 0 \quad$ and $\quad c_{,p} = \frac{2}{D-3}\, g^{mn} g_{m[n,u\|\|p]}$
III(b)	$g_{p[m,u\|\|q]} = \frac{1}{D-3}\, g^{os} \left(g_{pm} g_{o[s,u\|\|q]} - g_{pq} g_{o[s,u\|\|m]} \right)$
N	III(ab)
O	$g^{os} g_{o[s,u\|\|p]\|\|q} + \frac{1}{2(D-2)}\, {}^S R\, g_{pq,u} = \frac{g_{pq}}{D-2}\, g^{mn} \left(g^{os} g_{o[s,u\|\|m]\|\|n} + \frac{1}{2(D-2)}\, {}^S R\, g_{mn,u} \right)$ $d_{\|\|pq} + g_{pq,uu} - \frac{1}{2} g^{os} g_{op,u} g_{sq,u} - \frac{1}{2}\, c\, g_{pq,u}$ $= \frac{g_{pq}}{D-2}\, g^{mn} \left(d_{\|\|mn} + g_{mn,uu} - \frac{1}{2} g^{os} g_{om,u} g_{sn,u} - \frac{1}{2}\, c\, g_{mn,u} \right)$

Note: [a] The symbol "$_{||}$" denotes spatial covariant derivative ($d_{||pq} = d_{,pq} - \Gamma^m_{pq}\, d_{,m}$).

2. Field Equations

We insert the (non-gyratonic) Kundt metric into the QG field equations and observe the conditions they impose upon the metric functions. Firstly, we look at general Kundt metric, but our main focus will be on more special cases. Results for the simpler case of the Gauss–Bonnet gravity can be found in Refs. 10, 11.

2.1. *General case*

The rr-component $(2\alpha + \beta)\, g_{uu,rrrr} = 0$ implies that either $(2\alpha + \beta) = 0$ or g_{uu} is at most cubic in r, so it can be written as $g_{uu} = a\, r^3 + b\, r^2 + c\, r + d$, where a, b, c, d are functions of u and the transverse coordinates x^p. These functions will be constrained by the following components of the field equations.

The rp-component is $(2\alpha+\beta)\, g_{uu,rrrp}=0$. For $(2\alpha+\beta)\neq 0$, it implies that $a_{,p}=0$, so that $a=a(u)$.

The remaining components (ru, pq, up, uu) get more complicated, and longer, therefor we will not show them here. They can be found for example in Ref. 10. The equations give relations between the functions a, b, c, d and the metric g_{pq}.

2.2. *pp-waves with constant spatial curvature*

One of the most common examples of the Kundt spacetimes are plane-fronted waves with parallel rays, or *pp*-waves, see e.g., Ref. 12. They are defined as spacetimes admitting a covariantly constant vector field $\boldsymbol{k}=\partial_r$, which implies that the metric is r-independent. In the non-gyratonic case, we have

$$\mathrm{d}s^2 = g_{pq}(u,x)\,\mathrm{d}x^p\mathrm{d}x^q - 2\mathrm{d}u\,\mathrm{d}r + g_{uu}(u,x)\,\mathrm{d}u^2\,. \tag{4}$$

For further simplification, we assume that the transverse space is of a constant curvature, which means that its Riemann and Ricci tensors (denoted ${}^{S}R_{pqmn}$ and ${}^{S}R_{pq}$) are related to the Ricci scalar ${}^{S}R$ by

$${}^{S}R_{pqmn} = \frac{{}^{S}R}{(D-3)(D-2)}\left(g_{pm}g_{qn}-g_{pn}g_{qm}\right)\,, \quad {}^{S}R_{pq} = \frac{{}^{S}R}{D-2}\,g_{pq}\,. \tag{5}$$

The rr, rp and up-components of the field equations are satisfied identically. The ru and pq-components give a quadratic equation for the scalar curvature ${}^{S}R$ with two solutions

(a) Trivial solution

$${}^{S}R = 0 \Rightarrow g_{pq}=\delta_{pq}, \quad \Lambda = 0\,. \tag{6}$$

(b) Non-trivial solution

$${}^{S}R = -\frac{1}{2\kappa}\,\frac{(D-2)(D-3)}{(D-2)(D-3)\alpha+(D-4)(D-5)\gamma+(D-3)\beta} = 4\Lambda_0\,. \tag{7}$$

Since ${}^{S}R$ is constant, necessarily $g_{pq,u}=0$.

For both cases (6), (7), the uu-component can be written as

$$k\Delta g_{uu} + \beta\Delta\Delta g_{uu} = 0\,, \tag{8}$$

where the constant k differs for the two solutions mentioned above

$$\text{(a)}\ k=\kappa^{-1}\,, \quad \text{(b)}\ k = \frac{(D-3)\beta-2(D-4)\gamma}{\omega\kappa} + \frac{(D-2)(D-3)^2\beta}{2\omega^2\kappa^2}\,,$$
$$\omega \equiv (D-2)(D-3)\alpha+(D-4)(D-5)\gamma+(D-3)\beta\,. \tag{9}$$

This means that we need to solve Helmholtz-like equation for g_{uu} [13]

$$\Delta g_{uu} + \frac{k}{\beta}\,g_{uu} = f\,, \qquad \text{where} \qquad \Delta f = 0\,. \tag{10}$$

The interesting fact is that it is possible to find solution of (10) that does not solve $\Delta g_{uu} = 0$, so such a solution does not exist in Einstein's general relativity.

From Table 1, we see that the non-trivial solution (b) is of type II(bcd), and the non-zero Weyl scalars are

$$\Psi_{2S} = \frac{{}^S R}{(D-1)(D-2)}\,, \qquad \Psi_{4^{ij}} = \left[-\frac{1}{2} g_{uu||pq} + \frac{g_{pq}}{2(D-2)}\Delta g_{uu}\right] m_i^p\, m_j^q\,. \quad (11)$$

2.3. *VSI spacetimes*

The definition of VSI spacetimes[14] is that all scalar curvature invariants of all orders vanish. Their (non-gyratonic) line element has the form

$$\mathrm{d}s^2 = \delta_{pq}\,\mathrm{d}x^p \mathrm{d}x^q - 2\mathrm{d}u\,\mathrm{d}r + \big(c(u,x)\,r + d(u,x)\big)\mathrm{d}u^2\,, \quad (12)$$

so g_{uu} is at most linear in r. The rr, rp, ru, pq-components are satisfied identically, the only non-trivial components are up and uu. The up-component implies $(1/\kappa)c_{,p} + \beta\Delta c_{,p} = 0$, which in general relativity ($\beta = 0$) restricts $c = c(u)$.

The uu-component gives two constraints with respect to r, namely

$$\Delta\left(\tfrac{1}{\kappa}c + \beta\Delta c\right) = 0 \quad \text{and} \quad \Delta\left[-\tfrac{1}{\kappa}d + \beta\left(2c_{,u} - \tfrac{1}{2}c^2 - \Delta d\right)\right] = 0\,. \quad (13)$$

Looking at Table 1, we observe that VSI spacetimes are of type III(b) with non-zero scalar components of the Weyl tensor

$$\begin{aligned} \Psi_{3T^i} &= -m_i^p \frac{D-3}{2(D-2)} c_{,p}\,, \\ \Psi_{4^{ij}} &= -\frac{1}{2} m_i^p\, m_j^q \left[r\left(c_{||pq} - \frac{g_{pq}}{D-2}\Delta c\right) + d_{||pq} - \frac{g_{pq}}{D-2}\Delta d\right]. \end{aligned} \quad (14)$$

3. Geodesic Deviation

Possible way of physical and geometrical interpretation of spacetimes is to investigate local effects of curvature on freely falling test particles.[15,16] Such particles move along geodesics and due to the gravitational field, they can move closer or further away from each other. Let us have a geodesic congruence with central geodesic $\gamma(\tau)$. Its tangent vector $\boldsymbol{u}$ is a four-velocity of the particle moving along geodesic. The vector separating geodesics in the congruence is $\boldsymbol{Z}$. The equation of geodesic deviation then takes the form

$$\frac{\mathrm{D}^2 Z^a}{\mathrm{d}\tau^2} = R^a{}_{bcd}\, u^b\, u^c\, Z^d\,. \quad (15)$$

We choose an orthonormal frame $\boldsymbol{e}_{(a)}$ with $\boldsymbol{e}_{(0)} = \boldsymbol{u}$. The round brackets denote tetrad indices. The projection of Eq. (15) on spatial tetrad vectors becomes

$$\ddot{Z}^{(i)} = R^{(i)}{}_{(0)(0)(j)} Z^{(j)}, \quad i,j = 1,\dots D-1\,. \quad (16)$$

In the Kundt case $\boldsymbol{u} = \dot{r}\,\partial_r + \dot{u}\,\partial_u + \dot{x}^p\,\partial_p$ and the vector $\boldsymbol{e}_{(1)}$ is defined as $\boldsymbol{e}_{(1)} = \sqrt{2}\,\tilde{\boldsymbol{k}} - \boldsymbol{u}$, where $\tilde{\boldsymbol{k}} = 1/(\sqrt{2}\,\dot{u})\,\partial_r$. We call it longitudinal direction. The remaining vectors $\boldsymbol{e}_{(i)}$, $i = 2,\dots D-1$, are called transverse directions.

3.1. *Geodesic deviation for pp-waves*

The equations of geodesic deviations for *pp*-waves (with $g_{up} = 0$) are

$$\begin{aligned} \ddot{Z}^{(1)} &= 0\,, \\ \ddot{Z}^{(i)} &= \Big(\tfrac{1}{2}\dot{u}^2\left(g_{uu||pq} + g_{pq,uu} - \tfrac{1}{2}g^{os}g_{op,u}\,g_{sq,u}\right)m_i^p\,m_j^q \\ &\qquad +2\dot{u}\,g_{m[p,u||n]}m_{(i}^m\,m_{j)}^p\,\dot{x}^n - {}^S R_{mpnq}m_i^m m_j^n\,\dot{x}^p\dot{x}^q\Big)Z^{(j)}\,. \end{aligned} \tag{17}$$

These were calculated without any use of field equations. Due to the absence of the terms $R^{(1)}{}_{(0)(0)(i)}$, the effect of the waves on particles is only transverse, i.e., in the $\boldsymbol{e}_{(i)}$ directions. Unlike general relativity, the deformations caused by the waves are not always traceless. Since it is possible to choose a frame where $\dot{x}^i = 0$, the conditions for the waves to be traceless become

$$g^{pq}\left(g_{uu||pq} + g_{pq,uu} - \tfrac{1}{2}g^{os}g_{op,u}g_{sq,u}\right) = 0\,. \tag{18}$$

If we now restrict ourselves to the QG and transverse space with constant curvature, the equations simplify into

$$\begin{aligned} \ddot{Z}^{(1)} &= 0, \\ \ddot{Z}^{(i)} &= \left(\tfrac{1}{2}\dot{u}^2\,g_{uu||pq}m_i^p\,m_j^q + \frac{{}^S R}{(D-2)(D-3)}g_{pq}\big(g_{kl}\,\dot{x}^p\,\dot{x}^k\,m_i^q\,m_j^l - \dot{x}^p\,\dot{x}^q\,\delta_{ij}\big)\right)Z^j, \end{aligned} \tag{19}$$

and the waves are traceless when $\Delta g_{uu} = 0$, which corresponds to the solution of general relativity. If we solve Eq. (10) for g_{uu}, we can insert the resulting metric component into Eq. (19).

3.2. *Geodesic deviation for VSI-spacetimes*

For the VSI spacetimes (12), the equations of geodesic deviation are

$$\begin{aligned} \ddot{Z}^{(1)} &= \tfrac{1}{2}\dot{u}\,c_{,p}\,m_j^p Z^{(j)}, \\ \ddot{Z}^{(i)} &= \tfrac{1}{2}\dot{u}\,c_{,p}\,m_i^p Z^{(1)} + \left(\tfrac{1}{2}\dot{u}^2\,g_{uu||pq}m_i^p\,m_j^q + m_{(i}^p m_{j)}^q\,\dot{u}\,\dot{x}^k\,c_{,p}\,g_{qk}\right)Z^{(j)}\,. \end{aligned} \tag{20}$$

We can see that in general, there is a longitudinal effect due to the term $1/2\,\dot{u}\,c_{,p}$. This would be impossible in general relativity, because of $c_{,p} = 0$.

4. Conclusion

We studied Kundt spacetime in QG by an explicit analyses of the constraints imposed by the field equations. We investigated the special cases, the *pp*-waves and VSI spacetimes, and obtained solutions that are not allowed in general relativity. The equations of geodesic deviation shown that the effect of *pp*-waves is always transverse, but *generally not traceless* and the VSI spacetimes cause *longitudinal effect*. Both effects are not possible in general relativity.

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