

NEUTRINOS, GRAND UNIFIED THEORIES AND THE HIERARCHY PROBLEM

Francesco Caravaglios
*Dipartimento di Fisica, Università di Milano, Via Celoria 16,
I-20133 Milano, Italy*
and *INFN sezione di Milano*

Abstract

The Standard Model correctly describes all interactions at (and below) the electroweak scale. However it does not explain the peculiar pattern of quark, lepton and neutrino masses. Also charge quantization is not understood. These are well known motivations to go beyond the Standard Model and to build a Grand Unified Theory. This extension has several good predictions but the proton lifetime is huge compared to similar weak decays. This hierarchy problem suggests two possible extensions of the standard quantum field theory: a non linear version of the Schroedinger functional equation and Third Quantization.

1 Introduction

The theory that describes the strong, electromagnetic and weak interactions is based on the gauge group $SU(3) \times SU(2) \times U(1)$. The symmetry group is spontaneously broken and the gauge boson together with the matter fermions become massive. If and only if the scalar field responsible for electroweak symmetry breaking is a $SU(2)$ doublet with hypercharge $-1/2$ we get the well known relation

$$\frac{M_W^2}{M_Z^2} = \cos^2(\theta_W) \quad (1)$$

that relates the weak boson masses with the coupling constants in the interaction between weak boson and fermions. Also charge quantization comes from this peculiar choice for the Higgs hypercharge, and this choice is natural in Grand Unified Theories as we will see later. The Standard Model gives a correct description of all forces that act at and below the weak scale. In fact it provides us with several theoretical predictions for all the observables listed in Table 1.

Adding an extra Z' or additional Higgs doublets does not improve the fit of data; on the contrary these extensions of the Standard Model are strongly constrained by these data (Table 1). The the top mass obtained in this fit is in very good agreement with the direct experimental observation. The Higgs mass seems to be not very large, probably the Higgs is lighter than the top. When the top mass is very heavy, as proven by experiments, the radiative corrections to the effective potential are large. This theoretical extrapolation of the standard theory to values of the Higgs average field much higher than the weak scale, shows that the value of 246 GeV deduced from the weak boson masses is not a global minimum if M_H does not satisfy the inequality ^{1, 2)}

$$M_H \gtrsim 75 + 1.64(m_t - 140) - 3\left(\frac{\alpha_s - 0.117}{0.007}\right) \quad (2)$$

This limit holds in the standard theory. As we will see after, the effective potential is a theoretical extrapolation of the energy of the universe to quantum physical states very far from the present universe that we observe, however we know that theories often have a wide validity region that can often cover several order of magnitudes. The validity of maxwell equations, as well as quantum

mechanics have been proved in several extremely different experimental situations. If the effective potential of the standard theory has a validity extended over several order of magnitudes of the Higgs average field could be challenged not only by the limit (2) but also by the so called hierarchy problem that appears when the group $SU(3) \times SU(2) \times U(1)$ is embedded into a unified gauge group. We mention the following arguments that motivate us to embed the standard theory into a grand unified theory. The first motivation is the charge quantization and the quantum numbers of the matter fermions. In Table 2 we give a list of some reducible representation of $SU(3) \times SU(2) \times U(1)$ that are anomaly free. We observe that the unifying group $SU(5)$ predicts that matter fermions correspond to the choice of the last row. On the contrary, other rows are acceptable anomaly free representations that do not immediately lead to any unified group.

Without the assumption of a unified theory that includes a flavor symmetry, it remains a mystery why nature has chosen three times the last row (Table 2) for the three families ³⁾. Also the Higgs hypercharge, that is a completely arbitrary choice without unification hypotheses, find an obvious explanation within $SU(5)$. Among all possible groups of unification $SU(5)$, $SO(10)$ and E_6 are the most favored candidates. These are the arguments in favor of unification, but we have not yet understood why the proton lifetime is huge, if compared with the muon decay and the neutron decay lifetime. This is the hierarchy problem, *i.e.* the need of an explanation for the gauge lepto-quark boson masses and the weak boson masses. The effective potential responsible for the symmetry breaking pattern $SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3)_{\text{col}} \times U(1)_{\text{em}}$ is written

$$V = -\mu^2 H^2 - m^2 \Sigma^2 + \lambda_1 H^4 + \lambda_2 H \Sigma^2 H + \lambda_3 \Sigma^4 \quad (3)$$

where Σ and H are respectively the **24** and the **5** of $SU(5)$. We have to choose the arbitrary parameters μ, m, λ_2 , with an extreme fine tuning if we want the hierarchy $\tau_{\text{prot}} \gg \tau_{\mu}$ between the proton and the muon lifetime. We will see how it is possible to modify the standard theory in order to obtain a simple explanation of the hierarchy problem.

Table 1: *The electroweak data and the Standard Model fit* ⁴⁾.

observable	experimental value	SM prediction	pull
M_Z	91.1876 ± 0.0021	91.1874 ± 0.0021	0.1
Γ_Z	2.4952 ± 0.0023	2.4968 ± 0.0011	-0.7
σ_{had}^0 [nb]	41.541 ± 0.037	41.467 ± 0.009	2.0
R_e	20.804 ± 0.050	20.756 ± 0.011	1.0
R_μ	20.785 ± 0.033	20.756 ± 0.011	0.9
R_τ	20.764 ± 0.045	20.801 ± 0.011	-0.8
R_b	0.21629 ± 0.00066	0.21578 ± 0.00010	0.8
R_c	0.1721 ± 0.0030	0.17230 ± 0.00004	-0.1
A_{FB}^e	0.0145 ± 0.0025	0.01622 ± 0.00025	-0.7
A_{FB}^μ	0.0169 ± 0.0013		0.5
A_{FB}^τ	0.0188 ± 0.0017		1.5
A_{FB}^b	0.0992 ± 0.0016	0.1031 ± 0.0008	-2.4
A_{FB}^c	0.0707 ± 0.0035	0.0737 ± 0.0006	-0.8
A_{FB}^s	0.0976 ± 0.0114	0.1032 ± 0.0008	-0.5
\bar{s}_l^2	0.2324 ± 0.0012	0.23152 ± 0.00014	0.7
	0.2328 ± 0.0050		-1.5
A_e	0.15138 ± 0.00216	0.1471 ± 0.0011	2.0
	0.1544 ± 0.0060		1.2
	0.1498 ± 0.0049		0.6
A_μ	0.142 ± 0.015		-0.3
A_τ	0.136 ± 0.015		-0.7
	0.1439 ± 0.0043		-0.7
A_b	0.923 ± 0.020	0.9347 ± 0.0001	-0.6
A_c	0.670 ± 0.027	0.6678 ± 0.0005	0.1
A_s	0.895 ± 0.091	0.9356 ± 0.0001	-0.4
M_W			

Table 2: Representations of the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$. The last row corresponds to the $10 + \bar{5}$, the minimal and anomaly free chiral representation of $SU(5)$.

$(3,3)(-1) + (3,2)(4) + (3,1)(-5)$
$(1,1)(-5/6) + (1,1)(-5/6) + (1,1)(-1/6) + (1,1)(1/3) + (1,1)(1/2) + (1,1)(1)$
$(3,2)(4) + (3,2)(-4) + (1,2)(1) + (1,2)(-1)$ vectorlike
$(1,2)(-1/2) + (3,1)(1/3) + (1,1)(1) + (3,1)(-2/3) + (3,2)(1/6) \subset 10 + \bar{5}$

2 Non linear extension of quantum field theory

The free classical hamiltonian of a scalar real field is written

$$\mathcal{H} = \int d^3x \pi^2(x) + \phi(-\nabla^2 + m^2)\phi(x). \tag{4}$$

We have to replace the functions $\pi(x)$ and $\phi(x)$, defined in the three-dimensional space x , with two operators $\hat{\pi}(x)$ and $\hat{\phi}(x)$ that satisfy the commutation rules

$$\left[\hat{\pi}(x), \hat{\phi}(y) \right] = i \delta^3(x - y). \tag{5}$$

This quantizes the hamiltonian above (4). We can also give a representation of the algebra (5) of the operators $\hat{\pi}(x)$ and $\hat{\phi}(x)$ in the space of functionals $S[\phi]$, replacing $\hat{\pi}(x)$ and $\hat{\phi}(x)$ with

$$\begin{aligned} \hat{\phi}(x)|S > &\rightarrow \phi(x) S[\phi] \\ \hat{\pi}(x)|S > &\rightarrow i \frac{\delta}{\delta\phi(x)} S[\phi] \end{aligned} \tag{6}$$

It is easy to verify that they satisfy the algebra (5)

$$\left[i \frac{\delta}{\delta\phi(x)}, \phi(y) \right] = i \delta^3(x - y). \tag{7}$$

In the Schroedinger picture, the physical states of quantized field are described by the functional $S[\phi, t]$, whose time dependence t , is given by the Schroedinger equation

$$i \frac{\partial}{\partial t} S[\phi, t] = \int d^3x \left(-\frac{\delta^2}{\delta \phi^2(x)} - \phi(x) \nabla^2 \phi(x) + m^2 \phi^2(x) \right) S[\phi, t] \quad (8)$$

The equation (8) represent the quantized theory of a free scalar field. The mass m is a fundamental and arbitrary constant. In the case of a free particle, m coincides with the physical measured mass, but in the general case of an interacting field it does not coincide with the physical mass, because it also depends on the radiative corrections due to the presence of interactions, and on any possible vacuum expectation value of other scalar fields. For example in (3) the value of μ necessary to get a very light higgs at the weak scale, is around 10^{16} GeV, *i.e.* the order of magnitude of the vev of Σ . The fine tuning is needed to achieve a cancellation between several contributions. In other words this correspond to a very precise choice for μ , very close to the arrow depicted in Fig.1. Since μ is a free parameter, the choice of μ very close to the arrow (Fig.1) is accidental and would give not natural predictions. Now we will see how this odd fine tuning can be explained in a non linear extension of the equation (8). Let us assume that we add a non linear term that modifies eq. (8) as follows

$$i \frac{\partial}{\partial t} S[\phi, t] = \hat{H}S[\phi, t] + \int d^3x J(x, t) \phi^2(x) S[\phi, t] \quad (9)$$

$$J(x, t) = \int D\phi S^\dagger[\phi, t] \phi(x)^2 S[\phi, t]$$

When the non linear term $J(x)$ is very small and negligible the equation (9) reduces to a linear equation and it describes an ordinary quantum field theory. But in certain physical situations $J(x)$ could be not negligible¹. Let us consider the case when $J(x)$ is small but not negligible, and we can solve the equation (9) in perturbation theory. The simplest non trivial case is when $J(x, t)$ is a constant and does not depend on space and time. This happens when the functional S corresponds to physical systems where the field ϕ has constant and non zero vev. For any fixed value of J eq.(9) is linear, and we know that such a linear equation admits a stationary solution $S[\phi, t]$ when the expectation value of ϕ minimizes the effective potential (with J fixed). $S[\phi, t]$ is

¹When the physical state $S[\phi, t]$ is a system that contains one (or more) scalar particles ϕ , then J is proportional to the wave function squared of this particle.

the wave functional of the state with minimal energy. The vev of ϕ depends on the arbitrary choice of J , but also J (in the non linear case) is a function of the vev ϕ . Thus both the vev ϕ and the constant J are two variables determined by two equations (9). The non linear term in (9) can be replaced by any generic dependence on the vev ϕ , in fact the second eq.(9) is an arbitrary physical choice. An illustrative choice like

$$\mu^2(\phi) = \mu^2 + J = -M_{\text{unif}}^2 \log(\phi/M_{\text{unif}}) \quad (10)$$

could even explain the hierarchy problem. In fact in the linear theory the vacuum expectation value is a function of the arbitrary constant μ (see Fig.1), but in the non linear theory μ is not arbitrary and depends on ϕ (see eq.(9) and eq. (10)). The special dependence (10) explains why the intersection of both curves² (Figure 1) happens when the vev ϕ is very small *i.e.* close to the arrow. This explains the hierarchy problem.

However a non linear extension of the Schroedinger functional equation shows the lack of a theory of measurement. If a state S , evolves from being the superimposition of several eigenstates toward a single eigenstate of an observable, because of a measurement, then this time evolution also affects (9) and the probability distribution of the final states is automatically modified. In other words the time evolution deduced from equation (9) can be considered valid until when no measurement is performed³.

There is another extension of the field theory that does not violate the quantum mechanical principle of linear superimposition in the evolution of physical states and that could explain in a similar way the hierarchy problem. But before introducing this new theory we deepen briefly the safety of a collider like the LHC in the context of a non linear extension.

It is not hard to realize that if we abandon the request of linearity in eq. (8), various possible extensions are possible, each one with a phenomenology

²The first curve is the dependence of the vev from μ as from the minimization of (3); the second curve comes from the dependence of μ (or equivalently J) from the vev.

³Note that even the definition of measurement in quantum mechanics is rather ambiguous. And this put an ambiguity on the extent of validity of eq.(9).

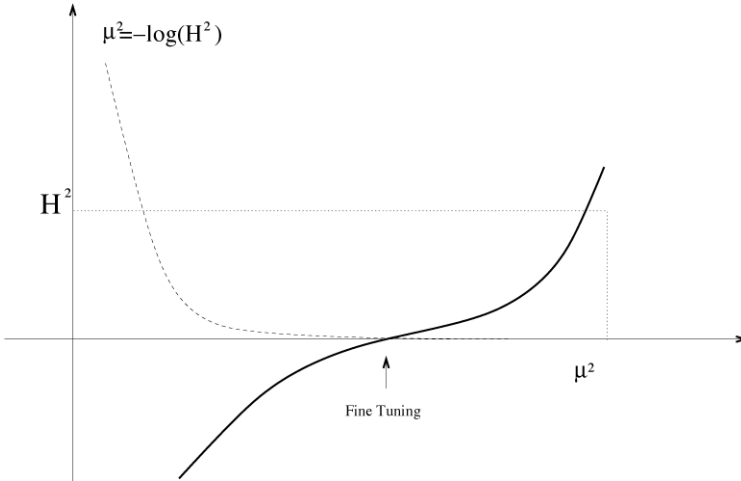


Figure 1: The Higgs doublet vev $H^2 = \langle \phi^2 \rangle$ as a function of the bare mass μ^2 (solid curve). The dashed curve comes from the non linear term and gives the bare mass μ^2 as a function of the vev H (see eq.(10 in the text)).

and with physical consequences that are completely unexpected. Even if an exhaustive discussion of all possible cases is very difficult or even impossible, we briefly draw our attention to few cases that probably deserve more attention. Firstly, let us note that the limit 1, 2) on the higgs mass due to the requirement of stability of the vacuum cannot be directly applied in a non linear extension of the standard theory. Let us now see some potential risks: the creation of a new exotic particle ϕ at the collider LHC locally changes the value of J (9), that is in the region occupied by the wave packet of this scalar particle. This could modify the fundamental bare constants of the linearized theory. It would also modify the physical masses and the couplings of the standard particles: for example the photon could become massive, and all electromagnetic interactions would be turned off in a region of finite volume⁴.

Another risk could come from the fact that the non linear theory violates the crossing symmetry and thus the production of very light particles with

⁴The theory of quantum mechanics does not put any bound on the size of a wave packet

strong interaction with matter is not incompatible with the observation of previous accelerators. We remind also that non linear interactions with the simultaneous presence of significant amount of dark matter in the solar systems adds other dangerous scenarios.

3 Third Quantization

A similar but alternative explanation of the hierarchy problem is obtained embedding second quantization into third quantization ^{5, 6}). The embedding of first quantization into second quantization proceeds as follows. The Schroedinger equation for one particle is written

$$i \frac{\partial}{\partial t} \psi(x, t) = H \psi(x, t) \quad (11)$$

and in fact the quantum state of a particle in the Schroedinger picture is a wave function $\psi(x)$. The wave function is replaced by an operator when we go to second quantization (quantum field theory)

$$\psi(x) \Rightarrow \hat{\psi}(x) \quad (12)$$

and we set the following anticommutation rules

$$\left\{ \hat{\psi}(x), \hat{\psi}^\dagger(y) \right\} = \delta^3(x - y). \quad (13)$$

The quantum field theory analogue of eq.(11) is eq.(8). This equation (8) tells us that the quantum state of the universe is described by a functional $S[\phi, t]$ where the variable t denotes the time evolution of the physical state. If we repeat the same steps as for going from first quantization to second quantization, and we want to embed second quantization into third quantization, then the functional $S[\phi]$ becomes an operator

$$S[\phi] \Rightarrow \hat{S}[\phi] \quad (14)$$

that satisfies the anticommutation rules

$$\left\{ \hat{S}[\phi], \hat{S}^\dagger[\phi'] \right\} = \delta(\phi - \phi'). \quad (15)$$

As an illustrative example, the simplest hamiltonian can be written

$$\mathcal{H} = \int D\phi \, d^3x \, \hat{S}^\dagger[\phi] \left(-\frac{\delta^2}{\delta \phi^2(x)} - \phi(x) \nabla^2 \phi(x) + m^2 \phi^2(x) \right) \hat{S}[\phi]. \quad (16)$$

The vacuum state $|0\rangle$ satisfies the condition

$$\mathcal{H}|0\rangle = 0 \quad (17)$$

and represents a state without fields and without space, while the state

$$|F\rangle = \int D\phi \, F[\phi] \, \hat{S}^\dagger[\phi] |0\rangle \quad (18)$$

with

$$F[\phi] = \exp\left(-\frac{1}{2} \int d^3x \, \phi(x) \sqrt{-\nabla^2 + m^2} \phi(x)\right) \quad (19)$$

represents the state of a universe with only one scalar field ϕ and with minimal energy. It is not difficult to verify the the functional (19) minimizes the energy E among all possible states $|F\rangle$

$$E = \langle F | \mathcal{H} | F \rangle. \quad (20)$$

Let us see why such a theory can explain the hierarchy problem.

We can add to the hamiltonian (16) new composite operators that contain a larger number of creation/annihilation \hat{S}, \hat{S}^\dagger operators. We add to the hamiltonian \mathcal{H} the following interaction

$$\mathcal{H}_{\text{int}} = \int D\phi \, d^3x \sum_{i=1}^n a_n \hat{S}^\dagger[\phi_1] \cdots \hat{S}^\dagger[\phi_n] \phi_1^2(x) \cdots \phi_n^2(x) \hat{S}[\phi_1] \cdots \hat{S}[\phi_n]. \quad (21)$$

We introduce the function $G(J)$ defined by the sequence of a_n as follows

$$G(J) = \sum_{n=1}^{\infty} a_n J^n \quad (22)$$

We have a considerable freedom in the $G(J)$, and almost any choice of $G(J)$ corresponds to a physically acceptable⁵ \mathcal{H}_{int} . In those cases where one can

⁵Unfortunately we have not yet (in third quantization) a highly constraining theoretical principle like “renormalizability”, that applies only in second quantization. Thus we have a lot of freedom in this embedding and in the choice of \mathcal{H}_{int} .

apply the mean field approximation, the vacuum does not satisfy the trivial relation

$$S[\phi] |0\rangle = 0. \tag{23}$$

On the contrary, the action of several annihilation operators S is the following

$$\hat{S}[\phi_1] \hat{S}[\phi_2] \cdots \hat{S}[\phi_n] |0\rangle \simeq F[\phi_1] F[\phi_2] \cdots F[\phi_n] |x\rangle \tag{24}$$

where F is a functional that must be determined by the minimization of E

$$E = \langle 0 | \mathcal{H} | 0 \rangle \tag{25}$$

that leads us to the equation

$$\left(-\frac{\delta^2}{\delta \phi^2(x)} - \phi(x) \nabla^2 \phi(x) + (m^2 + G(J)) \phi^2(x) + \gamma \phi^4 \right) F[\phi] = \lambda F[\phi]. \tag{26}$$

where J is given by

$$J = \int D\phi F^\dagger[\phi] \phi^2(x) F[\phi]. \tag{27}$$

The equation (26) is not linear in F but it can be solved as follows. Firstly, let us neglect eq.(27), and let us assume that J is an arbitrary constant (an external source) that does not depend on F . With this assumption, the equation (26) is much more simple, since it is linear and we know how to solve it, by means of ordinary quantum field theory methods. In fact the eq.(26) is the same equation that we solve to find out the state with minimal energy (the vacuum) in quantum field theory, we have to calculate and minimize the effective potential where $G(J)$ appears as an external source: it corrects the bare mass with the replacement

$$m^2 \rightarrow m^2 + G(J) \equiv \mu^2. \tag{28}$$

The field ϕ takes a vev if $\mu^2 \equiv m^2 + G(J)$ is negative; the vev will be a function of J , through the dependence (28). But also J is a function of the vev as predicted by the original exact equation (27). We have two variables and two equations: both the vev ϕ and J are determined. This clearly appears in Fig. 1, where the solid curve gives the dependence of the vev on the μ^2 , as predicted

by the minimization of the full effective potential (*i.e.* including all radiative corrections). The dashed curve gives the dependence of μ^2 on J , where we have assumed a logarithmic function for $G(J)$. In this case the intersection of the two curves occurs very close to the arrow: it is not a fine tuned and arbitrary choice, the hierarchy is enforced by the logarithmic function $G(J)$.

This theory of third quantization has another interesting direct prediction, concerning the flavor problem: it provides us with an explanation for the existence of fermion families. We have already mentioned that the existence of three fermion families with quantum numbers given by the last row in Table 1, hints a group of unification beyond the Standard Model. However the grand unified theory does not tell us why there are three identical families. In the past several unifying group have been studied, in the attempt to understand the three families. No convincing and significant result has been found. In third quantization our universe (made of three identical fermion families) is obtained applying three consecutive times the creation operator $S^\dagger[\psi]$ on the vacuum state

$$\int D\psi F[\psi_1, \psi_2, \psi_3] S^\dagger[\psi_1] S^\dagger[\psi_2] S^\dagger[\psi_3] |0\rangle. \quad (29)$$

The functional F identifies the physical quantum state of our universe, and the three functions ψ_i represent the fermionic fields of the three families. In the general case the functional operators S^\dagger create new families, and we can call them family creation operators. The anticommutation rules (15) tell us that the functional F is antisymmetric when we exchange the fields ψ_i , not only at $t = 0$, but for the full time evolution: the hamiltonian of second quantization that describes the time evolution of F must be symmetric under permutations of the fermions ψ_i .

We have obtained a simple explanation of the family problem and a clear prediction on the flavor symmetry group. Namely the flavor symmetry is the permutation group S_n where n is the number of families. We still have to understand if the functional S only depends on the fermion field ψ or it is preferable to add the dependence on the gauge boson A^μ too: in the last case the operator $S^\dagger[\psi, A^\mu]$ creates universe containing n families, with the following gauge group and flavor symmetry (7, 8)

$$G^n \times S_n \quad (30)$$

where G is a unified gauge group and the permutations S_n act both on the fermionic families and the gauge bosons families, exchanging the n factors in the group G^n . It remains to understand which gauge group G to choose. $SU(5)$ is a possible group ⁹⁾ but it is a symmetry that does not automatically contain righthanded neutrinos (*i.e.* gauge bosons ignore the righthanded neutrinos): we have not explored this possibility. $SO(10)$ is the most appealing candidate ⁸⁾, because it contains the righthanded neutrino in the **16**. In the simplest $SO(10)$ model where the Higgs doublet is in the **10**, we have yukawa unification between the Dirac neutrino masses and the up quark masses. This must be discarded. There are interesting exceptions to this unification if we put the Higgs into larger irreducible representations but this study is left for another work.

We have decided to focus on the gauge group E_6 . Differently from $SO(10)$, whose **16** contains only one Standard Model singlet, the **27** of E_6 contains two singlets of the Standard Model ($SU(3) \times SU(2) \times U(1)$). The lefthanded neutrino of the standard model can exchange a yukawa interaction with both singlets. While for the first singlet, precisely as in $SO(10)$, this interaction coincides with the yukawa interaction in the up quark sector, the coupling between the second singlet and the lefthanded neutrino does not unify with other yukawa fermion couplings. Namely, the scalar representation **351'** of E_6 contains various $SU(2)$ doublets with different quantum numbers, and a particular one gives a yukawa interaction for neutrinos only

$$\lambda_{27\ 27\ 351'} = \lambda_{\nu_L \nu_R} H \quad (31)$$

while all other fermions contained in the **27** have a combination of quantum numbers such that any yukawa coupling with the Higgs doublet in (31) is forbidden. The interaction (31) allows us to understand why the neutrino Dirac mass does not unify with up quark mass. After having chosen the group $G = E_6$ we must fix the number n in (30). The simplest and more obvious choice is $n = 3$, but this choice does not help us in understanding why the two almost degenerate states (the first two columns in (32)) in neutrino oscillations are the S_3 singlet and the component of the S_3 doublet that is even under the exchange of the two heaviest families (the S_2 symmetry). In other words the mass hierarchy between the even states and the odd state under S_2 suggests a S_2 symmetry and not S_3 ; but we need $n \geq 3$ in (30) if we want (at least)

three families. In fact even in those cases $n > 3$, some pattern of symmetry breaking of the group (30) lead us to the Standard Model with three families of fermionic matter. It is just in these cases that we also find an explanation for neutrino masses and mixing as observed in neutrino oscillations. For clarity, we study the case $n = 4$, because the generalization to the case with arbitrary $n > 3$ is trivial. Our aim is to explain how to attain in neutrino oscillations the mixing angle matrix ¹⁰⁾

$$\begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (32)$$

with $\Delta m_{\text{atm}}^2 \gg \Delta m_{\text{sol}}^2$. We have three distinct possibilities: the neutrino mass matrix is diagonal and the oscillations are due to an off-diagonal charged lepton mass matrix. The second case is when the charged lepton mass matrix is diagonal. The last possibility is when both matrices are not diagonal.

We will assume that the lepton charged matrix is diagonal, thus the columns of the matrix (32) coincide with the three mass eigenstates of neutrinos in the flavour basis. They are also eigenstates of the symmetry S_2 that exchanges the last two rows in the (32). The second column is a singlet of the S_3 symmetry that permutes the rows.

In the following model we will try to explain the matrix (32), and why $|\Delta m_{\text{atm}}^2| \gg |\Delta m_{\text{sol}}^2|$, but we will ignore the sign of Δm_{sol}^2 , because it requires a more detailed study. The S_n symmetry ($n \geq 3$) can hardly explain the pattern $m_3^2 \gg m_1^2 = m_2^2$, but it can more easily explain why

$$m_3^2 \gg m_1^2 \gg m_2^2. \quad (33)$$

In fact, the seesaw mechanism changes the (33) into $m_{\text{sing}}^R \gg m_{\text{doub}}^R$: the righthanded S_3 singlet becomes the heaviest state. So the S_3 symmetric righthanded neutrino matrix must be of the form

$$M_\nu^R \simeq \begin{pmatrix} m_d & m & m \\ m & m_d & m \\ m & m & m_d \end{pmatrix}. \quad (34)$$

with

$$m_d = m. \quad (35)$$

The matrix (34) descends from the S_3 symmetry, while eq. (35) does not. The reason why the S_3 doublet is much more light is obscure.

If we add a fourth family, we can write the following antisymmetric matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad (36)$$

that has the following properties: it is S_3 symmetric, *i.e.* it is invariant under the exchange of the first three families. It couples only with S_3 singlets, the only states acquiring a non zero mass. The doublet of S_3 is given by the two massless states $(-2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}, 0)$ and $(0, -1/\sqrt{2}, 1/\sqrt{2}, 0)$.

The matrix (36) is the only 4×4 matrix that is simultaneously S_3 symmetric and antisymmetric under transposition. Instead of majorana masses, we are forced to choose a dirac mass term

$$M_{ij} \nu_R^i X_R^j \quad (37)$$

with two distinct weyl spinors ν_R and X_R , otherwise the (37) would be identically zero, since $M_{ij} = -M_{ji}$. The **27** of E_6 contains two different weyl spinors, that we can call ν_R and X_R ; thus (36) and (37) are compatible with the choice of the group $E_6^4 \rtimes S_4$.

We complete this discussion, suggesting how to break the group S_3 into S_2 . We add a scalar field ϕ^i , with the family index $i = 1, 4$. Only the first component of this field takes a vev $\phi^1 = v$. The state $(-2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}, 0)$ takes a mass, while the orthogonal state $(0, -1/\sqrt{2}, 1/\sqrt{2}, 0)$ remains as the lightest righthanded neutrino. The seesaw mechanism through the diagonal yukawa interaction (31) will make the S_2 odd state (last column of (32)) the heaviest neutrino. A more detailed discussion of this model can be found in 7).

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