Tolman VII (TVII) Solution with Theory of Energy-Momentum Squared Gravity (EMSG) in Neutron Stars

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Abstract. We study the Tolman VII (TVII) solution for the Energy-Momentum Squared Gravity (EMSG) as well as the solution's compatibility with stars properties. We perform a systematic analysis of the suitability of the equation of state (EoS) for energy density and pressure profiles. In calculations, for convenience, all physical quantities are expressed in the form of dimensionless quantities by introducing the quantity $\zeta = r/R$. This dimensionless quantity will be used to determine the ratio between a star's mass and its radius. The TVII solution provides limits on compactness $C \in [0.01, 0.27]$ and neutron star's mass $2.60M_{\odot}$. Having α as a free parameter of the EMSG model gives the new range compactness $C \in [0.01, 0.386]$ for $\zeta[0, 1]$ with $\alpha \in [0.01, \infty]$ and shows a star mass of $2.97M\odot$. The metric e^{λ} effective and the metric e^{ν} effective produce the range of $\alpha \in [0.1, 0.6]$,which represents both metric to be physical.

1. Introduction

Neutron stars are astronomical objects formed from supernova explosions of collapsed giant stars [1]. When the core of a giant star collapses, the electrons and protons in the core combine to form neutrons. Furthermore, these neutrons then accumulate and form a neutron star which is very dense and has a very high density. Based on observations [2], the mass of the neutron star in EoS is $1.5 - 2.7M\odot$. This mass depends on the assumptions we make about the boundaries of possible states. Meanwhile, general relativity predicts the maximum limit for neutron stars at $\approx 3,5M\odot$. The density of a neutron star can also reach about one billion cubic centimeter tonnes, making it one of the densest objects known in the universe.

Another important thing that describes the interior of a neutron star is compactness. Compactness is defined as the ratio between a star's mass and its radius. The relation between the mass and radius of a NS depends strongly on the equation of state (EoS) of nuclear matter, the relationship between energy density and pressure. This, in turn, means that one can probe the EoS by measuring the NS mass and radius independently [3, 4, 5, 6, 7]. For example, the X-ray payload NICER at the International Space Station recently measured the mass and radius of a pulsar to $\approx 10\%$ accuracy [8, 9], which has been used to constrain the EoS and measure nuclear matter parameters [10, 11, 12]. Mass measurements of heavy NSs also help to constrain the EoS [13, 14, 15]. Neutron stars have very large compactness because they have quite a large

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mass but are very small in size. The compactness of a neutron star is only slightly smaller than the Schwarzschild radius, which is a measure of the radius of an object when its mass is in highly distorted spacetime such as a black hole. The dense nature and small size of neutron stars is closely related to the very extreme conditions in their cores.

In the general theory of relativity, one solution to Einstein's field equations in the description of neutron stars has been written in the Tolman solution [16]. The Tolman method was developed from a way of solving Einstein's field equations applied to static spherical systems to produce explicit solutions as analytical functions. The Tolman VII (TVII) analytical solution of Einstein's gravitational field equation has attracted much interest from astrophysicists to study it. According to [17, 18], one reason is that the analytical solution of Einstein's equations can explain the relationship between the structure of space-time and the interior of neutron stars. In addition, the solution from TVII is a realistic analytical model because it can be constructed into a simple quadratic function of the radial profile of energy density which can produce solutions for the radial profile of metric and pressure components. In understanding the TVII solution by predicting the energy density of the radial profile of energy based on a realistic equation of state (EoS), [19] introduced the Energy-Momentum Square Gravity (EMSG) solution by modifying the momentum energy tensor in the Hilbert-Einstein action equation and introducing the free parameter α to describe the energy density profile to be more realistic and accurate. Consideration of compact stars in braneworld scenarios, leading to Einstein's equations reminiscent of EMSG for $\alpha > 0$, is not new [20], and recently, it was claimed that the hyperon puzzle can be resolved, due to the pressure contribution nonlinear matter to the right side of Einstein's equation, in the Randall-Sundrum type-II brane world in [21] and in the brane world in Eddington-inspired Born-Infeld gravity in [22]. General relativity modification of the Lagrangian gravitational generalization of the linear function of scalar curvature, $\boldsymbol{R}$, it is possible to consider generalizing the Lagrangian matter form nonlinearly, for example, to some scalar analytic function $T^2 = T_{\mu\nu}T^{\mu\nu}$ formed from the energy-momentum tensor(EMT), $T_{\mu\nu}$, from material pressure, as first discussed in [19, 23, 24, 25].

In this paper, we focus on EMSG, which leads to a quadratic contribution to gravity in terms of matter, which can be effective at high energy densities and pressures. Tolman VII produced two equations that describe the pressure and energy density as well as the compactness range in neutron stars. The compactness range obtained on TVII becomes a reference for selecting compactness in the EMSG model. Another important thing, the pressure and energy density equations in TVII are substituted in the EMSG equation profile to produces effective energy density and effective pressure, which also produce several equations of state (EoS), mass equations and metrics space-time. Then, we need to checking the equation of state using the general physical requirements in [26] thereby generating a new compactness range for the EMSG. Not only that, this general physical requirement also produces the α maximum as a free parameter in EMSG.

In section II, first of all, we introduce the energy density equation and the pressure equation from TVII that are analytically produced through Einstein's gravitational field equations. Second, we introduce EMSG model that are produced through the Hilbert-Einstein action equations, thus also producing new equations for energy density and pressure in neutron stars. Third, we briefly describe the numerical method employed to determine the equations of energy density effective, pressure effective and some quantities that describe about the mass and metrics. Fourth, Equations of energy density effective and pressure effective directly explained the equation of state in neutron stars which needs to be check using general physical requirements to produce a new range of compactness. In section III, we conclude and give possible directions for future work. We use the geometric units of $c = 1$ and $G = 1$ throughout this paper.

2. Method

The methods that used used in this paper are analytical methods and numerical methods. First, we use the analytical method in the reproduced paper [27] explicitly using Einstein's gravitational equations to find the solution of EoS Tolman VII. Second, we use the numerical method in the paper [19], which reproduces explicitly the Einstein-Hilbert action with energy momentum and is already contracted, to determine the EMSG solution. Then, we will limit the solution obtained to check the physical meaning of the solution against the general physical requirements [28]. Finally we will plot the EMSG solution by varying the compactness C and the parameter α with the range of observational data carried out [29] which refers to the range of parameters compatible with the properties of neutron stars.

Tolman VII's original solution in [29] ends up with the energy density equation and the pressure equation can be written as,

$$\rho_{\text{Tolman}}(\zeta) = \frac{15C}{8\pi R^2}(1-\zeta^2), \tag{1}$$

$$P_{\text{Tolman}} = \frac{1}{4\pi R^2}\left[\sqrt{3Ce^{-\lambda_{\text{Tolman}}}}\tan\phi_{\text{Tolman}} - \frac{C}{2}(5-3\zeta^2)\right] \tag{2}$$

After getting the original equations for energy density and pressure from the Tolman VII solution. We also need to consider the expressions of the energy density equation and pressure equation of the EMSG in [19], which can be written as,

$$\rho_{effective} = \kappa\rho + \kappa\alpha\rho^2\left(1 + 8\frac{P}{\rho} + 3\frac{P^2}{\rho^2}\right), \tag{3}$$

$$P_{effective} = \kappa P + \kappa\alpha\rho^2\left(1 + 3\frac{P^2}{\rho^2}\right), \tag{4}$$

In this paper, the Tolman VII solution and the desired EMSG solution are written in dimensionless. In order to produce an equation that describes the relationship between TVII and EMSG. First of all, we need to multiply the energy density and pressure in TVII by their respective dimensional multipliers, resulting in a new equation for energy density which can be written as,

$$\begin{aligned}\bar{\rho}_{\text{Tolman}}(\zeta) &= \rho_{\text{Tolman}}(\zeta)\times 4\pi R^2 \\ &= \frac{15C}{8\pi R^2}(1-\zeta^2)\times 4\pi R^2 \\ &= \frac{15C}{2}(1-\zeta^2),\end{aligned} \tag{5}$$

for the value $\bar{P}$ to be,

$$\begin{aligned}\bar{P}_{\text{Tolman}} &= p_{\text{Tolman}}\times 4\pi R^2 \\ &= \frac{1}{4\pi R^2}\left[\sqrt{3Ce^{-\lambda_{\text{Tolman}}}}\tan\phi_{\text{Tolman}} - \frac{C}{2}(5-3\zeta^2)\right]\times 4\pi R^2 \\ &= \left[\sqrt{3Ce^{-\lambda_{\text{Tolman}}}}\tan\phi_{\text{Tolman}} - \frac{C}{2}(5-3\zeta^2)\right],\end{aligned} \tag{6}$$

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Next, we substitute the equations (5) and (6) into the equations(3) and (4) to produce the following equations which can be written using the effective subscript eff [19]. So, ρ_{eff} to be,

$$\rho_{eff}(\zeta) = \bar{\rho}_{Tolman}(\zeta) + \alpha\bar{\rho}^2_{Tolman}(\zeta)\left(1 + 8\frac{\bar{P}_{Tolman}(\zeta)}{\bar{\rho}_{Tolman}(\zeta)} + 3\frac{\bar{P}^2_{Tolman}(\zeta)}{\bar{\rho}^2_{Tolman}(\zeta)}\right), \tag{7}$$

for P_{eff}

$$P_{eff}(\zeta) = \bar{P}_{Tolman}(\zeta) + \alpha\bar{\rho}^2_{Tolman}(\zeta)\left(1 + 3\frac{\bar{P}^2_{Tolman}(\zeta)}{\bar{\rho}^2_{Tolman}(\zeta)}\right), \tag{8}$$

Here, we can get also the values of m_{eff} and $e^{-\lambda_{\text{eff}}}$, which can be written as,

$$\frac{dm_{eff}(\zeta)}{d\zeta} = \zeta^2\rho_{eff}(\zeta), \tag{9}$$

and for $e^{-\lambda_{\text{eff}}}$,

$$e^{-\lambda_{\text{eff}}(\zeta)} \equiv 1 - \frac{2m_{eff}(\zeta)}{\zeta}, \tag{10}$$

the expressions of equation $m_{eff}(\zeta)$ in the equation (9) and $e^{-\lambda_{\text{eff}}(\zeta)}$ in the equation (10) are both not again analytical, so it needs to be solved numerically. Identical to that, numerically we reach the value of $e^{\nu_{\text{eff}}}$ as,

$$e^{-\lambda_{\text{eff}}(\zeta)}\left(\frac{\nu'_{\text{eff}}(\zeta)}{\zeta} + \frac{1}{\zeta^2}\right) - \frac{1}{\zeta^2} = 8\pi P_{eff}(\zeta), \tag{11}$$

Equations 7 and equation 8 above will be used to express equation of state in neutron stars and equations $9-11$ are used to express neutron stars properties.

The solutions of the equations of state for TVII and EMSG have been successfully reproduced from the Schwarzschild metric ansatz, the momentum-energy tensor, and Einstein's gravitational field equations. In order to be physically meaningful, the interior solution for static fluid spheres of Einstein's gravitational field equations must satisfy some general physical requirements. The following conditions have been generally recognized to be crucial for anisotropic fluid spheres [26]:

(i) The energy density ρ and pressure P must be positive inside the star.

$$\rho > 0, \tag{12}$$
$$P > 0. \tag{13}$$

(ii) The gradients of $d\rho/d\zeta$ and $dP/d\zeta$ must be negative.

$$\frac{d\rho}{d\zeta} < 0, \tag{14}$$
$$\frac{dP}{d\zeta} < 0. \tag{15}$$

(iii) In a static configuration, the speed of sound must be slower than the speed of light ($c = 1$) to satisfy causality conditions and must be greater than zero due to microscopic stability conditions or

$$0 \leq \frac{dP}{d\rho} \leq 1. \tag{16}$$

(iv) Energy condition must comply with the conditions

$$3P + \rho \geq 0. \tag{17}$$

3. Results and Discussion

First we carry out a graphical analysis that explains the equation of state in TVII. We vary the compactness value, with $\zeta[0,1]$. Through the general physical requirements $(12)-(17)$, we finally get the compactness range in TVII.

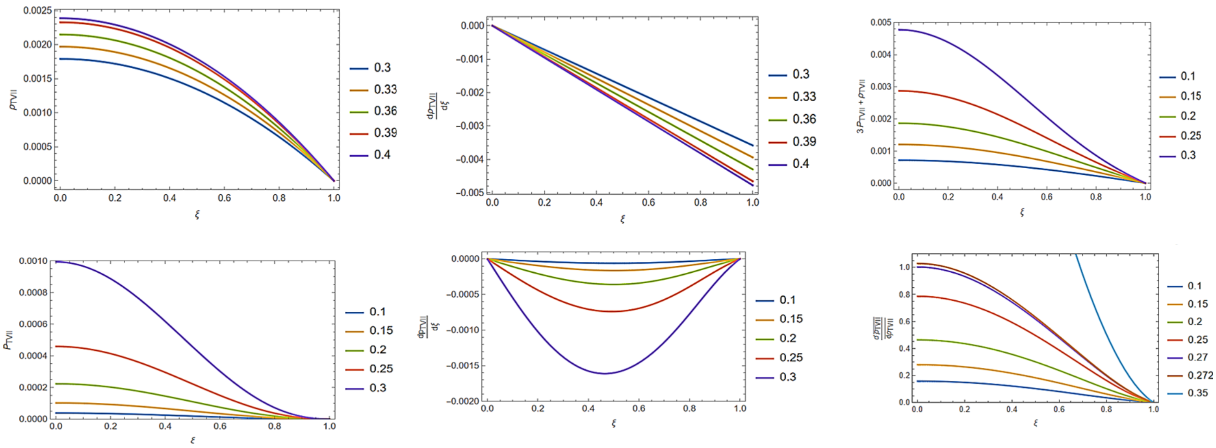

Figure 1. EoS profile of TVII with variations in Compactness.

All the graphs in figure 1 shows that the equation of state in TVII satisfies the general physical requirements. In the graph describing the speed of sound $dP/d\rho$, we also get the compactness range C [0.01,0.27]. This range as a reference as the compactness that will be used in the EMSG model. After that, we vary the parameter α to know whether the model is physical based on general physical requirements $(12)-(17)$. So, we also get a new range of compactness in the EMSG.

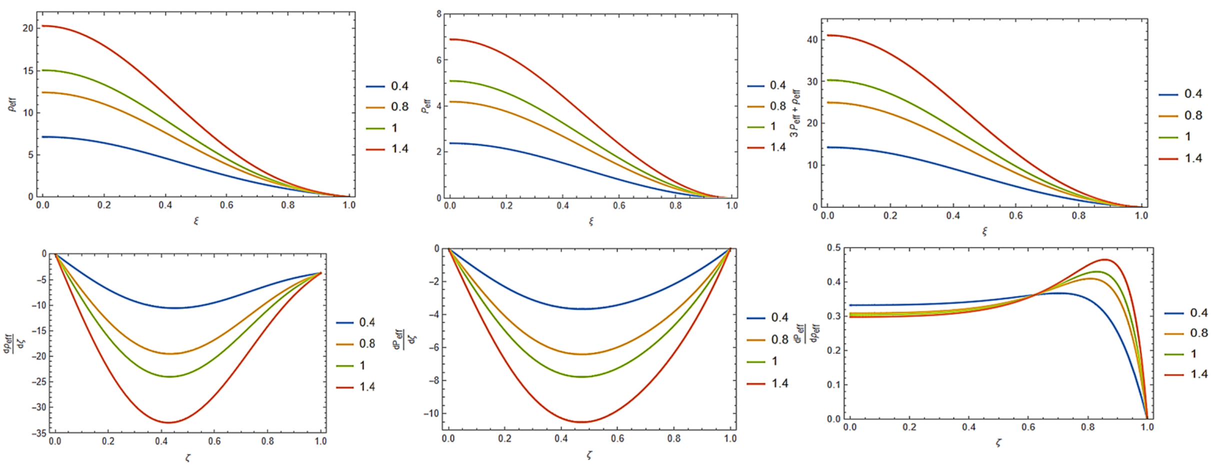

Figure 2. EoS profile of EMSG with $C = 0.25$ and variations in α.

All the graphs in figure 2 shows that the equation of state in EMSG satisfies the general physical requirements. In the graph describing the speed of sound $dP/d\rho$, we also get the range of α [0.01,∞].

Next, we need to analyze the characteristics of the star's properties in the form of the star's mass, the space metric e^{λ} and the time metric e^{ν}.

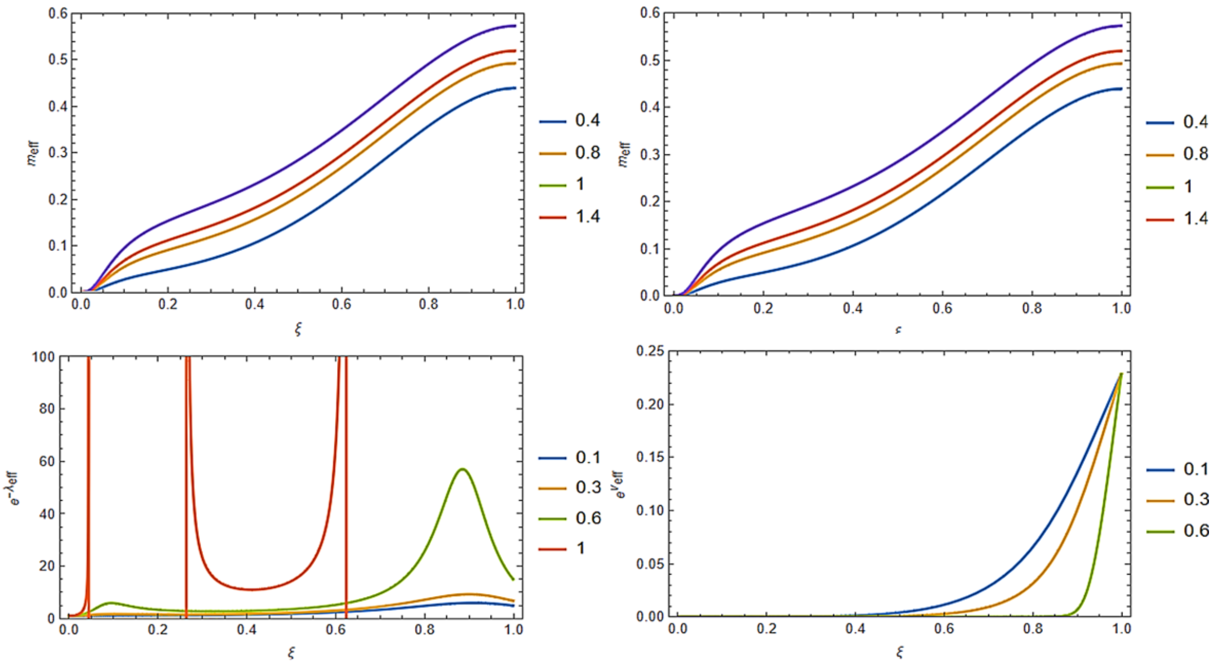

Figure 3. Mass and metric profile of neutron star with $C = 0.386$ and variations in α.

Based on the graph in figure 3 above, the value of the effective mass of a star that satisfy on the general physical requirements with $C = 0.27$ is obtained when $\alpha = 0.4$ is 0.439 (dimensionless mass). If calculated assuming R = 10 Km and mass $M\odot = 1.48Km$ it can reach $2.97M\odot$. Identical to that, the effective mass when it reaches the maximum compactness $C = 0.386$ with $\alpha = 0.4$, the graph shows the same effective mass 0.439 (dimensionless mass). This shows that the parameter α influences the effective mass value because the effective mass value continues to increase. .

Space metric e^{λ} and the time metric e^{ν} shows that, the greater the value of α, the curvature produced by the metric $e^{\lambda_{\text{eff}}}$ becomes infinite. The metric $e^{\lambda_{\text{eff}}}$ becomes physical in the range α [0.1,0.6]. Identical to that, the metric $e^{\nu_{\text{eff}}}$ also shows a value that decreases with changes in ζ until the metric $e^{\nu_{\text{eff}}}$ reaches 0.22. The metric $e^{\nu_{\text{eff}}}$ becomes physical in the range α [0.1,0.6]. This shows that variations in the parameter α also indicate the existence of a range α that satisfies the metric $e^{\nu_{\text{eff}}}$.

4. Conclusion

The Tolman VII solution provides nearly realistic physical information regarding stars, if all general physical requirements are satisfied. The speed of sound in a static configuration cannot be greater than the speed of light, limiting the range of compactness. In this paper we introduce a new range of compactness that satisfies general physical requirements in neutron stars. This is different in [29] which generally does not provide a compactness range that satisfy in general physical requirements. This new compactness range is the range used in EMSG models that feature an alpha parameter. In the frame of general physical conditions, the presence of the alpha parameter apparently expands the range of different compactness. This describe the difference in compactness produced in [19]. Finally, we obtain a range of compactness $C \in [0.01, 0.27]$ with respect to all changes in ζ. Meanwhile, in the EMSG model, we get range of $\alpha \in [0.01, \infty]$ and range of compactness $C \in [0, 0.386]$, both are satisfy in the general physical requirements.

In terms of stars properties, the mass profile of Tolman VII with *compactness* $C = 0.386$ is close to the observational mass of a neutron star ($2.60M_{\odot}$). Thus, with variations of α in the EMSG model, the star mass profile that meets the physical conditions with $C = 0.27$ and $C = 0.386$ can reach $2.97M\odot$. This shows that the α parameter produces a mass profile in the EMSG model that is larger than the Tolman VII mass profile. Meanwhile, variations of the parameter α in the metric $e^{\lambda_{\text{eff}}}$ and the metric $e^{\nu_{\text{eff}}}$ produce the range α [0.1,0.6] which shows the metric to be physical.

In general, the EMSG model can describe neutron stars physically by obtaining a new range of *compactness* for the model. Likewise with the mass profile, the metrics $e^{\lambda_{\text{eff}}}$ and $e^{\nu_{\text{eff}}}$ can also physically describe the properties of stars.The EMSG model has a significant impact on the *compactness* and mass profile of neutron stars. The greater the *compactness* value in the EMSG model, the greater the resulting mass.

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