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Article

Canonical Commutation Relation Derived from Witt Algebra

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Abstract: From an arbitrary definition of operators inspired by oscillators of Virasoro, an algebra is derived. It fits the structure of Virasoro algebra with null central charge or Witt algebra. The resulting formalism has yielded commutators with a dependence on integer numbers, and it follows the Witt-like algebra. Also, the quantum mechanics evolution operator for the case of the quantum harmonic oscillator was identified. Furthermore, the Schrödinger equation was systematically derived under the present framework. When operators are expressed in the framework of Hilbert space states, the resulting Witt algebra seems to be proportional to the well-known canonical commutation relation. This has demanded the development of a formalism based on arbitrary and physical operators as well as well-defined rules of commutation. The Witt-like was also redefined through the direct usage of the uncertainty principle. The results of the paper might suggest that Witt algebra encloses not only quantum mechanics' fundamental commutator but also other unexplored relations among quantum mechanics observables and Witt algebra.

Keywords: Witt algebra; Virasoro algebra; quantum mechanics; commutators

MSC: 11E81; 7B68; 18M40

1. Introduction

1.1. Motivation

It is well known the role of string theories in the direct development of mathematical physics and in the theory of elementary particles. String theories have influenced us to gain a deep understanding of cosmology and gravitation, black holes, for example, and other areas of theoretical physics. Although it has not been experimentally probed so far, a noteworthy strength and well-known capability of string theories constitutes its theoretical formalism exhibiting a robust mathematical structure. On the other hand, it is fair to wonder if such mathematical technology might have direct application to tangible quantum theory and solve realistic problems. For instance, abstract theories based on algebra applied to tangible QM are seen in the work of Girving et al. [1]. There, the density operator has been used in the study of variational estimates for the excitation energy of the quantum Hall effect. This operator has fulfilled the Lie algebra. P. Ginsparg [2], under the context of conformal field theory, has applied Laurent expansion to the stress-energy tensor to derive Virasoro's algebra, which was discovered inside the framework of string theories. J. Ellis et al. [3] have concluded that the whole string theory is quantum mechanics (QM) according to their results, by which it was found that massive string states are related to quantum light states. In the same way, S. Katagiri et al. [4] have tried to close the gap between abstract string theory formalism and QM observables. In fact, they have reformulated Virasoro's generators to address the modeling of Nth-order squeezing with operators' position and momentum. Indeed, the well-known quantum time-dependent

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harmonic oscillator was treated with Virasoro's oscillators. The spirit of this paper just goes over the avenue characterized by the lack of a direct link between string theory and quantum mechanics. This paper just explores this lack by using new definitions based on Virasoro's generators.

1.2. Background

In the context of the Veneziano model [5], new gauge conditions were formulated, giving origin to what is known as string theory. In this context emerged the so-called Virasoro algebra [6] with central charge c, and this algebra can be written below as follows:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \quad (1)$$

It is evident that $\delta_{m,-n}$ is well-known as the delta of Krönecker. Operators $\mathcal{L}_m$, also called oscillators of Virasoro, defined as a sum of products between no-commutative modes, can be written as

$$\mathcal{L}_n = \frac{1}{2}\sum_{p\in Z} \alpha_{n-p}\alpha_p. \quad (2)$$

The Operators written above have played a critical role in the formulation of dual models that express the covariant form of harmonic oscillator containing infinite modes satisfying $[\alpha_p^\mu, \alpha_q^\nu]$=$\delta_{p,q}g^{\mu,\nu}$ as seen in the work of R. C. Brower in [7]. Virasoro algebra from Lie generators and its link to Lie group because the $c-$number was studied by P. V. Alstine in [8], where it was also analyzed as to its direct relation to QM Jacobi theory, yielding interesting relationships involving the oscillators as new representations of $c-$number. In [9] R. Akhoury and Y. Okada have investigated the role of Virasoro oscillators as a function of classical variables inside the framework of Hamiltonian such as $H = \sum_i \pi_i \dot{q}_i - L(X^\mu, X^\nu)$ with the Lagrangian enclosing the action of closed bosonic string theory. The resulting oscillators have yielded a copy of Virasoro algebra at the flat space. Here, it was also reformulated oscillators that have been defined in order to annihilate ghosts, a fact that demanded the construction of a series of commutation relations (not like Virasoro algebra). Y. Saito in [10] has investigated the action of Virasoro oscillators inside the Becchi–Rouet–Stora (BRS) formalism in the form of subsidiary conditions specifically as seen in the action of them onto the Fock's space for $n > 0$:

$$(\mathcal{L}_n + \alpha_0\delta_{n,0})\,|\mathrm{phys}\rangle = 0. \quad (3)$$

Indeed, Saito observed that under conditions dictated by Virasoro algebra, they do not eliminate the negative states of a squeezed string. Such conditions have derived the redefinition of Virasoro oscillators in a new version, this being a kind of extension from the standard structure of algebra. In addition, complex forms in accordance with the ones obtained by Scherk, as seen in [11], were found. Variations on the Virasoro oscillators have been obtained in [12] through the modifications of Kac–Moody algebra, arriving at modified Virasoro oscillators containing also Kac–Moody generators and yielding other classes of Virasoro algebra (Equation (2.7), Ref. [11]). Here, it was seen the role of the central charge to interpret some properties of oscillators to some extent (see [13] for example).

1.3. Contribution of Paper

As mentioned above in [4], S. Katagiri and collaborators have postulated the idea that Virasoro generators, despite the fact that they are dimensionless entities in string theories, can be expressed as linear combinations of momentum and position operators (see in [4] Section 4.1, Equation (1), for instance). In the context of Katagiri's work the generators can be written as $\mathcal{L}_i = -\frac{i}{2}(\hat{\mathbf{x}}^{i+1}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{x}}^{i+1})$ with an i integer number and $\mathcal{L}_i$ satisfying algebra Equation (1). It is interesting to find possible links between dimensionless operators (Virasoro oscillators) and QM observables since it might yield important implications that cross an unproven string theory and tested QM theory [14]. As it is well-known, QM is based on a sophisticated mathematical formulation of physics operators acting on wave functions, in full agreement with QM postulates [15]. The central equation of QM theory, known as the Schrödinger equation, is written as

$$\mathbf{H}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) \left|\Psi(\mathbf{r}, t)\right\rangle = \mathcal{E} \left|\Psi(\mathbf{r}, t)\right\rangle, \tag{4}$$

being $\mathbf{H}$ is the Hamiltonian operator [16–19] often depending on $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$, the position and momentum operators, respectively, satisfying the well-known canonical commutation relation that can be written as

$$[\hat{\mathbf{x}}, \hat{\mathbf{p}}] = i\hbar, \tag{5}$$

and $\hbar$ the Planck constant [20–23]. This relation constitutes the main piece in the formulation of quantum theory. Moreover, other relations can be derived from it. An important aspect linked to Equation (5) is the unknown mathematical procedure that yields that. Some attempts have been performed in the past [24,25]. In this manner, it is interesting to identify a clear link (if any) between Equation (5) and the so-called Witt's algebra (or also centerless Virasoro algebra) [26–29] that is defined as

$$[\Xi_m, \Xi_n] = (m-n)\Xi_{m+n}, \tag{6}$$

with Ξ_q Virasoro oscillators that have played a noteworthy role in the development of string theory. While m and n are integer numbers, Equation (6) is also called Virasoro algebra without central extension [30–35] or Witt's algebra that can be rewritten again as

$$[\mathcal{L}(u, m), \mathcal{L}(u, n)] = g(m, n)\mathcal{A}(u, \Xi_q), \tag{7}$$

with $g(m, n)$ a function of integer numbers, and $\mathcal{A}(u, \Xi_q)$ a function of commutators such as the ones of Equation (5), and u being a physical observable. Therefore, this paper has as its central objective to demonstrate that there exists an alternative way that would allow us to arrive at the Canonical Commutation Relation (CCR) in a closed form from Witt algebra in a direct or indirect way. (By which new types of Virasoro oscillators $\mathcal{L}_q$ are proposed.) To accomplish this, it is argued that $\mathcal{L}_q(s)$ is proportional to the QM momentum operator defined as

$$\hat{\mathbf{p}} = \frac{\hbar}{i}\frac{d}{ds}, \tag{8}$$

in the sense that ($\mathcal{L}_q(s) \propto \mathbf{p}Y(s)$, with operator operating onto $Y(s)$ being an arbitrary integration as shall be seen below). Thus, all of the above allows us to write down the oscillator $\mathcal{L}_q(s)$ as

$$\mathcal{L}_q(s) = \frac{d}{ds}\int \delta(w-s) f(w) g_q(s, w) dw. \tag{9}$$

It should be noted that the derivative in Equation (8) comes from Equation (8), and the quantity $\frac{\hbar}{i}$ is absorbed by integration (only for pedagogical reasons). With respect to Equation (9), the derivative $\frac{d}{ds}$ with s having units of distance with q integer number and "$\delta(w-s)$" Dirac delta function and $g_q(s,w)$ arbitrary function. In this manner the hypothesis would consist that these oscillators are fulfilling the algebra:

$$[\mathcal{L}_p, \mathcal{L}_q] = (p-q)\mathcal{L}_{p+q}. \tag{10}$$

Therefore, the proposal of this paper is to establish that the left side of Equation (10) follows the rule:

$$[\mathcal{L}_p, \mathcal{L}_q] \equiv \mathcal{A}([\hat{\mathbf{x}}, \hat{\mathbf{p}}]), \tag{11}$$

by which one can see that Witt algebra might be an explicit function of CCR through the $\mathcal{A}(\cdot)$ function of QM commutators $[\hat{\mathbf{x}}, \hat{\mathbf{p}}]$. As shall be demonstrated in the next sections, $\mathcal{L}_q(s)$ is a pure QM expression in the sense that it has an imminent ground based on wave function and physical observables. Because of this, one would expect a set of QM relations such as CCR, for example. Indeed, extra commutators can also be derived, but without physics meaning.

Thus, oscillator Equation (9) can be seen as a kind of arbitrary expression instead of being perceived as one belonging to QM or having a well-defined physics profile. When $\frac{\hbar}{i}$ is restored into Equation (9):

$$\mathcal{L}_q(s) = \frac{\hbar}{i}\frac{d}{ds}.\int \delta(w-s) f(w) g_q(s,w) dw. \tag{12}$$

One can see that while $\int \delta(w-s) f(w) g_q(s,w) dw$ the whole integration is dimensionless, oscillator $\mathcal{L}_q(s)$ from Equation (12) is proportional to momentum operator $\mathcal{L}_q(s) \approx \mathbf{p}$, since $\hbar$ has units of momentum times distance. The rest of the paper is as follows: The second section presents the proposal of redefining Virasoro oscillators as a kind of Witt operator. Here are described the key pieces of paper. In the third section, the QM interpretation of Witt operators is explored by means of the derivation of the QM evolution operator as well as the Schrödinger equation. In the fourth section, the results of the paper are explicitly presented, and finally, the conclusion of the paper is given.

2. Definition and Properties of Witt Operators

We begin the present debate with the consideration of arbitrary operators: $\mathcal{A}_m$ and $\mathcal{A}_n$ with m and n integer numbers, then it is said that both satisfy Witt algebra [36–40] if

$$[\mathcal{A}_m, \mathcal{A}_n] = (m-n)\mathcal{A}_{m+n}. \tag{13}$$

From above it is clear that commutator is understood to follow the rule:

$$[\mathcal{A}_m, \mathcal{A}_n] = \mathcal{A}_m\mathcal{A}_n - \mathcal{A}_n\mathcal{A}_m, \tag{14}$$

so that it is obvious that

$$\mathcal{A}_m\mathcal{A}_n - \mathcal{A}_n\mathcal{A}_m = m\mathcal{A}_{m+n} - n\mathcal{A}_{m+n}. \tag{15}$$

So far, no information about operators $\mathcal{A}_m$ and $\mathcal{A}_n$ has been explicitly manifested. In other words, these operators are not necessarily to be subjected to being a real or complex quantity. Nevertheless, in virtue of Equation (12), one can see its complex character (although it can also be defined as pure real expression). It is important to observe that such operators are arbitrary entities without any concrete role. Based on the structure shown in Equation (12) with $\hbar = 1$, one can define a kind of Witt operator (being them complex quantities) as

$$\mathcal{L}_K = x_\ell \left(\frac{i}{\sqrt{2}} \right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^K dx_m, \tag{16}$$

and its corresponding "partner" (essentially follows the change $K \to L$) in the sense that

$$\mathcal{L}_L = x_\ell \left(\frac{i}{\sqrt{2}} \right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{17}$$

It should be noted that subindices K and L are seen inside integration through polynomials x_m^K and x_m^L. Now, written in that way, $\mathcal{L}_K$ and $\mathcal{L}_L$ are exhibiting their dimensionless character. Indeed, it is seen in both the presence of the delta of Krönecker as well as the Dirac delta function [41–43]. Indeed, it should be noted that $\frac{1}{\sqrt{2}}$ plays the role of a kind of normalization constant, although it is not exactly its genuine function, as shall be seen later. Because of the complex numbers, one can claim at first instance that both operators are pure complex entities. The reader can be aware that when Equations (16) and (17) are solved independently, $[\mathcal{L}_K, \mathcal{L}_L]$=$\mathcal{L}_K\mathcal{L}_L - \mathcal{L}_L\mathcal{L}_K$ = 0. Nevertheless, the case when we are operating on each other and vice versa, the commutator yields $[\mathcal{L}_K\mathcal{L}_L - \mathcal{L}_L\mathcal{L}_K] \neq 0$.

In this manner and by following the order seen in Equation (14), the next step is to calculate $[\mathcal{L}_K, \mathcal{L}_L]$ = $\mathcal{L}_K \otimes \mathcal{L}_L - \mathcal{L}_L \otimes \mathcal{L}_K$, that demands to calculate the products:

$$\mathcal{L}_K \otimes \mathcal{L}_L, \tag{18}$$

$$\mathcal{L}_L \otimes \mathcal{L}_K, \tag{19}$$

with $\otimes$ denoting the product of operators. In this manner one can carry out the products firstly in Equation (18) as follows:

$$\mathcal{L}_K \otimes \mathcal{L}_L = x_\ell \left(\frac{i}{\sqrt{2}} \right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^K dx_m \otimes x_\ell \left(\frac{i}{\sqrt{2}} \right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{20}$$

This can be also written as

$$\left[x_\ell \delta_\ell^m \left(\frac{i}{\sqrt{2}} \right) \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^K dx_m x_\ell \delta_\ell^m \right] \otimes \left(\frac{i}{\sqrt{2}} \right) \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{21}$$

Also that product $x_\ell \delta_\ell^m$ has passed to be inside the bracket on the left side. Further operations will do the direct usage of delta of Krönecker by affecting the quantity x_ℓ in both extremes, yielding for them $\delta_\ell^m x_\ell$ = x^m, so that subsequently ℓ opts for m in the whole Equation (21). Thus, one obtains

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left[\left(\frac{i}{\sqrt{2}} \right) \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^{K+2} dx_m \right] \otimes \left(\frac{i}{\sqrt{2}} \right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{22}$$

The same operation as done to the left-side bracket is applied to the right side. On the other side, it is obvious that $\left(\frac{i}{\sqrt{2}} \right) \left(\frac{i}{\sqrt{2}} \right)$ = $-\frac{1}{2}$ by which the product $\mathcal{L}_K \otimes \mathcal{L}_L$ is flipping to real space. In this manner, one arrives at

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left(-\frac{1}{2} \right) \left[\frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^{K+2} dx_m \right] \otimes \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{23}$$

From above it is clear that polynomial x^m was absorbed by integrations in both sides, so that one has below that

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left(-\frac{1}{2}\right) \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^{K+2} dx_m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{24}$$

Integrations containing the Dirac delta functions and their subsequent derivatives are solved in a straightforward manner. For the right side, one obtains

$$\frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m = \frac{d}{dx} x^L = L x^{L-1}. \tag{25}$$

The same procedure is applied to the left side as follows:

$$\frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^{K+2} dx_m = \frac{d}{dx} x^{K+2} = (K+2) x^{K+1}, \tag{26}$$

so that one arrives at

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left(-\frac{1}{2}\right) (K+2) x^{K+1} L x^{L-1} = \left(-\frac{1}{2}\right) L (K+2) x^{K+L}. \tag{27}$$

In order to solve product $\mathcal{L}_L \otimes \mathcal{L}_K$, the reader should be aware that it follows as done to $\mathcal{L}_K \otimes \mathcal{L}_L$. With this view, one can write down

$$\mathcal{L}_L \otimes \mathcal{L}_K = \left(-\frac{1}{2}\right) x^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m x^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^K dx_m. \tag{28}$$

Therefore, one arrives at

$$\mathcal{L}_L \otimes \mathcal{L}_K = \left(-\frac{1}{2}\right) (L+2) x^{L+1} K x^{K-1} = \left(-\frac{1}{2}\right) K (L+2) x^{K+L}. \tag{29}$$

Equations (27) and (29) allow us to calculate the commutator $[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_K \otimes \mathcal{L}_L - \mathcal{L}_L \otimes \mathcal{L}_K$. Therefore, by putting them altoobtainher, one can demonstrate that $[\mathcal{L}_K, \mathcal{L}_L]$ is in full agreement with Witt algebra $\mathcal{L}_K \otimes \mathcal{L}_L - \mathcal{L}_L \otimes \mathcal{L}_K =$

$$\left(-\frac{1}{2}\right) L (K+2) x^{K+L} - \left(-\frac{1}{2}\right) K (L+2) x^{K+L} = \left(-\frac{1}{2}\right) (LK + 2L - KL - 2K) x^{K+L}$$
$$= \left(-\frac{1}{2}\right) (2L - 2K) x^{K+L} \mathcal{L}_K \otimes \mathcal{L}_L - \mathcal{L}_L \otimes \mathcal{L}_K = (K - L) x^{K+L} \Rightarrow [\mathcal{L}_K, \mathcal{L}_L] = (K - L) x^{K+L}, \tag{30}$$

that satisfies the Witt-like algebra:

$$[\mathcal{L}_K, \mathcal{L}_L] = (K - L) x^{K+L}. \tag{31}$$

One can note that Equation (31) fits exactly to Witt algebra if the following proposal holds:

$$x^{K+L} \Rightarrow \mathcal{L}_{K+L}, \tag{32}$$

that can be validated through the usage of Equation (16), for example, with the substitution $K \to K + L$

$$\mathcal{L}_{K+L} = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^{K+L} dx_m \quad (33)$$

$$= \left(\frac{i}{\sqrt{2}}\right) \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^{K+L+1} dx_m = \left(\frac{i}{\sqrt{2}}\right)(K + L + 1) x^{K+L+1} \quad (34)$$

$$\Rightarrow \mathcal{L}_{K+L} = \left(\frac{i}{\sqrt{2}}\right)(K + L) x^{K+L+1} + \left(\frac{i}{\sqrt{2}}\right) x^{K+L+1}. \quad (35)$$

Aside, while Equation (35) is divided by $\frac{ix}{\sqrt{2}}$ one obtains

$$\Rightarrow \frac{\mathcal{L}_{K+L}}{\frac{ix}{\sqrt{2}}} = (K + L) x^{K+L} + x^{K+L} \Rightarrow \frac{\mathcal{L}_{K+L}}{\frac{ix}{\sqrt{2}}} - x^{K+L} = (K + L) x^{K+L}, \quad (36)$$

and one can see from above right-side Equation (36) one obtains directly the following:

$$\frac{\mathcal{L}_{K+L}}{\frac{ix}{\sqrt{2}}} = (K + L + 1) x^{K+L}. \quad (37)$$

Yielding that Equation (32) is satisfied under the condition derived above, written as

$$x = \frac{\sqrt{2}}{i(K + L + 1)}, \quad (38)$$

establishing the fact that under the proposal given by Equation (32), $K + L \neq 1$. One can see that if this inequality is violated, therefore, one can see in a straightforward manner inside Witt algebra the following:

$$[\mathcal{L}_K, \mathcal{L}_L]\Big|_{K+L=1} = (K - L) x^{K+L}\Big|_{K+L=1} \Rightarrow [\mathcal{L}_{1-L}, \mathcal{L}_L] = (1 - 2L) x, \quad (39)$$

yielding another type of algebra differing notably from Witt algebra. It is easy to check that the commutation is null only when $L = \frac{1}{2}$, which cannot be sustained in a scenario of Witt or Virasoro algebra that depends only on integer numbers.

Interestingly, the Witt commutator in the last term of Equation (29) can also be derived in a straightforward manner through the assumption that symmetry exists between operators, in the sense that when Equation (27) opts for $K \to L$ and $L \to K$, one obtains Equation (29). Thus, has that

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left(-\frac{1}{2}\right) L(K + 2) x^{K+L}, \quad (40)$$

$$K \to L$$

$$L \to K$$

$$\mathcal{L}_L \otimes \mathcal{L}_K = \left(-\frac{1}{2}\right) K(L + 2) x^{L+K}, \quad (41)$$

yielding directly $[\mathcal{L}_K, \mathcal{L}_L] = (K - L) x^{K+L}$ providing the fulfilling of Witt algebra. Aside, one can argue that this operation of symmetry might be considered a sufficient condition to arrive at Witt algebra in a straightforward manner.

3. Role of Witt Operators in Non-Relativistic QM

3.1. Identification of QM Structures

Since DeWitt operators Equations (16) and (17) exhibit the same structure, then one can carry out kind of "mathematical radiography" in order to interpret them inside QM territory. Thus, for example, one can rewrite Equation (16) as

$$\mathcal{L}_K = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^K dx_m, \tag{42}$$

moreover, by taking into account these definitions through the usage of bra and kets as states of Hilbert space, one obtains below:

$$\delta(x - x_m) = \langle x|x_m\rangle\,, \tag{43}$$

$$x_m^K = \langle x_m|K\rangle\,. \tag{44}$$

In virtue to properties of brat and ket of Dirac's formalism, $\Phi_K(x_m) = \langle x_m|K\rangle = x_m^K$, indicating now braket $\langle x_m|K\rangle$ is defined as a polynomial of Kth order, so that one can rewrite Equation (42) as

$$\mathcal{L}_K = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \langle x|x_m\rangle \langle x_m|K\rangle\,, dx_m. \tag{45}$$

Here, it should be noted that one can appeal to the well-known completeness relationship as commonly used in QM:

$$\int_{-\infty}^{\infty} |x_m\rangle \langle x_m|\, dx_m = \mathbb{I}, \tag{46}$$

that applies to integration Equation (45) as follows:

$$\mathcal{L}_K = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \langle x|\, |x_m\rangle \langle x_m|\, dx_m\, |K\rangle\,, = x_\ell \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) \frac{d}{dx} \langle x| \left[\int_{-\infty}^{\infty} |x_m\rangle \langle x_m|\, dx_m\right] |K\rangle\,. \tag{47}$$

In this way, operator can again be rewritten as

$$\mathcal{L}_K = x_\ell \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) \frac{d}{dx} \langle x|\, \mathbb{I}\, |K\rangle = \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) x_\ell \frac{d}{dx} \langle x|K\rangle\,. \tag{48}$$

With the definition of function $\langle x|K\rangle = \Phi_K(x)$, it allows to arrive to a form of operator as

$$\mathcal{L}_K = \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) x_\ell \frac{d}{dx} \Phi_K(x), \tag{49}$$

in conjunction with its partner, given by

$$\mathcal{L}_L = \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) x_\ell \frac{d}{dx} \Phi_L(x). \tag{50}$$

Subsequently, when we incorporate the Planck's constants in both Equations (49) and (50), we arrive at

$$\mathcal{L}_K = \left(-\frac{\delta_\ell^m}{\sqrt{2}\hbar}\right) x_\ell \frac{\hbar}{i} \frac{d}{dx} \Phi_K(x), \tag{51}$$

$$\mathcal{L}_L = \left(-\frac{\delta_\ell^m}{\sqrt{2}\hbar}\right) x_\ell \frac{\hbar}{i} \frac{d}{dx} \Phi_L(x). \tag{52}$$

It is easy to observe above that the momentum operator $\mathbf{p} = \frac{\hbar}{i}\frac{d}{dx}$ has been identified. With this, one can rewrite Equations (51) and (52) as follows:

$$\mathcal{L}_K = \left(-\frac{\delta_\ell^m}{\sqrt{2}\hbar}\right) x_\ell \mathbf{p} \Phi_K(x), \tag{53}$$

$$\mathcal{L}_L = \left(-\frac{\delta_\ell^m}{\sqrt{2}\hbar}\right) x_\ell \mathbf{p} \Phi_L(x). \tag{54}$$

Consider for example, Equation (53), so that both sides are multiplied by $\frac{\mathbf{p}}{M\sqrt{2}}$; in this manner, one obtains

$$\frac{\mathbf{p}}{M\sqrt{2}}\mathcal{L}_K = \left(-\frac{\delta_\ell^m}{\hbar}\right) x_\ell \frac{\mathbf{p}^2}{2M}\Phi_K(x) = \left(-\frac{x_m}{\hbar}\right)\frac{\mathbf{p}^2}{2M}\Phi_K(x), \tag{55}$$

the right side exhibits the form of kinetic energy. This fact can be harnessed in order to address quantum properties that presumably would emerge Witt operators in a direct manner.

3.2. Quantization of Harmonic Oscillator

One can write again Equation (55) under the argumentation of eigenvalue equations such as

$$\frac{\mathbf{p}^2}{2M}\Phi_K(x) = -\frac{\hbar \mathbf{p}}{M\sqrt{2}x_m}\mathcal{L}_K. \tag{56}$$

Thus, one can claim that Equation (56) is a kind of Schrödinger equation established for a harmonic oscillator whose Hamiltonian is operating onto $\Phi_K(x)$. Therefore, by incorporating the potential energy in the left side above, one obtains

$$\mathbf{H}\Phi_K(x) = \left[\frac{\mathbf{p}^2}{2M} + \frac{M\omega^2 x_m^2}{2}\right]\Phi_K(x). \tag{57}$$

From Equation (56), one arrives at

$$\mathbf{H}\Phi_K(x) = -\frac{\hbar \mathbf{p}}{M\sqrt{2}x_m}\mathcal{L}_K. \tag{58}$$

It can be noted that on the right side of Equation (58), a frequency emerges in the sense that

$$\frac{\mathbf{p}}{M\sqrt{2}x_m} \equiv \omega, \tag{59}$$

by encompassing the balance of physical units on both sides of Equation (58). Thus, one obtains below that

$$\mathbf{H}\Phi_K(x) = -\hbar\omega\mathcal{L}_K. \tag{60}$$

Equation (60) offers an interesting scenario to speculate about the physical meaning of that equation. Thus, while it is assumed, for instance, (It is allowed in virtue of the polynomial character of Equations (16) and (17).) $\Phi_K(x) = x^{-(\frac{K}{2}+1)}$ and guided by main definitions Equations (16) and (17) (having a derivative), then one can also assume the relationship:

$$\mathcal{L}_K = \frac{d}{dx}\Phi_K(x), \tag{61}$$

fact that allows us to arrive at quantized energies of harmonic oscillator given by

$$\mathbf{H}\Phi_K(x) = \hbar\omega(\frac{K}{2} + 1)\Phi_K(x). \tag{62}$$

3.3. Derivation of Schrödinger Equation

Despite the fact that to some extent the procedures above have been done inside a purely arbitrary scheme, some noteworthy aspects associated with tangible physics, such as the harmonic oscillator (e.g., Equation (62) presented above), might be consistently derived. For example, consider the case that states are canceling each other:

$$\mathcal{L}_K + \Phi_K(x) = 0, \tag{63}$$

implying the triviality $\mathcal{L}_K \Rightarrow -\Phi_K(x)$, then from Equation (58) one arrives at

$$\mathbf{H}\Phi_K(x) = \frac{\hbar\mathbf{p}}{M\sqrt{2}x_m}\Phi_K(x). \tag{64}$$

In this way one should be guaranteed that $\frac{\hbar\mathbf{p}}{M\sqrt{2}x_m}$ must be consistent with units of energy. It can be so only if

$$\frac{\mathbf{p}}{Mx_m\sqrt{2}} = \frac{1}{t} \equiv \omega, \tag{65}$$

having units of frequency. In this way, one can multiply by the left side of both terms of Equation (59) by $Mx_m^2\omega$, yielding

$$\sqrt{\frac{1}{8}}\omega x_m\mathbf{p} = \frac{1}{2}Mx_m^2\omega^2 = \mathcal{E}_{\text{HAR}}, \tag{66}$$

the classical energy of the harmonic oscillator again, and units are checked on both sides. In addition, from Equation (53), for example, one has again that

$$\mathcal{L}_K = \left(-\frac{\delta_\ell^m}{\sqrt{2}\hbar}\right)x_\ell\mathbf{p}\Phi_K(x) = \left(-\frac{1}{\sqrt{2}\hbar}\right)x_m\mathbf{p}\Phi_K(x). \tag{67}$$

With the usage of Equations (65) and (66) one arrives at

$$\mathcal{L}_K = \left(-\frac{2}{\hbar\omega}\right)\sqrt{\frac{1}{8}}\omega x_m\mathbf{p}\Phi_K(x). \tag{68}$$

Now one can see that now operator is proportional to the energy of the harmonic oscillator, so that one can write down that

$$\mathcal{L}_K = \left(-\frac{2}{\hbar\omega}\right)\mathcal{E}_{\text{HAR}}\Phi_K(x). \tag{69}$$

Based on the approximation Equation (65),

$$\omega = \frac{1}{\Delta t}. \tag{70}$$

Additionally, one can extend the meaning of Equation (63) so that one has two roots:

$$(\Phi_K(x) - \mathcal{L}_K)\left(\Phi_K(x) - \frac{i}{2}\Psi_K(x)\right) = 0, \tag{71}$$

by allowing to arrive in the reformulation of field $\Phi_K(x)$ as

$$\Phi_K(x) \Rightarrow \frac{i}{2}\Psi_K(x), \tag{72}$$

then, operator Equation (69) can be written as

$$\mathcal{L}_K = \left(-\frac{i\mathcal{E}_{\mathrm{HAR}}\Delta t}{\hbar}\right)\Psi_K(x). \tag{73}$$

From above, if Δt passes to the left side, then one has the below:

$$\frac{\mathcal{L}_K}{\Delta t} = \left(-\frac{i\mathcal{E}_{\mathrm{HAR}}}{\hbar}\right)\Psi_K(x). \tag{74}$$

In this way, the mathematical structure of the above Equation (74) is suggesting to accept that term in brackets on the right side is the outcome of the time-derivative of the well-known QM evolution operator given by $\mathbf{U}(t-t_0) = \mathrm{Exp}\left[-i\frac{i\mathcal{E}_{\mathrm{HAR}}(t-t_0)}{\hbar}\right]$. Thus from Equation (74) follows that

$$\frac{\mathcal{L}_K}{\Delta t} \equiv \left\{\frac{\partial}{\partial t}\mathbf{U}(t-t_0)\right\}\Psi_K(x) = \left\{\frac{\partial}{\partial t}\mathrm{Exp}\left[-i\frac{i\mathcal{E}_{\mathrm{HAR}}(t-t_0)}{\hbar}\right]\right\}\Psi_K(x) = \left(-\frac{i\mathcal{E}_{\mathrm{HAR}}}{\hbar}\right)\Psi_K(x), \tag{75}$$

by which it was assumed $\Delta = t - t_0$. By taking into account the first and last terms of Equation (75) (from left to right), it can also be written in a more familiar manner as

$$i\hbar\frac{1}{\Delta t}\mathcal{L}_K = \mathcal{E}_{\mathrm{HAR}}\Psi_K(x). \tag{76}$$

Since the kinetic part is missing, then it might be associated with the approximation under the assumption (That can also be understood as the inequality: $x_m \gg \frac{\mathbf{p}}{M\omega}$ in the classical limit applying systems with a large amplitude of oscillation) of $\frac{\mathcal{E}_{\mathrm{KI}}}{\mathcal{E}_{\mathrm{HAR}}} \approx 0$

$$\mathcal{E}_{\mathrm{HAR}} \approx \mathcal{E}_{\mathrm{HAR}}\left(1 + \frac{\mathcal{E}_{\mathrm{KI}}}{\mathcal{E}_{\mathrm{HAR}}}\right) \Rightarrow \mathcal{E}_{\mathrm{HAR}} + \mathcal{E}_{\mathrm{KI}}. \tag{77}$$

For infinitesimal times (commonly expected in quantum systems), one can approximate $\frac{1}{\Delta} \rightarrow \frac{\partial}{\partial t}$ so that one can employ the time derivative on the left side of Equation (57); therefore one arrives at

$$i\hbar\frac{\partial}{\partial t}\mathcal{L}_K = (\mathcal{E}_{\mathrm{HAR}} + \mathcal{E}_{\mathrm{KI}})\Psi_K(x) = \mathcal{E}\Psi_K(x). \tag{78}$$

Also, one can see that if operator $\mathcal{L}_K$ coincides with field $\Psi_K(x)$ (or $\mathcal{L}_K = \Psi_K(x)$), then, because of this, one can write down the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi_K(x) = \mathcal{E}\Psi_K(x). \tag{79}$$

It should be noted that under the equality $\mathcal{L}_K = \Psi_K(x)$ and Equation (72), one also arrives at

$$(\mathcal{L}_K - \Psi_k)(\mathcal{L}_K - \frac{i}{2}\Psi_k) = 0, \tag{80}$$

leading to quadratic equation:

$$\mathcal{L}_K^2 - \left(\frac{i}{2} + 1\right)\Psi_K\mathcal{L}_K - \frac{i}{2}\Psi_K^2 = 0. \tag{81}$$

by which holds the commutation $[\mathcal{L}_K, \Psi_K] = 0$, clearly not necessarily to be fulfilled.

4. Main Result

As shown above, both the evolution operator and the Schrödinger equation might be revealing some aspects that are not evident from Witt algebra as well as from the operators initially defined by Equations (13) and (14). In this manner, one can argue that these central operators and involved algebra would be more intrinsically associated with the aspects of quantum mechanics than being simple abstract operators belonging to unphysical Witt algebra. Under this view, Equations (49) and (50) can be harnessed to be used directly, with $\mathbf{p} = \frac{\hbar}{i}\frac{d}{dx}$, in commutators such as $[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_K \otimes \mathcal{L}_L - \mathcal{L}_L \otimes \mathcal{L}_K =$

$$[\mathcal{L}_K, \mathcal{L}_L] = \left[\frac{-\delta_\ell^m x_\ell}{i\sqrt{2}}\frac{d}{dx}\Phi_K(x)\right] \otimes \left[\frac{-\delta_\ell^m x_\ell}{i\sqrt{2}}\frac{d}{dx}\Phi_L(x)\right] - \left[\frac{-\delta_\ell^m x_\ell}{i\sqrt{2}}\frac{d}{dx}\Phi_L(x)\right]\left[\frac{-\delta_\ell^m x_\ell}{i\sqrt{2}}\frac{d}{dx}\Phi_K(x)\right] \quad (82)$$

$$= \left[\frac{\delta_\ell^m x_\ell}{i\sqrt{2}}\frac{\delta_\ell^m x_\ell}{i\sqrt{2}}\right] \otimes \left[\frac{d}{dx}\Phi_K(x)\frac{d}{dx}\Phi_L(x)\right] - \left[\frac{\delta_\ell^m x_\ell}{i\sqrt{2}}\frac{\delta_\ell^m x_\ell}{i\sqrt{2}}\right] \otimes \left[\frac{d}{dx}\Phi_L(x)\frac{d}{dx}\Phi_K(x)\right] \quad (83)$$

$$= \left[\frac{x_m^2}{2}\right] \otimes \left[\frac{d}{dx}\Phi_L(x)\frac{d}{dx}\Phi_K(x)\right] - \left[\frac{x_m^2}{2}\right] \otimes \left[\frac{d}{dx}\Phi_K(x)\frac{d}{dx}\Phi_L(x)\right] \quad (84)$$

$$= \left[\frac{x_m^2}{2}\right]\left(\frac{d}{dx}\Phi_L(x)\frac{d}{dx}\Phi_K(x) - \frac{d}{dx}\Phi_K(x)\frac{d}{dx}\Phi_L(x)\right). \quad (85)$$

Clearly while, not any rule of commutation between $\Phi_K(x)$ and $\Phi_L(x)$ has been explicitly established, then it is assumed that $[\Phi_K(x), \Phi_L(x)] \neq 0$.

On the other side, one can employ the momentum-based definitions as seen in Equation (67) for both cases: $\mathcal{L}_K = \frac{-\delta_\ell^m x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x)$ and $\mathcal{L}_L = \frac{-\delta_\ell^q x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x)$) inside Equation (82) so that one obtains

$$[\mathcal{L}_K, \mathcal{L}_L] = \frac{-\delta_\ell^m x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x)\frac{-\delta_\ell^q x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x) - \frac{-\delta_\ell^q x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x)\frac{-\delta_\ell^m x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x) \quad (86)$$

$$= \frac{x_m}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x)\frac{x_q}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x) - \frac{x_q}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x)\frac{x_m}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x) \quad (87)$$

$$= \frac{1}{2\hbar^2}\left[x_m\mathbf{p}x_q\mathbf{p}\Phi_K(x)\Phi_L(x) - x_q\mathbf{p}x_m\mathbf{p}\Phi_L(x)\Phi_K(x)\right]. \quad (88)$$

It should be remarked that the change of Equation (86) to Equation (87) has demanded to accept the validity of these commutators:

$$\left[\Phi_K, x_q\right] = 0, \quad (89)$$

$$[\Phi_L, x_m] = 0. \quad (90)$$

From above, one can argue that Equation (88) might also be written as (By which it was added and subtracted the term $x_q\mathbf{p}x_m\mathbf{p}\Phi_K(x)\Phi_L(x)$.):

$$[\mathcal{L}_K, \mathcal{L}_L] = \frac{1}{2\hbar^2}[x_m\mathbf{p}x_q\mathbf{p}\Phi_K(x)\Phi_L(x) - x_q\mathbf{p}x_m\mathbf{p}\Phi_K(x)\Phi_L(x)$$
$$+x_q\mathbf{p}x_m\mathbf{p}\Phi_K(x)\Phi_L(x) - x_q\mathbf{p}x_m\mathbf{p}\Phi_L(x)\Phi_K(x)] \quad (91)$$

$$= \frac{1}{2\hbar^2}\left[[x_m\mathbf{p}, x_q\mathbf{p}]\Phi_K(x)\Phi_L(x) + x_q\mathbf{p}x_m\mathbf{p}[\Phi_K(x), \Phi_L(x)]\right]. \quad (92)$$

Clearly, from $[x_m\mathbf{p}, x_q\mathbf{p}]$ in Equation (92), it is needed the usage of identity $[AB, CD] = A[B, C]D + [A, C]BD + CA[B, D] + C[A.D]B$. By applying this in a straightforward manner, one obtains the following:

$$[\mathcal{L}_K, \mathcal{L}_L] = \frac{1}{2\hbar^2}\left[(x_q[x_m, \mathbf{p}]\mathbf{p} - x_m[x_q, \mathbf{p}]\mathbf{p})\Phi_K(x)\Phi_L(x) + x_q\mathbf{p}x_m\mathbf{p}\{\Phi_K(x), \Phi_L(x)\}\right], \quad (93)$$

demonstrating the validity of the hypothesis formulated in Equation (7). Equation (93) becomes the main result of this paper. The general formulation of CCR given by $[x_I, p_J] = i\hbar\delta_{I,J}$ stops us from going through commutators in Equation (93), essentially because neither $x_{m,q}$ nor $\mathbf{p}$ have been explicitly specified. In this way, various scenarios might emerge for the choice of a concrete component of $\mathbf{p}$, as well as for x_m and x_q. The particular case when $m = q$ reduces Equation (93) to

$$[\mathcal{L}_K, \mathcal{L}_L] = \frac{1}{2}\left(\frac{x_m\mathbf{p}}{\hbar}\right)^2[\Phi_K(x), \Phi_L(x)]. \quad (94)$$

Because commutator $[\mathcal{L}_K, \mathcal{L}_L]$ has now acquired a certain physical meaning because the position and momentum operators, then Equation (94) can also be rewritten as a function of observables measurements such as

$$[\mathcal{L}_K, \mathcal{L}_L] = \frac{1}{2}\left(\frac{\Delta x_m\Delta\mathbf{p}}{\hbar}\right)^2[\Phi_K(x), \Phi_L(x)]. \quad (95)$$

Inspired at the uncertainty principle, one can impose the following restriction:

$$\Delta x_m\Delta\mathbf{p} \geq \sqrt{2}\hbar, \quad (96)$$

yielding an alternative redefinition of Witt algebra through new operators $\Phi_K(x)$ and $\Phi_L(x)$ as

$$[\mathcal{L}_K, \mathcal{L}_L] = [\Phi_K(x), \Phi_L(x)], \quad (97)$$

entering into a total contradiction with Equation (63). This suggests keeping Equation (96) as $\Delta x_m\Delta\mathbf{p} \geq \hbar$, which allows writing Equation (95) down as

$$[\mathcal{L}_K, \mathcal{L}_L] = \left[-i\frac{\Phi_K(x)}{\sqrt{2}}, i\frac{\Phi_L(x)}{\sqrt{2}}\right], \quad (98)$$

by which emerges the complex version of Equation (63) in the sense of

$$\mathcal{L}_K + i\frac{\Phi_K(x)}{\sqrt{2}} = 0, \quad (99)$$

$$\mathcal{L}_L - i\frac{\Phi_L(x)}{\sqrt{2}} = 0. \quad (100)$$

5. Discussion

Along the paper, it was employed the so-called Witt operators, essentially Equations (16) and (17), constructed in an artificial manner to demonstrate that they fulfill the Witt algebra or Virasoro centerless algebra. Although the whole procedure has been merely operative, it has planted the idea that these formulations have implications in a tangible QM. It is remarked on the role of integrals whose form has been relevant to deriving the QM evolution operator and the Schrödinger equation. The polynomial term Equation (44) has been crucial to claiming quantization of harmonic oscillators, even though no annihilation or creation operators have been considered. Because of this, one can wonder if all structures that fulfill Witt algebra can be seriously considered as unexplored territories and ends in a QM arena. Or results can be seen as fortuitous derivations that can also be ambiguities of a single formalism? The only fact that commutators of Witt operators yield the CCR is proof that Virasoro-like schemes might also be open windows to a general usage of string theories in concrete QM applications. As seen in [4], the Virasoro algebra has served to build a model of N-th order squeezing based on Virasoro oscillators through the definition of oscillators as functions of annihilation and creation operators as an indirect way to link the algebra to momentum and position operators. In the present paper, instead of opting for intermediate derivations involving extra operators, it was assumed the integral form as given at Equations (16) and (17), whose structure has been harnessed to derive QM energies (quantization of harmonic oscillator). For example, in [44], ideas about string theory as dissipative QM based at symmetries have yielded interesting similarities (both might be the same theory), far from common considerations as done in field theories. The proposal of this paper clearly is going in a novel direction with robust redefinitions of Virasoro oscillators that allow it to pass over the QM area without appealing to quantum symmetries or incorporating extra structures based on QM operators. On the other side, the spirit of this paper is to some extent inclined to view [45] in the sense that association of QM and general relativity can be only consistent through an inclusion of all spectrum of topological string states. The fact that Equations (16) and (17) have turned out to be fulfilling Witt algebra and are also a robust scheme for the derivation of QM relations and equations is, to some extent, comparable to the work of Bars and Rychkov [46] since they have taken advantage of Moyal string field theory formalism to derive commutators of operators' position and momentum. Indeed, Equation (57) of this paper might be aligned with the idea that the Moyal product yields products of string fields that finally can be recognized as ordinary QM Hamiltonians. Furthermore, Equation (44), which introduces the polynomial profile to Witt operators, would be in agreement with recent work [47], where Lie algebra based on polynomial algebra structures offers a new path to be applied in QM and related areas.

6. Conclusions

In this paper, it was presented a mathematical methodology that allows demonstrating that schemes based on Witt algebra are consistently linked to quantum mechanics canonical commutation relation (CCR), despite the fact that not any assumption associated with quantum variables or quantum mechanics formalism was deeply considered. As seen in Equations (16) and (17), operators have been defined built on the basis of the Krönecker delta and Dirac delta function, as well as the operations of derivative and integration. After closed-form operations, as seen in previous sections, these operators have turned out to fulfill the Witt algebra. Although some abstract procedures have been applied, it was seen that all that have acquired sense when quantum mechanics definitions were shortly applied. Particularly, the momentum operator emerged in a spontaneous manner. It has played a noteworthy role, as noted in the last section of the paper. In this manner, based rigorously on the presented formalism (by which it might be extended and improved), Witt algebra has turned out to be proportional to well-known CCR. Certainly, more formalism and operations would have to be added in order to claim a robust proportionality between the DeWitt (Virasoro central charge [48]) algebra and the quantum mechanics canonical commutation relation.

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