# Studies in Statistical Mechanics and Supersymmetric Lattice Models


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# Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted, or, is being concurrently submitted, for any degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

Chapter 3 is based on joint work with Rémy Poudevigne–Auboiron, and is published as

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> Ulrik Thinggaard Hansen, Frederik Ravn Klausen, and Peter Wildemann. "Non-uniqueness of phase transitions for graphical representations of the Ising model on tree-like graphs". In: *arXiv:2410.22061* (2024).

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# Abstract

This work concerns an array of probabilistic models in the context of lattice systems and statistical field theory. Firstly, motivated by predictions about the Anderson transition, we study two distinct but related models on regular tree graphs: the vertex-reinforced jump process (VRJP), a random walk that prefers to jump to previously visited sites, and the $\mathbb{H}^{2|2}$-model, a lattice spin system whose spins take values in a supersymmetric extension of the hyperbolic plane. Both models undergo a phase transition, and our work provides detailed information about the supercritical phase up to the critical point.

Moreover, we consider the rigorous construction of the Schwarzian field theory, a measure on the quotient $\mathrm{Diff}(S^1)/\mathrm{PSL}(2,\mathbb{R})$ of circle diffeomorphisms, which has gained popularity in recent theoretical physics literature. Its partition function is calculated by the rigorous implementation of an argument by Belokurov and Shavgulidze [1]. This method exploits a regularisation of the measure, motivated by the theory of Virasoro coadjoint orbits. We also provide motivation for the physical origins of the Schwarzian field theory and offer background on the theory of coadjoint orbits.

Furthermore, we consider the graphical representations of the Ising model, including the random cluster, loop $\mathrm{O}(1)$, and random current model. Considering these models as percolation-type random graph models in their own right, we are interested in their monotonicity behaviour. We construct some tree-like graphs for which the loop $\mathrm{O}(1)$ and random current model exhibit a non-unique phase transition. As a consequence there exist infinite graphs $\mathbb{G} \subseteq \mathbb{G}'$ such that the uniform even subgraph of $\mathbb{G}'$ percolates and the uniform even subgraph of $\mathbb{G}$ does not. Moreover, we show that in general the percolation thresholds of the models do not agree.

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# Chapter 1

# Introduction

*"Die meisten Menschen wollen nicht eher schwimmen, als bis sie es können."*
*(engl. "Most men don't want to swim until they can.")*

Statistical mechanics is one of the great unifying frameworks of modern theoretical physics. Historically, it was developed to provide a probabilistic and microscopic explanation for thermodynamics and, more specifically, phase transitions. In an interesting twist of scientific history, it later became apparent that the very same mathematical principles are central to our understanding of quantum field theory. The richness of the field is a treasure trove for probabilists, and rigorous arguments are of particular importance to the subject; particularly since finding convincing heuristics can often be a challenge, even within the physics literature.

This thesis will explore a variety of models that may initially appear disjointed, yet embody, in essence or in spirit, the principles of statistical mechanics. It is remarkable how much depth can hide behind an inconspicuous $e^{-\beta H}$, and I hope the reader is reminded of some of that wonder while exploring this text.

This thesis is organised as follows: The majority of the content consists of the publications [TREE], [SCHW], and [UNIQ], each presented in its entirety within its dedicated chapter. In Chapter 2, we offer "executive summaries" of these works, providing additional perspectives that complement the introductions of the original publications. Chapters 4 and 6 present unpublished material and further context related to the publications [TREE] and [SCHW], respectively.

# Chapter 2

# Executive summaries

This chapter aims at providing short and opinionated overviews on the context and the results of the publications that are part of this thesis. These are meant to complement the introductions of the articles, which are included in the appropriate chapters.

## 2.1 $\mathbb{H}^{2|2}$-model and VRJP on trees [TREE]

This works concerns the behaviour of two distinct but related models on the tree: The vertex-reinforced jump process (VRJP), a random walk preferring to jump to previously visited sites, and the $\mathbb{H}^{2|2}$-model, a lattice spin system whose spins take values in a supersymmetric extension of the hyperbolic plane. Before discussing context and motivation, let's introduce the models:

> **Definition** (Vertex-reinforced jump process): The VRJP is a continuous-time jump process $(X_t)_{t\geq 0}$ on a locally finite graph $G=(V,E)$. At time $t$ it jumps from its current location $X_t = x$ to a neighbour $y \sim x$ at rate
>
> $$\beta[1+L_t^y] \;\text{ where } L_t^y := \int_0^t \mathrm{d}s\, 1_{X_s=y} \;\text{ is the time spent at } y \text{ so far,} \tag{2.1.1}$$
>
> and $\beta > 0$ is a fixed *inverse temperature* parameter.

The *hyperbolic superplane* $\mathbb{H}^{2|2}$ is can be considered as the set of coordinate vectors $\mathbf{u} = (z,x,y,\xi,\eta)$, such that $\mathbf{u}\cdot\mathbf{u} = -z^2+x^2+y^2-2\xi\eta = -1$. Here, $z,x,y$ are *even*/bosonic coordinates, while $\xi,\eta$ are *odd*/fermionic. More details about this will be provided in Chapter 3, but for now it is enough to think of $\mathbb{H}^{2|2}$ as a manifold with a Haar measure d$\mathbf{u}$ and a scalar product $\mathbf{u}_i\cdot\mathbf{u}_j = -z_iz_j + x_ix_j + y_iy_j + \eta_i\xi_j - \xi_i\eta_j$. In fact, to make proper sense of the $\mathbb{H}^{2|2}$-model, we'll need to introduce Grassmann integration, but at a purely formal level one can write the following:

> **"Definition"** ($\mathbb{H}^{2|2}$-model): Consider a finite graph $G = (V, E)$. Fix an *inverse temperature* $\beta > 0$ and a *magnetic field* $h > 0$. For a functional $F \in C^\infty((\mathbb{H}^{2|2})^V)$ over *spin configurations* $\underline{\mathbf{u}} = (\mathbf{u}_i)_{i\in V} \in (\mathbb{H}^{2|2})^V$ we define the expectation of $F$ under the $\mathbb{H}^{2|2}$-model as
>
> $$\big\langle F(\underline{\mathbf{u}})\big\rangle_{\beta,h} := \int_{(\mathbb{H}^{2|2})^V} \mathrm{d}\underline{\mathbf{u}}\, F(\underline{\mathbf{u}})\, e^{\sum_{ij\in E}\beta(\mathbf{u}_i\cdot\mathbf{u}_j+1)-\sum_{i\in V}h(z_i-1)}, \tag{2.1.2}$$
>
> with Haar measure $\mathrm{d}\underline{\mathbf{u}} = \prod_{i\in V}\mathrm{d}\mathbf{u}_i$.

In other words, the $\mathbb{H}^{2|2}$ defines a prescription for calculating correlation functions. For example, one is interested in the covariance of the $x$-coordinate at different lattice points, that is $\langle x_i x_j\rangle_{\beta,h}$. While the $\mathbb{H}^{2|2}$-model itself cannot be interpreted as a probability measure, certain correlation functions and marginals have a direct probabilistic interpretation. One remarkable such example is the BFS-Dynkin isomorphism between the VRJP and the $\mathbb{H}^{2|2}$-model [2, 3]: Suppose that under $\mathbb{E}_{\beta;i}$ the process $(X_t)_{t\geq 0}$ denotes a VRJP started from vertex $i \in V$ at inverse temperature $\beta > 0$. Then, for any $\beta, h > 0$ and $j \in V$

$$\boxed{\langle x_i x_j\rangle_{\beta,h} = \int_0^\infty \mathbb{E}_{\beta;i}[\mathbb{1}_{X_t=j}]\,e^{-ht}\,\mathrm{d}t} \tag{2.1.3}$$

That is, the two-point function of the $\mathbb{H}^{2|2}$-model describes the expected local time of a VRJP with exponential killing rate $h > 0$. This relationship goes much further: Introducing the horospherical *t-field* coordinate on $\mathbb{H}^{2|2}$ via $e^t = z + x$, one can interpret the marginal of the $\mathbb{H}^{2|2}$-model onto this coordinate as a proper probability measure (in the sense that any correlation functions of these observables can equivalently be calculated under the expectation value of an appropriate $t$-field probability measure). This horospherical marginal turns out to be related to be directly related to the local time of the VRJP: Think of the $\mathbb{H}^{2|2}$-model on a finite graph $(V, E)$ with magnetic field as living on the extended graph containing the *ghost vertex* $\mathfrak{g}$, which is connected to every other vertex with an edge of weight $h > 0$. Consider a VRJP on this weighted graph, started at the ghost $\mathfrak{g}$. Then it holds that

$$\boxed{(T_i)_i := \Big(\lim_{t\to\infty}\log\frac{L_t^i}{L_t^{\mathfrak{g}}}\Big) \overset{\mathrm{d}}{=} (t_i)_i = \big(\log(z_i + x_i)\big)_i} \tag{2.1.4}$$

The last two equalities are to be understood "under the expectation value". However they highlight a remarkable fact: The asymptotic local time field of the VRJP (on a finite graph) is a random field, distributed as a marginal of the $\mathbb{H}^{2|2}$-model! The above

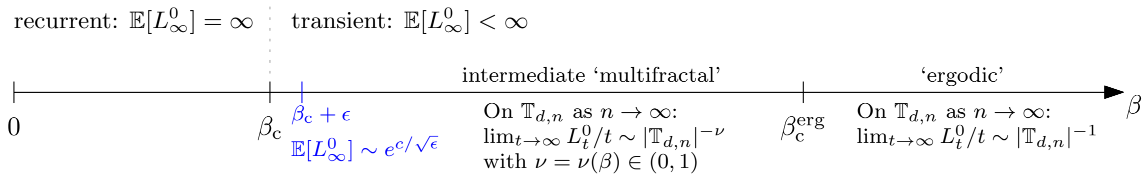


Figure 2.1: Sketch of the phase diagram for the VRJP on $\mathbb{T}_d$ with $d \geq 2$. The recurrence/transience transition at $\beta_c$ is phrased in terms of $\mathbb{E}[L^0_\infty]$, *i.e.* the expected total time the walk (on the infinite rooted $(d+1)$-regular tree $\mathbb{T}_d$) spends at the starting vertex. We obtain precise asymptotics for $\mathbb{E}[L^0_\infty]$ as $\beta \searrow \beta_c$. We show that there is an additional transition point $\beta_c^{\text{erg}} > \beta_c$. It is phrased in terms of the volume-scaling of the fraction of total time, $\lim_{t\to\infty} L^0_t/t$, the VRJP on the *finite* tree $\mathbb{T}_{d,n}$ spends at the origin. Here, the symbol "$\sim$" is understood loosely.

**Results on the regular tree.** In the publication [TREE] we focused on the models on the rooted $(d+1)$-regular tree $\mathbb{T}_d$ with $d \geq 2$. By Basdevant and Singh [4] both the $\mathbb{H}^{2|2}$-model and the VRJP exhibit a phase transition at a critical inverse temperature $\beta_c = \beta_c(d) > 0$. For the $\mathbb{H}^{2|2}$-model the transition is between a disordered high-temperature phase ($\beta < \beta_c$) and a symmetry-broken low-temperature phase ($\beta > \beta_c$) exhibiting long-range order. For the VRJP the transition is between a recurrent phase due to strong reinforcement effects and a transient phase due to low reinforcement effects. We analysed the behaviour of the models in the supercritical phase $\beta > \beta_c$: We show that their order parameter has an essential singularity as one approaches the critical point, in contrast to algebraic divergences typically expected for statistical mechanics models. Moreover, we identify a previously unexpected multifractal intermediate regime in the supercritical phase. We refer to Figure 2.1 for an illustration of the results for the VRJP. In the following we provide condensed versions of the main results, as discussed in more detail in Chapter 3.

> **Theorem** (Near-critical behaviour): Consider the VRJP and the $\mathbb{H}^{2|2}$-model on the infinite rooted regular tree $\mathbb{T}_d$ with $d \geq 2$ and let $\beta_c = \beta_c(d)$. There exist constants $c, C > 0$, such that for $\epsilon > 0$ sufficiently small we have
>
> $$e^{c/\sqrt{\epsilon}} \leq \mathbb{E}_{\beta_c+\epsilon}[L^0_\infty] = \langle x_0^2 \rangle^+_{\beta_c+\epsilon} \leq e^{C/\sqrt{\epsilon}}. \tag{2.1.5}$$

While the phase transition and above near-critical behaviour is seen on infinite trees $\mathbb{T}_d$, the *intermediate phase* is seen in the scaling behaviour over finite trees $\mathbb{T}_{d,n}$ of depth $n \to \infty$.

> **Theorem** (Intermediate Phase): For the VRJP on *finite* trees $\mathbb{T}_{d,n}$ we have
>
> $$\lim_{t\to\infty} \frac{L^0_t}{t} = \left|\mathbb{T}_{d,n}\right|^{-\nu(\beta)+o(1)} \qquad \text{w.h.p. as } n \to \infty,$$

with $\beta \mapsto \nu(\beta) \in [0,1]$ continuous and non-decreasing. There exists $\beta_c^{\mathrm{erg}} > \beta_c$, such that $\nu(\beta) \in (0,1)$ if and only if $\beta \in (\beta_c, \beta_c^{\mathrm{erg}})$.

Moreover, the intermediate phase exhibits *multifractal scaling* for certain observables of the $\mathbb{H}^{2|2}$-model and the VRJP:

**Theorem** (Multifractality in the intermediate phase): For $\beta_c < \beta < \beta_c^{\mathrm{erg}}$ and $\eta \in (0,1)$

$$\mathbb{E}_{\beta,\mathbb{T}_{d,n}}[(\lim_{t\to\infty}\tfrac{L_t^0}{t})^{-\eta}] = \lim_{h\searrow 0} h^{-\eta}\langle z_0 |x_0|^{-\eta}\rangle_{\beta,h;\mathbb{T}_{d,n}} \sim \left|\mathbb{T}_{d,n}\right|^{\tau_\beta(\eta)+o(1)} \quad \text{as} \quad n \to \infty,$$

where $\eta \mapsto \tau_\beta(\eta)$ is an increasing and non-linear function.

The exponents $\nu(\beta)$ and $\tau_\beta(\eta)$ can me made quite explicit and we refer to Chapter 3 for the appropriate details.

## 2.2 Probabilistic definition of the Schwarzian field theory [SCHW]

In recent years, the *Schwarzian field theory* has received an increasing amount of attention in the theoretical physics literature. From a mathematical perspective, it describes a measure on the space of circle reparametrisations[1] $\varphi \in \mathrm{Diff}(S^1)$, defined by the formal density

$$d\widetilde{\mathcal{M}}_{\sigma^2}(\varphi) = \exp\left[-\frac{1}{2\sigma^2}\int_0^1 d\tau\left[\left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)^2 - 4\pi^2\varphi'^2(\tau)\right]\right]\prod_\tau \frac{d\varphi(\tau)}{\varphi'(\tau)}. \tag{2.2.1}$$

This measure has a "hidden" $\mathrm{PSL}(2,\mathbb{R})$-invariance induced by Möbius transformations in the variable $\tan(\pi\varphi(\tau))$ and the name *Schwarzian field theory* typically refers to the corresponding quotient measure $\mathcal{M}_{\sigma^2} := \widetilde{\mathcal{M}}_{\sigma^2}/\mathrm{PSL}(2,\mathbb{R})$. It is instructive to think of the Schwarzian action as a penalty function quantifying how "non-Möbius" a certain circle reparametrisation is. The measure (2.2.1) can be seen as a replacement for a Haar-measure on $\mathrm{Diff}(S^1)$: As a topological group $\mathrm{Diff}(S^1)$ is not locally compact and hence does not admit a left-invariant Radon measure. However, (2.2.1) turns out to be a *quasi-invariant* Radon measure, meaning that left-translations of the measure are absolutely continuous with respect to each other (with an explicit Radon–Nikodym derivative). Furthermore, this model is of interest as a non-trivial field theory, for

[1] We consider the circle $S^1 \cong \mathbb{R}/\mathbb{Z} \cong [0,1]/\sim$ as the unit interval with identified endpoints. Alternatively sometimes we write $\mathbb{T}$ instead of $S^1$ to highlight that we're working with a parametrisation by $[0,1]$, rather than $[0,2\pi]$. Implicitly $\mathrm{Diff}(S^1) := \mathrm{Diff}_+(S^1)$, always refers to *orientation-preserving* diffeomorphisms. We represent any $\varphi \in \mathrm{Diff}(S^1)$ as a strictly increasing function $\varphi\colon \mathbb{R} \to \mathbb{R}$, such that $\varphi(0) \in [0,1)$ and $\varphi(\tau+1) = \varphi(\tau)+1$.

which the path integral measure can be made rigorous: In [SCHW] we construct the quotient measure and calculate its total mass (i.e. the partition function). Losev further extended the methods to calculate a class of natural correlation functions, and to establish a large-deviation principle [5, 6].

This work was motivated by a plethora of physics literature, in which the Schwarzian field theory has emerged as a reference model for low-dimensional quantum gravity and holography. Much of the attention is due to its relevance to the low-energy behaviour of the Sachdev–Ye–Kitaev (SYK) model [7, 8] and its connections to two-dimensional gravity theories like Jackiw–Teitelboim (JT) gravity [9–12]. In both cases, the Schwarzian theory describes the low-energy behaviour of theories for which a one-dimensional reparametrisation symmetry $\mathrm{Diff}(S^1)$ is broken down to $\mathrm{PSL}(2,\mathbb{R})$, and one expects it to be universal in such scenarios. Furthermore, the Schwarzian action appears in the semi-classical limit of Liouville field theory [13, 14], and in the context of coadjoint orbits for the Virasoro group [15–17].

Essentially all of the above mentioned physics literature is far from being rigorously understood, and our work aims at bringing the Schwarzian theory and its many connections within reach of the probabilistic community. In [SCHW] we follow an approach by Belokurov and Shavgulidze [1, 18, 19] in order to relate the Schwarzian theory to a reweighted Brownian bridge measure.

**Schwarzian measures.** In the following we introduce a one-parameter family of *Schwarzian measures*. These are quasi-invariant measures on $\mathrm{Diff}^1(S^1)$, which can be motivated via so-called *coadjoint orbits* of the Virasoro group, i.e. the central extension of $\mathrm{Diff}(S^1)$. Define the *Schwarzian derivative* of a function $\varphi(\tau)$ is defined as

$$\mathcal{S}_\varphi(\tau) = \mathcal{S}(\varphi,\tau) = \left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)' - \frac{1}{2}\left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)^2 . \tag{2.2.2}$$

For a parameter $\alpha \in \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}$, the (unquotiented) *Schwarzian measure for parameter* $\alpha$ is formally defined as a measure over $\varphi \in \mathrm{Diff}(S^1)$ given by

$$\mathrm{d}\widetilde{\mathcal{N}}^{\alpha}_{\sigma^2} = \exp\left[\frac{1}{\sigma^2}\int_0^1 \mathrm{d}\tau\left[\mathcal{S}(\varphi,\tau) + 2\alpha^2\varphi'^2(\tau)\right]\right]\prod_\tau \frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)} . \tag{2.2.3}$$

For $\alpha = \pi$ we recover the measure in (2.2.1). For convenience one can of the measures as parametrised by $\alpha^2 \in \mathbb{R}$ and implicitly assume that $\alpha \in \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}$ denotes the appropriately chosen root. In [SCHW] these measures make a somewhat less prominent appearance as a "regularisation" of the Schwarzian field theory measure ($\alpha = \pi$). In fact, for $\alpha < \pi$ these measures are finite and the Schwarzian field theory is constructed by controlling the limit $\alpha \nearrow \pi$.

In the following, we provide a rigorous definition of the measure by relating it to reweighted Brownian bridges. First, note that in terms of the variables $\xi(\tau) = \log \varphi'(\tau)$, the term $(\varphi''(\tau)/\varphi'(\tau))^2 = \xi'^2(\tau)$ and the "reference measure" $\prod_\tau \frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)} = \prod_\tau \mathrm{d}\xi(\tau)$ give rise to a formal Wiener measure with variance $\sigma^2$. This motivates the following definition: For a Brownian bridge $(\xi_\tau)_{\tau\in S^1} \in C(S^1)$ with variance $\sigma^2 > 0$, started at $\xi_0 = 0$, we define the diffeomorphism

$$\varphi_{\xi,0}(\tau) \coloneqq \frac{\int_0^\tau e^{\xi_s}\,\mathrm{d}s}{\int_0^1 e^{\xi_s}\,\mathrm{d}s} \in \mathrm{Diff}^1(S^1). \tag{2.2.4}$$

This prescription breaks the translation/rotation-invariance that one would expect from (2.2.3) (in fact $\varphi_{\xi,0}(0) = 0$). Hence, let $\Theta \sim \mathrm{Unif}(S^1)$ denote a uniformly random angle and define $\varphi_{\xi,\Theta} \coloneqq \varphi_{\xi,0} + \Theta$. For a functional $F\colon \mathrm{Diff}(S^1) \to \mathbb{R}$ we *define* the unnormalised expectation with respect to the Schwarzian measure (at variance $\sigma^2$ and parameter $\alpha$) as

$$[F(\varphi)]_{\alpha,\sigma^2} \coloneqq \int F(\varphi)\mathrm{d}\widetilde{\mathcal{N}}^\alpha_{\sigma^2}(\varphi) \coloneqq \sqrt{2\pi\sigma^2}\,\mathbb{E}_{\sigma^2}\Big[F(\varphi_{\xi,\Theta})\exp\Big[\frac{2\alpha^2}{\sigma^2}\int \varphi'^2_{\xi,\Theta}(\tau)\mathrm{d}\tau\Big]\Big], \tag{2.2.5}$$

where $\xi$ is a Brownian bridge with variance $\sigma^2$ under $\mathbb{E}_{\sigma^2}$. The factor $\sqrt{2\pi\sigma^2}$ accounts for the normalisation of the the formal Wiener integral.

We also define the *$(U(1)$-$)$quotiented Schwarzian measures* $\mathcal{N}^\alpha_{\sigma^2}$ via

$$\int F(\varphi)\,\mathrm{d}\mathcal{N}^\alpha_{\sigma^2}(\varphi) \coloneqq \sqrt{2\pi\sigma^2}\,\mathbb{E}_{\sigma^2}\Big[F(\varphi_{\xi,0})\exp\Big[\frac{2\alpha^2}{\sigma^2}\int \varphi'^2_{\xi,0}(\tau)\mathrm{d}\tau\Big]\Big]. \tag{2.2.6}$$

This defines an unnormalised measure on $\mathrm{Diff}(S^1)/U(1) \cong \{\varphi \in \mathrm{Diff}(S^1)\colon \varphi(0) = 0\}$. A major result in [SCHW] is the calculation of the partition function of these measures:

> **Theorem:** For $\alpha^2 < \pi^2$ the measure $\mathcal{N}^\alpha_{\sigma^2}$ is a finite Radon measure on $\mathrm{Diff}^1(S^1)/U(1)$ with total mass
>
> $$\mathcal{N}^\alpha_{\sigma^2}\big(\mathrm{Diff}^1(S^1)/U(1)\big) = \frac{\alpha}{\sin\alpha}\frac{e^{2\alpha^2/\sigma^2}}{\sqrt{2\pi\sigma^2}} \tag{2.2.7}$$

In the case of the (unquotiented) Schwarzian field theory, $\widetilde{\mathcal{M}}_{\sigma^2} \coloneqq \widetilde{\mathcal{N}}^\pi_{\sigma^2}$, the total mass is infinite as made evident by the divergence in (2.2.6) for $\alpha \nearrow \pi$. In fact, this divergence is due to the mentioned underlying $\mathrm{PSL}(2,\mathbb{R})$-invariance of the measure (more on this below). The quotient measure $\mathcal{M}_{\sigma^2} = \widetilde{\mathcal{M}}_{\sigma^2}/\mathrm{PSL}(2,\mathbb{R})$ turns out to be finite with an explicit partition function

**Theorem:** The Schwarzian measure $\mathcal{M}_{\sigma^2}$ is a finite Radon measure on $\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})$ with total mass

$$\mathcal{M}_{\sigma^2}\big(\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})\big) = \left(\frac{2\pi}{\sigma^2}\right)^{3/2} e^{2\pi^2/\sigma^2} \tag{2.2.8}$$

To prove above results we use a change-of-variables formula for the Schwarzian measures. A special case of the latter describes the quasi-invariance of the measures, which we describe first.

**Quasi-invariance of the Schwarzian measures.** After defining the formal measure (2.2.3), we should make sure that we are indeed working with the "correct" measure. One way to see this is by studying the transformation properties of the measure under left-composition with some diffeomorphism. For this, we recall some transformation properties of the Schwarzian derivative: Firstly, it satisfies the composition rule

$$\mathcal{S}(\psi\circ\varphi,\tau) = \mathcal{S}(\varphi,\tau) + \varphi'^2(\tau)\,\mathcal{S}(\psi,\varphi(\tau)). \tag{2.2.9}$$

Secondly, the Schwarzian derivative of any Möbius transformation vanishes:

$$\mathcal{S}\left(\frac{a\tau+b}{c\tau+d},\tau\right) = 0 \qquad \text{for any } \pm\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R}). \tag{2.2.10}$$

As a consequence the Schwarzian derivative is invariant under Möbius transformations, $\mathcal{S}(f(\tau),\tau) = \mathcal{S}\left(\frac{af(\tau)+b}{cf(\tau)+d},\tau\right)$. Furthermore, one can check that $\mathcal{S}(\tan(\alpha\tau),\tau) = 2\alpha^2$, hence the exponential in the density of the Schwarzian measures (2.2.3) can be rewritten using

$$\mathcal{S}(\tan(\alpha\varphi(\tau)),\tau) = \mathcal{S}(\varphi,\tau) + 2\alpha^2\varphi'^2(\tau). \tag{2.2.11}$$

It is sometimes convenient to write $\mathcal{S}(\frac{1}{\alpha}\tan(\alpha\varphi)) = \mathcal{S}(\tan(\alpha\varphi))$, as this emphasises the continuity in the parameter $\alpha^2 \in \mathbb{R}$. Here, we understand $\tau \mapsto \frac{1}{\alpha}\tan(\alpha\tau) = \frac{1}{\alpha}\frac{\sin(\alpha\tau)}{\cos(\alpha\tau)} \in \mathbb{R}P^1 \cong \mathbb{R}\cup\{\infty\}$ as a smooth map into the real projective line. The Schwarzian is in fact well-defined for any such map, despite potential singularities.

We can now consider how the exponential density in (2.2.3) changes under left-composition $\varphi \mapsto \psi\circ\varphi$ with some fixed diffeomorphism $\psi \in \mathrm{Diff}(S^1)$:

$$\mathcal{S}(\tan(\alpha[\psi\circ\varphi]),\tau) = \mathcal{S}(\tan(\alpha\varphi),\tau) + \left[\mathcal{S}(\tan(\alpha\psi),\varphi(\tau)) - 2\alpha^2\right]\varphi'^2(\tau). \tag{2.2.12}$$

The formal reference measure $\prod_\tau \frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)} = \prod_\tau \frac{\mathrm{d}[\psi\circ\varphi](\tau)}{[\psi\circ\varphi]'(\tau)}$ is invariant under left-translation.

**Theorem:** For $\alpha^2 \in \mathbb{R}$, $\sigma^2 > 0$ and any fixed $\psi \in \mathrm{Diff}^3(S^1)$ define $\mathrm{d}\psi^*\widetilde{\mathcal{N}}^{\alpha}_{\sigma^2}(\varphi) = \mathrm{d}\widetilde{\mathcal{N}}^{\alpha}_{\sigma^2}(\psi\tilde{\varphi})$. Then $\psi^*\widetilde{\mathcal{N}}^{\alpha}_{\sigma^2}$ is absolutely continuous with respect to $\widetilde{\mathcal{N}}^{\alpha}_{\sigma^2}$ with Radon–Nikodym derivative given by

$$\frac{\mathrm{d}\psi^*\widetilde{\mathcal{N}}^{\alpha}_{\sigma^2}}{\mathrm{d}\widetilde{\mathcal{N}}^{\alpha}_{\sigma^2}}(\varphi) = \exp\left[\frac{1}{\sigma^2}\int_0^1 \mathrm{d}\tau \left[\mathcal{S}(\tan(\alpha\psi), \varphi(\tau)) - 2\alpha^2\right]\varphi'^2(\tau)\right]. \tag{2.2.13}$$

This expression allows us to study the invariance properties of the measures $\widetilde{\mathcal{N}}^{\alpha}_{\sigma^2}$. In fact it is invariant under the subgroup of diffeomorphisms $\psi \in \mathrm{Diff}(S^1)$, such that $\mathcal{S}(\tan(\alpha\psi)) = 2\alpha^2$, or equivalently

$$\tfrac{1}{\alpha}\tan(\alpha\psi(\tau)) = \frac{a\frac{1}{\alpha}\tan(\alpha\tau)+b}{c\frac{1}{\alpha}\tan(\alpha\tau)+d} \qquad \text{for some } \pm\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R}). \tag{2.2.14}$$

Generically, the above is satisfied for the $U(1)$-subgroup of translations $\psi_\theta(\tau) = \tau - \theta$ corresponding to $\pm\left(\begin{smallmatrix} \cos\alpha\theta & -\frac{1}{\alpha}\sin\alpha\theta \\ \alpha\sin\alpha\theta & \cos\alpha\theta\end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R})$. We wrote this in terms of the variable $\frac{1}{\alpha}\tan(\alpha\tau)$ to highlight that this works for the whole parameter range $\alpha^2 \in \mathbb{R}$. We claim that this exhausts the invariance group of the measures, apart from the *exceptional* values $\alpha = k\pi$ with $k \in \mathbb{N}$, for which the symmetry group is enhanced to the $k$-fold covering group of $\mathrm{PSL}(2,\mathbb{R})$.

In fact, for $\alpha \neq \pi\mathbb{N}$ and any other diffeomorphism $\psi$, without loss of generality satisfying $\psi(0) = 0$ (achieved after potentially composing with a translation), the ranges including multiplicities of the two sides in (2.2.14) don't agree for non-trivial Möbius transformations[2]. However, for $\alpha = \pi\mathbb{N}$ the map $\frac{1}{\alpha}\tan(\alpha\cdot)\colon S^1 \to \mathbb{R}P^1 \cong \mathbb{R}\cup\{\infty\}$ is $k$-to-1 and (2.2.14) has $k$ solutions for any $M = \pm\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R})$.

**Partition functions: Sketch of the argument.** The first step in our approach to evaluating the partition functions is an extension of above quasi-invariance: One can drop the requirement that $\psi'$ and $\psi''$ are periodic, i.e. consider $\psi \in \mathrm{Diff}^3[0,1]$ (here we consider quasi-invariance for the quotiented measure $\mathcal{N}^{\alpha}_{\sigma^2}$, for which a similar statement holds, see Proposition 5.2.6). Then there is no reason to expect $\psi^*\mathcal{N}^{\alpha}_{\sigma^2}$ to be absolutely continuous with respect to $\mathcal{N}^{\alpha}_{\sigma^2}$ anymore. In fact, for $h \in \mathbb{R}$ define $\mathcal{N}^{\alpha,h}_{\sigma^2}$ as in (2.2.6) but with a Brownian bridge starting at 0 and ending at level $h$. This measure is supported on $\{\varphi \in \mathrm{Diff}^1[0,1] : \varphi'(1) = e^h\varphi'(0)\}$. An extension of our statement on quasi-invariance will show that for $\psi \in \mathrm{Diff}^3[0,1]$, the measure $\psi^*\mathcal{N}^{\alpha}_{\sigma^2}$ is absolutely continuous with respect to $\mathcal{N}^{\alpha,h_\psi}_{\sigma^2}$ with $h_\psi = \log\psi'(0)/\psi'(1)$.

[2] This can be checked easily using the Iwasawa-decomposition of $\mathrm{PSL}(2,\mathbb{R})$.

Now, the key "trick" to calculate the partition function, is that for the "hyperbolic/parabolic orbits" $\alpha^2 \leq 0$, there exists $\psi_\alpha \in \mathrm{Diff}^3[0,1]$ such that $\psi_\alpha^* \mathcal{N}_{\sigma^2}^{\alpha} = c_{\alpha,\sigma^2} \mathcal{N}_{\sigma^2}^{0,h_\alpha}$ for some explicit constants $c_{\alpha,\sigma^2} > 0$ and $h_\alpha \in \mathbb{R}$. This calculates the partition function for the hyperbolic orbits $\alpha^2 \leq 0$, as the total mass of $\mathcal{N}_{\sigma^2}^{0,h_\alpha}$ is fixed by convention. We can then calculate the total mass of $\mathcal{N}_{\sigma^2}^{\alpha}$ for all $\alpha^2 < \pi^2$ by analytic continuation. The case $\alpha = \pi$, corresponding to the Schwarzian field theory, is then obtained by controlling the divergence in the limit $\alpha \nearrow \pi$.

## 2.3 Non-uniqueness of phase transitions for graphical representations of Ising [UNIQ]

The summary for this publication will be rather short, as [UNIQ] is in itself already quite compressed: In this work we consider consider several well-known random graph models related to the Ising model. Among them are the random current, the loop $O(1)$, and the random-cluster model. The latter can be directly coupled to the Ising model and demonstrates useful monotonicity properties. The other models on the other hand generally appear in representations of the Ising partition functions and it is not at all clear if they are "well-behaved" as probabilistic percolation-type models themselves. For example, one may wonder if their percolation threshold (if it exist) is unique and agrees with the critical point for the Ising model, or one may ask about monotonicity of (say) the percolation probability. In this article we follow these line of questions and construct some elementary counterexamples: We give examples of tree-like graphs on which the loop $\mathrm{O}(1)$ and random current model exhibit non-unique phase transitions. A particular consequence of this is the existence of infinite graphs $\mathbb{G} \subseteq \mathbb{G}'$, such that the uniform even subgraph of $\mathbb{G}'$ percolates, while the uniform even subgraph of $\mathbb{G}$ does not. Furthermore, we see that the percolation thresholds on regular tree-like graphs don't agree for free boundary conditions (while they do for wired ones).

# Chapter 3

# $\mathbb{H}^{2|2}$-model and VRJP on trees [TREE]

**Abstract:** We explore the supercritical phase of the vertex-reinforced jump process (VRJP) and the $\mathbb{H}^{2|2}$-model on rooted regular trees. The VRJP is a random walk, which is more likely to jump to vertices on which it has previously spent a lot of time. The $\mathbb{H}^{2|2}$-model is a supersymmetric lattice spin model, originally introduced as a toy model for the Anderson transition.

On infinite rooted regular trees, the VRJP undergoes a recurrence/transience transition controlled by an inverse temperature parameter $\beta > 0$. Approaching the critical point from the transient regime, $\beta \searrow \beta_{\mathrm{c}}$, we show that the expected total time spent at the starting vertex diverges as $\sim \exp(c/\sqrt{\beta-\beta_{\mathrm{c}}})$. Moreover, on large *finite* trees we show that the VRJP exhibits an additional intermediate regime for parameter values $\beta_{\mathrm{c}} < \beta < \beta_{\mathrm{c}}^{\mathrm{erg}}$. In this regime, despite being transient in infinite volume, the VRJP on finite trees spends an unusually long time at the starting vertex with high probability.

We provide analogous results for correlation functions of the $\mathbb{H}^{2|2}$-model. Our proofs rely on the application of branching random walk methods to a horospherical marginal of the $\mathbb{H}^{2|2}$-model.

## 3.1 Introduction and Main Results

### 3.1.1 History and Introduction

Our work will focus on two distinct but related models: The $\mathbb{H}^{2|2}$-model, a lattice spin model which is related to the Anderson transition, and the *vertex-reinforced jump process* (VRJP), a random walk on graphs which is more likely to jump to vertices on which it has already spent a lot of time.

The $\mathbb{H}^{2|2}$-model was initially introduced by Zirnbauer [20] as a toy model for studying the Anderson transition. Formally, it is a lattice spin model taking values in the *hyperbolic superplane* $\mathbb{H}^{2|2}$, a supersymmetric analogue of hyperbolic space. Independently, the VRJP was introduced by Davis and Volkov [21] as a natural example of a reinforced (and consequently non-Markovian) continuous-time random walk. Somewhat surprisingly, Sabot and Tarrès [22] observed that these two models are intimately related. Namely, the time the VRJP asymptotically spends on vertices can be expressed in terms of the $\mathbb{H}^{2|2}$-model. This has been used to see the VRJP as a random walk in random environment, with the environment being given by the $\mathbb{H}^{2|2}$-model. Furthermore, the two models are linked by a Dynkin-type isomorphism theorem due to Bauerschmidt, Helmuth and Swan [2, 3], analogous to the connection between simple random walk and the Gaussian free field [23].

Both models are parametrised by an inverse temperature $\beta > 0$ and, depending on the background geometry of the graph under consideration, may exhibit a phase transition at some critical parameter $\beta_c \in (0, \infty]$. For the $\mathbb{H}^{2|2}$-model the expected transition is between a disordered high-temperature phase ($\beta < \beta_c$) and a symmetry-broken low-temperature phase ($\beta > \beta_c$) exhibiting long-range order. For the VRJP the transition is between a recurrent phase due to strong reinforcement effects and a transient phase due to low reinforcement effects.

On $\mathbb{Z}^D$ a fair bit is known about the phase diagram of the two models. In dimension $D \leq 2$ both models are never delocalised (*i.e.* they are always disordered and recurrent, respectively) [2, 21, 22, 24–26]. In dimensions $D \geq 3$, however, they exhibit a phase transition from a localised to a delocalised phase at a unique $\beta_c \in (0, \infty)$ [22, 25, 27–31].

In this article we consider both models on the geometry of a rooted $(d+1)$-regular tree $\mathbb{T}_d$ with $d \geq 2$ (see Figure 3.1). For the VRJP this setting was previously explored by various authors [4, 32–35]. In particular, Basdevant and Singh [4] showed that the VRJP on Galton-Watson trees with mean offspring $m > 1$ has a phase transition from recurrence to transience at some explicitly characterised $\beta_c \in (0, \infty)$. For simplicity, we focus on the "deterministic case", but our results should translate to Galton-Watson trees as well (up to some technical restrictions on the offspring distribution).

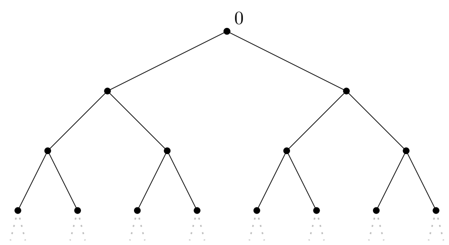


Figure 3.1: The rooted $(d+1)$-regular tree $\mathbb{T}_d$ for $d = 2$ shown up to its third generation, with the root vertex denoted as 0.

The main goal of this work is to provide new information on the supercritical phase ($\beta > \beta_c$) including the near-critical regime. Roughly speaking, we show that on the infinite rooted $(d+1)$-regular tree $\mathbb{T}_d$ the order parameters of the VRJP and the $\mathbb{H}^{2|2}$-model diverge as

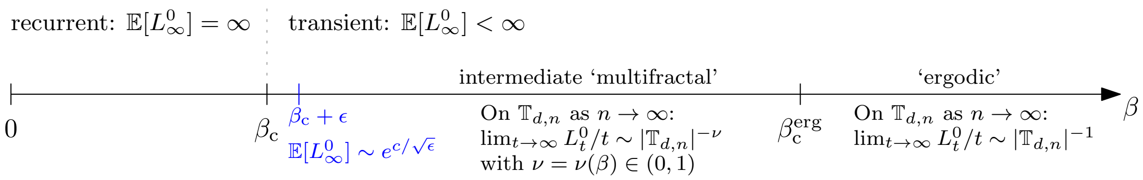


Figure 3.2: Sketch of the phase diagram for the VRJP on $\mathbb{T}_d$ with $d \geq 2$. The recurrence/transience transition at $\beta_c$ is phrased in terms of $\mathbb{E}[L^0_\infty]$, *i.e.* the expected total time the walk (on the infinite rooted $(d+1)$-regular tree $\mathbb{T}_d$) spends at the starting vertex. In this article, we obtain precise asymptotics for $\mathbb{E}[L^0_\infty]$ as $\beta \searrow \beta_c$. Second, we show that there is an additional transition point $\beta_c^{\mathrm{erg}} > \beta_c$. It is phrased in terms of the volume-scaling of the fraction of total time, $\lim_{t\to\infty} L^0_t/t$, the VRJP on the *finite* tree $\mathbb{T}_{d,n}$ spends at the origin. Here, the symbol "$\sim$" is understood loosely, and we refer to the text for precise error terms.

$\exp(c/\sqrt{\beta-\beta_c})$ as one approaches the critical point from the supercritical regime, $\beta \searrow \beta_c$ (see Theorem 3.1.2 and 3.1.5, respectively). Such behaviour has previously been predicted by Zirnbauer for Efetov's model [36]. This "infinite-order" behaviour towards the critical point is rather surprising, as it conflicts with usual scaling hypotheses in statistical mechanics, which predict algebraic singularities as one approaches the critical points. Moreover, we show that on *finite* rooted $(d+1)$-regular trees, the VRJP and the $\mathbb{H}^{2|2}$-model exhibit an additional *mulifractal* intermediate regime for $\beta \in (\beta_c, \beta_c^{\mathrm{erg}})$ (see Theorem 3.1.3, 3.1.4, and 3.1.6). An illustration of some of our results for the VRJP is given in Figure 3.2.

**Connection to the Anderson Transition and Efetov's Model.** Inspiration for our work originates from predictions in the physics literature on *Efetov's model* [36–41]. The latter is a supersymmetric lattice sigma model that is considered to capture the Anderson transition [42, 43]. To be more precise, Efetov's model can be derived from a *granular limit* (similar to a Griffiths-Simon construction [44]) of the random band matrix model, followed by a sigma model approximation [45, 46]. The connection to our work is due to Zirnbauer, who introduced the $\mathbb{H}^{2|2}$-model as a simplification of Efetov's model [20]. Namely, in Efetov's model spins take value in the symmetric superspace $U(1,1|2)/[U(1|1) \otimes U(1|1)]$. According to Zirnbauer, the essential features of this target space are its hyperbolic symmetry and its supersymmetry[1]. In this sense, $\mathbb{H}^{2|2}$ is the simplest target space with these two properties. Study of the $\mathbb{H}^{2|2}$-model may guide the analysis of supersymmetric field theories more closely related to the Anderson transition.

[1] Also referred to as "perfect grading". Roughly speaking, this refers to the fact that the space has the same number of bosonic and fermionic degrees of freedom (in this case four each), while these are also "exchangeable" under a symmetry of the space.

Moreover, the $\mathbb{H}^{2|2}$-model and the VRJP are directly and rigorously related to an Anderson-type model, which we refer to as the *STZ-Anderson model* (see Definition 3.1.8). This fact was already hinted at by Disertori, Spencer and Zirnbauer [27], but only fully appreciated by Sabot, Tarrès and Zeng [47, 48], who exploited the relationship to gain new insights on the VRJP. It is an interesting open problem to better understand the spectral properties of this model and how it relates to the VRJP and the $\mathbb{H}^{2|2}$-model.

Notably, the phase diagram of the $\mathbb{H}^{2|2}$-model is better understood than that of Efetov's model or the Anderson model on a lattice. For example, for the $\mathbb{H}^{2|2}$-model there is proven absence of long-range order in 2D [2] as well as proven existence of a phase transition in 3D [27, 28]. For the Anderson model on $\mathbb{Z}^D$, the existence of a phase transition in $D \geq 3$ and the absence of one in $D = 2$ are arguably among the most prominent open problems in mathematical physics. A good example of the Anderson model's intricacies is given by the work of Aizenman and Warzel [49, 50]. Despite many previous efforts, they were the first to gain a somewhat complete understanding of the model's spectral properties on the regular tree. However, many questions are still open, in particular there are no rigorous results on the Anderson model's (near-)critical behaviour. In this sense one might (somewhat generously) interpret this article as a step towards better understanding of the near-critical behaviour for a model in the "Anderson universality class".

We would also like to comment on the methods used in the physics literature on Efetov's model. The analysis of the model on a regular tree, initiated by Efetov and Zirnbauer [36, 37], relies on a recursion/consistency relation that is specific to the tree setting. Using this approach, Zirnbauer predicted the divergence of the order parameter (relevant for the symmetry-breaking transition of Efetov's model) for $\beta \searrow \beta_{\mathrm{c}}$. We should mention that Mirlin and Gruzberg [51] argued that this analysis should essentially carry through for the $\mathbb{H}^{2|2}$-model. In our case, we take a different path, exploiting a branching random walk structure in the "horospherical marginal" of the $\mathbb{H}^{2|2}$-model (the $t$-field).

After completion of this work, we were made aware by Martin Zirnbauer of recent numerical investigations for the Anderson transition on random tree-like graphs [52, 53]. The observed scaling behaviour near the transition point might suggest the need for a field-theoretic description beyond the supersymmetric approach of Efetov (also see [54, 55]). At this point, there does not seem to exist a consensus on the theoretical description of near-critical scaling for the Anderson transition of tree-like graphs and rigorous results would be of great value.

**Notation:** In multi-line estimates, we occasionally use "running constants" $c, C > 0$ whose precise value may vary from line to line. We denote by $[n] = 1, \ldots, n$ the range of positive

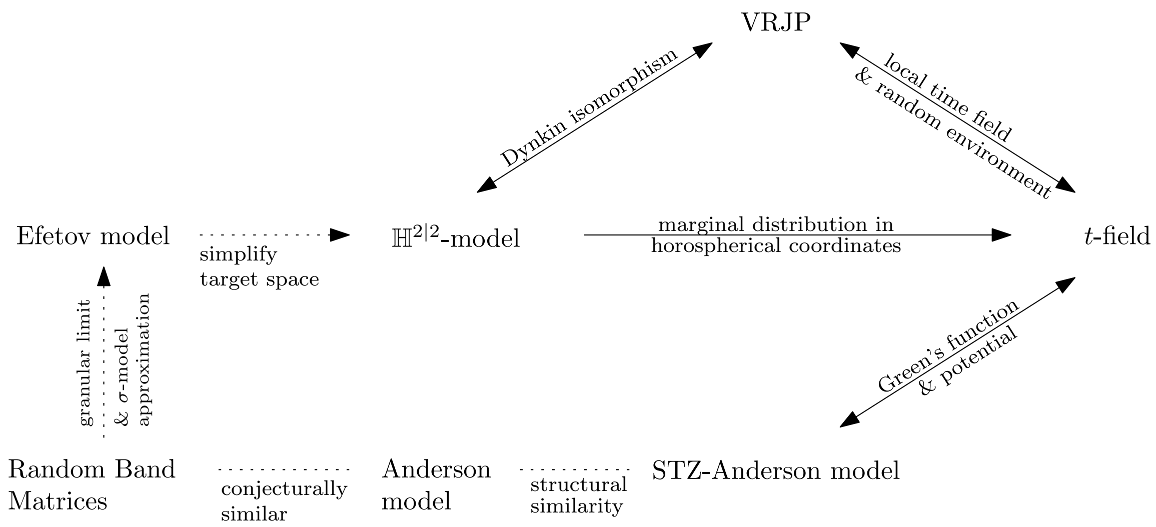


Figure 3.3: An illustration of various interconnected models, that we touch on. Solid lines denote rigorous connections, *i.e.* relevant quantities in one model can be expressed in terms of the other. Dashed lines signify conceptual/heuristic connections.

integers up to $n$. For a graph $G = (V, E)$ an unoriented edge $\{x, y\} \in E$ will be denoted by the juxtaposition $xy$, whereas an oriented edge is denoted by a tuple $(x, y)$, which is oriented from $x$ to $y$. Write $\vec{E}$ for the set of oriented edges. For a vertex $x$ in a rooted tree (or a particle of a branching random walk), we denote its *generation* (i.e. distance from the origin) by $|x|$. We use the short-hand $\sum_{|x|=n} \dots$ to denote summation over all vertices/particles at generation $n$. Variants of this convention will be used and the meaning should be clear from context. When our results concern the $(d+1)$-regular rooted tree $\mathbb{T}_d$, we assume $d \geq 2$ will typically suppress the $d$-dependence of all involved constants, unless specified otherwise. Mentions of $\beta_c$ implicity refer to the critical parameter $\beta_c = \beta_c(d)$ as given by Proposition 3.2.14.

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### 3.1.2 Model Definitions and Results

In this section, we define the VRJP, the $\mathbb{H}^{2|2}$-model, the $t$-field and the STZ-Anderson model. We are aware that spin systems with fermionic degrees of freedom, such as the $\mathbb{H}^{2|2}$-model, might be foreign to some readers. However, understanding this model is not necessary for the main results on the VRJP, and the reader can feel comfortable to skip references to the $\mathbb{H}^{2|2}$-model on a first reading. We also note that all models that we introduce are intimately related (as illustrated in Figure 3.3) and Section 3.2 will illuminate some of these connections.

#### 3.1.2.1 Vertex-Reinforced Jump Process.

**Definition 3.1.1:** Let $G = (V, E)$ be a locally finite graph equipped with positive edge-weights $(\beta_e)_{e\in E}$, and a starting vertex $i_0 \in V$. The VRJP $(X_t)_{t\geq 0}$ starting at $X_0 = i_0$ is the continuous-time jump process that at time $t$ jumps from a vertex $X_t = x$ to a neighbour $y$ at rate

$$\beta_{xy}[1+L_t^y] \quad \text{with} \quad L_t^y(t) := \int_0^t 1_{X_s=y} \mathrm{d}s. \tag{3.1.1}$$

We refer to $L_t^y$ as the *local time* at $y$ up to time $t$.

Unless specified otherwise, the VRJP on a graph $G$ refers to the case of constants weights $\beta_e \equiv \beta$ and the dependency on the weight $\beta$ is specified by a subscript, as in $\mathbb{E}_\beta$ or $\mathbb{P}_\beta$. By a slight abuse of language, we refer to $\beta$ as an *inverse temperature*.

**Results for the VRJP.** Note that Figure 3.2 gives a rough picture of our statements for the VRJP. In the following we provide the exact results.

In the following, $\beta_c = \beta_c(d)$ will denote the critical inverse temperature for the recurrence/transience transition of the VRJP on the infinite rooted $(d+1)$-regular tree $\mathbb{T}_d$ with $d \geq 2$. By Basdevant and Singh [4] this inverse temperature is well-defined and finite: $\beta_c \in (0,\infty)$ (*cf.* Proposition 3.2.14). Alternatively, $\beta_c$ is characterised in terms of divergence of the expected total local time at the origin: $\beta_c = \inf\{\beta > 0 : \mathbb{E}_\beta[L_\infty^0] < \infty\}$. The following theorem provides information about the divergence of $\mathbb{E}_\beta[L_\infty^0]$ as we approach the critical point from the transient regime.

**Theorem 3.1.2** (Local-Time Asymptotics as $\beta \searrow \beta_c$ for the VRJP on $\mathbb{T}_d$): Consider the VRJP, started at the root 0 of the infinite rooted $(d+1)$-regular tree $\mathbb{T}_d$ with $d \geq 2$. Let $\beta_c = \beta_c(d) \in (0,\infty)$ be as in Proposition 3.2.14. Let $L_\infty^0 = \lim_{t\to\infty} L_t^0$ denote the total time the

VRJP spends at the root. There are constants $c, C > 0$ such that for sufficiently small $\epsilon > 0$:

$$\exp(c/\sqrt{\epsilon}) \leq \mathbb{E}_{\beta_c+\epsilon}[L_\infty^0] \leq \exp(C/\sqrt{\epsilon}). \tag{3.1.2}$$

The above result concerned the *infinite* rooted $(d+1)$-regular tree $\mathbb{T}_d$. On a finite rooted $(d+1)$-regular tree $\mathbb{T}_{d,n}$ the total local time at the origin always diverges, but we may consider the fraction of time the walk spends at the starting vertex. In terms of this quantity we can identify both the recurrence/transience transition point $\beta_c$ as well as an additional intermediate phase inside the transient regime.

**Theorem 3.1.3** (Intermediate Phase for VRJP on Finite Trees): Consider the VRJP started at the root of the rooted $(d+1)$-regular tree of depth $n$, $\mathbb{T}_{d,n}$, with $d \geq 2$. Let $L_t^0$ denote the total time the walk spent at the root up until time $t$. We have

$$\lim_{t\to\infty} \tfrac{L_t^0}{t} = \left|\mathbb{T}_{d,n}\right|^{-\nu(\beta)+o(1)} \qquad \text{w.h.p. as } n \to \infty \tag{3.1.3}$$

with $\beta \mapsto \nu(\beta)$ continuous and non-decreasing such that

$$\nu(\beta) \begin{cases} = 0 & \text{for } \beta \leq \beta_c \\ \in (0,1) & \text{for } \beta_c < \beta < \beta_c^{\text{erg}} \\ = 1 & \text{for } \beta > \beta_c^{\text{erg}}, \end{cases} \tag{3.1.4}$$

for some $\beta_c^{\text{erg}} = \beta_c^{\text{erg}}(d) > \beta_c$. More precisely, we have

$$\nu(\beta) = \max\Big(0, \inf_{\eta\in(0,1]} \frac{\psi_\beta(\eta)}{\eta \log d}\Big) \tag{3.1.5}$$

with $\psi_\beta(\eta)$ given in (3.3.7).

Moreover, in the intermediate phase the inverse fraction of time at the origin shows a *multifractal* scaling behaviour:

**Theorem 3.1.4:** (Multifractality in the Intermediate Phase) Consider the setup of Theorem 3.1.3 and suppose $\beta \in (\beta_c, \beta_c^{\text{erg}})$. For $\eta \in (0,1)$ we have

$$\mathbb{E}_\beta[(\lim_{t\to\infty} \tfrac{L_t^0}{t})^{-\eta}] \sim \left|\mathbb{T}_{d,n}\right|^{\tau_\beta(\eta)+o(1)} \quad \text{as} \quad n \to \infty, \tag{3.1.6}$$

where

$$\tau_\beta(\eta) = \begin{cases} \frac{\eta}{\eta_\beta} \frac{\psi_\beta(\eta_\beta)}{\log d} & \text{for } \eta \leq \eta_\beta \\ \frac{\psi_\beta(\eta)}{\log d} & \text{for } \eta \geq \eta_\beta, \end{cases} \tag{3.1.7}$$

where $\psi_\beta$ is given in (3.3.7) and $\eta_\beta = \operatorname{argmin}_{\eta>0}\psi_\beta(\eta)/\eta \in (0,1)$.

#### 3.1.2.2 The $\mathbb{H}^{2|2}$-model.

**Definition of the $\mathbb{H}^{2|2}$-Model.** We start by writing down the formal expressions defining the $\mathbb{H}^{2|2}$-model, and then make sense out of it afterwards. Conceptually, we think of the *hyperbolic superplane* $\mathbb{H}^{2|2}$ as the set of vectors $\mathbf{u} = (z,x,y,\xi,\eta)$, satisfying

$$-1 = \mathbf{u}\cdot\mathbf{u} := -z^2 + x^2 + y^2 - 2\xi\eta. \tag{3.1.8}$$

Here, $z,x,y$ are *even/bosonic* coordinates and $\xi,\eta$ are *odd/fermionic*, a notion that will be explained shortly. For two vectors $\mathbf{u}_i = (z_i,x_i,y_i,\xi_i,\eta_i)$ and $\mathbf{u}_j = (z_j,x_j,y_j,\xi_j,\eta_j)$, we define the inner product

$$\mathbf{u}_i\cdot\mathbf{u}_j := -z_iz_j + x_ix_j + y_iy_j + \eta_i\xi_j - \xi_i\eta_j. \tag{3.1.9}$$

In other words, this pairing is of hyperbolic type in the even variables and of symplectic type in the odd variables.

Consider a finite graph $G = (V,E)$ with non-negative edge weights $(\beta_e)_{e\in E}$ and magnetic field $h > 0$. Morally, we think of the $\mathbb{H}^{2|2}$-model on $G$ as a probability measure on *spin configurations* $\underline{\mathbf{u}} = (\mathbf{u}_i)_{i\in V} \in (\mathbb{H}^{2|2})^V$, such that the formal expectation of a functional $F \in C^\infty((\mathbb{H}^{2|2})^V)$ is given by

$$\left\langle F(\underline{\mathbf{u}})\right\rangle_{\beta,h} := \int_{(\mathbb{H}^{2|2})^V} \prod_{i\in V} \mathrm{d}\mathbf{u}_i\, F(\underline{\mathbf{u}})\, e^{\sum_{ij\in E}\beta_{ij}(\mathbf{u}_i\cdot\mathbf{u}_j+1) - h\sum_{i\in V}(z_i-1)}, \tag{3.1.10}$$

with $\mathrm{d}\mathbf{u}$ denoting the Haar measure over $\mathbb{H}^{2|2}$. In other words, formally everything is analogous to the definition of spin/sigma models with "usual" target spaces, such as spheres $S^n$ or hyperbolic spaces $\mathbb{H}^n$. The only subtlety is that we still need to understand what a functional such as $F \in C^\infty((\mathbb{H}^{2|2})^V)$ means and how to interpret the integral above.

Rigorously, the space $\mathbb{H}^{2|2}$ is not understood as a set of points, but rather is defined in a dual sense by directly specifying its set of smooth functions to be

$$C^\infty(\mathbb{H}^{2|2}) := C^\infty(\mathbb{R}^2) \otimes \Lambda(\mathbb{R}^2) \tag{3.1.11}$$

In other words, this is the exterior algebra in two generators with coefficients in $C^\infty(\mathbb{R}^2)$ (which is the same as $C^\infty(\mathbb{R}^{2|2})$, analogous to the fact that $\mathbb{H}^2 \cong \mathbb{R}^2$ as smooth manifolds.). Note that this set naturally carries the structure of a graded-commutative algebra. More concretely, any *superfunction* $f \in C^\infty(\mathbb{H}^{2|2})$ can we written as

$$f = f_0(x,y) + f_\xi(x,y)\xi + f_\eta(x,y)\eta + f_{\xi\eta}(x,y)\xi\eta \tag{3.1.12}$$

with smooth functions $f_0, f_\xi, f_\eta, f_{\xi\eta} \in C^\infty(\mathbb{R}^2)$ and $\xi, \eta$ generating a Grassmann algebra, *i.e.* they satisfy the algebraic relations $\xi\eta = -\eta\xi$ and $\xi^2 = \eta^2 = 0$. We think of such $f$ as a smooth function in the variables $x, y, \xi, \eta$ and write $f = f(x,y,\xi,\eta)$. In particular, the *coordinate functions* $x, y, \xi, \eta$ are themselves superfunctions. In light of (3.1.8), we *define* the $z$-coordinate to be the (even) superfunction

$$z := (1 + x^2 + y^2 - 2\xi\eta)^{1/2} := (1 + x^2 + y^2)^{1/2} - \frac{\xi\eta}{(1 + x^2 + y^2)^{1/2}} \in C^\infty(\mathbb{H}^{2|2}). \tag{3.1.13}$$

In this sense the coordinate vector $\mathbf{u} = (z, x, y, \xi, \eta)$ satisfies $\mathbf{u} \cdot \mathbf{u} = -1$. By abuse of notation we write $\mathbf{u} \in \mathbb{H}^{2|2}$, but more correctly one might say that $\mathbf{u}$ *parametrises* $\mathbb{H}^{2|2}$. For a superfunction $f \in C^\infty(\mathbb{H}^{2|2})$ we write $f(\mathbf{u}) = f(x,y,\xi,\eta) = f$ and in line with physics terminology we might say that $f$ is a function of the *even/bosonic* variables $z, x, y$ and the *odd/fermionic* variables $\xi, \eta$.

The definition of $z$ in (3.1.13) shows a particular example of a more general principle: The composition of an ordinary function (the square root in the example) with a superfunction (in the example that is $1 + x^2 + y^2 - 2\xi\eta$) is defined by formal Taylor expansion in the Grassmann variables. Due to nilpotency of the Grassmann variables this is well-defined.

Next we would like to introduce a notion of integrating a superfunction $f(\mathbf{u})$ over $\mathbb{H}^{2|2}$. Expressing $f$ as in (3.1.12), we define the derivations $\partial_\xi, \partial_\eta$ acting via

$$\partial_\xi f = f_\xi(x,y) + f_{\xi\eta}(x,y)\eta \quad \text{and} \quad \partial_\eta f = f_\eta(x,y) - f_{\xi\eta}(x,y)\xi. \tag{3.1.14}$$

In particular, note that these derivations are *odd*: they anticommute, $\partial_\xi\partial_\eta = -\partial_\eta\partial_\xi$, and satisfy a graded Leibniz rule. The $\mathbb{H}^{2|2}$-integral of $f \in C^\infty(\mathbb{H}^{2|2})$ is then defined to be the linear functional

$$\int_{\mathbb{H}^{2|2}} \mathrm{d}\mathbf{u}\, f(\mathbf{u}) := \int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}y\, \partial_\eta \partial_\xi [\frac{1}{z} f]. \tag{3.1.15}$$

The factor $\frac{1}{z}$ plays the role of a $\mathbb{H}^{2|2}$-volume element in the coordinates $x, y, \xi, \eta$. Note that this integral evaluates to a real number.

In a final step to formalise (3.1.10) we define multivariate superfunctions over $\mathbb{H}^{2|2}$

$$C^\infty((\mathbb{H}^{2|2})^V) := \bigotimes_{i \in V} C^\infty(\mathbb{H}^{2|2}) \cong C^\infty(\mathbb{R}^{2|V|}) \otimes \Lambda(\mathbb{R}^{2|V|}), \tag{3.1.16}$$

that is the Grassmann algebra in $2|V|$ generators $\{\xi_i, \eta_i\}_{i\in V}$ with coefficients in $C^\infty(\mathbb{R}^{2|V|})$. An element of this algebra is considered a functional over spin configurations $\underline{\mathbf{u}} = \{\mathbf{u}_i\}_{i\in V}$ and we write $F = F(\underline{\mathbf{u}})$. Any superfunction $F \in C^\infty((\mathbb{H}^{2|2})^V)$ can be expressed, analogously to (3.1.12), as

$$\sum_{I,J\subseteq V} f_{I,J}(\{x_i, y_i\}_{i\in V}) \prod_{i\in I} \xi_i \prod_{j\in J} \eta_j. \tag{3.1.17}$$

The integral of such $F$ over $(\mathbb{H}^{2|2})^V$ is defined as

$$\int_{(\mathbb{H}^{2|2})^V} \mathrm{d}\underline{\mathbf{u}}\, F(\underline{\mathbf{u}}) := \int_{(\mathbb{H}^{2|2})^V} \prod_{i\in V} \mathrm{d}\mathbf{u}_i\, F(\underline{\mathbf{u}}) := \int_{\mathbb{R}^{2|V|}} \prod_{i\in V} \mathrm{d}x_i\, \mathrm{d}y_i \prod_{i\in V} \partial_{\eta_i} \partial_{\xi_i} [(\textstyle\prod_{i\in V} \frac{1}{z_i}) F(\underline{\mathbf{u}})]. \tag{3.1.18}$$

With this notion of integration, the definition of the $\mathbb{H}^{2|2}$-model in (3.1.10) can be understood in a rigorous sense: The "Gibbs factor" is the composition of a regular function (exponential) with a superfunction (the exponent). As such it is defined by expansion in the Grassmann variables.

**Results for the $\mathbb{H}^{2|2}$-Model.** In the following we will simply rephrase above theorems in terms of the $\mathbb{H}^{2|2}$-model.

**Theorem 3.1.5** (Asymptotics as $\beta \searrow \beta_c$ for the $\mathbb{H}^{2|2}$-model on $\mathbb{T}_d$): Consider the $\mathbb{H}^{2|2}$-model on $\mathbb{T}_{d,n}$. Suppose $\beta_c = \beta_c(d) \in (0,\infty)$ is as in Proposition 3.2.14. The quantity

$$\langle x_0^2 \rangle^+_{\beta_c+\epsilon} := \lim_{h\searrow 0} \lim_{n\to\infty} \langle x_0^2 \rangle_{\beta_c+\epsilon; h, \mathbb{T}_{d,n}} \tag{3.1.19}$$

is well-defined and finite for any $\epsilon > 0$. There exist constants $c, C > 0$ such that for sufficiently small $\epsilon > 0$

$$\exp(c/\sqrt{\epsilon}) \le \langle x_0^2 \rangle^+_{\beta_c+\epsilon} \le \exp(C/\sqrt{\epsilon}). \tag{3.1.20}$$

The above statement considered the *infinite-volume limit*, *i.e.* taking $n \to \infty$ before removing the magnetic field $h \searrow 0$. One may also consider a *finite-volume limit* (also referred to as *inverse-order thermodynamic limit* [56]): In that case, we consider scaling limits of observable as $h \searrow 0$ before taking $n \to \infty$. In this limit, we also demonstrate an intermediate multifractal regime for the $\mathbb{H}^{2|2}$-model.

**Theorem 3.1.6** (Intermediate Phase for the $\mathbb{H}^{2|2}$-Model on $\mathbb{T}_{d,n}$): There exist $0 < \beta_c < \beta_c^{\mathrm{erg}} < \infty$ as in Theorem 3.1.3, such that for $\beta_c < \beta < \beta_c^{\mathrm{erg}}$ we have for $\eta \in (0,1)$

$$\lim_{h\searrow 0} h^{-\eta} \langle z_0 |x_0|^{-\eta} \rangle_{\beta,h;\mathbb{T}_{d,n}} \sim \left|\mathbb{T}_{d,n}\right|^{\tau_\beta(\eta)+o(1)} \quad \text{as} \quad n\to\infty \tag{3.1.21}$$

with $\tau_\beta(\eta)$ as given in (3.1.7).

At first glance, the observable in (3.1.21) might seem somewhat obscure. However, in the physics literature on Efetov's model and the Anderson transition, analogous quantities are predicted to encode disorder-averaged (fractional) moments of eigenstates at a given vertex and energy level, see for example [41, Equation (6)]. The volume-scaling of these quantities provides information about the (de)localisation behaviour of the eigenstates.

#### 3.1.2.3 The $t$-field.

Despite the inconspicuous name, the $t$-field is the most relevant object for our analysis. It is directly related to both the VRJP, encoding the time the VRJP asymptotically spends on each vertex, as well as the $\mathbb{H}^{2|2}$-model, arising as a marginal in horospherical coordinates (see Section 3.2 for details).

**Definition 3.1.7** ($t$-field Distribution): Consider a finite graph $G = (V, E)$, a vertex $i_0 \in V$ and non-negative edge-weights $(\beta_e)_{e\in E}$. The law of the $t$-field, with weights $(\beta_e)_{e\in E}$, *pinned* at $i_0$, is a probability measure on configurations $\mathbf{t} = \{t_i\}_{i\in V} \in \mathbb{R}^V$ given by

$$Q_\beta^{(i_0)}(\mathrm{d}\mathbf{t}) := e^{-\sum_{ij\in E}\beta_{ij}[\cosh(t_i-t_j)-1]} D_\beta(\mathbf{t})^{1/2}\, \delta(t_{i_0}) \prod_{i\in V\setminus\{i_0\}} \frac{\mathrm{d}t_i}{\sqrt{2\pi/\beta}}, \tag{3.1.22}$$

with the determinantal term

$$D_\beta(\mathbf{t}) := \sum_{T\in\vec{\mathcal{T}}^{(i_0)}} \prod_{(i,j)\in T} \beta_{ij} e^{t_i - t_j}, \tag{3.1.23}$$

where $\vec{\mathcal{T}}^{(i_0)}$ is the set of spanning trees in $G$ oriented away from $i_0$.

Alternatively, one can write $D_\beta(\mathbf{t}) = \prod_{i\in V\setminus\{i_0\}} e^{-2t_i} \det_{i_0}(-\Delta_{\beta(\mathbf{t})})$, where $\det_{i_0}$ denotes the principal minor with respect to $i_0$ and $-\Delta_{\beta(\mathbf{t})}$ is the discrete Laplacian for edge-weights $\beta(\mathbf{t}) = (\beta_{ij}e^{t_i+t_j})_{ij}$.

In general the determinantal term renders the law $Q_\beta^{(i_0)}$ highly non-local. However, in case the underlying graph $G$ is a tree, only a single summand contributes to (3.1.23) and the measure factorises in terms of the oriented edge-increments $\{t_i - t_j\}_{(i,j)}$. This simplification is essential for this article and gives us the possibility to analyse the $t$-field on rooted $(d+1)$-regular trees in terms of a branching random walk.

#### 3.1.2.4 STZ-Anderson Model.

The following introduces a random Schrödinger operator, which is related to the previously introduced models. It will only be required for translating our results on the intermediate phase to the $\mathbb{H}^{2|2}$-model (Section 3.5.2), so the reader may skip this definition on a first reading. As Sabot, Tarrès and Zeng [47, 48] were the first to study this system in detail, we refer to it as the *STZ-Anderson model*.

**Definition 3.1.8** (STZ-Anderson Model): Consider a locally finite graph $G = (V, E)$, equipped with non-negative edge-weights $(\beta_e)_{e\in E}$. For $B = (B_i)_{i\in\Lambda} \subseteq \mathbb{R}_+^\Lambda$ define the Schrödinger-type operator

$$H_B := -\Delta_\beta + V(B) \quad \text{with} \quad [V(B)]_i = B_i - \textstyle\sum_j \beta_{ij}. \tag{3.1.24}$$

Define a probability distribution $\nu_\beta$ over configurations $B = (B_i)_{i\in\Lambda}$ by specifying the Laplace transforms of its finite-dimensional marginals: For any vector $(\lambda_i)_{i\in V} \in [0,\infty)^V$ with only finitely many non-zero entries, we have

$$\int e^{-(\lambda,B)} \nu_\beta(\mathrm{d}B) = \frac{1}{\prod_{i\in V}\sqrt{1+2\lambda_i}} \exp[-\sum_{ij\in E} \beta_{ij}(\sqrt{1+2\lambda_i}\sqrt{1+2\lambda_j}-1)]. \tag{3.1.25}$$

Subject to this distribution, we refer to $B$ as the *STZ-field* and to $H_B$ as the STZ-Anderson model.

One may note that on finite graphs, the density of $\nu_\beta$ is explicit:

$$\nu_\beta(\mathrm{d}B) \propto \frac{e^{-\frac{1}{2}\sum_i B_i}}{\sqrt{\det(H_B)}} \mathbb{1}_{H_B>0}\, \mathrm{d}B, \tag{3.1.26}$$

where $H_B > 0$ means that the matrix $H_B$ is positive definite. The definition via (3.1.25) is convenient, since it allows us to directly consider the infinite-volume limit. We also note that while the density (3.1.26) seems highly non-local, the Laplace transform in (3.1.25) only involves values of $\lambda$ at adjacent vertices and therefore implies 1-dependency of the STZ-field.

In the original literature the STZ-field is denoted by $\beta$ and referred to as the $\beta$-field. In order to be consistent with the statistical physics literature and avoid confusion with the inverse temperature, we introduced this slightly different notation. To be precise, we used this change of notation to also introduce a slightly more convenient normalisation: one has $B_i = 2\beta_i$ compared to the normalisation of the $\beta$-field $\{\beta_i\}$ used by Sabot, Tarrès and Zeng.

### 3.1.3 Further Comments

**Comments on Related Work.** As noted earlier, the VRJP on tree geometries was already studied by various authors [4, 32–35]. One notable difference to our work is that we do not consider the more general setting of Galton-Watson trees. While this is mostly to avoid unnecessary notational and technical difficulties, the Galton-Watson setting might be more subtle. This is due to an "extra" phase transition in the transient phase, observed by Chen and Zeng [34]. This phase transition depends on the probability of the Galton Watson tree having precisely one offspring. It is an interesting question how this would interact with our analysis.

In regard to our results, the recent work by Rapenne [35] is of particular interest. He provides precise quantitative information on the (sub-)critical phase $\beta \leq \beta_c$. The results are phrased in terms of a certain martingale, associated with the STZ-Anderson model, but they can be formulated in terms of the $\mathbb{H}^{2|2}$-model with wired boundary conditions (or analogously the VRJP started from the boundary) on a rooted $(d+1)$-regular tree of finite depth. In this sense, Rapenne's article can be considered as complementary to our work.

Another curious connection to our work is given by the Derrida-Retaux model [57–64]. The latter is a toy model for a hierarchical renormalisation procedure related to the depinning transition. It has recently been shown [64] that the free energy of this model may diverge as $\sim \exp(-c/\sqrt{p-p_c})$ approaching the critical point from the supercritical phase, $p \searrow p_c$. There are further formal similarities between their analysis and the present article. It would be of interest to shed further light on the universality of this type of behaviour.

**Debate on Intermediate Phase** We would like to highlight that the presence/absence of such an intermediate phase for the Anderson transition[2] on tree-geometries has been a recent topic of debate in the physics literature (see [56, 65] and references therein). In short, the debate concerns the question of whether the intermediate phase only arises due to finite-volume and boundary effects on the tree.

While the presence of a *non-ergodic delocalised* phase on finite regular trees has been established in recent years [40, 41, 66], it was not clear if this behaviour persists in the absence of a large "free" boundary. To study this, one can consider a system on a large random regular graphs (RRGs) as a "tree without boundary" (alternatively one could consider trees with wired boundary conditions). For the Anderson transition on RRGs, early numerical simulations [39, 67, 68] suggested existence of an intermediate phase, in conflict with existing theoretical predictions [38, 69–71]. Shortly afterwards, it was argued that the discrepancy was due to

[2] This may refer to the Anderson model, Efetov's model, or certain sparse random matrix models (such as random band matrices), all of which are largely considered equivalent in the theoretical physics community.

finite-size effects that vanish at very large system sizes [40, 65, 72], even though this does not seem to be the consensus[3] [56, 68].

We should note that Aizenman and Warzel [49, 73] have shown the existence of an energy-regime of "resonant delocalisation" for the Anderson model on regular trees. It would be interesting to understand if/how this phenomenon is related to the intermediate phase discussed here.

In accordance with the physics literature, we refer to the intermediate phase ($\beta_c < \beta < \beta_c^{\mathrm{erg}}$) as *multifractal* as opposed to the *ergodic* phase ($\beta > \beta_c^{\mathrm{erg}}$).

### 3.1.4 Structure of this Article

In **Section 3.2** we provide details on the connections between the various models and recall previously known results for the VRJP. In particular, we recall that the VRJP can be seen as a random walk in random conductances given in terms of a $t$-field (referred to as the *$t$-field environment*). On the tree, the $t$-field can be seen as a branching random walk (BRW) and we recall various facts from the BRW literature. In **Section 3.3** we apply BRW techniques to establish a statement on effective conductances in random environments given in terms of *critical* BRWs (Theorem 3.3.2). With Theorem 3.3.1 we prove a result on effective conductances in the *near-critical* $t$-field environment. We close the section by showing how the result on effective conductances implies Theorem 3.1.2 on expected local times for the VRJP. In **Section 3.4** we continue to use BRW techniques for the $t$-field to establish Theorem 3.1.3 on the intermediate phase for the VRJP. We also prove Theorem 3.1.4 on the multifractality in the intermediate phase. Moreover, we argue that Rapenne's recent work [35] implies the absence of such an intermediate phase on trees with wired boundary conditions. In **Section 3.5** we show how to establish the results for the $\mathbb{H}^{2|2}$-model. For the near-critical asymptotics (Theorem 3.1.5) this is an easy consequence of a Dynkin isomorphism between the $\mathbb{H}^{2|2}$-model and the VRJP. For Theorem 3.1.6 on the intermediate phase, we make use of the STZ-field to connect the observable for the $\mathbb{H}^{2|2}$-model with the observable $\lim_{t\to\infty} L_t^0/t$ that we study for the VRJP.

[3] To our understanding, the cited sources consider an *inverse-order thermodynamic limit*, in which they remove the level-broadening (resp. magnetic field) before taking the system size to infinity. This corresponds to a *finite-volume limit*, as opposed to the reversed limit order considered in other treatments of the Anderson transition. In this sense, the different statements are not directly comparable.

## 3.2 Additional Background

### 3.2.1 Dynkin Isomorphism for the VRJP and the $\mathbb{H}^{2|2}$-Model

Analogous to the connection between the Gaussian free field and the (continuous-time) simple random walk, there is a Dynkin-type isomorphism theorem relating correlation functions of the $\mathbb{H}^{2|2}$-model with the local time of a VRJP.

**Theorem 3.2.1** ([3, Theorem 5.6]): Suppose $G = (V, E)$ is a finite graph with positive edge-weights $\{\beta_{ij}\}_{ij\in E}$. Let $\langle\cdot\rangle_{\beta,h}$ denote the expectation of the $\mathbb{H}^{2|2}$-model and suppose that under $\mathbb{E}_i$, the process $(X_t)_{t\geq 0}$ denotes a VRJP started from $i$. Suppose $g\colon \mathbb{R}^V \to \mathbb{R}$ is a smooth bounded function. Then, for any $i, j \in V$

$$\langle x_i x_j g(\mathbf{z}-1)\rangle_{\beta,h} = \int_0^\infty \mathbb{E}_i[g(\mathbf{L}_t)\mathbb{1}_{X_t=j}]e^{-ht}\,\mathrm{d}t\,, \tag{3.2.1}$$

where $\mathbf{L}_t = (L_t^x)_{x\in V}$ denotes the VRJP's local time field.

This result will be key to deduce Theorem 3.1.5 from Theorem 3.1.2.

### 3.2.2 VRJP as Random Walk in a $t$-Field Environment

As a continuous-time process, there is some freedom in the time-parametrisation of the VRJP. While the definition in (3.1.1) (the *linearly reinforced* timescale) is the “usual” parametrisation, we also make use of the *exchangeable timescale* VRJP $(\tilde{X}_t)_{t\in[0,+\infty)}$:

$$\tilde{X}_t := X_{A^{-1}(t)} \quad \text{with} \quad A(t) := \textstyle\int_0^t 2(1+L_s^{X_s})\,\mathrm{d}s = \sum_{x\in V}[(1+L_t^x)^2 - 1] \tag{3.2.2}$$

Writing $\tilde{L}_t^x = \int_0^t \mathbb{1}\{\tilde{X}_s = x\}\,\mathrm{d}s$, the local times in the two timescales are related by

$$L_t^x = \sqrt{1+\tilde{L}_t^x} - 1. \tag{3.2.3}$$

Above reparametrisation is motivated by the following result of Sabot and Tarrès [22], showing that the VRJP in exchangeable timescale can be seen as a (Markovian) random walk in random conductances given in terms of the $t$-field.

**Theorem 3.2.2** (VRJP as Random Walk in Random Environment [22]): Consider a finite graph $G = (V, E)$, a starting vertex $i_0 \in V$ and edge-weights $(\beta_e)_{e\in E}$. The exchangeable timescale VRJP, started at $i_0$, equals in law an (annealed) continuous-time Markov jump process, with jump rates between from $i$ to $j$ given by

$$\tfrac{1}{2}\beta_{ij}e^{T_j-T_i}, \tag{3.2.4}$$

where $\mathbf{T}=(T_x)_{x\in V}$ are random variables distributed according to the law of the $t$-field (3.1.22) pinned at $i_0$.

As a consequence of Theorem 3.2.2, the $t$-field can be recovered from the VRJP's asymptotic local time:

**Corollary 3.2.3** ($t$-field from Asymptotic Local Time [47]): Consider the setting of Theorem 3.2.2. Let $(L_t^x)_{x\in V}$ and $(\tilde{L}_t^x)_{x\in V}$ denote the local time field of the VRJP in linearly reinforced and exchangeable timescale, respectively. Then

$$\begin{aligned} T_i &:= \lim_{t\to\infty}\log\left(L_t^i/L_t^{i_0}\right) && (i\in V)\\ \tilde{T}_i &:= \tfrac{1}{2}\lim_{t\to\infty}\log\left(\tilde{L}_t^i/\tilde{L}_t^{i_0}\right) && (i\in V) \end{aligned} \tag{3.2.5}$$

exist and follow the law $Q_\beta^{(i_0)}$ of the $t$-field in (3.1.22).

*Proof.* For the exchangeable timescale, Sabot, Tarrès and Zeng [47, Theorem 2] provide a proof. The statement for the usual (linearly reinforced) VRJP then follows by the time change formula for local times (3.2.3). □

Considering the VRJP as a random walk in random environment enables us to study its local time properties with the tools of random conductance networks. For a $t$-field $\mathbf{T}=(T_x)_{x\in V}$ pinned at $i_0$, we refer to the collection of random edge weights (or *conductances*)

$$\{\beta_{ij}e^{T_i+T_j}\}_{ij\in E} \tag{3.2.6}$$

as the *$t$-field environment*. This should be thought of as a symmetrised version of the VRJP's random environment (3.2.4). It is easier to study a random walk with symmetric jump rates, since its amenable to the methods of conductance networks. The following lemma relates local times in the $t$-field environment with the local times in the environment of the exchangeable timescale VRJP:

**Lemma 3.2.4:** Consider the setting of Theorem 3.2.2. Let $(\tilde{X}_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ denote two continuous-time Markov jump processes started from $i_0$ with rates given by (3.2.4) and (3.2.6), respectively. We write $\tilde{L}_t^x$ and $l_t^x$ for their respective local time fields. Let $B\subseteq V$ and write $\tilde{\mathcal{T}}_B$ and $\mathcal{T}_B$ for the respective hitting times of $B$. Then

$$L^x_{\tilde{\mathcal{T}}_B} \overset{\text{law}}{=} 2e^{T_x} l^x_{\mathcal{T}_B}, \tag{3.2.7}$$

for $x \in V$. In particular, $L^{i_0}_{\tilde{\mathcal{T}}_B} \overset{\text{law}}{=} 2l^{i_0}_{\mathcal{T}_B}$.

*Proof.* The discrete-time processes associated to $(\tilde{X}_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ apparently agree. In particular, they both visit a vertex $x$ the same number of times, before hitting $B$. Every time $\tilde{X}_t$ visits the vertex $x$, it spends an $\mathrm{Exp}(\sum_y \frac{1}{2}\beta_{xy}e^{T_y-T_x})$-distributed time there, before jumping to another vertex. $Y_t$ on the other hand will spend time distributed as $\mathrm{Exp}(\sum_y \beta_{xy}e^{T_x+T_y}) = \frac{1}{2}e^{-2T_x}\mathrm{Exp}(\sum_y \frac{1}{2}\beta_{xy}e^{T_y-T_x})$. This concludes the proof. □

### 3.2.3 Effective Conductance

Our approach to proving Theorem 3.1.2 will rely on establishing asymptotics for the *effective conductance* in the $t$-field environment (Theorem 3.3.1).

**Definition 3.2.5:** Consider a locally finite graph $G = (V,E)$ with edge weights (or *conductances*) $\{w_{ij}\}_{ij\in E}$. For two disjoint sets $A, B \subseteq V$, the *effective conductance* between them is defined as

$$C^{\text{eff}}(A,B) := \inf_{\substack{U:\, V\to\mathbb{R} \\ U|_A\equiv 0,\, U|_B\equiv 1}} \sum_{ij\in E} w_{ij}\,(U(i)-U(j))^2. \tag{3.2.8}$$

The variational definition (3.2.8) makes it easy to deduce monotonicity and boundedness properties:

**Lemma 3.2.6:** Consider the situation of Definition 3.2.5. Suppose $S \subseteq E$ is a edge-cutset separating $A, B$. Then

$$C^{\text{eff}}(A,B) \leq \sum_{ij\in S} w_{ij}. \tag{3.2.9}$$

Alternatively, suppose $C \subseteq V$ is a vertex-cutset separating $A, B$. Then

$$C^{\text{eff}}(A,B) \leq C^{\text{eff}}(A,C). \tag{3.2.10}$$

*Proof.* For the first statement, consider (3.2.8) for the function $U: V \to \mathbb{R}$ that is constant zero (resp. one) in the component of $A$ (resp. $B$) in $V\backslash S$. For the second statement, note that for any funcion $U: V \to \mathbb{R}$ with $U|_A \equiv 0$ and $U|_C \equiv 1$ we can define a function $\tilde{U}$ that agrees with $U$ on $C$ and the connected compenent of $V \setminus C$ containing $A$, and is constant equal to one on the component of $B$ in $V \setminus V$. Then, $\tilde{U}|_A \equiv 0$ and $\tilde{U}|_B \equiv 1$ and $\sum_{ij\in E} w_{ij}(U(i)-U(j))^2 \le \sum_{ij\in E} w_{ij}(\tilde{U}(i)-\tilde{U}(j))^2$, which proves the claim. □

The monotoniciy in (3.2.10) makes it possible to define an effective conductance *to infinity*. For an increasing exhaustion $V_1 \subseteq V_2 \subseteq \cdots$ of the vertex set $V = \bigcup_n V_n$ and a given finite set $A \subseteq V$, we define the *effective conductance from A to infinity* by

$$C_\infty^{\mathrm{eff}}(A) = \lim_{n\to\infty} C^{\mathrm{eff}}(A, V \setminus V_n). \tag{3.2.11}$$

One may check that this is independent from the choice of exhaustion. For us, the main use of effective conductances stems from their relation to *escape times*:

**Lemma 3.2.7:** Consider a locally finite graph $G = (V,E)$ with edge weights (or *conductances*) $\{w_{ij}\}_{ij\in E}$. Let $C^{\mathrm{eff}}(i_0, B)$ denote the effective conductance between the singleton $\{i_0\}$ and a disjoint set $B$. Consider a continuous-time random walk $(X_t)_{t\ge 0}$ on $G$, starting at $X_0 = i_0$ and jumping from $X_t = i$ to $j$ at rate $w_{ij}$. Let $L_{\mathrm{esc}}(i_0, B)$ denote the total time the walk spends at $i_0$ before visiting $B$ for the first time. Then $L_{\mathrm{esc}}(i_0, B)$ is distributed as an $\mathrm{Exp}(1/C^{\mathrm{eff}}(i_0,B))$-random variable.

For an infinite graph $G$, the above conclusions also hold for $B$ "at infinity": We let $L_{\mathrm{esc},\infty}(i_0)$ denote the total time spent at $i_0$ and understand $C_\infty^{\mathrm{eff}}(i_0)$ as in (3.2.11). Then $L_{\mathrm{esc},\infty}(i_0) \sim \mathrm{Exp}(1/C_\infty^{\mathrm{eff}}(i_0))$.

*Proof.* According to [23, Section 2.2], the walk's number of visits at $i_0$ before hitting $B$ is a geometric random variable $N \sim \mathrm{Geo}(p_{\mathrm{esc}})$ with the *escape probability* $p_{\mathrm{esc}} = C^{\mathrm{eff}}(i_0,B)/(\sum_{j\sim i_0} w_{i_0 j})$. Moreover, for the continuous-time process, every time we visit $i_0$ we spend an $\mathrm{Exp}(\sum_{j\sim i_0} w_{i_0 j})$-distributed time there, before jumping to a neighbour. Hence, $L_{\mathrm{esc}}(i_0,B)$ is distributed as the sum of $N$ independent $\mathrm{Exp}(\sum_{j\sim i_0} w_{i_0 j})$-distributed random variables. By standard results for the exponential distribution (easily checked via its moment-generating function), this implies the claim. Note that this argument also holds true for $B$ "at infinity", in which case $N \sim \mathrm{Geo}(p_{\mathrm{esc}})$ with $p_{\mathrm{esc}} = C_\infty^{\mathrm{eff}}(i_0)/(\sum_{j\sim i_0} w_{i_0 j})$ will simply denote the total number of visits at $i_0$ (see [23, Section 2.2] for more details). □

### 3.2.4 The $t$-Field from the $\mathbb{H}^{2|2}$- and STZ-Anderson Model

**$t$-Field as a Horospherical Marginal of the $\mathbb{H}^{2|2}$-model** First we introduce *horospherical coordinates* on $\mathbb{H}^{2|2}$. In these coordinates, $\mathbf{u} \in \mathbb{H}^{2|2}$ is parametrised by $(t, s, \bar{\psi}, \psi)$, with $t, s \in \mathbb{R}$

and Grassmann variables $\bar{\psi}, \psi$ via

$$\begin{pmatrix} z \\ x \\ y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cosh(t) + e^t(\frac{1}{2}s^2 + \bar{\psi}\psi) \\ \sinh(t) - e^t(\frac{1}{2}s^2 + \bar{\psi}\psi) \\ e^t s \\ e^t \bar{\psi} \\ e^t \psi \end{pmatrix}. \tag{3.2.12}$$

A particular consequence of this is that $e^t = z + x$. By rewriting the Gibbs measure for the $\mathbb{H}^{2|2}$-model, defined in (3.1.10), in terms of horospherical coordinates and integrating out the fermionic variables $\psi, \bar{\psi}$, one obtains a marginal density in $\underline{t} = \{t_x\}_{x \in V}$ and $\underline{s} = \{s_x\}_{x \in V}$, which can be interpreted probabilistically:

**Lemma 3.2.8** (Horospherical Marginal of the $\mathbb{H}^{2|2}$-Model [2, 27, 28].): Consider a finite graph $G = (V, E)$, a vertex $i_0 \in V$, and non-negative edge-weights $(\beta_{ij})_{ij \in E}$. There exist random variables $\underline{T} = \{T_x\}_{x \in V} \in \mathbb{R}^V$ and $\underline{S} = \{S_x\}_{x \in V} \in \mathbb{R}^V$, such that for any $F \in C_c^\infty(\mathbb{R}^V \times \mathbb{R}^V)$

$$\langle F(\underline{t}, \underline{s}) \rangle_\beta = \mathbb{E}[F(\underline{T}, \underline{S})]. \tag{3.2.13}$$

The law of $\underline{T}$ is given by the $t$-field pinned at $i_0$ (see Definition 3.1.7). Moreover, conditionally on $\underline{T}$, the *s-field* follows the law of a Gaussian free field in conductances $\{\beta_{ij} e^{T_i + T_j}\}_{ij \in E}$, pinned at $i_0$, $S_{i_0} = 0$.

***t*-Field and the STZ-Anderson Model.** It turns out that the (zero-energy) Green's function of the STZ-Anderson model is directly related to the $t$-field:

**Proposition 3.2.9** ([47]): For $H_B$ denoting the STZ-Anderson model as in Definition 3.1.8 define the Green's function $G_B(i,j) = [H_B^{-1}]_{i,j}$. For a vertex $i_0 \in V$, define $\{T_i\}_{i \in \Lambda}$ via

$$e^{T_i} := G_B(i_0, i) / G_B(i_0, i_0). \tag{3.2.14}$$

Then $\{T_i\}$ follows the law $Q_\beta^{(i_0)}$ of the $t$-field, pinned at $i_0$. Moreover, with $\{T_i\}$ as above we have $B_i = \sum_{j \sim i} \beta_{ij} e^{T_j - T_i}$ for all $i \in V \setminus \{i_0\}$.

This provides a way of coupling the STZ-field with the $t$-field, as well as a coupling of $t$-fields pinned at different vertices.

**Remark 3.2.10** (Natural Coupling): Lemma 3.2.8 and Proposition 3.2.9 give us a way to define a *natural coupling* of STZ-field, $t$-field and $s$-field as follows: Fix some pinning vertex $i_0 \in V$.

Sample an STZ-Anderson model $H_B$ with respect to edge weights $\{\beta_{ij}\}_{ij\in E}$. Then define the $t$-field $\{T_i\}_{i\in V}$, pinned at $i_0$ via (3.2.14). Then, conditionally on the $t$-field, sample the $s$-field $\{S_i\}_{i\in V}$ as a Gaussian free field in conductances $\{\beta_{ij}e^{T_i+T_j}\}_{ij\in E}$, pinned at $i_0$, $S_{i_0} = 0$.

### 3.2.5 Monotonicity Properties of the $t$-Field

A rather surprising property of the $t$-field, proved by the first author, is the monotonicity of various expectation values with respect to the edge-weights. The following is a restatement of [25, Theorem 6] after applying Proposition 3.2.9:

**Theorem 3.2.11** ([25, Theorem 6]): Consider a finite graph $G = (V, E)$ and fix some vertex $i_0 \in V$. Under $\mathbb{E}_{\boldsymbol{\beta}}$, we let $\mathbb{T} = \{T_i\}_{i\in V}$ denote a $t$-field pinned at $i_0$ with respect to non-negative edge weights $\boldsymbol{\beta} = \{\beta_e\}_{e\in E}$. Then, for any convex $f\colon [0,\infty) \to \mathbb{R}$ and non-negative $\{\lambda_i\}_{i\in V}$, the map

$$\boldsymbol{\beta} \mapsto \mathbb{E}_{\boldsymbol{\beta}}[f(\textstyle\sum_i \lambda_i e^{T_i})] \tag{3.2.15}$$

is decreasing.

A direct corollary of the above is that expectations of the form $\mathbb{E}_\beta[e^{\eta T_x}]$ are increasing in $\beta$ for $\eta \leq [0,1]$ and are decreasing for $\eta \geq 1$. This will be the extent to which we make use of the result.

### 3.2.6 The $t$-Field on $\mathbb{T}_d$

Consider the $t$-field measure (3.1.22) on $\mathbb{T}_{d,n} = (V_{d,n}, E_{d,n})$, the rooted $(d+1)$-regular tree of depth $n$, pinned at the root $i_0 = 0$. Only one term contributes to the determinantal term (3.1.23), namely the term corresponding to $\mathbb{T}_{d,n}$ itself, oriented away from the root:

$$Q^{(0)}_{\beta;\mathbb{T}_{d,n}}(\mathrm{d}\mathbf{t}) = e^{-\sum_{(i,j)\in\vec{E}_{d,n}}[\beta(\cosh(t_j-t_i)-1)+\frac{1}{2}(t_j-t_i)]}\delta(t_0)\prod_{i\in V_{d,n}\setminus 0}\frac{\mathrm{d}t_i}{\sqrt{2\pi/\beta}}, \tag{3.2.16}$$

where $\vec{E}_{d,n}$ is the set of edges in $\mathbb{T}_{d,n}$ oriented away from the root. In other words, the increments of the $t$-field along outgoing edges are i.i.d. and distributed according to the following:

**Definition 3.2.12** ($t$-field Increment Measure): For $\beta > 0$ define the probability distribution

$$Q^{\mathrm{inc}}_\beta(\mathrm{d}t) = e^{-\beta[\cosh(t)-1]-t/2}\frac{\mathrm{d}t}{\sqrt{2\pi/\beta}} \quad \text{with} \quad t\in\mathbb{R}. \tag{3.2.17}$$

We refer to this as the *t-field increment distribution* and if not specified otherwise, $T$ will always denote a random variable with distribution $Q_\beta^{\mathrm{inc}}$. The dependence on $\beta$ is either implicit or denoted by a subscript, such as in $\mathbb{E}_\beta$ or $\mathbb{P}_\beta$.

The density (3.2.17) implies that

$$e^T \sim \mathrm{IG}(1,\beta) \quad \text{and} \quad e^{-T} \sim \mathrm{RIG}(1,\beta), \tag{3.2.18}$$

where IG (RIG) denotes the (reciprocal) inverse Gaussian distribution (*cf.* (3.6.4)). Note that changing variables to $t \mapsto e^t$ and comparing to the density of the inverse Gaussian, we see that (3.2.17) is normalised.

**Definition 3.2.13** (Free Infinite Volume $t$-field on $\mathbb{T}_d$): For $\beta > 0$, associate to every edge $e$ of the infinite rooted $(d+1)$-regular tree $\mathbb{T}_d$ a $t$-field increment $\tilde{T}_e$, distributed according to (3.2.17). For every vertex $x \in \mathbb{T}_d$ let $\gamma_x$ denote the unique self-avoiding path from 0 to $x$ and define $T_x := \sum_{e\in\gamma_x} \tilde{T}_e$. The random field $\{T_x\}_{x\in\mathbb{T}_d}$ is the *free infinite volume t-field* on $\mathbb{T}_d$ at inverse temperature $\beta > 0$. In particular, its restriction $\{T_x\}_{x\in\mathbb{T}_{d,n}}$ onto vertices up to generation $n$ follows the law $Q^{(0)}_{\beta;\mathbb{T}_{d,n}}$.

By construction, $\{T_x\}_{x\in\mathbb{T}_d}$ can be considered a branching random walk (BRW) with a deterministic number of offsprings (every particle gives rise to $d$ new particles in the next generation). In Section 3.2.8 we will elaborate on this perspective.

### 3.2.7 Previous Results for VRJP on Trees.

As we have already noted in the introduction, the VRJP on tree graphs has received quite some attention [4, 32–35]. In particular, Basdevant and Singh [4] studied the VRJP on Galton-Watson trees with general offspring distribution, and exactly located the recurrence/transience phase transition:

**Proposition 3.2.14** (Basdevant-Singh [4]): Let $\mathcal{T}$ denote a Galton-Watson tree with mean offspring $b > 1$. Consider the VRJP started from the root of $\mathcal{T}$, conditionally on non-extinction of the tree. There exists a critical parameter $\beta_c = \beta_c(b)$, such that the VRJP is

- recurrent for $\beta \le \beta_c$,
- transient for $\beta > \beta_c$.

Moreover, $\beta_c$ is characterised as the unique positive solution to

$$\frac{1}{b} = \sqrt{\frac{\beta_c}{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-\beta_c(\cosh(t)-1)}. \tag{3.2.19}$$

We also take the opportunity to highlight Rapenne's recent results [35] concerning the (sub)critical phase, $\beta \le \beta_c$. His statements can be seen to complement our results, which focus on the supercritical phase $\beta > \beta_c$.

### 3.2.8 Background on Branching Random Walks

Let's quickly recall some basic results from the theory of branching random walks. For a more comprehensive treatment we refer to Shi's monograph [74].

A *branching random walk* (BRW) with offspring distribution $\mu \in \mathrm{Prob}(\mathbb{N}_0)$ and increment distribution $\nu$ is constructed as follows: We start with a "root" particle $x = 0$ at generation $|0| = 0$ and starting position $V(0) = v_0$. We sample its number of offsprings according to $\mu$. They constitute the particles at generation one, $\{|x| = 1\}$. Every such particle is assigned a position $v_0 + \delta V_x$ with $\{\delta V_x\}_{|x|=1}$ being i.i.d. according to the increment distribution $\nu$. This process is repeated recursively and we end up with a random collection of particles $\{x\}$, each equipped with a position $V(x) \in \mathbb{R}$, a *generation* $|x| \in \mathbb{N}_0$ and a *history* $0 = x_0, x_1, \ldots, x_{|x|} = x$ of predecessors. Unless otherwise stated, we assume from now on that a BRW always starts from the origin, $v_0 = 0$.

A particularly useful quantity for the study of BRWs is the log-Laplace transform of the offspring process:

$$\psi(\eta) := \log \mathbb{E}\Big[\sum_{|x|=1} e^{-\eta V(x)}\Big], \tag{3.2.20}$$

where the sum goes over all particles in the first generation. A priori, we have $\psi(\eta) \in [0, \infty]$, but we typically assume $\psi(0) > 0$ and $\inf_{\eta>0} \psi(\eta) < \infty$. The first assumption corresponds to supercriticality of the offspring distribution[4], whereas the second assumption enables us to study the average over histories of the BRW in terms of single random walk:

**Proposition 3.2.15** (Many-To-One Formula): Consider a BRW with log-Laplace transform $\psi(\eta)$. Choose $\eta > 0$ such that $\psi(\eta) < \infty$ and define a random walk $0 = S_0, S_1, \ldots$ with i.i.d. increments such that for any measurable $h \colon \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[h(S_1)] = \mathbb{E}\left[\textstyle\sum_{|x|=1} e^{-\eta V(x)} h(V(x))\right] \Big/ \mathbb{E}\left[\textstyle\sum_{|x|=1} e^{-\eta V(x)}\right]. \tag{3.2.21}$$

Then, for all $n \ge 1$ and $g \colon \mathbb{R}^n \to [0, \infty)$ measurable we have

$$\mathbb{E}\left[\textstyle\sum_{|x|=n} g(V(x_1), \ldots, V(x_n))\right] = \mathbb{E}\left[e^{n\psi(\eta) + \eta S_n} g(S_1, \ldots, S_n)\right]. \tag{3.2.22}$$

[4] Here we mean supercriticality in the sense of Galton-Watson trees. In other words, with positive probability the BRW consists of infinitely many particles. We also say that the BRW does not go extinct.

For a proof we refer to Shi's lecture notes [74, Theorem 1.1]. An application of the many-to-one formula is the following statement about the velocity of extremal particles (*cf.* [74, Theorem 1.3]).

**Proposition 3.2.16** (Asymptotic Velocity of Extremal Particles): Suppose $\psi(0) > 0$ and $\inf_{\eta>0} \psi(\eta) < \infty$. Then, almost surely under the event of non-extinction, we have

$$\lim_{n\to\infty} \frac{1}{n} \inf_{|x|=n} V(x) = -\inf_{\eta>0} \psi(\eta)/\eta. \tag{3.2.23}$$

**Critical Branching Random Walks.** A common assumption, under which BRWs exhibit various universal properties, is $\psi(1) = \psi'(1) = 0$. While not common terminology in the literature, we will refer to this as *criticality*:

$$\text{BRW with } \psi(\eta) = \log \mathbb{E}[\textstyle\sum_{|x|=1} e^{-\eta V(x)}] \text{ is } \textit{critical} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \psi(1) = \psi'(1) = 0 \tag{3.2.24}$$

This definition can be motivated by considering the many-to-one formula (Proposition 3.2.15) applied to a critical BRW for $\eta = 1$: In that case, the random walk $S_i$ has mean zero increments, $\mathbb{E}[S_1] = -\psi'(1) = 0$, and the exponential drift in (3.2.22) vanishes, $e^{n\psi(1)} = 1$. Consequently, as far as the many-to-one formula is concerned, critical BRWs inherit some of the universality of mean zero random walks (*e.g.* Donsker's theorem, say under an additional second moment assumption). Moreover, the notion of criticality is particularly useful, since in many cases we can reduce a BRW to the critical case by a simple rescaling/drift transformation:

**Lemma 3.2.17** (Critical Rescaling of a BRW): Consider a BRW with log-Laplace transform $\psi(\eta) = \log \mathbb{E}[\sum_{|x|=1} e^{-\eta V(x)}]$. Suppose there exists $\eta^* > 0$ solving the equation

$$\psi(\eta^*) = \eta^* \psi'(\eta^*). \tag{3.2.25}$$

Equivalently, $\eta^*$ is a critical point for $\eta \to \psi(\eta)/\eta$. Define a BRW with the same particles $\{x\}$ and rescaled positions

$$V^*(x) = \eta^* V(x) + \psi(\eta^*)|x|. \tag{3.2.26}$$

The resulting BRW is critical.

*Proof.* Write $\psi^*(\gamma) = \log \mathbb{E} \sum_{|x|=1} e^{-\gamma V^*(x)}$ for the log-Laplace transform of the rescaled BRW. We easily check

$$\begin{aligned}\psi^*(1) = \log \mathbb{E} \sum_{|x|=1} e^{-\eta^* V(x) - \psi(\eta^*)} &= -\psi(\eta^*) + \log \mathbb{E} \sum_{|x|=1} e^{-\eta^* V(x)} \\ &= -\psi(\eta^*) + \psi(\eta^*) = 0.\end{aligned} \tag{3.2.27}$$

Equivalently, $1 = \mathbb{E} \sum_{|x|=1} e^{-\eta^* V(x) - \psi(\eta^*)}$, which together with (3.2.25) yields

$$\begin{aligned}(\psi^*)'(1) &= -\frac{\mathbb{E} \sum_{|x|=1} (\eta^* V(x) + \psi(\eta^*)) e^{-\eta^* V(x) - \psi(\eta^*)}}{\mathbb{E} \sum_{|x|=1} e^{-\eta^* V(x) - \psi(\eta^*)}} \\ &= -\eta^* \mathbb{E} \sum_{|x|=1} V(x) e^{-\eta^* V(x)} - \psi(\eta^*) \\ &= \eta^* \psi'(\eta^*) - \psi(\eta^*) = 0,\end{aligned} \tag{3.2.28}$$

which concludes the proof. □

## 3.3 VRJP and the $t$-Field as $\beta \searrow \beta_c$

The main goal of this section is to prove Theorem 3.1.2 on the asymptotic escape time of the VRJP as $\beta \searrow \beta_c$. The main work will be in establishing the following result on the effective conductance in a $t$-field environment:

**Theorem 3.3.1** (Near-Critical Effective Conductance): Let $\{T_x\}_{x\in\mathbb{T}_d}$ denote the (free) $t$-field on $\mathbb{T}_d$, pinned at the origin. Let $C_\infty^{\mathrm{eff}}$ denote the effective conductance from the origin to infinity in the network given by conductances $\{\beta e^{T_i+T_j}\mathbb{1}_{i\sim j}\}_{i,j\in\mathbb{T}_d}$. There exist constants $c, C > 0$ such that

$$\exp[-(C+o(1))/\sqrt{\epsilon}] \le \mathbb{E}_{\beta_c+\epsilon}[C_\infty^{\mathrm{eff}}] \le \exp[-(c+o(1))/\sqrt{\epsilon}], \tag{3.3.1}$$

as $\epsilon \searrow 0$, where $\beta_c = \beta_c(d) > 0$ is given by Proposition 3.2.19.

For establishing this result, the BRW perspective onto the $t$-field is essential. The lower bound will follow from a mild modification of a result by Gantert, Hu and Shi [75] (see Theorem 3.3.8). For the upper bound we will consider the critical rescaling of the near-critical $t$-field (*cf.* Lemma 3.2.17). The bound will then follow by a perturbative argument applied to a result on effective conductances in a *critical* BRW environment. The latter we prove in a more general form, for which it is convenient to introduce some additional notions.

For a random variable $V$ and a fixed offspring degree $d$ we write

$$\psi_V(\eta) \coloneqq \log(d\,\mathbb{E}[e^{-\eta V}]). \tag{3.3.2}$$

Analogous to Definition 3.2.13, for an increment distribution given by $V$, we define a random field $\{V_x\}_{x\in\mathbb{T}_d}$ and refer to it as the *BRW with increments* $V$. We say that $V$ is a *critical increment* if $\{V_x\}_{x\in\mathbb{T}_d}$ is critical, *i.e.* $\psi_V(1) = \psi_V'(1) = 0$. Note that this implicitly depends on our choice of $d \ge 2$, but we choose to suppress this dependency. For a critical increment $V$ we write

$$\sigma_V^2 \coloneqq \psi_V''(1) = d\,\mathbb{E}[V^2 e^{-V}]. \tag{3.3.3}$$

Note that this is the variance of the (mean-zero) increments of the random walk $(S_i)_{i\ge 0}$ given by the many-to-one formula (Proposition 3.2.15 for $\eta = 1$).

**Theorem 3.3.2:** Fix some offspring degree $d \ge 2$ and consider a critical increment $V$ with $\sigma_V^2 < \infty$ and $\psi_V(1+2a) < \infty$ for some constant $a > 0$. Write $\{V_x\}_{x\in\mathbb{T}_d}$ for the BRW with increments $V$ and define the conductances $\{e^{-\gamma(V_x+V_y)}\}_{xy}$. Let $C_{n,\gamma}^{\mathrm{eff}}$ denote the effective conductance between the origin 0 and the vertices in the $n$-th generation. Then, for $\gamma \in$

$(1/2, 1/2+a)$, we have

$$\mathbb{E}[C^{\text{eff}}_{n,\gamma}] \leq \exp\Big[-\big[\min(\tfrac{1}{4},\gamma-\tfrac{1}{2})\,(\pi^2\sigma_V^2)^{1/3}+o(1)\big]n^{1/3}\Big] \quad \text{as} \quad n\to\infty. \tag{3.3.4}$$

Moreover, this is uniform with respect to $\gamma$, $\sigma_V^2$ and $\psi_V(1+2a)$ in the following sense: Suppose there is a family $V^{(k)}$, $k\in\mathbb{N}$, of critical increments and define $C^{\text{eff}}_{n,\gamma;k}$ as above. Further assume $0<\inf_k \sigma^2_{V^{(k)}} \leq \sup_k \sigma^2_{V^{(k)}} < \infty$ and $\sup_k \psi_{V^{(k)}}(1+2a)<\infty$. Then we have

$$\limsup_{n\to\infty}\sup_k \sup_{\frac{1}{2}<\gamma<\frac{1}{2}+a}\left(n^{-1/3}\log\mathbb{E}[C^{\text{eff}}_{n,\gamma;k}]+\min(\tfrac{1}{4},\gamma-\tfrac{1}{2})\,(\pi^2\sigma^2_{V^{(k)}})^{1/3}\right)\leq 0. \tag{3.3.5}$$

We note that random walk in (critical) multiplicative environments on trees has previously been studied, see for example [76–81]. In particular, Hu and Shi [79, Theorem 2.1] established bounds analogous to (3.3.4) for escape probabilities, instead of effective conductances. While the quantities are related, bounds on the expected escape probability do not directly translate into bounds for the expected effective conductance. Moreover, their setup for the random environment does not directly apply to our setting[5]. Last but not least, for our applications, we require additional uniformity of the bounds with respect to the underlying BRW.

### 3.3.1 The $t$-Field as a Branching Random Walk

Considered as a BRW, the $t$-field $\{T_x\}_{x\in\mathbb{T}_d}$ on the rooted $(d+1)$-regular tree $\mathbb{T}_d$ (or more precisely the negative $t$-field) has a log-Laplace transform given by

$$\psi_\beta(\eta) := \log\mathbb{E}\Big[\sum_{|x|=1} e^{\eta T_x}\Big] = \log(d\,\mathbb{E}_\beta[e^{\eta T}]) \qquad (\eta>0), \tag{3.3.6}$$

where $T$ denotes the $t$-field increment as introduced in Definition 3.2.12. One can check easily that $\psi_\beta(0)=\psi_\beta(1)=\log d$. More generally, using the density for $T$ we have

$$\psi_\beta(\eta) = \log\Big(d\int \frac{\mathrm{d}t}{\sqrt{2\pi/\beta}} e^{-\beta[\cosh(t)-1]-(\frac{1}{2}-\eta)t}\Big) = \log\Big(\frac{d\sqrt{2\beta}e^{\beta}}{\sqrt{\pi}} K_{\eta-\frac{1}{2}}(\beta)\Big) \tag{3.3.7}$$

where $K_\alpha$ denotes the modified Bessel function of second kind. An illustration of $\psi_\beta$ for different values of $\beta$ is given in Figure 3.4. In particular, it's a smooth function in $\beta,\eta>0$ and

[5] Roughly speaking, they are working with weights $\{e^{-\gamma V_x}\}_{(x,y)\in\vec{E}(\mathbb{T}_d)}$ while we consider the "symmetrised" variant $\{e^{-\gamma(V_x+V_y)}\}_{xy\in E(\mathbb{T}_d)}$.

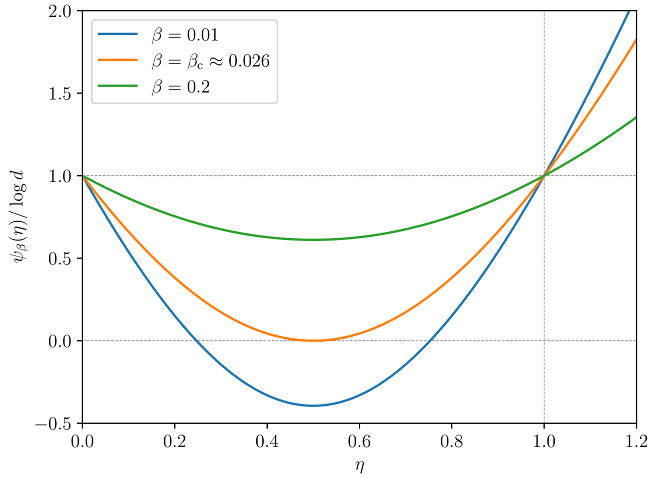



Figure 3.4: Illustration of $\psi_\beta(\eta)/\log(d)$ for $d = 2$ at different values of $\beta$. Its minimum is always at $\eta = 1/2$, and the value of this minimum is increasing with $\beta$. It is equal to zero at $\beta = \beta_c$.

one may check that it's strictly convex since

$$\psi''_\beta(\eta) = \frac{\mathbb{E}_\beta[T^2 e^{\eta T}]}{\mathbb{E}_\beta[e^{\eta T}]} - \frac{\mathbb{E}_\beta[T e^{\eta T}]^2}{\mathbb{E}_\beta[e^{\eta T}]^2} > 0 \tag{3.3.8}$$

equals the variance of a non-deterministic random variable. Moreover, by the symmetry and monotonicity properties of the Bessel function ($K_\alpha = K_{-\alpha}$ and $K_\alpha \le K_{\alpha'}$ for $0 \le \alpha \le \alpha'$), the infimum of $\psi_\beta(\eta)$ is attained at $\eta = 1/2$:

$$\inf_{\eta>0} \psi_\beta(\eta) = \psi_\beta(1/2) = \log(d\,\mathbb{E}_\beta[e^{T/2}]) = \log(\frac{\sqrt{2\beta}e^\beta d}{\sqrt{\pi}} K_0(\beta)) \tag{3.3.9}$$

The critical inverse temperature $\beta_c = \beta_c(d) > 0$, as given in Proposition 3.2.14, is equivalently characterised by the vanishing of this infimum:

$$\psi_{\beta_c}(1/2) = \inf_{\eta>0} \psi_{\beta_c}(\eta) = 0. \tag{3.3.10}$$

In particular, by Lemma 3.2.17, this implies that $\{-\frac{1}{2}T_x\}_{x\in\mathbb{T}_d}$ is a critical BRW at $\beta = \beta_c$. More generally, it will be useful to consider critical rescalings of $\{T_x\}$ for general $\beta > 0$. For this we write

$$\eta_\beta := \operatorname{argmin}_{\eta>0} \frac{\psi_\beta(\eta)}{\eta} \quad \text{and} \quad \gamma_\beta := \inf_{\eta>0} \frac{\psi_\beta(\eta)}{\eta} = \frac{\psi_\beta(\eta_\beta)}{\eta_\beta}. \tag{3.3.11}$$

An illustration of these quantities is given in Figure 3.5. If $\eta_\beta$ as above is well-defined, then it satisfies (3.2.25) and hence by Lemma 3.2.17 the rescaled field

$$\tau_x^\beta = -\eta_\beta T_x + \psi_\beta(\eta_\beta)|x| \tag{3.3.12}$$

defines a critical BRW. The following lemma lends rigour to this:

**Lemma 3.3.3:** $\eta_\beta$ as given in (3.3.11) is well-defined and the unique positive root of the strictly increasing map $\eta \mapsto \eta\psi_\beta'(\eta) - \psi_\beta(\eta)$. Consequently, the maps $\beta \mapsto \eta_\beta$ and $\beta \mapsto \gamma_\beta$ are continuously differentiable.

*Proof.* Recall the Bessel function asymptotics $K_\alpha(\beta) \sim \frac{1}{2}(2/\beta)^\alpha\Gamma(\alpha)$ as $\alpha \to \infty$, hence by (3.3.7) we have $\psi_\beta(\eta) \sim \eta\log\eta$ for $\eta \to \infty$. Consequently, $\psi_\beta(\eta)/\eta$ diverges as $\eta \to \infty$ (and it also diverges as $\eta \searrow 0$). Hence it attains its infimum at some finite value. We claim that there is a unique minimiser $\eta_\beta$. Since $\psi_\beta(\eta)/\eta$ is continuously differentiable in $\eta > 0$, at any minimum it will have vanishing derivative $\partial_\eta(\psi_\beta(\eta)/\eta) = [\eta\psi_\beta'(\eta) - \psi_\beta(\eta)]/\eta^2$. And in fact the map $\eta \mapsto \eta\psi_\beta'(\eta) - \psi_\beta(\eta)$ is strictly increasing, since its derivative equals $\eta\psi_\beta''(\eta) > 0$, see (3.3.8), and as such has at most one root. This implies that $\eta_\beta$ as in (3.3.11) is well-defined and the unique root of $\eta\psi_\beta'(\eta) - \psi_\beta(\eta)$.

Continuous differentiability of $\beta \mapsto \eta_\beta$ follows from the implicit function theorem applied to $f(\eta,\beta) := \eta\psi_\beta'(\eta) - \psi_\beta(\eta)$, noting that $\partial_\eta f(\eta,\beta) = \eta\psi_\beta''(\eta) > 0$. This directly implies continuous differentiability of $\beta \mapsto \gamma_\beta = \psi_\beta(\eta_\beta)/\eta_\beta$

□

Considering the graphs in Figure 3.5, one would conjecture that $\eta_\beta$ is strictly increasing in $\beta$. One can apply the implicit function theorem to $f(\eta,\beta) := \eta\psi_\beta'(\eta) - \psi_\beta(\eta)$ to obtain

$$\frac{\mathrm{d}\eta_\beta}{\mathrm{d}\beta} = -\frac{[\partial_\beta f](\eta_\beta,\beta)}{[\partial_\eta f](\eta_\beta,\beta)} = \frac{[\partial_\beta\psi_\beta](\eta_\beta) - \eta_\beta[\partial_\beta\psi_\beta'](\eta_\beta)}{\eta_\beta\psi_\beta''(\eta_\beta)}. \tag{3.3.13}$$

The denominator is positive by (3.3.8), but we are not aware how to show non-negativity of the numerator for general $\beta$. We can however make use of this for the special case $\beta = \beta_\mathrm{c}$, which will be relevant in Section 3.3.3, in order to prove Theorem 3.3.1.

**Proposition 3.3.4:** Let $\psi_\beta(\eta)$ and $\eta_\beta$ be as in (3.3.7) and (3.3.11), for some $d \geq 2$. For $\beta_\mathrm{c} = \beta_\mathrm{c}(d) > 0$, as given in Proposition 3.2.14, we have $\eta_{\beta_\mathrm{c}} = 1/2$ and

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Big|_{\beta=\beta_\mathrm{c}}\eta_\beta > 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\beta}\Big|_{\beta=\beta_\mathrm{c}}\psi_\beta(\eta_\beta) > 0 \tag{3.3.14}$$

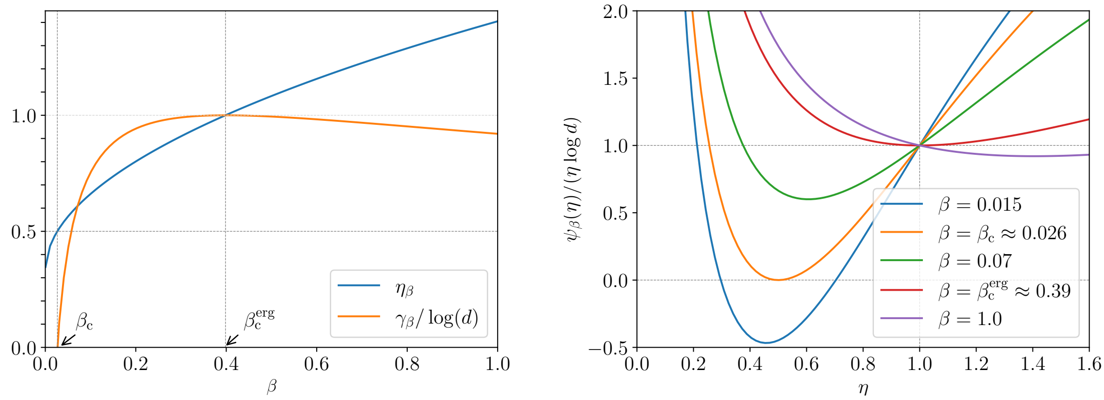



Figure 3.5: Illustration of $\eta_\beta$, $\gamma_\beta/\log d$ and $\psi_\beta(\eta)/(\eta \log d)$ for $d = 2$. For the figure on the left, note that $\gamma_\beta$ is positive for $\beta > \beta_c$ and attains its maximum at $\beta_c^{erg}$, at the same point at which $\eta_\beta = 1$. The right figure illustrates the same point: The minima of $\psi_\beta(\eta)/\eta$ move to the right with increasing $\beta$ and attain their highest value at $\beta = \beta_c^{erg}$.

*Proof.* By (3.3.10) we have $\frac{1}{2}\psi'_{\beta_c}(\frac{1}{2}) - \psi_{\beta_c}(\frac{1}{2}) = -\psi_{\beta_c}(\frac{1}{2}) = 0$. Lemma 3.3.3 therefore implies $\eta_{\beta_c} = 1/2$. Applying (3.3.13) and recalling $\psi'_\beta(\frac{1}{2}) = 0$, we get

$$\frac{\mathrm{d}\eta_\beta}{\mathrm{d}\beta}\bigg|_{\beta=\beta_c} = \frac{\partial_\beta|_{\beta=\beta_c}\psi_\beta(\frac{1}{2})}{\frac{1}{2}\psi''_\beta(\eta_\beta)}. \tag{3.3.15}$$

The denominator is positive by (3.3.8). As for the numerator, we recall (3.3.7) for $\eta = 1/2$:

$$\psi_\beta(\tfrac{1}{2}) = \log\left(d\int\sqrt{\frac{\beta}{2\pi}}e^{-\beta(\cosh(t)-1)}\,\mathrm{d}t\right). \tag{3.3.16}$$

To see monotonicity of the integral in $\beta$ it is convenient to apply the change of variables.

$$\begin{aligned} u &= e^{t/2} - e^{-t/2} = 2\sinh(t/2) \iff t = 2\,\mathrm{arsinh}(u/2) \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{1}{2}(e^{t/2} + e^{-t/2}) = \sqrt{1+u^2/4} \end{aligned} \tag{3.3.17}$$

Note that $u^2/2 = \frac{1}{2}(e^t + e^{-t}) - 1 = \cosh(t) - 1$, hence

$$\begin{aligned} \int\sqrt{\frac{\beta}{2\pi}}e^{-\beta(\cosh(t)-1)}\,\mathrm{d}t &= \int\sqrt{\frac{\beta}{2\pi}}e^{-\frac{\beta}{2}u^2}\frac{2}{\sqrt{u^2+4}}\,\mathrm{d}u \\ &= \int\sqrt{\frac{1}{2\pi}}e^{-\frac{1}{2}s^2}\frac{2}{\sqrt{s^2/\beta+4}}\,\mathrm{d}s\,. \end{aligned} \tag{3.3.18}$$

Clearly, the integrand in the last line is strictly increasing in $\beta$, hence $\partial_\beta \psi_\beta(\frac{1}{2}) > 0$. This implies the first statement in (3.3.14). For the second statement note that $\psi'_{\beta_c}(\frac{1}{2}) = 0$. Hence, $\partial_\beta|_{\beta=\beta_c}\psi_\beta(\eta_\beta) = \partial_\beta|_{\beta=\beta_c}\psi_\beta(\frac{1}{2}) > 0$. □

As already suggested in Figure 3.5, there is a second natural transition point $\beta_c^{\text{erg}} > \beta_c$, which is "special" due to $\gamma_\beta$ attaining its maximum there. This transition point will be relevant for the study of the intermediate phase in Section 3.4.

**Proposition 3.3.5** (Characterisation of $\beta_c^{\text{erg}}$): Let $\psi_\beta(\eta)$ and $\eta_\beta$ be as in (3.3.7) and (3.3.11), for some $d \geq 2$. The map $\beta \mapsto \psi'_\beta(1) - \psi_\beta(1)$ is strictly decreasing and there exists a unique $\beta_c^{\text{erg}} = \beta_c^{\text{erg}}(d) > 0$, such that

$$\psi_{\beta_c^{\text{erg}}}(1) = \psi'_{\beta_c^{\text{erg}}}(1). \tag{3.3.19}$$

Equivalently, $\beta_c^{\text{erg}} > 0$ is characterised by any of the following conditions:

$$\mathbb{E}_{\beta_c^{\text{erg}}}[T] = -\log d \quad \Longleftrightarrow \quad \eta_{\beta_c^{\text{erg}}} = 1 \quad \Longleftrightarrow \quad \gamma_{\beta_c^{\text{erg}}} = \sup_{\beta>0} \gamma_\beta = \log d. \tag{3.3.20}$$

Moreover, for $\beta < \beta_c^{\text{erg}}$ we have that $\eta_\beta < 1$ and that $\beta \mapsto \gamma_\beta$ is increasing, while for $\beta > \beta_c^{\text{erg}}$ one has $\eta_\beta > 1$ and $\beta \mapsto \gamma_\beta$ is decreasing.

*Proof.* By definition of $\psi_\beta$ and the $t$-field increment measure we have

$$\psi'_\beta(1) - \psi_\beta(1) = \mathbb{E}_\beta[Te^T] - \log d = -\mathbb{E}_\beta[T] - \log d. \tag{3.3.21}$$

We claim that $\beta \mapsto \mathbb{E}_\beta[T]$ is strictly increasing. In fact, using the change of variables in (3.3.17) and noting that $e^{-t/2} = \cosh(t/2) - \sinh(t/2) = \sqrt{1+(u/2)^2} - u/2$, we have

$$\begin{aligned}
\mathbb{E}_\beta[T] &= \int \sqrt{\frac{\beta}{2\pi}} e^{-\beta(\cosh(t)-1)} e^{-t/2} t \,\mathrm{d}t \\
&= \int \sqrt{\frac{\beta}{2\pi}} e^{-\frac{\beta}{2}u^2} \frac{2\operatorname{arsinh}(u/2)(\sqrt{1+(u/2)^2} - u/2)}{\sqrt{1+(u/2)^2}} \,\mathrm{d}u \\
&= -2\int \sqrt{\frac{\beta}{2\pi}} e^{-\frac{\beta}{2}u^2} \frac{u}{2} \frac{\operatorname{arsinh}(u/2)}{\sqrt{1+(u/2)^2}} \,\mathrm{d}u\,.
\end{aligned} \tag{3.3.22}$$

It is easy to check that $x \operatorname{arsinh}(x)/\sqrt{1+x^2}$ is strictly increasing in $|x|$. Consequently, rescaling $u = s/\sqrt{\beta}$ as in (3.3.18), we see that above integral is strictly increasing in $\beta$. Moreover, one also observes that that $\mathbb{E}_\beta[T] \to -\infty$ for $\beta \searrow 0$, whereas $\mathbb{E}_\beta[T] \to 0$ for $\beta \to \infty$. Hence by (3.3.21), there exists a unique $\beta_c^{\text{erg}} > 0$, such that $\psi'_{\beta_c^{\text{erg}}}(1) = \psi_{\beta_c^{\text{erg}}}(1)$. In particular, $\eta_{\beta_c^{\text{erg}}} = 1$.

The first two alternative characterisations in (3.3.20) follow from (3.3.21) and our previous considerations. Also, by Theorem 3.2.11, we have

$$\psi_\beta(1) \lesseqgtr \psi'_\beta(1) \quad \text{for} \quad \beta \lesseqgtr \beta_c^{\text{erg}}, \tag{3.3.23}$$

which by Lemma 3.3.3 implies that $\eta_\beta \lesseqgtr 1$ for $\beta \lesseqgtr \beta_c^{\text{erg}}$.

To show the last characterisation in (3.3.20), we calculate the derivative of $\beta \mapsto \gamma_\beta = \psi_\beta(\eta_\beta)/\eta_\beta$:

$$\begin{aligned}\partial_\beta \gamma_\beta &= \partial_\beta\Big[\frac{\psi_\beta(\eta_\beta)}{\eta_\beta}\Big] \\ &= \tfrac{1}{\eta_\beta}[\partial_\beta \psi_\beta](\eta_\beta) + \tfrac{1}{\eta_\beta}[\partial_\beta \eta_\beta]\psi'_\beta(\eta_\beta) - \tfrac{1}{\eta_\beta^2}[\partial_\beta \eta_\beta]\psi_\beta(\eta_\beta) \\ &= \tfrac{1}{\eta_\beta}[\partial_\beta \psi_\beta](\eta_\beta),\end{aligned} \tag{3.3.24}$$

where in the last line we used that $\eta_\beta \psi'_\beta(\eta_\beta) - \psi_\beta(\eta_\beta) = 0$. By Theorem 3.2.11, the last line in (3.3.24) is non-negative if $\eta_\beta \leq 1$ and non-positive for $\eta_\beta \geq 1$. Since $\eta_\beta \lesseqgtr 1$ for $\beta \lesseqgtr \beta_c^{\text{erg}}$ this implies the last statement in (3.3.20) as well as the stated monotonicity behaviour of $\beta \mapsto \gamma_\beta$. □

### 3.3.2 Effective Conductance in a Critical Environment (Proof of Theorem 3.3.2)

First we recall some results on small deviation of random walks. To be precise, we use an extension of Mogulskii's Lemma [82], due to Gantert, Hu and Shi [75].

**Lemma 3.3.6** (Triangular Mogulskii's Lemma [75, Lemma 2.1]): For each $n \geq 1$, let $X_i^{(n)}$, $1 \leq i \leq n$, be i.i.d. real-valued random variables. Let $g_1 < g_2$ be continuous functions on $[0,1]$ with $g_1(0) < 0 < g_2(0)$. Let $(a_n)$ be a sequence of positive numbers such that $a_n \to \infty$ and $a_n^2/n \to 0$ as $n \to \infty$. Assume that there exist constants $\eta > 0$ and $\sigma^2 > 0$ such that:

$$\sup_{n\geq 1} \mathbb{E}\left[|X_1^{(n)}|^{2+\eta}\right] < \infty, \qquad \mathbb{E}\left[X_1^{(n)}\right] = o\left(\frac{a_n}{n}\right), \qquad \operatorname{Var}\left[X_1^{(n)}\right] \to \sigma^2. \tag{3.3.25}$$

Consider the measurable event

$$E_n := \left\{ g_1\left(\frac{i}{n}\right) \leq \frac{S_i^{(n)}}{a_n} \leq g_2\left(\frac{i}{n}\right) \ \forall i \in [n] \right\}, \tag{3.3.26}$$

where $S_i^{(n)} := X_1^{(n)} + \cdots + X_i^{(n)}$, $1 \le i \le n$. We have

$$\frac{a_n^2}{n} \log(\mathbb{P}[E_n]) \xrightarrow[n\to\infty]{} -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{1}{(g_2(t)-g_1(t))^2}\,\mathrm{d}t\,. \tag{3.3.27}$$

**Lemma 3.3.7:** For each $k \ge 1$, let $X_i^{(k)}$, $i \in \mathbb{N}$, be i.i.d. real-valued random variables with $\mathbb{E}[X_i^{(k)}] = 0$ and $\sigma_k^2 := \mathbb{E}[(X_i^{(k)})^2]$. Suppose that $0 < \inf_k \sigma_k^2 \le \sup_k \sigma_k^2 < \infty$. Write $S_i^k = X_1^{(k)} + \cdots + X_i^{(k)}$. For $\gamma > 0$ and $\nu \in (0, \frac{1}{2})$, define the events

$$E_n^{(k)} := \{|S_i| \le \gamma n^\nu,\ \forall i \in [n]\}. \tag{3.3.28}$$

then we have

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \left| n^{1-2\nu} \log \mathbb{P}[E_n^{(k)}] + \left(\frac{\pi\sigma_k}{2\gamma}\right)^2 \right| = 0. \tag{3.3.29}$$

*Proof.* We proceed by contradiction. Write $b_n^{(k)} := -n^{1-2\nu} \log \mathbb{P}[E_n^{(k)}]$ and $b_\infty^{(k)} := \left(\frac{\pi\sigma_k}{2\gamma}\right)^2$ and suppose (3.3.29) does not hold. Then there exists $\epsilon > 0$, $(k_n)_{n\in\mathbb{N}}$, and a subsequence $\mathcal{N}_0 \subseteq \mathbb{N}$

$$\forall n \in \mathcal{N}_0 : \left| b_n^{(k_n)} - b_\infty^{(k_n)} \right| > \epsilon. \tag{3.3.30}$$

Since the $\sigma_k^2$ are bounded, we can refine to a subsequence $\mathcal{N}_1 \subseteq \mathcal{N}_0 \subseteq \mathbb{N}$, such that $\sigma_{k_n}^2 \to \tilde{\sigma} > 0$ along $\mathcal{N}_1$. But by Lemma 3.3.6 (with $a_n = n^\nu$, $g_1 = -\gamma$, and $g_2 = +\gamma$) we have $b_n^{(k_n)} \to -\left(\frac{\pi\tilde{\sigma}}{2\gamma}\right)^2$ along $\mathcal{N}_1$, in contradiction with (3.3.30). □

*Proof of Theorem 3.3.2.* Recall the notation in Theorem 3.3.2. We proceed by proving the statement for an individual increment $V$, but indicate at which steps care has to be taken to establish the uniformity (3.3.5).

Write $\partial\Lambda_n := \{x \in \mathbb{T}_d : |x| = n\}$ for the vertices at distance $n$ from the origin. Set $\alpha := \frac{1}{2}(\pi^2\sigma_V^2)^{1/3}$. Define the *stopping lines* of $\{V_x\}_{x\in\mathbb{T}_d}$ at level $\alpha n^{1/3}$:

$$\mathcal{L}^{(n)} := \{(x,y) \in \vec{E} : V_y \ge \alpha n^{1/3},\ \forall z \prec y : V_z < \alpha n^{1/3}\}, \tag{3.3.31}$$

where we write $\vec{E}$ for the set or edges oriented away from the origin and "$a \prec b$" means that $a$ is an ancestor of $b$. Let $A_n$ denote the event that $\mathcal{L}^{(n)}$ is a cut-set between the origin and $\partial\Lambda_n$. By (3.2.9), conditionally on the event $A_n$ we have the point-wise bound

$$C_{n,\gamma}^{\mathrm{eff}} \le \sum_{xy\in\mathcal{L}^{(n)}} e^{-\gamma(V_x+V_y)}. \tag{3.3.32}$$

We thus have:

$$\mathbb{E}\Big[C^{\text{eff}}_{n,\gamma}\Big] \le \mathbb{E}\Big[\sum_{xy\in\mathcal{L}^{(n)}} e^{-\gamma(V_x+V_y)}\Big] + \mathbb{E}\Big[C^{\text{eff}}_{n,\gamma}\mathbb{1}_{A_n^c}\Big] \tag{3.3.33}$$

*Bounding the second summand.* Clearly, we have

$$\begin{aligned}\mathbb{P}[A_n^c] \le \mathbb{P}[\exists |x| = n,\ \text{such that } \forall y \prec x, |V_y| \le \alpha n^{1/3}] \\ + \mathbb{P}[\exists |x| \le n,\ \text{such that } V_x \le -\alpha n^{1/3}].\end{aligned} \tag{3.3.34}$$

To bound the first summand on the right hand side, we apply the many-to-one formula (Proposition 3.2.15) with $\eta = 1$, and get a random walk $(S_i)_{i\ge 0}$, such that

$$\begin{aligned}\mathbb{P}[\exists |x| = n,\ &\text{such that } \forall y \prec x, |V_y| \le \alpha n^{1/3}] \\ &\le \mathbb{E}\Big[\textstyle\sum_{|x|=n} \mathbb{1}\{\forall y \prec x, |V_y| \le \alpha n^{1/3}\}\Big] \\ &= \mathbb{E}[e^{S_n}\mathbb{1}_{\forall i\in[n], |S_i|\le \alpha n^{1/3}}] \\ &\le e^{\alpha n^{1/3}}\mathbb{P}[\forall i \in [n], |S_i| \le \alpha n^{1/3}].\end{aligned} \tag{3.3.35}$$

In the third line we used that $\psi_V(1) = 0$. We recall that (since $\psi(1)_V = \psi'_V(1) = 0$) we have $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1^2] = \sigma_V^2$. Applying Lemma 3.3.7 (with $\gamma = \alpha$ and $\nu = 1/3$) yields

$$\mathbb{P}[\forall i \in [n], |S_i| \le \alpha n^{1/3}] = e^{-[2\alpha+o(1)]n^{1/3}}, \tag{3.3.36}$$

where we used that $(\frac{\pi\sigma_V}{2\alpha})^2 = 2\alpha$. Moreover, Lemma 3.3.7 states that the convergence in (3.3.36) is uniform over a family $V^{(k)}$, $k \in \mathbb{N}$, of critical increments given that $0 < \inf_k \sigma^2_{V^{(k)}} \le \sup_k \sigma^2_{V^{(k)}} < \infty$. In conclusion we have

$$\mathbb{P}[\exists |x| = n,\ \text{such that } \forall y \prec x, |V_y| \le \alpha n^{1/3}] \le e^{-[\alpha+o(1)]n^{1/3}}. \tag{3.3.37}$$

Fot the second summand in (3.3.34) we have

$$\begin{aligned}\mathbb{P}[\exists|x|\le n,\text{ such that } V_x\le-\alpha n^{1/3}]\le&\sum_{i=1}^n\mathbb{E}\Big[\sum_{|x|=i}\mathbb{1}_{V_x\le-\alpha n^{1/3}}\Big]\\=&\sum_{i=1}^n\sum_{|x|=i}\mathbb{E}[e^{-V_x}e^{V_x}\mathbb{1}_{V_x\le-\alpha n^{1/3}}]\\\le&\sum_{i=1}^n\sum_{|x|=i}\mathbb{E}[e^{-V_x}]e^{-\alpha n^{1/3}}\\=&\sum_{i=1}^n e^{i\psi_V(1)}e^{-\alpha n^{1/3}}\\=&\sum_{i=1}^n e^{-\alpha n^{1/3}}\\=&ne^{-\alpha n^{1/3}}.\end{aligned}\tag{3.3.38}$$

Where we used that $e^{i\psi_V(\eta)}=\sum_{|x|=i}\mathbb{E}[e^{-\eta V_x}]$, which one may check inductively. In conclusion, (3.3.34), (3.3.37) and (3.3.38) yield $\mathbb{P}(A_n^c)\le e^{-(\alpha+o(1))n^{1/3}}$. We proceed by controlling the second summand in (3.3.33) using Cauchy-Schwarz and properties of the effective conductance (Lemma 3.2.6):

$$\mathbb{E}[C^{\text{eff}}_{n,\gamma}\mathbb{1}_{A_n^c}]\le\sqrt{\mathbb{E}[(C^{\text{eff}}_{n,\gamma})^2]}\,e^{-\frac{\alpha}{2}[n^{1/3}+o(1)]}\tag{3.3.39}$$

To bound the first factor on the right hand side note that $C^{\text{eff}}_{n,\gamma}\le\sum_{|x|=1}e^{-\gamma V_x}$ by Lemma 3.2.6. By Jensen's and Hölder's inequality

$$\begin{aligned}\mathbb{E}[(\textstyle\sum_{|x|=1}e^{-\gamma V_x})^2]&\le d\,\mathbb{E}[\textstyle\sum_{|x|=1}e^{-2\gamma V_x}]\\&=d^2\,\mathbb{E}[e^{-2\gamma V}]\\&\le d^2\mathbb{E}[e^{-V}]^{2\gamma(1-\frac{2\gamma-1}{2a})}\mathbb{E}[e^{-(1+2a)V}]^{\frac{2\gamma}{1+2a}\frac{2\gamma-1}{2a}}\\&\le d^{2-2\gamma(1-\frac{2\gamma-1}{2a})}[\tfrac{1}{d}e^{\psi_V(1+2a)}]^{\frac{2\gamma}{1+2a}\frac{2\gamma-1}{2a}},\end{aligned}\tag{3.3.40}$$

where we used $1=e^{\psi_V(1)}=d\,\mathbb{E}[e^{-V}]$. The last line in (3.3.40) is continuous in $\gamma\in\mathbb{R}$, hence uniformly bounded for $\gamma\in(1/2,1/2+a)$. In conclusion, we have

$$\sup_{1/2<\gamma<1/2+a}\mathbb{E}[C^{\text{eff}}_{n,\gamma}\mathbb{1}_{A_n^c}]\le C(\psi_V(1+2a))\,e^{-[\frac{\alpha}{2}+o(1)]n^{1/3}},\tag{3.3.41}$$

for a constant $C(\psi_V(1+2a)) > 0$ depending continuously on $\psi_V(1+2a)$. In particular, this yields a uniform bound over a family of critical increments $V^{(k)}$ with $0 < \inf_k \sigma^2_{V^{(k)}} \leq \sup_k \sigma^2_{V^{(k)}} < \infty$ and $\sup_k \psi_{V^{(k)}}(1+2a)\infty$.

*Bounding the first summand.* For a vertex $x \in \partial\Lambda_n$ we write $(x_k)_{k=0,\dots,n}$ for its sequence of predecessors ($x_0 = 0, x_n = x$). For a walk $X = (X_i)_{i\geq 0}$, analogously to our stopping lines, we introduce the stopping time at level $\alpha n^{1/3}$:

$$T_X^{(n)} = \inf\{i \geq 0\colon\ X_i \geq \alpha n^{1/3}\} \tag{3.3.42}$$

Note that on the event $A_n$, we know for every $x \in \partial\Lambda_n$ that the sequence $(V_{x_i})_{i=0,\dots,n}$ crosses level $\alpha n^{1/3}$. In other words, $T^{(n)}_{(V_{x_i})} \leq n$.
Consequently, the first summand in (3.3.33) is bounded via

$$\mathbb{E}\Big[\sum_{xy\in\mathcal{L}^{(n)}} e^{-\gamma(V_x+V_y)}\Big] \leq \sum_{k=1}^{n} \mathbb{E}\Big[\sum_{|x|=k} \mathbb{1}\{T^{(n)}_{(V_{x_i})} = k\} e^{-\gamma(V_{x_{k-1}}+V_{x_k})}\Big]. \tag{3.3.43}$$

The last line is amenable to the many-to-one formula (Theorem 3.2.15). Write $(S_i)_{i\geq 0}$ for the associated random walk (choosing $\eta = 1$), then the last line in (3.3.43) is equal to

$$\sum_{k=1}^{n} \mathbb{E}\Big[\mathbb{1}\{T_S^{(n)} = k\} e^{S_k} e^{-\gamma(S_{k-1}+S_k)}\Big] = \sum_{k=1}^{n} \mathbb{E}\Big[\mathbb{1}\{T_S^{(n)} = k\} e^{-(2\gamma-1)S_{k-1}} e^{(1-\gamma)(S_k-S_{k-1})}\Big]. \tag{3.3.44}$$

Now, since $S_k \geq \alpha n^{1/3}$ for $T_S^{(n)} = k$, and since $\gamma > 1/2$ by assumption, we can bound the right hand side and obtain

$$\begin{aligned}\mathbb{E}\Big[\sum_{xy\in\mathcal{L}^{(n)}} e^{-\gamma(V_x+V_y)} \mathbb{1}_{A_n}\Big] &\leq e^{-(2\gamma-1)\alpha n^{1/3}} \times \sum_{k=1}^{n} \mathbb{E}\Big[\mathbb{1}\{T_S^{(n)} = k\} e^{(1-\gamma)(S_k-S_{k-1})}\Big] \\ &\leq e^{-(2\gamma-1)\alpha n^{1/3}} \times n\mathbb{E}\Big[e^{(1-\gamma)S_1}\Big]\end{aligned} \tag{3.3.45}$$

Now by using the definition of $(S_i)$ in (3.2.21) we have

$$\mathbb{E}[e^{(1-\gamma)S_1}] = d\,\mathbb{E}[e^{-\gamma V}] \leq d\,\mathbb{E}[e^{-(1+2a)V}]^{\frac{\gamma}{1+2a}} \leq d\,[\tfrac{1}{d} e^{\psi_V(1+2a)}]^{\frac{\gamma}{1+2a}} \leq C(\psi_V(1+2a)), \tag{3.3.46}$$

for a constant $C(\psi_V(1+2a)) > 0$ that is independent of $\gamma \in (1/2, 1/2+a)$ and continuous with respect to $\psi_V(1+2a)$. Hence,

$$\mathbb{E}\Big[\sum_{xy\in\mathcal{L}^{(n)}} e^{-\gamma(V_x+V_y)}\Big] \le e^{-[(2\gamma-1)\alpha+o(1)]n^{1/3}}, \tag{3.3.47}$$

and this bound holds uniformly with respect to $\gamma \in (1/2, 1/2+a)$ and over family of critical increments $V^{(k)}$, given that $\sup_k \psi_{V^{(k)}}(1+2a) < \infty$. In conclusion (3.3.32), (3.3.41) and (3.3.47) yield

$$\begin{aligned}\mathbb{E}[C^{\text{eff}}_{n,\gamma}] &\le e^{-[\alpha/2+o(1)]n^{1/3}} + e^{-[(2\gamma-1)\alpha+o(1)]n^{1/3}}\\ &\le e^{-[\min(\frac{1}{2},2\gamma-1)\alpha+o(1)]n^{1/3}}\\ &= e^{-[\min(\frac{1}{4},\gamma-\frac{1}{2})(\pi^2\sigma_V^2)^{1/3}+o(1)]n^{1/3}}\end{aligned} \tag{3.3.48}$$

uniformly over $\gamma \in (1/2, 1/2+a)$ as $n \to \infty$. And as noted, this bound is also uniform over a family of critical increments $V^{(k)}$, given the assumptions in the theorem. This concludes the proof. □

### 3.3.3 Near-Critical Effective Conductance (Proof of Theorem 3.3.1)

The upper bound in Theorem 3.3.1 will follow from Theorem 3.3.2 and a perturbative argument. For the lower bound, we will apply a modification of a result due to Gantert, Hu and Shi [75]. In their work they give the asymptotics for the probability that some trajectory of a critical branching random walk stays below a slope $\delta|i|$ when $\delta \searrow 0$. We are interested in this result applied to the critical rescaling of $t$-field $\{\tau_x^\beta\}_{x\in\mathbb{T}_d}$ as given in (3.3.12). Comparing to Gantert, Hu and Shi's result, we will require additional uniformity in $\beta$:

> **Theorem 3.3.8:** Let $\{\tau_x^\beta\}_{x\in\mathbb{T}_d}$ be as in (3.3.12). For any $a > 0$ small enough, there exists a constant $C > 0$ such that for all $\beta \in [\beta_c, \beta_c + a]$, for $\delta$ small enough:
>
> $$\mathbb{P}_\beta[\exists \text{a path } \gamma\colon 0 \to \infty \text{ s.t. } \forall i \in \mathbb{N},\ \tau^\beta_{\gamma_i} \le \delta i] \ge e^{-C/\sqrt{\delta}}.$$

This theorem will be proven in Appendix 3.7, as it closely follows the arguments of Gantert, Hu and Shi, while taking some extra care to ensure the required uniformity.

*Proof of Theorem 3.3.1.* The main idea is to consider, for $\beta = \beta_c + \epsilon$, the critical rescaling of the $t$-field (see Lemma 3.2.17, (3.3.11) and Lemma 3.3.3)

$$\tau_i^\beta = -\eta_\beta T_i + \psi_\beta(\eta_\beta)|i|. \tag{3.3.49}$$

We remind the reader of the definition of the rescaled field with the following near-critical behaviour for the constants (Proposition 3.3.4):

$$\begin{aligned} \eta_{\beta_c+\epsilon} &= \tfrac{1}{2} + c_\eta \epsilon + O(\epsilon^2) \quad \text{with} \quad c_\eta > 0 \\ \psi_{\beta_c+\epsilon}(\eta_{\beta_c+\epsilon}) &= c_\psi \epsilon + O(\epsilon^2) \quad \text{with} \quad c_\psi > 0. \end{aligned} \tag{3.3.50}$$

Together with these asymptotics, application Theorem 3.3.8 and Theorem 3.3.2 to $\{\tau_i^\beta\}_{i\in\mathbb{T}_d}$, will yield the lower and upper bound, respectively.

*Lower Bound:* According to Theorem 3.3.8 we have that there exist constants $a, C > 0$, such that for all sufficiently small $\delta > 0$:

$$\inf_{\beta_c<\beta<\beta_c+a} \mathbb{P}_\beta[\exists \text{a path } \gamma\colon 0 \to \infty \text{ s.t. } \forall i \in \mathbb{N},\ \tau_{\gamma_i}^\beta \le \delta i] \ge e^{-C/\sqrt{\delta}}. \tag{3.3.51}$$

Note that $\tau_{\gamma_i} \le \delta i$ is equivalent to $T_{\gamma_i} \ge \eta_\beta^{-1}[\psi_\beta(\eta_\beta) - \delta] i$. Choosing $\delta(\epsilon) = \frac{1}{2} c_\psi \epsilon$, we have $\eta_{\beta_c+\epsilon}^{-1}[\psi_{\beta_c+\epsilon}(\eta_{\beta_c+\epsilon}) - \delta(\epsilon)] = c_\psi \epsilon + O(\epsilon^2)$. Hence, for $\epsilon > 0$ small enough

$$\mathbb{P}_{\beta_c+\epsilon}[\exists \text{a path } \gamma\colon 0 \to \infty \text{ s.t. } \forall i \in \mathbb{N},\ T_{\gamma_i} \ge \tfrac{1}{2} c_\psi \epsilon i] \ge e^{-C/\sqrt{\epsilon}}. \tag{3.3.52}$$

Write $A_\epsilon$ for the event in brackets. Conditionally on this event, we can bound $C_\infty^{\text{eff}}$ from below by the conductance along the path $\gamma$ (which is given by Kirchhoff's rule for conductors in series):

$$\text{On } A_\epsilon\colon \quad C_\infty^{\text{eff}} \ge \Big[ \sum_{i=0}^\infty \frac{1}{\beta} e^{-2\frac{1}{2} c_\psi \epsilon i} \Big]^{-1} = \beta(1 - e^{-c_\psi \epsilon}). \tag{3.3.53}$$

Consequently, (3.3.52) and (3.3.53) yield

$$\mathbb{E}_{\beta_c+\epsilon}[C_\infty^{\text{eff}}] \ge (\beta_c + \epsilon)(1 - e^{-c_\psi \epsilon}) e^{-C/\sqrt{\epsilon}} = e^{-[C+o(1)]/\sqrt{\epsilon}} \text{ as } \epsilon \to 0. \tag{3.3.54}$$

This concludes the proof of the lower bound in (3.3.1).

*Upper Bound:* Recalling the definition (3.3.49), we have for any $i, j \in \mathbb{T}_{d,n} \subseteq \mathbb{T}_d$ that

$$e^{T_i+T_j} = e^{(|i|+|j|)\psi_\beta(\eta_\beta)/\eta_\beta} e^{-\eta_\beta^{-1}(\tau_i^\beta+\tau_j^\beta)} \le e^{2n\psi_\beta(\eta_\beta)/\eta_\beta} e^{-\eta_\beta^{-1}(\tau_i^\beta+\tau_j^\beta)}. \tag{3.3.55}$$

Hence, if we write $\tilde{C}_n^{\text{eff}}$ for the effective conductance between the origin and $\partial\Lambda_n = \{x \in \mathbb{T}_d : |x| = n\}$ in the electrical network with conductances $\{e^{-\eta_\beta^{-1}(\tau_i^\beta + \tau_j^\beta)}\}_{ij\in E}$, we have

$$\mathbb{E}_\beta[C_n^{\text{eff}}] \leq e^{2n\psi_\beta(\eta_\beta)/\eta_\beta}\, \mathbb{E}_\beta[\tilde{C}_n^{\text{eff}}]. \tag{3.3.56}$$

For any $\beta > 0$, the field $\tau_i^\beta$ is the BRW for the critical increment $\tau^\beta := -\eta_\beta T + \psi_\beta(\eta_\beta)$, with $T$ is distributed as a $t$-field increment (at inverse temperature $\beta$). Hence, Theorem 3.3.2 implies

$$\mathbb{E}_\beta[\tilde{C}_n^{\text{eff}}] \leq \exp[-\big[\min(\tfrac{1}{4}, \eta_\beta^{-1} - 1/2)\,(\pi^2\sigma_{\tau^\beta}^2)^{1/3} + o(1)\big]n^{1/3}] \quad \text{as} \quad n \to \infty, \tag{3.3.57}$$

and moreover this holds uniformly as $\beta \searrow \beta_c$. Note that by (3.3.50) we have $\min(\frac{1}{4}, \eta_\beta^{-1} - 1/2) = \frac{1}{4}$ for $\beta$ sufficiently close to $\beta_c$. In the following write $\beta = \beta_c + \epsilon$. By (3.3.50) we have $\psi_{\beta_c+\epsilon}(\eta_{\beta_c+\epsilon})/\eta_{\beta_c+\epsilon} \sim 2c_\psi\epsilon$ as $\epsilon \searrow 0$. Hence, choosing $n = n(\epsilon) = c'\epsilon^{-3/2}$ we have

$$2n(\epsilon)\,\psi_{\beta_c+\epsilon}(\eta_{\beta_c+\epsilon})/\eta_{\beta_c+\epsilon} \sim 4c_\psi c'\epsilon^{-1/2} \quad \text{and} \quad n(\epsilon)^{1/3} = c'^{1/3}\epsilon^{-1/2}, \tag{3.3.58}$$

consequently for $c' > 0$ sufficiently small, (3.3.56) and (3.3.57) together with Lemma 3.2.6 yield

$$\mathbb{E}_{\beta_c+\epsilon}[C_\infty^{\text{eff}}] \leq \mathbb{E}_{\beta_c+\epsilon}[C_{n(\epsilon)}^{\text{eff}}] \leq e^{-(C+o(1))\,\epsilon^{-1/2}} \quad \text{as} \quad \epsilon \searrow 0, \tag{3.3.59}$$

for some constant $C > 0$. □

A corollary of the proof above, in particular (3.3.52), (3.3.53) is the following

**Lemma 3.3.9:** In the setting of Theorem 3.3.1 one has, for some constants $c, C > 0$

$$\mathbb{P}_{\beta_c+\epsilon}[C_\infty^{\text{eff}} > c\epsilon] \geq \exp[-(C + o(1))/\sqrt{\epsilon}], \tag{3.3.60}$$

as $\epsilon \searrow 0$.

### 3.3.4 Average Escape Time of the VRJP as $\beta \searrow \beta_c$ (Proof of Theorem 3.1.2)

**Lemma 3.3.10** (Local Time and Effective Conductance): Let $L_\infty^0$ denote the time the VRJP spends at the origin. Let $C_\infty^{\text{eff}}$ be the effective conductance between the origin and infinity in the $t$-field environment. Also suppose $Z$ is an independent exponential random variable of unit mean. Then we have

$$L_\infty^0 \overset{\text{law}}{=} \sqrt{1 + 2Z/C_\infty^{\text{eff}}} - 1. \tag{3.3.61}$$

*Proof.* Write $\tilde{L}^0_\infty$ for the total time the exchangeable timescale VRJP spends at the origin. By the time change formula for the local times (3.2.3), we have:

$$L^0_\infty = \sqrt{1+\tilde{L}^0_\infty} - 1. \tag{3.3.62}$$

By Theorem 3.2.2, Lemma 3.2.4, and Lemma 3.2.7, $\tilde{L}^0_\infty$ is $\mathrm{Exp}(2/C^{\mathrm{eff}}_\infty)$-distributed. □

**Lemma 3.3.11:** Let $C^{\mathrm{eff}}_\infty$ be as in Theorem 3.3.1. For any $\alpha > 0$, there exists a constant $c = c(d,\alpha) > 0$, such that for $\epsilon > 0$ small enough and $x \geq e^{c/\sqrt{\epsilon}}$

$$\mathbb{P}_{\beta_c+\epsilon}[\tfrac{1}{C^{\mathrm{eff}}_\infty} > x] \leq x^{-\alpha}. \tag{3.3.63}$$

In particular, there exists a constant $C > 0$ such that

$$\mathbb{E}_{\beta_c+\epsilon}\left[\frac{1}{C^{\mathrm{eff}}_\infty}\right] \leq e^{\frac{C}{\sqrt{\epsilon}}} \tag{3.3.64}$$

*Proof.* Recall that the $t$-field environment is given by edge-weights $\{\beta_{ij}e^{T_i+T_j}\}_{ij\in E(\mathbb{T}_d)}$, where the $t$-field $T_i$ has independent increments along outgoing edges and is defined to equal 0 at the origin. In particular, the environment on the subtree emanating from $x$ (which is isomorphic to $\mathbb{T}_d$) is distributed as a $t$-field environment on $\mathbb{T}_d$ multiplied by $e^{2T_x}$ (which is the same as requiring that the $t$-field equals $T_x$ at the "origin" $x$). For any $n \in \mathbb{N}$, and a vertex $x$ at generation $n$, write $\omega_{n,x}$ for the effective conductance from $x$ to infinity. By the above we have that $\{e^{-2T_x}\omega_{n,x}\}_{|x|=n}$ are independently distributed as $C^{\mathrm{eff}}_\infty$. Also, they are independent from the $t$-field up to generation $n$.

In the following, we replace each of the $d^n$ subtrees emanating from the vertices $x$ at generation $n$ by a single edge "to infinity" with weight $\omega_{n,x}$. The resulting network has the same effective conductance between 0 and infinity.

Define the event

$$A_n \coloneqq \{\exists |x| = n : e^{-2T_x}\omega_{n,x} > 2c\epsilon\}. \tag{3.3.65}$$

By Lemma 3.3.9 we have $\mathbb{P}_{\beta_c+\epsilon}[e^{-2T_x}\omega_{n,x} > 2c\epsilon] \geq e^{-2C/\sqrt{\epsilon}}$ and hence

$$\mathbb{P}_{\beta_c+\epsilon}[A_n^c] = 1 - \mathbb{P}_{\beta_c+\epsilon}[A_n] \leq (1-e^{-2C/\sqrt{\epsilon}})^{d^n} \leq e^{-d^n e^{-2C/\sqrt{\epsilon}}}, \tag{3.3.66}$$

which is small for appropriately chosen $n$.

Hence, suppose we are working under the event $A_n$, and let $x_0$ be a vertex at generation $n$, such that $e^{-2T_{x_0}}\omega_{n,x_0} > 2c\epsilon$. The effective conductance on the tree is larger than the effective conductance on the subgraph where we only keep the edges between 0 and $x_0$, as well as an edge between $x_0$ and infinity with conductance $e^{2T_{x_0}}2c\epsilon < \omega_{n,x_0}$. Denote the conductance of this reduced graph by $C^{\text{red}}$. We write $y_0 = 0, \dots, y_n = x_0$ for the vertices along the path from 0 to $x_0$. The series formula for conductances yields

$$\frac{1}{C_\infty^{\text{eff}}} \le \frac{1}{C^{\text{red}}} = \frac{1}{\beta}\sum_{i=0}^{n-1} e^{-(T_{y_i}+T_{y_{i+1}})} + \frac{1}{2c\epsilon}e^{-2T_{y_n}}. \tag{3.3.67}$$

We bound $T_{y_i} + T_{y_{i+1}} \ge 2\min(T_{y_i}, T_{y_{i+1}})$. Recall that $T_{y_i} \overset{\text{law}}{=} \sum_{k=0}^{i} T^{(k)}$ with i.i.d. samples $\{T^{(k)}\}_{k\ge 0}$ from the $t$-field increment measure (3.2.17). This yields

$$\frac{1}{C^{\text{red}}} \le (\tfrac{n}{\beta} + \tfrac{1}{2c\epsilon})e^{-2\min(T_{y_0},\dots,T_{y_n})}. \tag{3.3.68}$$

For fixed $\tau > 0$ we apply a union bound and Chernoff's bound (resp. Lemma 3.6.1)

$$\begin{aligned} \mathbb{P}_\beta[\min(T_{y_0},\dots,T_{y_n}) < -n\tau] &\le \sum_{i=0}^{n} \mathbb{P}[\textstyle\sum_{k=0}^{i} T^{(k)} < -n\tau] \\ &\le \sum_{i=0}^{n} \exp(-i\Psi_\beta^*(\tfrac{n}{i}\tau)), \end{aligned} \tag{3.3.69}$$

where $\Psi_\beta^*(\tau) = \sup_{\lambda\ge 0}(\lambda\tau - \log\mathbb{E}_\beta[e^{-\lambda T}])$ is the Fenchel-Legendre dual of the (negative) $t$-field increment's log-MGF. Convexity of $\Psi_\beta^*$ (and $\Psi_\beta^*(0) = 0$) implies $\Psi_\beta^*(\frac{n}{i}\tau) \ge \frac{n}{i}\Psi_\beta^*(\tau)$. Consequently, (3.3.69) yields

$$\mathbb{P}_\beta[\min(T_{y_0},\dots,T_{y_n}) < -n\tau] \le (n+1)e^{-n\Psi_\beta^*(\tau)} \quad \text{for} \quad \tau > 0 \tag{3.3.70}$$

which by (3.3.67) and (3.3.68) implies

$$\mathbb{P}_{\beta_c+\epsilon}[\tfrac{1}{C_\infty^{\text{eff}}} > (\tfrac{n}{\beta} + \tfrac{1}{2c\epsilon})e^{2n\tau}|A_n] \le (n+1)\exp[-n\Psi_{\beta_c+\epsilon}^*(\tau)], \tag{3.3.71}$$

In Appendix 3.6 we obtain lower bounds on $\Psi_\beta^*$ (Lemma 3.6.1). By (3.6.3), we have that for fixed $\alpha > 0$ and sufficiently small $\epsilon > 0$, any sufficiently large $\tau > 0$ will satisfy $\Psi_{\beta_c+\epsilon}^*(\tau) \ge 7\alpha\tau$, uniformly as $\epsilon \searrow 0$. To conclude, we choose $n \ge N(\epsilon) := \frac{4C}{\log(d)\sqrt{\epsilon}}$, such that $\mathbb{P}[A_n] \le e^{-d^{n/2}}$. In conclusion, with above choices, (3.3.66) and (3.3.71) yield

$$\mathbb{P}_{\beta_c+\epsilon}\Big[\tfrac{1}{C_\infty^{\text{eff}}} > e^{3n\tau}\Big] \le e^{-6n\alpha\tau} + e^{-d^{n/2}} \tag{3.3.72}$$

This implies the claim. □

*Proof of Theorem 3.1.2.* We start with the lower bound: By Lemma 3.3.10 there exists an exponential random variable $Z$ of expectation 1 such that:

$$\begin{aligned}
\mathbb{E}[L_\infty^0] &= \mathbb{E}\Big[\sqrt{1+2Z/C_\infty^{\text{eff}}}\Big]-1 \\
&\geq \mathbb{E}\Big[\sqrt{1+2Z/\mathbb{E}(C_\infty^{\text{eff}})}\Big]-1 \text{ by cond. Jensen inequality} \\
&\geq \mathbb{E}[\sqrt{Z}]/\mathbb{E}[C_\infty^{\text{eff}}]-1 \\
&\geq \exp(c/\sqrt{\epsilon})-1 \text{ by Theorem 3.3.1.}
\end{aligned} \tag{3.3.73}$$

For the upper bound, we start with Jensen's inequality:

$$\begin{aligned}
\mathbb{E}[L_\infty^0] &= \mathbb{E}\Big[\sqrt{1+2Z/C_\infty^{\text{eff}}}-1\Big] \\
&\leq \sqrt{1+2\mathbb{E}\big[Z/C_\infty^{\text{eff}}\big]}-1 \\
&= \sqrt{1+2\mathbb{E}\big[1/C_\infty^{\text{eff}}\big]}-1 \\
&\leq \sqrt{2}\sqrt{\mathbb{E}\big[1/C_\infty^{\text{eff}}\big]}.
\end{aligned} \tag{3.3.74}$$

The result now follows by Lemma 3.3.11. □

## 3.4 Intermediate Phase of the VRJP

In this section we show that the VRJP on large finite regular trees exhibits an intermediate phase. We also argue that Rapenne's recent results [35] imply the *absence* of such an intermediate phase on regular trees with *wired* boundary conditions.

### 3.4.1 Existence of an Intermediate Phase on $\mathbb{T}_{d,n}$ (Proof of Theorem 3.1.3)

The intermediate phase is characterised by the VRJP, despite being transient, spending "unusually much" time at the root. To be precise, on finite trees the fraction of time spent at the origin scales with the system size as a *fractional power* of the inverse system volume. At the second transition point the walk then reverts to the behaviour that one expects by comparison with simple random walk, spending time inversely proportional to the tree's volume at the starting vertex.

We will see that the different scalings will be due to different regimes for the log-Laplace transform of the $t$-field increments, $\psi_\beta(\eta) = \log[d\,\mathbb{E}_\beta e^{\eta T}]$, as elaborated in Section 3.3.1.

Before starting the proof, we show how the observable in Theorem 3.1.3 can be rephrased in terms of a $t$-field. The proof will then proceed by analysing the resulting $t$-field quantity via branching random walk methods.

**Lemma 3.4.1:** Consider the situation of Theorem 3.1.3. Further consider a $t$-field $\{T_x\}$ on $\mathbb{T}_{d,n}$, rooted at the origin 0. We then have

$$\lim_{t\to\infty} \frac{L_t^0}{t} \overset{\text{law}}{=} \Big[\sum_{|x|\le n} e^{T_x}\Big]^{-1}, \tag{3.4.1}$$

*Proof.* Trivially one has $t = \sum_{|x|\le n} L_t^x$. Consequently,

$$\lim_{t\to\infty} \frac{L_t^0}{t} = \lim_{t\to\infty} \Big[\sum_{|x|\le n} L_t^x / L_t^0\Big]^{-1}. \tag{3.4.2}$$

Hence, the claim follows from Corollary 3.2.3. □

*Proof of Theorem 3.1.3.* In light of Lemma 3.4.1 we consider a $t$-field $\{T_x\}$ on $\mathbb{T}_d$, rooted at the origin. In the following we analyse the asymptotic behaviour of the random variable $\sum_{|x|\le n} e^{T_x}$.

*Case* $\beta_c < \beta < \beta_c^{\mathrm{erg}}$*:* We note that it suffices to show

$$\sum_{|x|\leq n} e^{T_x} = e^{n\gamma_\beta + o(n)} \quad \text{a.s. for} \quad n \to \infty \quad \text{with} \quad \gamma_\beta = \inf_{\eta>0} \psi_\beta(\eta)/\eta > 0, \tag{3.4.3}$$

since we have $0 < \gamma_\beta < \log(d)$ by Proposition 3.3.5. The lower bound in (3.4.3) follows from Theorem 3.2.16:

$$\sum_{|x|\leq n} e^{T_x} \geq \sum_{|x|=n} e^{T_x} \geq e^{\max_{|x|=n} T_x} = e^{n\gamma_\beta + o(n)}. \tag{3.4.4}$$

For the upper bound in (3.4.3) note that for $\eta \in (0,1)$ and $\epsilon > 0$ we have

$$\begin{aligned} \mathbb{P}\Big[\sum_{|x|\leq n} e^{T_x} > e^{n(\gamma_\beta+\epsilon)}\Big] &\leq e^{-n\eta(\gamma_\beta+\epsilon)}\mathbb{E}\Big[\Big(\sum_{|x|\leq n} e^{T_x}\Big)^\eta\Big] \\ &\leq e^{-n\eta(\gamma_\beta+\epsilon)}\mathbb{E}\Big[\sum_{|x|\leq n} e^{\eta T_x}\Big] \\ &= e^{-n\eta(\gamma_\beta+\epsilon)} \sum_{k=0}^{n} e^{\psi(\eta)k} \end{aligned} \tag{3.4.5}$$

Now let $\eta = \eta_\beta$ as in Lemma 3.3.3, *i.e.* such that $\gamma_\beta = \psi_\beta(\eta_\beta)/\eta_\beta > 0$. Note that by Proposition 3.3.5, we have $\gamma_\beta \in (0, \log(d))$. With this choice (3.4.5) implies $\limsup_{n\to\infty} \frac{1}{n} \log \sum_{|x|\leq n} e^{T_x} \leq \gamma_\beta + \epsilon$ almost surely for any $\epsilon > 0$. This yields the lower bound in (3.4.3).

*Case* $\beta \leq \beta_c$*:* This proceeds similarly to the previous case. For the lower bound we simply use $\sum_{|x|\leq n} e^{T_x} \geq e^{T_0} = 1$. For the lower bound we use (3.4.5) with $\gamma_\beta \mapsto 0$ and $\eta = 1/2$, which implies that $\limsup_{n\to\infty} \frac{1}{n} \log \sum_{|x|\leq n} e^{T_x} \leq \epsilon$. almost surely for any $\epsilon > 0$.

*Case* $\beta > \beta_c^{\mathrm{erg}}$*:* First note that the quantity $W_n \coloneqq d^{-n} \sum_{|x|=n} e^{T_x}$ is a martingale. In the branching random walk literature this is referred to as the *additive martingale* associated with the BRW $\{T_x\}_{x\in\mathbb{T}_d}$. Since $W_n$ is non-negative it converges almost surely to a random variable $W_\infty = \lim_{n\to\infty} W_n$. Biggin's martingale convergence theorem [74, Theorem 3.2] implies that for $\beta > \beta_c^{\mathrm{erg}}$ (equivalently $\psi'_\beta(1) < \psi_\beta(1)$, see Proposition 3.3.5), the sequence is uniformly integrable and the limit $W_\infty$ is almost surely strictly positive. Consequently we also get convergence for the weighted average

$$\frac{1}{|\mathbb{T}_{d,n}|} \sum_{|x|\leq n} e^{T_x} = \frac{1}{|\mathbb{T}_{d,n}|} \sum_{k=0}^{n} d^k W_k \to W_\infty > 0 \quad \text{a.s. for} \quad n \to \infty. \tag{3.4.6}$$

In other words,

$$\sum_{|x|\le n} e^{T_x} \sim \left|\mathbb{T}_{d,n}\right| W_\infty = d^{n+O(1)} \quad \text{as} \quad n\to\infty, \tag{3.4.7}$$

which implies the claim for $\beta > \beta_c^{\text{erg}}$. □

### 3.4.2 Multifractality of the Intermediate Phase (Proof of Theorem 3.1.4)

For the proof we will make use of explicit large deviation asymptotics for the maximum of the $t$-field. These follow (as an easy special case) from results due to Gantert and Höfelsauer on the large deviations of the maximum of a branching random walk [83, Theorem 3.2]:

> **Lemma 3.4.2:** Consider the $t$-field $\{T_x\}_{x\in\mathbb{T}_d}$ on $\mathbb{T}_d$, pinned at the origin 0. Let $\gamma_\beta = \inf_{\eta>0}\psi_\beta(\eta)/\eta$ as in (3.3.11). For any $\gamma > \gamma_\beta$ we have
>
> $$\liminf_{n\to\infty} \tfrac{1}{n}\log\mathbb{P}[\max_{|x|=n} T_x \ge n\gamma] = -\sup_{\eta\in\mathbb{R}}[\gamma\eta - \psi_\beta(\eta)] < 0. \tag{3.4.8}$$

*Proof.* As noted, this is a direct consequence of [83, Theorem 3.2]. To be precise, we consider the special case of a deterministic offspring distribution (instead of Galton-Watson trees) and fluctuations *above* the asymptotic velocity $\gamma_\beta$ (corresponding to the case $x > x^*$ in [83]). In this case, the rate function given by Gantert and Höfelsauer (denoted by $x \mapsto I(x) - \log(m)$ in their article) is equal to

$$\gamma \mapsto \sup_{\eta\in\mathbb{R}}(\gamma\eta - \log\mathbb{E}[e^{\eta T}]) - \log d = \sup_{\eta\in\mathbb{R}}[\gamma\eta - \psi_\beta(\eta)]. \tag{3.4.9}$$

This concludes the proof. □

*Proof of Theorem 3.1.4.* By Lemma 3.4.1, we would like to understand fractional moments of

$$[\lim_{t\to\infty} L_t^0/t]^{-1} \overset{\text{law}}{=} \sum_{|x|\le n} e^{T_x}, \tag{3.4.10}$$

where $\{T_x\}_{x\in\mathbb{T}_d}$ denotes a $t$-field on the rooted $(d+1)$-regular tree, pinned at the origin. Recall the definition of $\eta_\beta$ in (3.3.11) and Lemma 3.3.3. For $\beta \in (\beta_c, \beta_c^{\text{erg}})$ we have $\eta_\beta \in (0,1)$ by Proposition 3.3.5.

*Case* $\eta \in (0, \eta_\beta]$*:* We recall Proposition 3.2.16, which implies that

$$\lim_{n\to\infty} \frac{1}{n} \max_{|x|=n} T_x = \gamma_\beta = \psi_\beta(\eta_\beta)/\eta_\beta. \tag{3.4.11}$$

By Jensen's inequality and Fatou's lemma we get

$$\begin{aligned}\liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}[(\sum_{|x|\le n}e^{T_x})^{\eta}] &\ge \liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}[e^{\eta\max_{|x|=n}T_x}]\\ &\ge \liminf_{n\to\infty}\frac{\eta}{n}\mathbb{E}[\max_{|x|=n}T_x]\\ &\ge \eta\psi_\beta(\eta_\beta)/\eta_\beta.\end{aligned}\tag{3.4.12}$$

On the other hand, since $\eta/\eta_\beta\le 1$

$$\mathbb{E}[(\sum_{|x|\le n}e^{T_x})^{\eta}]\le\mathbb{E}[(\sum_{|x|\le n}e^{T_x})^{\eta_\beta}]^{\eta/\eta_\beta}\tag{3.4.13}$$

For any $\eta\in(0,1)$ and $\beta>\beta_c$ we can bound

$$\mathbb{E}[(\sum_{|x|\le n}e^{T_x})^{\eta}]\le\mathbb{E}[\sum_{|x|\le n}e^{\eta T_x}]\le\sum_{k=0}^{n}e^{k\psi_\beta(\eta)}\le e^{n\psi_\beta(\eta)+o(n)},\tag{3.4.14}$$

where we used that $\inf_{\eta>0}\psi_\beta(\eta)=\psi_\beta(1/2)>0$ for $\beta>\beta_c$ (*cf.* (3.3.10), (3.3.9) and (3.3.16)). Applying this to the last line of (3.4.13), we obtain

$$\mathbb{E}[(\sum_{|x|\le n}e^{T_x})^{\eta}]\le e^{n\eta\psi_\beta(\eta_\beta)/\eta_\beta+o(n)}\tag{3.4.15}$$

*Case* $\eta\in\left[\eta_\beta,1\right)$*:* The upper bound already follows from (3.4.14). For the lower bound we start with

$$\begin{aligned}\mathbb{E}[(\sum_{|x|\le n}e^{T_x})^{\eta}] &\ge \mathbb{E}[e^{\eta\max_{|x|=n}T_x}]\\ &\ge e^{n\eta\gamma}\,\mathbb{P}[\max_{|x|=n}T_x\ge n\gamma]\quad\text{for any}\quad\gamma>0.\end{aligned}\tag{3.4.16}$$

We get that for any $\gamma\in\mathbb{R}$:

$$\liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}[(\sum_{|x|\le n}e^{T_x})^{\eta}]\ge\eta\gamma+\liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}[\max_{|x|=n}T_x\ge n\gamma].\tag{3.4.17}$$

By Lemma 3.4.2, we have

$$\liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}[(\sum_{|x|\le n}e^{T_x})^{\eta}]\ge\sup_{\gamma>\gamma_\beta}\Big(\eta\gamma-\sup_{\tilde\eta\in\mathbb{R}}[\gamma\tilde\eta-\psi_\beta(\tilde\eta)]\Big).\tag{3.4.18}$$

We claim that the right hand side of (3.4.18) is equal to $\psi_\beta(\eta)$. For the upper bound simply choose $\tilde{\eta} = \eta$. For the lower bound first note that the supremum of $\tilde{\eta} \mapsto \gamma\tilde{\eta} - \psi_\beta\tilde{\eta}$ is attained at the unique $\tilde{\eta}$, such that $\psi'_\beta(\tilde{\eta}) = \gamma$ (uniqueness follow from convexity of $\eta \mapsto \psi_\beta(\eta)$). Since we assumed $\eta > \eta_\beta$, we may choose $\gamma = \psi'_\beta(\eta)$, satisfying $\gamma > \gamma_\beta = \psi'_\beta(\eta_\beta)$. Together with previous observation this shows that the right hand side is larger or equal to $\psi_\beta(\eta)$. This concludes the proof. □

### 3.4.3 On the Intermediate Phase for Wired Boundary Conditions

We recall that for the Anderson transition it was debated whether an intermediate multifractal phase persists in the infinite volume and on tree-like graphs without free boundary conditions (see Section 3.1.3).

We *conjecture* that there is no intermediate phase for the VRJP on regular trees with wired boundary conditions. In this section, we would like to provide some evidence for this claim, based on recent work by Rapenne [35].

Let $\overline{\mathbb{T}}_{d,n}$ denote the rooted $(d+1)$-regular tree of depth $n$ with *wired* boundary, *i.e.* all vertices at generation $n$ have an outgoing edge to a single *boundary ghost* $\mathfrak{g}$. We consider $\mathbb{T}_{d,n} \subset \overline{\mathbb{T}}_{d,n}$ as a the subgraph induced by the vertices excluding the ghost. Let $\{\overline{T}^{\mathfrak{g}}_x\}_{x\in\overline{T}_{d,n}}$ denote a $t$-field on the wired tree $\overline{\mathbb{T}}_{d,n}$, pinned at the ghost $\mathfrak{g}$, and at inverse temperature $\beta$. We define

$$\psi_n(x) = e^{\overline{T}^{\mathfrak{g}}_x} \text{ for } x \in \mathbb{T}_{d,n}, \tag{3.4.19}$$

where we use the index $n$ to make the dependence on the underlying domain $\overline{\mathbb{T}}_{d,n}$ more explicit. This coincides with the (vector) martingale $\{\psi_n(x)\}_{x\in\mathbb{T}_{d,n}}$ considered by Rapenne (see [48, Lemma 2] for a proof that these are in fact the same). By [35, Theorem 2] we have for $\beta > \beta_c$ and $p \in (1,\infty)$

$$\sup_{n\geq 1} \mathbb{E}_\beta[\psi_n(0)^p] < \infty. \tag{3.4.20}$$

Our statement about the absence of an intermediate phase, will be conditional on a (conjectural) extension of this result:

$$\textit{Conjecture:}\ \sup_{n\geq 1} \frac{1}{|\mathbb{T}_{d,n}|} \sum_{x\in\mathbb{T}_{d,n}} \mathbb{E}_\beta[\psi_n(x)^p] < \infty \quad \text{for} \quad p > 1 \text{ and } \beta > \beta_c. \tag{3.4.21}$$

We believe this statement to be true due to the following heuristic: Given that the origin of $\overline{\mathbb{T}}_{d,n}$ is furthest away from the ghost $\mathfrak{g}$, at which the $t$-field in (3.4.19) is pinned, we expect the fluctuations of $\psi_n(x)$ to be largest at $x = 0$. Hence, we expect the moments of $\psi_n(x)$ to be comparable with the ones of $\psi_n(0)$, in which case (3.4.20) would imply (3.4.21).

**Proposition 3.4.3:** Consider a VRJP started from the root of $\overline{\mathbb{T}}_{d,n}$ and let $L_t^0$ denote the time it spent at root up until time $t$. Assume (3.4.21) holds true. Then, for any $\beta > \beta_c$

$$\lim_{t\to\infty} \frac{L_t^0}{t} \le \left|\mathbb{T}_{d,n}\right|^{-1+o(1)} \quad \text{w.h.p. as} \quad n \to \infty. \tag{3.4.22}$$

This is to be contrasted with the behaviour in Theorem 3.1.3.

*Proof.* Let $\{\overline{T}_x\}_{x\in\overline{\mathbb{T}}_{d,n}}$ denote the $t$-field on $\overline{\mathbb{T}}_{d,n}$, pinned at the origin 0. We stress that this is different from $\overline{T}_x^{\mathfrak{g}}$, as used in (3.4.19), which is pinned at the ghost $\mathfrak{g}$. However, we can sample the former from the latter: First consider an STZ-Anderson operator $H_B$ on the infinite graph $\mathbb{T}_d$, as defined in Definition 3.1.8. Define $\hat{G}_n \coloneqq (H_B|_{\mathbb{T}_{d,n}})^{-1}$ and also define $\{\psi_n(x)\}_{x\in\mathbb{T}_d}$ by

$$(H_B\psi_n)|_{\mathbb{T}_{d,n}} = 0 \text{ and } \psi|_{\mathbb{T}_d\setminus\mathbb{T}_{d,n}} \equiv 1. \tag{3.4.23}$$

By [48, Lemma 2], the $\psi_n$ so defined (and restriced to $\mathbb{T}_{d,n}$) agree in law with the definition in (3.4.19). Then define $\overline{T}_x$ for $x \in \mathbb{T}_{d,n}$ via

$$e^{\overline{T}_x} = \frac{\hat{G}_n(0,x) + \frac{1}{2\gamma}\psi_n(0)\psi_n(x)}{\hat{G}_n(0,0) + \frac{1}{2\gamma}\psi_n(0)^2}, \tag{3.4.24}$$

where $\gamma \sim \mathrm{Gamma}(\frac{1}{2}, 1)$ is independent of $H_B$. By [48, Proposition 8], $\{\overline{T}_x\}_{x\in\mathbb{T}_{d,n}}$ has the law of a $t$-field on $\overline{\mathbb{T}}_{d,n}$, pinned at the origin 0 (and restricted to $\mathbb{T}_{d,n}$). Note that $\overline{\mathbb{T}}_{\mathfrak{g}}$ is not defined by (3.4.24). Using the conditional law of the $t$-field on $\overline{\mathbb{T}}_{d,n}$ given its values away from the ghost, we can however define it such that $\{\overline{T}_x\}_{x\in\overline{\mathbb{T}}_{d,n}}$ is the "full" $t$-field on $\overline{\mathbb{T}}_{d,n}$, pinned at the origin. Then, as in (3.4.1), we have that

$$\lim_{t\to\infty} \frac{L_t^0}{t} \overset{\text{law}}{=} \left[\sum_{x\in\overline{\mathbb{T}}_{d,n}} e^{\overline{T}_x}\right]^{-1}. \tag{3.4.25}$$

By (3.4.24) and positivity of $\hat{G}$ we get

$$\sum_{x\in\overline{\mathbb{T}}_{d,n}} e^{\overline{T}_x} \ge \sum_{x\in\mathbb{T}_{d,n}} e^{\overline{T}_x} \ge \frac{\psi_n(0)}{2\gamma\hat{G}_n(0,0) + \psi_n(0)^2} \sum_{x\in\mathbb{T}_{d,n}} \psi_n(x). \tag{3.4.26}$$

By [48, Theorem 1], for $\beta > \beta_c$ the fraction on the right hand side converges a.s. to a (random) positive number as $n \to \infty$. Hence, the claim in (3.4.22) follows if we show that $\sum_{x\in\mathbb{T}_{d,n}} \psi_n(x) \geq |\mathbb{T}_{d,n}|^{1-o(1)}$ a.s. as $n \to \infty$. For any $s > 0$ and $q \geq 1$ we have

$$\begin{aligned} \mathbb{P}[\sum_{x\in\mathbb{T}_{d,n}} \psi_n(x) \leq s|\mathbb{T}_{d,n}|] &= \mathbb{P}[(\frac{1}{|\mathbb{T}_{d,n}|}\sum_{x\in\mathbb{T}_{d,n}} \psi_n(x))^{-q} \geq s^{-q}] \\ &\leq s^q\, \mathbb{E}[(\frac{1}{|\mathbb{T}_{d,n}|}\sum_{x\in\mathbb{T}_{d,n}} \psi_n(x))^{-q}] \\ &\leq s^q \frac{1}{|\mathbb{T}_{d,n}|}\sum_{x\in\mathbb{T}_{d,n}} \mathbb{E}[\psi_n(x)^{-q}] \\ &= s^q \frac{1}{|\mathbb{T}_{d,n}|}\sum_{x\in\mathbb{T}_{d,n}} \mathbb{E}[\psi_n(x)^{1+q}], \end{aligned} \tag{3.4.27}$$

where in the last line we used the reflection property of the $t$-field (see Lemma 3.8.1). Subject to the assumption that (3.4.21) holds true, we may choose $q = 1$ and $s = n^{-2}$ in (3.4.27). An application of the Borel-Cantelli lemma then yields that $\sum_{x\in\mathbb{T}_{d,n}} \psi_n(x) \geq |\mathbb{T}_{d,n}|^{1-o(1)}$ a.s. as $n \to \infty$. Together with (3.4.25) and (3.4.26), this implies (3.4.22). □

## 3.5 Results for the $\mathbb{H}^{2|2}$-Model

### 3.5.1 Asymptotics for the $\mathbb{H}^{2|2}$-Model as $\beta \searrow \beta_c$ (Proof of Theorem 3.1.5)

*Proof of Theorem 3.1.5.* By Theorem 3.1.2 it suffices to show that

$$\langle x_0^2 \rangle_\beta^+ = \lim_{h \searrow 0} \lim_{n \to \infty} \langle x_0^2 \rangle_{\beta; h, \mathbb{T}_{d,n}} = \mathbb{E}_\beta [L_\infty^0]. \tag{3.5.1}$$

For this, we use the $\mathbb{H}^{2|2}$-Dynkin isomorphism (Theorem 3.2.1):

$$\langle x_0^2 \rangle_{\beta; h, \mathbb{T}_{d,n}} = \int_0^\infty \mathrm{d}t \, \mathbb{E}_{\beta; \mathbb{T}_{d,n}} \left[ e^{-ht} \mathbb{1}_{X_t = 0} \right], \tag{3.5.2}$$

where, subject to $\mathbb{E}_{\beta; \mathbb{T}_{d,n}}$, $(X_t)_{t \geq 0}$ is a VRJP on $\mathbb{T}_{d,n}$ started at 0. Coupling the VRJP on $\mathbb{T}_{d,n}$ with a VRJP on the infinite tree $\mathbb{T}_d$ up to the time they first visit the leaves of $\mathbb{T}_{d,n}$, we get

$$\left| \mathbb{E}_{\beta; \mathbb{T}_{d,n}} [\mathbb{1}_{X_t = 0}] - \mathbb{E}_{\beta; \mathbb{T}_d} [\mathbb{1}_{X_t} = 0] \right| \leq \mathbb{P}_{\beta; \mathbb{T}_d} [T_n \leq t], \tag{3.5.3}$$

with $T_n$ being the VRJP's hitting time of $\partial \mathbb{T}_{d,n} = \{x \in \mathbb{T}_{d,n} : |x| = n\}$. By definition of the VRJP, the time it takes to reach $\partial \mathbb{T}_{d,n}$ is stochastically lower bounded by an exponential random variable of rate $d\beta / n$. Consequently, the right hand side of (3.5.3) converges to zero as $n \to \infty$. By this observation and the monotone convergence theorem we have

$$\langle x_0^2 \rangle_\beta^+ = \lim_{h \searrow 0} \int_0^\infty \mathrm{d}t \, e^{-ht} \mathbb{E}_{\beta; \mathbb{T}_d} \left[ \mathbb{1}_{X_t = 0} \right] = \int_0^\infty \mathrm{d}t \, \mathbb{E}_{\beta; \mathbb{T}_d} \left[ \mathbb{1}_{X_t = 0} \right] = \mathbb{E}_{\beta; \mathbb{T}_d} [L_\infty^0], \tag{3.5.4}$$

which proves the claim. □

### 3.5.2 Intermediate Phase for the $\mathbb{H}^{2|2}$-Model (Proof of Theorem 3.1.6)

In this section, we want to prove Theorem 3.1.6 on the intermediate phase of the $\mathbb{H}^{2|2}$-model. We will make use of the STZ-Anderson model, as defined in Definition 3.1.8, making use of its restriction properties as discussed in [25, 84].

The proof consists of three parts: First we evaluate the quantity on the left hand side of (3.1.21) on a graph consisting of a single vertex (and a coupling to a ghost vertex). Then we reduce the actual quantity in (3.1.21) onto the case of a single vertex with a random *effective* *magnetic field* $h^{\text{eff}}$. As $h \searrow 0$, the law of $h^{\text{eff}}$ can be expressed in terms of the $t$-field and we can deduce Theorem 3.1.6 from Theorem 3.1.4 on the VRJP's multifractality.

**Lemma 3.5.1:** Consider the $\mathbb{H}^{2|2}$-model on a single vertex 0 with magnetic field $h > 0$. For $\eta \in (0,1)$ we have

$$\langle z_0 |x_0|^{-\eta} \rangle_{h;\{0\}} = h^{\eta} \times g_\eta(h) \tag{3.5.5}$$

with

$$g_\eta(h) := \frac{1}{\pi} e^h (2h)^{(1-\eta)/2} \Gamma(\tfrac{1}{2} - \tfrac{\eta}{2}) K_{(1-\eta)/2}(h). \tag{3.5.6}$$

In particular

$$c_\eta := \frac{1}{\pi} 2^{-\eta} \Gamma(\tfrac{1}{2} - \tfrac{\eta}{2})^2 = \lim_{h \searrow 0} g_\eta(h) \tag{3.5.7}$$

*Proof.* For convenience, lets write $\langle \cdot \rangle = \langle \cdot \rangle_{h;\{0\}}$. By $e^{t_0} = z_0 + x_0$ and $y_0 = s_0 e^{t_0}$, see (3.2.12), we have

$$\begin{aligned} \langle z_0 |x_0|^{-\eta} \rangle = \langle z_0 |y_0|^{-\eta} \rangle = \langle (e^{t_0} + x_0) |y_0|^{-\eta} \rangle = \langle e^{t_0} |y_0|^{-\eta} \rangle = \langle e^{t_0} |s_0|^{-\eta} e^{-\eta t_0} \rangle \\ = \langle e^{(1-\eta)t_0} |s_0|^{-\eta} \rangle. \end{aligned} \tag{3.5.8}$$

The last line can be interpreted in purely probabilistic terms: $t_0$ follows the law of a $t$-field increment with inverse temperature $h > 0$ and conditionally on $t_0$, $s_0$ is a Gaussian random variable with variance $e^{-t_0}/h$. Consequently,

$$\begin{aligned} \mathbb{E}[|s_0|^{-\eta} | t_0] &= \sqrt{\frac{h e^{t_0}}{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}s \, |s|^{-\eta} e^{-h e^{t_0} s^2/2} \\ &= (h e^{t_0})^{\eta/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \, |x|^{-\eta} e^{-x^2/2} \\ &= (h e^{t_0})^{\eta/2} \frac{2^{-\eta/2}}{\sqrt{\pi}} \Gamma(\tfrac{1}{2} - \tfrac{\eta}{2}). \end{aligned} \tag{3.5.9}$$

With (3.5.8) we obtain

$$\langle z_0 |x_0|^{-\eta} \rangle = h^{\eta/2} \frac{2^{-\eta/2}}{\sqrt{\pi}} \Gamma(\tfrac{1}{2} - \tfrac{\eta}{2}) \mathbb{E}_h[e^{(1-\eta/2)T}], \tag{3.5.10}$$

where $T$ denotes a $t$-field increment at inverse temperature $h$. Expressing the exponential moments of $T$ in terms of the modified Bessel function of second kind $K_\alpha$, as in (3.3.7), and using small-argument asymptotics for the latter, we obtain

$$\mathbb{E}_h[e^{(1-\eta/2)T}] = \frac{\sqrt{2h}e^h}{\sqrt{\pi}} K_{(1-\eta)/2}(h) \sim h^{\eta/2} \times \frac{2^{(1-\eta)/2}\Gamma(\frac{1}{2}-\frac{\eta}{2})}{\sqrt{2\pi}} \quad \text{as} \quad h \searrow 0. \tag{3.5.11}$$

Inserting this into (3.5.10) yields the claim. □

**Effective Weight.** Before proceeding, we need to introduce the notion of *effective weight* for the STZ-field: Consider an STZ-Anderson model $H_B$ as in 3.1.8 and suppose the underlying graph $G = (V, E)$ is finite. Write $G_B = (H_B)^{-1}$. Then, for $i_0, j_0 \in V$, the effective weight between these two vertices is defined by

$$\beta^{\text{eff}}_{i_0 j_0} := \frac{G_B(i_0, j_0)}{G_B(i_0, i_0)G_B(j_0, j_0) - G_B(i_0, j_0)^2}. \tag{3.5.12}$$

Another expression can be deduced using Schur's complement: Write $V_0 = \{i_0, j_0\}$ and $V_1 = V \setminus \{i_0, j_0\}$ and decompose $H_B$ as

$$H_B = \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix}, \tag{3.5.13}$$

with $H_{00}$ being the restriction of $H_B$ to entries with indices in $V_0$ and analogously for the other submatrices. By Schur's decomposition we have

$$\begin{aligned} G_B|_{V_0} &= H_B^{-1}|_{V_0} \\ &= (H_{00} - H_{01}H_{11}^{-1}H_{10})^{-1} \\ &= \begin{pmatrix} B_{i_0} - [H_{01}H_{11}^{-1}H_{10}](i_0, i_0) & -\beta_{i_0 j_0} - [H_{01}H_{11}^{-1}H_{10}](i_0, j_0) \\ -\beta_{j_0 i_0} - [H_{01}H_{11}^{-1}H_{10}](j_0, i_0) & B_{j_0} - [H_{01}H_{11}^{-1}H_{10}](j_0, j_0) \end{pmatrix}^{-1}. \end{aligned} \tag{3.5.14}$$

Note that (3.5.12) reads as $\beta^{\text{eff}}_{i_0 j_0} = G_B(i_0, j_0)/\det(G_B|_{V_0}) = G_B(i_0, j_0)\det([G_B|_{V_0}]^{-1})$. Hence using the familiar formula for the inverse of a $2\times 2$-matrix we get

$$\beta^{\text{eff}}_{i_0 j_0} = \beta_{i_0 j_0} + [H_{01}H_{11}^{-1}H_{10}](i_0, j_0), \tag{3.5.15}$$

which is measurable with respect to $B|_{V_1}$. The relevance of the effective weight stems from the following Lemma (see [25, Section 6])

**Lemma 3.5.2:** For a finite graph $G = (V, E)$ with positive edge-weights $\{\beta_{ij}\}_{ij\in E}$ and a pinning vertex $i_0$, consider the natural coupling of an STZ-field $(B_i)_{i\in V}$ and a $t$-field $(T_i)_{i\in V}$ (see Remark 3.2.10). For a vertex $j_0 \in V \setminus \{i_0\}$ write $V_0 := \{i_0, j_0\}$ and $V_1 := V \setminus \{i_0, j_0\}$.

Then, conditionally on $B|_{V_1}$, the $t$-field $T|_{V_0} = (T_{i_0}, T_{j_0})$ is distributed as a $t$-field on $V_0$, pinned at $i_0$, with edge-weight given by $\beta^{\text{eff}}_{i_0 j_0} = \beta^{\text{eff}}_{i_0 j_0}(B|_{V_1})$.

Moreover, the notion of effective weight and effective conductance are directly related:

**Lemma 3.5.3** (Effective Conductance vs. Weight): Consider the setting of Lemma 3.5.2. For $j_0 \in V \setminus \{i_0\}$, let $C^{\text{eff}}_{i_0 j_0}$ denote the effective conductance between $i_0$ and $j_0$ in the $t$-field environment $\{\beta_{ij} e^{T_i + T_j}\}_{ij\in E}$. Then

$$C^{\text{eff}}_{i_0 j_0} = e^{T_{j_0}} \beta^{\text{eff}}_{i_0 j_0}. \tag{3.5.16}$$

This statement is proved in Appendix 3.8. In the following, we will come back to the setting of the regular tree.

**Reduction to Two Vertices on the Tree.** We denote by $\tilde{\mathbb{T}}_{d,n}$ the graph obtained by adding an additional *ghost vertex* $\mathfrak{g}$ connected to every vertex of the graph $\mathbb{T}_{d,n}$. For the $\mathbb{H}^{2|2}$-model (and consequently the $t$-/$s$-field) we refer to the model on $\mathbb{T}_{d,n}$ *with magnetic field* $h > 0$ as the model on $\tilde{\mathbb{T}}_{d,n}$, pinned at the ghost $\mathfrak{g}$, with weights $\beta_{x\mathfrak{g}} = h$ between the ghost and any other vertex.

**Lemma 3.5.4** (Effective Magnetic Field at the Origin): Consider the natural coupling of $t$-field, $s$-field and STZ-field on $\tilde{\mathbb{T}}_{d,n}$, at inverse temperature $\beta > 0$ and with magnetic field $h > 0$, pinned at the ghost $\mathfrak{g}$. The random fields are denoted by $T_x$, $S_x$ and $B_x$, respectively ($x \in \tilde{\mathbb{T}}_{d,n}$). Write $V_0 := \{0, \mathfrak{g}\}$ and $V_1 := \tilde{\mathbb{T}}_{d,n} \setminus \{0, \mathfrak{g}\}$ and define $H_{11} := H_B|_{V_1}$.

Conditionally on $B|_{V_1}$, the $t$-/$s$-field at the origin $(T_0, S_0)$ follows the law of a $t$-/$s$-field on $\{0, \mathfrak{g}\}$ with *effective magnetic field*

$$h^{\text{eff}} := \beta^{\text{eff}}_{0\mathfrak{g}} = h + h\beta \sum_{x,y\in V_1 : y\sim 0} H_{11}^{-1}(y, x). \tag{3.5.17}$$

*Proof.* By Lemma 3.5.2, conditionally on $B|_{V_1}$, the $t$-field at the origin $T_0$ has the law of a $t$-field increment at inverse temperature $h^{\text{eff}}$. We claim that the analogous fact is true for the joint measure of $(T_0, S_0)$.

Recall that, conditionally on $\{T_x\}$, the law of $\{S_x\}$ is that of Gaussian free field, pinned at $\mathfrak{g}$, edge-weights given by the $t$-field environment $\{\beta_{ij} e^{T_i + T_j}\}$ over edges in $\tilde{\mathbb{T}}_{d,n}$ with $\beta_{x\mathfrak{g}} = h$. Let $C^{\text{eff}}_{0\mathfrak{g}}$ denote the effective conductance between the origin 0 and the ghost $\mathfrak{g}$ in the $t$-field environment. Then, conditionally on $\{T_x\}$, we have that $S_0$ is a centred normal random variable with variance given by the effective resistance $1/C^{\mathrm{eff}}_{0\mathfrak{g}}$ (see [23, Proposition 2.24]). By Lemma 3.5.3 we have $C^{\mathrm{eff}}_{0\mathfrak{g}} = e^{T_0}\beta^{\mathrm{eff}}_{0\mathfrak{g}} = e^{T_0}h^{\mathrm{eff}}$. To conclude, it suffices to note that $h^{\mathrm{eff}}$ is measurable with respect to $B|_{V_1}$. □

**Lemma 3.5.5** (Law of Effective Magnetic Field as $h \searrow 0$.): Consider the setting of Lemma 3.5.4. Further consider a $t$-field $\{T^{(0)}_x\}$ on $\mathbb{T}_{d,n}$, pinned at the origin, at the same inverse temperature $\beta$. Then we have that

$$\frac{h^{\mathrm{eff}}}{h} \xrightarrow{\text{law}} \sum_{x\in\mathbb{T}_{d,n}} e^{T^{(0)}_x} \quad \text{as} \quad h \searrow 0. \tag{3.5.18}$$

*Proof.* By (3.5.17) it suffices to show that

$$\beta \sum_{y\in V_1:\, y\sim 0} H^{-1}_{11}(y,x) \xrightarrow{\text{law}} e^{T^{(0)}_x} \quad \text{as} \quad h \searrow 0. \tag{3.5.19}$$

We start by decomposing the restriction of $H_B$ to $\mathbb{T}_{d,n}$, *i.e.* without the ghost vertex $\mathfrak{g}$, as follows

$$H_B|_{\mathbb{T}_{d,n}} = \begin{pmatrix} B_0 & -\beta_0^\top \\ -\beta_0^\top & H_{11} \end{pmatrix}, \tag{3.5.20}$$

where we write $\beta_0 = [\beta \mathbb{1}_{y\sim 0}]_{y\in V_1}$. By Schur's complement we have

$$(H_B|_{\mathbb{T}_{d,n}})^{-1} = \begin{pmatrix} (B_0 - \beta_0^\top H_{11}^{-1}\beta_0)^{-1} & (B_0 - \beta_0^\top H_{11}^{-1}\beta_0)^{-1}\beta_0^\top H_{11}^{-1} \\ \cdots & \cdots \end{pmatrix}. \tag{3.5.21}$$

As a consequence, for any $x \in V_1$

$$\frac{(H_B|_{\mathbb{T}_{d,n}})^{-1}(0,x)}{(H_B|_{\mathbb{T}_{d,n}})^{-1}(0,0)} = (\beta_0^\top H_{11}^{-1})(0,x) = \beta \sum_{y\in V_1:\, y\sim 0} H^{-1}_{11}(y,x). \tag{3.5.22}$$

We now note that as $h \searrow 0$ the law of $B|_{\mathbb{T}_{d,n}}$ converges to that of a STZ-field on $\mathbb{T}_{d,n}$, as can be seen from (3.1.25). Consequently, by Proposition 3.2.9, the law of the left hand side in (3.5.22) converges to that of $e^{T^{(0)}_x}$, which proves the claim. □

*Proof of Theorem 3.1.6.* Combining Lemma 3.5.1 and 3.5.4 we have

$$\lim_{h\searrow 0} h^{-\eta}\langle z_0|x_0|^{-\eta}\rangle_{\beta,h;\mathbb{T}_{d,n}} = \lim_{h\searrow 0} \mathbb{E}_{\beta,h}\Big[\Big(\frac{h^{\mathrm{eff}}}{h}\Big)^{\eta} g_\eta(h^{\mathrm{eff}})\Big] \tag{3.5.23}$$

We note that by [25, Proposition 6.1.2] we have $\mathbb{E}[h^{\text{eff}}] \leq h|\mathbb{T}_{d,n}|$. Hence, for any fixed $C > 0$ we have $h^{\text{eff}} \leq C$ with probability $1 - o(1)$ as $h \searrow 0$. Lemma 3.5.5 therefore implies

$$\lim_{h\searrow 0} h^{-\eta} \langle z_0 |x_0|^{-\eta} \rangle_{\beta,h;\mathbb{T}_{d,n}} = c_\eta \mathbb{E}_\beta\Big[\Big(\sum_{x\in\mathbb{T}_{d,n}} e^{T_x^{(0)}}\Big)^\eta\Big], \tag{3.5.24}$$

with $c_\eta > 0$ given in (3.5.7). Consequently, application of Lemma 3.4.1 and Theorem 3.1.4 concludes the proof. □

## 3.6 Appendix: Tail Bounds for the $t$-field increments.

In this section, we apply the Cramér-Chernoff method [85] to prove a doubly-exponential lower tail-bound for sums of independent (negative) $t$-field increments. Consider the Fenchel-Legendre dual of the $t$-field's log-moment-generating function:

$$\Psi_\beta^*(\tau) = \sup_{\lambda\geq 0}(\lambda\tau - \log \mathbb{E}_\beta[e^{-\lambda T}]). \tag{3.6.1}$$

**Lemma 3.6.1** (Lower Tail bound for sums of $t$-Field Increments): Let $\{T_i\}_{i=1,\ldots,n}$ denote independent random variables distributed according to the $t$-field increment measure $Q_\beta^{\text{inc}}$ (see Definition 3.2.12). For any $\tau > 0$ we have

$$\mathbb{P}_\beta[\textstyle\sum_{i=1}^n T_i \leq -n\tau] \leq \exp\Big[-n\Psi_\beta^*(\tau)\Big], \tag{3.6.2}$$

Moreover, $\Psi_\beta^*$ is bounded from below as

$$\begin{aligned}\Psi_\beta^*(\tau) &> \sup_{0<\rho<1}\Big[\rho\frac{\beta e^\tau}{2} - \beta(1-\sqrt{1-\rho}) + \tfrac{1}{2}\log(1-\rho)\Big] \\ &\geq (\tfrac{3}{8}\beta e^\tau - \log[2e^{\beta/2}]).\end{aligned} \tag{3.6.3}$$

To prove this, we note that for a $t$-field increment $T$, the random variable $e^{\pm T}$ follows a (reciprocal) inverse Gaussian distribution. For completeness, recall that a random variable $X > 0$ is said to follow an *inverse Gaussian* distribution, $X \sim \text{IG}(\mu,\beta)$, if it has density

$$\frac{e^{\beta/\mu}}{\sqrt{2\pi/\beta}} e^{-\frac{\beta}{2}(\frac{x}{\mu^2}+\frac{1}{x})} \frac{\mathrm{d}x}{x^{3/2}} \tag{3.6.4}$$

over the positive real numbers. Similarly, $Y > 0$ follows *reciprocal inverse Gaussian* distribution, $Y \sim \mathrm{RIG}(\mu,\beta)$, if it has density

$$\frac{e^{\beta/\mu}}{\sqrt{2\pi/\beta}} e^{-\frac{\beta}{2}(y+\frac{1}{\mu^2 y})} \frac{\mathrm{d}y}{\sqrt{y}} \tag{3.6.5}$$

over the positive real numbers. With this convention, we have $e^T \sim \mathrm{IG}(1,\beta)$ and $e^{-T} \sim \mathrm{RIG}(1,\beta)$. Also recall the moment-generating functions (MGF):

$$\begin{aligned} \mathbb{E}[e^{\lambda X}] &= e^{\frac{\beta}{\mu}\left(1-\sqrt{1-2\mu^2\lambda/\beta}\right)} \quad \text{for} \quad \lambda < \beta/(2\mu^2), \\ \mathbb{E}[e^{\lambda Y}] &= \frac{e^{\frac{\beta}{\mu}\left(1-\sqrt{1-2\lambda/\beta}\right)}}{\sqrt{1-2\lambda/\beta}} \quad \text{for} \quad \lambda < \beta/2. \end{aligned} \tag{3.6.6}$$

With this, we have everything we need:

*Proof of Lemma 3.6.1.* By Markov's inequality one easily derives the Chernoff bound

$$\mathbb{P}[T \leq -\tau] \leq e^{-\Psi_\beta^*(\tau)}. \tag{3.6.7}$$

Similarly, for independent $t$-field increments $\{T_i\}$ one obtains

$$\mathbb{P}[\textstyle\sum_{i=1}^n T_i \leq -n\tau] \leq e^{-n\Psi_\beta^*(\tau)}. \tag{3.6.8}$$

In the following, we establish lower bounds on $\Psi_\beta^*$. We start by bounding $\mathbb{E}_\beta[e^{-\lambda t}]$, using the elementary inequality $x^\lambda \leq (\lambda/e)^\lambda e^x$ for $x > 0$:

$$\begin{aligned} \mathbb{E}[e^{-\lambda T}] &= \rho^{-\lambda}\mathbb{E}[(\rho e^{-T})^\lambda] \\ &\leq \left(\tfrac{\lambda}{\rho e}\right)^\lambda \mathbb{E}[e^{\rho e^{-T}}], \end{aligned} \tag{3.6.9}$$

with the right hand side being finite and explicit for $0 < \rho < \beta/2$ by the MGF for the reciprocal inverse Gaussian distribution (3.6.6). Consequently, for any $\lambda, \tau > 0$ and $0 < \rho < \beta/2$ we have

$$\lambda\tau - \log\mathbb{E}[e^{-\lambda T}] \geq \lambda(\tau - \log(\lambda/\rho) + 1) - \log\mathbb{E}[e^{\rho e^{-T}}]. \tag{3.6.10}$$

In $\lambda$, the right hand side is maximised for $\lambda = \rho e^\tau$, which yields

$$\Psi_\beta^*(\tau) \geq \sup_{\rho>0}(\rho e^\tau - \log\mathbb{E}[e^{\rho e^{-T}}]). \tag{3.6.11}$$

After inserting (3.6.6) and rescaling $\rho \mapsto \frac{\beta}{2}\rho$, first bound in (3.6.3) follows. For the second bound, one may simply choose $\rho = 3/4$. □

## 3.7 Appendix: Uniform Gantert-Hu-Shi Asymptotics for $\tau_x^\beta$: Proof of Theorem 3.3.8

We will stay close to the original proof by Gantert, Hu ans Shi [75], but get rid of some of the technical details as we only require a lower bound not a precise limit. Also note that Gantert *et al.* prove their result for general branching random walks, whereas we only show the result for a deterministic offspring distribution. A crucial technical ingredient to Gantert *et al.*'s proof is their extension of Mogulskii's Lemma (Lemma 3.3.6), which we also make use of.

**Definition 3.7.1:** Let $\rho_\beta(\delta, n)$ be the probability that there exists $|x| = n$ such that for all $i \in [n]$, $\tau_{x_i} \le \delta i$.

**Definition 3.7.2:** Let $\tau = \tau^\beta$ be a random variable distributed as the increment of $\{\tau_x^\beta\}_{x\in\mathbb{T}_d}$. Let $M_\beta$ be such that $\mathbb{P}_\beta\left(\tau \ge M_\beta\right) = 2/3$ and let $p_d > 0$ be the probability that a Galton-Watson tree where the reproduction law is given by a binomial $\mathrm{Bin}(d, 2/3)$ survives. We now define for any $\delta > 0$ small enough and for any $n \in \mathbb{N}$ the set $G_{n,\delta}$ as follows:

$$\begin{aligned} G_{n,\delta} := \{|x| = n \text{ such that } \tau_{x_i} \le \frac{1}{2}\delta i, \forall i \in [(1-\delta/(2M_\beta)n] \text{ and} \\ \text{for all } \left(1 - \frac{\delta}{2M_\beta}\right)n + 1 \le k \le n,\ \tau_{x_k} - \tau_{x_{k-1}} \le M_\beta\}. \end{aligned} \tag{3.7.1}$$

The idea is that if $G_{n,\delta}$ is not empty, it means that there is a vertex $x$ such that $|x| = n$ and $\forall i \in [n], \tau_{x_i} \le \delta i$. Then started at all the vertices of $G_{n,\delta}$ we can see if the corresponding sets $G_{n,\delta}$ are not empty. This allows us to create a Galton-Watson tree. The exact definition of $G_{n,\delta}$ is chosen to ensure that if it is not empty it contains many vertices. In turn this means that if the Galton-Watson tree we construct is not empty then it is infinite with high probability. To compute everything precisely we will use 3.3.6 but first we need a preliminary result. The following results allows us to show that if $G_{n,\delta}$ is not empty then with high probability it has many vertices.

**Lemma 3.7.3** (Lemma 1 of [86]): Let $(Z_n)_{n\in\mathbb{N}}$ be a supercritical Galton Watson tree. There exists $\eta > 1$ such that:

$$\mathbb{P}[Z_n < \eta^n] = \mathbb{P}[Z \text{ is finite}] + o(\eta^{-n}). \tag{3.7.2}$$

**Corollary 3.7.4:** Let $(Z_n)_{n\in\mathbb{N}}$ be a supercritical Galton Watson tree. There exists $\eta > 1$ such that:

$$\mathbb{P}[1 \le Z_n \le \eta^n] = o(\eta^{-n}). \tag{3.7.3}$$

*Proof.* The Galton-Watson tree conditioned on dying is a sub-critical Galton Watson tree and thus the probability that it survives up to time $n$ decreases exponentially in $n$. This coupled with 3.7.3 gives the desired result. □

Now the goal is to give a lower bound on the probability that $G_{n,\delta}$ is not empty. First we express this in terms of $\rho_\beta$.

**Lemma 3.7.5** ([75, Lemma 4.3]): Let $\delta > 0$. We have:

$$\mathbb{P}_\beta[G_{n,\delta} \neq \emptyset] \ge p_d \rho_\beta(\delta/2, n). \tag{3.7.4}$$

*Proof.* Let $L := \left\lfloor \left(1 - \frac{\delta}{2M_\beta}\right) n \right\rfloor$.

$$\begin{aligned}\mathbb{P}_\beta[G_{n,\delta} \neq \emptyset] =& \mathbb{P}_\beta\left[\exists |x| = L \text{ such that } \tau_{x_i} \le \frac{1}{2}\delta i, \forall i \in [L]\right] \dots \\ & \dots \times \mathbb{P}_\beta\left[\exists |x| = n - L \text{ such that } \max_{1\le k\le n-L} \tau_{x_i} - \tau_{x_{i-1}} \le M_\beta\right] \\ \ge& \rho_\beta(\delta/2, n) p_d.\end{aligned} \tag{3.7.5}$$

□

Once we have this lower bound, we need to show that with high probability if $|G_{n,\delta}|$ is not empty then it has many children with high probability.

**Lemma 3.7.6:** Let $L := \left\lfloor \left(1 - \frac{\delta}{2M_\beta}\right) n \right\rfloor$. There exists $\eta > 1$ such that for $n - L$ large enough (this only depends on $d$):

$$\mathbb{P}_\beta\left[1 \le |G_{n,\delta}| \le \eta^{n-L} \big| |G_{n,\delta}| \ge 1\right] = o\left(\frac{1}{\eta^{n-L}}\right). \tag{3.7.6}$$

*Proof.* If $|G_{n,\delta}| \ge 1$, it means that there exists $x$ such that $|x| = n$ and

$$\tau_{x_i} \le \alpha\delta i, \forall i \in [L] \text{ and } \max_{L+1\le k\le n} \tau_{x_i} - \tau_{x_{i-1}} \le M. \tag{3.7.7}$$

Now, if we restrict $G_{n,\delta}$ to the descendant of $x_L$, we get a Galton-Watson tree conditioned to survive up to time $n - L$ and where the reproduction law is a binomial $B\big(n, \mathbb{P}_\beta(\tau \le M_\beta)\big)$ which does not depend on $\beta$. Then, by 3.7.4, we have the desired result. □

What is left is to give a lower bound $\rho$. The goal of the next lemmata is to give a lower bound of $\rho$ by terms for which we can apply Lemma 3.3.6.

**Lemma 3.7.7** (Lemma 4.5 of [75]): For any $n \geq 1$ and any $i \in [n]$, let $I_{i,n} \subset \mathbb{R}$ be a Borel set. We have:

$$\mathbb{P}_\beta\left[\exists |x| = n \text{ such that } \forall i \in [n], \tau_{x_i} \in I_{i,n}\right] \geq \frac{\mathbb{E}_\beta\left[e^{S_n} 1_{\forall i \in [n], S_i \in I_{i,n}}\right]}{1 + (d-1)\sum_{j=1}^{n} h_{j,n}}, \tag{3.7.8}$$

where $h_{j,n}$ is defined by:

$$h_{j,n} := \sup_{u \in I_{j,n}} \mathbb{E}_\beta\left[e^{S_{n-j}} 1_{\forall l \in [n-j], S_l + u \in I_{l+j,n}}\right]. \tag{3.7.9}$$

**Lemma 3.7.8:** For any $\beta > \beta_c$ we have:

$$\rho_\beta(n^{-2/3}, n) \geq \frac{\mathbb{P}_\beta\left[\frac{i}{n} - 1 \leq \frac{S_i}{n^{1/3}} \leq \frac{i}{n}\ \forall i \in [n]\right]}{1 + (d-1)ne^{2n^{1/3}}}. \tag{3.7.10}$$

*Proof.* Let $I_{i,n} := \left[\frac{i}{n^{2/3}} - n^{1/3}, \frac{i}{n^{2/3}}\right]$. We have:

$$\begin{aligned}\rho_\beta(n^{-2/3}, n) \geq& \mathbb{P}_\beta[\exists |x| = n \text{ such that } \tau_{x_i} \in I_{i,n} \forall i \in [n]] \\ \geq& \frac{\mathbb{E}_\beta\left[e^{S_n} 1_{\forall i \in [n], S_i \in I_{i,n}}\right]}{1 + (d-1)\sum_{j=1}^{n} h_{j,n}} \text{ by lemma 3.7.7,}\end{aligned} \tag{3.7.11}$$

where $h_{j,n}$ is as in lemma 3.7.7. The numerator of 3.7.11 can be bounded as follows:

$$\mathbb{E}_\beta\left[e^{S_n} 1_{\forall i \in [n], S_i \in I_{i,n}}\right] \geq e^{(1-1)n^{1/3}} \mathbb{P}\left[\forall i \in [n], S_i \in I_{i,n}\right]. \tag{3.7.12}$$

As for the denominator, we have:

$$\begin{aligned}h_{j,n} =& \sup_{u \in I_{j,n}} \mathbb{E}\left[e^{S_{n-j}} 1_{\forall i \in [n-j], S_i \in [(i+j)/n^{2/3} - \lambda n^{1/3} - u, (i+j)/n^{2/3} - u]}\right] \\ \leq& e^{(i+j)/n^{2/3} - j/n^{2/3} + n^{1/3}} \\ \leq& e^{2n^{1/3}}.\end{aligned} \tag{3.7.13}$$

From this we get the desired result.

□

Now we have everything we need to prove the result we want.

*Proof of Theorem 3.3.8.* Given the tree $\mathbb{T}^d$ and the $\tau$-field on it we create the new tree $\tilde{\mathbb{T}}$ as follows: we look at all the vertices $x$ at distance $n$ of the origin, and we only keep those that are in $G_{n,\delta}$. Then we look at the trees started at those vertices and we apply the same procedure recursively. The tree we obtain is thus a Galton-Watson tree with reproduction law given by the law of $|G_{n,\delta}|$. Furthermore, by definition of $G_{n,\delta}$, if $\tilde{\mathbb{T}}$ is infinite then there exists an infinite path $\gamma$ in $\mathbb{T}^d$ such that for all $i \in \mathbb{N}$, $\tau_{\gamma_i} \le \delta i$. Now we just need to give a lower bound on the probability that $\tilde{\mathbb{T}}$ is infinite. By the lemmata 3.7.7 and 3.7.8, we have by taking $\delta_n := 2n^{-2/3}$:

$$\mathbb{P}_\beta[G_{n,\delta_n} \neq \emptyset] \ge p_d \frac{\mathbb{P}_\beta\left[\frac{i}{n} - 1 \le \frac{S_i}{n^{1/3}} \le \frac{i}{n}\ \forall i \in [n]\right]}{1 + (d-1)ne^{2n^{1/3}}}. \tag{3.7.14}$$

Now we want to apply 3.3.6 but unfortunately we are not exactly in the conditions of the theorem, we would need $\frac{S_i}{n^{1/3}} \le \frac{i}{n} +$ something. To get that, we say that there exists some constant $c$ such that uniformly on some interval $[\beta_c, \beta_c + a]$ we have:

$$\mathbb{P}_\beta[S_1 \in (-2,-1)] \ge c. \tag{3.7.15}$$

Therefore for any $\delta > 0$:

$$\mathbb{P}_\beta[\forall i \in [\delta n^{1/3}] (S_i - S_{i-1}) \in (-2,-1)] \ge e^{\log(c)\delta n^{1/3}}. \tag{3.7.16}$$

Now, we get for any $\epsilon > 0$ small enough:

$$\begin{aligned}
&\mathbb{P}_\beta\left[\frac{i}{n} - 1 \le \frac{S_i}{n^{1/3}} \le \frac{i}{n}\ \forall i \in [n]\right] \\
\ge &\mathbb{P}_\beta\left[\forall i \in [\epsilon n^{1/3}],\ (S_i - S_{i-1}) \in (-2,-1)\right] \mathbb{P}_\beta\left[\frac{i}{n} - 1 + 2\epsilon \le \frac{S_i}{n^{1/3}} \le \frac{i}{n} + \epsilon\ \forall i \in [n - \epsilon n^{1/3}]\right] \\
\ge &e^{\log(c)\epsilon n^{1/3}} \mathbb{P}_\beta\left[\frac{i}{n} - 1 + 2\epsilon \le \frac{S_i}{n^{1/3}} \le \frac{i}{n} + \epsilon\ \forall i \in [n]\right].
\end{aligned} \tag{3.7.17}$$

Finally we satisfy the condition of our lemma 3.3.6. We have by lemma 3.3.6 that for any interval of the form $[\beta_c, \beta_c + a]$ there exists some explicit constant $C_a$ such that :

$$\limsup_{n\to\infty} \sup_{\beta \in [\beta_c, \beta_c + a]} n^{-1/3} \log \mathbb{P}_\beta\left[\frac{i}{n} - 1 + 2\epsilon \le \frac{S_i}{n^{1/3}} \le \frac{i}{n} + \epsilon\ \forall i \in [n]\right] \le C_\delta. \tag{3.7.18}$$

Define $f_\beta$ by $f_\beta := \mathbb{E}_\beta[s^{|G_{n,\delta}(\alpha)|}]$ and let $q_{\beta,n}$ be the extinction probability. We have $q_{\beta,n} = f_\beta(\beta, n)$. For any $r < q_{\beta,n}$ we have:

$$q_{\beta,n} = f_\beta(0) + \int_0^{q_{\beta,n}} f_\beta^{'}(s)\mathrm{d}s = f_\beta(0) + \int_0^{q_{\beta,n}-r} f_\beta^{'}(s)\mathrm{d}s + \int_{q_{\beta,n}-r}^{q_{\beta,n}} f_\beta^{'}(s)\mathrm{d}s. \tag{3.7.19}$$

Now, using that $f_\beta$ is convex and therefore $f_\beta^{'}$ is non-decreasing, we get:

$$q_{\beta,n} \leq f_\beta(0) + (q_{\beta,n} - r) f_\beta^{'}(q_{\beta,n} - r) + r f_\beta^{'}(q_{\beta,n}) \leq f_\beta(0) + (1-r) f_\beta^{'}(1-r) + r. \tag{3.7.20}$$

Now, $f_\beta(0) = \mathbb{P}_\beta[G_{n,\delta_n} = \emptyset]$ and $f_\beta^{'}(1-r) = \mathbb{E}_\beta[|G_{n,\delta_n}|(1-r)^{|G_{n,\delta_n}|-1}]$ which is bounded from above by $\frac{1}{1-r}\mathbb{E}_\beta(|G_{n,\delta_n}|e^{-r|G_{n,\delta_n}|})$. Now if we take $r < 1/2$ we get:

$$1 - q_{\beta,n} \geq \mathbb{P}_\beta[G_{n,\delta_n} \neq \emptyset] - 2\mathbb{E}_\beta[|G_{n,\delta_n}|e^{-r|G_{n,\delta_n}|}] - r. \tag{3.7.21}$$

From this we get:

$$\begin{aligned} 1 - q_{\beta,n} \geq & \mathbb{P}_\beta[G_{n,\delta_n} \neq \emptyset] - \frac{2}{r^2}\mathbb{P}_\beta\left[1 \leq |G_{n,\delta_n}| \leq r^2\right] - \frac{2e^{-1/r}}{r^2} - r \\ \geq & \mathbb{P}_\beta[G_{n,\delta_n} \neq \emptyset] - \frac{2}{r^2}\mathbb{P}_\beta\left[1 \leq |G_{n,\delta}| \leq r^2\right] - 2r \text{ for } r \text{ small enough.} \end{aligned} \tag{3.7.22}$$

By taking $r = \eta^{-n}$ we get that for $n$ large enough, for some constant $C > 0$, for any $\beta \in [\beta_c, \beta_c + a]$:

$$1 - q_{\beta,n} \geq e^{-Cn^{1/3}}. \tag{3.7.23}$$

Then by noticing that $n = (2/\delta_n)^{3/2}$ we get the desired result. □

## 3.8 Appendix: Effective Conductance and Effective Weight

Before starting with the proof of Lemma 3.5.3, we would like to remind the reader of the definition of the effective weight (3.5.12) as well as the discussion following it.

*Proof of Lemma 3.5.3.* Let $D_T$ denote the graph Laplacian on $G$ with weights given by the $t$-field environment $\{\beta_{ij}e^{T_i+T_j}\}_{ij\in E}$. The effective resistance (*i.e.* inverse effective conductance) can be expressed as

$$1/C_{i_0j_0}^{\text{eff}} = (-D_T|_{V\setminus\{i_0\}})^{-1}(j_0, j_0), \tag{3.8.1}$$

where $D_T|_{V\setminus\{i_0\}}$ denotes $D_T$ with deletion of the row and column corresponding to $i_0$. Recall that on $V\setminus\{i_0\}$ we have $B_x = \sum_{y\sim x}\beta_{xy}e^{T_y-T_x}$. Defining the diagonal matrices $L_T = \mathrm{diag}(\{e^{T_x}\}_{x\in T\setminus\{i_0\}})$, one may check that

$$-D_T|_{V\setminus\{i_0\}} = L_T H_B|_{V\setminus\{i_0\}} L_T. \tag{3.8.2}$$

Inserting this into (3.8.1) yields

$$e^{-T_{j_0}}C^{\mathrm{eff}}_{i_0j_0} = \frac{e^{T_{j_0}}}{(H_B|_{V\setminus\{i_0\}})^{-1}(j_0,j_0)} = \frac{H_B^{-1}(i_0,j_0)}{H_B^{-1}(i_0,i_0)\,(H_B|_{V\setminus\{i_0\}})^{-1}(j_0,j_0)} \tag{3.8.3}$$

Using (3.5.14) and the familiar expression for the inverse of a 2x2-matrix, we have

$$\frac{H_B^{-1}(i_0,j_0)}{H_B^{-1}(i_0,i_0)} = \frac{\beta_{i_0j_0}+[H_{01}H_{11}^{-1}H_{10}](i_0,j_0)}{B_{j_0}-[H_{01}H_{11}^{-1}H_{10}](j_0,j_0)}. \tag{3.8.4}$$

Note that the numerator equals $\beta^{\mathrm{eff}}_{i_0j_0}$. On the other hand, using a Schur decomposition for $H_B|_{V\setminus\{i_0\}}$, decomposing $V\setminus\{i_0\}$ into $\{j_0\}$ and $V_1$, one may compute

$$(H_B|_{V\setminus\{i_0\}})^{-1}(j_0,j_0) = 1/(B_{j_0}-[H_{01}H_{11}^{-1}H_{10}](j_0,j_0)). \tag{3.8.5}$$

Inserting (3.8.4) and (3.8.5) into (3.8.3) we obtain

$$e^{-T_{j_0}}C^{\mathrm{eff}}_{i_0j_0} = \beta_{i_0j_0}+[H_{01}H_{11}^{-1}H_{10}](i_0,j_0) = \beta^{\mathrm{eff}}_{i_0j_0}, \tag{3.8.6}$$

which proves the claim. □

**Lemma 3.8.1** (Reflection Property of the $t$-Field): Consider a finite graph $G=(V,E)$ with positive edge weights $\{\beta_{ij}\}_{ij\in E}$. Let $\{T_x\}_{x\in V}$ denote a $t$-field on $G$ with weights $\{\beta_{ij}\}$, pinned at some vertex $i_0$. For any $q\in\mathbb{R}$ and $x\in V$ we have

$$\mathbb{E}[e^{qT_x}] = \mathbb{E}[e^{(\frac{1}{2}-q)T_x}]. \tag{3.8.7}$$

*Proof.* On a graph with two vertices, the claim follows from the density of the $t$-field increment measure (Definition 3.2.12). On a larger graph, we consider the natural coupling of the STZ-field $\{B_x\}_{x\in V}$ and the $t$-field. By [25, Section 6.1] we know that conditionally on $B_y$ for $y\in V\setminus\{i_0,x\}$, the $t$-field on $\{i_0,x\}$ follows the law of a $t$-field on this reduced graph (still pinned at $i_0$) with edge-weights given by an effective weight $\beta_{i_0x}$ (the latter being measurable with respect to the STZ-field outside $\{i_0, x\}$). Consequently, the claim follows from the statement on two vertices. □

# Chapter 4

# More on the $\mathbb{H}^{2|2}$-model on trees

## 4.1 Tree-Recursion for the $\mathbb{H}^{2|2}$-Model

In [TREE] most of our insight into the $\mathbb{H}^{2|2}$-model and the VRJP came from analysis of the $t$-field, which is simply a marginal of the $\mathbb{H}^{2|2}$-model in horospherical coordinates. Alternatively, following a technique that is more common in the physics literature, one can exploit the recursive structure of regular trees to produce *consistency equations* for, say, the single-site marginal of the $\mathbb{H}^{2|2}$-model. Such an approach to "solving" the model has been successful for other systems, such as the Ising model [87]. Zirnbauer [36] followed this path in a (non-rigorous) treatment of Efetov's model on the regular tree, predicting the model's near-critical behaviour, analogous to Theorem 3.1.5. Mirlin and Gruzberg argued that Zirnbauer's reasoning should translate to the $\mathbb{H}^{2|2}$-model [51] and in particular imply Theorem 3.1.5. Relying on the recursive approach, Mirlin, Tikhonov and Sonner [40, 41] have predicted an intermediate multifractal phase for Efetov's model on finite regular trees. Their analysis relies on a partially heuristic study of *travelling wave* solutions to the recursion equations.

The goal of this section is to provide details on the recursive approach to the $\mathbb{H}^{2|2}$-model. We start by stating a recursion relation for the marginal "density" of the spin at the origin $\mathbf{u}_0$ on $\mathbb{T}_{d,n}$. This results in an integral equation (4.1.2) for superfunctions over $\mathbb{H}^{2|2}$. We show that, after passing to polar coordinates on $\mathbb{H}^{2|2}$, the problem reduces to the study of an integral for functions of a single real variable. In the finite-volume limit (taking a scaling limit in $h \searrow 0$ before passing to infinite volume), the recursion equation further simplifies and we show that its solution can be given explicitly in terms of the $t$-field (Proposition 4.1.5). Furthermore, our results on the intermediate phase for the VRJP (Theorem 3.1.3) imply a certain travelling-wave behaviour of the solutions. This provides rigorous evidence for behaviour analogous to such predicted by Mirlin, Tikhonov and Sonner for Efetov's Model [40, 41].

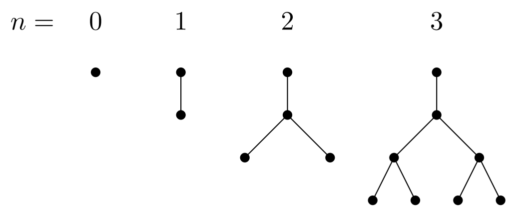



Figure 4.1: Illustration of $\mathbb{T}^*_{d,n}$ for $d = 2$ and various values of $n$. The topmost vertex is considered the root and denoted by 0.

### 4.1.1 A Supersymmetric Recursion Relation for the $\mathbb{H}^{2|2}$-Model

In this and the following sections, we consider the $\mathbb{H}^{2|2}$-model on $\mathbb{T}^*_{d,n}$, the rooted $(d+1)$-regular tree of depth $n \geq 0$ with an additional "dangling edge", as illustrated in Figure 4.1. Note that the root 0 denotes the topmost vertex in Figure 4.1.

Consider the $\mathbb{H}^{2|2}$-model on $\mathbb{T}^\star_{d,n}$ with magnetic field $h > 0$. We are interested in observables at the origin 0. The expectation of some observable $F(\mathbf{u}_0)$ for some superfunction $F \in C^\infty(\mathbb{H}^{2|2})$ can be expressed as

$$\begin{aligned}\langle F(\mathbf{u}_0)\rangle_{\beta,h}^{\mathbb{T}^\star_{d,n}} &= \int_{(\mathbb{H}^{2|2})^{\mathbb{T}^\star_{d,n}}} \prod_{i\in\mathbb{T}^\star_{d,n}} d\mathbf{u}_i\, F(\mathbf{u}_0)\, e^{\beta\sum_{ij}(\mathbf{u}_i\cdot\mathbf{u}_j+1)-h\sum_i(z_i-1)} \\ &= \int_{\mathbb{H}^{2|2}} d\mathbf{u}\, F(\mathbf{u}) f_n(\mathbf{u}) e^{-h(z-1)},\end{aligned} \tag{4.1.1}$$

where $f_n = f_{n;\beta,h}$ is the, up to a magnetic field factor, the marginal "density" of $\mathbf{u}_0$ after integrating out all spins away from the origin. In the following we will typically suppress the dependency of $f_n$ on $\beta$ and $h$.

**Proposition 4.1.1:** The *marginal functions* $f_n = f_{n;\beta,h} \in C^\infty(\mathbb{H}^{2|2})$, defined by the relation (4.1.1), satisfy the recursion relation

$$f_{n+1}(\mathbf{u}') = \int_{\mathbb{H}^{2|2}} d\mathbf{u}\, e^{\beta(\mathbf{u}\cdot\mathbf{u}'+1)-h(z-1)} f_n^d(\mathbf{u}) \quad \text{with} \quad f_0(\mathbf{u}) = 1. \tag{4.1.2}$$

*Proof.* For $n = 0$ the graph $\mathbb{T}^*_{d,n}$ is the singleton $\{0\}$. In that case, $f_0(\mathbf{u}) = 1$ is trivial. For $n \geq 1$, consider the unique neighbour $\bar{0}$ of the root 0. From $\bar{0}$ we have $d$ outgoing copies of $\mathbb{T}^*_{d,n-1}$ and integrating out all spins on these (sub)trees yields $d$ factors of $f_{n-1}(\mathbf{u}_{\bar{0}})$. There is an additional factor $e^{-h(z_{\bar{0}}-1)}$ for the magnetic field at $\bar{0}$. Together, this implies the claim. □

In previous work by Efetov and Zirnbauer, an analogous recursion for Efetov's model has been studied [36, 37, 88]. In their work, in particular for the study of the symmetry-broken phase, they considered fixed points of the recursion, which are formally obtained in the $n \to \infty$ limit:

$$f(\mathbf{u}') = \lim_{n\to\infty} f_n(\mathbf{u}') = \int_{\mathbb{H}^{2|2}} \mathrm{d}\mathbf{u}\, e^{\beta(\mathbf{u}\cdot\mathbf{u}'+1)-h(z-1)} f^d(\mathbf{u}). \tag{4.1.3}$$

Note that such a solution $f$ implicitly depends on $\beta, h$. Observe that for vanishing magnetic field $h = 0$ the constant function $f \equiv 1$ is always a solution, reflecting the global symmetry under isometries of $\mathbb{H}^{2|2}$, *i.e.* the supergroup $\mathrm{OSp}(2,1|2)$. However, a non-zero magnetic field $h > 0$ breaks this symmetry, in particular the invariance under Lorentz boost transformations. Spontaneous symmetry-breaking is said to occur if the solution $f \equiv 1$ is unstable under the perturbation by a magnetic field. Hence, the symmetry-broken phase is characterised by the existence of a non-trivial fixed point as $h \searrow 0$, while in the disordered phase any fixed point should converge to $f \equiv 1$ as $h \searrow 0$.

While above observations give a neat characterisation for the occurrence of a symmetry-breaking transition, mathematical justification for the study of the fixed point equation is not entirely clear. In particular, since the $f_n$ are superfunctions over $\mathbb{H}^{2|2}$, it is not obvious in which sense to take their limit as $n \to \infty$.

In order to put above reasoning on a more rigorous footing, we would like to get rid of the fermionic degrees of freedom in (4.1.2). In fact, in the following we will rewrite (4.1.2) in terms of polar coordinates, which will enable us to perform the integral over the fermionic and angular degrees of freedom, reduce the equation onto an integral equation in a single radial variable.

### 4.1.2 Polar Coordinates on $\mathbb{H}^{2|2}$.

Note that the integral operator in (4.1.2) is invariant under $\mathrm{OSp}(2,1|2)$-transformations which fix $z$, in other words under the stabiliser $\mathrm{Stab}(\mathbf{u}_{(0)}) = \mathrm{OSp}(2|2)$ of the *origin* $\mathbf{u}_{(0)} := (1,0,0,0,0) \in \mathbb{H}^{2|2}$. This motivates the use of generalised polar coordinates on $\mathbb{H}^{2|2}$ in order to simplify the recursion in (4.1.2). In order to define polar coordinates most easily, we consider the following elements of the Lie super-algebra $\mathfrak{osp}(2,1|2)$:

$$
H := \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & & & & \\ & 0 & 1 & & \\ & -1 & 0 & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}
$$
$$
Y_1 := \begin{pmatrix} 0 & & & & \\ & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_2 := \begin{pmatrix} 0 & & & & \\ & 0 & 0 & -1 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.1.4}
$$

$H$ and $X$ are even elements of $\mathfrak{osp}(2,1|2)$ and generate $xz$-boosts and $xy$-rotations, respectively. $Y_1$ and $Y_2$ are odd elements and generate supersymmetries between $x$ and the two fermionic coordinates $\xi, \eta$.

The following result is due to Zirnbauer and yields an explicit formula for the Grassmann integral of a superfunction over $\mathbb{H}^{2|2}$ in terms of polar coordinates.

**Theorem 4.1.2** (Polar Coordinates on $\mathbb{H}^{2|2}$ [20]): Let $H, X, Y_1, Y_2$ denote the elements of $\mathfrak{osp}(2,1|2)$ as defined in (4.1.4). Then the mapping

$$
(r, \varphi, \psi, \bar{\psi}) \mapsto e^{\varphi X} e^{\psi Y_1 + \bar{\psi} Y_2} e^{rH} \mathbf{u}_{(0)} \quad \text{with} \quad r > 0, \varphi \in [0, 2\pi), \psi, \bar{\psi} \text{ Grassmann}, \tag{4.1.5}
$$

gives a parametrisation of $\mathbb{H}^{2|2} \setminus \{\mathbf{u}_{(0)}\}$. For any $g \in C_c^\infty(\mathbb{H}^{2|2})$ it holds that

$$
\int_{\mathbb{H}^{2|2}} \mathrm{d}\mathbf{u}\, g(\mathbf{u}) = g(\mathbf{u}_{(0)}) + \int_0^\infty \frac{\mathrm{d}r}{\sinh(r)} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\, \partial_{\bar{\psi}} \partial_\psi\, g\left(e^{\varphi X} e^{\psi Y_1 + \bar{\psi} Y_2} e^{rH} \mathbf{u}_{(0)}\right). \tag{4.1.6}
$$

More explicitly, we have

$$
\mathbf{u} = \begin{pmatrix} z \\ x \\ y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cosh(r) \\ (1+\psi\bar{\psi})\sinh(r)\cos(\varphi) \\ (1+\psi\bar{\psi})\sinh(r)\sin(\varphi) \\ \psi \sinh(r) \\ \bar{\psi}\sinh(r) \end{pmatrix} = e^{\varphi X} e^{\psi Y_1 + \bar{\psi} Y_2} e^{rH} \mathbf{u}_{(0)}. \tag{4.1.7}
$$

For illustration, we compare this to polar coordinates for the usual hyperbolic plane $\mathbb{H}^2$, where $H$ and $X$ generate $xz$-boosts and $xy$-rotations, respectively. In that case, one would have, for any $g \in C_c^\infty(\mathbb{H}^2)$

$$\int_{\mathbb{H}^2} \mathrm{d}\mathbf{u}\, g(\mathbf{u}) = \int_0^\infty \mathrm{d}r \sinh(r) \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\, g\left(e^{\varphi X} e^{rH} \mathbf{u}_{(0)}\right). \tag{4.1.8}$$

The additional fermionic degrees of freedom in $\mathbb{H}^{2|2}$ modify the radial volume factor and introduce less intuitive "angular" supersymmetry generators $Y_1, Y_2$. The most important difference however, is the appearance of the constant "boundary term" $g(\mathbf{u}_{(0)})$ in (4.1.6). This singular contribution is due to a localisation phenomenon, which is unique to the supersymmetric case.

### 4.1.3 Reduced Recursion Relation in Polar Coordinates

In this subsection we rewrite the recursion (4.1.2) in terms of polar coordinates for $\mathbb{H}^{2|2}$ and integrate out the angular variables (including the fermionic degrees of freedom) to obtain a recursion for a function in one real coordinate.

**Proposition 4.1.3** (Reduced Recursion in Polar Coordinates): For $\beta, h > 0$, let $f_n(\mathbf{u})$ denote the superfunctions over $\mathbb{H}^{2|2}$ recursively defined in (4.1.2). There exist smooth functions $f_n^{\mathrm{rad}} \colon [1, \infty) \to \mathbb{R}$ such that

$$f_n^{\mathrm{rad}}(z) = f_n^{\mathrm{rad}}(\sqrt{1 + x^2 + y^2 - 2\xi\eta}) = f_n(\mathbf{u}), \tag{4.1.9}$$

as superfunctions over $\mathbb{H}^{2|2}$, that is as elements of the superalgebra $C^\infty(\mathbb{H}^{2|2})$. For $\lambda > 1$ we let $\mu = \mu(\lambda) = \sqrt{\lambda^2 - 1}$. That is, in terms of polar coordinates $\lambda = \cosh(r)$ and $\mu = \sinh(r)$. In the following, we let $\mu$ implicitly depend on $\lambda$. Then, the $f_n^{\mathrm{rad}}$ satisfy, for $\lambda' > 1$,

$$f_{n+1}^{\mathrm{rad}}(\lambda') = e^{-\beta(\lambda'-1)} + \int_1^\infty \frac{\mathrm{d}\lambda}{\mu^2} L_\beta(\lambda, \lambda') D_h(\lambda)\, (f_n^{\mathrm{rad}})^d(\lambda) \quad \text{and} \quad f_0^{\mathrm{rad}}(\lambda') = 1 \tag{4.1.10}$$

with the *kernel* and the *symmetry-breaking term* given respectively by

$$L_\beta(\lambda, \lambda') = \beta\mu\mu' I_1(\beta\mu\mu') e^{-\beta(\lambda\lambda'-1)} \quad \text{and} \quad D_h(\lambda) = e^{-h(\lambda-1)}, \tag{4.1.11}$$

with $I_1$ denoting the modified Bessel function of first kind.

*Proof.* We start with a crucial observation: Formally, the integral operator in (4.1.2) is invariant with respect to the reduced symmetry group $K = \mathrm{OSp}(2|2)$ and moreover $f_0 \equiv 1$ is $K$-invariant. Hence, all $f_n$ are $K$-invariant. In other words, at least formally the $f_n$ should only depend on the radial coordinate. In rigorous terms, $K$-invariance is defined in terms the action of its super-Lie algebra $\mathfrak{osp}(2|2)$ on $C^\infty(\mathbb{H}^{2|2})$. Namely the $f_n$ are $K$-invariant in the sense that

$$R.f_n(\mathbf{u}) = 0 \quad \text{for} \quad R \in \mathfrak{osp}(2|2), n \geq 0, \tag{4.1.12}$$

where the dot denotes the action of $R$ as a derivation. By Coulembier-Bie-Sommen [89, Theorem 3] a function $f_n$, which is rotationally invariant in the sense of (4.1.12), can be expressed as a function of the "radius" $x^2 + y^2 - 2\xi\eta = z^2 - 1$ and in particular as a function of $z$. In other words, there exist $f_n^{\text{rad}}$ as in (4.1.9). Hence, rewriting the recursion in (4.1.2) in polar coordinates using Theorem 4.1.2, we obtain

$$\begin{aligned} f_{n+1}^{\text{rad}}(\cosh r') = e^{-\beta(\cosh(r')-1)} + \int_0^\infty \frac{\mathrm{d}r}{\sinh(r)} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\, \partial_{\bar\psi}\partial_\psi \,(f_n^{\text{rad}})^d(\cosh r) \times \cdots \\ \cdots \times \exp[\beta \Big([e^{\varphi X} e^{\psi Y_1 + \bar\psi Y_2} e^{rH} u_{(0)}] \cdot [e^{r'H} u_{(0)}] + 1\Big) - h(\cosh(r) - 1)]. \end{aligned} \tag{4.1.13}$$

By (4.1.7) the interaction term in the exponential equals

$$\begin{aligned} &[e^{\varphi X} e^{\psi Y_1 + \bar\psi Y_2} e^{rH} u_{(0)}] \cdot [e^{r'H} u_{(0)}] \\ &= \begin{pmatrix} \cosh(r) \\ (1+\bar\psi\psi)\sinh(r)\cos(\varphi) \\ (1+\bar\psi\psi)\sinh(r)\sin(\varphi) \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cosh(r') \\ \sinh(r') \\ 0 \\ 0 \\ 0 \end{pmatrix} + [\text{odd terms}] \\ &= -\cosh(r)\cosh(r') + (1+\psi\bar\psi)\sinh(r)\sinh(r')\cos(\varphi) + [\text{odd terms}]. \end{aligned} \tag{4.1.14}$$

We suppressed the odd terms in above calculation as they do not contribute to the integral in (4.1.13). The nilpotent part of the exponent in (4.1.13) can be expanded

$$\exp[\beta\psi\bar\psi \sinh(r)\sinh(r')\cos(\varphi) + [\text{odd terms}]] = 1 + \beta\psi\bar\psi \sinh(r)\sinh(r')\cos(\varphi). \tag{4.1.15}$$

Plugging (4.1.14) and (4.1.15) into the recursion relation (4.1.13), we can perform the fermionic integral and isolate the angular average:

$$
\begin{aligned}
&f_{n+1}^{\mathrm{rad}}(\cosh r') \\
&= e^{-\beta(\cosh(r')-1)} + \int_0^\infty \frac{\mathrm{d}r}{\sinh(r)} \sinh(r)\sinh(r') e^{-\beta\cosh(r)\cosh(r')} e^{-h\cosh(r)} (f_n^{\mathrm{rad}})^d(\cosh r) \times \cdots \\
&\qquad \cdots \times \underbrace{\int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \cos(\varphi) e^{\beta\sinh(r)\sinh(r')\cos(\varphi)}}_{I_1(\beta\sinh(r)\sinh(r'))},
\end{aligned} \tag{4.1.16}
$$

where $I_1$ denotes the modified Bessel function of first kind. Passing to $\lambda = \cosh(r)$ and $\mu = \sinh(r)$ concludes the proof. □

### 4.1.4 Finite Volume Limit and Relation to a $t$-Field Martingale

Recall that a symmetry-breaking phase transition for the $\mathbb{H}^{2|2}$ is most easily characterised by considering the limit $\lim_{h\searrow 0}\lim_{n\to\infty} f_{n;\beta,h}$ of the marginal functions $f_n = f_{n;\beta,h}$ as defined by the recursion relation (4.1.2) or (4.1.10). This order of the limit corresponds to the *infinite volume limit*. In this section, we consider the *finite volume limit*, namely for fixed $n$ we extract a non-trivial scaling limit for $f_{n;\beta,h}$ as $h \searrow 0$. The resulting *finite-volume marginal functions* satisfy a simplified recursion relation which involves the $t$-field increment measure. An analogous recursion for Efetov's model has previously been studied in the literature [36, 40, 41].

**Proposition 4.1.4** (Finite Volume Limit for the Recursion Relation): Consider the radial marginal functions $f_n^{\mathrm{rad}} = f_{n;\beta,h}^{\mathrm{rad}}$ as defined in (4.1.11). Then

$$\psi_n(t) = \lim_{h\searrow 0} f_{n;\beta,h}^{\mathrm{rad}}(e^t/h) \quad \text{with} \quad t \in \mathbb{R} \tag{4.1.17}$$

exists point-wise. The functions $\psi_n$ satisfy the recursion relation

$$\psi_{n+1}(t') = \int_{\mathbb{R}} \mathrm{d}t\, l_\beta(t'-t) e^{-e^t} \psi_n^d(t) \quad \text{with} \quad \psi_0(t) = 1 \tag{4.1.18}$$

where

$$l_\beta(t) = \sqrt{\frac{\beta}{2\pi}}\, e^{-\beta(\cosh(t)-1)} e^{t/2} \tag{4.1.19}$$

coincides with the density of negative $t$-field increments (*cf.* Definition 3.2.12).

*Proof.* For convenience we write $f_{n;\beta,h} = f^{\mathrm{rad}}_{n;\beta,h}$. We proceed by induction over $n$: For $n=0$ one has $f_{0;\beta,h}(e^t/h) = 1 = \psi_0(t)$ for all $h \geq 0$. Now assume that $\psi_n(t)$ as in (4.1.17) is well-defined. We want to use the radial recursion relation (4.1.10) to obtain the scaling limit for $\psi_{n+1}$. For this, we first claim that for fixed $\omega, \omega' > 0$ one has

$$L_\beta(\omega/h, \omega'/h) \sim \frac{e^\beta}{h}\sqrt{\frac{\beta\omega\omega'}{2\pi}}\exp[-\frac{\beta}{2}\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}\right)]. \tag{4.1.20}$$

To see this, we use asymptotics $I_1(z) \sim e^z/\sqrt{2\pi z}$ as $z \to \infty$. This yields

$$\begin{aligned} L_\beta(\omega/h, \omega'/h) \sim \sqrt{\tfrac{\beta}{2\pi}}\,[(\tfrac{\omega}{h})^2-1]^{1/2}\,[(\tfrac{\omega'}{h})^2-1]^{1/2} \times \cdots \\ \cdots \times \exp[-\beta\Big((\tfrac{\omega}{h})^2(\tfrac{\omega'}{h})^2 - [(\tfrac{\omega}{h})^2-1]^{1/2}\,[(\tfrac{\omega'}{h})^2-1]^{1/2} - 1\Big)] \end{aligned} \tag{4.1.21}$$

To obtain (4.1.20) one simply expands the square roots in the exponent. Changing variables $\lambda \mapsto e^t/h$ in (4.1.10), we get

$$f_{n+1;\beta,h}(e^{t'}/h) = e^{-\beta(e^{t'}/h-1)} + \int_{-\log(h)}^{\infty} \mathrm{d}t\left[\frac{he^t}{e^{2t}-h^2}L_\beta(e^t/h, e^{t'}/h)\right]D_h(e^t/h)(f_{n;\beta,h})^d(e^t/h) \tag{4.1.22}$$

The first summand on the right hand side converges to zero as $h \searrow 0$. By (4.1.20), one may check that the term in brackets converges point-wise to $l_\beta(t'-t)$ as $h \searrow 0$. For the symmetry-breaking term we have $D_h(e^t/h) \to e^{-e^t}$ and by our inductive assumption we also have $f_{n;\beta,h}(e^t/h) \to \psi_n(t)$. By induction over (4.1.18) it is clear that $\psi_n(t) \leq 1$ for all $t \in \mathbb{R}$. The claim then follows by an application of the dominated convergence theorem. □

**Proposition 4.1.5:** Consider $\psi_n = \psi_{n;\beta}$ as defined in Proposition 4.1.4. Further, consider a $t$-field $\{T_x\}$ on $\mathbb{T}^*_d$, rooted at the origin. Then, we have

$$\psi_n(t) = \mathbb{E}_\beta[\exp(-e^t \textstyle\sum_{x\in\mathbb{T}^*_{d,n}\setminus\{0\}} e^{T_x})] \tag{4.1.23}$$

for all $n \geq 0$.

*Proof.* Write $G_n(t)$ for the right hand side of (4.1.23). For $n=0$, the sum in (4.1.23) is empty, hence we have $\psi_0(t) = 1 = G_0(t)$. For $n \geq 1$, we claim that $G_n$ satisfies the recursion equation

$$G_{n+1}(t) = \mathbb{E}[e^{-e^{t+T}}G_n^d(t+T)], \tag{4.1.24}$$

where $T$ is a random variable with law of the $t$-field increment (see Definition 3.2.12). In fact, by (4.1.18), $\psi_n$ satisfies the same recursion relation, hence proving (4.1.24) is sufficient.

Write $\bar{0}$ for the unique neighbour of 0. Let $\{T_x^{(i)}\}_{x\in\mathbb{T}^*_{d,n}}$ with $i = 1, \ldots, d$ denote $d$ independent copies of a $t$-field on $\mathbb{T}^*_{d,n}$. Then we have

$$G_{n+1}(t) = \mathbb{E}[\exp(-e^{t+T_{\bar{0}}}[1 + \textstyle\sum_{i=1}^{d} \sum_{x\in\mathbb{T}^*_{d,n}\setminus\{0\}} e^{T_x^{(i)}}])], \tag{4.1.25}$$

where $T_{\bar{0}}$ is distributed as a $t$-field increment, independent of the $\{T_x^{(i)}\}$. Using independence, the claim follows. □

**Remark 4.1.6** (Travelling-Wave Behaviour): By our analysis in the proof of Theorem 3.1.3, in particular (4.2.3), we have

$$\sum_{x\in\mathbb{T}^*_{d,n}\setminus\{0\}} e^{T_x} = e^{n[\max(\gamma_\beta,0)+o(1)]} \quad \text{a.s. as} \quad n\to\infty. \tag{4.1.26}$$

Consequently $\psi_n(t)$ should behave like a wave-front, moving to the left with velocity $\max(\gamma_\beta, 0)$. To make this precise, one needs finer control of the errors in (4.1.26). Outside the intermediate phase this is rather easy (see the proof of Theorem 3.1.3 for details). For the intermediate phase, this would need further analysis. This does however, provide a rigorous approach to statements found in the physics literature on Efetov's model [40, 41].

### 4.1.5 Addendum: Group-Theoretic Background on Polar Coordinates for $\mathbb{H}^{2|2}$.

The main goal of this section is to justify the first statement of Theorem 4.1.2, namely that (4.1.5) provides a parametrisation of $\mathbb{H}^{2|2}\setminus\{0\}$. In order to prove formula (4.1.6) in Theorem 4.1.2, one is only left with calculating the Berezinian (generalised Jacobian) of this parametrisation and deal with the "boundary term" at $\mathbf{u}_{(0)}$, which will give rise to the singular contribution in (4.1.6).

We follow Zirnbauer's exposition and refer to his work for more details [20]. In the following we will treat the supermanifold $\mathbb{H}^{2|2}$ and its isometry supergroup $\mathrm{OSp}(2,1|2)$ as if they were manifolds and Lie groups, respectively.

We assure the reader that details justifying this simplification are well-understood in the superanalysis literature and refer to Berezin's monograph for more information [90].

Let us write $G \coloneqq \mathrm{OSp}(2,1|2)$ for the group of isometries on $\mathbb{H}^{2|2}$ and $K \coloneqq \mathrm{OSp}(2|2) \subseteq G$ for its maximal compact subgroup. The latter can be identified with the stabiliser of the "origin" $\mathbf{u}_{(0)} = (1,0,0,0,0) \in \mathbb{H}^{2|2}$. Hence, we have

$$G/K \xrightarrow{\cong} \mathbb{H}^{2|2} \quad \text{via} \quad gK \mapsto g\mathbf{u}_{(0)}. \tag{4.1.27}$$

This is simply stating that $\mathbb{H}^{2|2}$ is a symmetric space with respect to the supergroup $G$. However, (4.1.27) also yields a parametrisation of $\mathbb{H}^{2|2}$. In order to get the polar decomposition from this, we make use of the KAK-decomposition for $G$.

Write $A \coloneqq \{e^{rH} : r \in \mathbb{R}\}$, for the subgroup consisting of $xz$-boosts. $A$ is a maximal abelian subgroup and we write $A^+ \coloneqq \{e^{rH} : r \in \mathbb{R}_+\}$ for its "positive" part. If we write $M$ for the centraliser of $A$ in $K$, then the quotient group $K/M$ is generated by the super-Lie algebra elements $X, Y_1, Y_2 \in \mathfrak{osp}(2,1|2)$, as defined in (4.1.4):

$$K/M = \left\{ e^{\varphi X} e^{\psi Y_1 + \bar{\psi} Y_2} M : \varphi \in \mathbb{R}, \psi, \bar{\psi} \text{ Grassmann} \right\}. \tag{4.1.28}$$

This fact is relevant since the following map is a diffeomorphism onto its image:

$$K/M \times A^+ \to G/K \cong \mathbb{H}^{2|2}, (kM, a) \mapsto kaK \tag{4.1.29}$$

Indeed, injectivity follows from uniqueness of the $KAK$-decomposition and differentiability is clear from the definition. We can think of $A^+$ as parametrising a non-compact *radial* coordinate, while the compact degrees of freedom $K/M$ are *angular*. The image of (4.1.29) is $\mathbb{H}^{2|2} \setminus \{\mathbf{u}_{(0)}\}$. In conclusion, this shows what we initially claimed, namely that the mapping in (4.1.5) is a parametrisation of $\mathbb{H}^{2|2} \setminus \{\mathbf{u}_{(0)}\}$.

## 4.2 Heuristics and $\mathbb{H}^{2|2}$-Fourier analysis

In this section we introduce the basics of Fourier analysis over the hyperbolic superplane $\mathbb{H}^{2|2}$. As a byproduct, we can diagonalise the integral operator that appears in the recursion (4.1.2), and characterise the phase transition in terms of spectral linear stability. Moreover, we give a rough heuristic for the $e^{C(\beta-\beta_c)^{-1/2}}$ behaviour from a Fourier perspective.

### 4.2.1 Fourier analysis and Harish-Chandra functions on $\mathbb{H}^{2|2}$

In this section we introduce a family of radially symmetric eigenfunctions of the Laplacian on $\mathbb{H}^{2|2}$. These so-called *Harish-Chandra functions* form a Fourier-type basis to analyse the recursion relation (4.1.2). In particular, for vanishing magnetic field $h = 0$ the integral kernel in (4.1.2) is diagonalised by these functions. The Fourier theory over $\mathbb{H}^{2|2}$ was developed by Zirnbauer [20] and we refer to his work for additional details on the harmonic analysis over $\mathbb{H}^{2|2}$.

**Harish-Chandra spherical functions on $\mathbb{H}^{2|2}$.** Let $t(\mathbf{u})$ denote the horospherical coordinate of $\mathbf{u} \in \mathbb{H}^{2|2}$. We define the *Harish-Chandra Spherical Functions* as angular averages of $\exp[(-\frac{1}{2}+i\rho)\,t(\mathbf{u})]$:

$$\varphi_\rho(\mathbf{u}) := \int \mathrm{d}\varphi\,\partial_\psi\partial_{\bar\psi}\exp[(-\tfrac{1}{2}+i\rho)t(e^{\varphi X}e^{\psi Y_1+\bar\psi Y_2}\mathbf{u})] \quad \text{with} \quad \rho\in\mathbb{R}. \tag{4.2.1}$$

The motivation behind this definition is that $\exp[(-\frac{1}{2}+i\rho)\,t(\mathbf{u})]$ are eigenfunctions of the $\mathbb{H}^{2|2}$-Laplacian and after angular averaging, they are *radial* eigenfunctions [20]. In particular, they take a more explicit expression in terms of the radial coordinate:

**Proposition 4.2.1:** Let $r$ denote the radial coordinate over $\mathbb{H}^{2|2}$. Then, for $\rho \in \mathbb{R}$ we have

$$\varphi_\rho(\mathbf{u}) = \varphi_\rho(r) = (-\tfrac{1}{2}+i\rho)\sinh(r)\int \mathrm{d}\varphi\frac{\cos(\varphi)}{(\cosh(r)+\cos(\varphi)\sinh(r))^{3/2-i\rho}}. \tag{4.2.2}$$

Moreover they satisfy the asymptotic

$$\varphi_\rho(r) \sim -\frac{1}{2\sqrt{\pi}}\frac{\Gamma(1-i\rho)}{\Gamma(1/2-i\rho)}e^{(3/2-i\rho)r} \quad \text{as} \quad r\to\infty. \tag{4.2.3}$$

We give a proof for this further below (Section 4.2.4). The Harish-Chandra functions form a Fourier-type basis in which to expand radial superfunctions and to diagonalise $\mathrm{OSp}(2,1|2)$-invariant operators over $\mathbb{H}^{2|2}$. A concrete example, that we will make use of, is the diagonalisation of the integral operator with kernel $e^{\beta(\mathbf{u}\cdot\mathbf{u}'+1)}$:

**Proposition 4.2.2:** For $\beta > 0, \rho \in \mathbb{R}$ we have

$$\kappa_\beta(\rho)\varphi_\rho(\mathbf{u}') = [L_\beta\varphi_\rho](\mathbf{u}') = \int_{\mathbb{H}^{2|2}}\mathrm{d}\mathbf{u}\,e^{\beta(\mathbf{u}\cdot\mathbf{u}'+1)}\varphi_\rho(\mathbf{u}). \tag{4.2.4}$$

with eigenvalues $\kappa_\beta(\rho)$ given by

$$\kappa_\beta(\rho) = \sqrt{\frac{\beta}{2\pi}} \int \mathrm{d}t\, e^{-\beta(\cosh(t)-1)} e^{i\rho t}. \tag{4.2.5}$$

In particular, they are real-valued and for any fixed $\beta > 0$, $\kappa_\beta(\rho)$ is maximal at $\rho = 0$.

In order to prove above statement, we need the following *addition theorem*, which we will not prove here.

**Lemma 4.2.3** (Addition Theorem for Harish-Chandra Functions [20]): For $g \in \mathrm{OSp}(2,1|2)$ and $\mathbf{u} \in \mathbb{H}^{2|2}$

$$\begin{aligned} &\varphi_\rho(g^{-1}\mathbf{u}) \\ &= \int \mathrm{d}\varphi\, \partial_\psi \partial_{\bar\psi} \exp[(-\tfrac{1}{2}+i\rho)t(e^{\varphi X} e^{\psi Y_1 + \bar\psi Y_2} g\mathbf{u}_{(0)})] \exp[(-\tfrac{1}{2}-i\rho)t(e^{\varphi X} e^{\psi Y_1+\bar\psi Y_2}\mathbf{u})], \end{aligned} \tag{4.2.6}$$

in the sense of superfunctions over $\mathrm{OSp}(2,1|2) \times \mathbb{H}^{2|2}$.

*Proof of Proposition 4.2.2.* Note, by radial symmetry of $\varphi_\rho$, that it suffices to check (4.2.4) for $\mathbf{u}' = e^{r'H}\mathbf{u}_{(0)}$. We start with the right hand side of (4.2.4) for this special case and apply boost-invariance of the integral kernel and the addition theorem (4.2.12):

$$\begin{aligned} &\int \mathrm{d}\mathbf{u}\, e^{\beta(\mathbf{u}\cdot[e^{r'H}\mathbf{u}_{(0)}]+1)} \varphi_\rho(\mathbf{u}) \\ &= \int \mathrm{d}\mathbf{u}\, e^{\beta(\mathbf{u}\cdot\mathbf{u}_{(0)}+1)} \varphi_\rho(e^{-r'H}\mathbf{u}) \\ &= \int \mathrm{d}\mathbf{u}\, e^{-\beta(z-1)} \int \mathrm{d}\varphi\, \partial_\psi \partial_{\bar\psi} \exp[(-\tfrac{1}{2}+i\rho)t(e^{\varphi X} e^{\psi Y_1+\bar\psi Y_2} e^{r'H}\mathbf{u}_{(0)})]\times \\ &\qquad\qquad \times \exp[(-\tfrac{1}{2}-i\rho)t(e^{\varphi X} e^{\psi Y_1+\bar\psi Y_2}\mathbf{u})] \end{aligned} \tag{4.2.7}$$

In the last line, we may interchange the order of integration and note that the $\mathbf{u}$-integral

$$\begin{aligned} &\int \mathrm{d}\mathbf{u}\, e^{-\beta(z-1)} \exp[(-\tfrac{1}{2}-i\rho)t(e^{\varphi X} e^{\psi Y_1+\bar\psi Y_2}\mathbf{u})] \\ &= \int \mathrm{d}\mathbf{u}\, e^{-\beta(z-1)} \exp[(-\tfrac{1}{2}-i\rho)t(\mathbf{u})] =: \kappa_\beta(\rho) \end{aligned} \tag{4.2.8}$$

is independent of $\varphi, \psi, \bar\psi$. The remaining integral in (4.2.7) equals $\varphi_\rho(e^{r'H}\mathbf{u}_{(0)})$ by definition of the Harish-Chandra functions in (4.2.1). Hence, we have

$$\int \mathrm{d}\mathbf{u}\, e^{\beta(\mathbf{u}\cdot[e^{r'H}\mathbf{u}_{(0)}]+1)} \varphi_\rho(\mathbf{u}) = \kappa_\beta(\rho)\varphi_\rho(e^{r'H}) \tag{4.2.9}$$

This verifies (4.2.4) together with an explicit expression for the eigenvalues $\kappa_\beta(\rho)$ given in (4.2.8). The latter can be further simplified by passing to horospherical coordinates:

$$\begin{aligned}\kappa_\beta(\rho) &= \int \mathrm{d}\mathbf{u}\, e^{-\beta(z-1)} e^{(-\frac{1}{2}-i\rho)t(\mathbf{u})} \\ &= \int \frac{\mathrm{d}t}{e^t} \frac{\mathrm{d}s}{2\pi} \partial_\psi \partial_{\bar\psi} e^{-\beta(\cosh(t)+e^t(\frac{1}{2}s^2+\bar\psi\psi)-1)} e^{-(\frac{1}{2}-i\rho)t} \\ &= \sqrt{\frac{\beta}{2\pi}} \int \mathrm{d}t\, e^{-\beta(\cosh(t)-1)} e^{i\rho t}\end{aligned} \tag{4.2.10}$$

This expression is clearly maximal at $\rho = 0$, due to positivity of $e^{-\beta(\cosh(t)-1)}$. □

### 4.2.2 Characterisation of $\beta_c$: instability of the symmetric solution

As we discussed in Section 4.1.1, one may understand a symmetry-breaking phase transition of the $\mathbb{H}^{2|2}$-model on the $d$-ary tree in terms of the fixed points of the recursion relation (4.1.2), *i.e.* solutions $f = f_{\beta,h}$ to (4.1.3). In this section we show that linear stability of the trivial solution for $h = 0$ indeed characterises the subcritical phase. In particular, the critical inverse temperature agrees with the one for the VRJP/$t$-field.

**Theorem 4.2.4:** Let $\beta_c > 0$ denote the critical inverse temperature as defined in Proposition 3.2.14. The symmetric solution $f \equiv 1$ to the fixed point equation

$$f(\mathbf{u}') = [L_\beta f^d](\mathbf{u}') = \int_{\mathbb{H}^{2|2}} \mathrm{d}\mathbf{u}\, e^{\beta(\mathbf{u}\cdot\mathbf{u}'+1)} f^d(\mathbf{u}). \tag{4.2.11}$$

is linearly stable under radially symmetric perturbations if and only $\beta < \beta_c$.

*Proof.* Consider the linearisation of the right hand side (4.2.11) around the $f \equiv 1$ for $h = 0$:

$$\int_{\mathbb{H}^{2|2}} \mathrm{d}\mathbf{u}\, e^{\beta(\mathbf{u}\cdot\mathbf{u}'+1)} (1+\epsilon g(\mathbf{u}))^d = 1 + \epsilon d \int_{\mathbb{H}^{2|2}} \mathrm{d}\mathbf{u}\, e^{\beta(\mathbf{u}\cdot\mathbf{u}'+1)} g(\mathbf{u}) + O(\epsilon^2). \tag{4.2.12}$$

Hence, linear stability of the fixed point $f \equiv 1$ is governed by the operator

$$g(\mathbf{u}) \mapsto d \int_{\mathbb{H}^{2|2}} \mathrm{d}\mathbf{u}'\, e^{\beta(\mathbf{u}\cdot\mathbf{u}'+1)} g(\mathbf{u}'), \tag{4.2.13}$$

for radially symmetric perturbations $g(\mathbf{u})$. Recall Proposition 4.2.2: The spectrum of the operator (4.2.13) over radial functions is given by $\{d\kappa_\beta(\rho)\}_{\rho\in\mathbb{R}}$. Recall that the $\kappa_\beta(\rho)$ are real-valued and (for fixed $\beta$) attain their maximum at $\rho = 0$. The fixed point $f \equiv 1$ is linearly

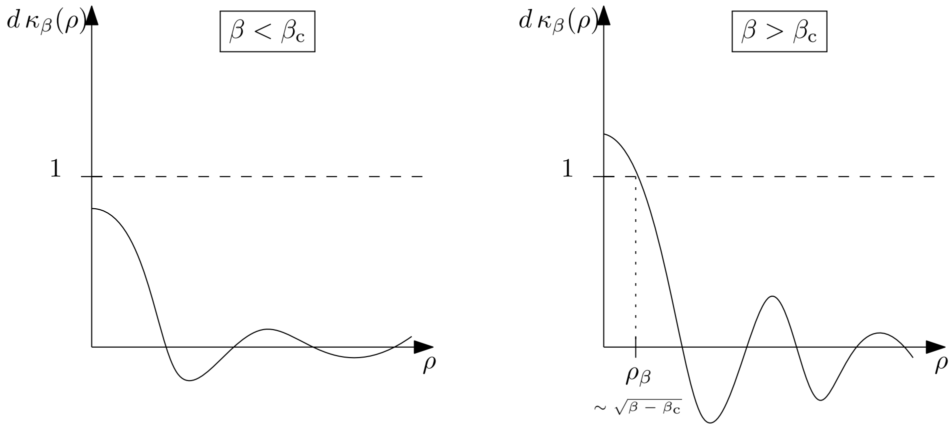



Figure 4.2: Illustration of the spectrum of the operator $L_\beta$ in the subcritical and supercritical phases, respectively. In the subcritical phase, $\beta < \beta_c$ the spectrum is stricly below 1. Consequently, iterative applications of $L_\beta$ converge to zero. In the supercritical phase $\beta > \beta_c$, the spectrum of $L_\beta$ is positive for $|\rho| < \rho_\beta \sim (\beta - \beta_c)^{1/2}$. In other words, iterative application of $L_\beta$ will suppress Fourier modes $\rho > \rho_\beta$ and amplify modes $\rho < \rho_\beta$.

stable if and only if the spectrum of (4.2.13) is contained in the unit disk. Equivalently, if and only if

$$\max_{\rho\in\mathbb{R}} \kappa_\beta(\rho) = \kappa_\beta(0) = \sqrt{\frac{\beta}{2\pi}} \int \mathrm{d}t\, e^{-\beta(\cosh(t)-1)} < 1/d. \tag{4.2.14}$$

This is equivalent to $\beta < \beta_c$. □

Note that the restriction to radially symmetric perturbations in Theorem 4.2.4 is natural. In fact, any solution obtained by a limit of the recursion (4.1.2) will necessarily be radially symmetric.

### 4.2.3 Heuristic derivation of near-critical behaviour for the $\mathbb{H}^{2|2}$-model

The marginal functions satisfy the recursion in (4.1.2), in other words

$$f_{n+1} = L_\beta[e^{-h(z-1)} f_n^d(\mathbf{u})] \quad \text{with} \quad f_0(\mathbf{u}) \equiv 1. \tag{4.2.15}$$

For fixed $n$ as $h \searrow 0$ we have

$$f_n = 1 - L_\beta^n[e^{-h(z-1)} - 1] + O(h^2). \tag{4.2.16}$$

By Proposition 4.2.2, the spectrum of the operator $L_\beta$ is given by $d\kappa_\beta(\rho)$. In order to say something about $f_n$, we can ask for which $\rho$ the modulus $d\kappa_\beta(\rho)$ is larger or smaller than 1. In fact, the existence of values $\rho$ for which the modulus is larger than 1 characterises the supercritical phase, and the width of the interval for which this holds is relevant for the near-critical behaviour. The qualitative picture is given in Figure 4.2. More quantitatively, we have

$$d\kappa_\beta(\rho) - 1 = a\,[\beta - \beta_c] - b(\beta)\,\rho^2 + O([\beta - \beta_c]^2 + \rho^4) \quad \text{as} \quad \beta \to \beta_c, \tag{4.2.17}$$

with $a, b > 0$. In the slightly supercritical regime, $0 < \beta_c - \beta \ll 1$ and disregarding error terms, the right hand side of (4.2.17) is positive for $|\rho| \le (1 + o(1))\rho_\beta$ with

$$\rho_\beta := \sqrt{a/b(\beta_c)}\,(\beta - \beta_c)^{1/2}. \tag{4.2.18}$$

In other words, $L_\beta$ amplifies Fourier modes in that range (since $d\kappa_\beta(\rho) > 1$), while it suppresses the ones outside (since $d\kappa_\beta(\rho) < 1$). Hence, we expect the Fourier modes of $f_n$ to be concentrated in $0 < \rho < \rho_\beta \sim (\beta - \beta_c)^{1/2}$. Consequently, in the conjugate variable, namely the horospherical $t$-variable (see (4.2.1)), we expect $f_n$ to be delocalised on a scale $\sim (\beta - \beta_c)^{-1/2}$. In other words, we would expect $\lim_{h \searrow 0} \lim_{n\to\infty} \langle e^{2t_0} \rangle_{n,h}$ to be of order $e^{C(\beta - \beta_c)^{-1/2}}$, which is precisely what we show.

### 4.2.4 Addendum: Harish-Chandra functions in radial coordinates

*Proof of Proposition 4.2.1.* Recalling that $e^t = z + x$ and using the explicit representation (4.1.7) for polar coordinates, we have

$$e^t = \cosh(r) + (1 + \bar\psi\psi)\sinh(r)\cos(\varphi). \tag{4.2.19}$$

Hence,

$$\varphi_\rho(\mathbf{u}) = \int \mathrm{d}\varphi\, \partial_\psi \partial_{\bar\psi} \frac{1}{[\cosh(r) + (1 + \bar\psi\psi)\sinh(r)\cos(\varphi)]^{1/2 - i\rho}}. \tag{4.2.20}$$

Expanding the integrand in the nilpontent variable $\bar\psi\psi$ one has

$$\begin{aligned} &\frac{1}{[\cosh(r) + (1 + \bar\psi\psi)\sinh(r)\cos(\varphi)]^{1/2 - i\rho}} \\ &= \frac{1}{[\cosh(r) + \sinh(r)\cos(\varphi)]^{1/2 - i\rho}} - \bar\psi\psi \frac{(1/2 - i\rho)\sinh(r)\cos(\varphi)}{[\cosh(r) + \sinh(r)\cos(\varphi)]^{3/2 - i\rho}}. \end{aligned} \tag{4.2.21}$$

Plugging (4.2.21) into (4.2.20) we can perform the fermionic integration:

$$\varphi_\rho(\mathbf{u}) = (-\tfrac{1}{2} + i\rho)\sinh(r) \int \mathrm{d}\varphi \frac{\cos(\varphi)}{[\cosh(r) + \sinh(r)\cos(\varphi)]^{3/2 - i\rho}}, \tag{4.2.22}$$

which proves the claim.

In order to derive the asymptotic (4.2.3), we follow the line of thought of Zirnbauer in the Appendix of [36]. We start by using the elementary identity

$$x^{-3/2+i\rho} = \frac{1}{\Gamma(3/2-i\rho)} \int_0^\infty \frac{\mathrm{d}z}{z} z^{3/2-i\rho} e^{-xz} \quad \text{for} \quad x \in \mathbb{R} \tag{4.2.23}$$

in above expression. This yields

$$\begin{aligned} \varphi_\rho(r) &= -\frac{\frac{1}{2}-i\rho}{\Gamma(3/2-i\rho)} \sinh(r) \int_0^\infty \frac{\mathrm{d}z}{z} z^{3/2-i\rho} e^{-z\cosh(r)} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \cos(\varphi) e^{-z\sinh(r)\cos(\varphi)} \\ &= -\frac{1}{\Gamma(1/2-i\rho)} \sinh(r) \int_0^\infty \frac{\mathrm{d}z}{z} z^{3/2-i\rho} e^{-z\cosh(r)} I_1(z\sinh(r)), \end{aligned} \tag{4.2.24}$$

where $I_\alpha$ is the modified Bessel function of first kind. We may use the well-known asymptotic $I_1(x) \sim e^x/\sqrt{2\pi x}$ for $x \to \infty$, which is easily derived by Laplace's method. Using this, for $r \to \infty$ we have

$$\begin{aligned} \varphi_\rho(r) &\sim -\frac{1}{\Gamma(1/2-i\rho)} \sinh(r) \int_0^\infty \frac{\mathrm{d}z}{z} z^{3/2-i\rho} \frac{e^{-z(\cosh(r)-\sinh(r))}}{\sqrt{2\pi z\sinh(r)}} \\ &\sim -\frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(1/2-i\rho)} \sqrt{\sinh(r)} \int_0^\infty \frac{\mathrm{d}z}{z} z^{1-i\rho} e^{-ze^{-r}} \end{aligned} \tag{4.2.25}$$

In fact, we had to be a bit more careful in replacing $I_1$ by its asymptotic under the integral: This approximation is only valid for $z \gg e^{-r}$. However, the contribution from the rest of the domain is negligible for $r \to \infty$, so above approximation is valid. To conclude, we note that by rescaling $z \mapsto e^r z$ the integral in the last line of (4.2.25) is evaluated to $e^{(1-i\rho)r}\Gamma(1-i\rho)$. In conclusion

$$\varphi_\rho(r) \sim -\frac{1}{\sqrt{2\pi}} \frac{\Gamma(1-i\rho)}{\Gamma(1/2-i\rho)} e^{(3/2-i\rho)r} \quad \text{as} \quad r \to \infty. \tag{4.2.26}$$

□

# Chapter 5

# Probabilistic definition of the Schwarzian field theory [SCHW]

**Abstract:** We define the Schwarzian Field Theory as a finite measure on $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ and compute its generalised partition functions exactly using methods of stochastic analysis. Our results rigorously implement an approach by Belokurov–Shavgulidze. The Schwarzian Field Theory has attracted recent attention due to its role in the analysis of the Sachdev–Ye–Kitaev model and as the proposed holographic dual to Jackiw–Teitelboim gravity. In two companion papers by Losev, the predicted exact cross-ratio correlation functions for non-crossing Wilson lines and the large deviations are derived from the probability measure.

## 5.1 Introduction and main results

### 5.1.1 Introduction

The Schwarzian Field Theory arose in the study of Sachdev–Ye–Kitaev (SYK) random matrix model, see [91] and [92] for introductions, and it also appears interesting with different motivation. In particular, it is an example of a highly nonlinear but, at least formally, exactly solvable Euclidean field theory in $0+1$ dimensions, with connections to Liouville Field Theory, infinite dimensional symplectic geometry, two-dimensional Yang-Mills theory, and other topics; references are given below. Perhaps most intriguingly, it has been proposed as the holographic dual to Jackiw–Teitelboim (JT) gravity on the Poincaré disk, see for example [93–95]. In [96] the partition function of the Schwarzian Field Theory was computed exactly by a formal application of the Duistermaat–Heckman theorem on the infinite dimensional space $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$, and in [97] the natural cross-ratio correlation functions of the

Schwarzian Field Theory were obtained via an application of the conformal bootstrap and the DOZZ formula to a degenerate limit of the two-dimensional Liouville Field Theory. Further perspectives on the Schwarzian Field Theory, which are discussed in relation to our results below, were proposed in [98] and [99].

The goal of this paper is to define a finite Borel measure on $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ that corresponds to the Schwarzian Field Theory and then compute the partition function of it and its generalisations. This measure should formally be given by (see [96, (1.1)])

$$\mathrm{d}\mathscr{M}_{\sigma^2}(\varphi) = \exp\left\{+\frac{1}{\sigma^2}\int_0^1 \left[\mathcal{S}_\varphi(\tau) + 2\pi^2\varphi'^2(\tau)\right]\mathrm{d}\tau\right\}\frac{\prod_{\tau\in[0,1)}\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}}{\mathrm{PSL}(2,\mathbb{R})}, \tag{5.1.1}$$

where $\mathcal{S}_\varphi(\tau)$ is the Schwarzian derivative of $\varphi$ defined by

$$\mathcal{S}_\varphi(\tau) = \mathcal{S}(\varphi,\tau) = \left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)' - \frac{1}{2}\left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)^2, \tag{5.1.2}$$

and the measure $\mathscr{M}_{\sigma^2}$ should be supported on the topological space $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$, where $\mathbb{T} = [0,1]/\{0\sim 1\}$ is the unit circle parametrised by the angle $[0,1)$, and $\mathrm{Diff}^1(\mathbb{T})$ is the space of $C^1$ orientation-preserving diffeomorphisms of $\mathbb{T}$, see Section 5.1.4. The $\mathrm{PSL}(2,\mathbb{R})$ action on $\mathrm{Diff}^1(\mathbb{T})$ implicit in the quotient in (5.1.1) is described in the next paragraph. Heuristically, the formal density (5.1.1) only depends on the orbit of this action and the quotient by $\mathrm{PSL}(2,\mathbb{R})$ therefore makes sense.

The $\mathrm{PSL}(2,\mathbb{R})$ action on $\mathrm{Diff}^1(\mathbb{T})$ arises from left composition by conformal diffeomorphisms of the unit disk restricted to the boundary, which is identified with $\mathbb{T}$. Explicitly, it is instructive to map the circle to the real line and consider $f(\tau) = \tan(\pi\varphi(\tau) - \frac{\pi}{2})$ instead of $\varphi(\tau)$, where $\tau \in \mathbb{T}$ remains on the circle. In terms of this variable, the exponential in the formal density (5.1.1) of the measure can be written as

$$\exp\left\{+\frac{1}{\sigma^2}\int_0^1 \mathcal{S}(\tan(\pi\varphi - \tfrac{\pi}{2}),\tau)\,\mathrm{d}\tau\right\}, \tag{5.1.3}$$

again see Section 5.1.4, explaining the name Schwarzian Field Theory. The bijection $\varphi \in (0,1) \mapsto f = \tan(\pi\varphi - \frac{\pi}{2}) \in \mathbb{R}$ is the restriction to the boundary of the standard conformal map $z \mapsto i\frac{1+z}{1-z}$ from the unit disk in $\mathbb{C}$ to the upper half plane, on which $\mathrm{PSL}(2,\mathbb{R})$ acts by the fractional linear transformation

$$f \mapsto M \circ f = \frac{af+b}{cf+d}, \qquad M = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}). \tag{5.1.4}$$

Even though this $\mathrm{PSL}(2,\mathbb{R})$ action is by left composition, we will call it the right action on $\mathrm{Diff}^1(\mathbb{T})$, following the discussion in [96, Section 2.1], where this action is interpreted as an action on the inverse of $\varphi$. The Schwarzian is invariant under this $\mathrm{PSL}(2,\mathbb{R})$ action, i.e.,

$$\mathcal{S}(M \circ f, \tau) = \mathcal{S}(f, \tau) \qquad \text{for any } M \in \mathrm{PSL}(2,\mathbb{R}), \tag{5.1.5}$$

and it vanishes if and only if $f$ itself is a fractional linear transformation. It can further be argued that $\prod_{\tau\in[0,1)} \frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}$ in the formal expression (5.1.1) should be the non-existent Haar measure on the group $\mathrm{Diff}^1(\mathbb{T})$, and in particular be invariant under the $\mathrm{PSL}(2,\mathbb{R})$ action, see [96, Section 2.2]. We abuse the notation throughout the paper and use the same symbol for $\varphi \in \mathrm{Diff}^1(\mathbb{T})$ and its conjugacy class $\varphi \in \mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ when only the latter is relevant.

### 5.1.2 Main results

We follow the plan proposed in [98, 100], and interpret the measure (5.1.1) as an appropriate change of variables of a reweighted Brownian bridge and a Lebesgue measure quotiented by $\mathrm{PSL}(2,\mathbb{R})$, and then verify that the obtained measure satisfies the desired properties.
Essentially, using the parametrisation

$$\varphi(\tau) = \Theta + \frac{\int_0^\tau e^{\xi(t)}\,\mathrm{d}t}{\int_0^1 e^{\xi(t)}\,\mathrm{d}t}, \tag{5.1.6}$$

with $\xi\colon [0,1] \to \mathbb{R}$ and $\Theta \in \mathbb{R}$ a constant, one has

$$-\mathcal{S}(\tan(\pi\varphi - \tfrac{\pi}{2}), \tau) = \frac{1}{2}\xi'(\tau)^2 - \xi''(\tau) - 2\pi^2 \left( \frac{e^{\xi(\tau)}}{\int_0^1 e^{\xi(t)}\,\mathrm{d}t} \right)^2, \tag{5.1.7}$$

and under this change of variable, heuristically,

$$\prod_\tau \frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)} = \mathrm{d}\Theta \prod_\tau \mathrm{d}\xi(\tau). \tag{5.1.8}$$

Thus it is natural to interpret the measure with action given by (5.1.7) in terms of a reweighted Brownian bridge, whose formal action is $\frac{1}{2}\int \xi'(\tau)^2\,\mathrm{d}\tau$, and a constant $\Theta$ (the zero mode), distributed according to the Lebesgue measure. In Section 5.2, we give a precise version of this construction and show that it indeed leads to an $\mathrm{PSL}(2,\mathbb{R})$-invariant infinite Borel measure $\widetilde{\mathscr{M}}_{\sigma^2}$ on $\mathrm{Diff}^1(\mathbb{T})$ which can then be quotiented to obtain a finite measure $\mathscr{M}_{\sigma^2}$ on $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$.

We prove that the unquotiented measure $\widetilde{\mathscr{M}}_{\sigma^2}$ we construct satisfies a change of measure formula that is consistent with the goal that the action functional corresponding to the measure should be proportional to the Schwarzian derivative. Indeed, for $f,g \in C^3$, the Schwarzian derivative satisfies the following chain rule:

$$\mathcal{S}(g\circ f,\tau) = \mathcal{S}(f,\tau) + \mathcal{S}(g, f(\tau)) f'(\tau)^2. \tag{5.1.9}$$

It follows that $\mathcal{S}(\tan(\pi\psi - \frac{\pi}{2}),\psi) = 2\pi^2$ and

$$\mathcal{S}(\tan(\pi(\psi\circ\varphi) - \tfrac{\pi}{2}),\tau) = \mathcal{S}(\tan(\pi\varphi - \tfrac{\pi}{2}),\tau) + \Big(\mathcal{S}(\tan(\pi\psi - \tfrac{\pi}{2}),\varphi(\tau)) - 2\pi^2\Big)\varphi'(\tau)^2, \tag{5.1.10}$$

see Section 5.1.4. Thus changing variables in (5.1.1) and (5.1.3) from $\varphi$ to $\psi\circ\varphi$ for a fixed $\psi \in \mathrm{Diff}^3(\mathbb{T})$, we expect that, if the action is given by the Schwarzian (5.1.3), then the Radon–Nikodym derivative of the measures should be given by the exponential of the second term on the right-hand side of (5.1.10). That this is indeed the case is the content of the following theorem. It can be regarded as the analogue of the Girsanov formula for Brownian motion under the change of variable from $B$ to $B+h$.

In the following statements, diffeomorphisms $\psi \in \mathrm{Diff}^3(\mathbb{T})$ act on $\varphi \in \mathrm{Diff}^1(\mathbb{T})$ by left composition, i.e. $\psi\circ\varphi \in \mathrm{Diff}^1(\mathbb{T})$, and $\psi^*\widetilde{\mathscr{M}}_{\sigma^2}$ denotes the pullback of the measure $\widetilde{\mathscr{M}}_{\sigma^2}$ under the action of $\psi$,

$$\psi^*\widetilde{\mathscr{M}}_{\sigma^2}(A) = \psi_*^{-1}\widetilde{\mathscr{M}}_{\sigma^2}(A) = \widetilde{\mathscr{M}}_{\sigma^2}(\psi\circ A), \tag{5.1.11}$$

where $\psi\circ A := \big\{\psi\circ\varphi \,\big|\, \varphi\in A\big\}$. We further identify $M \in \mathrm{PSL}(2,\mathbb{R})$ with its action $\psi$ via

$$f \mapsto M\circ f = \frac{af+b}{cf+d}, \qquad f = \tan(\pi\varphi - \tfrac{\pi}{2}), \qquad M\circ f = \tan(\pi(\psi\circ\varphi) - \tfrac{\pi}{2}). \tag{5.1.12}$$

For any such $\psi$ one has $\mathcal{S}(\tan(\pi\psi - \frac{\pi}{2}),\varphi) = \mathcal{S}(M\circ f,\varphi) = \mathcal{S}(f,\varphi) = \mathcal{S}(\tan(\pi\varphi - \frac{\pi}{2}),\varphi) = 2\pi^2$.

**Theorem 5.1.1:** The constructed measure $\widetilde{\mathscr{M}}_{\sigma^2}$ is supported on $\mathrm{Diff}^1(\mathbb{T})$ and satisfies the following change of variable formula: for any $\psi \in \mathrm{Diff}^3(\mathbb{T})$,

$$\frac{\mathrm{d}\psi^*\widetilde{\mathscr{M}}_{\sigma^2}(\varphi)}{\mathrm{d}\widetilde{\mathscr{M}}_{\sigma^2}(\varphi)} = \exp\left\{\frac{1}{\sigma^2}\int_0^1 \left[\mathcal{S}(\tan(\pi\psi - \tfrac{\pi}{2}), \varphi(\tau)) - 2\pi^2\right]\varphi'(\tau)^2\,\mathrm{d}\tau\right\}. \tag{5.1.13}$$

As a consequence, $\widetilde{\mathscr{M}}_{\sigma^2}$ is $\mathrm{PSL}(2,\mathbb{R})$-invariant, where we recall that the action of $M \in \mathrm{PSL}(2,\mathbb{R})$ is identified with the action $\psi \circ \varphi$ via (5.1.12) and that then $\mathcal{S}(\tan(\pi\psi - \frac{\pi}{2})) = 2\pi^2$.

We also confirm the exact formula for the partition function, i.e., the total mass of the quotient measure, computed in [96] by using a formal application of the Duistermaat–Heckman theorem. The normalisation of the partition function is explained in Remark 5.1.4 below.

**Theorem 5.1.2:** The measure $\widetilde{\mathscr{M}}_{\sigma^2}$ from Theorem 5.1.1 can be quotiented by $\mathrm{PSL}(2,\mathbb{R})$ and the resulting measure $\mathrm{d}\mathscr{M}_{\sigma^2}$ on $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ is finite and has total mass

$$Z(\sigma^2) = \left(\frac{2\pi}{\sigma^2}\right)^{3/2}\exp\left(\frac{2\pi^2}{\sigma^2}\right) = \int_0^\infty e^{-\sigma^2 E}\sinh(2\pi\sqrt{2E})\,2\,\mathrm{d}E. \tag{5.1.14}$$

The right-hand side of (5.1.14) has the form of a Laplace transform of a spectral density $\rho(E) = 2\sinh(2\pi\sqrt{2E})$. It is expected that it approximates the thermal partition function $\mathbb{E}[\mathrm{tr}(e^{-\beta H})]$ of the SYK model $H$ (and with $\sigma^2$ corresponding to inverse temperature $\beta$). For further discussion, see for example [96, Section 2.4].
The above computation of the partition function also applies to generalised measures on $\mathrm{Diff}^1(\mathbb{T})$ which are similar to $\widetilde{\mathscr{M}}_{\sigma^2}$ but not $\mathrm{PSL}(2,\mathbb{R})$ invariant (and thus cannot be quotiented by $\mathrm{PSL}(2,\mathbb{R})$), see Section 5.3 around (5.3.3). These measures correspond to other Virasoro coadjoint orbits, discussed in [99, 101, 102].

The previous theorems can further be generalised by introduction of a non-constant metric $\rho^2 \colon \mathbb{T} \to \mathbb{R}_+$. Similar to [96, Appendix C], the Schwarzian Field Theory with background metric $\rho^2$ on $\mathbb{T}$ is formally given by

$$\mathrm{d}\mathscr{M}_\rho(\varphi) = \exp\left\{\int_0^1 \mathcal{S}(\tan(\pi\varphi - \tfrac{\pi}{2}), \tau)\,\frac{\mathrm{d}\tau}{\rho(\tau)}\right\}\frac{\prod_{\tau\in[0,1)}\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}}{\mathrm{PSL}(2,\mathbb{R})}, \tag{5.1.15}$$

where $\rho = \sqrt{\rho^2}$ is the positive square root of the metric $\rho^2$. Thus the constant choice $\rho(\tau) = \sigma^2$ for all $\tau \in \mathbb{T}$ corresponds to (5.1.1). We again define this measure precisely in Section 5.2.

**Theorem 5.1.3:** For $\rho\colon \mathbb{T}\to\mathbb{R}_+$ in $C^1(\mathbb{T})$, there is a $\mathrm{PSL}(2,\mathbb{R})$-invariant positive measure $\widetilde{\mathscr{M}_\rho}$ on $\mathrm{Diff}^1(\mathbb{T})$ satisfying the following change of variable formula: for any $\psi\in\mathrm{Diff}^3(\mathbb{T})$,

$$\frac{\mathrm{d}\psi^*\widetilde{\mathscr{M}_\rho}(\varphi)}{\mathrm{d}\widetilde{\mathscr{M}_\rho}(\varphi)} = \exp\left\{\int_0^1 \left[\mathcal{S}(\tan(\pi\psi-\tfrac{\pi}{2}),\varphi(\tau)) - 2\pi^2\right]\varphi'(\tau)^2\frac{\mathrm{d}\tau}{\rho(\tau)}\right\}, \tag{5.1.16}$$

and its quotient $\mathrm{d}\mathscr{M}_\rho$ by $\mathrm{PSL}(2,\mathbb{R})$ has total mass

$$Z(\rho) = \exp\left\{\frac{1}{2}\int\frac{\rho'(\tau)^2}{\rho(\tau)^3}\,d\tau\right\}Z(\sigma_\rho^2), \qquad \text{where } \sigma_\rho^2 = \int\rho\,\mathrm{d}\tau, \tag{5.1.17}$$

and $Z(\sigma^2)$ denotes the partition function (5.1.14).

**Remark 5.1.4:** The normalisation of measures $\mathscr{M}_{\sigma^2}$ and $\mathscr{M}_\rho$ uses the following convention. Essentially, we *define* the partition function of the unnormalised Brownian bridge with boundary conditions $\xi(0)=0$ and $\xi(1)=a$ and metric $\rho^2:[0,1]\to\mathbb{R}_+$, corresponding to the action

$$\frac{1}{2}\int_0^1 \xi'(\tau)^2\frac{\mathrm{d}\tau}{\rho(\tau)}, \tag{5.1.18}$$

see Section 5.2.1 for the precise definition, by

$$Z^{\mathrm{BB}}(\rho) = \left(2\pi\int_0^1\rho(t)\,\mathrm{d}t\right)^{-1/2}\exp\left\{-\frac{a^2}{2\int_0^1\rho(t)\,\mathrm{d}t}\right\}. \tag{5.1.19}$$

This normalisation implies a natural composition property for unnormalised Brownian bridges (see Section 5.2.1) and is also proportional to the square root of the $\zeta$-function regularised determinant of the Laplacian. The latter is a standard definition of the partition function of the free field, used for example in the context of Liouville CFT, see [103] for a review.
Since the measure of the Schwarzian Field Theory will be defined in terms of Brownian bridges, the partition function is essentially determined uniquely with this convention.

**Remark 5.1.5:** The quadratic variation of $(\log\varphi'(\tau))_{\tau\in[0,1)}$ does not depend on the representative of $\varphi\in\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ because the action by $\mathrm{PSL}(2,\mathbb{R})$ is by left-composition with a smooth function. It follows from the construction of the measure $\mathscr{M}_\rho$ that, almost surely under the normalised version of $\mathscr{M}_\rho$, this quadratic variation is given by $(\int_0^\tau\rho(t)\,\mathrm{d}t)_{\tau\in[0,1)}$.

**Remark 5.1.6:** In [96, Appendix C], the 'correlation functions' of the Schwarzian Field Theory are formally defined by differentiation of the partition function with respect to the metric:

$$\begin{aligned} \langle \mathcal{S}(\tau_1)\cdots\mathcal{S}(\tau_n)\rangle_{\sigma^2} &= \frac{1}{Z(\sigma^2)}\frac{\partial^n}{\partial\epsilon_1\cdots\partial\epsilon_n}Z(\rho(\epsilon_1,\dots,\epsilon_n)),\\ \frac{1}{\rho}(\epsilon_1,\dots,\epsilon_n) &= \frac{1}{\sigma^2}+\epsilon_1\delta_{\tau_1}+\cdots+\epsilon_n\delta_{\tau_n}. \end{aligned} \tag{5.1.20}$$

These 'correlation functions' can be computed from the formula (5.1.17). For example,

$$\langle \mathcal{S}(0)\rangle_{\sigma^2} = 2\pi^2+\tfrac{3}{2}\sigma^2 \tag{5.1.21}$$

and

$$\langle \mathcal{S}(0)\mathcal{S}(\tau)\rangle_{\sigma^2} = [4\pi^4+10\pi^2\sigma^2+\tfrac{15}{4}\sigma^4]-2\sigma^2[2\pi^2+\tfrac{3}{2}\sigma^2]\,\delta(\tau)-\sigma^2\delta''(\tau). \tag{5.1.22}$$

See the appendix in Section 5.5. In the relation to Liouville Field Theory, these correlation correspond to stress-energy tensor correlation functions, see [97, Appendix A].

**Remark 5.1.7:** Since the definition of the partition functions involves $\zeta$-function regularisation (see Remark 5.1.4), the correlation functions defined by (5.1.20) are not obviously expectations of random variables. In fact, the Schwarzian $\mathcal{S}_f(\tau)$ of a function $f\colon \mathbb{T}\to\mathbb{R}$ is only defined if $f\in C^3$ while the support of the Schwarzian Field Theory measure $\mathrm{d}\mathcal{M}_{\sigma^2}$ only has $C^1$-regularity.
*Cross-ratios*

$$\mathcal{O}(\varphi;s,t) = \frac{\pi\sqrt{\varphi'(t)\varphi'(s)}}{\sin(\pi[\varphi(t)-\varphi(s)])}, \tag{5.1.23}$$

where $s\neq t$, provide a finite-difference-type regularisation of the Schwarzian derivative, which still respects the $\mathrm{PSL}(2,\mathbb{R})$-invariance and is well-defined for $C^1$-functions. For $C^3$-functions, in the limit of infinitesimally close end-points, the cross-ratios approximate the Schwarzian derivative.
In [104], the (probabilistically well-defined) correlation functions of cross-ratios are explicitly computed, confirming the predictions of [97] obtained using the conformal bootstrap. It is further shown that in the limit $t-s\to 0$ these coincide with the above Schwarzian correlation functions obtained by differentiating the partition function.
In [94, 95], these cross-ratio observables are related to Wilson lines in the gauge theory formulation of JT gravity.

**Remark 5.1.8:** The change of measure formulas (5.1.13) and (5.1.16) are consistent with the goal that the action of the measures is the desired Schwarzian action. It would be interesting to show that these change of measure formulas indeed characterise the measure uniquely. Our results show that the partition functions of the measures agree with those from [96]. In [104] it is further shown that the cross-ratio correlation functions, which agree with those of [97], in fact characterise the measure uniquely. Finally, in [105], it is shown that the large deviations of the measure as $\sigma^2 \to 0$ are indeed described by the Schwarzian action as expected.

### 5.1.3 Related probabilistic literature

We will not survey the vast literature in physics related to the Schwarzian Field Theory and the SYK model, but refer to [91, 92] for a starting point on the SYK model and [93–95] for a starting point on its relation to JT gravity. Further physical perspectives on the construction carried out in this paper can be found in [98, 100, 106, 107]. In the following, we do mention some related more probabilistic references.

The Schwarzian Field Theory is formally related to a degenerate limit of Liouville Field Theory [97, 108], and the conformal bootstrap and the DOZZ formula applied in this context has been used to predict the correlation functions of the Schwarzian Field Theory [97]. While much progress has been made on the mathematical justification of Liouville Field Theory [109, 110], see [103] for a review, and it would be very interesting to explore this connection, this paper, [104], and [105] only use standard stochastic analysis.

Random homeomorphisms of $\mathbb{T}$ have also been studied in the context of random conformal welding [111, 112], where given a random homeomorphism of the circle one constructs an associated random Jordan curve in the plane. The Schwarzian Field Theory provides a different natural random diffeomorphism of the circle, and it would be interesting to explore the associated random conformal welding. We also remark that the space $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ has also appeared in the study of large deviations of SLE [113].

Some motivation for the construction of quasi-invariant measures on diffeomorphism groups (and loop groups), of which the Schwarzian Theory is an example, has been the construction of unitary representations of those. The earliest references concerning quasi-invariant measures on $\mathrm{Diff}^1(\mathbb{T})$ appear to go back to Shavgulidze and collaborators such as [114], and we refer to [115] for further discussion and references. The Malliavins also considered such measures [116], and diffusion on such spaces (and quotients) and associated Wiener measures were studied in follow-up works such as [117, 118].

Finally, we mention that the theory of path integrals for coadjoint orbits of loop group extension of compact Lie groups is somewhat well-developed. On a formal level, these orbits again carry a natural symplectic structure and the path integral associated to the Hamiltonian generating the

$U(1)$ action is of Duistermaat–Heckman form. However, this case looks simpler as the relevant Hamiltonian is simply the Dirichlet energy associated with a Lie algebra valued Brownian bridge. Bismut has developed an analytical approach to the calculation of the corresponding heat kernels via a rigorous Duistermaat–Heckman-type deformation involving hypoelliptic Laplacians, see, e.g., [119].

### 5.1.4 Preliminaries and notation

We write $\mathbb{D}_r = \{z \in \mathbb{C} \colon |z| < r\}$ for the open disk of radius $r$ and $\mathbb{D} = \mathbb{D}_1$ for the open unit disk. The unit circle is denoted by $\mathbb{T} = [0,1]/\{0 \sim 1\}$, and $\mathrm{Diff}^k(\mathbb{T})$ is the set of oriented $C^k$-diffeomorphisms of $\mathbb{T}$, i.e. $\varphi \in \mathrm{Diff}^k(\mathbb{T})$ can be identified with a $k$-times continuously differentiable function $\varphi \colon \mathbb{R} \to \mathbb{R}$ satisfying $\varphi(\tau+1) = \varphi(\tau)+1$ and $\varphi'(\tau) > 0$ for all $\tau \in \mathbb{R}$. Note that $\mathrm{Diff}^k(\mathbb{T})$ is not a linear space. The topology on $\mathrm{Diff}^k(\mathbb{T})$ is the natural one given by the identification of $\varphi$ with $\xi \in C^{k-1}[0,1]$ and $\Theta \in \mathbb{R}$ as in (5.1.6), which makes $\mathrm{Diff}^k(\mathbb{T})$ is a Polish (separable completely metrisable) space as well as a topological group. The same topology is induced by viewing $\mathrm{Diff}^k(\mathbb{T})$ as a subspace of $C^k(\mathbb{T})$.

It is also useful to consider diffeomorphisms of $[0,1]$ (or more generally of $[0,T]$) that are not periodic, and we denote by $\mathrm{Diff}^k[0,T]$ the set of oriented $C^k$-diffeomorphisms of $[0,T]$ satisfying $\varphi'(t) > 0$, $\varphi(0) = 0$, and $\varphi(T) = T$. Thus the derivatives do not have to match at the endpoints. We further set $C_{0,\mathrm{free}}[0,T] = \{f \in C[0,T] \,|\, f(0) = 0\}$, and $C_0[0,T] = \{f \in C[0,T] \,|\, f(0) = f(T) = 0\}$.

The projective special linear group is $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ where $\mathrm{SL}(2,\mathbb{R})$ consists of all matrices $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with real entries and determinant 1. The action of $\pm M \in \mathrm{PSL}(2,\mathbb{R})$ on $\varphi \in \mathbb{T}$ is

$$f \mapsto M \circ f = \frac{af+b}{cf+d}, \qquad \text{where } f = \tan(\pi\varphi - \tfrac{\pi}{2}). \tag{5.1.24}$$

We may identify $\mathrm{PSL}(2,\mathbb{R})$ with its orbit at $\mathrm{id}_{\mathbb{T}} \in \mathrm{Diff}^1(\mathbb{T})$, equivalently parametrised by[1]

$$\varphi_{z,a}(t) = a - \frac{i}{2\pi} \ln \frac{e^{i2\pi t} - z}{1 - \bar{z} e^{i2\pi t}} \pmod 1, \qquad \text{for } z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, a \in [0,1). \tag{5.1.25}$$

[1]This parametrisation is the restriction to the boundary of the action of $\mathrm{PSL}(2,\mathbb{R})$ as a conformal map of the unit disk onto itself, see for example [120, Section 6.2].

Up to normalisation, the Haar measure on $\mathrm{PSL}(2,\mathbb{R})$ then takes the form[2]

$$\mathrm{d}\nu_H(\varphi_{z,a}) = \frac{4\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}a}{(1-\rho^2)^2}, \qquad \text{where } z = \rho\, e^{i2\pi\theta}, \tag{5.1.26}$$

and we always assume this normalisation for the Haar measure. One may check that the subspace topology on the $\{\varphi_{z,a}\}_{z\in\mathbb{D},a\in[0,1)}$ inherited from $\mathrm{Diff}^1(\mathbb{T})$ agrees with the topology on $\mathrm{PSL}(2,\mathbb{R})$. Hence, the $\mathrm{PSL}(2,\mathbb{R})$-orbit at $\mathrm{id}_{\mathbb{T}}$ is a faithful embedding of $\mathrm{PSL}(2,\mathbb{R})$ as a subgroup of $\mathrm{Diff}^1(\mathbb{T})$. As a consequence, the right action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathrm{Diff}^1(\mathbb{T})$ is a *proper* group action.

Finally, we recall the definition of the Schwarzian derivative (5.1.2) and the chain rule (5.1.9). The chain rule implies that the Schwarzian action can be written as

$$\mathcal{S}(\tan(\pi\varphi - \tfrac{\pi}{2}), \tau) = \mathcal{S}(\varphi,\tau) + 2\pi^2\varphi'(\tau)^2 \tag{5.1.27}$$

where we used that $\mathcal{S}(\tan(\pi\varphi-\frac{\pi}{2}),\varphi) = 2\pi^2$. In particular,

$$\begin{aligned}\mathcal{S}(\tan(\pi(\psi\circ\varphi)-\tfrac{\pi}{2}),\tau) &= \mathcal{S}(\psi\circ\varphi,\tau) + 2\pi^2(\psi\circ\varphi)'(\tau)^2\\ &= \mathcal{S}(\varphi,\tau) + \Big[\mathcal{S}(\psi,\varphi(\tau)) + 2\pi^2(\psi'(\varphi(\tau))^2\Big]\varphi'(\tau)^2\\ &= \mathcal{S}(\tan(\pi\varphi-\tfrac{\pi}{2}),\tau) + \Big(\mathcal{S}(\tan(\pi\psi-\tfrac{\pi}{2}),\varphi(\tau)) - 2\pi^2\Big)\varphi'(\tau)^2,\end{aligned} \tag{5.1.28}$$

where we used (5.1.27) on the first and third line and the chain rule (5.1.9) on the second line.

## 5.2 Definition of the Schwarzian measure

In this section, we define the Schwarzian Field Theory as a finite Borel measure $\mathcal{M}_{\sigma^2}$ supported on $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$, and its unquotiented version $\widetilde{\mathcal{M}}_{\sigma^2}$ which is an infinite Borel measure on $\mathrm{Diff}^1(\mathbb{T})$.

In Section 5.2.1, we begin with the definition of the unnormalised Brownian bridge measure $\mathcal{B}^{a,T}_{\sigma^2}$ which is the starting point for the definition of the former measures. In Section 5.2.2, we then define measures $\mu_{\sigma^2}$ through a change of variable of the product of an unnormalised Brownian bridge and a Lebesgue measure. The Schwarzian Field Theory measure is finally defined in Section 5.2.3.

[2] See for example [121, Lemma 9.16] which states that the Haar measure on $\mathrm{PSL}(2,\mathbb{R})$ is given as the uniform measure on circle (corresponding to $\mathrm{d}a$) and the hyperbolic measure on the upper half plane $\mathbb{H}$. In (5.1.26) we have parametrised the hyperbolic measure by the Poincaré disk $\mathbb{D}$ instead of $\mathbb{H}$.

### 5.2.1 Unnormalised Brownian Bridge measure

The unnormalised version of the Brownian bridge measure is defined in Definition 5.2.1 below. It should be a finite measure on $C_{0,\text{free}}[0,T] = \{f \in C[0,T] \mid f(0) = 0\}$ formally represented as

$$\mathrm{d}\mathcal{B}^{a,T}_{\sigma^2}(\xi) = \exp\left\{-\frac{1}{2\sigma^2}\int_0^T \xi'^2(t)\,\mathrm{d}t\right\}\delta(\xi(0))\delta(\xi(T)-a)\prod_{\tau\in(0,T)}\mathrm{d}\xi(\tau). \tag{5.2.1}$$

More generally, with a metric $\rho^2\colon \mathbb{T} \to \mathbb{R}_+$, the unnormalised Brownian bridge measure should be

$$\mathrm{d}\mathcal{B}^{a,T}_{\rho}(\xi) = \exp\left\{-\frac{1}{2}\int_0^T \xi'^2(t)\frac{\mathrm{d}t}{\rho(t)}\right\}\delta(\xi(0))\delta(\xi(T)-a)\prod_{\tau\in(0,T)}\mathrm{d}\xi(\tau). \tag{5.2.2}$$

For any $T_1, T_2 > 0$, $a \in \mathbb{R}$ and any positive continuous functional $F$ on $C[0,T_1+T_2]$, we then expect

$$\int F(\xi)\,\mathrm{d}\mathcal{B}^{a,T_1+T_2}_{\rho}(\xi) = \int_{\mathbb{R}}\int\int F(\xi_1\sqcup\xi_2)\,\mathrm{d}\mathcal{B}^{b,T_1}_{\rho}(\xi_1)\,\mathrm{d}\mathcal{B}^{a-b,T_2}_{\rho(T_1+\cdot)}(\xi_2)\,\mathrm{d}b, \tag{5.2.3}$$

where for $f \in C_{0,\text{free}}[0,T_1]$ and $g \in C_{0,\text{free}}[0,T_2]$, we denote by $f \sqcup g \in C_{0,\text{free}}[0,T_1+T_2]$ the function

$$(f\sqcup g)(t) = \begin{cases} f(t) & \text{if } t \in [0,T_1], \\ f(T_1)+g(t-T_1) & \text{if } t \in (T_1, T_1+T_2]. \end{cases} \tag{5.2.4}$$

The precise definition achieving these properties is as follows.

**Definition 5.2.1:** The unnormalised Brownian bridge measure with variance $\sigma^2 > 0$ is a finite Borel measure $\mathrm{d}\mathcal{B}^{a,T}_{\sigma^2}$ on $C_{0,\text{free}}[0,T]$ such that

$$\sqrt{2\pi T\sigma^2}\exp\left\{\frac{a^2}{2T\sigma^2}\right\}\mathrm{d}\mathcal{B}^{a,T}_{\sigma^2}(\xi) \tag{5.2.5}$$

is the distribution of a Brownian bridge $(\xi(t))_{t\in[0,T]}$ with variance $\sigma^2$ and $\xi(0) = 0$, $\xi(T) = a$. More generally, given $\rho\colon [0,T] \to \mathbb{R}_+$, let $\mathrm{d}\mathcal{B}^{a,T}_{\rho}$ be a measure on $C_{0,\text{free}}[0,T]$ such that

$$\sqrt{2\pi\int_0^T \rho(t)\,\mathrm{d}t}\,\exp\left\{\frac{a^2}{2\int_0^T \rho(t)\,\mathrm{d}t}\right\}\mathrm{d}\mathcal{B}^{a,T}_{\rho}(\xi) \tag{5.2.6}$$

is the distribution of a Brownian bridge $(\xi(t))_{t\in[0,T]}$ with quadratic variation $(\int_0^t \rho(\tau)\,\mathrm{d}\tau)_{t\in[0,T]}$ and $\xi(0) = 0$, $\xi(T) = a$.

**Remark 5.2.2:** A Brownian bridge on $[0,T]$ with quadratic variation $(\int_0^t \rho(\tau)\,\mathrm{d}\tau)_{t\in[0,T]}$ can be obtained from a Brownian bridge with quadratic variation $\sigma^2 t$ via the reparametrisation

$$t \mapsto h(t) = \int_0^t \frac{\rho(\tau)}{\sigma^2}\,\mathrm{d}\tau, \qquad \text{where } \sigma^2 T = \int_0^T \rho(\tau)\,\mathrm{d}\tau, \tag{5.2.7}$$

and the normalisations in (5.2.5) and (5.2.6) are also compatible. In other words,

$$\xi \circ h \sim \mathcal{B}_\rho^{a,T} \qquad \text{if } \xi \sim \mathcal{B}_{\sigma^2}^{a,T}. \tag{5.2.8}$$

Heuristically, this corresponds to the following change of variable in the action of the Brownian bridge:

$$\frac{1}{\sigma^2}\int_0^T \xi'(\tau)^2\,\mathrm{d}\tau = \frac{1}{\sigma^2}\int_0^T \xi'(h(\tau))^2 h'(\tau)\,d\tau = \frac{1}{\sigma^2}\int_0^T (\xi\circ h)'(\tau)^2 \frac{d\tau}{h'(\tau)} = \int_0^T (\xi\circ h)'(\tau)^2 \frac{d\tau}{\rho(\tau)}. \tag{5.2.9}$$

The remaining definitions in Section 5.2 which are expressed in terms of constant $\sigma^2$ can therefore be transferred in a straightforward way by reparametrisation.

The above normalisation of the normalised Brownian bridge in Definition 5.2.1 is exactly the one that is needed to ensure that the composition property (5.2.3) holds. We note that, up to a constant, it coincides with the $\zeta$-function regularisation of the determinant, see Remark 5.2.4.

**Proposition 5.2.3:** The unnormalised Brownian bridge measures $\mathrm{d}\mathcal{B}_{\sigma^2}^{a,T}$ satisfy the property (5.2.3).

*Proof.* Notice that $\mathrm{d}\mathcal{B}_\rho^{a,T}(\xi)\otimes \mathrm{d}a$ is a probability measure and that the distribution of $\xi$ under this measure that of a Brownian motion restricted to $[0,T]$, see [RevuzYor, Exercise (3.16)]. Therefore, the Markov property for Brownian motion implies that the distribution of $\xi_1 \sqcup \xi_2$ under $\mathrm{d}\mathcal{B}_\rho^{b,T_1}(\xi_1)\otimes \mathrm{d}b \otimes \mathrm{d}\mathcal{B}_\rho^{a-b,T_2}(\xi_2)\otimes \mathrm{d}a$ is a Brownian motion restricted to $[0,T_1+T_2]$. On the other hand, $\xi$ under $\mathrm{d}\mathcal{B}_\rho^{a,T_1+T_1}(\xi)\otimes \mathrm{d}a$ is also a Brownian motion on $[0,T_1+T_2]$. Since $(\xi_1\sqcup\xi_2)(T_1+T_2) = a$ under the first measure and $\xi(T_1+T_2)=a$ under the second measure, the distributions of $\xi_1\sqcup\xi_2$ and $\xi$ under $\mathrm{d}\mathcal{B}_\rho^{b,T_1}(\xi_1)\otimes \mathrm{d}b\otimes \mathrm{d}\mathcal{B}_\rho^{a-b,T_2}(\xi_2)$ respectively $\mathrm{d}\mathcal{B}_\rho^{a,T_1+T_1}(\xi)$ must be the same, i.e. (5.2.3) holds. □

**Remark 5.2.4:** The normalisation in (5.2.6) coincides with square root of the $\zeta$-function normalised determinant, up to an overall constant. Indeed, if $\Delta_\rho = \rho^{-1}\frac{\partial}{\partial\tau}(\rho^{-1}\frac{\partial}{\partial\tau})$ is the

Laplace–Beltrami operator on $[0, T]$ with metric $\rho^2$ and Dirichlet boundary condition, then

$$\det{}'\left(-\frac{1}{2\pi}\Delta_\rho\right) = C\int_0^T \rho(t)\,\mathrm{d}t, \tag{5.2.10}$$

where $\det'$ is the $\zeta$-regularised determinant and $C$ is a constant independent of $\rho$. The determinant is defined by (see for example [122])

$$\det{}'\left(-\frac{1}{2\pi}\Delta_\rho\right) = e^{-\zeta'(0)}, \tag{5.2.11}$$

where $\zeta$ is the spectral $\zeta$-function, i.e. the analytic continuation of $\zeta(s) = \sum_n \lambda_n^{-s}$ where $\lambda_n$ are the eigenvalues of $-\frac{1}{2\pi}\Delta_\rho$. To see the equality (5.2.10) one can adapt the argument leading to [122, Equation (1.13)] to $d = 1$.

### 5.2.2 Unnormalised Malliavin–Shavgulidze measure

Towards defining the Schwarzian measure $\mathscr{M}_{\sigma^2}$, we next define a finite measure $\mu_{\sigma^2}$ on $\mathrm{Diff}^1(\mathbb{T})$ that is similar to what is known as the Malliavin–Shavgulidze measure, see [115, Section 11.5]. This measure is defined as a push-forward of an unnormalised Brownian bridge on $[0, 1]$ with respect to a suitable change of variables, and should formally correspond to

$$\mathrm{d}\mu_{\sigma^2}(\varphi) = \exp\left\{-\frac{1}{2\sigma^2}\int_0^1\left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)^2\mathrm{d}\tau\right\}\prod_{\tau\in[0,1)}\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}. \tag{5.2.12}$$

To motivate the actual definition, recall the formal density (5.2.1) of the unnormalised Brownian bridge measure $\mathcal{B}^{0,1}_{\sigma^2}$. Thus, formally, under the measure $\mu_{\sigma^2}$ the process $(\log\varphi'(\tau) - \log\varphi'(0))_{\tau\in[0,1)}$ has the same density as $(\xi_t)_{t\in[0,1)}$ under $\mathcal{B}^{0,1}_{\sigma^2}$. We *define* $\mu_{\sigma^2}$ by

$$\mathrm{d}\mu_{\sigma^2}(\varphi) := \mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi)\otimes\mathrm{d}\Theta, \qquad \text{with } \varphi(t) = \Theta + \mathsf{P}_\xi(t) \pmod 1, \text{ for } \Theta\in[0,1), \tag{5.2.13}$$

where $\mathrm{d}\Theta$ is the Lebesgue measure on $[0, 1)$ and with the change of variables

$$\mathsf{P}(\xi)(t) := \mathsf{P}_\xi(t) := \frac{\int_0^t e^{\xi(\tau)}\,\mathrm{d}\tau}{\int_0^1 e^{\xi(\tau)}\,\mathrm{d}\tau}, \tag{5.2.14}$$

The variable $\Theta$ corresponds to the value of $\varphi(0)$. Note that the map $\xi \mapsto \mathsf{P}(\xi)$ is a bijection between $C_{0,\mathrm{free}}[0,1]$ and $\mathrm{Diff}^1[0,1]$ with inverse map

$$\begin{aligned} \mathsf{P}^{-1}: \mathrm{Diff}^1[0,1] &\to C_{0,\mathrm{free}}[0,1] \\ \phi &\mapsto \log\phi'(\cdot) - \log\phi'(0). \end{aligned} \tag{5.2.15}$$

With $\mu_{\sigma^2}$ defined as above, the following change of variables formula holds:

**Proposition 5.2.5:** Let $f \in \mathrm{Diff}^3[0,1]$ be such that $f'(0) = f'(1)$ and $f''(0) = f''(1)$, and denote by $f^*\mu_{\sigma^2} = f_*^{-1}\mu_{\sigma^2}$ the push-forward of $\mu_{\sigma^2}$ under left composition with $f^{-1}$, i.e.,

$$f^*\mu_{\sigma^2}(A) = \mu_{\sigma^2}(f \circ A), \tag{5.2.16}$$

where $f \circ A \coloneqq \{f \circ \varphi \,|\, \varphi \in A\}$. Then

$$\frac{\mathrm{d}f^*\mu_{\sigma^2}(\varphi)}{\mathrm{d}\mu_{\sigma^2}(\varphi)} = \exp\left\{\frac{1}{\sigma^2}\int_0^1 \mathcal{S}_f\big(\varphi(t)\big)\,\varphi'^2(t)\,\mathrm{d}t\right\}. \tag{5.2.17}$$

The proposition is a consequence of the following change of variable formula for the unnormalised Brownian bridge. For $f \in \mathrm{Diff}^3[0,1]$ denote by $L_f$ the left composition operator on $\mathrm{Diff}^1[0,1]$:

$$L_f(\varphi) = f \circ \varphi. \tag{5.2.18}$$

**Proposition 5.2.6:** Let $f \in \mathrm{Diff}^3[0,1]$ and set $b = \log f'(1) - \log f'(0)$. Let $f^\sharp\mathcal{B}^{a,1}_{\sigma^2} = f_\sharp^{-1}\mathcal{B}^{a,1}_{\sigma^2}$ be the push-forward of $\mathcal{B}^{a,1}_{\sigma^2}$ under $\mathsf{P}^{-1} \circ L_{f^{-1}} \circ \mathsf{P} = \big(\mathsf{P}^{-1} \circ L_f \circ \mathsf{P}\big)^{-1}$. Then for any $a \in \mathbb{R}$, $f^\sharp\mathcal{B}^{a,1}_{\sigma^2}$ is absolutely continuous with respect to $\mathcal{B}^{a-b,1}_{\sigma^2}$ and

$$\frac{\mathrm{d}f^\sharp\mathcal{B}^{a,1}_{\sigma^2}(\xi)}{\mathrm{d}\mathcal{B}^{a-b,1}_{\sigma^2}(\xi)} = \frac{1}{\sqrt{f'(0)f'(1)}}\exp\left\{\frac{1}{\sigma^2}\left[\frac{f''(0)}{f'(0)}\mathsf{P}'_\xi(0) - \frac{f''(1)}{f'(1)}\mathsf{P}'_\xi(1)\right] + \frac{1}{\sigma^2}\int_0^1 \mathcal{S}_f\big(\mathsf{P}_\xi(t)\big)\,\mathsf{P}'_\xi(t)^2\,\mathrm{d}t\right\}. \tag{5.2.19}$$

We prove this statement in an appendix in Section 5.6. A similar statement (for the Wiener measure instead of the unnormalised Brownian bridge) can be found in [115, Theorem 11.5.1], for example. For now, we show how it implies Proposition 5.2.5:

*Proof of Proposition 5.2.5.* Let

$$\varphi(t) = \Theta + \mathsf{P}_\xi(t) \pmod 1, \tag{5.2.20}$$

$$\big(f \circ \varphi\big)(t) = \widetilde{\Theta} + \mathsf{P}_{\widetilde{\xi}}(t) \pmod 1. \tag{5.2.21}$$

In other words,

$$\widetilde{\Theta} = f(\Theta), \qquad \widetilde{\xi} = \Big(\mathsf{P}^{-1} \circ L_{f_\Theta} \circ \mathsf{P}\Big)(\xi), \tag{5.2.22}$$

where $f_\Theta(\tau) = f(\tau + \Theta) - \widetilde{\Theta}$. From (5.2.13) we see that,

$$\mathrm{d}f^* \mu_{\sigma^2}(\varphi) = \mathrm{d}f_\Theta^\sharp \mathcal{B}^{0,1}_{\sigma^2}(\xi) \times \mathrm{d}f(\Theta), \tag{5.2.23}$$

with notation as in Proposition 5.2.6. Hence, Proposition 5.2.6 implies

$$\frac{\mathrm{d}f_\Theta^\sharp \mathcal{B}^{0,1}_{\sigma^2}(\xi)}{\mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi)} = \frac{1}{f'(\Theta)} \exp\left\{ \frac{1}{\sigma^2} \int_{\mathbb{T}} \mathcal{S}_f\big(\varphi(t)\big)\, \varphi'^2(t)\, \mathrm{d}t \right\}, \tag{5.2.24}$$

and using

$$\frac{\mathrm{d}f(\Theta)}{\mathrm{d}\Theta} = f'(\Theta) \tag{5.2.25}$$

the proof is finished. □

We define the measure $\mu_\rho$ in which $\sigma^2$ is generalised to $\rho\colon \mathbb{T} \to \mathbb{R}_+$ analogously to (5.2.13) by

$$\mathrm{d}\mu_\rho(\varphi) := \exp\left\{ \int_0^1 \left( \frac{\varphi''(\tau)}{\varphi'(\tau)} \right)' \frac{\mathrm{d}\tau}{\rho(\tau)} \right\} \mathrm{d}\mathcal{B}^{0,1}_\rho(\xi) \otimes \mathrm{d}\Theta, \tag{5.2.26}$$

again with $\varphi(t) = \Theta + \mathsf{P}_\xi(t) \pmod 1$ and $\mathsf{P}_\xi(t)$ given by (5.2.14), and where the term in the exponential is interpreted as the Itô integral

$$\int_0^1 \left( \frac{\varphi''(\tau)}{\varphi'(\tau)} \right)' \frac{\mathrm{d}\tau}{\rho(\tau)} = \int_0^1 \xi''(\tau) \frac{\mathrm{d}\tau}{\rho(\tau)} = \int_0^1 \frac{\rho'(\tau)}{\rho(\tau)^2}\, \mathrm{d}\xi(\tau). \tag{5.2.27}$$

Thus $\mu_\rho$ has formal density

$$\mathrm{d}\mu_\rho(\varphi) = \exp\left\{ \int_0^1 \mathcal{S}(\varphi, \tau) \frac{\mathrm{d}\tau}{\rho(\tau)} \right\} \prod_{\tau \in [0,1)} \frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}. \tag{5.2.28}$$

**Lemma 5.2.7:** Let $h : [0,1] \to [0,1]$ be as in (5.2.7). Then

$$\int F(\varphi)\, \mathrm{d}\mu_\rho(\varphi) = \exp\left\{ \frac{1}{2} \int_0^1 \frac{\rho'(\tau)^2}{\rho(\tau)^3}\, \mathrm{d}\tau \right\} \int F(\varphi \circ h)\, \mathrm{d}\mu_{\sigma^2}(\varphi). \tag{5.2.29}$$

*Proof.* Giranov's theorem, using that the quadratic variation of $d\xi(\tau)$ is $\rho(\tau)\,d\tau$, gives

$$
\begin{aligned}
&\exp\left\{-\frac{1}{2}\int_0^1 \frac{\rho'(\tau)^2}{\rho(\tau)^3}\,d\tau\right\}\int F(\varphi)\,d\mu_\rho(\varphi) \\
&= \int F(\varphi)\exp\left\{\int_0^1 \psi'(\tau)\,d\xi(\tau) - \frac{1}{2}\int_0^1 \psi'(\tau)^2\rho(\tau)\,d\tau\right\} d\mathcal{B}_\rho^{0,1}(\xi)\otimes d\Theta \\
&= \int F(\varphi^\psi)\,d\mathcal{B}_\rho^{0,1}(\xi)\otimes d\Theta
\end{aligned}
\tag{5.2.30}
$$

where $\psi(t) = -1/\rho(\tau)$ and

$$
\varphi^\psi(t) = \Theta + \frac{\int_0^t e^{\xi(\tau)+g(\tau)}\,d\tau}{\int_0^1 e^{\xi(\tau)+g(\tau)}\,d\tau} = \Theta + \frac{\int_0^t e^{\xi(\tau)}\rho(\tau)\,d\tau}{\int_0^1 e^{\xi(\tau)}\rho(\tau)\,d\tau},
\tag{5.2.31}
$$

with the following drift resulting from Girsanov's theorem:

$$
g(\tau) = \int_0^\tau \psi'(t)\rho(t)\,dt = \int_0^\tau (\log\rho)'(t)\,dt = \log(\rho(\tau)/\rho(0)).
\tag{5.2.32}
$$

Let $\tilde{\xi} \sim \mathcal{B}_{\sigma^2}^{0,1}$ and recall that $\xi = \tilde{\xi}\circ h \sim \mathcal{B}_\rho^{0,1}$. Let $\Theta \sim d\Theta$ on $[0,1)$. Then since $\rho(\tau) = \sigma^2 h'(\tau)$,

$$
\varphi^\psi(t) = \Theta + \frac{\int_0^t e^{(\tilde{\xi}\circ h)(\tau)}\,h'(\tau)\,d\tau}{\int_0^1 e^{(\tilde{\xi}\circ h)(\tau)}\,h'(\tau)\,d\tau} = \Theta + \frac{\int_0^{h(t)} e^{\tilde{\xi}(\tau)}\,d\tau}{\int_0^1 e^{\tilde{\xi}(\tau)}\,d\tau} = (\tilde{\varphi}\circ h)(t)
\tag{5.2.33}
$$

where $\tilde{\varphi} \sim \mu_{\sigma^2}$. In summary, we have

$$
\int F(\varphi)\exp\left\{-\frac{1}{2}\int_0^1 \frac{\rho'(\tau)^2}{\rho(\tau)^3}\,d\tau\right\} d\mu_\rho(\varphi) = \int F(\tilde{\varphi}\circ h)\,d\mu_{\sigma^2}(\tilde{\varphi})
\tag{5.2.34}
$$

which is the claim. □

**Remark 5.2.8:** It is instructive to verify the identity (5.2.29) in terms of the formal actions of $\mu_{\sigma^2}$ and $\mu_\rho$, see (5.2.12) and (5.2.28). Let $y : [0,1] \to [0,1]$ be the inverse to $h$. Then, in view of (5.2.8), the statement for the formal actions is equivalent to

$$
\frac{1}{2\sigma^2}\int_0^1 \left(\frac{(\varphi\circ y)''(\tau)}{(\varphi\circ y)'(\tau)}\right)^2 d\tau = \frac{1}{2}\int_0^1 \left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)^2 \frac{d\tau}{\rho(\tau)} - \int_0^1 \left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)' \frac{d\tau}{\rho(\tau)} + \frac{1}{2}\int_0^1 \frac{\rho''(\tau)^2}{\rho'(\tau)^3}\,d\tau.
\tag{5.2.35}
$$

To verify this, note

$$\frac{1}{2}\int_0^1 \left(\frac{(\varphi\circ y)''(\tau)}{(\varphi\circ y)'(\tau)}\right)^2 \mathrm{d}\tau = \frac{1}{2}\int_0^1 \left(\frac{(\varphi''\circ y)(\tau)y'(\tau)^2+(\varphi'\circ y)(\tau)y''(\tau)}{(\varphi'\circ y)(\tau)y'(\tau)}\right)^2 \mathrm{d}\tau$$
$$= \frac{1}{2}\int_0^1 \left(\frac{(\varphi''\circ y)(\tau)}{(\varphi'\circ y)(\tau)}\right)^2 y'(\tau)^2\,\mathrm{d}\tau \tag{5.2.36}$$
$$+\int_0^1 \frac{(\varphi''\circ y)(\tau)}{(\varphi'\circ y)(\tau)}\, y''(\tau)\,\mathrm{d}\tau + \frac{1}{2}\int_0^1 \left(\frac{y''(\tau)}{y'(\tau)}\right)^2 \mathrm{d}\tau.$$

Changing variables from $\tau$ to $h(\tau)$ and using $y'(h(\tau)) = 1/h'(\tau)$ and $y''(h(\tau))h'(\tau) = -h''(\tau)/h'(\tau)^2$, the right-hand side equals

$$\frac{1}{2}\int_0^1 \left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)^2 \frac{\mathrm{d}\tau}{h'(\tau)} + \int_0^1 \frac{\varphi''(\tau)}{\varphi'(\tau)}\left(\frac{1}{h'(\tau)}\right)' \mathrm{d}\tau + \frac{1}{2}\int_0^1 \frac{h''(\tau)^2}{h'(\tau)^3}\,\mathrm{d}\tau. \tag{5.2.37}$$

The claim is obtained by dividing by $\sigma^2$ and using $\sigma^2 h'(\tau) = \rho(\tau)$.

### 5.2.3 Schwarzian measure

In view of (5.1.1) and (5.2.12), the unquotiented Schwarzian measure is defined by

$$\mathrm{d}\widetilde{\mathscr{M}}_{\sigma^2}(\varphi) = \exp\left\{\frac{2\pi^2}{\sigma^2}\int_0^1 \varphi'^2(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}\mu_{\sigma^2}(\varphi). \tag{5.2.38}$$

Since $\mu_{\sigma^2}$ is supported on $\mathrm{Diff}^1(\mathbb{T})$, this defines a Borel measure on $\mathrm{Diff}^1(\mathbb{T})$. In Proposition 5.2.9 below, it is verified that this measure is invariant under the right action of $\mathrm{PSL}(2,\mathbb{R})$. In particular, we remark that $\widetilde{\mathscr{M}}_{\sigma^2}$ is an infinite measure since $\mathrm{PSL}(2,\mathbb{R})$ has infinite Haar measure. The Schwarzian measure on $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ with formal density (5.1.1) will be defined as the quotient of $\widetilde{\mathscr{M}}_{\sigma^2}$ by $\mathrm{PSL}(2,\mathbb{R})$, see Proposition 5.2.10 and Definition 5.2.11 below.

The change of measure formula from Theorem 5.1.1 is a consequence of Proposition 5.2.5. We recall the statement as the following proposition.

**Proposition 5.2.9:** The measure $\widetilde{\mathscr{M}}_{\sigma^2}$ satisfies the following change of variable formula for any $\psi \in \mathrm{Diff}^3(\mathbb{T})$:

$$\frac{\mathrm{d}\psi^*\widetilde{\mathscr{M}}_{\sigma^2}(\varphi)}{\mathrm{d}\widetilde{\mathscr{M}}_{\sigma^2}(\varphi)} = \exp\left\{\frac{1}{\sigma^2}\int_0^1 \left[\mathcal{S}(\tan(\pi\psi + \tfrac{\pi}{2}), \varphi(t)) - 2\pi^2\right]\varphi'^2(t)\,\mathrm{d}t\right\}. \tag{5.2.39}$$

In particular, $\widetilde{\mathscr{M}}_{\sigma^2}$ is invariant under the right action of $\mathrm{PSL}(2,\mathbb{R})$. In other words, for any $\psi \in \mathrm{PSL}(2,\mathbb{R})$ and Borel $A \subset \mathrm{Diff}^1(\mathbb{T})$ we have

$$\widetilde{\mathscr{M}}_{\sigma^2}(\psi \circ A) = \widetilde{\mathscr{M}}_{\sigma^2}(A), \tag{5.2.40}$$

where $\psi \circ A := \{\psi \circ \varphi \,|\, \varphi \in A\}$.

*Proof.* By Proposition 5.2.5,

$$\frac{\mathrm{d}\psi^*\mu_{\sigma^2}}{\mathrm{d}\mu_{\sigma^2}} = \exp\left\{\frac{1}{\sigma^2}\int_0^1 \mathcal{S}_\psi(\varphi(\tau))\,\varphi'^2(\tau)\,\mathrm{d}\tau\right\} \tag{5.2.41}$$

and

$$\frac{\exp\left\{\frac{2\pi^2}{\sigma^2}\int_0^1 (\psi\circ\varphi)'^2(\tau)\,\mathrm{d}\tau\right\}}{\exp\left\{\frac{2\pi^2}{\sigma^2}\int_0^1 \varphi'^2(\tau)\,\mathrm{d}\tau\right\}} = \exp\left\{\frac{2\pi^2}{\sigma^2}\int_0^1 \left(\psi'(\varphi(\tau))^2 - 1\right)\varphi'^2(\tau)\,\mathrm{d}\tau\right\}. \tag{5.2.42}$$

Using the identity (5.1.27), i.e.,

$$\mathcal{S}(\tan(\pi\psi - \tfrac{\pi}{2}), \varphi) = \mathcal{S}(\psi,\varphi) + 2\pi^2\psi'^2(\varphi), \tag{5.2.43}$$

therefore

$$\frac{\mathrm{d}\psi^*\widetilde{\mathscr{M}}_{\sigma^2}(\varphi)}{\mathrm{d}\widetilde{\mathscr{M}}_{\sigma^2}(\varphi)} = \exp\left\{\frac{1}{\sigma^2}\int_0^1 \left[\mathcal{S}(\tan(\pi\psi - \tfrac{\pi}{2}), \varphi(\tau)) - 2\pi^2\right]\varphi'^2(\tau)\,\mathrm{d}\tau\right\} \tag{5.2.44}$$

as claimed. □

From the $\mathrm{PSL}(2,\mathbb{R})$-invariance of the measure, it follows that $\widetilde{\mathscr{M}}_{\sigma^2}$ can be decomposed into a product of the Haar measure of $\mathrm{PSL}(2,\mathbb{R})$ with its quotient by $\mathrm{PSL}(2,\mathbb{R})$. Since $\mathrm{PSL}(2,\mathbb{R})$ is not compact, we need to choose the normalisation of the Haar measure $\nu_H$, and we work with the normalisation (5.1.26). The precise statement is as follows.

**Proposition 5.2.10:** There exists a unique Borel measure $\mathscr{M}_{\sigma^2}$ on $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ such that for any continous $F\colon \mathrm{Diff}^1(\mathbb{T}) \to [0,\infty]$,

$$\int_{\mathrm{Diff}^1(\mathbb{T})} \mathrm{d}\widetilde{\mathscr{M}}_{\sigma^2}(\varphi)\,F(\varphi) = \int_{\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})} \mathrm{d}\mathscr{M}_{\sigma^2}(\varphi) \int_{\mathrm{PSL}(2,\mathbb{R})} \mathrm{d}\nu_H(\psi)\,F(\psi\circ\varphi), \tag{5.2.45}$$

where the right hand side is well-defined since the second integral only depends on $\varphi \in \mathrm{Diff}^1(\mathbb{T})$ through the conjugacy class of $\varphi$ in $\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$. (We recall the abuse of notation to use $\varphi$ both for an element of $\mathrm{Diff}^1(\mathbb{T})$ and its conjugacy class.)

*Proof.* Since the space $\mathrm{Diff}^1(\mathbb{T})$ is not locally compact, we could not locate a reference for the existence of the quotient measures (which would be standard in the locally compact situation). Therefore, in the appendix in Section 5.7, we establish a sufficiently general result (Proposition 5.7.1) about such quotient measures. Note that the assumptions of that result are satisfied in our context: $\widetilde{\mathscr{M}}_{\sigma^2}$ is a Radon measure on $\mathrm{Diff}^1(\mathbb{T})$ (because the Brownian bridge is a Radon measure on $C[0,1]$), $\mathrm{Diff}^1(\mathbb{T})$ is a complete separable metric space, and $\mathrm{PSL}(2,\mathbb{R})$ acts continuously and properly from the right (note the discussion after (5.1.26)). Moreover, $\mathrm{PSL}(2,\mathbb{R})$ is unimodular and $\widetilde{\mathscr{M}}_{\sigma^2}$ is invariant under its right action. □

**Definition 5.2.11:** The Schwarzian measure is given by $\mathscr{M}_{\sigma^2}$.

Finally, we generalise the above definition to a nontrivial metric $\rho^2 : \mathbb{T} \to \mathbb{R}_+$. First, define the unquotiented Schwarzian measure with metric $\rho^2 : \mathbb{T} \to \mathbb{R}_+$ analogously by

$$\mathrm{d}\widetilde{\mathscr{M}}_\rho(\varphi) = \exp\left\{2\pi^2 \int_0^1 \varphi'^2(\tau)\frac{\mathrm{d}\tau}{\rho(\tau)}\right\} \mathrm{d}\mu_\rho(\varphi), \tag{5.2.46}$$

where we recall that $\rho$ is the positive square root of $\rho^2$. The change of variable formula (5.2.29) relating $\mu_{\sigma^2}$ and $\mu_\rho$, together with the identity (see (5.4.5) for the computation)

$$\int_0^1 \mathcal{S}(h,\tau)\frac{\mathrm{d}\tau}{\rho(\tau)} = \frac{1}{2}\int_0^1 \frac{\rho'(\tau)^2}{\rho(\tau)^3}\,\mathrm{d}\tau, \tag{5.2.47}$$

where $h$ is defined in terms of $\rho$ by (5.2.7), then imply the following relation between $\widetilde{\mathscr{M}}_{\sigma^2}$ and $\widetilde{\mathscr{M}}_\rho$ and in particular the generalisation of Proposition 5.2.9.

**Proposition 5.2.12:** For any bounded continuous $F\colon \mathrm{Diff}^1(\mathbb{T}) \to \mathbb{R}$,

$$\int F(\varphi)\,\mathrm{d}\widetilde{\mathscr{M}}_\rho(\varphi) = \exp\left\{\int_0^1 \mathcal{S}(h,\tau)\frac{\mathrm{d}\tau}{\rho(\tau)}\right\}\int F(\varphi\circ h)\,\mathrm{d}\widetilde{\mathscr{M}}_{\sigma_\rho^2}(\varphi), \qquad \sigma_\rho^2 = \int_0^1 \rho(\tau)\,\mathrm{d}\tau. \tag{5.2.48}$$

In particular, the change of variable formula (5.1.16) follows from (5.1.13), i.e.

$$\frac{\mathrm{d}\psi^*\widetilde{\mathscr{M}}_\rho(\varphi)}{\mathrm{d}\widetilde{\mathscr{M}}_\rho(\varphi)} = \exp\left\{\int_0^1 \left[\mathcal{S}(\tan(\pi\psi - \tfrac{\pi}{2}),\varphi(\tau)) - 2\pi^2\right]\varphi'(\tau)^2\frac{\mathrm{d}\tau}{\rho(\tau)}\right\}. \tag{5.2.49}$$

As a consequence of the change of variable formula, also $\widetilde{\mathcal{M}_\rho}$ is $\mathrm{PSL}(2,\mathbb{R})$ invariant and the quotient of $\widetilde{\mathcal{M}_\rho}$ by $\mathrm{PSL}(2,\mathbb{R})$ exists as in Proposition 5.2.10. We define $\mathcal{M}_\rho$ as this quotient.

> **Definition 5.2.13:** The Schwarzian measure $\mathcal{M}_\rho$ with metric $\rho^2$ is the quotient of $\widetilde{\mathcal{M}_\rho}$ by $\mathrm{PSL}(2,\mathbb{R})$.

**Remark 5.2.14:** The change of variable formula (5.2.48) is consistent with the chain rule for the Schwarzian derivative, i.e. for any $f,h \in C^3$,

$$\mathcal{S}(f \circ h,\tau) = \mathcal{S}(f,h(\tau))(h'(\tau))^2 + \mathcal{S}(h,\tau). \tag{5.2.50}$$

Indeed, the chain rule implies that

$$\frac{1}{\sigma^2}\int_0^1 \mathcal{S}(f\circ y,\tau)\,\mathrm{d}\tau = \int_0^1 \mathcal{S}(f,\tau)\frac{\mathrm{d}\tau}{\rho(\tau)} - \int_0^1 \mathcal{S}(h,\tau)\,\frac{\mathrm{d}\tau}{\rho(\tau)}, \tag{5.2.51}$$

where $y$ is the inverse to $h$.

## 5.3 Expectation via regularisation

In this section we will introduce an approximation of the Schwarzian measure by finite measures with formal density

$$\exp\left\{-\frac{1}{2\sigma^2}\int_0^1\left[\left(\frac{\varphi''(\tau)}{\varphi'(\tau)}\right)^2 - 4\alpha^2\varphi'^2(\tau)\right]\mathrm{d}\tau\right\}\prod_{\tau\in[0,1)}\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}, \tag{5.3.1}$$

where we allow $\alpha^2$ to take real values in $(-\infty,\pi^2)$, i.e., $\alpha \in i\mathbb{R}\cup(0,\pi)$. For $\alpha=\pi$ this measure would correspond to the unquotiented Schwarzian measure $\widetilde{\mathcal{M}_{\sigma^2}}$, which is infinite as remarked below (5.2.38). Below we will see that the measure is finite for $\alpha^2 < \pi^2$ and then compute its partition function (i.e. total mass) as a function of $\alpha^2$. The latter is accessible due to a diagonalisation (also referred to as a *bosonisation* in the literature) of the measure that is available for $\alpha^2 < 0$: Indeed, in that case one can see that $\xi := 2\sqrt{-\alpha^2}\,\varphi + \log\varphi'$ satisfies $\xi(1) = \xi(0) + 2\sqrt{-\alpha^2}$ and has formal density

$$\frac{\alpha}{\sin\alpha}\exp\left\{-\frac{1}{2\sigma^2}\int_0^1 \xi'^2(\tau)\,\mathrm{d}\tau\right\}\prod_{\tau\in[0,1)}\mathrm{d}\xi(\tau). \tag{5.3.2}$$

Thus it describes the law on an unnormalised Brownian bridge (plus a uniform shift according to the Lebesgue measure on $\mathbb{R}$).
Upon quotienting by the $\mathbb{R}$-translation (which corresponds to quotienting by a $\mathrm{U}(1)$-symmetry in (5.3.1)), the partition function of (5.3.2) can be evaluated in terms of that of the unnormalised Brownian bridge and equals

$$\frac{\alpha}{\sin\alpha}\frac{e^{2\alpha^2/\sigma^2}}{\sqrt{2\pi\sigma^2}}. \tag{5.3.3}$$

A similar reasoning is explained in [99], see also [102] and Remark 5.3.9 below. For comparison, we emphasise that the former reference uses a different normalisation, without the prefactor $\alpha/\sin\alpha$ in (5.3.2). The prefactor is important in the limit $\alpha \to \pi$ that we are interested in, however, because $\alpha/\sin\alpha$ diverges. We will show that this formal calculation indeed provides the total mass of (5.3.1) for all $\alpha^2 < \pi^2$, by relying on the described bosonisation for $\alpha^2 < 0$ and using an additional analytic extension to access the parameter range $\alpha^2 \in [0,\pi^2)$.

**Remark 5.3.1:** The measures in (5.3.1) can be motivated in the context of Virasoro coadjoint orbits. Any $\alpha^2 \in \mathbb{R}$ corresponds to an orbit of $\mathrm{Diff}(\mathbb{T})$ acting on $\mathfrak{vir}^*$. The action has geometrical meaning in that its Hamiltonian flow (with respect to the natural Kirillov-Kostant-Souriau symplectic form on the orbit) generates the $\mathrm{U}(1)$-action $\psi(\cdot) \mapsto \psi(\cdot + t)$.

### 5.3.1 Measure regularisation

In order to evaluate expectations with respect to the (finite) quotient measure $\mathscr{M}_{\sigma^2}$, it is helpful to approximate the (infinite) unquotiented measure $\widetilde{\mathscr{M}}_{\sigma^2}$ by finite measures. Since these measures are finite, they necessarily break the $\mathrm{PSL}(2,\mathbb{R})$-invariance. The following convenient regularisation was proposed in [98]. For $\alpha \in (0,\pi) \cup i\mathbb{R}$ consider the measures given by

$$\mathrm{d}\mathscr{N}^{\alpha}_{\sigma^2}(\varphi) = \exp\left\{\frac{2\alpha^2}{\sigma^2}\int_0^1 \varphi'^2(t)\,\mathrm{d}t\right\}\mathrm{d}\mathscr{B}^{0,1}_{\sigma^2}(\xi), \qquad \text{where } \varphi = \mathsf{P}(\xi). \tag{5.3.4}$$

By definition, this measure is supported on functions $\varphi$ with $\varphi(0) = 0$. In particular, $\widetilde{\mathscr{M}}_{\sigma^2}$ differs from the limiting case $\alpha \nearrow \pi$ of $\mathscr{N}^{\alpha}_{\sigma^2}$ only by rotation by the random angle $\Theta$, which is chosen independently and uniformly on $\mathbb{T}$. As hinted at earlier, these measures are finite and we can explicitly determine their partition function (i.e. total mass):

**Proposition 5.3.2:** For any $\alpha \in (0,\pi) \cup i\mathbb{R}$ we have

$$\mathscr{N}^{\alpha}_{\sigma^2}\Big(\mathrm{Diff}^1(\mathbb{T})\Big) = \frac{\alpha}{\sin\alpha}\frac{e^{2\alpha^2/\sigma^2}}{\sqrt{2\pi\sigma^2}}. \tag{5.3.5}$$

Note that this mass diverges as $\alpha \nearrow \pi$, which is expected due to the $\mathrm{PSL}(2,\mathbb{R})$-invariance of the limiting measure (resp. the limiting measure with an additional uniform rotation). However, we are able to obtain precise control over the divergence of mass along $\mathrm{PSL}(2,\mathbb{R})$-orbits (see Lemma 5.3.5). As a consequence, we obtain the following approximation result for expectations of non-negative functionals of the *quotiented* Schwarzian measure:

**Proposition 5.3.3:** Let $F\colon \mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R}) \to [0,\infty]$ be a continuous function. Then

$$\int_{\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})} F(\varphi)\,\mathrm{d}\mathscr{M}_{\sigma^2}(\varphi) = \lim_{\alpha\to\pi-} \frac{4\pi(\pi-\alpha)}{\sigma^2} \int_{\mathrm{Diff}^1(\mathbb{T})} F(\varphi)\,\mathrm{d}\mathscr{N}^{\alpha}_{\sigma^2}(\varphi), \tag{5.3.6}$$

where, by slight abuse of notation, on the right-hand side we denote the lift of $F$ along the quotient map $\mathrm{Diff}^1(\mathbb{T}) \twoheadrightarrow \mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})$ by $F$ as well.

#### 5.3.1.1 Regularised measures $\mathscr{N}^{\alpha}_{\sigma^2}$ as $\alpha \nearrow \pi$: Proof of Proposition 5.3.3

The proposition follows from the following lemmas. Recall the parametrisation (5.1.25) of $\mathrm{PSL}(2,\mathbb{R})$ and that in this parametrisation, the Haar measure takes the form (5.1.26).

**Lemma 5.3.4:** For $\varphi_{z,a}$ as in (5.1.25), with $\rho = |z| < 1$,

$$\int_0^1 \varphi'^2_{z,a}(s)\,\mathrm{d}s = \frac{1+\rho^2}{1-\rho^2}. \tag{5.3.7}$$

Moreover, as $\rho \nearrow 1$ the functions $\frac{1-\rho^2}{1+\rho^2}\varphi'^2_{z,0}$ are an approximate identity on $\mathbb{T}$, i.e. for any $f \in C(\mathbb{T})$,

$$\lim_{\rho\nearrow 1}\left[\frac{1-\rho^2}{1+\rho^2}\int \varphi'^2_{z,0}(s) f(s)\,\mathrm{d}s\right] = f(\theta), \qquad z = \rho\, e^{i2\pi\theta}, \tag{5.3.8}$$

uniformly in $\theta \in \mathbb{T}$.

*Proof.* Since

$$e^{i2\pi\varphi_{z,a}(t)} = e^{i2\pi a}\frac{e^{i2\pi t} - z}{1 - e^{i2\pi t}\bar{z}}, \tag{5.3.9}$$

it suffices to consider Möbius transformations of $\mathbb{D}_1$, given by

$$w \to G_z(w) = \frac{w - z}{1 - w\bar{z}}. \tag{5.3.10}$$

In order to prove (5.3.7) we then need to show that

$$\frac{1}{(2\pi)^2}\int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{i2\pi t} - z}{1 - e^{i2\pi t}\bar{z}}\right)\right|^2 \mathrm{d}t = \frac{1+|z|^2}{1-|z|^2}. \tag{5.3.11}$$

To see this, expand

$$\frac{e^{i2\pi t}-z}{1-e^{i2\pi t}\bar{z}} = -z + \sum_{n=1}^{\infty} e^{i2\pi nt}\bar{z}^{n-1}(1-z\bar{z}). \tag{5.3.12}$$

Therefore,

$$\frac{1}{(2\pi)^2}\int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{i2\pi t}-z}{1-e^{i2\pi t}\bar{z}}\right)\right|^2 \mathrm{d}t = \sum_{n=1}^{\infty} n^2|z|^{2n-2}(1-|z|^2)^2. \tag{5.3.13}$$

The right-hand side equals

$$\sum_{n=0}^{\infty}(n+1)^2|z|^{2n} - 2\sum_{n=1}^{\infty} n^2|z|^{2n} + \sum_{n=2}^{\infty}(n-1)^2|z|^{2n} = 1+4|z|^2-2|z|^2+\sum_{n=2}^{\infty}2|z|^{2n} = \frac{1+|z|^2}{1-|z|^2}, \tag{5.3.14}$$

which gives (5.3.11). The claim (5.3.8) follows similarly. Since $C^\infty(\mathbb{T})$ is dense in $C(\mathbb{T})$, it suffices to assume that $f \in C^\infty(\mathbb{T})$. Then

$$f(t) = \sum_{k\in\mathbb{Z}} e^{i2\pi kt}\hat{f}_k, \tag{5.3.15}$$

with $(\hat{f}_k)_{k\in\mathbb{Z}} \in \ell^1$. Therefore

$$\begin{aligned}\frac{1}{(2\pi)^2}\int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{i2\pi t}-z}{1-e^{i2\pi t}\bar{z}}\right)\right|^2 f(t)\,\mathrm{d}t &= \sum_k \hat{f}_k \sum_{n=1}^{\infty}\mathbf{1}_{n\geq -k}\, n(n+k)z^k|z|^{2n-2}(1-|z|^2)^2 \\ &= \sum_k \hat{f}_k e^{i2\pi k\theta} q_k(\rho)\end{aligned} \tag{5.3.16}$$

with

$$q_k(\rho) = \sum_{n=1}^{\infty}\mathbf{1}_{n\geq -k}\, n(n+k)\rho^{2n-2+k}(1-\rho^2)^2. \tag{5.3.17}$$

Analogous to (5.3.14),

$$\begin{aligned}q_k(\rho) = \sum_{n=0}^{\infty}\mathbf{1}_{n\geq -k-1}\,(n+1)(n+1+k)\rho^{2n+\rho} - 2\sum_{n=1}^{\infty}\mathbf{1}_{n\geq -k}\, n(n+k)\rho^{2n+k} \\ + \sum_{n=2}^{\infty}\mathbf{1}_{n\geq -k+1}\,(n-1)(n-1+k)\rho^{2n+k} = \sum_{n=2}^{\infty}\mathbf{1}_{n\geq -k}\, 2\rho^{2n+k} + O(1).\end{aligned} \tag{5.3.18}$$

Therefore as $\rho \nearrow 1$,

$$(1-\rho^2)q_k(\rho) = O(1), \qquad \lim_{\rho\nearrow 1}(1-\rho^2)q_k(\rho) = 2. \tag{5.3.19}$$

By dominated convergence, (5.3.8) follows. □

**Lemma 5.3.5:** For $\varphi \in \mathrm{Diff}^1(\mathbb{T})$, define

$$D^\alpha(\varphi) = \frac{4\pi(\pi-\alpha)}{\sigma^2}\int_{\mathrm{PSL}(2,\mathbb{R})} \exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\int_0^1 (\psi\circ\varphi)'^2(t)\,\mathrm{d}t\right\}\mathrm{d}\nu_H(\psi). \tag{5.3.20}$$

Then the following holds.

1. For any $\varphi \in \mathrm{Diff}^1(\mathbb{T})$,
$$D^\alpha(\varphi) \le \frac{2\pi}{\pi+\alpha}. \tag{5.3.21}$$

2. For any $\varphi \in \mathrm{Diff}^1(\mathbb{T})$,
$$\lim_{\alpha\to\pi-} D^\alpha(\varphi) = 1. \tag{5.3.22}$$

*Proof.* First, we prove (5.3.21). By (5.1.26),

$$\int_{\mathrm{PSL}(2,\mathbb{R})} F(\psi)\,\mathrm{d}\nu_H(\psi) = \int_{\substack{0\le\rho<1\\0\le\theta<1\\0\le a<1}} F(\varphi_{z,a})\frac{4\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}a}{(1-\rho^2)^2}. \tag{5.3.23}$$

Recall the parameterisation (5.1.25). Clearly one has $(\varphi_{z,a}\circ\varphi)' = (\varphi_{z,0}\circ\varphi)' = (\varphi'_{z,0}\circ\varphi)\,\varphi'$. Upon the reparametrisation $s = \varphi(t)$, we get

$$\int_0^1 (\varphi_{z,a}\circ\varphi)'^2(t)\,\mathrm{d}t = \int_0^1 \frac{\varphi'^2_{z,0}(s)}{(\varphi^{-1})'(s)}\,\mathrm{d}s. \tag{5.3.24}$$

We insert this into (5.3.20) and apply Jensen's inequality to obtain

$$\begin{aligned} D^\alpha(\varphi) &= \frac{4\pi(\pi-\alpha)}{\sigma^2}\int_{\substack{0\le\rho<1\\0\le\theta<1}} \exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\int_0^1\frac{\varphi'^2_{z,0}(s)}{(\varphi^{-1})'(s)}ds\right\}\frac{4\rho\,\mathrm{d}\rho\,\mathrm{d}\theta}{(1-\rho^2)^2}, \\ &\le \frac{4\pi(\pi-\alpha)}{\sigma^2}\int_{\substack{0\le\rho<1\\0\le\theta<1}}\int_0^1 \frac{1-\rho^2}{1+\rho^2}\varphi'^2_{z,0}(s)\exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\cdot\frac{1+\rho^2}{1-\rho^2}\cdot\frac{1}{(\varphi^{-1})'(s)}\right\}\mathrm{d}s\,\frac{4\rho\,\mathrm{d}\rho\,\mathrm{d}\theta}{(1-\rho^2)^2}. \end{aligned} \tag{5.3.25}$$

Then, by Tonelli's theorem,

$$\begin{aligned} D^\alpha(\varphi) &\le \frac{4\pi(\pi-\alpha)}{\sigma^2}\int_{0\le\rho<1}\int_0^1 \exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\cdot\frac{1+\rho^2}{1-\rho^2}\cdot\frac{1}{(\varphi^{-1})'(s)}\right\}\mathrm{d}s\,\frac{4\rho\,\mathrm{d}\rho}{(1-\rho^2)^2} \\ &= \frac{4\pi(\pi-\alpha)}{\sigma^2}\int_0^1 \exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2(\varphi^{-1})'(s)}\right\}\frac{\sigma^2(\varphi^{-1})'(s)}{2(\pi^2-\alpha^2)}\,\mathrm{d}s \\ &\le \frac{4\pi(\pi-\alpha)}{\sigma^2}\int_0^1 \frac{\sigma^2(\varphi^{-1})'(s)}{2(\pi^2-\alpha^2)}\,\mathrm{d}s \\ &= \frac{2\pi}{\pi+\alpha}. \end{aligned} \tag{5.3.26}$$

To find the limit of $D^\alpha(\varphi)$, recall that $\left\{\frac{1-\rho^2}{1+\rho^2}\varphi'^2_{z,0}\right\}_{\rho\nearrow 1}$ is an approximate identity. More precisely,

$$\int_0^1 \frac{\varphi'^2_{z,0}(s)}{(\varphi^{-1})'(s)}\,\mathrm{d}s = \frac{1+\rho^2}{1-\rho^2}\cdot\left(\frac{1}{(\varphi^{-1})'(\theta)}+o(1)\right) \qquad \text{as } \rho\nearrow 1, \tag{5.3.27}$$

uniformly in $\theta\in\mathbb{T}$, by Lemma 5.3.4. Therefore, as $\alpha\nearrow\pi$,

$$\begin{aligned} &\int_{0\le\rho<1}\exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\int_0^1\frac{\varphi'^2_{z,0}(s)}{(\varphi^{-1})'(s)}\,\mathrm{d}s\right\}\frac{4\rho\,\mathrm{d}\rho}{(1-\rho^2)^2} \\ &\quad = \exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2\left[(\varphi^{-1})'(\theta)+o(1)\right]}\right\}\frac{\sigma^2\left[(\varphi^{-1})'(\theta)+o(1)\right]}{2(\pi^2-\alpha^2)}+O(1) \\ &\quad = \frac{\sigma^2(\varphi^{-1})'(\theta)}{2(\pi^2-\alpha^2)}\big(1+o(1)\big). \end{aligned} \tag{5.3.28}$$

Integration in $\theta$ finishes the proof. □

*Proof of Proposition 5.3.3.* It follows from the definition of $\widetilde{\mathscr{M}}_{\sigma^2}$ that

$$\exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\int_0^1\varphi'^2(t)\,\mathrm{d}t\right\}\mathrm{d}\widetilde{\mathscr{M}}_{\sigma^2}(\varphi) = \mathrm{d}\mathscr{N}^\alpha_{\sigma^2}(\varphi_0)\otimes\mathrm{d}\Theta, \tag{5.3.29}$$

where

$$\varphi(\cdot) = \varphi_0(\cdot)+\Theta, \text{ for } \Theta\in[0,1). \tag{5.3.30}$$

Therefore,

$$\int_{\mathrm{Diff}^1(\mathbb{T})} F(\varphi)\,\mathrm{d}\mathscr{N}^\alpha_{\sigma^2}(\varphi) = \int_{\mathrm{Diff}^1(\mathbb{T})} F(\varphi)\exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\int_0^1\varphi'^2(t)\,\mathrm{d}t\right\}\mathrm{d}\widetilde{\mathscr{M}}_{\sigma^2}(\varphi). \tag{5.3.31}$$

Using the definition of $D^\alpha$ given in (5.3.20) and the factorisation of $\widetilde{\mathscr{M}}_{\sigma^2}$ as stated in (5.2.45),

$$\frac{4\pi(\pi-\alpha)}{\sigma^2}\int_{\mathrm{Diff}^1(\mathbb{T})} F(\varphi)\exp\left\{-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\int_0^1 \varphi'^2(t)\,\mathrm{d}t\right\}\mathrm{d}\widetilde{\mathscr{M}}_{\sigma^2}(\varphi) = \int_{\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})} D^\alpha(\varphi)F(\varphi)\,\mathrm{d}\mathscr{M}_{\sigma^2}(\varphi). \quad (5.3.32)$$

If $\int F(\varphi)\,\mathrm{d}\mathscr{M}_{\sigma^2}(\varphi)$ is finite, then using Lemma 5.3.5 and the Dominated Convergence Theorem we obtain the desired result. If, on the other hand, $\int F(\varphi)\,\mathrm{d}\mathscr{M}_{\sigma^2}(\varphi)$ is infinite, then we get the desired by Fatou's Lemma. □

#### 5.3.1.2 Partition function of $\mathscr{N}^\alpha_{\sigma^2}$: Proof of Proposition 5.3.2

The first step of the proof of Proposition 5.3.2 is the following application of the change of measure formula for the unnormalised Malliavin–Shavgulidze measure.

**Proposition 5.3.6:** For any $\alpha \in (0,\pi)\cup i\mathbb{R}$ we have

$$\mathscr{N}^\alpha_{\sigma^2}\left(\mathrm{Diff}^1(\mathbb{T})\right) = \frac{\alpha}{\sin\alpha}\int_{\mathrm{Diff}^1(\mathbb{T})} \exp\left\{\frac{8\sin^2\frac{\alpha}{2}}{\sigma^2}\cdot\varphi'(0)\right\}\mathrm{d}\mathscr{N}^0_{\sigma^2}(\varphi). \quad (5.3.33)$$

**Remark 5.3.7:** All functions of $\alpha$ in Proposition 5.3.6 are even. Therefore, their values are real.

*Proof.* For $\alpha = 0$ the statement is obvious. First we consider $\alpha \in (0,\pi)$. Take

$$f(t) = \frac{1}{2}\left[\frac{1}{\tan\frac{\alpha}{2}}\tan\left(\alpha\left(t-\frac{1}{2}\right)\right)+1\right]. \quad (5.3.34)$$

It is easy to check that

$$\mathcal{S}_f(t) = 2\alpha^2, \qquad f'(0) = f'(1) = \frac{\alpha}{\sin\alpha}, \qquad -\frac{f''(0)}{f'(0)} = \frac{f''(1)}{f'(1)} = 2\alpha\tan\frac{\alpha}{2}. \quad (5.3.35)$$

Thus, it follows from Proposition 5.2.6 and the definition of $\mathscr{N}^\alpha_{\sigma^2}$ which was given in (5.3.4) that for any non-negative continuous functional $F$ on $\mathrm{Diff}^1(\mathbb{T})$ we have

$$\int_{\mathrm{Diff}^1(\mathbb{T})} F(\varphi)\,\mathrm{d}\mathscr{N}^0_{\sigma^2}(\varphi) = \frac{\sin\alpha}{\alpha}\int_{\mathrm{Diff}^1(\mathbb{T})} F(f\circ\varphi)\exp\left\{-\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\cdot\varphi'(0)+\frac{2\alpha^2}{\sigma^2}\int_0^1\varphi'^2(t)\,\mathrm{d}t\right\}\mathrm{d}\mathscr{N}^0_{\sigma^2}(\varphi). \quad (5.3.36)$$

Now we choose $F$ to be

$$F(\varphi) = \exp\left\{\frac{8\sin^2\frac{\alpha}{2}}{\sigma^2}\cdot\varphi'(0)\right\}, \tag{5.3.37}$$

which guarantees that

$$F(f\circ\varphi) = \exp\left\{\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\cdot\varphi'(0)\right\}, \tag{5.3.38}$$

and the claim follows. Finally, for $\alpha\in i\mathbb{R}$ the proof is exactly the same if we take

$$f(t) = \frac{1}{2}\left[\frac{1}{\tanh\frac{i\alpha}{2}}\tanh\left(i\alpha\left(t-\frac{1}{2}\right)\right)+1\right]. \tag{5.3.39}$$

□

Now we make sense of and prove [98, Equation (20)]. For $\lambda\geq 1$ this is a direct change of variable and related to the construction in [99] of global equivariant Darboux charts, see Remark 5.3.9.

**Lemma 5.3.8:** For any $\lambda\in(-1,+\infty)$ we have

$$\int\exp\left\{\frac{-2\lambda^2}{\sigma^2(\lambda+1)}\cdot\frac{1}{\int_0^1 e^{\xi(t)}\,\mathrm{d}t}\right\}\mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{2\big(\log(\lambda+1)\big)^2}{\sigma^2}\right\}. \tag{5.3.40}$$

*Proof.* We follow the argument from Appendix B in [100]. Consider $g\in\mathrm{Diff}^3_+[0,1]$ given by

$$g(t) = \frac{(\lambda+1)t}{\lambda t+1}. \tag{5.3.41}$$

It is easy to see that

$$g'(t) = \frac{\lambda+1}{(\lambda t+1)^2},\qquad g''(t) = -\frac{2(\lambda+1)\lambda}{(\lambda t+1)^3},\qquad \mathcal{S}_g(t) = 0. \tag{5.3.42}$$

Then, according to Proposition 5.2.6,

$$\frac{\mathrm{d}g^\sharp\mathcal{B}^{-2\log(\lambda+1),1}_{\sigma^2}(\xi)}{\mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi)} = \exp\left\{\frac{-2\lambda}{\sigma^2}\left(1-\frac{1}{\lambda+1}\right)\frac{1}{\int_0^1 e^{\xi(t)}dt}\right\}. \tag{5.3.43}$$

Therefore,

$$\int \exp\left\{\frac{-2\lambda}{\sigma^2}\left(1-\frac{1}{\lambda+1}\right)\frac{1}{\int_0^1 e^{\xi(t)}\,\mathrm{d}t}\right\}\mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{\left(2\log(\lambda+1)\right)^2}{2\sigma^2}\right\}, \tag{5.3.44}$$

which finishes the proof. □

**Remark 5.3.9:** The construction from the proof above is related to the construction of global equivariant Darboux charts proposed in [99] as follows: For $\alpha \in i\mathbb{R}$ take $f$ as in the proof of Proposition 5.3.6 and $g$ as in the proof of Lemma 5.3.8 with $\lambda = e^{|\alpha|}-1$. Then $g\circ f\circ \mathrm{P}$ corresponds to map $q^{-1}$ as defined in [99] under suitable normalisation. However, this map does not generalize to the case $\alpha > 0$.

We use the following analytic continuation lemma to access $\lambda \in \mathbb{C}$ satisfying $|\lambda+1| = 1$.

**Lemma 5.3.10:** Let $\mathcal{P}$ be a non-negative measure on $\mathbb{R}_+$. Assume that there exists $\varepsilon > 0$ such that the exponential moment generating function $F(z) = \int \exp(zX)\,\mathrm{d}\mathcal{P}(X)$ exists for all $z \in [0,\varepsilon)$. Assume further that for some $R > 0$, $F(z)$ can be analytically continued for all $z \in \mathbb{D}_R$. Then $\int \exp(zX)\,\mathrm{d}\mathcal{P}(X)$ converges absolutely for all $z \in \mathbb{D}_R$, and is equal to the analytic continuation of $F(z)$ to $\mathbb{D}_R$.

*Proof.* Since $\mathcal{P}$ supported on $\mathbb{R}_+$, by Tonelli's Theorem,

$$F(z)=\sum_{n\geq 0} z^n\frac{\int X^n\,\mathrm{d}\mathcal{P}(X)}{n!}, \tag{5.3.45}$$

for $z \in [0,\varepsilon)$. Given that $F(z)$ is analytic in $\mathbb{D}_R$, we conclude that the right-hand side converges absolutely for $z \in \mathbb{D}_R$. Thus, by Tonelli's Theorem again, $\mathbb{E}\exp(zX)$ converges for $z \in [0,R)$ and is equal to $F(z)$. We can continue the equality for the whole disc $\mathbb{D}_R$, since $|z^nX^n| \leq |z|^nX^n$, and all expressions are absolutely convergent in $\mathbb{D}_R$. □

**Lemma 5.3.11:** For any $\lambda \in \mathbb{C}$ with $|\lambda+1| = 1$ we have

$$\int \exp\left\{\frac{-2\lambda^2}{\sigma^2(\lambda+1)}\cdot\frac{1}{\int_0^1 e^{\xi(t)}\,\mathrm{d}t}\right\}\mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{2\left(\log(\lambda+1)\right)^2}{\sigma^2}\right\}. \tag{5.3.46}$$

Here $\log(\lambda+1)$ is taken in $[-i\pi, i\pi)$, making the right-hand side of (5.3.46) continuous for $\lambda$ in question.

Observe that the right-hand side of (5.3.46) is not a well-defined analytic function on the whole complex plane. Thus, the left-hand side of (5.3.46) does not converge for some values of $\lambda$.

*Proof.* We prove that we can analytically continue the equality from Lemma 5.3.8. Consider the function

$$F(z) = \int \exp\left\{ \frac{z}{\sigma^2 \int_0^1 e^{\xi(t)}\, \mathrm{d}t} \right\} \mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi). \tag{5.3.47}$$

We are interested in taking $z = -\frac{2\lambda^2}{\lambda+1}$. Observe that since $z = -2((\lambda+1) - 2 + \frac{1}{\lambda+1})$, we have that $\lambda$ in question correspond to $z \in [0,8]$, and $\lambda \in (-1,\infty)$ corresponds to $z \in (-\infty, 0]$.
First, we notice that it follows from Lemma 5.6.1 that there exists $\varepsilon > 0$ such that the integral in (5.3.47) converges absolutely for all $z \in \mathbb{D}_\varepsilon$, defining an analytic function in $\mathbb{D}_\varepsilon$.
Secondly, observe that from (5.3.40) for $\lambda \in (-1,+\infty)$ we obtain that for all $z \in (-\epsilon, 0]$, we have

$$F(z) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{2}{\sigma^2} \left( \log\left( \frac{-z+4+\sqrt{z^2-8z}}{4} \right) \right)^2 \right\}. \tag{5.3.48}$$

Notice that the right-hand side in (5.3.48) defines an analytic function for $|z| < 8$. Indeed, the only problematic point in $\mathbb{D}_8$ is $z = 0$, but around $z = 0$ the expression in (5.3.48) is analytic, since different branches of the square root give rise to the values of log which differ only by a sign which, in turn, is cancelled by taking the square. Now we use Lemma 5.3.10 and obtain that for $z \in \mathbb{D}_8$ the expectation in (5.3.47) converges and

$$\int \exp\left\{ \frac{z}{\sigma^2 \int_0^1 e^{\xi(t)}\, \mathrm{d}t} \right\} \mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{2}{\sigma^2} \left( \log\left( \frac{-z+4+\sqrt{z^2-8z}}{4} \right) \right)^2 \right\}, \tag{5.3.49}$$

for all $z \in \mathbb{D}_8$. Also note that by taking $z \to 8$ and monotone convergence we can continue identity for $z = 8$ as well. We finish the proof by using the fact that values of $\lambda$ with $|\lambda+1| = 1$ correspond to $z \in [0,8]$. □

*Proof of Proposition 5.3.2.* By Proposition 5.3.6 and (5.3.4) we have

$$\mathcal{N}^{\alpha}_{\sigma^2}\left(\mathrm{Diff}^1(\mathbb{T})\right) = \frac{\alpha}{\sin\alpha} \int_{C_0[0,1]} \exp\left\{ \frac{8\sin^2\frac{\alpha}{2}}{\sigma^2} \cdot \frac{1}{\int_0^1 e^{\xi(t)}\, \mathrm{d}t} \right\} \mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi) \tag{5.3.50}$$

where we used that for $\varphi = \mathsf{P}(\xi)$, we have

$$\varphi'(0) = \frac{1}{\int_0^1 e^{\xi(t)}\,\mathrm{d}t}. \tag{5.3.51}$$

Applying Lemma 5.3.11 with $\lambda = e^{i\alpha} - 1$ for $\alpha \in (0, \pi)$ or Lemma 5.3.8 with $\lambda = e^{|\alpha|} - 1$ for $\alpha \in i\mathbb{R}$ we get

$$\int_{C_0[0,1]} \exp\left\{\frac{8\sin^2\frac{\alpha}{2}}{\sigma^2} \cdot \frac{1}{\int_0^1 e^{\xi(t)}\,\mathrm{d}t}\right\} \mathrm{d}\mathcal{B}^{0,1}_{\sigma^2}(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{2\alpha^2}{\sigma^2}\right\}, \tag{5.3.52}$$

which concludes the proof. □

## 5.4 Partition function and proofs of main theorems

Theorem 5.1.1 was already proved in Proposition 5.2.9, and the corresponding change of measure formula with general metric of Theorem 5.1.3 was proved in Proposition 5.2.12. Thus it remains to prove Theorem 5.1.2 and the corresponding statement for the partition function with general metric in Theorem 5.1.3.

*Proof of Theorem 5.1.2.* Our goal is to prove

$$\begin{aligned} Z(\sigma^2) := \mathscr{M}_{\sigma^2}\left(\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})\right) &= \left(\frac{2\pi}{\sigma^2}\right)^{3/2} \exp\left(\frac{2\pi^2}{\sigma^2}\right) \\ &= \int_0^\infty \exp\left(-\frac{\sigma^2 k^2}{2}\right) \sinh(2\pi k)\, 2k\,\mathrm{d}k \\ &= \int_0^\infty e^{-\sigma^2 E} \sinh(2\pi\sqrt{2E})\, 2\,\mathrm{d}E. \end{aligned} \tag{5.4.1}$$

For the first line in (5.4.1) we apply Proposition 5.3.3 and Proposition 5.3.2:

$$\begin{aligned} \mathscr{M}_{\sigma^2}\left(\mathrm{Diff}^1(\mathbb{T})/\mathrm{PSL}(2,\mathbb{R})\right) &= \lim_{\alpha\to\pi-} \frac{4\pi(\pi-\alpha)}{\sigma^2} \mathscr{N}^{\alpha}_{\sigma^2}\left(\mathrm{Diff}^1(\mathbb{T})\right). \\ &= \lim_{\alpha\to\pi-} \frac{4\pi(\pi-\alpha)}{\sigma^2} \frac{\alpha}{\sin\alpha} \frac{e^{2\alpha^2/\sigma^2}}{\sqrt{2\pi\sigma^2}} \\ &= \left(\frac{2\pi}{\sigma^2}\right)^{3/2} \exp\left(\frac{2\pi^2}{\sigma^2}\right). \end{aligned} \tag{5.4.2}$$

The second line in (5.4.1) follows from an elementary integration by parts:

$$\begin{aligned}\int_0^\infty \exp\left(-\frac{\sigma^2k^2}{2}\right)\sinh(2\pi k)\,2k\,\mathrm{d}k &= \frac{4\pi}{\sigma^2}\int_0^\infty \exp\left(-\frac{\sigma^2k^2}{2}\right)\cosh(2\pi k)\,\mathrm{d}k \\ &= \frac{2\pi}{\sigma^2}\int_{-\infty}^\infty \exp\left(-\frac{\sigma^2k^2}{2}+2\pi k\right)\mathrm{d}k \\ &= \left(\frac{2\pi}{\sigma}\right)^{3/2}\exp\left(\frac{2\pi^2}{\sigma^2}\right).\end{aligned} \tag{5.4.3}$$

The third line then follows by changing variables to $E = \frac{1}{2}k^2$. □

*Proof of* (5.1.17) *in Theorem 5.1.3.* By the definition of the Schwarzian measure with metric and the representation (5.2.48), the total mass of $\widetilde{\mathscr{M}}_\rho$ is simply

$$Z(\sigma_\rho^2)\exp\left\{\int_0^1 \mathcal{S}(h,\tau)\frac{\mathrm{d}\tau}{\rho(\tau)}\right\}, \qquad \text{where } \sigma_\rho^2 = \int_0^1 \rho(\tau)\,\mathrm{d}\tau. \tag{5.4.4}$$

As already remarked in (5.2.47), by integrating by parts, the term inside the exponential can be written

$$\begin{aligned}\int_0^1 \mathcal{S}(h,\tau)\frac{\mathrm{d}\tau}{h'(\tau)} &= \int_0^1 \left(\left(\frac{h''(\tau)}{h'(\tau)}\right)' - \frac{1}{2}\left(\frac{h''(\tau)}{h'(\tau)}\right)^2\right)\frac{\mathrm{d}\tau}{h'(\tau)} \\ &= \int_0^1 \left(-\left(\frac{h''(\tau)}{h'(\tau)}\right)\left(\frac{1}{h'(\tau)}\right)' - \frac{1}{2}\left(\frac{h''(\tau)^2}{h'(\tau)^3}\right)\right)\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^1 \frac{h''(\tau)^2}{h'(\tau)^3}\,\mathrm{d}\tau,\end{aligned} \tag{5.4.5}$$

which gives the claim. □

## 5.5 Appendix: Calculation of formal correlation functions

The *truncated* correlation functions are formally defined as the functional derivatives

$$\langle \mathcal{S}(\tau_1);\cdots;\mathcal{S}(\tau_k)\rangle_{\sigma^2} := \frac{\delta}{\delta(1/\rho)(\tau_1)}\cdots\frac{\delta}{\delta(1/\rho)(\tau_k)}\Big|_{\rho=\sigma^2}\log Z(\rho). \tag{5.5.1}$$

In the following we make sense of this expression in a distributional sense by considering the $k$-th variation of $\log Z(\rho)$: For $h_1,\dots,h_k \in C^\infty(\mathbb{T})$ define $\rho_{\epsilon_1,\dots,\epsilon_k}$ via $1/\rho_{\epsilon_1,\dots,\epsilon_k} = 1/\rho + \sum_{i=1}^k \epsilon_i h_i$.

Then

$$
\begin{aligned}
[D^k_{1/\rho}\log Z(\rho)](h_1,\dots,h_k) &:= \frac{\partial^k}{\partial\epsilon_1\cdots\partial\epsilon_k}\Big|_0 \log(Z(\rho_{\epsilon_1,\dots,\epsilon_k})) \\
&= \int \prod_{i=1}^k \mathrm{d}\tau_i\, h(\tau_1)\cdots h(\tau_k)\,\langle \mathcal{S}(\tau_1);\cdots;\mathcal{S}(\tau_k)\rangle_\rho,
\end{aligned}
\tag{5.5.2}
$$

where the last line is understood as defining (5.5.1) as a $k$-variate distribution.

**Proposition 5.5.1:** For $\sigma^2 > 0$, $k \geq 1$ and $h_1, \cdots, h_k \in C^\infty(\mathbb{T})$ we have

$$
\begin{aligned}
&[D^k_{1/\rho}\log Z(\rho)]\Big|_{\rho=\sigma^2}(h_1,\dots,h_k) \\
&\quad= (-1)^k k!\,\sigma^{2(k-1)} \sum_{1\leq i<j\leq k} \int \mathrm{d}\tau\, h_i' h_j' \textstyle\prod_{l\neq i,j} h_l \\
&\qquad+ (-1)^k \sigma^{2k} \sum_{\pi\in\mathrm{Part}[k]} \sigma^{2|\pi|}[\log Z]^{(|\pi|)}(\sigma^2) \prod_{B\in\pi}\left(|B|!\int \mathrm{d}\tau \prod_{b\in B} h_b(\tau)\right),
\end{aligned}
\tag{5.5.3}
$$

where the sum in the last line is over all partitions $\pi = \{B_1, \dots, B_{|\pi|}\}$ of $\{1, \dots, k\}$.

**Corollary 5.5.2:** For non-coinciding $\tau_1, \cdots, \tau_k \in \mathbb{T}$ we have

$$
\langle \mathcal{S}(\tau_1);\cdots;\mathcal{S}(\tau_k)\rangle_{\sigma^2} = (-1)^k\sigma^{4k}[\log Z]^{(k)}(\sigma^2) = 2\pi^2 k!\,\sigma^{2(k-1)} + \tfrac{3}{2}(k-1)!\,\sigma^{2k}. \tag{5.5.4}
$$

Thus the Schwarzian correlators are constant away from coinciding points and their values (up to factors of $\sigma^2$) are given by the cumulants of the Boltzmann-weighted spectral density $e^{-\sigma^2 E}\rho(E)\,\mathrm{d}E$, see (5.1.14). The untruncated Schwarzian correlators for non-coinciding points are therefore equal to the moments of the spectral measure $\rho(E)$ (again up to factors of $\sigma^2$). We note that this relationship has been predicted by Stanford and Witten [96, Appendix C]. More generally, by (5.5.2) and (5.5.3) we can express the truncated Schwarzian correlators completely in terms of multivariate (derivatives of) $\delta$-functions in the variables $\tau_i$. While the general expression is somewhat messy, we can easily derive (5.1.21) and (5.1.22):

$$
\langle \mathcal{S}(\tau)\rangle_{\sigma^2} = -\sigma^4[\log Z]'(\sigma^2) = 2\pi + \tfrac{3}{2}\sigma^2 \tag{5.5.5}
$$

and

$$
\begin{aligned}
\langle \mathcal{S}(0)\mathcal{S}(\tau)\rangle_{\sigma^2} &= \langle \mathcal{S}(0);\mathcal{S}(\tau)\rangle_{\sigma^2} + \langle \mathcal{S}(0)\rangle_{\sigma^2}\langle \mathcal{S}(\tau)\rangle_{\sigma^2} \\
&= [2\pi^2\sigma^2 + \tfrac{3}{2}\sigma^4 + (2\pi + \tfrac{3}{2}\sigma^2)^2] - 2\sigma^2[(2\pi + \tfrac{3}{2}\sigma^2)]\delta(\tau) - 2\sigma^2\delta''(\tau).
\end{aligned}
\tag{5.5.6}
$$

*Proof of Proposition 5.5.1.* We treat the first and second summand in

$$\log Z(\rho) = \frac{1}{2}\int \frac{\rho'^2}{\rho^3} + \log Z(\sigma_\rho^2) \tag{5.5.7}$$

separately. For the second term, we simply apply the multivariate version Faà di Bruno's formula:

$$[D^k_{1/\rho}\log Z(\sigma_\rho^2)]\Big|_{\rho=\sigma^2}(h_1,\dots,h_k) = \sum_{\pi\in\mathrm{Part}[k]}[\log Z]^{(|\pi|)}(\sigma^2)\prod_{B\in\pi}\left([D^{|B|}_{1/\rho}\sigma_\rho^2]\Big|_{\rho=\sigma^2}(\{h_b\}_{b\in C})\right). \tag{5.5.8}$$

Recall that $\sigma_\rho^2 = \int\rho$ and check that

$$[D^{|B|}_{1/\rho}\sigma_\rho^2]\Big|_{\rho=\sigma^2}(\{h_b\}_{b\in C}) = (-1)^{|B|}|B|!\,\sigma^{2(|B|+1)}\int\prod_{b\in B}h_b. \tag{5.5.9}$$

This yields the second summand on the right hand side of (5.5.3). For the first summand write

$$F(\tfrac{1}{\rho}) := \frac{1}{2}\int\frac{\rho'^2}{\rho^3} = \frac{1}{2}\int\left(\log\frac{1}{\rho}\right)'\left(\frac{1}{\rho}\right)'\mathrm{d}\tau. \tag{5.5.10}$$

For smooth functions $f, h \in C^\infty(\mathbb{T})$ and $f>0$ we have for $\epsilon>0$ sufficiently small

$$\begin{aligned}\log(f+\epsilon h)'(f+\epsilon h)' &= (f'+\epsilon h')\left(\log f + \sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\epsilon^k\left(\frac{h}{f}\right)^k\right)' \\ &= (f'+\epsilon h')\left(\frac{f'}{f} + \sum_{k\geq 1}(-1)^{k-1}\epsilon^k\left(\frac{h}{f}\right)^{k-1}\left(\frac{h}{f}\right)'\right),\end{aligned} \tag{5.5.11}$$

with the series converging uniformly on $\mathbb{T}$. Hence, we have for sufficiently small $\epsilon>0$ that

$$F(\tfrac{1}{\sigma^2}+\epsilon h) = \frac{1}{2}\sum_{k\geq 2}(-\epsilon)^k\sigma^{2(k-1)}\int h'^2h^{k-2}. \tag{5.5.12}$$

In other words

$$[D^k_{1/\rho}F(1/\rho)]\Big|_{\rho=\sigma^2}(h,\dots,h) = (-1)^k k!\,\sigma^{2(k-1)}\frac{1}{2}\int h'^2h^{k-2}. \tag{5.5.13}$$

The general derivative $[D^k_{1/\rho}F(1/\rho)]\big|_{\rho=\sigma^2}(h_1,\dots,h_k)$ follows by polarisation of this identity and yields the first summand on the right hand side (5.5.3). □

## 5.6 Appendix: Change of variables: Proof of Proposition 5.2.6

Let $\mathcal{W}_{\sigma^2}$ denote the Wiener measure with variance $\sigma^2$ on $C_{0,\text{free}}[0,1] = \{f \in C[0,1] \mid f(0) = 0\}$, and recall the definition of the unnormalised Brownian bridge $\mathcal{B}^{a,1}_{\sigma^2}$ from 0 to $a$ in time 1 from Definition 5.2.1. Then by [RevuzYor, Exercise (3.16)], for any $X \subset C_{0,\text{free}}[0,1]$,

$$\mathcal{W}_{\sigma^2}(X) = \int_{\mathbb{R}} \mathcal{B}^{a,1}_{\sigma^2}(X)\,\mathrm{d}a. \tag{5.6.1}$$

**Lemma 5.6.1:** Let $\mathcal{P}$ be either $\mathcal{B}^{a,T}_{\sigma^2}$ for some $a \in \mathbb{R}$ and $T > 0$, or $\mathcal{W}_{\sigma^2}$ (in this case we put $T = 1$). Then there exists $\varepsilon > 0$ such that

$$\int \exp\left(\frac{\varepsilon}{\int_0^T e^{\xi(\tau)}\,\mathrm{d}\tau}\right)\mathrm{d}\mathcal{P}(\xi) < \infty. \tag{5.6.2}$$

*Proof.* If $\mathcal{P} = \mathcal{B}^{a,T}_{\sigma^2}$, then let $\widetilde{\xi}$ be a Brownian bridge distributed according to the probability measure $\sqrt{2\pi T}\sigma \exp\left(\frac{a^2}{2T\sigma^2}\right)\mathrm{d}\mathcal{B}^{a,T}_{\sigma^2}(\widetilde{\xi})$, and according to $\mathcal{W}_{\sigma^2}$ otherwise. Then

$$\mathbb{P}\left[\frac{1}{\int_0^T e^{\widetilde{\xi}(\tau)}\,\mathrm{d}\tau} > \lambda\right] \leq \mathbb{P}\left[\min_{t\in[0,e\lambda^{-1})]} \widetilde{\xi}(t) < -1\right] \leq C^{-1}e^{-C\lambda}, \tag{5.6.3}$$

for some $C > 0$, independent of $\lambda > 10/T$. □

Recall the left-composition operator $L_f$ as in (5.2.18). The following lemma is key for the calculations. It verifies formula (7) in [98] and (2.7) in [100], which are key for the calculations.

**Lemma 5.6.2:** Let $f \in \mathrm{Diff}^3[0,1]$, and let $f^\sharp\mathcal{W}_{\sigma^2} = f_\sharp^{-1}\mathcal{W}_{\sigma^2}$ be the push-forward of $\mathcal{W}_{\sigma^2}$ under $\mathsf{P}^{-1} \circ L_{f^{-1}} \circ \mathsf{P} = \left(\mathsf{P}^{-1} \circ L_f \circ \mathsf{P}\right)^{-1}$. Then

$$\begin{aligned}\frac{\mathrm{d}f^\sharp\mathcal{W}_{\sigma^2}(W)}{\mathrm{d}\mathcal{W}_{\sigma^2}(W)} &= \frac{1}{\sqrt{f'(0)f'(1)}} \\ &\times \exp\left\{\frac{1}{\sigma^2}\left[\frac{f''(0)}{f'(0)}\mathsf{P}'_W(0) - \frac{f''(1)}{f'(1)}\mathsf{P}'_W(1)\right] + \frac{1}{\sigma^2}\int_0^1 \mathcal{S}_f\left(\mathsf{P}_W(t)\right)\left(\mathsf{P}'_W(t)\right)^2\mathrm{d}t\right\}.\end{aligned} \tag{5.6.4}$$

*Proof.* Let $B(t)=\sigma^{-1}W(t)$ be a standard Brownian motion. The transformation $W\mapsto \mathsf{P}^{-1}\circ L_f\circ \mathsf{P}(W)$ corresponds to the transformation of $B(t)$ given by

$$B(t)\mapsto B(t)+\sigma^{-1}\log f'\left(\frac{\int_0^t e^{\sigma B(\tau)}\,\mathrm{d}\tau}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}\right)-\sigma^{-1}\log f'(0). \tag{5.6.5}$$

The problem reduces to the calculation of Radon-Nikodym derivative of the probability measure, corresponding to the standard Brownian motion under the transformation inverse to (5.6.5). Denote

$$\phi(t)=\frac{\int_0^t e^{\sigma B(\tau)}\,\mathrm{d}\tau}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}, \tag{5.6.6}$$

and

$$h(x)=\log f'(x)-\log f'(0). \tag{5.6.7}$$

Using [123, Theorem 4.1.2], the Radon-Nikodym derivative is given by

$$\det\nolimits_2(1+K)\exp\left(-\delta(u)-\frac{1}{2}\int_0^1 u^2(t)\,\mathrm{d}t\right), \tag{5.6.8}$$

where $\delta$ denotes Skorokhod integral, $\det_2$ is a Hilbert-Carleman (or Carleman-Fredholm) determinant (see, e.g. [124, Chapter X]), $u(t)=u[\phi](t)$ is given by

$$u(t)=\frac{\mathrm{d}}{\mathrm{d}t}\sigma^{-1}h\big(\phi(t)\big)=\sigma^{-1}h'\big(\phi(t)\big)\phi'(t), \tag{5.6.9}$$

and $K=K[\phi]$ is the Fréchet derivative of the map

$$(B(t))_{t\in[0,1]}\mapsto(\sigma^{-1}h(\phi(t)))_{t\in[0,1]}=\left(\sigma^{-1}\log f'\left(\frac{\int_0^t e^{\sigma B(\tau)}\,\mathrm{d}\tau}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}\right)-\sigma^{-1}\log f'(0)\right)_{t\in[0,1]} \tag{5.6.10}$$

with respect to the Cameron-Martin space of Brownian motion, which we associate with the Sobolev space $\tilde{H}^1=\left\{g:\|g\|_{\tilde{H}^1}=\int_0^1\left(g'(t)\right)^2\mathrm{d}t<\infty,\, g(0)=0\right\}$. Direct calculation shows that

$$(Kg)(t)=\int_0^1 k(t,s)g(s)\,\mathrm{d}s,\qquad \text{with } k(t,s)=-h'\left(\phi(t)\right)\phi(t)\phi'(s)+\mathbb{1}_{s<t}\,h'\left(\phi(t)\right)\phi'(s). \tag{5.6.11}$$

**Integral.** First, we calculate Skorokhod integral

$$\delta(u) = \sigma^{-1}\delta\Big(h'(\phi(t))\phi'(t)\Big) = \sigma^{-1}\delta\left(h'\left(\frac{\int_0^t e^{\sigma B(\tau)}\,\mathrm{d}\tau}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}\right)\frac{e^{\sigma B(t)}}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}\right). \tag{5.6.12}$$

The process is not adapted because of the term $\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau$. We use [123, Theorem 3.2.9] in order to reduce the Skorokhod integral to a Itô integral. It follows from Lemma 5.6.1 that the random variable

$$\frac{1}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau} \tag{5.6.13}$$

is Malliavin smooth and that

$$D_t\left(\frac{1}{\int_0^1 e^{\sigma B(\tau)}d\tau}\right) = \frac{\sigma\int_t^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}{\left(\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau\right)^2}. \tag{5.6.14}$$

Thus, since $u(t)$ is also Malliavin smooth, using [123, Theorem 3.2.9] we get

$$\begin{aligned}\delta\left(h'\left(\frac{\int_0^t e^{\sigma B(\tau)}\,\mathrm{d}\tau}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}\right)\frac{e^{\sigma B(t)}}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}\right) &= \int_0^1 h'\left(F\int_0^t e^{\sigma B(\tau)}\,\mathrm{d}\tau\right)F\,e^{\sigma B(t)}\,\mathrm{d}B(t)\Big|_{F=\left(\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau\right)^{-1}} \\ &+ \int_0^1\Big[h''\big(\phi(t)\big)\phi(t)\phi'(t) + h'\big(\phi(t)\big)\phi'(t)\Big]\cdot\left(\frac{\sigma\int_t^1 e^{\sigma B(\tau)}d\tau}{\int_0^1 e^{\sigma B(\tau)}d\tau}\right)\mathrm{d}t.\end{aligned} \tag{5.6.15}$$

We observe that the expression in square brackets is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[h'(\phi(t))\phi(t)\Big], \tag{5.6.16}$$

and calculate the second term in the right-hand side of (5.6.15) by integrating by parts

$$\begin{aligned}\int_0^1\Big[h''\big(\phi(t)\big)\phi(t)\phi'(t) + h'\big(\phi(t)\big)\phi'(t)\Big]\cdot\left(\frac{\sigma\int_t^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}\right)\mathrm{d}t& \\ = \sigma\int_0^1 h'(\phi(t))\phi(t)\frac{e^{\sigma B(t)}}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}\,\mathrm{d}t.&\end{aligned} \tag{5.6.17}$$

Integrating by parts once again we obtain

$$\begin{aligned}\sigma\int_0^1 h'(\phi(t))\phi(t)\frac{e^{\sigma B(t)}}{\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau}\,\mathrm{d}t &= \sigma\int_0^1 h'(\phi(t))\phi'(t)\phi(t)\,\mathrm{d}t\\ &= \sigma\, h(1)-\sigma\int_0^1 h(\phi(t))\phi'(t)\,\mathrm{d}t\\ &= \sigma\, h(1)-\sigma\int_0^1 h(s)\,\mathrm{d}s. \end{aligned}\tag{5.6.18}$$

We calculate the first term in the right-hand side of (5.6.15) by replacing $e^{\sigma B(t)}\,\mathrm{d}B(t) = \sigma^{-1}\,\mathrm{d}e^{\sigma B(t)} - \frac{\sigma}{2}e^{\sigma B(t)}\,\mathrm{d}t$, and using Itô integration by parts

$$\begin{aligned}&\int_0^1 h'\left(F\int_0^t e^{\sigma B(\tau)}\,\mathrm{d}\tau\right)F\,e^{\sigma B(t)}\,\mathrm{d}B(t)\Big|_{F=\left(\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau\right)^{-1}}\\ &= \frac{h'(1)e^{\sigma B(1)}}{\sigma\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau} - \frac{h'(0)}{\sigma\int_0^1 e^{\sigma B(\tau)}\,\mathrm{d}\tau} - \frac{\sigma}{2}\int_0^1 h'(\phi(t))\phi'(t)\,\mathrm{d}t - \sigma^{-1}\int_0^1 h''(\phi(t))\,\phi'^2(t)\,\mathrm{d}t\\ &\qquad = \sigma^{-1}h'(1)\phi'(1) - \sigma^{-1}h'(0)\phi'(0) - \frac{\sigma}{2}h(1) - \sigma^{-1}\int_0^1 h''(\phi(t))\,\phi'^2(t)\,\mathrm{d}t. \end{aligned}\tag{5.6.19}$$

Therefore,

$$\delta(u) = \sigma^{-2}\big(h'(1)\phi'(1) - h'(0)\phi'(0)\big) + \frac{1}{2}h(1) - \int_0^1 h(s)\,\mathrm{d}s - \sigma^{-2}\int_0^1 h''(\phi(t))\,\phi'^2(t)\,\mathrm{d}t. \tag{5.6.20}$$

**Determinant.** To calculate the Hilbert-Carleman determinant $\det_2(1+K)$, with $K$ given by (5.6.11), we use [124, Chapter XIII, Corollary 1.2]. In the notations from [124, Chapter XIII] we have

$$k(t,s) = \begin{cases} F_1(t)G_1(s), & 0\le s<t\le 1;\\ -F_2(t)G_2(s), & 0\le t<s\le 1,\end{cases}\tag{5.6.21}$$

with

$$F_1(t) = h'(\phi(t))\,(1-\phi(t)), \qquad F_2(t) = h'(\phi(t))\phi(t), \qquad G_1(s) = G_2(s) = \phi'(s). \tag{5.6.22}$$

According to [124, Chapter XIII, Corollary 1.2],

$$\det_2(1+K) = \det\left(N_1 + N_2 U(1)\right)\cdot\exp\left\{\int_0^1 F_2(s)G_2(s)\,\mathrm{d}s\right\}, \tag{5.6.23}$$

where

$$N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.6.24}$$

and $U(t)$ is a $2\times 2$ matrix which satisfies the differential equation

$$U(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U'(t) = -\begin{pmatrix} G_1(t)F_1(t) & G_1(t)F_2(t) \\ G_2(t)F_1(t) & G_2(t)F_2(t) \end{pmatrix} U(t). \tag{5.6.25}$$

We start by solving the differential equation (5.6.25). Note that

$$\begin{pmatrix} G_1(t)F_1(t) & G_1(t)F_2(t) \\ G_2(t)F_1(t) & G_2(t)F_2(t) \end{pmatrix} = h'\big(\phi(t)\big)\phi'(t) \begin{pmatrix} 1-\phi(t) & \phi(t) \\ 1-\phi(t) & \phi(t) \end{pmatrix}. \tag{5.6.26}$$

Therefore, by rewriting (5.6.25), we see that $U(t) = V\big(\phi(t)\big)$ with

$$V'(s) = -h'(s)\begin{pmatrix} 1-s & s \\ 1-s & s \end{pmatrix} V(s) = -h'(s)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{2s-1}{2}\begin{pmatrix} -1 & 1 \end{pmatrix}\right) V(s). \tag{5.6.27}$$

Observe that if

$$v(t) = v_1(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + v_2(t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{5.6.28}$$

is a representation of $v(t) = V(t)v(0)$ in terms of the orthogonal basis, then

$$v_2(t) = v_2(0), \tag{5.6.29}$$

$$v_1(t) = v_1(0)e^{-h(t)} - \int_0^t e^{h(s)-h(t)} h'(s)(2s-1)v_2(s)\,\mathrm{d}s \tag{5.6.30}$$

$$= v_1(0)e^{-h(t)} - v_2(0)\left((2t-1) + e^{-h(t)} - 2e^{-h(t)}\int_0^t e^{h(s)}\,\mathrm{d}s\right). \tag{5.6.31}$$

In particular,

$$v_2(1) = v_2(0), \qquad v_1(1) = v_1(0)e^{-h(1)} + v_2(0)\left(2e^{-h(1)}\int_0^1 e^{-h(s)}\,\mathrm{d}s - e^{-h(1)} - 1\right). \tag{5.6.32}$$

Therefore,

$$V(1)\begin{pmatrix}1\\1\end{pmatrix} = e^{-h(1)}\begin{pmatrix}1\\1\end{pmatrix} \tag{5.6.33}$$

$$V(1)\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} + \left(2e^{-h(1)}\int_0^1 e^{-h(s)}\,\mathrm{d}s - e^{-h(1)} - 1\right)\begin{pmatrix}1\\1\end{pmatrix} \tag{5.6.34}$$

which is equivalent to

$$U(1) = V(1) = \begin{pmatrix} 1+e^{-h(1)} - e^{-h(1)}\int_0^1 e^{h(s)}\,\mathrm{d}s & -1+e^{-h(1)}\int_0^1 e^{h(s)}\,\mathrm{d}s \\ e^{-h(1)} - e^{-h(1)}\int_0^1 e^{h(s)}\,\mathrm{d}s & e^{-h(1)}\int_0^1 e^{h(s)}\,\mathrm{d}s \end{pmatrix}. \tag{5.6.35}$$

Thus,

$$\det\left(N_1 + N_2 U(1)\right) = e^{-h(1)}\int_0^1 e^{h(s)}\,\mathrm{d}s. \tag{5.6.36}$$

Also,

$$\int_0^1 F_2(s)G_2(s)\,\mathrm{d}s = \int_0^1 h'\left(\phi(s)\right)\phi(s)\phi'(s)ds = \int_0^1 h'(t)t\,\mathrm{d}t = h(1) - \int_0^1 h(t)\,\mathrm{d}t. \tag{5.6.37}$$

Combining (5.6.23), (5.6.36), and (5.6.37) we obtain

$$\det\nolimits_2(1+K) = \exp\left(-\int_0^1 h(t)\,\mathrm{d}t\right)\cdot\int_0^1 e^{h(s)}\,\mathrm{d}s. \tag{5.6.38}$$

**Final result.** Bringing together (5.6.38), (5.6.20), (5.6.7), and the formula for the Radon-Nikodym derivative (5.6.8) we get the desired result. □

Proposition 5.2.6 is a straightforward consequence of the previous lemma. For convenience, we restate it as the following corollary.

**Corollary 5.6.3:** Suppose $f \in \mathrm{Diff}^3[0,1]$. Denote $b = \log f'(1) - \log f'(0)$. Let $f^\sharp\mathcal{B}^{a,1}_{\sigma^2} = f^{-1}_\sharp\mathcal{B}^{a,1}_{\sigma^2}$ be the push-forward of $\mathcal{B}^{a,1}_{\sigma^2}$ under $\mathsf{P}^{-1}\circ L_{f^{-1}}\circ\mathsf{P} = \left(\mathsf{P}^{-1}\circ L_f\circ\mathsf{P}\right)^{-1}$. Then for any $a\in\mathbb{R}$, $f^\sharp\mathcal{B}^{a,1}_{\sigma^2}$ is absolutely continuous with respect to $\mathcal{B}^{a-b,1}_{\sigma^2}$ and

$$\frac{\mathrm{d}f^\sharp\mathcal{B}^{a,1}_{\sigma^2}(\xi)}{\mathrm{d}\mathcal{B}^{a-b,1}_{\sigma^2}(\xi)} = \frac{1}{\sqrt{f'(0)f'(1)}}\exp\left\{\frac{1}{\sigma^2}\left[\frac{f''(0)}{f'(0)}\mathsf{P}'_\xi(0) - \frac{f''(1)}{f'(1)}\mathsf{P}'_\xi(1)\right] + \frac{1}{\sigma^2}\int_0^1 \mathcal{S}_f\left(\mathsf{P}_\xi(t)\right)\left(\mathsf{P}'_\xi(t)\right)^2\mathrm{d}t\right\}. \tag{5.6.39}$$

*Proof.* The corollary follows immediately by combining Lemma 5.6.2, the decomposition (5.6.1), the fact that the right-hand side in Lemma 5.6.2 is continuous in $W$, and

$$\Big([\mathsf{P}^{-1}\circ L_g\circ \mathsf{P}]\xi\Big)(1)=\log g'(1)-\log g'(0)+\xi(1). \tag{5.6.40}$$

□

## 5.7 Appendix: Quotients of measures: Proof of Proposition 5.2.10

Suppose $(X,d)$ is a complete separable metric space (not necessarily locally compact). Suppose a locally compact Hausdorff group $G$ is acting properly and continuously on $X$ from the right. Then $X/G$ is a Polish space[3]. Let $\nu$ denote a left-invariant Haar measure on $G$ and $\Delta_G$ denote the modular function, such that $\nu(\cdot g)=\Delta_G(g)\nu(\cdot)$. Write $\pi\colon X\to X/G$ for the canonical projection. Write $C_b(X)$ for the space of continuous bounded functions equipped with the compact-open topology. Denote by $C_b^{G-\mathrm{inv}}(X)$ the subspace of $G$-invariant functions equipped with the subspace topology. Observe that $\pi^*\colon C_b(X/G)\to C_b^{G-\mathrm{inv}}(X), h\mapsto h\circ\pi$ is a bijection. We say that a set $A\subseteq X$ is *$G$-(pre)compact* if $G_A:=\{g\in G\colon A\cap Ag\neq\emptyset\}$ is (pre)compact in $G$. Furthermore, we say that a set $A$ is *$G$-tempered* if it is $G$-precompact and moreover has a $G$-precompact open neighbourhood $U\supseteq A$, such that $UG\supseteq \mathrm{Cl}(AG)$. Write $C_b^{G-\mathrm{temp}}(X)$ for the space of bounded continuous functions whose support is $G$-tempered (note that this space is not necessarily linear). For $f\in C_b^{G-\mathrm{temp}}(X)$ write

$$f^\flat(x)=\int_G f(xg)\,\mathrm{d}\nu(g). \tag{5.7.1}$$

Note that $f^\flat$ is $G$-invariant (i.e. $f^\flat(\cdot g)=f^\flat$) and satisfies $[f(\cdot g)]^\flat=\Delta_G(g)^{-1}f^\flat$, for $g\in G$. The main goal of this appendix is to prove the following

[3] Separability is clear. Using continuity of the projection $\pi\colon X\to X/G$ and paracompactness of $X$, one may check that $X/G$ is paracompact. Since $G$ acts properly, it also is Hausdorff. As a consequence, it is normal (Theorem of Jean Dieudonné). Moreover, since $\pi$ is an open map, $X/G$ is second countable and Urysohn's metrisation theorem implies that it is metrisable. Finally, complete metrisability follows from [125].

**Proposition 5.7.1:** Suppose $\mu$ is a Radon measure on $X$ such that $\mu(\cdot g) = \Delta_G(g)\,\mu$ for $g \in G$. Then there exists a unique Radon measure $\lambda$ on $X/G$ such that

$$\int_{X/G} f^\flat(x)\,\mathrm{d}\lambda(xG) = \int_X f(x)\,\mathrm{d}\mu(x), \tag{5.7.2}$$

for any $f \in C_b^{G-\mathrm{temp}}(X)$. This extends to all non-negative $f \in C(X)$.

The modular function makes an appearance in our condition for $G$-covariance of $\mu$, since we are working with a left-Haar measure $\nu$, but a right-action on $X$. Requiring that $\mu(\cdot g) = \Delta_G(g)\,\mu$, tells us that the measure transforms like a left-Haar measure under the right action. Also, note that in our application, the group $G = \mathrm{PSL}(2,\mathbb{R})$ is unimodular, hence $\Delta_G \equiv 1$, and the requirement simply reduces to $G$-invariance of $\mu$.

We follow [126, Section 2], with appropriate modifications to allow for $X$ not locally compact.

**Lemma 5.7.2:** For $f \in C_b^{G-\mathrm{temp}}(X)$, the function $f^\flat$ is well-defined as an element of $C_b^{G-\mathrm{inv}}(X)$.

*Proof.* Obviously, we have $|f(xg)| \leq \|f\|_\infty \mathbf{1}_{\mathrm{spt}(g\mapsto f(xg))}$. Moreover, note that $\mathrm{spt}(g \mapsto f(xg))$ is contained in a $G$-translate of $G_{\mathrm{spt} f}$, hence $f^\flat(x) \leq \|f\|_\infty \nu(\mathrm{spt}(g \mapsto f(xg))) \leq \|f\|_\infty \nu(G_{\mathrm{spt} f})$. Consequently, $f^\flat$ is well-defined and uniformly bounded. $G$-invariance is clear from invariance of the Haar-measure $\nu$. Regarding continuity, suppose $x_n \to x$ in $X$. Write $\mathcal{X} := \{x\} \cup \{x_n\}_{n\in\mathbb{N}}$. Let $U \supseteq \mathrm{spt}(f)$ denote a $G$-precompact open neighbourhood of $\mathrm{spt}(f)$ such that $UG \supseteq \mathrm{Cl}(\mathrm{spt}(f)G)$. We may assume $x \subseteq UG$. In fact, otherwise $f^\flat(x_n) = 0 = f^\flat(x)$ for sufficiently large $n$ and continuity is clear. Moreover, choosing $g_0 \in G$ such that $x \in Ug_0$, we can assume without loss of generality that $\mathcal{X} \subseteq Ug_0$. Now consider

$$|f^\flat(x) - f^\flat(x_n)| \leq \int_G |f(xg) - f(x_n g)|\,\mathrm{d}\nu(g). \tag{5.7.3}$$

Since $f$ is bounded, it suffices to show that we can restrict the integral to a compact subset of $G$ (independently from $n$). Note for $g \in G$ we have $|f(xg) - f(x_n g)| \neq 0$ only if $U \cap \mathcal{X}g \neq \emptyset$. Since $\mathcal{X}g \subseteq Ug_0 g$, the set of such $g$ is precompact. Consequently, the right hand side of (5.7.3) goes to zero as $n \to \infty$ by dominated convergence, proving continuity of $f^\flat$. □

**Lemma 5.7.3:** For any $x \in X$, there exists an open neighbourhood $U \ni x$ which is $G$-tempered.

*Proof.* First we show that there exist arbitrarily small $G$-compact neighbourhoods: Consider a decreasing sequence of neighbourhoods $\{U_n\}_{n\in\mathbb{N}}$, such that $\bigcap_n U_n = \{x\}$. Assume that none of the $U_n$ is $G$-precompact, hence there exists a sequence $x_n \in U_n$ and $g_n \in G$, such that $x_n g_n \in U_n$, but with $\{g_n\}$ non-precompact. However, by definition of the $U_n$, we have $x_n \to x$ and $x_n g_n \to x$, and since $G$ acts properly on $X$ this implies that $g_n$ is precompact, leading to a contradiction. Now suppose $\tilde{U}$ is a $G$-compact neighbourhood of $x$. Then $\tilde{U}G$ is a neighbourhood of $xG$ in $X/G$. Since $X/G$ is metrisable, we may choose a sufficiently small ball $\mathcal{B}$ around $[x]$ in $X/G$, such that $\mathrm{Cl}(\mathcal{B}) \subseteq \tilde{U}G$. Set $U := \tilde{U} \cap \pi^{-1}(\mathcal{B})$. This is a $G$-tempered neighbourhood of $x$. □

**Lemma 5.7.4:** For any $x_0 \in X$ and $\epsilon > 0$ sufficiently small, there exists $0 < \epsilon' < \epsilon$ and a non-negative function $v \in C_b^{G-\text{temp}}(X)$ with $\mathrm{spt}(v) \subseteq B_\epsilon(x_0)$, such that

$$0 < \inf_{x' \in B_{\epsilon'}(x_0)} v^\flat(x') < \sup_{x' \in B_{\epsilon'}(x_0)} v^\flat(x') < \infty. \tag{5.7.4}$$

*Proof.* It suffices to show the following: for any $x_0 \in X$ and $\epsilon > 0$ sufficiently small, there exists $0 < \epsilon' < \epsilon$, such that

$$0 < \inf_{x' \in B_{\epsilon'}(x_0)} (\mathbf{1}^\flat_{B_\epsilon(x_0)})(x') < \sup_{x' \in B_{\epsilon'}(x_0)} (\mathbf{1}^\flat_{B_\epsilon(x_0)})(x') < \infty, \tag{5.7.5}$$

where we note that $(\mathbf{1}^\flat_{B_\epsilon(x_0)})(x') = \nu(\{g \in G : x'g \cap B_\epsilon(x_0) \neq \emptyset\})$. Indeed if above holds then we can choose $v = \max(0, 1 - \frac{4}{3\epsilon}\mathrm{dist}(x, B_{\epsilon/2}(x_0)))$. Since $\frac{1}{3}\mathbf{1}_{B_{\epsilon/2}} \leq v \leq \mathbf{1}_{B_\epsilon}$, for small enough $\epsilon > 0$ the claim in (5.7.4) follows.

Now we address the claim (5.7.5). By continuity of the group action $X \times G \to X, (x,g) \mapsto xg$, the preimage of $B_\epsilon(x_0)$ under this map is an open neighbourhood of $(x_0, \mathrm{id})$. In particular, it contains a neighbourhood of the form $B_{\epsilon'}(x_0) \times U$, with $U \subseteq G$ a neighbourhood of id. In particular, for $x' \in B_{\epsilon'}(x_0)$ we have $U \subseteq \{g \in G : x'g \cap B_\epsilon(x_0) \neq \emptyset\}$ and hence $(\mathbf{1}^\flat_{B_\epsilon(x_0)})(x') \geq \nu(U) > 0$. This proves the lower bound in (5.7.5). For the upper bound note that we may assume $\epsilon > 0$ sufficiently small, such that $B_\epsilon(x_0)$ is $G$-tempered. As a consequence $\{g \in G : x'g \cap B_\epsilon(x_0) \neq \emptyset\} \subseteq G_{B_\epsilon(x_0)}$ for $x' \in B_\epsilon(x_0)$, so $(\mathbf{1}^\flat_{B_\epsilon(x_0)})(x') \leq \nu(G_{B_\epsilon(x_0)}) < \infty$, which proves the upper bound in (5.7.5). □

**Lemma 5.7.5:** Consider above setting and consider a Radon measure $\mu$ on $X$. There exists a countable family of non-negative functions $u_i \in C_b^{G-\text{temp}}(X)$, $i \in \mathbb{N}$, with bounded support, such that $\mu(u_i) < \infty$ and such that $\{u_i^\flat\}$ is a partition of unity.

Moreover, writing $X_i := \mathrm{spt}(u_i) \cdot G$, the map $f \mapsto f^\flat$ is surjective from $C^{G-\text{temp}}(X_i)$ onto $C^{G-\text{inv}}(X_i)$.

*Proof.* Recall that by assumption $X/G$ is Polish, hence Hausdorff and paracompact. This implies that any open cover of $X/G$ admits a subordinate partition of unity. We choose

$\{x_i\}_{i\in\mathbb{N}} \subseteq X$ and $\epsilon_i > 0$, with $\{B_{\epsilon_i}(x_i)\}_{i\in\mathbb{N}}$ covering the whole space, such that $B_{\epsilon_i}(x_i)$ is $G$-tempered and $\mu(B_{\epsilon_i}(x_i)) < \infty$ (which can be done by Lemma 5.7.3 and since $\mu$ is Radon). Furthermore, choose $\epsilon_i$ sufficiently small, such that the conclusion of Lemma 5.7.4 holds true: There are $0 < \epsilon_i' < \epsilon_i$ and non-negative functions $v_i \in C_b^{G-\mathrm{temp}}(X)$ supported in $B_{\epsilon_i}(x_i)$, such that $v_i^\flat$ is uniformly bounded away from zero and infinity on $B_{\epsilon_i'}(x_i)$.
The saturated balls $\{B_{\epsilon_i'}(x_i)G\}$ form an open cover of the quotient space $X/G$, hence admit a subordinate partition of unity $\{\bar{u}_i\}_{i\in\mathbb{N}}$, with $\bar{u}_i \in C_b(X/G)$ and $\mathrm{spt}(\bar{u}_i) \subseteq B_{\epsilon_i'}(x_i)G$. The lifts $\pi^*\bar{u}_i \in C_b^{G-\mathrm{inv}}(X)$ along $\pi\colon X \to X/G$ form a partition of unity on $X$. Define the functions

$$u_i = \begin{cases} \frac{v_i}{v_i^\flat}\pi^*\bar{u}_i \text{ on } B_{\epsilon_i'}(x_i)G \\ 0 \text{ otherwise.} \end{cases} \tag{5.7.6}$$

Notice that $u_i \geq 0$ is a well-defined bounded continuous function, is supported in $B_{\epsilon_i}(x_i)$, and satisfies $u_i^\flat = \pi^*\bar{u}_i$. This concludes the construction of the $u_i$. Moreover, by the same construction as in (5.7.6), we see the surjectivity of $f \mapsto f^\flat$ from $C^{G-\mathrm{temp}}(X_i)$ onto $C^{G-\mathrm{inv}}(X_i)$. □

*Proof of Proposition 5.7.1.* In the following we write $\mu(f) = \int_X f\,\mathrm{d}\mu$. We first note that the condition $\mu(\cdot g) = \Delta_G(g)\,\mu$ implies that $\mu(f_1 f_2^\flat) = \mu(f_1^\flat f_2)$ for any $f_1, f_2 \in C_b^{G-\mathrm{temp}}(X)$. Indeed,

$$\begin{aligned} \mu(f_1 f_2^\flat) &= \int_X \mathrm{d}\mu(x) f_1(x) \int_G \mathrm{d}\nu(g) f_2(xg) \\ &= \int_G \mathrm{d}\nu(g) \int_X \mathrm{d}\mu(xg)\Delta_G(g)^{-1} f_1(x) f_2(xg) \\ &= \int_G \mathrm{d}\nu(g)\Delta_G(g)^{-1} \int_X \mathrm{d}\mu(x) f_1(xg^{-1}) f_2(x) \\ &= \int_X \mathrm{d}\mu(x)\Big[\int_G \mathrm{d}\nu(g)\Delta_G(g)^{-1} f_1(xg^{-1})\Big] f_2(x) \\ &= \int_X \mathrm{d}\mu(x)\Big[\int_G \mathrm{d}\nu(g^{-1}) f_1(xg^{-1})\Big] f_2(x). \\ &= \mu(f_1^\flat f_2), \end{aligned} \tag{5.7.7}$$

where in second to last line we used that $\mathrm{d}\nu(g)\Delta_G(g)^{-1} = \mathrm{d}\nu(g^{-1})$.
Consider $\{u_i\}$ as in Lemma 5.7.5 and write $X_i \coloneqq \mathrm{spt}(u_i)\cdot G$ for the saturation of the support. For any $i \in \mathbb{N}$, define a linear functional $I_i\colon C_b^{G-\mathrm{inv}}(X_i) \to \mathbb{R}$ by

$$I_i(h) = \mu(u_i h) \text{ for for } h \in C_b^{G-\mathrm{inv}}(X_i). \tag{5.7.8}$$

By (5.7.7) we have $I_i(f^\flat) = \mu(u_i^\flat f)$ for any $f \in C_b^{G-\mathrm{temp}}(X_i)$. Note that this characterises $I_i$ due to the surjectivity of $f \mapsto f^\flat$ from $C_b^{G-\mathrm{temp}}(X_i)$ onto $C_b^{G-\mathrm{inv}}(X_i)$. Each $I_i$ is monotone, since $\mu u_i$ is a positive measure, so $I_i$ is continuous with respect to the compact-open topology on $C_b^{G-\mathrm{inv}}(X_i)$. Considering the push-forward along $\pi\colon X_i \to X_i/G$, we get $\tilde{I}_i \coloneqq \pi_* I_i \in C_b(X_i/G)^*$. Continuity of $\tilde{I}_i$ follows from continuity of $\pi$ and definition of the compact-open topologies. Since $\mu u_i$ is a finite Radon measure, for any $\epsilon > 0$ there exists a compact set $K_\epsilon \subseteq X_i$ such that $\mu(u_i h) \leq \epsilon \|h\|_\infty$ for $h \in C_b^{G-\mathrm{inv}}(X_i)$ with $h|_{K_\epsilon G} \equiv 0$. Consequently, application of a variant of the Riesz–Markov–Kakutani representation theorem [127, Theorem 7.10.6] to $\tilde{I}_i$ and the fact that $C_b(X_i/G) \cong C_b^{G-\mathrm{inv}}(X_i)$ imply that there exists a unique finite Radon measure $\lambda_i$ on $X_i/G$, such that

$$\int_{X_i/G} f^\flat(x)\, \mathrm{d}\lambda_i(xG) = \int_{X_i} f(x) u_i^\flat(x)\, \mathrm{d}\mu(x) \text{ for } f \in C_b^{G-\mathrm{temp}}(X_i). \tag{5.7.9}$$

In the following, also write $\lambda_i$ for the push-forward along the inclusion $X_i \hookrightarrow X$. Since $\sum_i u_i^\flat = 1$, we define $\lambda = \sum_i \lambda_i$ and note that it satisfies (5.7.2) (using monotone convergence and positivity of $f \mapsto f^\flat$). This defines a locally finite Borel measure and since $X/G$ is strongly Radon (as a completely separable metric space), $\lambda$ is a Radon measure. Regarding uniqueness, note that for any $\lambda$ satisfying (5.7.2), we have $\lambda \cdot u_i^\flat = \lambda_i$. In particular, any other candidate for $\lambda$ constructed using a different partition of unity will therefore agree, which implies that the so constructed measure is unique.

The extension from $f \in C_b^{G-\mathrm{temp}}(X)$ to non-negative $f \in C(X)$ follows from the existence of a partition of unity on $X$, consisting of functions with $G$-tempered supports, and monotone convergence. □

# Chapter 6

# More on the Schwarzian field theory

This chapter provides additional context for the Schwarzian field theory and the mathematical structures in relation to it. In Section 6.1 we follow discussions from the physics literature, and argue how the Schwarzian action relates to other well-known models, namely Liouville field theory, the SYK model, and JT-gravity. In Section 6.2 we revisit the Schwarzian derivative and motivate it using cross-ratios and the real projective line. Then we discuss the theory of coadjoint orbits and their classification in the context of loop groups and the Virasoro group. In the latter case we can see how the family of Schwarzian measures appears naturally in this geometric language.

## 6.1 Origins of the Schwarzian theory

In the following subsections we discuss how the Schwarzian action arises from other well-known models in theoretical physics. Our discussion will follow a "physics style" discussion: We explicitly do not follow rigorous arguments, but aim to provide a condensed presentation of physics folklore.

### 6.1.1 Liouville field theory

From a high-level perspective, both Liouville field theory as well as the Schwarzian theory can be constructed as a "geometric action" related to the Virasoro group [16, 128]. More concretely, one can argue that the Schwarzian action arises in a simultaneous semiclassical and "thin cylinder" limit of Liouville field theory [14].

Liouville theory has been extensively studied in the recent probabilistic literature. In particular, correlation functions under the Euclidean path integral measure $\mathcal{D}[\phi]e^{-S_{\mathrm{E}}[\phi]}$ have been constructed, where

$$\frac{1}{\hat{c}}\mathcal{L}_{\mathrm{E}}[\phi] = \tfrac{1}{2}(\partial_\tau\phi)^2 + \tfrac{1}{2}(\partial_\sigma\phi)^2 + 2e^{2\phi} - \partial_\sigma^2\phi. \tag{6.1.1}$$

To discuss the connection with the Schwarzian, we consider the action in Lorentzian signature:

$$\frac{1}{\hat{c}}\mathcal{L}_{\mathrm{L}}[\phi] := -\tfrac{1}{2}(\partial_\tau\phi)^2 + \tfrac{1}{2}(\partial_\sigma\phi)^2 + 2e^{2\phi} - \partial_\sigma^2\phi. \tag{6.1.2}$$

The theories are connection by a formal Wick rotation $\tau \mapsto -i\tau$. For example, one formally expects that the partition functions agree

$$\int \mathcal{D}[\phi]e^{-S_{\mathrm{E}}[\phi]} = \int \mathcal{D}[\phi]e^{iS_{\mathrm{L}}[\phi]}. \tag{6.1.3}$$

Similar statements hold true for correlation functions, as elaborated by the Osterwalder–Schrader theory. In the limit $\hat{c} \to \infty$, the Lorentzian theory localises on critical points of the action. In the next paragraph we study the classical solutions in this situation.
We consider the Lorentzian Liouville theory on a strip $(\tau,\sigma) \in \mathbb{R}\times[0,\frac{1}{2}]$. In the limit of large central charge $\hat{c} \to \infty$, the theory should be dominated by critical points (i.e. classical solutions) of (6.1.2). Thermal correlators of the Lorentzian theory can be expressed in terms of an Euclidean path integral on a cylinder $(\tau,\sigma) \in [0,\beta]\times[0,\frac{1}{2}]$ with periodic boundary conditions in the time variable.

$$\mathrm{tr}(e^{-\beta\hat{H}}) = \int \mathcal{D}[\phi]e^{-\int_0^\beta \mathrm{d}\tau\int_0^{1/2}\mathrm{d}\sigma\mathcal{L}_E[\phi]}. \tag{6.1.4}$$

**Classical solutions of Lorentzian Liouville theory on the strip.** We are interested in the classical solutions (i.e. the Euler-Lagrange equations) corresponding to the Lagrangian $\mathcal{L}_{\mathrm{L}}$ in (6.1.2) on a strip $(\tau,\sigma) \in \mathbb{R}\times[0,\frac{1}{2}]$, equipped with boundary conditions at $\sigma = 0, \frac{1}{2}$:

$$-\partial_\tau^2\phi + \partial_\sigma^2\phi + 4e^{2\phi} = 0. \tag{6.1.5}$$

The boundary conditions of interest are Dirichlet–von Neumann in terms of the variable $V(\tau,\sigma) := e^{-\phi(\tau,\sigma)}$:

$$V|_{\sigma=0} = 0 \quad \text{and} \quad \partial_\sigma|_{\sigma=1/2}V = 2\cos(\alpha) \quad \text{with} \quad \alpha \in \mathbb{R}_{\geq 0}\cup i\mathbb{R}_{\geq 0} \tag{6.1.6}$$

These are referred to as *ZZ–FZZT*$_\alpha$ boundary conditions. Note that the boundary behaviour of $\phi$ is singular in that $\phi(\tau,\sigma) \to \infty$ as $\sigma \searrow 0$. In order to eliminate ambiguities that arise due to this singularity, one can make the additional assumptions

$$\partial_\sigma|_{\sigma=0} V = 2 \quad \text{and} \quad \lim_{\tau \searrow 0} \partial_\sigma \partial_\tau V/V = 0. \tag{6.1.7}$$

Solutions to (6.1.5), subject to boundary conditions (6.1.6) have been extensively studied in the physics literature and are connected to the coadjoint orbits of the Virasoro group (cf. [129]). Following the treatment in [130], it is convenient to rephrase the Liouville equations (6.1.5) in terms of $V = e^{-\phi}$ and to use light-cone coordinates $x = \tau + \sigma$, $\bar{x} = \tau - \sigma$. Then the Liouville equation is equivalent to

$$V \partial_x \partial_{\bar{x}} V - \partial_x V \partial_{\bar{x}} V = 1 \quad \text{and} \quad V > 0. \tag{6.1.8}$$

The light-cone coordinates highlight the conformal symmetry of Liouville's equation: For any smooth function $\xi\colon \mathbb{R} \to \mathbb{R}$ with $\xi(\tau+1) = \xi(\tau)+1$, the reparametrisation

$$V_\xi(x,\bar{x}) := \xi'(x)^{-1/2} \xi'(\bar{x})^{-1/2} V(\xi(x), \xi(\bar{x})) \tag{6.1.9}$$

is also a solution. In other words, every solution comes with an associated Diff$(S^1)$-orbit of solutions. It turns out that this is the only "degree of freedom":

**Proposition 6.1.1** ([130]): Consider $0 \le \alpha \le \pi$ or $\alpha \in i\mathbb{R}_{\ge 0}$. Subject to ZZ-FZZT$_\alpha$ boundary conditions (6.1.6) and the regularity constraint (6.1.7), the solutions to Liouville's equation (6.1.8) are given by the Diff$(S^1)$-orbit $\xi \mapsto V_\xi^{(\alpha)}$, where

$$V^{(\alpha)}(x,\bar{x}) := \frac{1}{\alpha} \sin(\alpha[x - \bar{x}]) = \frac{1}{\alpha} \sin(2\alpha\sigma). \tag{6.1.10}$$

For $\alpha > \pi$, (6.1.10) and its associated orbits still defines a solution to the *global* Liouville equation (6.1.8) with the positivity constraint for $V$ dropped. However, these solutions don't correspond to regular solutions of (6.1.5).

**The Hamiltonian for classical solutions.** By a standard Legendre transformation, the Hamiltonian corresponding to (6.1.2) is given by

$$H\big[\{\phi(\sigma)\}_\sigma, \{\pi_\phi(\sigma)\}_\sigma\big] = \int_0^{\frac{1}{2}} \mathrm{d}\sigma\, [\pi_\phi^2 + \tfrac{1}{2}(\partial_\sigma \phi)^2 + 2e^{2\phi} - 2\partial_\sigma^2 \phi], \tag{6.1.11}$$

where $\pi_\phi$ is a momentum variables conjugate to $\phi$, hence the field space is equipped with the canonical symplectic form $\omega = \int \mathrm{d}\sigma\, \mathrm{d}\phi(\sigma) \wedge \mathrm{d}\pi_\phi(\sigma)$. We can evaluate the Hamiltonian of the solutions given in Proposition 6.1.1:

**Proposition 6.1.2:** Let $V_\xi^{(\alpha)}$ be as in Proposition 6.1.1 and write $\phi_\xi^{(\alpha)}(\tau,\sigma) = -\log(V_\xi^{(\alpha)})$. Then we have

$$\begin{aligned} \mathcal{H}[\phi_\xi^{(\alpha)}] &\coloneqq H[\{\phi_\xi^{(\alpha)}(\tau,\sigma)\}_\sigma, \{\partial_\tau \phi_\xi^{(\alpha)}(\tau,\sigma)\}_\sigma] \\ &= -\tfrac{1}{2}\int_0^1 \mathrm{d}\sigma [\mathcal{S}(\xi(\sigma),\sigma) + 2\alpha^2 \xi'^2(\sigma)]. \end{aligned} \tag{6.1.12}$$

In other words, we see that the Hamiltonian of classical solutions with ZZ–FZZT$_\alpha$ boundary conditions is the action of the Schwarzian measure at parameter $\alpha$. Also, note that the Hamiltonian (6.1.12) is constant in time $\tau$, as it should for solutions to the Hamiltonian equations.

*Proof.* For $V = e^\phi$ one can use $\partial_\sigma = \partial_x - \partial_{\bar{x}}$ and $\partial_\tau = \partial_x + \partial_{\bar{x}}$ to show that generally

$$\frac{\partial_x^2 V}{V} + \frac{\partial_{\bar{x}}^2 V}{V} = \tfrac{1}{2}(\partial_\tau \phi)^2 + \tfrac{1}{2}(\partial_\sigma \phi)^2 + 2e^{2\phi} - \partial_\sigma^2 \phi. \tag{6.1.13}$$

Now, consider a solution $V_\xi^{(\alpha)}$ as in Proposition 6.1.1. We have

$$\partial_x V_\xi = -\tfrac{1}{2}\log(\xi')'(x) V_\xi + \xi'(x)(\partial_x V)_\xi. \tag{6.1.14}$$

And hence

$$\partial_x^2 V_\xi = -\tfrac{1}{2}[\log(\xi')''(x) - \tfrac{1}{2}\log(\xi')'^2(x)]V_\xi + \xi'^2(x)(\partial_x^2 V)_\xi. \tag{6.1.15}$$

Note that the term in brackets is the Schwarzian of $\xi$. Also, according to (6.1.10) we have $\partial_x^2 V = \partial_{\bar{x}}^2 V = -\alpha^2 V$. Consequently

$$\frac{\partial_x^2 V_\xi(x,\bar{x})}{V_\xi} = -\tfrac{1}{2}[\mathcal{S}(\xi(x),x) + 2\alpha^2\xi'^2(x)], \tag{6.1.16}$$

and analogously for $(\partial_{\bar{x}}^2 V_\xi)/V_\xi$, with $x$ replaced by $\bar{x}$ on the right hand side. Note that (6.1.16) is *chiral*, in that it only depends on $x = \tau + \sigma$ and the same goes for $(\partial_{\bar{x}}^2 V_\xi)/V_\xi$, which only depends on $\bar{x} = \tau - \sigma$. Since $\xi(x+1) = \xi(x) + 1$ is periodic with period one, we can use the "doubling trick"

$$\begin{aligned} H &= \int_0^{1/2} \mathrm{d}\sigma \frac{\partial_x^2 V_\xi}{V_\xi}(\tau+\sigma) + \frac{\partial_{\bar{x}}^2 V_\xi}{V_\xi}(\tau-\sigma) \\ &= \int_0^1 \mathrm{d}\sigma \frac{\partial_x^2 V_\xi}{V_\xi}(\sigma) \\ &= -\tfrac{1}{2}\int_0^1 \mathrm{d}\sigma [\mathcal{S}(\xi(\sigma),\sigma) + 2\alpha^2 \xi'^2(\sigma)]. \end{aligned} \tag{6.1.17}$$

□

**Schwarzian theory in the thin cylinder limit.**

$$\hat{c}\int \mathrm{d}\tau \left\{ \int_0^{\frac{1}{2}} \mathrm{d}\sigma\, \pi_\phi \partial_\tau \phi + \int_0^{\frac{1}{2}} \mathrm{d}\sigma\, [\pi_\phi^2 + \tfrac{1}{2}(\partial_\sigma \phi)^2 + 2e^{2\phi} - \partial_\sigma^2 \phi] \right\}. \tag{6.1.18}$$

At the formal path integral-level this is seen by a Hubbard-Stratonovich transform

$$\exp[-\hat{c}\int (\partial_\tau \phi)^2/2] \propto \int \mathcal{D}[\pi_\phi] \exp\left[\hat{c}\int [i\pi_\phi \partial_\tau \phi - \pi_\phi^2/2]\right]. \tag{6.1.19}$$

In the following we make use of the above (rigorous) study of the classical Liouville equation to given a (very much heuristic) "physics-style" explanation for the connection between Liouville field theory and Schwarzian actions.

We have seen in Proposition 6.1.2 that the Schwarzian action (resp. more generally the family of actions associated to coadjoint Virasoro orbits) appears as the Hamiltonian of classical solutions (6.1.12) given appropriate boundary conditions. As such, the Schwarzian appears in the Hamiltonian formulation of the action (6.1.18) in the semiclassical limit $\hat{c} \to \infty$, where the quantum theory localises on classical solutions.

Suppose there is appropriate Hilbert-space formulation for Liouville field theory at central charge $\hat{c}$ and with ZZ-FZZT$_\alpha$ boundary conditions. That is, suppose there is Hamiltonian $\hat{H}_{\hat{c},\alpha}$ together with an appropriate Hilbert space and a representation of field operators $\hat{\phi}(\sigma), \hat{\pi}_\phi(\sigma)$. For some (time-ordered) observable $O[\hat{\phi}]$, typically a product of vertex operators $e^{-l\hat{\phi}(\tau,\sigma)}$, one can formally rewrite unnormalised thermal expectations as

$$\mathrm{tr}(O[\hat{\phi}]e^{-\beta \hat{H}_{\hat{c},\alpha}}) = \mathcal{N}\int_{\text{ZZ–FZZT}_\alpha} \mathcal{D}[\phi]\mathcal{D}[\pi_\phi] O[\phi] \exp\left( -\hat{c}\int_0^\beta \mathrm{d}\tau \left\{ \int_0^{\frac{1}{2}} \mathrm{d}\sigma\, i\pi_\phi \partial_\tau \phi + \mathcal{H}[\phi,\pi_\phi] \right\} \right), \tag{6.1.20}$$

where the integral runs over field configurations satisfying ZZ–FZZT$_\alpha$ boundary conditions, and where $\mathcal{N}$ denotes an unspecified (potentially divergent) normalisation factor. For $\hat{c} \to \infty$ we expect the path integral to be dominated by field configurations $\phi, \pi_\phi$ solving the classical equations of motion, subject to ZZ–FZZT$_\alpha$ boundary conditions. On the strip $(\tau, \sigma) \in \mathbb{R} \times [0, 1/2]$, these are given by

$$\begin{aligned} \phi_\xi^{\text{strip}}(\tau,\sigma) &= \tfrac{1}{2}\log(\xi')(\tau+\sigma) + \tfrac{1}{2}\log(\xi')(\tau-\sigma) - \log[\tfrac{1}{\alpha}\sin(\alpha[\xi(\tau+\sigma)-\xi(\tau-\sigma)])] \\ \pi_{\phi,\xi}^{\text{strip}}(\tau,\sigma) &= \tfrac{1}{2}\log(\xi')'(\tau+\sigma) + \tfrac{1}{2}\log(\xi')'(\tau-\sigma) - \alpha\frac{\xi'(\tau+\sigma)-\xi'(\tau-\sigma)}{\tan(\alpha[\xi(\tau+\sigma)-\xi(\tau-\sigma)])}, \end{aligned} \tag{6.1.21}$$

for $\xi \in \widetilde{\mathrm{Diff}}(S^1)$. Here, we are working on the geometry of a cylinder with circumference $\beta$. The claim(!) is now as follows: Taking $\beta = \frac{2}{\hat{c}\sigma^2} \to 0$ simultaneously, the dominant configurations become time-independent and are of the form

$$\begin{aligned} \phi_\xi(\sigma) = \phi_\xi(\tau,\sigma) &= \tfrac{1}{2}\log(\xi')(\sigma) + \tfrac{1}{2}\log(\xi')(-\sigma) - \log[\tfrac{1}{\alpha}\sin(\alpha[\xi(\sigma)-\xi(-\sigma)])] \\ \pi_{\phi,\xi}(\sigma) = \pi_{\phi,\xi}(\tau,\sigma) &= \tfrac{1}{2}\log(\xi')'(\sigma) + \tfrac{1}{2}\log(\xi')'(-\sigma) - \alpha\frac{\xi'(\sigma)-\xi'(-\sigma)}{\tan(\alpha[\xi(\sigma)-\xi(-\sigma)])}. \end{aligned} \tag{6.1.22}$$

For these the kinetic terms vanishes and the Hamiltonian is evaluated by Proposition 6.1.2, which leaves us with

$$\lim_{\hat{c}\to\infty} \frac{\mathrm{tr}(O[\hat{\phi}]e^{-\frac{2}{\hat{c}\sigma^2}\hat{H}_{\hat{c},\alpha}})}{\mathrm{tr}(e^{-\frac{2}{\hat{c}\sigma^2}\hat{H}_{\hat{c},\alpha}})} \quad \text{“}=\text{”} \quad \frac{1}{Z_{\alpha,\sigma^2}}\int_{\Xi_\alpha} \mathcal{D}[\xi] O[\phi_\xi] e^{\frac{1}{\sigma^2}\int_0^1 \mathrm{d}\sigma [S(\xi(\sigma),\sigma)+2\alpha^2\xi'^2(\sigma)]}, \tag{6.1.23}$$

for some partition function $Z_{\alpha,\sigma^2}$. More concretely, (6.1.23) relates operator insertions in Liouville theory to such in Schwarzian field theory: From (6.1.22) one sees that

$$e^{l\phi(\tau,\sigma)} \mapsto e^{l\phi_\xi(\sigma)} = O_\alpha(-\sigma,\sigma)^l = \left[\frac{\alpha\sqrt{\xi'(\sigma)\xi'(-\sigma)}}{\sin(\alpha[\xi(\sigma)-\xi(-\sigma)])}\right]^l. \tag{6.1.24}$$

This also holds true for $\alpha = 0$, where it is understood that $\frac{1}{\alpha}\sin(\alpha[\xi(\sigma)-\xi(-\sigma)])|_{\alpha=0} = \xi(\sigma)-\xi(-\sigma)$. In other words, vertex operator insertions $e^{l\phi(\tau,\sigma)}$ in Liouville theory become cross-ratio observables on the Schwarzian side.

In (6.1.23) we suppressed details regarding the integration measure of $\xi$. In order to really identify the right-hand side as the Schwarzian field theory, we would like to show that the integration measure for $\xi$ agrees with the one in (2.2.1). In geometric parlance, we would like it to be the symplectic volume measure on the appropriate Virasoro coadjoint orbit. In fact, recall that in the Hamiltonian path integral (6.1.20), the measure $\mathcal{D}[\phi]\mathcal{D}[\pi_\phi]$ is interpreted as the volume form associated to the canonical symplectic form $\omega = \int_0^{1/2} \mathrm{d}\sigma\, \mathrm{d}\phi(\sigma) \wedge \mathrm{d}\pi_\phi(\sigma)$. Inserting (6.1.22), using the "doubling trick" and dismissing total derivatives, one obtains

$$\omega = \tfrac{1}{4}\int_0^1 [\mathrm{d}\log(\xi'(\sigma)) \wedge \mathrm{d}\log(\xi'(\sigma))' - 2\alpha^2 \mathrm{d}\xi(\sigma) \wedge \mathrm{d}\xi'(\sigma)], \tag{6.1.25}$$

which agrees with the KKS-form on the Virasoro coadjoint orbits (6.2.38). This should reassure us that the right-hand side of (6.1.23) really is the Schwarzian field theory. Since the latter is invariant under rotations/translations along the circle, we can translate $\sigma \mapsto \sigma + 1/2$ to ease the comparison with out definition in (2.2.1). Consequently, we end up with

**Claim 6.1.3:** Consider the Hamiltonian operator $\hat{H}_{\hat{c},\alpha}$ for Lorentzian Liouville field theory on a strip $(\tau, \sigma) \in \mathbb{R} \times [0, \frac{1}{2}]$ with ZZ-FZZT$_\alpha$ boundary conditions. For a finite family $l_i \in \mathbb{R}$ and $\sigma_i \in [0, \frac{1}{2}]$, $i \in \mathcal{I}$, and $\sigma^2 > 0$, we expect

$$\lim_{\hat{c}\to\infty} \mathrm{tr}\left(\prod_i e^{l_i \hat{\phi}(\sigma_i)}\, e^{-\frac{2}{\hat{c}\sigma^2}\hat{H}_{\hat{c},\alpha}}\right) \propto \left[\prod_i \mathcal{O}_\alpha^{l_i}(\tfrac{1}{2} - \sigma_i, \tfrac{1}{2} + \sigma_i)\right]_{\alpha,\sigma^2} \tag{6.1.26}$$

We reiterate that all of the above is conjectural. In order to make this rigorous, one would require a Hilbert space formulation of Lorentzian Liouville field theory on a strip. Alternatively, one can reinterpret the left hand side in terms of the thermal (Euclidean) path integral, in which case it reduces to a questions about Liouville field theory on the cylinder/annulus. The latter has been studied in the recent mathematical literature [131, 132], which may serve as a starting point for a rigorous study.

**Boundary operator.** Instead of imposing the ZZ-FZZT$_\alpha$ boundary conditions as part of the integral measure we can impose them for critical points of the action by adding a "boundary term" to the action. We keep the Dirichlet (ZZ) boundary condition at $\sigma = 0$, i.e. $V|_{\sigma=0} = e^{-\phi}|_{\sigma=0} = 0$, but drop the von Neumann (FZZT$_\alpha$) boundary condition $\partial_\sigma V|_{\sigma=1/2} = 2\cos(\alpha)$ (or equivalently $\partial_\sigma \phi = -2\cos(\alpha) e^{\phi}$ at $\sigma = 1/2$), while adding a term

$$S^{(\alpha)}_{\mathrm{bdry}}[\phi] = \hat{c} \int \mathrm{d}\tau\, [-2\cos(\alpha) e^{\phi(\tau, 1/2)}] \tag{6.1.27}$$

to the action. we can impose the boundary condition as part of the variational equations. We refer to $e^{S^{(0)}_{\mathrm{bdry}}[\phi] - S^{(\alpha)}_{\mathrm{bdry}}[\phi]}$ as a "boundary operator insertion" for Liouville field theory: In the semiclassical limit, this insertion "shifts" the boundary condition at $\sigma = 1/2$ from FZZT$_0$ to FZZT$_\alpha$. According to (6.1.24), we can interpret the boundary operator in the Schwarzian limit.

In that case, using using (6.1.24) with $\alpha = 0$ and $\xi(\sigma+1) = \xi(\sigma)+1$, we have

$$e^{\phi(\tau,1/2)} \mapsto \frac{\sqrt{\xi'(1)\xi'(0)}}{\xi(1)-\xi(0)} = \xi'(0). \tag{6.1.28}$$

Consequently, using $1-\cos(\alpha) = 2\sin^2(\frac{\alpha}{2})$, the boundary operator insertion becomes

$$e^{S^{(0)}_{\mathrm{bdry}}[\phi]-S^{(\alpha)}_{\mathrm{bdry}}[\phi]} \mapsto e^{4\sin^2(\frac{\alpha}{2})\xi'(0)}. \tag{6.1.29}$$

**Claim 6.1.4:** In other words, for observables $\mathcal{O}^{(i)}_\alpha = \mathcal{O}_\alpha(s_i,t_i)$, we expect

$$\boxed{\frac{[\prod_i \mathcal{O}^{(i)}_\alpha]^{\mathrm{Sch}}_\alpha}{[1]^{\mathrm{Sch}}_\alpha} = \frac{[\prod_i \mathcal{O}^{(i)}_0 e^{4\sin^2(\frac{\alpha}{2})\xi'(0)}]^{\mathrm{Sch}}_0}{[e^{4\sin^2(\frac{\alpha}{2})\xi'(0)}]^{\mathrm{Sch}}_0}} \tag{6.1.30}$$

In fact, a very similar statement is the "key lemma" used by Losev in [6] in order to calculate Schwarzian correlation functions. There is it proved using a change-of-variables formula for the Schwarzian field theory. It is satisfying to see that it has a somewhat clear interpretation from the Liouville perspective.

**Heuristic for the solution of the Schwarzian** Let $\langle\cdot\rangle_\alpha$ and $[\cdot]_\alpha$ denote the normalised and unnormalised expectations of Schwarzian field theory at parameter $\alpha \in \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}$, respectively. Here we'd like to present a heuristic calculation of the partition function of Schwarzian theory, using the "boundary operator" representation in (6.1.30).

**Proposition 6.1.5:** Suppose (6.1.30) holds for $\alpha^2 < \pi$. Then we have

$$\boxed{[1]_\alpha = \frac{\alpha}{\sin(\alpha)} e^{\alpha^2}[1]_0} \tag{6.1.31}$$

The partition function $[1]_0$ is usually fixed by convention (in our case we impose it to be equal to the $\zeta$-regularised determinant of Dirichlet Laplacian).

*Proof.* First step:

$$
\begin{aligned}
\partial_\alpha \log[e^{4\sin^2(\alpha/2)\varphi'(0)}]_0 &= 2\sin(\alpha)\frac{[\varphi'(0)e^{4\sin^2(\alpha/2)\varphi'(0)}]_0}{[e^{4\sin^2(\alpha/2)\varphi'(0)}]_0} \\
&= 2\sin(\alpha)\frac{[O_0(0,1)e^{4\sin^2(\alpha/2)\varphi'(0)}]_0}{[e^{4\sin^2(\alpha/2)\varphi'(0)}]_0} \\
&= 2\sin(\alpha)\frac{[O_\alpha(0,1)]_\alpha}{[1]_\alpha} \\
&= 2\sin(\alpha)\frac{\alpha}{\sin(\alpha)}\frac{[\varphi'(0)]_\alpha}{[1]_\alpha} \\
&= 2\alpha
\end{aligned}
\tag{6.1.32}
$$

Consequently

$$
[e^{4\sin^2(\alpha/2)\varphi'(0)}]_0 = e^{\alpha^2}[1]_0 \tag{6.1.33}
$$

By translational invariance, application of (6.1.30) and (6.1.33) we get

$$
\begin{aligned}
\partial_\alpha \log[1]_\alpha &= 2\alpha\frac{[\varphi'^2(0)]_\alpha}{[1]_\alpha} \\
&= 2\alpha\frac{\sin^2(\alpha)}{\alpha^2}\frac{[\varphi'^2(0)e^{4\sin^2(\alpha/2)\varphi'(0)}]_0}{[e^{4\sin^2(\alpha/2)\varphi'(0)}]_0} \\
&= 2\alpha\frac{\sin^2(\alpha)}{\alpha^2}\frac{1}{e^{\alpha^2}}\frac{1}{2\sin(\alpha)}\partial_\alpha\Big(\frac{1}{2\sin(\alpha)}\partial_\alpha e^{\alpha^2}\Big) \\
&= \partial_\alpha \log\Big[\frac{\alpha}{\sin(\alpha)}e^{\alpha^2}\Big].
\end{aligned}
\tag{6.1.34}
$$

Consequently

$$
[1]_\alpha = \frac{\alpha}{\sin(\alpha)}e^{\alpha^2}[1]_0. \tag{6.1.35}
$$

This determined the partition function. □

**Channel duality and boundary states.** In the context boundary conformal field theory, there is a relatively well-established principle of *channel duality* (or *open/closed string duality*), which relates thermal expectations in a theory with spatial boundary conditions (such as the left hand side of (6.1.20)) to quantum transition amplitudes between appropriate *boundary states* (or *branes*). Roughly speaking, this amounts to swapping the role of space- and time-variables. In the context of Liouville theory with ZZ–FZZT$_\alpha$ boundary conditions, the statement for the partition function would be

$$\operatorname{tr}(e^{-\beta\hat{H}_{\hat{c},\alpha}}) = \langle \mathrm{ZZ}_{\hat{c}}|\, e^{-\frac{a_0}{\beta}\hat{H}^{\mathrm{clo}}_{\hat{c}}} \left|\mathrm{FZZT}_{\alpha,\hat{c}}\right\rangle, \tag{6.1.36}$$

where $a_0 > 0$ is some universal constant, $\hat{H}^{\mathrm{clo}}_{\hat{c}}$ is a Hamiltonian for field configurations on a circle, and the bra/ket-states are certain "boundary states". We mention this, since it provides an alternative starting point to studying the Schwarzian limit from the Liouville field theory, which is perhaps closer to currently available methods in constructive field theory.

### 6.1.2 SYK model

The Sachdev–Ye–Kitaev (SYK) model resembles a family of fermions with mean-field spin-glass-type interactions. The typical setup consists of $2N$ Majorana fermions $\{\chi_i\}_{i=1,\dots,2N}$ interacting via a quartic action with i.i.d. Gaussian couplings $J_{ijkl}$, where $1 \le i < j < k < l \le 2N$:

$$H = -\sum_{1\le i<j<k<l\le 2N} J_{ijkl}\, \chi_i\chi_j\chi_k\chi_l. \tag{6.1.37}$$

Algebraically, referring to $\chi_i$ as Majorana fermions means that we impose the $*$-algebra $\chi_i^* = \chi_i$ and $\{\chi_i, \chi_j\} = 2\delta_{ij}$, i.e. the real Clifford algebra over $2N$ generators. Using a matrix realisation of this algebra, we can consider $H$ as a $2^N$-dimensional random matrix.

An explicit representation in terms of Pauli matrices on the Hilbert space $(\mathbb{C}^2)^{\otimes n}$ is given, up to normalisation, by generators of the form $\sigma_0^{\otimes k} \otimes \sigma_{1|2} \otimes \sigma_3^{\otimes(N-k-1)}$. Another explicit representation can be given in terms of fermionic creation/annihilation operators $a_1, \dots, a_N$:

$$\chi_{2k} = \frac{1}{\sqrt{2}}[a_k + a_k^*] \quad \text{and} \quad \chi_{2k+1} = \frac{i}{\sqrt{2}}[a_k - a_k^*]. \tag{6.1.38}$$

The canonical anticommutation relations ($\{a_i, a_j\} = 0, \{a_i, a_j^*\} = \delta_{ij}$) imply the Clifford algebra. A Hilbert-space representation is given by the fermionic Fock space spanned by the basis $|j_1 \cdots j_N\rangle \coloneqq (a_1^\dagger)^{j_1} \cdots (a_N^\dagger)^{j_N} |0\rangle$ with $j_k \in \{0, 1\}$.

**Physical observables.** The physically relevant quantities (in terms of linear response theory) are the retarded real-time Green's functions at some inverse temperature $\beta > 0$:

$$G^{\mathrm{R}}_{ij}(t,t') = -i\Theta(t-t')\frac{1}{Z}\operatorname{tr}(e^{-\beta H}[\chi_i(it), \chi_j(it')]), \tag{6.1.39}$$

where we define $\chi_i(\tau) = e^{\tau H}\chi_i e^{-\tau H}$. After Fourier transformation in the time/frequency domain and analytic continuation, it turns out that one may equivalently focus on the thermal Green's function (also: *imaginary-time time-ordered* Green's function):

$$G^{\text{th}}_{ij}(\tau,\tau') = \langle \mathbf{T}\chi_i(\tau)\chi_j(\tau')\rangle := \frac{1}{Z}\operatorname{tr}(e^{-\beta H}\mathbf{T}\chi_i(\tau)\chi_j(\tau')) \tag{6.1.40}$$

with $\mathbf{T}$ denoting a formal time-ordering operator defined via

$$\langle \mathbf{T}\chi_i(\tau)\chi_j(\tau')\rangle := \begin{cases} \langle \chi_i(\tau)\chi_j(\tau')\rangle & \text{if } \tau \geq \tau' \\ -\langle \chi_j(\tau')\chi_i(\tau)\rangle & \text{if } \tau < \tau'. \end{cases} \tag{6.1.41}$$

The thermal Green's function is antiperiodic with period $\beta$. All above quantities are understood in terms of finite-dimensional matrices and as such well-defined. Above definitions were with respect to fixed/quenched disorder. In the following we denote the disorder-averaging by an expectation $\mathbb{E}$. In relation to the Schwarzian theory we are interested in disorder-averaged Green's functions

$$\mathcal{G}(\tau,\tau') = \mathcal{G}^{(n)}(\tau,\tau') := \mathbb{E}[G^{\text{th}}_{ii}(\tau,\tau')], \tag{6.1.42}$$

in the limit of $n \to \infty$. More generally, we are interested in disorder-averages of products of Green's functions, which form the relevant family of "correlation functions".

**Coherent state path integral for observables.** Thermal (imaginary-time) correlators can be rewritten using the (Euclidean) coherent state path integral over the Grassmannian fields $\chi_i(\tau)$, $i = 1,\dots,N$:

$$\mathbb{E}\int \mathcal{D}[\chi(\tau)]F[\chi]\exp[-\int_0^\beta \mathrm{d}\tau \Big(\frac{1}{2}\sum_i \chi_i(\tau)\partial_\tau\chi_i(\tau) + \sum_{ijkl} J_{ijkl}\chi_i(\tau)\chi_j(\tau)\chi_k(\tau)\chi_l(\tau)\Big)], \tag{6.1.43}$$

where $F[\chi]$ stands for some field insertions, such as $\chi_i(\tau)\chi_i(\tau')$ in the case of the thermal Green's function (6.1.42). One can formally perform the average over the disorder variables

$J_{ijkl}$:

$$\begin{aligned}
&\mathbb{E}\exp\Big[-\sum_{ijkl} J_{ijkl}\int_0^\beta \mathrm{d}\tau [\chi_i\chi_j\chi_k\chi_l](\tau)\Big] \\
&= \exp\Big[\frac{\mathbb{E}[J^2]}{2}\sum_{ijkl}\int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' [\chi_i\chi_j\chi_k\chi_l](\tau)[\chi_i\chi_j\chi_k\chi_l](\tau')\Big] \\
&= \exp\Big[\frac{\mathbb{E}[J^2]}{2}\frac{N^4}{4!}\sum_{ijkl}\int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' (-\frac{1}{N}\sum_i \chi_i(\tau)\chi_i(\tau'))^4\Big] \\
&= \exp\Big[\frac{N}{2}\frac{\mathcal{J}^2}{4}\int_0^\beta\int_0^\beta \mathrm{d}\tau\mathrm{d}\tau' G_\chi(\tau,\tau')^4\Big],
\end{aligned} \tag{6.1.44}$$

where we defined $G_\chi(\tau,\tau') = \frac{1}{N}\sum_i \chi_i(\tau)\chi_i(\tau')$ and set $\mathbb{E}[J^2] = 3!\mathcal{J}^2/N^3$ for $\mathcal{J}^2 > 0$ in order to obtain a non-trivial $N \to \infty$ limit.

**$G$-$\Sigma$ field theory for SYK.** In the spirit of mean-field theory, we would now like to treat $G_\chi(\tau,\tau')$ as a "self-sufficient" bilocal field. Following the presentation in [7, 133], we proceed as follows: we introduce a new field $G(\tau,\tau')$, which is set to $G_\chi(\tau,\tau')$ via a Lagrange multiplier $\Sigma(\tau,\tau')$. That is, we would like to insert the following into our integral:

$$1 = \int \mathcal{D}[\Sigma(\tau,\tau')]\mathcal{D}[G(\tau,\tau')]\, e^{i\frac{N}{2}\int \mathrm{d}\tau\mathrm{d}\tau'\Sigma(\tau,\tau')[G(\tau,\tau')-G_\chi(\tau,\tau')]}. \tag{6.1.45}$$

Formally, for every $\tau,\tau' \in [0,\beta]^2$ the variable $\Sigma(\tau,\tau')$ is integrated along the real axis (or a contour parallel to it). It is typically assumed that one can "rotate" this integration contour onto the imaginary axis, in effect dropping the imaginary unit in the exponential. To justify this, one would need to insert (6.1.45) into the path integral, integrate out the fermionic variables and then consider a careful analysis of the integrand's singularities with respect to the $\Sigma$-field. In the following, we just assume that this works and drop the imaginary unit, $i\Sigma \mapsto \Sigma$. Frankly, (6.1.45) contains another subtlety that is much more fundamental: For any fixed number $N$ of Majorana fermions, the quantity $G_\chi$ is nilpotent: $G_\chi(\tau,\tau')^{N+1} = 0$. This is problematic as we think of $G(\tau,\tau')$ as a real variable (even and non-nilpotent in the context of Grassmann integrals). Consequently, integrating over $\Sigma$ cannot really set $G = G_\chi$, but this might still be true "under the integral" in that for example all correlation functions involving $G^{N+1}$ vanish. We refer to the appendix in [12] for a more detailed discussion of a toy model including the mentioned subtleties.

Now we proceed with the evaluation of (6.1.43). For this assume that the observable $F[\chi] = F[G_\chi]$ is actually a function of $G_\chi$. Inserting (6.1.45) and setting $G_\chi = G$ in the disorder-averaged interaction term (6.1.44) we get

$$\int \mathcal{D}[\Sigma(\tau,\tau')]\mathcal{D}[G(\tau,\tau')]\; F[G]e^{N/2\int \mathrm{d}\tau\mathrm{d}\tau'[\Sigma(\tau,\tau')G(\tau,\tau')+\frac{\mathcal{J}^2}{4}G(\tau,\tau')^4]}\times$$
$$\times\int \mathcal{D}[\chi(\tau)]e^{-\frac{1}{2}\sum_i\int \mathrm{d}\tau\mathrm{d}\tau'\chi_i(\tau)[\delta(\tau-\tau')\partial_\tau+\Sigma(\tau,\tau')]\chi_i(\tau)}$$
$$=\int \mathcal{D}[\Sigma,G]\,F[G]e^{N/2\left[\log\det(\partial_\tau+\Sigma)+\int \mathrm{d}\tau\mathrm{d}\tau'[\Sigma(\tau,\tau')G(\tau,\tau')+\frac{\mathcal{J}^2}{4}G(\tau,\tau')^4]\right]} \tag{6.1.46}$$

In the $N\to\infty$ limit this integral should be dominated by the maxima of the action in the exponential. In fact, the saddle point equations are

$$G = (\partial_\tau+\Sigma)^{-1} \quad \text{and} \quad \Sigma(\tau,\tau') = \mathcal{J}^2 G(\tau,\tau')^3, \tag{6.1.47}$$

where we understand $G$ and $\Sigma$ as integral operators, defined by their kernels $G(\tau,\tau')$ and $\Sigma(\tau,\tau')$, respectively. In [7, 133] it is argued that for $\beta\mathcal{J}\gg 1$ the $\partial_\tau$-term in (6.1.47) becomes negligible. It is claimed that solutions of this time-independent equation are of the form

$$\mathcal{G}_\varphi(\tau_1,\tau_2) \propto \frac{\mathrm{sgn}(\tau_1-\tau_2)}{[\beta\mathcal{J}]^{1/2}}\left[\frac{\pi\sqrt{\varphi'(\tau_1/\beta)\varphi'(\tau_2/\beta)}}{\sin(\pi|\varphi(\tau_1/\beta)-\varphi(\tau_2/\beta)|)}\right]^{1/2}, \tag{6.1.48}$$

for any $\varphi\in\mathrm{Diff}(S^1)$ and some universal constant of proportionality. Defining $\Sigma_\varphi = \mathcal{G}_\varphi^{-1}$ we have $\Sigma_\varphi(\tau,\tau') = \mathcal{J}^2\mathcal{G}_\varphi(\tau,\tau')^3$.

To reiterate, the above is an infinite-dimensional family of saddle points (with the same action) in the limit $N\to\infty$ and $\beta\mathcal{J}\to\infty$. One expects that for finite but large values of $N$ and $\beta\mathcal{J}$, the path integral in (6.1.46) is still dominated by the integral over the functions $\mathcal{G}_\varphi$, but the error in neglecting the $\partial_\tau$-term may give these terms a finite $\varphi$-dependent action. In fact, on the level of the action, the approximation error is $\frac{N}{2}[\mathrm{tr}\log(\partial_\tau+\Sigma_\varphi)-\mathrm{tr}\log(\Sigma_\varphi)] = \frac{N}{2}\mathrm{tr}\log(1+\partial_\tau G_\varphi)$, where we assumed (6.1.47). Note that here $\partial_\tau G_\varphi$ is a composition of operators. This can formally be expanded in powers of $\partial_\tau G_\varphi$ with the first non-vanishing term being

$$\frac{N}{4}\mathrm{tr}[\partial_\tau G_\varphi\partial_\tau G_\varphi] \propto -\frac{N}{\beta\mathcal{J}}\iint \mathrm{d}\tau\mathrm{d}\tau'\left[\frac{\pi\sqrt{\varphi'(\tau)\varphi'(\tau')}}{\sin\left(\pi|\varphi(\tau)-\varphi(\tau')|\right)}\right]^3. \tag{6.1.49}$$

Note that this expression is inherently non-local. In a further approximation step it is conjectured that the theory defined by such an action is dominated by the singular diagonal $\tau \approx \tau'$ contribution

$$\left[\frac{\pi\sqrt{\varphi'(\tau)\varphi'(\tau')}}{\sin\left(\pi|\varphi(\tau)-\varphi(\tau')|\right)}\right]^3 = \frac{1}{|\tau-\tau'|^3} + 9|\tau-\tau'|\mathcal{S}(\tan(\pi\varphi(\tau)),\tau) + O(|\tau-\tau'|^2). \tag{6.1.50}$$

The divergent contribution is independent of $\varphi$, and can be absorbed into a normalisation constant. Consequently, the claim is, that the leading-order action for the low-energy excitation $\mathcal{G}_\varphi$ is

$$S^{\text{eff}}[G_\varphi] \propto -\frac{N}{\beta\mathcal{J}}\int \mathrm{d}\tau\mathcal{S}(\tan(\pi\varphi(\tau)),\tau), \tag{6.1.51}$$

which is precisely the Schwarzian action. Furthermore, disorder-averaged Green's function correlators should correspond to correlators of Schwarzian cross-ratio observables.

### 6.1.3 JT-gravity

Jackiw-Teitelboim (JT) gravity is defined over a surface $\Sigma$ with boundary $\partial\Sigma$. We will consider the case where $\Sigma$ is the upper half-plane $\mathbb{H}^2 = \{z = t + iy\colon\, t \in \mathbb{R}, y > 0\}$. The formal action is given for a metric $g$ and a scalar "dilaton field" $\phi$:

$$I[g,\phi] = -\frac{1}{16\pi G}\left[\int_\Sigma (R_g+2)\phi\sqrt{g}\,\mathrm{d}x + 2\int_{\partial\Sigma} K_g\phi\sqrt{\gamma}\,\mathrm{d}u\right], \tag{6.1.52}$$

where $\sqrt{g}$ denotes the square root determinant of the metric tensor $g_{\mu\nu}$, $R_g$ is its Ricci scalar curvature, $\gamma$ is the induced metric on the boundary and $K_g$ the extrinsic curvature.

Ideally, one would like to make sense of the formal path integral measure $\mathcal{D}g\mathcal{D}\phi e^{-I[g,\phi]}$. As there are various issues[1] with this, one may resort to making sense of a "regularised on-shell action". Here, *on-shell* refers to the fact that we derive an action for configurations that satisfy the classical equations of motion. The latter are given by

$$\begin{aligned} 0 &= R_g + 2 \\ 0 &= \nabla_\mu\nabla_\nu\phi + g_{\mu\nu}(-\Delta+1)\phi. \end{aligned} \tag{6.1.53}$$

[1] For real-valued $\phi$ the integration is unbounded and generally the integration space for metrics is not clear. In the physics literature one typically assumes that we can deform the contour onto the imaginary axis, in which case the integral "localises" on solutions to $R_g + 2 = 0$, which is essentially equivalent to the on-shell action.

In particular, all "on-shell geometries" are of constant negative curvature. Ideally, one would now like to parametrise all such solutions and insert them into the action. The bulk term vanishes, but the boundary term might yield a non-trivial quantity. In this case the solution space includes unbounded geometries, for which the boundary term doesn't make sense, and one imposes an additional form of regularisation: We restrict onto a subset of geometries for which the boundary term can be evaluated, namely finite volume and boundary-length subsets of the hyperbolic plane enscribed by a particular boundary curve:

Recall that the geodesic boundary of the hyperbolic plane $\mathbb{H}^2$ can be identified with the real projective line $\mathbb{R}P^1 \cong \mathbb{R}\cup\{\infty\}$. For any orientation-preserving diffeomorphism $f\colon \mathbb{R}P^1 \to \mathbb{R}P^1$ of the boundary[2], given in affine coordinates $t = f(u)$ with $t,u \in \mathbb{R}\cup\{\infty\}$, and any fixed $\epsilon > 0$ we define the curve in the upper half-plane

$$p_{f,\epsilon}(u) = f(u) + i\epsilon f'(u). \tag{6.1.54}$$

The area above this curve defines a (simply-connected) finite-volume portion of the hyperbolic plane with smooth boundary. By the Riemann mapping theorem, we can map this back onto $\mathbb{H}^2$ and consider the induced metric as a metric on the upper half-plane. To be precise, this mapping is up to a global $\mathrm{PSL}(2,\mathbb{R})$-transformation, which acts via Möbius transformations on $f$. In other words, every $\epsilon > 0$ and $f \in \mathrm{Diff}(\mathbb{R}P^1)/\mathrm{PSL}(2,\mathbb{R})$ determines a metric on $\mathbb{H}$. Given a boundary condition $\phi|_{\partial\Sigma}$, we can evaluate the boundary term in (6.1.52). For $(n^\mu)$ denoting the normal vector to the curve $p_{f,\epsilon}$, the extrinsic/geodesic curvature is

$$K_g = \nabla_\mu n^\mu = \partial_\mu n^\mu + \Gamma^\mu_{\mu\nu} n^\nu. \tag{6.1.55}$$

For the upper half-plane model, the Christoffel symbols are given by $\Gamma^t_{ty} = \Gamma^t_{yt} = \Gamma^y_{yy} = -\Gamma^y_{tt} = \frac{1}{y}$ and hence

$$\begin{aligned} K_g &= \frac{y(t'y'' - y't'') + t'(y'^2 + t'^2)}{(t'^2 + y'^2)^{3/2}} \\ &= 1 + \epsilon^2 \frac{f'''}{f'^2} - \frac{3}{2}\epsilon^2 \frac{f''^2}{f'^2} + O(\epsilon^4) \\ &= 1 + \epsilon^2 \mathrm{Sch}(f,u) + O(\epsilon^4) \end{aligned} \tag{6.1.56}$$

The induced metric $\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial u^a} \frac{\partial x^\nu}{\partial u^b}$ can be evaluated to

$$\gamma_{uu} = \frac{1}{\epsilon^2} + \frac{f''^2}{f'^2}. \tag{6.1.57}$$

[2] Any such diffeomorphism or the real projective line can be related to a circle diffeomorphism $\varphi \in \mathrm{Diff}(S^1)$ via $f(\tan(\pi\tau)) = \tan(\pi\varphi(\tau))$.

To evaluate the boundary term we have to impose some boundary conditions for the dilaton field:

$$\phi(p_{f,\epsilon}(u)) = \frac{1}{\epsilon}\phi_{\mathrm{r}}(u) \tag{6.1.58}$$

for some fixed function $\phi_{\mathrm{r}}$. Inserting (6.1.56), (6.1.57), (6.1.58) into (6.1.52), we get

$$\begin{aligned} I(g[p_{f,\epsilon}]) &= -\frac{1}{8\pi G}\int_{-\infty}^{\infty} \mathrm{d}u\, \frac{1}{\epsilon}\phi_{\mathrm{r}}(u)[1+\epsilon^2 \mathcal{S}(f,u)+O(\epsilon^4)]\sqrt{\frac{1}{\epsilon^2}+\frac{f''^2}{f'^2}} \\ &= -\frac{1}{8\pi G}\frac{1}{\epsilon^2}\int_{-\infty}^{\infty} \mathrm{d}u\, \phi_{\mathrm{r}}(u) - \frac{1}{8\pi G}\int \mathrm{d}u\, \phi_{\mathrm{r}}(u)\mathcal{S}(f,u)+O(\epsilon) \\ &= -\frac{1}{8\pi G}\int_{-\infty}^{\infty} \mathrm{d}u\, \phi_{\mathrm{r}}(u)\mathcal{S}(f,u) + S_{\mathrm{counter},\epsilon}+O(\epsilon), \end{aligned} \tag{6.1.59}$$

where we absorbed the divergent part into a “counter-term” that is discarded as part of the regularisation. In other words we find that the Schwarzian action appears as the leading $f$-dependent contribution for the on-shell JT-action for a specific set of geometries. The boundary datum $\phi_{\mathrm{r}}$ can be chosen freely and for $\phi_{\mathrm{r}}(u) = [\pi(1+u^2)]^{-1}$ we can reparametrise via $u = \tan(\pi\tau)\colon S^1 \to \mathbb{R}P^1$ to obtain the usual Schwarzian measure (writing $f(\tan(\pi\tau)) = \tan(\pi\varphi(\tau))$ for a circle diffeomorphism $\varphi \in \mathrm{Diff}(S^1)$).

## 6.2 Additional Background

This section provides a kaleidoscopic view over various objects which are directly or indirectly related to the Schwarzian field theory. We mention the real projective line and how the Schwarzian derivative arises as an infinitesimal cross-ratio. Furthermore, we mention the Schwarzian’s relevance in the context of Sturm-Liouville (/Hill’s) operators. Then, we discuss *coadjoint orbits* and their classification in the context of two infinite-dimensional cases, the loop group and the Virasoro group. The path integral of Schwarzian measures can be seen to appear naturally in this context. In particular, this point of view provides a way to “guess” the partition function by formally applying an infinite-dimensional analogue of the Duistermaat-Heckman theorem [17].

### 6.2.1 Cross-ratios and the Schwarzian derivative

In the following we recall some basic facts about the geometry of the real projective line, including the notion of cross-ratios and how the Schwarzian derivative appears naturally in this context.

**Projective line, cross-ratios and** $\mathrm{PSL}(2,\mathbb{R})$**.** Recall the definition of the real projective line $\mathbb{R}P^1$ as equivalence classes $[x:y]$ of points in $\mathbb{R}^2\backslash\{0\}$ under rescaling: $[x:y]=[tx:ty]$ for $t\neq 0$. Topologically, $\mathbb{R}P^1$ is just the circle, with $S^1\ni\tau\mapsto[\sin(\pi\tau):\cos(\pi\tau)]$ providing a homeomorphism. We typically refer to a point $[x:y]=[x/y:1]\in\mathbb{R}P^1$ via the *affine coordinate* $x/y\in\mathbb{R}\cup\{\infty\}$, with the understanding that this is equal to $\infty$ if $y=0$. Consequently, we typically understand $\mathbb{R}P^1\cong\mathbb{R}\cup\{\infty\}$. In these coordinates, the identification with the circle is given by $\tan(\pi\cdot)\colon S^1\mapsto\mathbb{R}P^1\cong\mathbb{R}\cup\{\infty\}$.

The action of $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}(2,\mathbb{R})$ on $\mathbb{R}^2$ induces an action on $\mathbb{R}P^1$:

$$M[x:y]\coloneqq[ax+by:cx+dy]=\left[\frac{ax/y+d}{cx/y+d}:1\right]. \tag{6.2.1}$$

In particular, on the affine coordinate this acts via fractional linear (*i.e.* Möbius) transformations and it descends to a faithful transitive action of $\mathrm{PSL}(2,\mathbb{R})=\mathrm{SL}(2,\mathbb{R})/\{\pm\mathbb{1}\}=\mathrm{GL}(2,\mathbb{R})/\mathbb{R}^\times$ on the real projective line $\mathbb{R}P^1$.

Consider four points $a,b,c,d\in\mathbb{R}P^1$. We write $a=[a_0:a_1]$, $b=[b_0:b_1]$ et cetera. For two vectors in $x=(x_0,x_1)$ and $y=(y_0,y_1)$ we define the "cross product" $x\times y=x_0y_1-x_1y_0$. The *cross-ratio* is defined as

$$[a,b;c,d]\coloneqq[(a\times c)(b\times d):(a\times d)(b\times c)]\in\mathbb{R}P^1. \tag{6.2.2}$$

Note that the cross product of two vectors in $\mathbb{R}^2$ is invariant under $\mathrm{SL}(2,\mathbb{R})$ transformations (as it measures the area of the parallelogram spanned by those vectors). Consequently, the cross-ratio is manifestly invariant under the $\mathrm{PSL}(2,\mathbb{R})$-action on $\mathbb{R}P^1$. Using affine coordinates, we can rewrite (6.2.2) into the more familiar form

$$\begin{aligned}[a,b;c,d]&=[(a_0c_1-a_1c_0)(b_0d_1-b_1d_0):(a_0d_1-a_1d_0)(b_0c_1-b_1c_0)]\\&=\left[\left(\frac{a_0}{a_1}-\frac{c_0}{c_1}\right)\left(\frac{b_0}{b_1}-\frac{d_0}{d_1}\right):\left(\frac{a_0}{a_1}-\frac{d_0}{d_1}\right)\left(\frac{b_0}{b_1}-\frac{c_0}{c_1}\right)\right]\\&=\frac{(a-c)(b-d)}{(a-d)(b-c)},\end{aligned} \tag{6.2.3}$$

where we identify points in $\mathbb{R}P^1$ with their affine coordinate.

**Schwarzian as an infinitesimal cross-ratio.** Consider a map $f\colon \mathbb{R} \to \mathbb{R}P^1 \cong \mathbb{R}\cup\{\infty\}$. Expanding the infinitesimal cross-ratio in $\epsilon > 0$ one finds

$$[f(t), f(t+\epsilon); f(t+2\epsilon), f(t+3\epsilon)] = 1 + \frac{\epsilon^2}{6}\mathcal{S}(f,t) + o(\epsilon^2). \tag{6.2.4}$$

As a consequence of this, the $\mathrm{PSL}(2,\mathbb{R})$-invariance of the Schwarzian is clear. In the context of the Schwarzian we are also interested in observables

$$\begin{aligned}[f(t), f(s); f(t+\epsilon), f(s+\epsilon)] &= \frac{(f(t+\epsilon)-f(t))(f(s+\epsilon)-f(s))}{(f(t)-f(s))(f(t+\epsilon)-f(s+\epsilon))} \\ &= \epsilon^2 \frac{f'(t)f'(s)}{(f(t)-f(s))^2} + O(\epsilon^3).\end{aligned} \tag{6.2.5}$$

Consequently, for $t \to s$, we see

$$\frac{f'(t)f'(s)}{(f(t)-f(s))^2} = \frac{1}{|t-s|^2} + 6|t-s|^2\mathcal{S}(f,\tau) + o(|t-s|^2). \tag{6.2.6}$$

**Cross-ratios, hyperbolic distances and JT-observables.** Consider the upper half plane-model for the hyperbolic plane $\mathbb{H}^2 = \{z = t+iy\colon t \in \mathbb{R}, y > 0\}$ with the metric $g = \frac{1}{y^2}[\mathrm{d}t^2 + \mathrm{d}y^2]$. Its geodesic boundary is $\mathbb{R}\cup\{\infty\}$ and can be identified with the real projective plane $\mathbb{R}P^1$. In fact, the hyperbolic distance is directly related to the cross-ratio:

> **Lemma 6.2.1** ([134]): Consider two points $z_1, z_2 \in \mathbb{H}^2$. Suppose the unique geodesic through these points hits the boundary at $t_1, t_2 \in \mathbb{R}\cup\{\infty\} \cong \mathbb{R}P^1$. Then we have
>
> $$\mathrm{dist}(z_1, z_2) = \Big|\log[t_1, t_2; z_1, z_2]\Big|, \tag{6.2.7}$$
>
> where we understand the cross-ratio as in the last line of (6.2.3), extended to complex (affine) coordinates.

This is of particular interest in view of the relationship between JT-gravity and the Schwarzian theory (Section 6.1.3): Consider an orientation-preserving diffeomorphism $f\colon \mathbb{R}P^1 \to \mathbb{R}P^1$ and the associated curve $p_{f,\epsilon}(u) = f(u) + i\epsilon f'(u)$ as in (6.1.45). Lemma 6.2.1 gives us a way of calculating distances between different points on the curve $p_{f,\epsilon}$ as $\epsilon \searrow 0$: For sufficiently small $\epsilon$, the geodesic through $p_{f,\epsilon}(t)$ and $p_{f,\epsilon}(s)$ will be a circular arc hitting the boundary at points $f(t)+O(\epsilon^2)$ and $f(s)+O(\epsilon^2)$. Hence, we calculating the cross-ratio

$$\begin{aligned}[f(t),f(s);p_{f,\epsilon}(t),p_{f,\epsilon}(s)] &= [f(t),f(s);f(t)+i\epsilon f'(t),f(s)+i\epsilon f'(s)] \\ &= \frac{-\epsilon^2 f'(t)f'(s)}{(f(t)-f(s)-i\epsilon f'(s))(f(s)-f(t)-i\epsilon f'(t))} \\ &= \epsilon^2 \frac{f'(t)f'(s)}{|f(t)-f(s)|^2}+O(\epsilon^3)\end{aligned} \tag{6.2.8}$$

we obtain that for $\epsilon \searrow 0$

$$\operatorname{dist}\big(p_{f,\epsilon}(t),p_{f,\epsilon}(s)\big) = \log\Big(\frac{1+o(1)}{\epsilon^2}\Big) - \log\Big(\frac{f'(t)f'(s)}{|f(t)-f(s)|^2}\Big). \tag{6.2.9}$$

In other words, exponential of the *renormalised distance* (removing the universal divergence as $\epsilon \to 0$) between boundary points in JT-gravity (or, more honestly, the "approximation" constructed in Section 6.1.3) are expressed in terms of Schwarzian cross-ratio observables.

**Hill's operators and the Schwarzian.** The Schwarzian derivative makes an appearance in the, seemingly, entirely separate context of Sturm-Liouville/Hill's operators. This perspective is of particular interest for the classification of coadjoint orbits of the Virasoro group (Section 6.2.5).

**Proposition 6.2.2:** Consider a smooth potential $b\colon \mathbb{R} \to \mathbb{R}$ and two linearly independent solutions $f,g\colon \mathbb{R} \to \mathbb{R}$ to Hill's equation

$$0 = [\partial_\tau^2 + \tfrac{1}{2}b(\tau)]f(\tau) = [\partial_\tau^2 + \tfrac{1}{2}b(\tau)]g(\tau) \tag{6.2.10}$$

Then the Wronskian $W = \det\begin{pmatrix} f & g \\ f' & g' \end{pmatrix}$ is constant. Without loss of generality, after potential rescaling, it can be chosen to be constant equal to one, $W \equiv 1$. In particular $f,g$ don't have coinciding zeros and we can define the *projective solution*

$$\eta(\tau) = \frac{g(\tau)}{f(\tau)}\colon \mathbb{R} \to \mathbb{R}P^1 \cong \mathbb{R} \cup \{\infty\}. \tag{6.2.11}$$

Then, $\eta' = 1/f^2 > 0$ is monotone and satisfies

$$\mathcal{S}(\eta,\tau) = b(\tau). \tag{6.2.12}$$

*Proof.* The Wronskian is constant since Hill's equation has no first-order term. To be more explicit, one can check that

$$\partial_\tau \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2}b(\tau) & 0 \end{pmatrix} \begin{pmatrix} f & g \\ f' & g' \end{pmatrix}. \tag{6.2.13}$$

Note that $\operatorname{tr}\begin{pmatrix} 0 & 1 \\ -\frac{1}{2}b(\tau) & 0 \end{pmatrix} = 0$, hence the flow (6.2.13) leaves the determinant constant. Moreover, rescaling and/or interchanging the role of $f$ and $g$ we can make sure the Wronskian is constant equal to 1. To calculate $\mathcal{S}(\eta) = \log(\eta)'' - \frac{1}{2}\log(\eta)'^2$, we first note that

$$\eta' = \left(\frac{g}{f}\right)' = \frac{fg' - f'g}{f^2} = \frac{1}{f^2}. \tag{6.2.14}$$

Consequently,

$$\begin{aligned} \mathcal{S}(\eta) &= -2\log(f)'' - \tfrac{1}{2}[-2\log(f)]'^2 \\ &= -2f''/f \\ &= b(\tau). \end{aligned} \tag{6.2.15}$$

□

**The Schwarzian of the tangent is special.** The tangent function $\tan\theta = \sin\theta/\cos\theta \in \mathbb{R}P^1 \cong \mathbb{R}\cup\{\infty\}$ is conveniently considered to take values in the real projective line. We think of it as a "uniform rotation" in $\mathbb{R}P^1 \cong S^1$. It turns out that its Schwarzian derivative is remarkably simple, a fact that will be relevant in the classification of Virasoro coadjoint orbits (Section 6.2.4 and Section 6.2.5):

> **Lemma 6.2.3:** For any $\alpha^2 \in \mathbb{R}$, that is $\alpha \in \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}$, we have
>
> $$\mathcal{S}(\tan(\alpha\tau), \tau) = \mathcal{S}(\tfrac{1}{\alpha}\tan(\alpha\tau), \tau) = 2\alpha^2. \tag{6.2.16}$$

*Proof.* We demonstrate two separate derivations. The first, fast and indirect, is a consequence of Proposition 6.2.2: Consider the Sturm-Liouville operator $\partial_\tau^2 + \alpha^2$. Both $f = \cos(\alpha\tau)$ and $g = \frac{1}{\alpha}\sin(\alpha\tau)$ are (real-valued) solutions, such that $fg' - f'g = \cos^2(\alpha\tau) + \sin^2(\alpha\tau) = 1$. Hence, by (6.2.11), we have $\mathcal{S}(\tan(\alpha)) = \mathcal{S}(g/f) = 2\alpha^2$, which proves the claim. The second one is by direct calculation. Recall that $\tan(\alpha\tau)' = \alpha/\cos^2(\alpha\tau)$. Hence

$$\log\left(\tfrac{1}{\alpha}\tan(\alpha\tau)'\right)' = \log(\tan(\alpha\tau)')' = -2\log(\cos(\alpha\tau))' = 2\alpha\tan(\alpha\tau). \tag{6.2.17}$$

Consequently, for $\alpha \in \mathbb{R}$ we have (recall $\tan' = 1 + \tan^2$)

$$\begin{aligned} \mathcal{S}(\tan(\alpha\tau), \tau) &= \log(\tan(\alpha\tau)')'' - \frac{1}{2}\log(\tan(\alpha\tau)')'^2 \\ &= 2\alpha \tan(\alpha\tau)' - 2\alpha^2 \tan^2(\alpha\tau) \\ &= 2\alpha^2. \end{aligned} \tag{6.2.18}$$

□

**Schwarzian as a 1-cocycle.** A central property of the Schwarzian is its composition rule

$$\mathcal{S}(f \circ g) = \mathcal{S}(g) + |g'|^2 \mathcal{S}(f) \circ g \tag{6.2.19}$$

This can be stated in more flamboyant terms: Suppose a group $G$ acts an a vector space $V$. A map $\rho\colon G \to V$ is a 1-*cocycle* (or crossed homomorphism) if

$$\rho(gh) = \rho(g) + g \cdot \rho(h) \qquad \text{for } g, h \in G. \tag{6.2.20}$$

In our case, the group is $G = \mathrm{Diff}(S^1)$ and $V = \mathcal{F}_2(S^1) = \{F(\tau)(\mathrm{d}\tau)^2 \colon F\colon S^1 \to \mathbb{R}\}$ is the space of 2-tensor densities on $S^1$, i.e. functions $F(\tau)$ that transform via $(g \cdot F)(\tau) := |g'|^2 F(g(\tau))$ under the action of a reparametrisation $g \in \mathrm{Diff}(S^1)$. In other words, (6.2.19) says that *the Schwarzian is a* 1*-cocycle* on $\mathrm{Diff}(S^1)$ with coefficients in $\mathcal{F}_2(S^1)$. For this reason, the Schwarzian derivative is relevant for the central extension of $\mathrm{Diff}(S^1)$, namely the Virasoro group (see Section 6.2.4).

### 6.2.2 Coadjoint orbits

Consider a Lie group $G$ with Lie algebra $\mathfrak{g}$. $G$ acts on $\mathfrak{g}$ via the *adjoint representation* Ad, which (for matrix groups) we can write as $\mathrm{Ad}_g(X) = gXg^{-1}$, with $g \in G$ and $X \in \mathfrak{g}$. By duality, there is a *coadjoint representation* $\mathrm{Ad}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$ on the vector-space dual $\mathfrak{g}^*$ of the Lie algebra, such that $\langle \xi, X \rangle = \langle \mathrm{Ad}^*_g \xi, \mathrm{Ad}_g X \rangle$ for any $g \in G, \xi \in \mathfrak{g}^*, X \in \mathfrak{g}$ and where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing. For an element $\xi \in \mathfrak{g}^*$ we write $\mathcal{O}_\xi := \mathrm{Ad}_G \xi \cong G/\mathrm{Stab}(\xi)$ for the orbit of $\xi$ under this action. These spaces are referred to as *coadjoint orbits*. Mathematically these spaces are of particular interest as they carry a natural symplectic structure: On $\mathcal{O}_\xi$ we can define the Kirillov–Kostant–Souriau (KKS) 2-form $\omega \in \Omega^2(\mathcal{O}_\xi)$:

$$\omega_\nu(\mathrm{ad}^*_X \nu, \mathrm{ad}^*_Y \nu) := \langle \nu, [X, Y] \rangle \qquad \text{with } \nu \in \mathcal{O}_\xi \text{ and } X, Y \in \mathfrak{g}. \tag{6.2.21}$$

This defines a closed, non-degenerate and $G$-invariant 2-form, hence an invariant symplectic form. Moreover, the canonical embedding $\mu \colon \mathcal{O}_\xi \hookrightarrow \mathfrak{g}^*$ defines a moment map, i.e. for any Lie algebra element $X \in \mathfrak{g}$ the gradient flow generated by the Hamiltonian $\langle \mu, X \rangle \colon \mathcal{O}_\xi \to \mathbb{R}$ agrees with the $G$-action generated by $X$.

**What is this good for?** From a mathematical point of view, the above construction opens the door to Kirillov's *orbit method*, relating geometric properties of the coadjoint orbits with the irreducible unitary representations of $G$. For example, Kirillov's character formula expresses the characters of irreducible representations in terms *orbital integrals*

$$\int_{\mathcal{O}_\xi} e^{i\langle \mu, X \rangle} \mathrm{dvol}_{\mathrm{O}_\xi} \qquad \text{where } \mathrm{dvol}_{\mathrm{O}_\xi} = \omega^n / n!, \tag{6.2.22}$$

for $\dim \mathrm{O}_\xi = 2n$ is the volume form induced by the KKS-form $\omega$.

From a physical point of view, the orbit method produces a wealth of examples for *geometric quantisation*: As a symplectic manifold, a coadjoint orbit can be considered as the phase space (i.e. the space of positions and momenta) of a classical mechanical system, with a family of potential Hamiltonian functions provided by the moment map. Geometric quantisation then constructs a Hilbert space of quantum states, where classical observables (such as the Hamiltonian) are promoted to operators, and which is equipped with a unitary $G$-representation.

Alternatively, the orbital integrals (6.2.22) provide natural candidates for the construction of *path integrals*. This is particularly true for the case of infinite-dimensional groups and orbits, in which case one may try to give probabilistic meaning to formal measures such as

$e^{-\beta\langle\mu,X\rangle}\mathrm{dvol}_{O_\xi}$. A central point of the following chapters is to discuss this perspective for the Schwarzian theory.

**Orbital integrals and the Duistermaat–Heckman theorem.** Without further knowledge, one might expect orbital integrals (6.2.22) to be rather hard to evaluate. However, if $X \in \mathfrak{g}$ generates a $U(1)$-action, a somewhat magical statement in symplectic geometry states that the integral "localises" on critical points of the action $\langle\mu, X\rangle$: Suppose $G$ is finite-dimensional and $H_X$ had finitely many non-degenerate critical points. Then Write $H_X(\nu) = \langle\mu, X\rangle(\nu) \in \mathbb{R}$ for $\nu \in O_\xi$

$$\int_{O_\xi} e^{i\lambda H_X(\nu)}\mathrm{dvol}_{O_\xi}(\nu) = \sum_{\nu_0\,:\,\mathrm{D}H_X(\nu_0)=0} \frac{e^{i\lambda H_X(\nu_0)}}{\sqrt{\det(2\pi\lambda\,\mathrm{D}^2 H_X(\nu_0))}}. \tag{6.2.23}$$

In other words, the saddle point approximation for the orbital integral is exact! Witten and Stanford suggested that the equivalent statement holds true in the infinite-dimensional setting [17], which allowed them to "guess" the partition function of the Schwarzian field theory, by interpreting it as a orbital integral.

**A note on the infinite-dimensional setting.** In this section we are mostly interested in the coadjoint orbits of the Virasoro group and so-called loop groups, both of which are infinite-dimensional and understood as Fréchet manifolds. In this scenario, various functional-analytic and geometric subtleties come into play. For example, the exponential map from the Virasoro algebra to its group is not even locally surjective. We will mostly ignore such issues, as our interest in the coadjoint orbits is mostly for "guessing" interesting action functionals. We refer to the monograph [135] for a more careful treatment.

Moreover, in the context of an infinite-dimensional group $G$, we will typically consider the coadjoint orbits of its *central extension* $\widehat{G}$, assuming it is unique. It's Lie algebra $\widehat{\mathfrak{g}}$ is a central extension of the Lie algebra $\mathfrak{g}$, with dual space $\widehat{\mathfrak{g}}^*$. However, since central elements acts trivially via the (co)adjoint representation, the coadjoint orbits of $\widehat{G}$ simply correspond to the $G$-orbits in $\widehat{\mathfrak{g}}^*$. As a consequence, if we refer to the coadjoint orbits of $G$, we usually refer to this generalisation[3].

[3]Nita that this modification is typically irrelevant in the finite-dimensional case, as semi-simple finite-dimensional Lie algebras don't have any central extensions. On the other hand, for our infinite-dimensional groups, the coadjoint actions of their central extensions tend to be more "regular" and offer a richer structure.

### 6.2.3 Loop groups and their coadjoint orbits

Consider a finite-dimensional Lie group $G$ with Lie algebra $\mathfrak{g}$. We define the *loop group* $LG = C^\infty(S^1, G)$ as the smooth maps from to circle into $G$ with pointwise multiplication, as well as the *loop algebra* $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ as smooth maps with pointwise Lie brackets[4]. In the following we recall the theory of coadjoint orbits for these groups, roughly following the monograph [135].

Mathematically, loop groups are of interest as one of the easiest and best-understood examples of infinite-dimensional Lie groups, with a well-understood Lie algebra theory (Kac-Moody algebras), and natural connections to topology and homotopy theory. From a physical perspective, they find use in the context of string theory (*e.g.* strings propagating in a group manifold) and in certain integrable systems, such as the Korteweg–de Vries (KdV) equations of shallow water waves[5].

To ease the notation, we will treat $G$ as a matrix group, identifying $\mathfrak{g}$ as a space of matrices as well. We quickly state the main result of this chapter, before going into more details: We will see that the (smooth) dual of the centrally extended loop algebra $\widehat{L\mathfrak{g}}^*$ can be identified with "covariant derivatives" $-c\partial_\tau + A(\tau)$, where $c \in \mathbb{R}$ and $A \in C^\infty(S^1, \mathfrak{g})$. The coadjoint action of $g(\tau) \in LG$ leaves $c$ invariant and maps $A \mapsto gAg^{-1} + c\partial_\tau g g^{-1}$, where multiplication is understood pointwise.

**Theorem 6.2.4** ([135–137]): Let $G$ denote a compact connected Lie group. There is a one-to-one correspondence between the coadjoint orbits of (the central extension of) $LG$ at $c = 1$ and the conjugacy classes of $G$. In particular, every orbit contains a constant element $(A(\tau), 1)$, with $A(\tau) \equiv X \in \mathfrak{g}$, and is of the form $LG/H$ for $H = \mathrm{Stab}_{\mathrm{Ad}}(X) \subseteq G$.

**Central extension of the loop algebra.** Given an Ad-invariant inner product on $\mathfrak{g}$ (denoted by tr), we define the central extension $\widehat{L\mathfrak{g}}$: Elements of $\widehat{L\mathfrak{g}}$ are of the form $(X(\tau), k)$, where $X(\tau)$ is a $\mathfrak{g}$-valued function of $S^1$ and $k \in \mathbb{R}$ is central. The Lie bracket on $\widehat{L\mathfrak{g}}$ is defined by

$$\mathrm{ad}_{(X,k)}(v, l) = [(X(\tau), k), (Y(\tau), l)] = \left( [X(\tau), Y(\tau)]_{\mathfrak{g}}, \int_{S^1} \mathrm{tr}\left(Y(\tau) X'(\tau)\right) \mathrm{d}\tau \right). \tag{6.2.24}$$

[4] $L\mathfrak{g}$ naturally carries a Fréchet-space structure, and the pointwise exponential map $\exp\colon L\mathfrak{g} \to LG$ is locally surjective onto a neighbourhood of the identity. Via left-translation, this endows $LG$ with the structure of a Fréchet Lie group with Lie algebra $L\mathfrak{g}$.

[5] Generally, the geodesic/Hamiltonian flows on natural infinite-dimensional groups and coadjoint orbits give rise to well-known partial differential equations. A notable example are the Euler equations, which can be seen as the geodesic flow on the group of volume-preserving diffeomorphisms [135].

One may check that this bracket still defines the Jacobi identity[6]. Note that the scalar term on the right hand side does not depend on $k, l$. Consequently, the added element is *central*: $[(0,1),(Y,l)] = 0$ for all $(Y,l) \in \widehat{L\mathfrak{g}}$.

**Adjoint and coadjoint action.** The infinitesimal adjoint action (6.2.24) "integrates" to the following adjoint action of $g \in LG$ on $\widehat{L\mathfrak{g}}$:

$$\mathrm{Ad}_g(Y,l) = \Big(g(\tau)Y(\tau)g(\tau)^{-1},\ l - \int_{S^1} \mathrm{d}\tau\, \mathrm{tr}[g^{-1}\partial_\tau g Y]\Big). \tag{6.2.25}$$

From this we would like to derive the coadjoint action: The (smooth subspace[7] of the) dual Lie algebra $\widehat{L\mathfrak{g}}^*$ can be described as tuples $(A(\tau), c) \in C^\infty(S^1, \mathfrak{g}) \times \mathbb{R}$ with the paring

$$\langle (A(\tau),c),(X(\tau),k)\rangle = \mathrm{tr}\int_{S^1} A(\tau)X(\tau)\mathrm{d}\tau - ck. \tag{6.2.26}$$

From (6.2.25) and (6.2.26), one can see that the coadjoint action of $g(\tau) \in LG$ on $\widehat{L\mathfrak{g}}^*$ is given by

$$\boxed{\mathrm{Ad}^*_{g(\tau)}(A(\tau),c) = (A_{g,c}(\tau),c) := \big(g(\tau)A(\tau)g(\tau)^{-1} + c\partial_\tau g(\tau)g(\tau)^{-1},\, c\big)} \tag{6.2.27}$$

In other words, $c^{-1}A(\tau)$, transforms like a $G$-connection (i.e. a gauge field):

$$g(\tau)[-c\partial_\tau + A(\tau)]g(\tau)^{-1} = -c\partial_\tau + A_{g,c}(\tau). \tag{6.2.28}$$

**Monodromy and classification of coadjoint orbits.** The interpretation of the elements of the coadjoint orbit as connections(/gauge fields) suggests to consider their *monodromy*: Consider the parallel transport

$$\big[-c\partial_\tau + A(\tau)\big]\psi(\tau) = 0 \quad \text{for } \psi: \mathbb{R} \to G \text{ with } \psi(0) = \mathbb{1}. \tag{6.2.29}$$

Define the *monodromy* $M_{(A,c)} := M_\psi = \psi(0)^{-1}\psi(1) \in G$.

> **Lemma 6.2.5:** Suppose $(A,c) \in \widehat{L\mathfrak{g}}^*$ for $c \neq 0$. The conjugacy class $\mathrm{Ad}_G M_{(A,c)}$ of its monodromy is invariant under the coadjoint action.

[6] This is equivalent to the statement that $\eta\colon L\mathfrak{g}\times L\mathfrak{g} \to \mathbb{R}, \eta(X,Y) := \int \mathrm{tr}[Y(\tau)X'(\tau)]$ is a Lie algebra 2-cocycle: $\eta([X,Y],Z) + \eta([Z,X],Y) + \omega([Y,Z],X) = 0$.

[7] The topological dual consists of $\mathfrak{g}$-valued Schwartz distributions. For our purposes, we implicitly restrict to the regular dense subspace spanned by smooth functions.

*Proof.* We consider $\psi(\tau)$ as in (6.2.29). Dropping the condition $\psi(0) = \mathbb{1}$, any other solution to (6.2.29) is of the form $\tilde{\psi}(\tau) = \psi(\tau)h$ for $h \in G$. The monodromy $\tilde{\psi}(0)^{-1}\tilde{\psi}(1) = h^{-1}\psi(0)^{-1}\psi(1)h$ is conjugate to $M_{(A,c)}$. Hence, for determining the conjugacy class of the monodromy, it doesn't matter which solution to the parallel transport equations we consider. Now, recall the transformation behaviour (6.2.28) of the covariant derivative. For any $g \in LG$, we have that $\psi_g(\tau) := g(\tau)\psi(\tau)$ is a solution to $[-c\partial_\tau + A_{g,c}]\psi_g = 0$ with monodromy $\psi(0)^{-1}g(0)^{-1}g(1)\psi(1) = \psi(0)^{-1}\psi(1)$. Hence $M_{(A_{g,c},c)}$ and $M_{(A,c)}$ are conjugate (with respect to $\psi_g(0) = g(0)$). □

In short, the conjugacy class of the monodromy is an invariant of the coadjoint orbits. On the other hand, suppose that two elements $(A,c),(B,c) \in \widehat{L\mathfrak{g}}^*$ give rise to the same conjugacy class. In other words, there exist solutions $\psi, \varphi$

$$[-c\partial_\tau + A]\psi = 0 \quad \text{and} \quad [-c\partial_\tau + B]\varphi = 0, \tag{6.2.30}$$

and without loss of generality $M_\psi = M_\varphi$ (otherwise just translate by a constant element). Then $g(\tau) := \varphi(\tau)\psi(\tau)^{-1}$ is periodic and smooth. Hence it is an element of $LG$ such that $\psi_g = \varphi$ and $A_g = B$. We arrive at the following:

> **Lemma 6.2.6:** Fix $c \neq 0$, then the map $\mathcal{O}_{(A,c)} \mapsto \mathrm{Ad}_G M_{(A,c)}$ is an injective map from the set of coadjoint orbits of $\widehat{LG}$ into the conjugacy classes of $G$.

*Proof of Theorem 6.2.4.* By Lemma 6.2.6, it suffices to ask the "classical" question which group elements $g$ can be obtained as monodromies of (6.2.29). For $G$ compact and connected, the exponential map is surjective, and consequently for every $g \in G$ there exists $X \in \mathfrak{g}$, such that $A(\tau) \equiv X$ has monodromy $g$. This completes the proof. □

**Remark 6.2.7:** One often assumes that $G$ is simply connected as well. While this is not essential for the classification, one should note that otherwise $LG$ is disconnected and consequently the orbits consists of several connected components.

**Remark 6.2.8:** In Theorem 6.2.4, compactness of $G$ was only required for the surjectivity of the exponential map. In special cases, such as $G = \mathbb{R}$, this is clear despite lack of compactness. Hence, the same classification result holds true.

**Symplectic structure.** Recall that coadjoint orbits are naturally equipped with a symplectic form, see (6.2.21). On the coadjoint orbits of the loop group, the KKS-form is defined by

$$\begin{aligned}&\omega_{(A(\tau),c)}(\mathrm{ad}^*_{(X,0)}(A,c),\mathrm{ad}^*_{(Y,0)}(A,c))\\&=\int \mathrm{tr}\left(A(\tau)[X(\tau),Y(\tau)]_{\mathfrak{g}}\right)+\frac{c}{2}\int \mathrm{tr}\left(X(\tau)Y'(\tau)-X'(\tau)Y(\tau)\right).\end{aligned} \tag{6.2.31}$$

Still thinking of $G$ as a matrix group, we can formally consider the collection $(g(\tau))_{\tau\in S^1}$ as a family of coordinates for $LG$ (and therefore for a given orbit). Associated to the matrix coordinate $g(\tau)$, there is a differential form $\mathrm{d}g(\tau)$. In these coordinates, the Maurer-Cartan form is given by $g(\tau)^{-1}\,\mathrm{d}g(\tau)$. The KKS-form can be written as

$$\begin{aligned}\omega_{(A,c)}=&\frac{1}{2}\int \mathrm{d}\tau\,\mathrm{tr}\Big(A(\tau)\big[g(\tau)^{-1}\,\mathrm{d}g(\tau)\,,\,g(\tau)^{-1}\,\mathrm{d}g(\tau)\big]\Big)\\&+\frac{c}{2}\int \mathrm{d}\tau\,\mathrm{tr}\Big(g(\tau)^{-1}\,\mathrm{d}g(\tau)\wedge\big(g(\tau)^{-1}\,\mathrm{d}g(\tau)\big)'\Big),\end{aligned} \tag{6.2.32}$$

where the bracket in the first line is understood as the bracket between Lie-algebra valued one-forms.

**The example of $L\mathbb{R}$.** We write $\xi(\tau)\in L\mathbb{R}$ for an element of the loop group. The Lie algebra is $\mathrm{Lie}(L\mathbb{R})=L\mathbb{R}$. The coadjoint action of $\xi(\tau)$ on $(A(\tau),c)\in\widehat{L\mathbb{R}^*}$ is

$$\mathrm{Ad^*}_{\xi(\tau)}(A(\tau),c)=(A(\tau)+c\xi'(\tau),c). \tag{6.2.33}$$

Consequently, all coadjoint orbits are simply of the form $\mathcal{O}_{a_0}=\{(a_0+c\xi'(\tau),c):\xi\in L\mathbb{R}\}$ for a constant $a_0$. For the KKS-form, the first term in (6.2.32) vanishes (as $\mathbb{R}$ is abelian). And $g(\tau)^{-1}\,\mathrm{d}g(\tau)=\mathrm{d}\xi(\tau)$. Consequently

$$\omega=\frac{c}{2}\int \mathrm{d}\tau\,\mathrm{d}\xi(\tau)\wedge\mathrm{d}\xi'(\tau). \tag{6.2.34}$$

### 6.2.4 Virasoro group and its coadjoint action

In the following consider the group $\mathrm{Diff}(S^1)$ of (orientation-preserving) smooth reparametrisations of the circle. The (unique) central extension of this group is the so-called *Virasoro group* Vir, whose Lie algebra is the *Virasoro algebra* $\mathfrak{vir}$. We are interested in the coadjoint action of $\mathrm{Diff}(S^1)$ on the (smooth) dual of the Virasoro algebra $\mathfrak{vir}^*$. Before giving more details, we give a short overview of the main result: The smooth dual $\mathfrak{vir}^*$ of the Virasoro algebra will be identified with *Hill's operators* $c\partial_\tau^2 + \frac{1}{2}b(\tau)$, with $c \in \mathbb{R}$ and $b \in C^\infty(S^1, \mathbb{R})$. The coadjoint action of $\varphi^{-1} \in \mathrm{Diff}(S^1)$ leaves $c$ invariant and acts on the potential via $b \mapsto b_\varphi = \varphi'^2 b + c\mathcal{S}(\varphi)$. Note the structural similarity with the case of loop groups. In fact, perhaps somewhat surprisingly, the classification of coadjoint orbits of the Virasoro group closely resembles that of loop groups (Theorem 6.2.4) for the case $G = \mathrm{PSL}(2,\mathbb{R})$. To be precise, we need to consider the universal covering group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$, which will be discussed in Section 6.2.5.

> **Theorem 6.2.9** ([135, 137]): For $c = 1$, there is a one-to-one correspondence between the coadjoint orbits of the Virasoro-Bott group and the $\mathrm{GL}(2,\mathbb{R})$-conjugacy classes of $\widetilde{\mathrm{SL}}(2,\mathbb{R})\backslash\{\mathrm{id}\}$. Any orbit $\varphi \mapsto (b_\varphi, c)$ is of the form $b_\varphi(\tau) = c\mathcal{S}(\eta\circ\varphi, \tau)$ for some smooth $\eta\colon \mathbb{R} \to \mathbb{R}P^1$ with $\eta' > 0$. Moreover, $\eta(\tau+1) = \mathcal{M}\eta(\tau)$ for some *monodromy* $\mathcal{M} \in \mathrm{PSL}(2,\mathbb{R})$ acting via Möbius transformations, and this monodromy is a representative of the (projection onto $\mathrm{PSL}(2,\mathbb{R})$ of the) conjugacy class.

By $\mathrm{GL}(2,\mathbb{R})$-conjugacy class we mean there exists an action of $\mathrm{GL}(2,\mathbb{R})$ on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, induced by conjugation on $\mathrm{SL}(2,\mathbb{R})$, and the conjugacy classes are the orbits of this action. Note that these are "almost" $\mathrm{PSL}(2,\mathbb{R})$-conjugacy classes: Conjugation with $\mathrm{GL}(2,\mathbb{R})$ factors through $\mathrm{PGL}(2,\mathbb{R}) = \mathrm{GL}(2,\mathbb{R})/\mathbb{R}^\times \cong \mathrm{PSL}(2,\mathbb{R})\times\mathbb{Z}^2$. In fact, any element from $\mathrm{PGL}(2,\mathbb{R})$ is a product of an element from $\mathrm{PSL}(2,\mathbb{R})$ and $\{\mathbb{1}, \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\} \cong \mathbb{Z}_2$.

**Remark 6.2.10** (Orbits of constant potentials): Of particular interest for the Schwarzian measures are the orbits of constant potential $b^{(\alpha)} \equiv 2\alpha^2$ for $\alpha \in \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}$ (we usually write $\alpha^2 \in \mathbb{R}$). In this case the orbit $\mathcal{O}_\alpha := \mathcal{O}_{b^{(\alpha)}}$ is given by (see Proposition 6.2.2 and use the composition rule (6.2.19))

$$\boxed{b_\varphi^{(\alpha)}(\tau) = c\mathcal{S}(\tan(\alpha\varphi), \tau) = c[\mathcal{S}(\varphi,\tau) + 2\alpha^2\varphi'^2(\tau)].} \tag{6.2.35}$$

The corresponding monodromy ($\eta = \frac{1}{\alpha}\tan(\alpha\varphi)$) is

$$\mathcal{M}_\alpha = \begin{pmatrix} \cos(\alpha) & \frac{1}{\alpha}\sin(\alpha) \\ -\alpha\sin(\alpha) & \cos(\alpha) \end{pmatrix}. \tag{6.2.36}$$

Following the classification of $\mathrm{GL}(2,\mathbb{R})$-conjugacy classes of $\mathrm{PSL}(2,\mathbb{R})$ (see Proposition 6.2.16), we refer to *elliptic*, *hyperbolic* and *parabolic* orbits/monodromies respectively:

$$\mathcal{M}_\alpha = \begin{pmatrix} \cos(\alpha) & \frac{1}{\alpha}\sin(\alpha) \\ -\alpha\sin(\alpha) & \cos(\alpha) \end{pmatrix}, \qquad \mathcal{M}_{i\lambda} = \begin{pmatrix} \cosh(\lambda\tau) & \frac{1}{\lambda}\sinh(\lambda\tau) \\ \lambda\sinh(\lambda\tau) & \cosh(\lambda\tau) \end{pmatrix} \qquad \mathcal{M}_0 = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \tag{6.2.37}$$

where $\alpha \in \mathbb{R}_+\backslash\pi\mathbb{N}$, $\lambda \in \mathbb{R}_+$. For $\alpha \in \pi\mathbb{N}$ we have $\mathcal{M}_{k\pi} = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, which are referred to as *exceptional* orbits/monodromies.

**Remark 6.2.11** (Schwarzian measures as orbital integrals of $\mathcal{O}_\alpha$): In the beginning of Section 6.2.2 we discussed how coadjoint orbits are naturally equipped with a symplectic structure and an associated moment map. In the infinite-dimensional setup, these things are more subtle[8]. In the following, we ignore any such details and hope that the finite-dimensional setting generalises appropriately. In (6.2.49) we will see that $\mathcal{O}_\alpha$ formally carries a symplectic structure

$$\boxed{\omega_{b_\varphi^{(\alpha)}} = c\int \mathrm{d}\tau\left[\tfrac{1}{2}\mathrm{d}\log(\varphi'(\tau))\wedge\mathrm{d}\log(\varphi'(\tau))' - 2\alpha^2\mathrm{d}\varphi(\tau)\wedge\mathrm{d}\varphi'(\tau)\right]}. \tag{6.2.38}$$

Here one understands $\varphi(\tau)\in\mathbb{R}$ and $\log(\varphi'(\tau))\in\mathbb{R}$ as coordinates on the orbit, with $\mathrm{d}\varphi(\tau)$ and $\mathrm{d}\log(\varphi'(\tau))$ as corresponding 1-forms. With respect to the symplectic structure, $\langle b_\varphi^{(\alpha)},\cdot\rangle : \mathcal{O}_\alpha \mapsto \mathfrak{vec}(S^1)^*$ is the moment map for the $\mathrm{Diff}(S^1)$-action. In particular, pairing with the constant vector field $v(\tau)\cong 1\in\mathfrak{vec}(S^1)$, we obtain the Hamiltonian generating the $U(1)$-action $\varphi\mapsto\varphi(\cdot+\tau_0)$

$$\boxed{H^{(\alpha)}(\varphi) = \int \mathrm{d}\tau\, b_\varphi^{(\alpha)}(\tau) = -c\int \mathrm{d}\tau\,\mathcal{S}(\tan(\alpha\varphi),\tau)}. \tag{6.2.39}$$

This is the action of the Schwarzian measure (2.2.3). For $\alpha^2\le\pi^2$, the action has a unique minimiser $\varphi_0 = \mathrm{id}_{S^1}$ with value $H^{(\alpha)}(\mathrm{id}_{S^1}) = -2c\alpha^2$. A formal application of the Duistermaat-Heckman formula (see [17] for more details) yields

$$\int_{\mathcal{O}_\alpha} e^{-H^{(\alpha)}(\varphi)}\mathrm{dvol}_{\mathcal{O}_\alpha}(\varphi) \quad \text{“}\propto\text{”} \quad (2\pi c)^{\frac{1}{2}\dim\mathrm{Stab}(b^{(\alpha)})} e^{2c\alpha^2}, \tag{6.2.40}$$

where $\dim\mathrm{Stab}(b^\alpha)$ is equal to 3 for $\alpha=\pi$ and 1 for $\alpha^2<\pi^2$ and where the constant of proportionality may depend on $\alpha$ but not $c$.

**Virasoro algebra and (co)adjoint action.** The Lie algebra of $\mathrm{Diff}(S^1)$ is identified with the *Witt algebra* $\mathrm{diff}(S^1)=\mathfrak{vec}(S^1)$, the space of vector fields $v(\tau)\partial_\tau$ on the circle, equipped with the Lie bracket $[v_1,v_2]_{\mathfrak{vec}} = v_1v_2' - v_1'v_2$. The Witt algebra has a unique central extension, the Virasoro algebra $\mathfrak{vir}=\mathfrak{vec}(S^1)\oplus\mathbb{R}$ with Lie bracket defined by

$$[(v,a),(w,b)] = \left([v,w]_{\mathfrak{vec}},\ \frac{1}{2}\int \mathrm{d}\tau[v'''w - vw''']\right). \tag{6.2.41}$$

[8] In certain cases, in particular for the exceptional orbit $\mathcal{O}_\pi$ which is relevant to the Schwarzian field theory, one can understand the orbits (resp. a completion thereof) as a proper infinite-dimensional Hilbert-manifold. In that case, the symplectic form can be understood in a “classical” sense as a 2-form over this manifold, see [138, 139] for details.

This "integrates" to the following adjoint action of $\mathrm{Diff}(S^1)$ on $\mathfrak{vir}$:

$$\mathrm{Ad}_{\varphi^{-1}}(v(\tau),a) = \Big(\frac{v(\varphi(\tau))}{\varphi'(\tau)},\ a + \int \mathrm{d}\tau \frac{v(\varphi(\tau))}{\varphi'(\tau)} \mathcal{S}(\varphi,\tau)\Big) \qquad \text{for } \varphi \in \mathrm{Diff}(S^1), \tag{6.2.42}$$

with the Schwarzian derivative $\mathcal{S}(\varphi,\tau) = \log(\varphi')'' - \frac{1}{2}\log(\varphi')'^2$ For reasons of notational convenience we expressed the action in terms of the inverse element $\varphi^{-1}$. To obtain (6.2.41) from (6.2.42), one may note $\mathcal{S}(\tau - \epsilon w(\tau), \tau) = -\epsilon w'''(\tau) + O(\epsilon^2)$. Note that (6.2.42) has a clear geometric meaning in that $v$ simply transforms like a vector field under reparametrisation. The regular subspace of the dual $\mathfrak{vir}^*$ can be identified with pairs $(b(\tau),c) \in C^\infty(S^1)\times\mathbb{R}$, together with the pairing

$$\langle (b,c),(v,a)\rangle = -ca + \int \mathrm{d}\tau\, b(\tau)v(\tau). \tag{6.2.43}$$

The minus sign above is chosen purely for notational convenience. By (6.2.42) and (6.2.43) one deduces the coadjoint action

$$\boxed{\mathrm{Ad}^*_{\varphi^{-1}}(b,c) = (b_\varphi,c) \quad \text{with} \quad b_\varphi(\tau) = b_{\varphi,c}(\tau) \coloneqq \varphi'^2(\tau)b(\tau) + c\mathcal{S}(\varphi,\tau)} \tag{6.2.44}$$

We will often drop the $c$-dependence in our notation, as it is left invariant under the coadjoint action. For $c = 0$ the formula (6.2.44) has a geometric interpretation in that $b$ transforms like a 2-tensor density $b(\tau)(\mathrm{d}\tau)^2$. For general $c \in \mathbb{R}$, this transformation behaviour is that of a stress-energy tensor in a (chiral two-dimensional) CFT with central charge $12c$ (for notational convenience we departed from physics conventions for the normalisation of the central charge). We see that for different values of $c \neq 0$, the orbits of (6.2.44) can be related by rescaling. Consequently, to classify the coadjoint orbits it is sufficient to consider $c = 0$ and $c = 1$. The former case is not particularly relevant to us, however, and we refer to [15] for a short treatment.

**Symplectic structure.** The action of an infinitesimal diffeomorphism $\varphi(\tau) = \tau + \epsilon v(\tau) + O(\epsilon^2)$, respectively its inverse $\varphi^{-1}(\tau) = \tau - \epsilon v(\tau) + O(\epsilon^2)$ follows from (6.2.44):

$$[\mathrm{ad}^*_{-v} b](\tau) = v(\tau)b'(\tau) + 2v'(\tau)b(\tau) - cv'''(\tau). \tag{6.2.45}$$

By (6.2.21) and (6.2.41) the KKS form is given by

$$\begin{aligned}\omega_{(b,c)}\Big(\mathrm{ad}^*_{-v}(b,c),\mathrm{ad}^*_{-w}(b,c)\Big) &= \langle (b,c),\big([v,w]_{\mathfrak{vec}},\ \frac{1}{2}\int \mathrm{d}\tau[v'''w - vw''']\big)\rangle \\ &= \int \mathrm{d}\tau\Big[b(\tau)(vw'-v'w)+\frac{c}{2}(v'''w-vw''')\Big] \\ &= \int \mathrm{d}\tau\Big[b(\tau)(vw'-v'w)+\frac{c}{2}(v'w''-v''w')\Big]\end{aligned} \tag{6.2.46}$$

We can also write it more globally: Formally consider the collection $(\varphi(\tau))_{\tau\in S^1}$ as a family of coordinates for $\varphi\in \mathrm{Diff}(S^1)$ (and therefore for the orbit). Associated to these coordinates, there are 1-forms $d\varphi(\tau)$. Then $\mathrm{d}\varphi(\tau)/\varphi'(\tau)$ acts as (a coordinate of) the Maurer-Cartan form for the $\mathrm{Diff}(S^1)$-action: The action of $v(\tau)\in\mathfrak{vec}(S^1)$ on $\varphi$ is via the infinitesimal flow $(L_{\varphi *}v)(\tau) = \partial_\epsilon|_{\epsilon=0}\varphi(\tau+\epsilon v(\tau)) = \varphi'(\tau)v(\tau)$, where $L_\varphi\colon \psi\mapsto\varphi\circ\psi$ denotes left-translation. Consequently, $\frac{1}{\varphi'(\tau)}\langle \mathrm{d}\varphi(\tau), L_{\varphi *}v\rangle = v(\tau)$, which is the defining property of the Maurer-Cartan form. Consequently, by (6.2.46), we can write

$$\omega_{b_\varphi} = \int \mathrm{d}\tau\left[b_\varphi(\tau)\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}\wedge\left(\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}\right)' + \frac{c}{2}\left(\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}\right)'\wedge\left(\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}\right)''\right] \tag{6.2.47}$$

We can further simplify this for the orbits of constant potentials, $b^{(\alpha)}_\varphi(\tau) = -c\mathcal{S}(\tan(\alpha\varphi),\tau) = -c[\mathcal{S}(\varphi,\tau)-2\alpha^2\varphi'^2(\tau)]$ for $\alpha^2\in\mathbb{R}$. In that case, using that $(\mathrm{d}\varphi(\tau))' = \mathrm{d}\varphi'(\tau)$ and $\mathrm{d}\varphi(\tau)\wedge\mathrm{d}\varphi(\tau) = 0$ we expand (and drop total derivatives) to obtain

$$\begin{aligned}&\int \mathrm{d}\tau\left(\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}\right)'\wedge\left(\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}\right)'' \\ &\quad= \int \mathrm{d}\tau\left[\frac{2\mathcal{S}(\varphi,\tau)}{\varphi'^2(\tau)}\mathrm{d}\varphi(\tau)\wedge\mathrm{d}\varphi'(\tau) + \frac{1}{\varphi'^2(\tau)}\mathrm{d}\varphi'(\tau)\wedge\mathrm{d}\varphi''(\tau)\right] \\ &\quad= \int \mathrm{d}\tau\left[\frac{2\mathcal{S}(\varphi,\tau)}{\varphi'^2(\tau)}\mathrm{d}\varphi(\tau)\wedge\mathrm{d}\varphi'(\tau) + \mathrm{d}\log(\varphi'(\tau))\wedge\mathrm{d}\log(\varphi'(\tau))'\right]\end{aligned} \tag{6.2.48}$$

Consequently, the $\mathcal{S}(\varphi)$-contributions from $b^{(\alpha)}_\varphi$ and (6.2.48) cancel and we obtain

$$\boxed{\omega_{b^{(\alpha)}_\varphi} = c\int \mathrm{d}\tau\Big[\mathrm{d}\log(\varphi'(\tau))\wedge\mathrm{d}\log(\varphi'(\tau))' - 2\alpha^2\mathrm{d}\varphi(\tau)\wedge\mathrm{d}\varphi'(\tau)\Big]}. \tag{6.2.49}$$

**Hill's operators and monodromy.** In the orbit theory of loop groups, we considered the first order operators $-c\partial_\tau + A(\tau)$, as the coadjoint action on $A$ had a convenient interpretation in terms of the (adjoint) transformation behaviour of this operator and its solutions. Perhaps somewhat surprisingly, it turns out that a similar picture exists in the context of the Virasoro group. For $(b(\tau), c) \in \mathfrak{vir}^*$ we define the Hill's operator

$$H_{b,c} := c\partial_\tau^2 + \tfrac{1}{2}b(\tau). \tag{6.2.50}$$

We consider this as an operator acting on functions over $\mathbb{R}$, with $b$ extended periodically.

**Proposition 6.2.12:** Suppose $f\colon \mathbb{R} \to \mathbb{R}$ solves Hill's equation for some potential $b$: $H_{b,c} f = 0$. For $\varphi \in \mathrm{Diff}(S^1)$ we define

$$f_\varphi(\tau) := \varphi'(\tau)^{-1/2} f(\varphi(\tau)). \tag{6.2.51}$$

Then $f_\varphi$ solves Hill's equation for the potential $b_\varphi$:

$$H_{b_\varphi,c}\, f_\varphi = [c\partial_\tau^2 + \tfrac{1}{2}b_{\varphi,c}] f_\varphi = 0. \tag{6.2.52}$$

*Proof.* To start off, we note that

$$\partial_\tau^2[\varphi'(\tau)^{-1/2}] = -\tfrac{1}{2}S(\varphi,\tau)\,\varphi'(\tau)^{-1/2}. \tag{6.2.53}$$

We find that

$$\partial_\tau^2 f_\varphi(\tau) = \partial_\tau^2\big[\varphi'(\tau)^{-1/2} f(\varphi(\tau))\big] = -\tfrac{1}{2}S(\varphi,\tau) f_\varphi(\tau) + \varphi'(\tau)^{3/2} f''(\varphi(\tau)). \tag{6.2.54}$$

Consequently, using $b_{\varphi,c} - c\mathcal{S}(\varphi,\tau) = \varphi'(\tau)^2 b(\varphi(\tau))$

$$\begin{aligned}
[c\partial_\tau^2 + b_{\varphi,c}] f_\varphi &= \varphi'(\tau)^{3/2} c f''(\varphi(\tau)) + [\tfrac{1}{2}b_{\varphi,c} - \tfrac{c}{2}\mathcal{S}(\varphi,\tau)] f_\varphi(\tau) \\
&= \varphi'(\tau)^{3/2}\big([c\partial_\tau^2 + \tfrac{1}{2}b] f\big)(\varphi(\tau)) \\
&= 0
\end{aligned} \tag{6.2.55}$$

□

In other words, the coadjoint action on the potential agrees with the transformation behaviour of Hill's operators considered as maps from $-1/2$-densities into $3/2$-densities[9].

**Proposition 6.2.13:** For $(b,c) \in \mathfrak{vir}^*$ consider the Hill's operator $c\partial_\tau + \frac{1}{2}b(\tau)$. Let $\eta\colon \mathbb{R} \to \mathbb{R}P^1 \cong \mathbb{R} \cup \{\infty\}$ denote the projective solution, see Proposition 6.2.2. Under the coadjoint

[9] If $f(\tau)(\mathrm{d}\tau)^{-1/2}$ transforms as a $-1/2$-tensor density, then $\partial_\tau^2 f(\tau)(\mathrm{d}\tau)^{3/2}$ transforms as a $3/2$-tensor density.

action of $\varphi^{-1} \in \mathrm{Diff}(S^1)$, it transforms as $\eta \mapsto \eta \circ \varphi$. To be precise, $\eta \circ \varphi$ is a projective solution for the Hill's equation corresponding to $(b_{\varphi,c}, c)$, meaning that $\mathcal{S}(\eta \circ \varphi) = b_{\varphi,c}/c$.

*Proof.* Consider two $f, g$ to Hill's equation with $\eta = g/f$. By Proposition 6.2.12 these transform as $-1/2$-tensor densities. Consequently $\eta = g/f$ transforms as a 0-tensor density, in other words as $\eta \mapsto \eta \circ \varphi$. □

### 6.2.5 Classification of Virasoro coadjoint orbits

Topologically, $\mathrm{PSL}(2,\mathbb{R})$ is a solid torus and as such homotopic to $S^1$. The appearance of the universal covering can be explained by the difference between $\mathrm{Diff}(S^1)$ and $LS^1$. The latter contains maps $g(\tau) \in S^1$ that wind around $S^1$ arbitrarily often. Thinking about the parallel transport in (6.2.29), the map $\psi(\tau)$ naturally lifts to a path in the universal covering of $G = S^1$. Conjugation with $g(\tau)$ won't change the (conjugation class of the) monodromy, but it changes the lift. Now $\mathrm{Diff}(S^1)$ does not contain "multiply winding" maps and consequently cannot change the lift of the monodromy, which is why we need to control the full covering group.

Also, as opposed to the case of loop groups $LG$ with compact $G$, we don't have that any orbits contains a constant element. For $LG$ this was a consequence of the fact that any element in $G$ is a Lie algebra exponential, in other words lies in a 1-parameter subgroup. Segal claims [137] that the same is true for Virasoro coadjoint orbits: Orbits contains a constant element if and only if their conjugacy class lies in a 1-parameter subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

**On the universal covering group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.** We provide some intuition for the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. This is somewhat difficult, since it has no matrix representation. We can first understand it topologically and then algebraically. According to the Iwasawa decomposition, we can parametrise $A \in \mathrm{SL}(2,\mathbb{R})$ via

$$A(\theta, r, x) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r & \\ & 1/r \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \quad \text{with} \quad \theta \in [0, 2\pi), r > 0, x \in \mathbb{R}. \tag{6.2.56}$$

In other words $\mathrm{SL}(2,\mathbb{R})$ is diffeomorphic to $S^1 \times \mathbb{H}^2$ and has fundamental group $\pi_1(\mathrm{SL}(2,\mathbb{R})) \cong \pi_1(S^1) = \mathbb{Z}$. The universal covering group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ will be obtained by "unwinding" the $S^1$-variable into a real line $\mathbb{R}$. Formally, this will be obtained via a central extension by $\mathbb{Z}$. In the following we are following the construction by Rawnsley [140]. Suppose $\varphi\colon \mathrm{SL}(2,\mathbb{R}) \to S^1$ is a smooth function with $\varphi(\mathbb{1}) = 1$ and $\varphi(A^{-1}) = \varphi(A)^{-1}$, inducing an isomorphism of fundamental groups. Then the relation $\varphi(A_1A_2) = \varphi(A_1)\varphi(A_2)e^{i\eta(A_1,A_2)}$ defines a 2-cocycle $\eta\colon \mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$. In other words, we can define the set

$$\widetilde{G} \coloneqq \{(A,c) \in \mathrm{SL}(2,\mathbb{R})\colon \varphi(A) = e^{ic}\} \tag{6.2.57}$$

and equip it with the operation

$$(A_1,c_1)\cdot(A_2,c_2) \coloneqq (A_1A_2, c_1+c_2+\eta(A_1,A_2)). \tag{6.2.58}$$

In fact, $\widetilde{G}$ turns out to be a Lie group with identity element $(\mathbb{1},0)$, inverse $(A,c)^{-1} = (A^{-1},-c)$ and a surjective homomorphisms $(A,c) \mapsto A$ onto $\mathrm{SL}(2,\mathbb{R})$ with kernel $(\mathbb{1},2\pi\mathbb{Z})$. In other words $\widetilde{G} \cong \widetilde{\mathrm{SL}}(2,\mathbb{R})$. This group carries a natural $\mathbb{Z}^2$-action given by $(A,c) \mapsto ((A^{-1})^\top, c)$. This is the "outer automorphism" on $\mathrm{SL}(2,\mathbb{R})$ induced by conjugation with $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \notin \mathrm{SL}(2,\mathbb{R})$. To complete above construction, one needs to give concrete examples for $\varphi$ and $\eta$, for which (6.2.58) is manageable. We refer to [140] for details.

**Monodromy of Hill's operators and classification of orbits.** In Section 6.2.4 we discussed how the coadjoint action can be seen as acting on Hill's operators. In the following we introduce the notion of monodromy for those operators, and show that a picture similar to that for coadjoint orbits of loop groups in Section 6.2.3

Consider a potential $b$ with corresponding Hill's operator $H_{b,c} = H_b = \partial_\tau^2 + \frac{1}{2}b(\tau)$ (by rescaling we can focus on the case $c=1$). For any two solutions $H_b f = 0 = H_b g$ we can define $M(\tau) = \begin{pmatrix} f & g\\ f' & g'\end{pmatrix}$. Then, $M(\tau)$ satisfies the following parallel transport equation

$$\partial_\tau M(\tau) = \partial_\tau \begin{pmatrix} f & g\\ f' & g'\end{pmatrix} = \begin{pmatrix} 0 & 1\\ -\frac{1}{2}b(\tau) & 0\end{pmatrix}\begin{pmatrix} f & g\\ f' & g'\end{pmatrix} \eqqcolon A(\tau)M(\tau). \tag{6.2.59}$$

Note that $A(\tau) \in \mathfrak{sl}(2,\mathbb{R})$, namely $\operatorname{tr} A(\tau) = 0$. Choosing, say, $M(0) = \mathbb{1}$, this defines a path $M(\tau) \in \mathrm{SL}(2,\mathbb{R})$. We can lift this path $\widetilde{M}(\tau)$ to the universal covering $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Define $M_b = M(1)$ (and $\widetilde{M}_b = \widetilde{M}(1)$) as the *monodromy* (resp. *lifted monodromy*) of the Hill's operator $H_b$.

By periodicity of the potential $b(\tau)$, we have that $M(\tau+1) = M(\tau)M(1) = M(\tau)M_b$. In particular $(f(\tau+1), g(\tau+1)) = (f(\tau), g(\tau))M_b$. For the projective solution $\eta = g/f \in \mathbb{R}P^1$ we similarly have $\eta(\tau+1) = \eta(\tau)M_b$, with the right action via Möbius transformations. Note that due to sharp 3-transitivity of $\mathrm{PSL}(2,\mathbb{R})$ acting on $\mathbb{R}P^1$ this uniquely characterises $\pm M_b \in \mathrm{PSL}(2,\mathbb{R})$.

Note that given $\eta$ and an assumption on the Wronksian (which we set to 1), we can reconstruct $f, g$: We have $\eta' = 1/f^2$. Consequently, $\eta(0), \eta'(0), \eta''(0)$ and the Wronskian determine the initial conditions for $f, g$. If $M(0) = \mathbb{1}$, then we call $f, g$ the *fundamental solutions* . The associated $\eta$ is a *fundamental projective solution*.

**Lemma 6.2.14:** For a diffeomorphism $\varphi \in \mathrm{Diff}(S^1)$, the monodromies $M_b$ and $M_{b_\varphi}$ are conjugate in $\mathrm{PSL}(2,\mathbb{R})$.

*Proof.* Let $\eta = g/f$ denote the fundamental projective solution associated to $H_b$. By Proposition 6.2.13 we have that $\eta \circ \varphi$ is a projective solution of $H_{b_\varphi}$. Then $\eta(\varphi(\tau+1)) = \eta(\varphi(\tau)+1) = \eta(\varphi(\tau))M_b$ However, $\eta \circ \varphi$ does not yet satisfy the initial conditions of a fundamental projective solution. By (sharp) 3-transitivity, there exists a unique $N_\varphi \in \mathrm{PSL}(2,\mathbb{R})$, such that $\eta_\varphi(\tau) := \eta(\varphi(\tau))N_\varphi$ is a fundamental projective solution at $\tau = 0$. Consequently

$$\eta_\varphi(\tau+1) = \eta(\varphi(\tau))M_b N_\varphi = \eta_\varphi(\tau)N_\varphi^{-1}M_b N_\varphi = \eta_\varphi(\tau)M_{b_\varphi} \tag{6.2.60}$$

In other words, $M_{b_\varphi} = N_\varphi^{-1}M_b N_\varphi$ as elements in $\mathrm{PSL}(2,\mathbb{R})$. □

**Lemma 6.2.15:** Suppose we have two Hill's operators $c\partial_\tau^2 + b_i(\tau)$ with $i = 0, 1$, such that the associated lifted monodromies $\widetilde{M}_i(1)$ in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ are the same. Then there exists a diffeomorphism $\varphi \in \mathrm{Diff}(S^1)$, such that $(b_1, c) = \mathrm{Ad}^*_{\varphi^{-1}}(b_0, c)$.

*Proof-sketch.* In the following, we sketch the argument following Ovsienko [141]. We refer to [135, 137] for different approaches. Since $\widetilde{M}_0(1) = \widetilde{M}_1(1)$, the paths $M_i(\tau)$ in $\mathrm{PSL}(2,\mathbb{R})$ are homotopic. In a first step one shows that there exists a smooth homotopy of Hill's operators $(\partial_\tau^2 + \frac{1}{2}b_s(\tau))_{s\in[0,1]}$ with associated monodromies $M_s(\tau)$, such that $(M_s(\tau))_{s\in[0,1]}$ is a homotopy between $M_0$ and $M_1$.

In a second step one shows that this homotopy can be generated by the $\mathrm{Diff}(S^1)$-action on Hill's operators. In fact, consider (6.2.45) for the infinitesimal coadjoint action, and solve the ODE $\partial_s b_s = \mathrm{ad}^*_{-v_s} b_s$. We can find a smoothly parametrised family of vector fields $v_s(\tau)$, such that the the flow $\partial_s f_s(\tau) = v_s(\tau)\partial_\tau f_s(\tau)$ integrates to a diffeomorphism $\varphi \in \mathrm{Diff}(S^1)$, i.e. $f_1 = f_0 \circ \varphi$. In particular, the infinitesimal coadjoint action (6.2.45) integrates to $(b_1, c) = \mathrm{Ad}^*_{\varphi^{-1}}(b_0, c)$. □

**Conjugacy classes of** $\mathrm{PSL}(2,\mathbb{R})$**.** For convenience we recall the classification of $\mathrm{PSL}(2,\mathbb{R})$-conjugacy classes:

**Proposition 6.2.16:** The $\mathrm{GL}(2,\mathbb{R})$-conjugacy classes of $\pm M \in \mathrm{PSL}(2,\mathbb{R})\backslash\{\pm\mathbb{1}\}$ are characterised by $|\mathrm{tr}(M)|$. The same is true for $\mathrm{PGL}(2,\mathbb{R})$-conjugacy classes.

Elements with $|\mathrm{tr}(M)| < 2$ are called *elliptic*, while ones with $|\mathrm{tr}(M)| > 2$ are *hyperbolic*. The case $|\mathrm{tr}(M)| = 2$ is referred to as *parabolic*. We will use the convention to explicitly exclude the case $M = \pm\mathbb{1}$, which we refer to as *exceptional*.

*Proof.* Note that for $M \in \mathrm{SL}(2,\mathbb{R})$, its conjugacy class in $\mathrm{GL}(2,\mathbb{C})$ is uniquely determined by its Jordan normal form. In fact, this is even true for conjugacy with respect to $\mathrm{GL}(2,\mathbb{R})$ since for any complex eigenvector, its real and complex parts are real eigenvectors. Moreover, conjugation with $\mathrm{GL}(2,\mathbb{R})$ factorises through $\mathrm{PGL}(2,\mathbb{R}) = \mathrm{GL}(2,\mathbb{R})/\mathbb{R}^\times$ Finally, note that the characteristic polynomial of $M \in \mathrm{SL}(2,\mathbb{R})$ is given by $\lambda^2 - \mathrm{tr}(M)\lambda + 1$. In other words, the trace fully characterises the eigenvalues. If $|\mathrm{tr}(M)| \neq 2$, these are non-degenerate and hence $M$ is diagonalisable and its Jordan normal form is determined. If $|\mathrm{tr}(M)| = 2$, then the Jordan normal form is $\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)$ for $x \in \{0,1\}$, however we explicitly excluded the identity, so the Jordan normal form is also fixed. This completes the classification of conjugacy classes. □

### 6.2.6 Mapping between Virasoro and loop group orbits

In the previous sections we considered the orbit theory for loop groups and the Virasoro group. Here we would like to show that hyperbolic orbits of the Virasoro group are "isomorphic" (as symplectic manifolds with an $S^1$-flow generating Hamiltonian) to coadjoint orbits of $\widehat{L\mathbb{R}}$. Orbital integrals for the latter can be identified with the Wiener measure. Consequently, hyperbolic Virasoro group orbits (resp. the associated orbital path integral) are "just Brownian bridges". This is referred to as "bosonisation" of these orbits and was first demonstrates by Alekseev and Shatashvili [142].

We consider the coadjoint orbits of the (central extension of the) loop group $L\mathbb{R}$. We denote the centrally extended loop algebra by $\widehat{L\mathfrak{r}}^*$, where $\mathfrak{r} = \mathrm{Lie}(\mathbb{R}) \cong \mathbb{R}$. For simplicity we restrict to $c = 1$. For simplicity, we fix the central charge $c = 1$ and drop it from our notation. The coadjoint orbits are parameterised by a real parameter $a_0 \in \mathbb{R}$:

$$\mathcal{O}_{a_0} = \{A_\xi := a_0 + \xi'(\tau) : \xi \in C^\infty(S^1,\mathbb{R})\}. \tag{6.2.61}$$

According to (6.2.34), the symplectic form is formally expressed as

$$\omega_{A_\xi} := \frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}\xi(\tau) \wedge \mathrm{d}\xi'(\tau). \tag{6.2.62}$$

Moreover, the $S^1$-action $\xi' \mapsto \xi'(\cdot + \tau_0)$ is generated by the Hamiltonian

$$H(A_\xi) := \frac{1}{2}\int A_\xi^2 \mathrm{d}\tau = \frac{1}{2}\int (a_0^2 + \xi'^2)\mathrm{d}\tau. \tag{6.2.63}$$

Indeed, let $X_H$ denote the vector field generating the $S^1$-action, that is $X_H F[\xi] = \partial_\epsilon|_{\epsilon=0} F[\xi + \epsilon\xi'(\tau)]$. In particular $\langle \mathrm{d}\xi(\tau), X_H\rangle = \xi'(\tau)$. Then we have that

$$\mathrm{d}H(A_\xi) = \int \mathrm{d}\tau \xi' \mathrm{d}\xi' = \omega_{A_\xi}(X_{S^1}, \cdot), \tag{6.2.64}$$

meaning that the Hamiltonian flow generated by $H$ agrees with the $S^1$-action.

However, identifying $\mathrm{diff}(S^1) \cong L\mathfrak{r}$ , there is a natural action of $\mathrm{Diff}(S^1)$ on $\widehat{L\mathfrak{r}}^*$ with orbits given by

$$\widetilde{O}_{a_0} := \{\tilde{a}_\varphi := a_0\varphi'(\tau) + (\log\varphi'(\tau))' : \varphi \in \mathrm{Diff}(S^1)\}. \tag{6.2.65}$$

In fact, we have $O_{a_0} = \widetilde{O}_{a_0}$, in fact this is just another parametrisation of the same loop group orbit. The symplectic structure on these Virasoro orbits is

$$\tilde{\omega}_{a_\varphi} = \frac{1}{2}\int [a_0^2\delta\varphi \wedge \delta\varphi' + \delta\log\varphi' \wedge \delta\log(\varphi')']\mathrm{d}\tau, \tag{6.2.66}$$

which agrees with the Virasoro KKS-form (6.2.49) for $2\alpha = ia_0$. Moreover, the $S^1$-action is generated by

$$\tilde{H}(a_\varphi) = \frac{1}{2}\int [a_0^2\varphi'^2 + (\log\varphi')'^2]\mathrm{d}\tau, \tag{6.2.67}$$

which is the moment map (6.2.39) for $2\alpha = ia_0$.

# Chapter 7

# Non-uniqueness of phase transitions for graphical representations of Ising [UNIQ]

**Abstract:** We consider the graphical representations of the Ising model on tree-like graphs. We construct a class of graphs on which the loop $\mathrm{O}(1)$ model and the single random current exhibit a non-unique phase transition with respect to the inverse temperature, highlighting the non-monotonicity of both models. It follows from the construction that there exist infinite graphs $\mathbb{G} \subseteq \mathbb{G}'$ such that the uniform even subgraph of $\mathbb{G}'$ percolates and the uniform even subgraph of $\mathbb{G}$ does not. We also show that on the wired $d$-regular tree, the phase transitions of the loop $\mathrm{O}(1)$, the single random current, and the random-cluster models are all unique and coincide.

## 7.1 Introduction

The Ising model needs no introduction as one of the cornerstones of statistical mechanics, and over the past 50 years its so-called graphical representations have become one of the main tools for its rigorous study [143–145]. Consequently, they are increasingly regarded as objects of independent study [146–151]. The most prominent of these is the *random-cluster model*[1] $\varphi_x$, introduced in [152] as an interpolation between Potts models. The *loop* $\mathrm{O}(1)$ *model* $\ell_x$, was introduced by Van der Waerden [153] as the high-temperature expansion of the Ising model. On finite graphs, $\ell_x$ can be defined as Bernoulli percolation $\mathbb{P}_p$ at parameter $p = \frac{x}{1+x}$ conditioned on being even (that is every vertex has even degree). The *random current* representation $\mathbf{P}_x$ was first introduced in [154] and given a useful probabilistic interpretation in [143]. While usually

[1] We only consider the case of cluster weight $q = 2$ in which case it is also referred to as the FK-Ising model.

considered as a multigraph, we will be concerned with its traced version (i.e. the induced simple graph), which can be defined as $\mathbf{P}_x = \ell_x \cup \mathbb{P}_{1-\sqrt{1-x^2}}$, where $\mu_1 \cup \mu_2$ is the distribution of $\omega_1 \cup \omega_2$ under $\mu_1 \otimes \mu_2$ ($\omega_i \sim \mu_i$). We also refer to $\mathbf{P}_x \cup \mathbf{P}_x$ as the *double random current*.
Similarly, we note that the random-cluster can be defined as $\varphi_x = \ell_x \cup \mathbb{P}_x$. As all models are obtained from the loop O(1) model via "sprinkling" with Bernoulli percolation, much of our analysis will focus on the former.
In this paper, we investigate a number of natural questions regarding the interplay between the graphical representations which, while by no means ground-breaking, offer some conceptual clarification which is at present not well-represented in the literature.

### 7.1.1 Results

In this paper, we prove non-uniqueness of the percolative phase transition of the (free) loop O(1) model $\ell^0_{x,\mathbb{G}}$. Here, the index $\mathbb{G}$ denotes the underlying graph and the superscript 0 denotes free boundary conditions. We also let $C_\infty$ denote the event that there exists an infinite cluster.

**Theorem 7.1.1:** There exists a graph $\mathbb{M}$ where $x \mapsto \ell^0_{x,\mathbb{M}}[C_\infty]$ is not monotone.

In Theorem 7.2.8 we prove the same result for the (traced, sourceless) single random current model $\mathbf{P}^0_x$. Next, consider the *uniform even subgraph* $\mathrm{UEG}_{\mathbb{G}}$, defined to be the uniform measure on even subgraphs of $\mathbb{G}$. This model is intimately related to the Ising model [150, 155, 156] and can be understood as a special case of $\ell^0_{x,G}$ for $x = 1$. Using Theorem 7.1.1, we prove that percolation of the uniform even subgraph is not monotone in the graph.

**Corollary 7.1.2:** There exist graphs $\mathbb{G}' \subset \mathbb{G}$ such that $\mathrm{UEG}_{\mathbb{G}}[C_\infty] = 0$ and $\mathrm{UEG}_{\mathbb{G}'}[C_\infty] = 1$.

In [150], it was proven that on the hypercubic lattice $\mathbb{Z}^d$, the regime of exponential decay for the loop $O(1)$ and random current model coincides with the high temperature phase of the Ising model. Here we establish the phase diagrams for the $d$-regular tree $\mathbb{T}^d$ as well as the graph obtained from the $d$-regular tree by replacing every edge with a cycle of length $2n$ (and glued through opposite points of the cycle), henceforth denoted $\mathtt{C}^d_n$. For a boundary condition $\xi \in \{0,1\}$, corresponding respectively to free and wired boundaries, we define the critical point of the loop $O(1)$ model via $x_c(\ell^\xi_{\mathbb{G}}) = \inf_{x\in[0,1]}\{\ell^\xi_{x,\mathbb{G}}[C_\infty] > 0\}$. Analogous definitions are used for the other models.

**Theorem 7.1.3:** For any $d \geq 2$ and $n \geq 1$, it holds that

$$x_c(\ell^1_{\mathbb{T}^d}) = x_c(\mathbf{P}^1_{\mathbb{T}^d}) = x_c(\mathbf{P}^1_{\mathbb{T}^d} \cup \mathbf{P}^1_{\mathbb{T}^d}) = x_c(\varphi^1_{\mathbb{T}^d})$$
$$x_c(\ell^0_{\mathbb{T}^d}) > x_c(\mathbf{P}^0_{\mathbb{T}^d}) > x_c(\mathbf{P}^0_{\mathbb{T}^d} \cup \mathbf{P}^0_{\mathbb{T}^d}) > x_c(\varphi^0_{\mathbb{T}^d}).$$

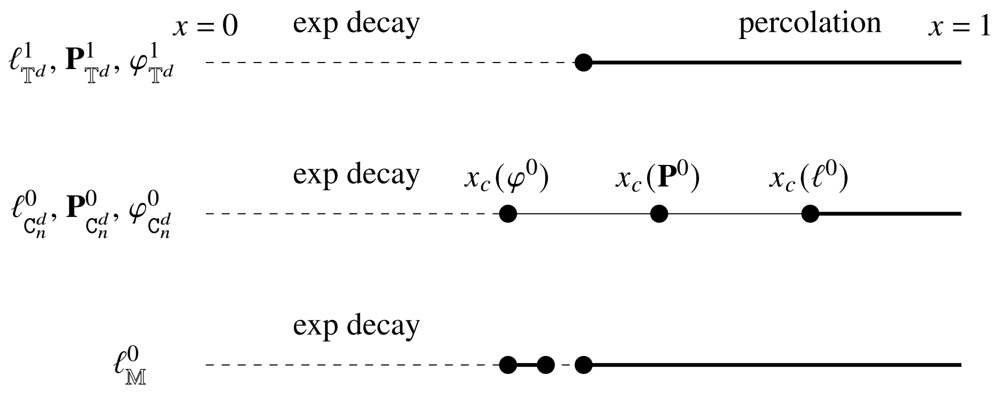



Figure 7.1: The phase diagrams of the loop O(1), single random current, and random-cluster measures on the $d$-regular wired tree $\mathbb{T}^d$ coincide. The free measures on $\mathtt{C}^d_n$, the $d$-regular tree where every edge is substituted by a cycle, have different phase transitions. Finally, the free loop O(1) model on the monster $\mathbb{M}$ (constructed in the proof of Theorem 7.1.1) has a non-unique phase transition. This is to be contrasted with the corresponding table for the hypercubic and hexagonal lattices in [150, Figure 1].

The same statements are true for the graph $\mathtt{C}^d_n$. In both cases, all phase transitions are unique.

This theorem is the most basic illustration of the mechanism first investigated in [157]: While it is surprising that the single random current, double random current and random-cluster model should share a single phase transition[2], this phenomenon ultimately boils down to the existence of long loops in the ambient graph. In the absence of loops (as in the free tree), the phase transitions should be distinct, and if there are only long loops (as in the wired tree), we expect them to be one and the same. This carries over to the situation where all loops are of uniformly bounded length (as for the free loops on $\mathtt{C}^d_n$).

In Figure 7.1, we provide a graphical overview of the results. Finally, we prove that the phase transitions of the uniform even subgraph and Bernoulli percolation are not in any way related:

**Theorem 7.1.4:** For any $\varepsilon > 0$, there exists a graph $\mathbb{G}^\varepsilon$ with $p_c(\mathbb{P}_{p,\mathbb{G}^\varepsilon}) \in (1-\varepsilon, 1)$ and $\mathrm{UEG}_{\mathbb{G}^\varepsilon}[C_\infty] = 1$.

### 7.1.2 The graphical representations of Ising

We define the random-cluster and random current model as in [158, 159] through the couplings to the loop O(1) model. Given a finite graph $G = (V, E)$, an *even subgraph* $(V, F)$ of $G$ is a spanning subgraph where each $v \in V$ is incident to an even number of edges in $F$. We let $\Omega_\emptyset(G)$ denote the set of even subgraphs of $G$. The loop O(1) model $\ell_{x,G}$ is a natural probability

[2] See the introduction in [150] for further explication.

measure on $\Omega_\emptyset(G)$:

$$\ell_{x,G}[\eta] = \frac{1}{Z_G} x^{|\eta|}, \text{ for each } \eta \in \Omega_\emptyset(G) \tag{7.1.1}$$

with $Z_G = \sum_{\eta \in \Omega_\emptyset(G)} x^{|\eta|}$. Here $|\eta|$ denotes the number of (open) edges in $\eta$ and $x = \tanh(\beta) \in (0,1)$ as in [160], where $\beta$ is the inverse temperature. For $G$ a graph with boundary, we denote $\ell^1_{x,G} = \ell_{x,G/\sim}$, where $\sim$ identifies the boundary vertices of $G$. We refer to $\ell^1_{x,G}$ as the *wired* loop O(1) model.

We denote Bernoulli edge percolation with parameter $x \in [0,1]$ by $\mathbb{P}_x$ and define the (traced, sourceless) *single random current* at parameter $x$ and boundary condition $\xi \in \{0,1\}$ as

$$\mathbf{P}^\xi_x = \ell^\xi_x \cup \mathbb{P}^\xi_{1-\sqrt{1-x^2}}, \tag{7.1.2}$$

where $\ell^0_x = \ell_x$. This definition of the model is equivalent to the standard definition due to a result by Lupu and Werner [161].

Similarly, we define the random-cluster model via

$$\varphi^\xi_x = \ell^\xi_x \cup \mathbb{P}^\xi_x, \tag{7.1.3}$$

which is equivalent to the standard definition of the model by a result due to Grimmet and Janson [156].

The random-cluster model satisfies several useful monotonicity properties which are not enjoyed by the loop O(1) and random current models [158]. We endow $\{0,1\}^E$ with the pointwise partial order $\preceq$, and say that an event $A$ is increasing if $\omega \in A$ and $\omega \preceq \omega'$ implies that $\omega' \in A$. The following monotonicity properties will be of use in this paper (see [145, Theorem 1.6]):

1. The FKG inequality: $\varphi_{x,G}[A \cap B] \geq \varphi_{x,G}[A]\varphi_{x,G}[B]$ for $A$ and $B$ increasing.
2. Monotonicity in boundary conditions: $\varphi^1_{x,G}[A] \geq \varphi^0_{x,G}[A]$ for $A$ increasing.
3. Stochastic monotonicity: $\varphi_{x_2,G}[A] \geq \varphi_{x_1,G}[A]$, whenever $x_1 < x_2$ and $A$ is increasing. We write $\varphi_{x_1,G} \preceq \varphi_{x_2,G}$.

The last property is equivalent to the existence of an increasing coupling - that is, a probability measure $\mu$ with marginals $\omega_1 \sim \varphi_{x_1}$ and $\omega_2 \sim \varphi_{x_2}$ such that $\omega_1 \preceq \omega_2$ almost surely (i.e. $\omega_1$ is a subgraph of $\omega_2$).

We refer to the lecture notes of Duminil-Copin [145] for an overall introduction to the Ising model and its graphical representations. Furthermore, we stick with the parametrisation in terms of the loop O(1) parameter $x \in [0, 1]$ throughout this paper and refer to [150, Table 1] for an overview of the standard parametrisations.

### 7.1.3 Graphical representations and uniform even subgraphs.

In the following, we will consider the uniform even subgraph UEG, which not only serves as the limiting case of the loop $O(1)$ model $\ell_x$ for $x = 1$, but also yields (perhaps surprising) connections between the different graphical representations. For a finite graph $G$, the uniform even subgraph $\mathrm{UEG}_G$ is a uniform element of $\Omega_\emptyset(G)$, the set of even subgraphs of $G$. In [150], an abstract view of the uniform even subgraph was taken as the Haar measure on the group of even graphs[3], and its percolative properties were studied. Before that, the uniform even subgraph and its infinite volume measures were studied in detail by Angel, Ray and Spinka in [155], where the free $\mathrm{UEG}^0$ and wired uniform even subgraphs $\mathrm{UEG}^1$ were introduced and it was shown that they coincide on one-ended graphs [155, Lemma 3.9]. In this article, we are concerned with tree-like graphs (as opposed to, say, $\mathbb{Z}^d$), which in general have infinitely many ends. Hence, the distinction between the free and wired measures, $\mathrm{UEG}^0$ and $\mathrm{UEG}^1$, plays a bigger role than in [150, 155].

For an infinite graph $\mathbb{G}$ the wired uniform even subgraph $\mathrm{UEG}^1_{\mathbb{G}}$ can be defined as the Haar measure (normalised to probability) on $\Omega_\emptyset$, the group of all even subgraphs of $\mathbb{G}$. In particular, it pushes forward to Haar measures under group homomorphisms and as a consequence, its marginals are also Haar measures on their supports.

The set of all finite even graphs $\Omega_\emptyset^{<\infty}(\mathbb{G}) = \{\eta \in \Omega_\emptyset(\mathbb{G}) \mid |\eta| < \infty\}$ is a subgroup of $\Omega_\emptyset$. Its closure

$$\Omega^0(\mathbb{G}) = \overline{\Omega_\emptyset^{<\infty}(\mathbb{G})}$$

is a (compact) subgroup of $\Omega_\emptyset(\mathbb{G})$, and the free uniform even subgraph $\mathrm{UEG}^0_{\mathbb{G}}$ is the Haar measure on that group. For more details on the construction of the free and wired uniform even subgraphs for infinite graphs see [150, Section 3.2].

In [156, Theorem 3.5] it was realised that the loop O(1) model arises as the uniform even subgraph of the random-cluster model and in [158, Theorem 4.1], that it is also the uniform even subgraph of the double random current.

Thus, on any graph, the loop $O(1)$ measure can be written as follows:

$$\ell^\xi_{\mathbb{G}}[\,\cdot\,] = \varphi^\xi_{\mathbb{G}}\Big[\mathrm{UEG}^\xi_\omega[\,\cdot\,]\Big], \tag{7.1.4}$$

[3] With the group operation given by pointwise addition mod 2 in the space $\{0, 1\}^E$ — or, equivalently, taking symmetric differences of edge sets.

where $\xi = 0$ in the free case and $\xi = 1$ in the wired case[4], and $\varphi^{\xi}_{\mathbb{G}}$ is defined as a thermodynamic limit (see e.g. [150, Sec. 2.1.3.]) when $\mathbb{G}$ is infinite. In infinite volume, this may be taken as the definition of the loop O(1) model (cf. [150, (4)]). This can then be used to define the single and double random current,

$$\mathbf{P}^{\xi}_{x} = \ell^{\xi}_{x} \cup \mathbb{P}_{1-\sqrt{1-x^2}} \quad \text{and} \quad \mathbf{P}^{\xi}_{x} \cup \mathbf{P}^{\xi}_{x} = \ell^{\xi}_{x} \cup \ell^{\xi}_{x} \cup \mathbb{P}_{x^2}. \tag{7.1.5}$$

### 7.1.4 Percolation regimes

Since the graphs we will work on are not vertex-transitive, we will use the following definition of percolation: We say that a percolation measure $\mu_{x,\mathbb{G}}$ on an infinite graph $\mathbb{G}$ *percolates* if $\mu_{x,\mathbb{G}}[C_\infty] > 0$ (recall that $C_\infty$ denotes the event that there exists an infinite cluster) and we define the *percolation regime*

$$\mathcal{P}(\mu_{x,\mathbb{G}}) = \{x \in (0,1) \mid \mu_{x,\mathbb{G}}[C_\infty] > 0\}. \tag{7.1.6}$$

We say that the phase transition on $\mathbb{G}$ is unique if both $\mathcal{P}(\mu_{x,\mathbb{G}})$ and $(0,1) \setminus \mathcal{P}(\mu_{x,\mathbb{G}})$ are connected. In that case, we define the critical parameter $x_c(\mu_{x,\mathbb{G}}) = \inf \mathcal{P}(\mu_{x,\mathbb{G}})$. By stochastic monotonicity, the phase transition of the random-cluster model $\varphi$ is unique on any graph.

#### 7.1.4.1 Bernoulli percolation on a tree.

We denote by $\mathbb{T}^d$ the $d$-regular tree and by $\mathbb{T}^d_n$ the ball of size $n$ for the graph distance on $\mathbb{T}^d$. Observing that the cluster of the origin can be described in terms of a Galton-Watson process (and with the observation that vertex-transitivity implies that $\mathbb{P}_{p,\mathbb{T}^d}[0 \leftrightarrow \infty] > 0$ if and only if $\mathbb{P}_{p,\mathbb{T}^d}[C_\infty] = 1$), one sees that the critical parameter for Bernoulli percolation on the $d$-regular tree is

$$p_c(\mathbb{P}_{\mathbb{T}^d}) = \frac{1}{d-1}. \tag{7.1.7}$$

#### 7.1.4.2 Percolation versus long-range order

For readers more familiar with models on lattices, a brief word of caution might be in order: One might wonder why models that essentially share correlation functions nonetheless have different critical parameters for percolation.

[4] For finite graphs we write $\ell$, omitting the boundary condition.

In particular, we have the following agreement of two-point functions (see e.g. [145, Corollary 1.4, Lemma 4.3]):

$$\varphi^0_{x,G}[v \leftrightarrow w]^2 = \langle \sigma_v \sigma_w \rangle^2_{G,\beta} = \mathbf{P}^0_{x,G} \cup \mathbf{P}^0_{x,G}[v \leftrightarrow w] \tag{7.1.8}$$

for all finite graphs $G$ and vertices $v, w$ (here $\langle \sigma_v \sigma_w \rangle_{G,\beta}$ is the Ising correlation function and $\beta = \operatorname{arctanh}(x)$). However, this is not an obstruction to percolation setting in at different values of $x$ because percolation does not, in general, imply anything for the two-point function. One is tempted to write that the bound $\varphi^0[v \leftrightarrow w] \geq \varphi^0[v \leftrightarrow \infty]\varphi^0[w \leftrightarrow \infty]$ follows from the FKG inequality, but this only holds if the infinite cluster is unique, which, as may be checked, is never true for trees.

## 7.2 Non-uniqueness of percolation

In this section we tackle the following question.

**Question 7.2.1:** Let $G \subseteq G'$ be two infinite graphs and suppose that the uniform even subgraph of $G'$ almost surely percolates. Does the uniform even subgraph of $G$ almost surely percolate?

The following counterexample answers the question negatively and at the same time constructs a class of graphs for which $\mathcal{P}(\ell_x)$ has multiple connected components.

We use that the loop $O(1)$ model factorises on graphs lacking certain cycles: A cycle denotes a path of vertices $v_1, v_2, \ldots, v_n$ such that $v_1 = v_n$. We say that a cycle is *simple* if $v_j \neq v_k$ for distinct $1 \leq j, k < n$.

**Definition 7.2.2:** For a graph $G$ and two subgraphs $G_1, G_2 \subseteq G$, we say that $(G_1, G_2)$ is a **cut-point factorisation** of $G$ if $E(G) = E(G_1) \dot\cup E(G_2)$ and it holds that there is no simple cycle in $G$ which contains edges from both $E(G_1)$ and $E(G_2)$.

**Definition 7.2.3:** We say that a graph-indexed family of percolation measures $\nu_G$ **cut-point factorises** if $\nu_G = \nu_{G_1} \otimes \nu_{G_2}$ whenever $(G_1, G_2)$ is a cut-point factorisation of $G$.

The following lemma will be useful.

**Lemma 7.2.4** (Cut point factorisation): Each of the measures $\ell_x, \varphi_x, \mathbf{P}_x, \mathbf{P}_x \cup \mathbf{P}_x$ cut-point factorises.

*Proof.* We first prove the statement for the loop O(1) model. For any even subgraph $\eta$ of $G$, its restrictions $\eta_1, \eta_2$ to the subgraphs $G_1$ and $G_2$ are even graphs since $(G_1, G_2)$ is a cut-point factorisation of $G$. Thus, writing $\Omega_i = \{0,1\}^{E_i}$, one may simply rewrite

$$\ell_{x,G}[\eta] = \frac{x^{|\eta|}}{\sum_{\eta' \in \Omega} 1_{\partial \eta' = \emptyset} x^{|\eta'|}} = \frac{x^{|\eta_1|} x^{|\eta_2|}}{\sum_{\eta_i' \in \Omega_i} 1_{\partial \eta_1' = \emptyset} 1_{\partial \eta_2' = \emptyset} x^{|\eta_1'|} x^{|\eta_2'|}} = \ell_{x,G_1}[\eta_1] \ell_{x,G_2}[\eta_2].$$

For the other models, the statement follows from the couplings of $\mathbf{P}, \mathbf{P} \cup \mathbf{P}$ and $\varphi$ to the loop O(1) model (which we used to define said models in Section 7.1.2). Indeed, note that for any two product measures, $\mu_{x,G} = \mu_{x,G_1} \otimes \mu_{x,G_2}$ and $\nu_{x,G} = \nu_{x,G_1} \otimes \nu_{x,G_2}$ that

$$\mu_{x,G} \cup \nu_{x,G} = (\mu_{x,G_1} \cup \nu_{x,G_1}) \otimes (\mu_{x,G_2} \cup \nu_{x,G_2}),$$

and Bernoulli percolation clearly cut-point factorises. □

In Theorem 7.2.4 we proved the cut-point factorisation property for finite graphs. The infinite volume measures also have the cut-point factorisation property as they are either limits or (sprinkled) uniform even subgraphs of a cut-point factorising measure.

**Lemma 7.2.5** (Non-monotonicity of loop $O(1)$ two-point function): There exist parameter values $0 < x_1 < x_2 < 1$, and a finite graph $G^\diamond$ with vertices $a, b$ such that

$$\ell_{x_2,G^\diamond}[a \leftrightarrow b] < \frac{1}{4} < \ell_{x_1,G^\diamond}[a \leftrightarrow b].$$

*Proof.* Consider the graph $G^\diamond$, described in the right-most column of Figure 7.2 (which was previously used as an example in [158]). In this graph, the probability that the two marked vertices are connected is

$$\ell_{x,G^\diamond}[a \leftrightarrow b] = \frac{x^{2m} + x^{2m+2n}}{1 + x^{2n} + x^{2m} + 4x^{n+m} + x^{2m+2n}}.$$

Setting $n = 12, m = 2, x_1 = 0.85$, and $x_2 = 0.965$ yields

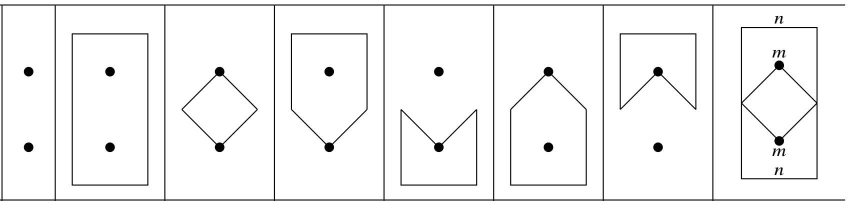
$$\ell_{x_2,G^\diamond}[a \leftrightarrow b] \leq 0.245 < \frac{1}{4}, \text{ and } \ell_{x_1,G^\diamond}[a \leftrightarrow b] \geq 0.27 > \frac{1}{4}.$$

We have also plotted the graph of $x \mapsto \ell_{x,G^\diamond}[a \leftrightarrow b]$ in Figure 7.3. □

With this in hand, we can construct our counterexample for Theorem 7.1.1: We note that the proof of the theorem also implies that the percolation regime $\mathcal{P}(\ell^0_{x,\mathbb{G}}) \subset [0,1]$ is not connected.

*Proof.* We construct $\mathbb{M}$ from the $d$-regular tree $\mathbb{T}^d$ with root 0 for an appropriate choice of $d$. Consider the natural orientation of the tree where edges are oriented away from the root and

Figure 7.2: The graph $G^\diamond$ (pictured to the right) along with its eight even subgraphs (including $G^\diamond$ itself). We let the outer paths be $n$ edges long and the inner paths be $m$ edges long. The nodes $a$ and $b$ are marked with dots. We list the number of edges of each subgraph, the corresponding weights and whether $a$ and $b$ are connected in the subgraph. (Sketch and text partially revised from [158].)




| Subgraph | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Edges | 0 | $2n$ | $2m$ | $n+m$ | $n+m$ | $n+m$ | $n+m$ | $2m+2n$ |
| Weight | 1 | $x^{2n}$ | $x^{2m}$ | $x^{n+m}$ | $x^{n+m}$ | $x^{n+m}$ | $x^{n+m}$ | $x^{2m+2n}$ |
| $\{a \leftrightarrow b\}$ | $\times$ | $\times$ | $\checkmark$ | $\times$ | $\times$ | $\times$ | $\times$ | $\checkmark$ |

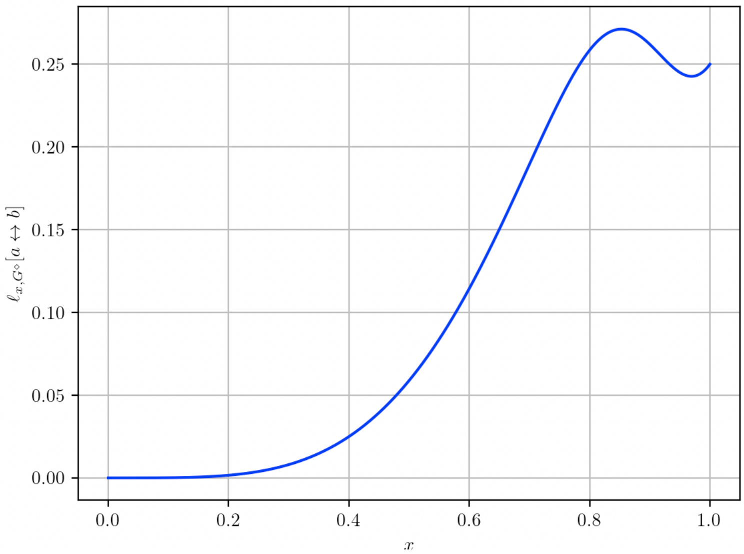



Figure 7.3: Graph of the connection probability for the loop O(1) model on the graph $G^\diamond$, described in Figure 7.2, for $n = 12$ and $m = 2$. See also [158, Figure 2.3] for similar figures.

replace every such oriented edge $e=(v,w)$ by a copy of the graph $G^\diamond$ where $a$ is identified with $v$ and $b$ is identified with $w$. More formally, let $\mathbb{M} = (\coprod_{e\in E(\mathbb{T}^d)} G_e^\diamond)/\sim$, where $\sim$ is the equivalence relation such that $a_e \sim b_{e'}$ whenever the source of $e$ is equal to the sink of $e'$. See Figure 7.4 for an illustration. Now, since the macroscopic structure of $\mathbb{M}$ is that of a tree, Lemma 7.2.4 applies, and it holds that

$$\ell^0_{x,\mathbb{M}} = \otimes_{e\in E(\mathbb{T}^d)} \ell^0_{x,G_e^\diamond}. \tag{7.2.1}$$

As a consequence, analysing percolation of $\ell_{x,\mathbb{M}}$ just boils down to Bernoulli percolation on $\mathbb{T}^d$: For given $\eta \in \Omega^0_\emptyset(\mathbb{M})$ and $e \in E(\mathbb{T}^d)$, we define $m_e = 1$ if $a \overset{\eta}{\leftrightarrow} b$ in $G_e^\diamond$ and $m_e = 0$ otherwise. In other words, $m = (m_e)_{e\in E(\mathbb{T}^d)}$ maps a percolation configuration in $\mathbb{M}$ onto one in $\mathbb{T}^d$ such that $0 \overset{m(\eta)}{\longleftrightarrow} \infty$ if and only if $0 \overset{\eta}{\longleftrightarrow} \infty$. Furthermore, by (7.2.1), the image measure of $\ell_{x,\mathbb{M}}$ is Bernoulli percolation: $m(\ell^0_{x,\mathbb{M}}) = \mathbb{P}_{f(x),\mathbb{T}^d}$ where $f(x) = \ell_{x,G^\diamond}[a \leftrightarrow b]$.
As a consequence, it holds that $\ell^0_{x,M}[C_\infty] = \mathbb{P}_{f(x),\mathbb{T}^d}[C_\infty]$. By (7.1.7), we know that $\mathbb{P}_{f(x),\mathbb{T}^d}[C_\infty] > 0$ if and only if $f(x) > \frac{1}{d+1}$. Now, by Lemma 7.2.5 there exist $x_1 < x_2$ and $d$ such that $f(x_2) < \frac{1}{d+1} < f(x_1)$, which proves the desired. □

As a result, we obtain a negative answer to Question 7.2.1, that is a proof of Corollary 7.1.2:

*Proof of Corollary 7.1.2 (in the free case).* First, recall that the loop O(1) model can be obtained by sampling a uniform even subgraph from a random-cluster configuration, see (7.1.4). For the random-cluster model $\varphi^0_{x,\mathbb{M}}$ we consider the increasing coupling in $x$. For $x_1 < x_2$ as above, the uniform even subgraph of $\varphi^0_{x_1,\mathbb{M}}$ almost surely percolates and the uniform even subgraph of $\varphi^0_{x_2,\mathbb{M}}$ almost surely does not. Since the coupling is increasing, there must exist at least one pair $\omega_1 \preceq \omega_2$ where the uniform even subgraph of $\omega_1$ percolates while that of $\omega_2$ does not. □

### 7.2.1 Corollary 7.1.2 in the wired case.

In this section, we will construct a supergraph[5] of $\mathbb{Z}$ for which the wired UEG does not percolate.
Define $\mathbb{G} = (\mathbb{Z}, \hat{\mathbb{E}})$, where $(n,m) \in \hat{E}$ if either $|n-m| = 1$ or $n = -m$. Let $\hat{\mathbb{E}}_{\text{arc}} := \hat{\mathbb{E}} \setminus \mathbb{E}(\mathbb{Z})$ and $\mathbb{G}_{\text{arc}} = (\mathbb{Z}, \hat{E}_{\text{arc}})$.

[5]One may note that our trick is basically to pick a one-ended supergraph of a two-ended graph. One may check that the wired UEG of a graph with multiple ends percolates (see also [150, Eq. (14)]). Therefore, our construction does not have the flavour of an optimal solution to the problem.

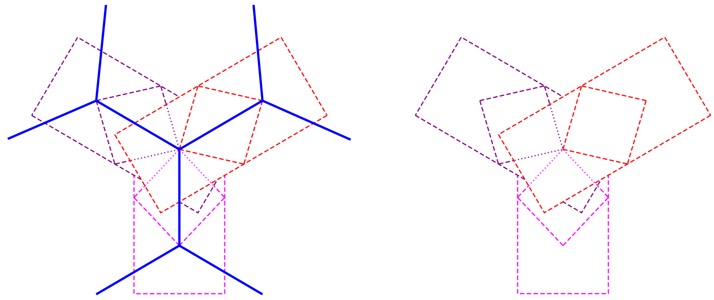


Figure 7.4: An example of the graph $\mathbb{M}$ built from the graphs $G^\diamond$ (see Figure 7.2) when $d = 2$. To the left with the $d$-regular tree overlaid. To the right, the geometry of $\mathbb{M}$ at a single vertex of the initial tree.

**Lemma 7.2.6:** The marginal $\mathrm{UEG}^1_{\mathbb{G}}|_{\hat{\mathbb{E}}_{\mathrm{arc}}} = \mathbb{P}_{\frac{1}{2},\mathbb{G}_{\mathrm{arc}}}$.

*Proof.* This follows from the fact that the restriction group homomorphism $\Omega_\emptyset(G) \to \{0,1\}^{\hat{\mathbb{E}}_{\mathrm{arc}}}$ is surjective. To see that the map is surjective, it suffices that all single-edge configurations lie in the image. For a given $e = (-n,n) \in \hat{\mathbb{E}}_{\mathrm{arc}}$, we see that $1\!\!1_e$ is the image of the loop containing $[-n,n] \cap \mathbb{Z}$ and $e$. □

*Proof of Corollary 7.1.2 (for the wired case).* Let $\mathbb{G}$ be as above. Let $v \in \mathbb{Z}$ and note that on the event that there is an open edge in $\hat{\mathbb{E}}_{\mathrm{arc}}$ outside of $[-|v|,|v|]$, the cluster of $v$ has to be finite. By Lemma 7.2.6, with probability 1, infinitely many edges in $\hat{\mathbb{E}}_{\mathrm{arc}}$ are open in $\mathrm{UEG}_{\mathbb{G}}$. By the previous comment, on this event, all clusters are finite. Hence $\mathrm{UEG}_{\mathbb{G}}$ does not percolate. However, $\mathbb{Z}$ has exactly two even subgraphs, $\eta \equiv 0$ and $\eta \equiv 1$, so $\mathrm{UEG}^1_{\mathbb{Z}}$ does percolate. □

### 7.2.2 Generalisations and non-uniqueness of random current phase transitions

We can adapt the earlier construction of a disconnected percolation regime from the loop $O(1)$ model to more general cut-point factorising measures:

**Proposition 7.2.7:** Let $F$ be a finite graph and that $v, w$ are two vertices. Let $\{\mu_{x,F}\}_{x\in[0,1]}$ be a family of cut point factorising percolation measures. Suppose that there exists a finite graph $F$ such that $x \mapsto \mu_{x,F}[v \leftrightarrow w]$ is not monotone. Then, there exists an infinite graph $\mathbb{M}$ such that $\mathcal{P}(\mu_{x,\mathbb{M}})$ is disconnected.

*Proof.* By assumption, there exist $x_1 < x_2 < x_3$ such that

$$\max\{\mu_{x_1,F}[v \leftrightarrow w], \mu_{x_3,F}[v \leftrightarrow w]\} < \mu_{x_2,F}[v \leftrightarrow w].$$

For $p \in (0,1)$ let $\mathbb{G}_p \sim \mathbb{P}_{p,\mathbb{T}^d}$. Notice that almost surely, $x_c(\mathbb{P}_{x,\mathbb{G}_p}) = \frac{1}{p(d-1)}$. By tuning the parameters $p$ and $d$ appropriately, we can make sure that

$$\max\{\mu_{x_1,F}[v \leftrightarrow w], \mu_{x_3,F}[v \leftrightarrow w]\} < \frac{1}{p(d-1)} < \mu_{x_2,F}[v \leftrightarrow w].$$

Now, construct $\mathbb{M}$ by sampling $\mathbb{G}_p$ and substituting each edge of $\mathbb{G}_p$ by a copy of $F$, gluing in the same way as in our construction in the proof of Theorem 7.1.1. □

The proposition shows that we do not need to "fine-tune" the parameters for the transition points to fit the phase transition of the $d$-regular tree.
As a consequence, we can see that the single random current also admits a disconnected percolation regime: Recall that [158, Figure 2.3] gives an example of a graph for which connection probabilities are not monotone, hence, by the above proposition, it follows that we get:

**Corollary 7.2.8:** There exists an infinite graph $\mathbb{G}$ such that $\mathcal{P}(\mathbf{P}_G)$ is not connected.

**Remark 7.2.9:** We note that since single site connection probabilities are monotone for $\varphi_x$ and $\mathbf{P}_x \cup \mathbf{P}_x$, the counterexamples do not work for these models.

Another model of statistical mechanics which cut-point factorises is the arboreal gas model. The simplest definition of this model in finite volume is as Bernoulli percolation conditioned to be a forest (i.e. conditioned not to contain any cycles):

$$\nu_{\beta,G}[\omega] = \frac{1}{Z_{G,\beta}} \beta^{|\omega|} 1\!\!1_{\Omega_\emptyset(\omega)=\{0\}},$$

where 0 denotes the empty graph, which is even. It is immediate that if $(G_1, G_2)$ is a cut-point factorisation of $G$ and $F_1, F_2$ are subforests of $G_1$ and $G_2$ respectively, then $F_1 \cup F_2$ is a subforest of $G$. Thus, $\nu_{\beta,G}$ cut-point factorises. At present, the connection probabilities of the model are conjectured to be monotone in $\beta$ on all finite graphs $G$ [162, p.2], but supposing that they are not, our construction would go through for this model as well, yielding a graph on which the percolation phase transition of the model is not unique.

## 7.3 Phase transitions of the wired models on the $d$-regular tree coincide

In this section, we prove the first part of Theorem 7.1.3 on trees. The main tool in the proof is the observation that all wired cycles in $\mathbb{T}^d$ are infinite. In particular, $\ell^1_{x,\mathbb{T}^d}$ is empty if it does not percolate.

First, we will need a small bookkeeping result. Let $\tilde{\mathbb{T}}^d$ denote the rooted $d$-regular tree. Cycles of $\mathbb{T}^d$ containing an open edge $e$ will consist of two infinite paths in isomorphic copies of $\tilde{\mathbb{T}}^d$. It will therefore be important (and easily demonstrated) that changing the degree of a single vertex does not change the phase transition.

**Lemma 7.3.1:** For any $d \geq 2$, we have that $x_c(\varphi^1_{\mathbb{T}^d}) = x_c(\varphi^1_{\tilde{\mathbb{T}}^d})$.

*Proof.* Fix one edge $e$ of $\mathbb{T}^d$ and note that for any $x \in (0,1)$,

$$\varphi^1_{x,\mathbb{T}^d}[C_\infty] = \varphi^1_{x,\mathbb{T}^d}[C_\infty |\, \omega_e = 1]\,\varphi^1_{x,\mathbb{T}^d}[\omega_e = 1] \;+\; \varphi^1_{x,\mathbb{T}^d}[C_\infty |\, \omega_e = 0]\,\varphi^1_{x,\mathbb{T}^d}[\omega_e = 0].$$

By tail-triviality (see [149, Theorem 10.67]), $\varphi^1_{x,\mathbb{T}^d}[C_\infty] \in \{0,1\}$ and since $\varphi^1_{x,\mathbb{T}^d}[\omega_e = 0] \in (0,1)$, we conclude that

$$\varphi^1_{x,\mathbb{T}^d}[C_\infty |\, \omega_e = 0] = \varphi^1_{x,\mathbb{T}^d}[C_\infty].$$

Furthermore, $(\mathbb{V}(\mathbb{T}^d), \mathbb{E}(\mathbb{T}^d) \setminus \{e\})$ has two connected components, both of which are isomorphic to $\tilde{\mathbb{T}}^d$. Permitting ourselves a natural abuse of notation, remark that by the Domain Markov Property [145, p.8], $\varphi^1_{x,\mathbb{T}^d}[\,\cdot\,|\, \omega_e = 0] = \varphi^1_{x,\tilde{\mathbb{T}}^d} \otimes \varphi^1_{x,\tilde{\mathbb{T}}^d}$. Thus,

$$\varphi^1_{x,\mathbb{T}^d}[C_\infty] = \varphi^1_{x,\mathbb{T}^d}[C_\infty |\, \omega_e = 0] = \varphi^1_{x,\tilde{\mathbb{T}}^d} \otimes \varphi^1_{x,\tilde{\mathbb{T}}^d}[C_\infty] = 1 - (1 - \varphi^1_{x,\tilde{\mathbb{T}}^d}[C_\infty])^2,$$

which finishes the proof. □

**Theorem 7.3.2:** For $x > x_c(\varphi^1_{\mathbb{T}^d})$, there exists $c > 0$ such that for any vertex $v$,

$$\ell^1_{x,\mathbb{T}^d}[v \leftrightarrow \infty] \geq c.$$

In particular, $x_c(\ell^1_{\mathbb{T}^d}) = x_c(\varphi^1_{\mathbb{T}^d})$.

*Proof.* Consider two neighbouring vertices $v, w$ and define the event $A_v = \{v \longleftrightarrow \infty \text{ in } \mathbb{T}^d \setminus \{v, w\}\}$ and analogously $A_w$. Define the event $L_{v,w} := A_v \cap A_w \cap \{(v,w) \text{ open}\}$. In words, this is the event that that there is a loop which contains the edge $(v,w)$ goes through the wired boundary at infinity. Now, suppose $x > x_c(\varphi^1_{\mathbb{T}^d})$. In this case, by the FKG inequality, we obtain a lower bound that a random-cluster configuration $\omega$ satisfies $L_{v,w}$:

$$\varphi^1_{x,\mathbb{T}^d}[L_{v,w}] \geq \varphi^1_{x,\mathbb{T}^d}[A_v]\varphi^1_{x,\mathbb{T}^d_n}[A_w]\varphi^1_{x,\mathbb{T}^d}[(v,w) \text{ open}] \geq \varphi^1_{x,\tilde{\mathbb{T}}^d}[0 \leftrightarrow \infty]^2 x \geq c > 0,$$

where, in the second inequality, we have used monotonicity in boundary conditions, the fact that the probability of a given edge being open is at least $x$ (which follows from (7.1.3)), as well as Lemma 7.3.1.

Conditionally on $L_{vw}$, there exist two disjoint infinite paths in $\omega$ starting from $v$ and $w$ respectively. Let us argue that, for such a configuration $\omega$, $\mathrm{UEG}_\omega[(v,w) \text{ open}, v \leftrightarrow \infty] = \frac{1}{2}$. This boils down to two observations: First, because all components of an even subgraph of $\mathbb{T}^d$ are either trivial or infinite,

$$\mathrm{UEG}^1_\omega[(v,w) \text{ open}, v \leftrightarrow \infty] = \mathrm{UEG}^1_\omega[(v,w) \text{ open}]. \tag{7.3.1}$$

Second, the probability of a given edge, which is part of a loop in $\omega$, being open in $\mathrm{UEG}_\omega$ is $\frac{1}{2}$ (see the much more general statement [150, Lemma 3.5]).

In conclusion, for $x > x_c(\varphi^1_{\mathbb{T}^d})$,

$$\ell^1_{x,\mathbb{T}^d}[v \leftrightarrow \infty] \geq \varphi^1_{x,\mathbb{T}^d}[\mathrm{UEG}^1_\omega[v \leftrightarrow \infty]] > c/2 > 0.$$

□

To finish the proof of the first statement in Theorem 7.1.3, we make a short aside to discuss the subcritical regime of the random-cluster and random current models on the tree. It is classical that, for $x < x_c(\varphi^1_{\mathbb{T}^d})$, we have that $\varphi^1_{x,\mathbb{T}^d} = \mathbb{P}_{x,\mathbb{T}^d}$ (see [149, Theorem 10.67]). A similar result holds for the double random current:

**Lemma 7.3.3:** For $x < x_c(\varphi^1_{\mathbb{T}^d})$, then $\mathbf{P}^1_{x,\mathbb{T}^d} \cup \mathbf{P}^1_{x,\mathbb{T}^d} = \mathbb{P}_{x^2,\mathbb{T}^d}$. Moreover, $x_c(\varphi^1_{\mathbb{T}^d}) \leq x_c(\mathbf{P}^1_{\mathbb{T}^d} \cup \mathbf{P}^1_{\mathbb{T}^d})$.

*Proof.* Since $x < x_c(\varphi^1_{\mathbb{T}^d})$ and $\ell^1_{x,\mathbb{T}^d} \preceq \varphi^1_{x,\mathbb{T}^d}$, we get that $\ell^1_{x,\mathbb{T}^d}[C_\infty] \leq \varphi^1_{x,\mathbb{T}^d}[C_\infty] = 0$. Therefore, since all trivial components of an even subgraph of $\mathbb{T}^d$ are infinite, we have that

$$\ell^1_{x,\mathbb{T}^d}[\eta \equiv 0 \mid \Omega_\emptyset(\mathbb{T}^d) \setminus C_\infty] = 1.$$

By (7.1.5), this implies that $\mathbf{P}^1_{x,\mathbb{T}^d} \cup \mathbf{P}^1_{x,\mathbb{T}^d} = \mathbb{P}_{x^2,\mathbb{T}^d}$. For the second statement, note that $\mathbb{P}_{x^2,\mathbb{T}^d} \preceq \mathbb{P}_{x,\mathbb{T}^d} = \varphi^1_{x,\mathbb{T}^d}$. □

We can now put (7.1.5), Theorem 7.3.2 and Lemma 7.3.3 together to obtain

$$x_c(\ell^1_{\mathbb{T}^d}) \geq x_c(\mathbf{P}^1_{\mathbb{T}^d}) \geq x_c(\mathbf{P}^1_{\mathbb{T}^d} \cup \mathbf{P}^1_{\mathbb{T}^d}) \geq x_c(\varphi^1_{\mathbb{T}^d}) = x_c(\ell^1_{\mathbb{T}^d}).$$

Hence, we arrive at the following corollary:

**Corollary 7.3.4:** For $d \geq 2$, then

$$x_c(\ell^1_{\mathbb{T}^d}) = x_c(\mathbf{P}^1_{\mathbb{T}^d}) = x_c(\mathbf{P}^1_{\mathbb{T}^d} \cup \mathbf{P}^1_{\mathbb{T}^d}).$$

## 7.3.1 Modifications for $\mathtt{C}^d_n$.

In the following, we comment on how to adapt the previous proof strategy to yield the analogue of Corollary 7.3.4 (resp./ the first part of Theorem 7.1.3) on $\mathtt{C}^d_n$. This requires two ingredients:

a) For $x < x_c(\varphi^1_{\mathtt{C}^d_n})$, all models reduce to explicitly comparable independent models. In particular, we will argue that

$$x_c(\mathbf{P}^1_{x,\mathtt{C}^d_n} \cup \mathbf{P}^1_{x,\mathtt{C}^d_n}) \geq x_c(\varphi^1_{x,\mathtt{C}^d_n}).$$

b) For $x > x_c(\varphi^1_{\mathtt{C}^d_n})$, the loop $\mathrm{O}(1)$ model $\ell^1_{x,\mathtt{C}^d_n}$ percolates.

For $a$) if $x < x_c(\varphi^1_{\mathtt{C}^d_n})$, rather than $\ell^1_{x,\mathtt{C}^d_n}$ being deterministically empty, it includes each simple cycle of $\mathtt{C}^d_n$ independently since the free loops cut-point factorise. Accordingly, $\mathbf{P}^1_{x,\mathtt{C}^d_n} \cup \mathbf{P}^1_{x,\mathtt{C}^d_n}$ is a union of two independent cycle measures and a Bernoulli percolation and therefore, it percolates only if it has better connection probabilities in finite volume than $\varphi^1_{x,\mathtt{C}^d_n}$. But the finite-volume two-point function of $\varphi^1_{x,\mathtt{C}^d_n}$ is always larger than that of $\mathbf{P}^1_{x,\mathtt{C}^d_n} \cup \mathbf{P}^1_{x,\mathtt{C}^d_n}$ by (7.1.8). Since $x < x_c(\varphi^1_{\mathtt{C}^d_n})$, we conclude that $\mathbf{P}^1_{x,\mathtt{C}^d_n} \cup \mathbf{P}^1_{x,\mathtt{C}^d_n}$ does not percolate.
Now, for $b$), if $x > x_c(\varphi^1_{\mathtt{C}^d_n})$, we want to make an observation that infinite paths can be deduced in $\ell^1_{x,\mathtt{C}^d_n}$ from a local configuration. On a tree, it is true that any open edge is part of an infinite cluster (this is what we used in the proof for $\mathbb{T}^d$, see (7.3.1)). On $\mathtt{C}^d_n$, instead, it is true that if $e, e'$ are edges belonging to the same simple cycle, and $\eta \in \Omega_\emptyset(\mathtt{C}^d_n)$ with $\eta_e = 1$ and $\eta_{e'} = 0$, then $e$ lies on an infinite cluster in $\eta$ (in which case $e'$ and $e$ are on opposite paths between the glued vertices). By the same argument as previously, conditionally on $e$ being cyclic and lying in an infinite component of $\omega \sim \varphi^1_{x,\mathtt{C}^d_n}$, the probability that $\eta_e = 1$ and $\eta_{e'} = 0$ is at least $\frac{1}{4}$ for $\eta \sim \mathrm{UEG}^1_\omega$, which concludes the argument.

## 7.4 Explicit computation of critical points

In this section we explicitly compute the critical points for the free models on $\mathtt{C}_n^d$, the $d$-regular tree where every edge is replaced by a cycle of length $2n$ (and glued through opposite point of the cycle).

**Proposition 7.4.1:** For any $n \geq 1$ and $d \geq 2$, it holds that

$$x_c(\ell^0_{\mathtt{C}_n^d}) = (d-2)^{-\frac{1}{2n}}$$
$$x_c(\varphi^0_{\mathtt{C}_n^d}) = \sqrt[n]{(d-1) - \sqrt{(d-1)^2 - 1}}$$
$$x_c(\mathbf{P}^0_{\mathtt{C}_n^d} \cup \mathbf{P}^0_{\mathtt{C}_n^d}) = \sqrt[2n]{(2d-5) - \sqrt{(2d-5)^2 - 1}}\,.$$

In particular, the three different models have three different phase transitions.

A graphical presentation of the functions is given in Figure 7.5.

*Proof.* Let $C_{2n}$ denote cycle graph of length $2n$ and let $a$ and $b$ be two antipodal points. As in the proof Theorem 7.1.1, we use cut-point factorisation (Lemma 7.2.4) to reduce everything to Bernoulli percolation on $\mathbb{T}^d$ with parameter $\nu_{x,C_{2n}}[a \leftrightarrow b]$ (with $\nu$ denoting one of the models under consideration). The rest follows by direct computation:
*Loop* $\mathrm{O}(1)$ *model:* It holds that $\ell_{x,C_{2n}}[a \leftrightarrow b] = \frac{x^{2n}}{1+x^{2n}}$. Thus,

$$\frac{1}{d-1} = \frac{x_c^{2n}}{1+x_c^{2n}}$$

can be solved to obtain $x_c(\ell^0_{\mathtt{C}_n^d}) = (d-2)^{-\frac{1}{2n}}$.

*Random-cluster model:* Now, for $(\eta,\omega) \sim \ell_{x,C_{2n}} \otimes \mathbb{P}_{p,C_{2n}}$, we see that

$$\ell_{x,C_{2n}} \otimes \mathbb{P}_{p,C_{2n}}[\eta \cup \omega \in (a \leftrightarrow b) \,|\, \eta] = \begin{cases} 1 & \eta \equiv 1 \\ 2p^n - p^{2n} & \eta \equiv 0. \end{cases}$$

Since the cycle graph has exactly two even subgraphs (the full and the empty graph) we get that

$$(\ell_{x,C_{2n}} \cup \mathbb{P}_{p,C_{2n}})[a \leftrightarrow b] = \frac{x^{2n} + 2p^n - p^{2n}}{1+x^{2n}}. \tag{7.4.1}$$

Thus, for the random-cluster model (where we choose $p = x$) we obtain

$$\frac{1}{d-1} = \varphi_{x_c,C_{2n}}[a \leftrightarrow b] = \frac{x_c^{2n} + 2x_c^n - x_c^{2n}}{1 + x_c^{2n}} = \frac{2x_c^n}{1 + x_c^{2n}},$$

which can be solved with the substitution $z = x_c^n$ and we obtain that

$$x_c(\varphi^0_{\mathbb{C}_n^d}) = \sqrt[n]{(d-1) - \sqrt{(d-1)^2 - 1}}.$$

*Double random current:* Analogously to (7.4.1) we obtain

$$(\ell_{x,C_{2n}} \cup \ell_{x,C_{2n}} \cup \mathbb{P}_{p,C_{2n}})[a \leftrightarrow b] = \frac{2x^{2n} + x^{4n} + 2p^n - p^{2n}}{(1 + x^{2n})^2}.$$

Hence, choosing $p = x^2$ for the double random current we obtain the following equation

$$\frac{1}{d-1} = \frac{4x^{2n}}{(1 + x^{2n})^2}$$

which can be solved with the substitution $z = x^{2n}$, giving rise to

$$x_c(\mathbf{P}^0_{\mathbb{C}_n^d} \cup \mathbf{P}^0_{\mathbb{C}_n^d}) = \sqrt[2n]{(2d-5) - \sqrt{(2d-5)^2 - 1}}.$$

□

The same argument for the single current does not lead to a closed formula, but the following separate argument allows us to conclude Theorem 7.1.3.

*Proof of Theorem 7.1.3.* The first part of the theorem is given by combining Theorem 7.3.2 and Theorem 7.3.4. We focus on the single random current: From (7.4.1) we sprinkle with $p(x) = 1 - \sqrt{1 - x^2}$ to obtain the single current. One may check that for any increasing (differentiable) function $p(x)$ taking values in $[0,1]$, the function $x \mapsto (\ell_x \cup \mathbb{P}_{p(x)})[a \leftrightarrow b]$ is increasing. Hence, there exists a unique solution $x_c$ to the equation

$$\frac{1}{d-1} = \frac{x_c^{2n} + 2p(x_c)^n - p(x_c)^{2n}}{1 + x_c^{2n}}.$$

Thus, if $p$ and $q$ are increasing differentiable functions that take values in $[0,1]$ such that $p(x) < q(x)$ for all $x \in [0,1]$ then $x_c(\ell^0_{x,\mathbb{C}_n^d} \cup \mathbb{P}_{p(x),\mathbb{C}_n^d}) > x_c(\ell^0_{x,\mathbb{C}_n^d} \cup \mathbb{P}_{q(x),\mathbb{C}_n^d})$.

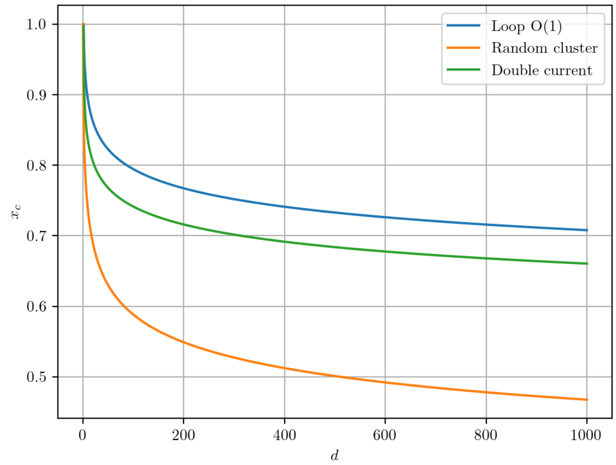



Figure 7.5: The critical $x_c$ on the graph $\mathtt{C}_{d,n}$ as a function of $d$ for the loop O(1), and double random current.

Using this for the functions $r(x) = 0$, $p(x) = 1 - \sqrt{1-x^2}$, and $q(x) = x^2$ together with stochastic domination yields

$$x_c(\ell^0_{x,\mathbf{C}^d_n}) > x_c(\underbrace{\ell^0_{x,\mathtt{C}^d_n} \cup \mathbb{P}_{p(x),\mathtt{C}^d_n}}_{\mathbf{P}^0_{x,\mathtt{C}^d_n}}) > x_c(\ell^0_{x,\mathtt{C}^d_n} \cup \mathbb{P}_{q(x),\mathtt{C}^d_n}) \geq x_c(\underbrace{\ell^0_{x,\mathtt{C}^d_n} \cup \ell^0_{x,\mathtt{C}^d_n} \cup \mathbb{P}_{q(x),\mathtt{C}^d_n}}_{\mathbf{P}^0_{x,\mathtt{C}^d_n} \cup \mathbf{P}^0_{x,\mathtt{C}^d_n}}).$$

$\square$

## 7.5 The critical probability for Bernoulli percolation is no obstruction for the UEG

In the previous section, we considered graphs where the free uniform even subgraph is intimately tied to the behaviour of ordinary Bernoulli percolation on another graph. One might wonder about general links between the behaviour of Bernoulli percolation and that of the UEG. For instance, one might have a suspicion that if a graph $\mathbb{G}$ is easily disconnected in the sense that the percolation threshold $p_c(\mathbb{P}_{\mathbb{G}})$ is very close to 1, this lack of connectivity might also impact the UEG. This turns out to be false even for one-ended graphs.
We are going to give two counterexamples:

- The first is a construction that can be applied to just about any graph and which admits an easy proof. However, the graphs thus produced are not of inherent interest otherwise.

- The second is the infinite cluster of $\varphi_{x_c+\varepsilon,\mathbb{Z}^2}$ as $\varepsilon \to 0^+$, which is a more natural object, but for which the proof is more involved.

The UEG of the latter model always percolates by [157, Theorem 1.3], and it is reasonable to believe that continuity of the phase transition implies that breaking even a small fraction $\delta = \delta(\varepsilon)$ of the edges breaks the infinite cluster. However, the proof we give relies on [163], the results of which are (conjecturally) not valid for all planar percolation models with a continuous phase transition (see discussion after [163, Theorem 7.5]). Therefore, the matter is much more subtle than one might expect and restating all necessary prerequisites is beyond the scope of the present paper. As such, we shall settle for referring to suitable places in the literature. However, we find the example to be important in the sense that the graph produced is, in some sense, a much more natural negative resolution to the question.

### 7.5.1 The edge-halving construction

For a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, define $\mathbb{G}^{1/2}$ with $\mathbb{V}(\mathbb{G}^{1/2}) = \mathbb{V} \cup \mathbb{E}$ and $\mathbb{E}(\mathbb{G}^{1/2})$ consisting of pairs $(v, e)$ with $v \in \mathbb{V}$ and $e$ an edge in $\mathbb{G}$ with $v$ as its one end-point. One may note that $\mathbb{G}^{1/2}$ is bipartite with bi-partition $\mathbb{V} \cup \mathbb{E}$. In pictures, $\mathbb{G}^{1/2}$ is obtained from $\mathbb{G}$ by dividing each edge in two. The point is that doing so does not change the behaviour of the uniform even subgraph at all, while it makes it strictly harder for Bernoulli percolation to percolate.
We note that since $\mathbb{G}^{1/2}$ is bipartite, a subgraph thereof will have an infinite component if and only if it has a connected component containing infinitely many vertices from $\mathbb{V}$.

**Lemma 7.5.1:** For any graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ there is a group isomorphism $\psi : \Omega_\emptyset(\mathbb{G}^{1/2}) \to \Omega_\emptyset(\mathbb{G})$ such that

$$v \overset{\eta}{\longleftrightarrow} w \quad \text{if and only if} \quad v \overset{\psi(\eta)}{\longleftrightarrow} w,$$

for every $\eta \in \Omega_\emptyset(\mathbb{G})$ and $v, w \in \mathbb{V}$. In particular, $\mathrm{UEG}^1_{\mathbb{G}}$ percolates if and only if $\mathrm{UEG}^1_{\mathbb{G}^{1/2}}$ does.

**Remark 7.5.2:** We note that the lemma also holds for $\mathrm{UEG}^0$, but omit it from the statement for notational ease. The same proof carries through.

*Proof.* Any $e = (v, w) \in \mathbb{E}$ has degree two in $\mathbb{G}^{1/2}$. Therefore, for any $\eta \in \Omega_\emptyset(\mathbb{G}^{1/2})$, $(v, e) \in \eta$ if and only if $(w, e) \in \eta$. Accordingly,

$$\psi(\eta) = \{e \in \mathbb{E} | \deg_\eta(e) = 2\}$$

defines an even subgraph of $\mathbb{G}$ with the desired connectivity property. One checks that $\psi$ is a group homomorphism and, furthermore, that its inverse is given by

$$\psi^{-1}(\eta) = \{(v,e) |\ e \in \eta\}.$$

Since $\psi$ is a group homomorphism and $\mathrm{UEG}^1_{\mathbb{G}^{1/2}}$ is the Haar measure on $\Omega_\emptyset(\mathbb{G}^{1/2})$, it pushes forward to Haar measure on its image under $\psi$, which is $\mathrm{UEG}^1_{\mathbb{G}}$, since $\psi$ is surjective. □

**Lemma 7.5.3:** For any graph $\mathbb{G} = (\mathbb{V},\mathbb{E})$, it holds that $p_c(\mathbb{P}_{\mathbb{G}^{1/2}}) = \sqrt{p_c(\mathbb{P}_{\mathbb{G}})}$.

*Proof.* The proof proceeds by coupling $\omega_p \sim \mathbb{P}_{p^2,\mathbb{G}}$ and $\omega_{p,1/2} \sim \mathbb{P}_{p,\mathbb{G}^{1/2}}$ for every $p \in [0,1]$ in such a way that $\omega_p \in \{v \leftrightarrow w\}$ if and only if $\omega_{p,1/2} \in \{v \leftrightarrow w\}$ for every pair $v,w \in \mathbb{V}$. The coupling itself declares that $e = (v,w) \in \omega_p$ if and only if $(v,e) \in \omega_{p,1/2}$ and $(w,e) \in \omega_{p,1/2}$. The process $\omega_p$ thus defined is i.i.d. since $\omega_{p,1/2}$ is, and its marginals may be checked to be Bernoulli variables of parameter $p^2$. The desired connectivity property also follows by construction. □

This allows us to prove Theorem 7.1.4:

*Proof.* Let $\varepsilon > 0$ and let $\mathbb{G}_0 = \mathbb{Z}^2$, the uniform even subgraph of which percolates by [157] and for which $p_c(\mathbb{P}_{\mathbb{Z}^2}) = \frac{1}{2}$ by Kesten's Theorem [164]. Inductively, define $\mathbb{G}_{j+1} = \mathbb{G}_j^{1/2}$. By Lemma 7.5.1, we have that $\mathrm{UEG}_{\mathbb{G}_j}[0 \leftrightarrow \infty] > 0$ for every $j$ and by Lemma 7.5.3, we have that $p_c(\mathbb{G}_j) = 2^{-2^{-j}}$. Picking $j$ sufficiently large proves the desired. □

### 7.5.2 The infinite cluster of the slightly supercritical random-cluster model.

For our second example, for $x > x_c$ we let $\mathbb{G}^x$ denote the infinite cluster of $\varphi_{x,\mathbb{Z}^2}$. By [157, Theorem 3.1], we have that the uniform even subgraph of $\mathbb{G}^x$ percolates almost surely. Therefore, we obtain a second proof of Theorem 7.1.4 if we can prove the following:

**Proposition 7.5.4:** Almost surely, under the increasing coupling of $\varphi_{x,\mathbb{Z}^2}$, we have

$$\lim_{x \downarrow x_c} p_c(\mathbb{P}_{\mathbb{G}^x}) = 1.$$

*Proof.* We start by fixing parameters and notation. Fix $\delta \in (0,1)$ and let $\rho > 0$ be small enough. For finite $G \subseteq \mathbb{Z}^2$, let $(\omega_G, \xi_{G,\delta}) \sim \varphi^1_{x_c,G} \otimes \mathbb{P}_{1-\delta,G}$. Furthermore, let $R_k = [-k,k] \times [-3k,3k] \cap$

$\mathbb{Z}^2$, $\Lambda_k = [-k,k]^2 \cap \mathbb{Z}^2$ and let $\mathscr{C}_k$ denote the event that there is a crossing in $R_k$ between its left and right faces.
We claim that if $k = k(\delta,\rho)$ is large enough, then

$$\varphi^1_{x_c,R_k} \otimes \mathbb{P}_{1-\delta,R_k}[\omega_{R_k} \cap \xi_{\delta,R_k} \in \mathscr{C}_k] < \rho. \tag{7.5.1}$$

Before we indicate how to prove (7.5.1), let us see how it finishes the proof. As any crossing between $\Lambda_k$ and $\Lambda_{3k}$ must cross at least one rotated translate of $R_k$, monotonicity in boundary conditions and a union bound implies that

$$\varphi^1_{x_c,\Lambda_{3k}} \otimes \mathbb{P}_{1-\delta,\Lambda_{3k}}[\omega_{\Lambda_{3k}} \cap \xi_{\delta,\Lambda_{3k}} \in \{\Lambda_k \leftrightarrow \Lambda_{3k}\}] < 4\rho. \tag{7.5.2}$$

It is well-known that if $\rho$ is sufficiently small, an estimate of the form (7.5.2) for some $k$ is enough to imply non-percolation by techniques that go back to [165] (see e.g. the proof of [150, Proposition 2.11]). However, by continuity of the finite volume measures, (7.5.2) remains true if $\omega_{\Lambda_{3k}}$ is replaced by $\tilde{\omega}_{\Lambda_{3k}} \sim \varphi^1_{x_c+\varepsilon,\Lambda_{3k}}$ for $\varepsilon$ sufficiently small. Upon inspection, this is the same as saying that $p_c(\mathbb{P}_{\mathbb{G}^x}) \geq 1-\delta$ almost surely for $x \in (x_c, x_c+\varepsilon)$, which is what we wanted, since, under the increasing coupling, $x \mapsto p_c(\mathbb{P}_{\mathbb{G}^x})$ is almost surely decreasing.
Now, to see that (7.5.1) holds provided $k$ is large enough, we refer to [166, Lemma 5.2]. This lemma is stated in the context of Boolean percolation, but as is remarked on in that paper, its proof only relies on the techniques of [163, Theorem 7.5]. Thus, it is also valid for the random-cluster model. Combining [166, Lemma 5.2] with the fact that the four-arm exponent of the random-cluster model is smaller than 2 [167, Page 11] yields (7.5.1). □

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