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Coupling sum rules and oblique corrections in gauge-Higgs unification

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In grand-unified-theory-inspired $SO(5) \times U(1)_X \times SU(3)_C$ gauge-Higgs unification (GHU) in the Randall–Sundrum warped spacetime, the W- and Z-couplings of all 4D fermion modes become nontrivial. The W- and Z-couplings of zero-mode quarks and leptons slightly deviate from those in the SM, and the couplings take the matrix form in the space of Kaluza–Klein (KK) states. In particular, the 4D couplings and mass spectra in the KK states depend on the Aharonov–Bohm phase θ_H in the fifth dimension. Nevertheless there emerge three astonishing sum rules among those coupling matrices, which guarantees the finiteness of certain combinations of corrections to vacuum polarization tensors. We confirm by numerical evaluation that the equality in the sum rules holds with 5- to 7-digit accuracy. Based on the sum rules we propose improved oblique parameters in GHU. Oblique corrections due to fermion 1-loop diagrams are found to be small.

Subject Index B00, B33, B40, B43

1. Introduction

Although the standard model (SM), $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory, has been successful in describing almost all of the phenomena at low energies, it has a severe gauge hierarchy problem when embedded in a larger theory such as grand unification. One possible answer to this problem is the gauge-Higgs unification (GHU) scenario in which gauge symmetry is dynamically broken by an Aharonov–Bohm (AB) phase, θ_H, in the fifth dimension. The 4D Higgs boson is identified with a 4D fluctuation mode of θ_H [1–14].

Many GHU models have been proposed, among which $SO(5) \times U(1)_X \times SU(3)_C$ GHU models in the Randall–Sundrum (RS) warped space [15] turn out to be promising candidates for describing physics beyond the SM. The $SO(5) \times U(1)_X \times SU(3)_C$ gauge symmetry naturally

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incorporates the custodial symmetry in the Higgs boson sector [9]. The orbifold boundary condition breaks $SO(5)$ to $SO(4) \simeq SU(2)_L \times SU(2)_R$. The $SU(2)_R \times U(1)_X$ symmetry is spontaneously broken to $U(1)_Y$ by a brane scalar on the ultraviolet (UV) brane in the RS space. The resultant $SU(2)_L \times U(1)_Y$ SM symmetry is dynamically broken to $U(1)_{\rm EM}$ by the Hosotani mechanism. The 4D Higgs boson is a zero mode of the fifth-dimensional component of gauge fields in $SO(5)/SO(4)$ which generates an AB phase in the fifth dimension. The finite Higgs boson mass $m_H \sim 125.1$ GeV is generated at the quantum level by the dynamics of the AB phase θ_H.

A realistic GHU model has been first proposed with quark-lepton multiplets in the vector (**5**) representation of $SO(5)$, which is referred to as the GHU A-model [13]. It is recognized, however, that it is difficult to embed the A-model in the grand unification scheme. A natural grand-unified-theory (GUT) containing $SO(5) \times U(1)_X \times SU(3)_C$ GHU is $SO(11)$ GHU with fermion multiplets in the spinor (**32**) and vector (**11**) representations [16,17]. The GUT-inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU model is defined with fermion multiplets in the bulk in the (**3**, **4**), (**1**, **4**), (**1**, **5**), and (**3**, **1**) representations of ($SU(3)_C$, $SO(5)$) [18]. [(**3**, **4**), (**1**, **4**)] is contained in **32** of $SO(11)$, whereas [(**1**, **5**), (**3**, **1**)] is contained in **11** of $SO(11)$. In addition, Majorana fermions in the singlet (**1**, **1**) representation are introduced on the UV brane which provide the inverse seesaw mechanism for neutrinos. The GUT-inspired GHU model is referred to as the GHU B-model.

The GHU B-model is successful in many respects. It reproduces the quark, lepton, gauge, and Higgs boson spectra (except for the small mass of the up quark), and yields nearly the same gauge couplings of the SM particles. It can incorporate the Cabbibo–Kobayashi–Maskawa (CKM) matrix structure in the W-couplings with Flavour-Changing-Neutral-Currents naturally suppressed [19]. Many of the physical quantities at low energies are described mainly by the AB phase θ_H, being mostly independent of the parameters in the dark fermion sector [20]. Both A- and B-models predict Z' particles as KK excited states of the photon, Z boson, and Z_R boson. Z'-couplings of quarks and leptons exhibit large parity violation which can be explored and tested at the 250 GeV e^-e^+ International Linear Collider (ILC) [21]. By examining the dependence of event numbers on the polarization of incident electron and positron beams the A- and B-models can be clearly distinguished [22–26]. Effects of Z' bosons can be seen in single Higgs boson production processes as well [27].

It has been established that useful and convenient quantities to investigate new physics beyond the SM are the oblique parameters S, T, and U of Peskin–Takeuchi, which represent corrections to vacuum polarization tensors of W, Z, and photon [28–31]. In the early study of gauge theory in the RS warped space it was argued that $SO(5) \times U(1)_X$ GHU models may yield appreciable corrections to S and T [32]. To evaluate oblique corrections at the 1-loop level in GHU, one has to know the mass spectrum and gauge couplings of all KK modes. In Ref. [32] S and T were expressed in terms of truncated propagators of gauge bosons and fermions, by adopting a perturbative expansion in θ_H.

In this paper we use exactly determined mass spectra of fermion KK states at general θ_H at the tree level, and evaluate W- and Z-couplings of the fermion KK states by making use of exactly determined wave functions of both gauge bosons and fermions in the $SO(5) \times U(1)_X$ space. It is known that the W- and Z-couplings of quarks and leptons are nearly the same as in the SM. The W- and Z-couplings of the KK modes of quark and lepton multiplets become highly nontrivial, however. They are not diagonal in the KK states at $\theta_H \neq 0$, taking the matrix form with nontrivial off-diagonal elements. Further, the wave functions of the W and Z bosons have substantial components not only in the $SU(2)_L \times U(1)_Y$ space but also in the entire $SO(5) \times U(1)_X$ space, which necessitates refinement of the definition of oblique parameters.

One-loop corrections to the vacuum polarization tensors of W, Z, and photon contain divergences. In the 4D $SU(2)_L \times U(1)_Y$ gauge theory certain combinations of those vacuum polarization tensors, represented by S, T, and U, are finite. In the GUT-inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU various combinations of the KK states of fermions run along the loops in the propagators of gauge bosons. It will be shown that there appear sum rules among the W- and Z-coupling matrices. We shall confirm those sum rules by numerically evaluating the mass spectra of the KK modes and their W- and Z-coupling matrices. The sum rules are found to hold with 5- to 7-digits accuracy. With these sum rules certain combinations of the 1-loop corrections to the vacuum polarization tensors become finite, which leads to improved oblique parameters. It will be seen that corrections to the improved oblique parameters are small. The total corrections are found to be $S \sim 0.01$, $T \sim 0.12$, and $U \sim 0.00004$ for $\theta_H = 0.1$ and the KK mass scale $m_{\rm KK} = 13$ TeV, which is consistent with the current experimental data [33]

Although the gauge couplings in GHU in the RS space have a highly nontrivial matrix structure, they satisfy remarkable identities. The identities in the sum rules discussed in the present paper are associated with two-point functions of gauge fields. It is known that the exact identities hold in the combinations of the couplings appearing in three-point functions of gauge fields in orbifold gauge theory [34,35]. Triangle loop diagrams generally give rise to chiral anomalies in 4D gauge theory. In 5D orbifold gauge theory anomaly coefficients for the three legs of various 4D KK modes of gauge fields vary with the AB phase in the fifth dimension. This phenomenon is called the anomaly flow by an AB phase. Along triangle loops all possible KK modes of fermions run. The sum of all those loop contributions leads to the total anomaly coefficient, which is expressed in terms of the values of the wave functions of gauge fields at the UV and infrared (IR) branes in the RS space and orbifold boundary conditions (BCs) of the fermions. In other words there hold sum rules for gauge coupling matrices of the third order.

In Sect. 2 the GUT-inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU is described. We explain how to determine the mass spectrum and wave functions of gauge bosons and quark-lepton multiplets. In Sect. 3 the W- and Z-couplings of all fermion modes are determined. In Sect. 4 fermion 1-loop corrections to the vacuum polarization tensors of the W boson, Z boson, and photon are evaluated. In Sect. 5 we show that there appear three sum rules among the W- and Z-coupling matrices of quarks, leptons, and their KK modes. The sum rules are confirmed by numerical evaluation. Based on the coupling sum rules the improved oblique parameters are introduced in Sect. 6. The finite corrections to the S, T, and U parameters are evaluated, and are found to be small. Section 7 is devoted to a summary and discussions. In Appendix A basis functions used to express wave functions of gauge bosons and fermions are summarized. In Appendix B wave functions of KK modes of down-type quarks and neutrinos are given.

2. GUT-inspired GHU

$SO(5) \times U(1)_X \times SU(3)_C$ GHU is defined in the RS space whose metric is given by [15]

$$ds^2 = g_{MN} dx^M dx^N = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{1}$$

where $M, N = 0, 1, 2, 3, 5$, $\mu, \nu = 0, 1, 2, 3$, $y = x^5$, $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, $\sigma(y) = \sigma(y + 2L) = \sigma(-y)$, and $\sigma(y) = ky$ for $0 \le y \le L$. In terms of the conformal coordinate

Table 1. The matter field content in the GUT-inspired GHU model. The $(SU(3)_C, SO(5))_{U(1)_X}$ content of each field is shown in the last column.

in the bulk	quark	$(\mathbf{3}, \mathbf{4})_{\frac{1}{6}}$ $(\mathbf{3}, \mathbf{1})^+_{-\frac{1}{3}}$ $(\mathbf{3}, \mathbf{1})^-_{-\frac{1}{3}}$
	lepton	$(\mathbf{1}, \mathbf{4})_{-\frac{1}{2}}$
	dark fermion Ψ^D	$(\mathbf{3}, \mathbf{4})_{\frac{1}{6}}$ $(\mathbf{1}, \mathbf{5})^+_0$ $(\mathbf{1}, \mathbf{5})^-_0$
on the UV brane	Majorana fermion $\hat{\chi}$	$(\mathbf{1}, \mathbf{1})_0$
	brane scalar $\hat{\Phi}$	$(\mathbf{1}, \mathbf{4})_{\frac{1}{2}}$

$z = e^{ky}$ $(0 \le y \le L, 1 \le z \le z_L = e^{kL})$

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{dz^2}{k^2}\right). \tag{2}$$

The bulk region $0 < y < L$ is anti-de Sitter spacetime with a cosmological constant $\Lambda = -6k^2$, which is sandwiched by the UV brane at $y = 0$ and the IR brane at $y = L$. The KK mass scale is $m_{\rm KK} = \pi k/(z_L - 1) \simeq \pi k z_L^{-1}$ for $z_L \gg 1$.

Gauge fields $A_M^{SO(5)}$, $A_M^{U(1)_X}$, and $A_M^{SU(3)_C}$ of $SO(5) \times U(1)_X \times SU(3)_C$ satisfy the orbifold BCs

$$\begin{pmatrix} A_\mu \\ A_y \end{pmatrix}(x, y_j - y) = P_j \begin{pmatrix} A_\mu \\ -A_y \end{pmatrix}(x, y_j + y) P_j^{-1} \tag{3}$$

where $(y_0, y_1) = (0, L)$. In terms of

$$P_{\mathbf{4}}^{SO(5)} = \text{diag}\,(I_2, -I_2)\,,$$
$$P_{\mathbf{5}}^{SO(5)} = \text{diag}\,(I_4, -I_1)\,, \tag{4}$$

$P_0 = P_1 = P_{\mathbf{5}}^{SO(5)}$ for $A_M^{SO(5)}$ in the vector representation and $P_0 = P_1 = 1$ for $A_M^{U(1)_X}$ and $A_M^{SU(3)_C}$. The 4D Higgs field is contained in the $SO(5)/SO(4)$ part of $A_y^{SO(5)}$. The orbifold BCs break $SO(5)$ to $SO(4) \simeq SU(2)_L \times SU(2)_R$.

The matter content in the GUT-inspired GHU (B-model) is summarized in Table 1 [18]. Quark and lepton multiplets are introduced in three generations. They satisfy

$$\begin{aligned}
\Psi_{(\mathbf{3},\mathbf{4})}(x, y_j - y) &= -P_{\mathbf{4}}^{SO(5)}\gamma^5\Psi_{(\mathbf{3},\mathbf{4})}(x, y_j + y),\\
\Psi_{(\mathbf{3},\mathbf{1})^\pm}(x, y_j - y) &= \mp\gamma^5\Psi_{(\mathbf{3},\mathbf{1})^\pm}(x, y_j + y),\\
\Psi_{(\mathbf{1},\mathbf{4})}(x, y_j - y) &= -P_{\mathbf{4}}^{SO(5)}\gamma^5\Psi_{(\mathbf{1},\mathbf{4})}(x, y_j + y).
\end{aligned} \tag{5}$$

Here 5D Dirac matrices γ^a $(a = 0, 1, 2, 3, 5)$ satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}(\eta^{ab} = \text{diag}(-I_1, I_4))$, and $\gamma^5 = \text{diag}(1, 1, -1, -1)$. Dark fermion fields satisfy

$$\begin{aligned}
\Psi^D_{(\mathbf{3},\mathbf{4})}(x, y_j - y) &= (-1)^j P_{\mathbf{4}}^{SO(5)}\gamma^5\Psi^D_{(\mathbf{3},\mathbf{4})}(x, y_j + y),\\
\Psi^D_{(\mathbf{1},\mathbf{5})^\pm}(x, y_j - y) &= \pm P_{\mathbf{5}}^{SO(5)}\gamma^5\Psi^D_{(\mathbf{1},\mathbf{5})^\pm}(x, y_j + y).
\end{aligned} \tag{6}$$

In addition, Majorana fermion fields $(\hat{\chi}(x))$ and a brane scalar field $(\hat{\Phi}(x))$ are introduced on the UV brane (at $y = 0$). The brane scalar field $\hat{\Phi}$ spontaneously breaks $SU(2)_R \times U(1)_X$ to $U(1)_Y$ with a vacuum expectation value (VEV) much larger than the KK mass scale $m_{\rm KK}$. The Majorana fermion field $(\hat{\chi} = \hat{\chi}^c)$ in each generation, combining with $\Psi_{(\mathbf{1},\,\mathbf{4})}$ and $\hat{\Phi}$, induces the inverse seesaw mechanism to account for a very small mass of the observed neutrino.

The action of the GUT-inspired GHU has been given in Ref. [18]. Let Ψ^J collectively denote all fermion fields in the bulk. Then the action of the fermions in the bulk becomes

$$S_{\rm bulk}^{\rm fermion} = \int d^5x\sqrt{-\det G}\,\bigg\{\sum_J \overline{\Psi^J}\mathcal{D}(c_J)\Psi^J - \sum_\alpha\Big(m_D^\alpha\overline{\Psi}^\alpha_{(\mathbf{3},\mathbf{1})^+}\Psi^{-\alpha}_{(\mathbf{3},\mathbf{1})^-}+\text{h.c.}\Big) - \sum_\beta\Big(m_V^\beta\overline{\Psi}^\beta_{(\mathbf{1},\mathbf{5})^+}\Psi^\beta_{(\mathbf{1},\mathbf{5})^-}+\text{h.c.}\Big)\bigg\}, \tag{7}$$

where $\overline{\Psi} = i\Psi^\dagger\gamma^0$ and

$$\mathcal{D}(c) = \gamma^A e_A{}^M\Big(D_M + \frac{1}{8}\omega_{MBC}[\gamma^B,\gamma^C]\Big) - c\sigma'(y),$$
$$D_M = \partial_M - ig_S A_M^{SU(3)} - ig_A A_M^{SO(5)} - ig_B Q_X A_M^{U(1)}. \tag{8}$$

The dimensionless parameter c in $\mathcal{D}(c)$ is called the bulk mass parameter, which controls the wave functions of the zero modes of the fermions. m_D^α and m_V^β are pseudo-Dirac mass terms. The action for the Majorana fermion field ($\hat\chi^\alpha$) is

$$S_{\rm brane}^{\hat\chi} = \int d^5x\sqrt{-\det G}\,\delta(y)\left\{\frac{1}{2}\overline{\hat\chi}^\alpha\gamma^\mu\partial_\mu\hat\chi^\alpha - \frac{1}{2}M^{\alpha\beta}\overline{\hat\chi}^\alpha\hat\chi^\beta\right\} \tag{9}$$

where $M^{\alpha\beta}$ represents Majorana masses. In the present paper we take $M^{\alpha\beta} = M^\alpha\delta^{\alpha\beta}$ for simplicity.

In addition there are gauge-invariant brane interactions given by

$$S_{\rm brane}^{\rm int} = \int d^5x\sqrt{-\det G}\,\delta(y)\,(\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3),$$
$$\mathcal{L}_1 = -\Big\{\kappa^{\alpha\beta}\,\overline{\Psi}^\alpha_{(\mathbf{3},\mathbf{4})}\hat\Phi_{(\mathbf{1},\mathbf{4})}\cdot\Psi^\beta_{(\mathbf{3},\mathbf{1})^+}+\text{h.c.}\Big\},$$
$$\mathcal{L}_2 = -\Big\{\tilde\kappa'^{\alpha\beta}\,\overline{\Psi}^\alpha_{(\mathbf{1},\mathbf{4})}\Gamma^a\,\tilde{\hat\Phi}_{(\mathbf{1},\mathbf{4})}\cdot\Big(\Psi^\beta_{(\mathbf{1},\mathbf{5})^-}\Big)_a+\text{h.c.}\Big\},$$
$$\mathcal{L}_3 = -\Big\{\tilde\kappa_{\mathbf{1}}^{\alpha\beta}\,\overline{\chi}^\beta\tilde{\hat\Phi}^\dagger_{(\mathbf{1},\mathbf{4})}\Psi^\alpha_{(\mathbf{1},\mathbf{4})}+\text{h.c.}\Big\}, \tag{10}$$

where κ's are coupling constants. $\tilde\Phi_{(\mathbf{1},\mathbf{4})}$ denotes a conjugate field in $(\mathbf{1},\mathbf{4})$ formed from $\Phi^*_{(\mathbf{1},\mathbf{4})}$. The brane field $\hat\Phi$ develops a nonvanishing expectation value $\langle\hat\Phi\rangle \neq 0$.

In the electroweak sector there are two 5D gauge couplings, g_A and g_B, corresponding to the gauge groups $SO(5)$ and $U(1)_X$, respectively. The 5D gauge coupling $g_Y^{\rm 5D}$ of $U(1)_Y$ is given by

$$g_Y^{\rm 5D} = g_A s_\phi,\quad s_\phi = \frac{g_B}{\sqrt{g_A^2+g_B^2}}. \tag{11}$$

The 4D $SU(2)_L$ and $U(1)_Y$ gauge coupling constants are given by

$$g_w = \frac{g_A}{\sqrt{L}},\quad g_Y = \frac{g_Y^{\rm 5D}}{\sqrt{L}}. \tag{12}$$

The bare weak mixing angle θ_W^0 is given by

$$\sin\theta_W^0 = \frac{g_Y}{\sqrt{g_w^2+g_Y^2}} = \frac{s_\phi}{\sqrt{1+s_\phi^2}}. \tag{13}$$

As is seen below, the mixing angle determined from the ratio of m_W to m_Z slightly differs from the one in Eq. (13) even at the tree level in GHU in the RS space; $m_W/m_Z|_{\rm tree} \neq \cos\theta_W^0$.

The 4D Higgs boson field is a part of $A_y^{SO(5)}$. $A_z^{SO(5)} = (kz)^{-1} A_y^{SO(5)}$ $(1 \le z \le z_L)$ in the tensor representation is expanded as

$$A_z^{(j5)}(x,z) = \frac{1}{\sqrt{k}} \phi_j(x) u_H(z) + \cdots, \quad u_H(z) = \sqrt{\frac{2}{z_L^2 - 1}}\, z,$$

$$\Phi_H(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2 + i\phi_1 \\ \phi_4 - i\phi_3 \end{pmatrix}. \tag{14}$$

$\Phi_H(x)$ corresponds to the doublet Higgs field in the SM. Φ_H develops a nonvanishing expectation value at the quantum level by the Hosotani mechanism. Without loss of generality we take $\langle\phi_1\rangle$, $\langle\phi_2\rangle$, $\langle\phi_3\rangle = 0$, and $\langle\phi_4\rangle \neq 0$. The AB phase θ_H in the fifth dimension is given by

$$\hat{W} = P \exp\left\{ i g_A \int_{-L}^{L} dy \left\langle A_y^{SO(5)} \right\rangle \right\} = \exp\left\{ i\theta_H \cdot 2T^{(45)} \right\}. \tag{15}$$

Here $\langle A_y^{SO(5)} \rangle = (2k)^{-1/2} \langle\phi_4\rangle v_H(y) T^{(45)}$, $v_H(y) = k e^{ky} u_H(z)$ for $0 \le y \le L$, and $v_H(-y) = v_H(y) = v_H(y + 2L)$. In terms of θ_H, $A_z^{(45)}$ is expanded as

$$A_z^{(45)}(x,z) = \frac{1}{\sqrt{k}} \left\{ \theta_H f_H + H(x) \right\} u_H(z) + \cdots,$$

$$f_H = \frac{2}{g_A} \sqrt{\frac{k}{z_L^2 - 1}} = \frac{2}{g_w} \sqrt{\frac{k}{L\left(z_L^2 - 1\right)}}. \tag{16}$$

The 4D neutral Higgs field $H(x)$ is the fluctuation mode of the AB phase θ_H.

2.1. *Spectrum and wave functions of gauge fields*

When the VEV $|\langle\hat{\Phi}\rangle| = w$ is sufficiently large, $w \gg m_{\rm KK}$, the spectra of the W and Z towers, $\{m_{W^{(n)}} = k\lambda_{W^{(n)}}, m_{Z^{(n)}} = k\lambda_{Z^{(n)}}\}$, are determined by the zeros of

$$2S(1; \lambda_{W^{(n)}}) C'(1; \lambda_{W^{(n)}}) + \lambda_{W^{(n)}} \sin^2\theta_H = 0,$$

$$2S(1; \lambda_{Z^{(n)}}) C'(1; \lambda_{Z^{(n)}}) + \left(1 + s_\phi^2\right) \lambda_{Z^{(n)}} \sin^2\theta_H = 0, \tag{17}$$

where the functions $C(z;\lambda)$ and $S(z;\lambda)$ are given in Appendix A. Note $(1 + s_\phi^2)^{-1} = \cos^2\theta_W^0$. The lowest modes are $W = W^{(0)}$ and $Z = Z^{(0)}$. For $z_L \gg 1$ their masses at the tree level are approximately given by

$$m_W^{\rm tree} \simeq \frac{\sin\theta_H}{\pi\sqrt{kL}}\, m_{\rm KK},$$

$$m_Z \simeq \sqrt{1 + s_\phi^2}\, \frac{\sin\theta_H}{\pi\sqrt{kL}}\, m_{\rm KK} \simeq \frac{m_W^{\rm tree}}{\cos\theta_W^0}. \tag{18}$$

In this paper $m_Z = m_Z^{\rm tree} = 91.1876\,\text{GeV}$ is taken as one of the input parameters. As typical values we take $m_{\rm KK} = 13\,\text{TeV}$, $\theta_H = 0.1$, $\sin^2\theta_W^0 = 0.230634$, and $\alpha_{\rm EM}(m_Z) = 1/128$, which implies that $z_L = 3.86953 \times 10^{11}$ and $kL = 26.6816$. The precise values determined from Eq. (17) give $m_{Z^{(0)}} \cos\theta_W^0 / m_{W^{(0)}} = 1.00002$.

Each mode of the gauge boson tower has components in the $SO(5) \times U(1)_X$ space. Let us decompose the $SO(5)$ generators $\{T^{jk} = -T^{kj}\,; j,k = 1 \sim 5\}$ into $SU(2)_L$ and $SU(2)_R$ generators $\{T^a_{L/R} = \frac{1}{2}(\frac{1}{2}\epsilon^{ajk} T^{jk} \pm T^{a4})\,; a, j, k = 1 \sim 3\}$ and $\{\hat{T}^p = 2^{-1/2} T^{p5}\,; p = 1 \sim 4\}$. We denote the generator of $U(1)_X$ as Q_X. Then 5D gauge potentials $(\sqrt{k})^{-1}\big[A_\mu^{SO(5)}(x,z) +$

$(g_B/g_A)A_\mu^{U(1)_X}(x,z)Q_X\big]$ have expansion

$$\sum_n W_\mu^{(n)}(x)\left\{h^L_{W^{(n)}}(z)\frac{T_L^1+iT_L^2}{\sqrt{2}}+h^R_{W^{(n)}}(z)\frac{T_R^1+iT_R^2}{\sqrt{2}}+\hat{h}_{W^{(n)}}(z)\frac{\hat{T}^1+i\hat{T}^2}{\sqrt{2}}\right\} \tag{19}$$

for the W tower component,

$$\sum_n Z_\mu^{(n)}(x)\left\{h^L_{Z^{(n)}}(z)T_L^3+h^R_{Z^{(n)}}(z)T_R^3+\hat{h}_{Z^{(n)}}(z)\hat{T}^3+h^B_{Z^{(n)}}(z)\frac{g_B}{g_A}Q_X\right\} \tag{20}$$

for the Z tower component, and

$$\sum_n A_\mu^{\gamma(n)}(x)\left\{h^L_{\gamma^{(n)}}(z)T_L^3+h^R_{\gamma^{(n)}}(z)T_R^3+h^B_{\gamma^{(n)}}(z)\frac{g_B}{g_A}Q_X\right\} \tag{21}$$

for the photon tower component. Here

$$\begin{pmatrix} h^L_{W^{(n)}}(z)\\ h^R_{W^{(n)}}(z)\\ \hat{h}_{W^{(n)}}(z)\end{pmatrix}=\frac{1}{\sqrt{2\,r_{W^{(n)}}}}\begin{pmatrix}(1+\cos\theta_H)\,C(z,\lambda_{W^{(n)}})\\ (1-\cos\theta_H)\,C(z,\lambda_{W^{(n)}})\\ -\sqrt{2}\sin\theta_H\hat{S}(z,\lambda_{W^{(n)}})\end{pmatrix},$$

$$\begin{pmatrix} h^L_{Z^{(n)}}(z)\\ h^R_{Z^{(n)}}(z)\\ \hat{h}_{Z^{(n)}}(z)\\ h^B_{Z^{(n)}}(z)\end{pmatrix}=\frac{1}{\sqrt{2\,r_{Z^{(n)}}}}\left\{\begin{pmatrix}(1+\cos\theta_H)\,C(z,\lambda_{Z^{(n)}})\\ (1-\cos\theta_H)\,C(z,\lambda_{Z^{(n)}})\\ -\sqrt{2}\sin\theta_H\hat{S}(z,\lambda_{Z^{(n)}})\\ 0\end{pmatrix}\right.$$

$$\left.-2\sin\theta_W^0\begin{pmatrix}\sin\theta_W^0\\ \sin\theta_W^0\\ 0\\ \sqrt{1-2\sin^2\theta_W^0}\end{pmatrix}C(z,\lambda_{Z^{(n)}})\right\},$$

$$\begin{pmatrix} h^L_{\gamma^{(n)}}(z)\\ h^R_{\gamma^{(n)}}(z)\\ h^B_{\gamma^{(n)}}(z)\end{pmatrix}=\begin{pmatrix}\sin\theta_W^0\\ \sin\theta_W^0\\ \sqrt{1-2\sin^2\theta_W^0}\end{pmatrix}\begin{cases}\dfrac{1}{\sqrt{r_{\gamma^{(n)}}}}C(z,\lambda_{\gamma^{(n)}}) & \text{for } n\ge 1,\\ \dfrac{1}{\sqrt{kL}} & \text{for } n=0,\end{cases} \tag{22}$$

where the normalization factors $\{r_{W^{(n)}}, r_{Z^{(n)}}, r_{\gamma^{(n)}}\}$ are determined by

$$\int_1^{z_L}\frac{dz}{z}\sum_\alpha |h_n^\alpha(z)|^2=1 \tag{23}$$

in each mode. $\hat{S}(z,\lambda)$is given in Eq. (A1). The photon ($\gamma^{(0)}$) coupling is $e\,Q_{\rm EM}$ where $e=g_w\sin\theta_W^0$ and $Q_{\rm EM}=T_L^3+T_R^3+Q_X$.

2.2. *Spectrum and wave functions of fermion fields*

The spectra $\{m_n=k\lambda_n\}$ of the KK towers of up-type quarks and charged leptons are determined by the zeros of

$$S_L(1;\lambda_n,c)S_R(1;\lambda_n,c)+\sin^2\frac{\theta_H}{2}=0 \tag{24}$$

where $S_{L/R}(z;\lambda,c)$ is defined in Eq. (A3). The bulk mass parameter c of each doublet multiplet is determined such that the lowest value λ_0 reproduces $m_u, m_c, m_t, m_e, m_\mu$, or m_τ. For $\theta_H=0.1$

and $m_{\rm KK} = 13\,{\rm TeV}$, $(c_u, c_c, c_t) = (-0.859, -0.719, -0.275)$ and $(c_e, c_\mu, c_\tau) = (-1.01, -0.793, -0.675)$. Although Eq. (24) is satisfied by either positive or negative c, the negative values for c are chosen in the B-model to have the spectra of the KK towers of down-type quarks and neutrinos consistent with observation as explained below.

Wave functions of $\{u^{(n)}\}$ and $\{e^{(n)}\}$ in the first generation are contained in the $SO(5)$ spinor multiplets $\Psi_{(\mathbf{3},\,\mathbf{4})}$ and $\Psi_{(\mathbf{1},\,\mathbf{4})}$, which are denoted as (u, d, u', d') and (ν_e, e, ν'_e, e'), respectively. (u, d) and (ν_e, e) are $SU(2)_L$ doublets, whereas (u', d') and (ν'_e, e') are $SU(2)_R$ doublets. 5D $(u, u')(x, z)$ fields are expanded as

$$\frac{1}{z^2}\begin{pmatrix} u \\ u' \end{pmatrix} = \sqrt{k}\sum_{n=0}^{\infty}\left\{ u_L^{(n)}(x)\begin{pmatrix} f_L^{u^{(n)}}(z) \\ g_L^{u^{(n)}}(z) \end{pmatrix} + u_R^{(n)}(x)\begin{pmatrix} f_R^{u^{(n)}}(z) \\ g_R^{u^{(n)}}(z) \end{pmatrix}\right\} \tag{25}$$

where, in terms of functions defined in Eq. (A3),

$$\begin{pmatrix} f_L^{u^{(n)}}(z) \\ g_L^{u^{(n)}}(z) \end{pmatrix} = \frac{1}{\sqrt{r_{u^{(n)}L}}}\begin{pmatrix} \cos\frac{1}{2}\theta_H C_L(z, \lambda_{u^{(n)}}, c_u) \\ i\sin\frac{1}{2}\theta_H \hat{S}_L(z, \lambda_{u^{(n)}}, c_u) \end{pmatrix},$$

$$\begin{pmatrix} f_R^{u^{(n)}}(z) \\ g_R^{u^{(n)}}(z) \end{pmatrix} = \frac{1}{\sqrt{r_{u^{(n)}R}}}\begin{pmatrix} \cos\frac{1}{2}\theta_H S_R(z, \lambda_{u^{(n)}}, c_u) \\ i\sin\frac{1}{2}\theta_H \hat{C}_R(z, \lambda_{u^{(n)}}, c_u) \end{pmatrix}. \tag{26}$$

The normalization factor in each mode is determined by the condition

$$\int_1^{z_L} dz\, \left\{ |f_n(z)|^2 + |g_n(z)|^2 \right\} = 1 \quad \text{for} \begin{pmatrix} f_n(z) \\ g_n(z) \end{pmatrix}. \tag{27}$$

One can show that $r_{u^{(n)}L} = r_{u^{(n)}R} \equiv r_{u^{(n)}}$. Note that with the use of Eq. (24) one can express $(f_R^{u^{(n)}}, g_R^{u^{(n)}})$ as

$$\begin{pmatrix} f_R^{u^{(n)}}(z) \\ g_R^{u^{(n)}}(z) \end{pmatrix} = \frac{1}{\sqrt{r'_{u^{(n)}R}}}\begin{pmatrix} \sin\frac{1}{2}\theta_H \hat{S}_R(z, \lambda_{u^{(n)}}, c_u) \\ -i\cos\frac{1}{2}\theta_H C_R(z, \lambda_{u^{(n)}}, c_u) \end{pmatrix}. \tag{28}$$

At $\theta_H = 0$, $\lambda_{u^{(0)}} = 0$ so that the zero mode $u^{(0)}$ has a purely chiral structure; $(u_L^{(0)}, d_L^{(0)})$ becomes an $SU(2)_L$ doublet, whereas $u_R^{(0)}$ and $d_R^{(0)}$ become $SU(2)_L$ singlets.

Wave functions of $\{e^{(n)}\}$ have the same structure as those of $\{u^{(n)}\}$. Formulas for $\{e^{(n)}\}$ are obtained from Eqs. (25)–(28) by replacing $u^{(n)}$ and c_u by $e^{(n)}$ and c_e.

For mass spectra and wave functions of down-type quark and neutrino multiplets the $SO(5)$ singlet fields $\Psi_{(\mathbf{3},\mathbf{1})^\pm}$ and Majorana brane fermions $\hat{\chi}$ intertwine. In general, the coupling constants κ's in the brane interactions given in Eq. (10) are not diagonal in the generation space. The $\mathcal{L}_1$ term in Eq. (10), with $\langle\hat{\Phi}\rangle \neq 0$, leads to the Kobayashi–Maskawa mixing matrix in the quark sector. Further complex $\kappa^{\alpha\beta}$'s give rise to CP-violation phases. In the present paper we analyze the case in which the brane interactions given in Eq. (10) are diagonal in the generation space.

In the first generation the d and d' components in $\Psi_{(\mathbf{3},\,\mathbf{4})}$ and $\Psi_{(\mathbf{3},\mathbf{1})^\pm} \equiv D^\pm$ intertwine with each other. The mass spectrum $\{m_n = k\lambda_n\}$ is determined by the zeros of

$$\left(S_L^Q S_R^Q + \sin^2\frac{\theta_H}{2}\right)\left(\mathcal{S}_{L1}^D\mathcal{S}_{R1}^D - \mathcal{S}_{L2}^D\mathcal{S}_{R2}^D\right) + |\mu|^2 C_R^Q S_R^Q\left(\mathcal{S}_{L1}^D\mathcal{C}_{L1}^D - \mathcal{S}_{L2}^D\mathcal{C}_{L2}^D\right) = 0 \tag{29}$$

where $S_{L/R}^Q = S_{L/R}(1; \lambda, c_u)$, $\mathcal{S}_{Lj}^D = \mathcal{S}_{Lj}(1; \lambda, c_D, \tilde{m}_D)$, etc. The functions $\mathcal{S}_{L/Rj}, \mathcal{C}_{L/Rj}$ are given in Eq. (A5). c_D is the bulk mass parameter of the $\Psi_{(\mathbf{3},\mathbf{1})^\pm}$ field, and $\tilde{m}_D = m_D/k$ where m_D is a Dirac mass connecting D^+ and D^-. The parameter μ represents the strength of a brane interaction among $\Psi_{(\mathbf{3},\,\mathbf{4})}$, $\Psi_{(\mathbf{3},\mathbf{1})^\pm}$, and $\hat{\Phi}$, which is necessary to reproduce a mass of each down-type quark. As typical values we take $(\tilde{m}_d, \tilde{m}_s, \tilde{m}_b) = (1, 1, 1)$ and $(\mu_d, \mu_s, \mu_b) = (0.1, 0.1, 1)$. We determine bulk mass parameters c_D's to reproduce a down-type quark mass in each generation, finding $(c_{D_d}, c_{D_s}, c_{D_b}) = (0.6244, 0.6563, 0.8725)$. We have chosen the negative values for (c_u, c_c, c_t). With positive $c_u, c_c > \frac{1}{2}$ there would arise an exotic extra light mode of charge $Q_{\rm EM} = -\frac{1}{3}$ with a mass much less than $m_{\rm KK}$ in the first and second generations from Eq. (29), which contradicts with the observation. One comment is in order. Eqs. (24) and (29) imply that the up-type quark mass is larger than the corresponding down-type quark mass, although in the first generation $m_u < m_d$. The resolution of this problem is left for future investigation. In this paper we take $(m_u, m_d) = (20, 2.9)$ MeV at the m_Z scale. This does not affect gauge couplings and KK spectra of (u, d) multiplets in the discussions below as $m_u, m_d \ll m_{\rm KK} \sim 13$ TeV.

In the first generation there are two types of series, $\{d^{(n)}; n \geq 0\}$ and $\{D^{(n)}; n \geq 1\}$.[1] For $\theta_H = 0.1$ and $m_{\rm KK} = 13$ TeV, the mass spectra of the KK excited states are $(m_{d^{(1)}}, m_{d^{(2)}}, m_{d^{(3)}}, \cdots) = (12.2, 17.8, 25.1, \cdots)$ TeV and $(m_{D^{(1)}}, m_{D^{(2)}}, m_{D^{(3)}}, \cdots) = (8.4, 16.7, 22.8, \cdots)$ TeV. The (d, d', D^+, D^-) fields are expanded as

$$\frac{1}{z^2}\begin{pmatrix} d \\ d' \\ D^+ \\ D^- \end{pmatrix} = \sqrt{k}\sum_{n=0}^{\infty}\left\{ d_L^{(n)}(x)\begin{pmatrix} f_L^{d^{(n)}}(z) \\ g_L^{d^{(n)}}(z) \\ h_L^{d^{(n)}}(z) \\ k_L^{d^{(n)}}(z) \end{pmatrix} + d_R^{(n)}(x)\begin{pmatrix} f_R^{d^{(n)}}(z) \\ g_R^{d^{(n)}}(z) \\ h_R^{d^{(n)}}(z) \\ k_R^{d^{(n)}}(z) \end{pmatrix}\right\}$$
$$+ \sqrt{k}\sum_{n=1}^{\infty}\left\{ D_L^{(n)}(x)\begin{pmatrix} f_L^{D^{(n)}}(z) \\ g_L^{D^{(n)}}(z) \\ h_L^{D^{(n)}}(z) \\ k_L^{D^{(n)}}(z) \end{pmatrix} + D_R^{(n)}(x)\begin{pmatrix} f_R^{D^{(n)}}(z) \\ g_R^{D^{(n)}}(z) \\ h_R^{D^{(n)}}(z) \\ k_R^{D^{(n)}}(z) \end{pmatrix}\right\}. \tag{30}$$

Wave functions are normalized by

$$\int_1^{z_L} dz\,\left\{|f|^2 + |g|^2 + |h|^2 + |k|^2\right\} = 1 \tag{31}$$

in each mode. The explicit forms of the wave functions are given in Eqs. (B1) and (B2).

In the neutrino sector the ν and ν' components in $\Psi_{(\mathbf{1},\mathbf{4})}$ and the brane Majorana fermion $\hat{\chi}$ mix with each other. $\hat{\chi}(x) = \hat{\chi}^c(x)$ has a Majorana mass M. The brane interaction $\mathcal{L}_3$ in Eq. (10) generates a mixing brane mass term $(m_B/\sqrt{k})(\bar{\hat{\chi}}\nu'_R + \bar{\nu}'_R\hat{\chi})$. Because of the Majorana mass term eigen modes in the neutrino sector have both left- and right-handed components. The mass spectra $\{m_{\nu^{\pm(n)}} = k\lambda_{\nu^{\pm(n)}}\}$ are determined by

$$K_\nu^{\pm} \equiv (k\lambda \mp M)\left\{S_L^L S_R^L + \sin^2\frac{\theta_H}{2}\right\} + \frac{m_B^2}{k}S_R^L C_R^L = 0 \tag{32}$$

where $S_{L/R}^L = S_{L/R}(1; \lambda, c_e)$ etc. in the first generation. For $c_e < -\frac{1}{2}$ and $M > 0$ the gauge-Higgs seesaw mechanism, similar to the inverse seesaw mechanism, is at work in the K_ν^+ series to generate a small neutrino mass [37]:

$$m_{\nu_e} = k\lambda_{\nu^{+(0)}} \simeq \frac{m_e^2 M}{(-2c_e - 1)m_B^2}. \tag{33}$$

There arises no light mode in the K_ν^- series. For the KK excited modes the two series are nearly degenerate; $\lambda_{\nu^{-(n)}} \simeq \lambda_{\nu^{+(n)}}$ for $n \geq 1$. We note that with $c_e > \frac{1}{2}$ the mass of the lightest mode

[1] In Ref. [36] the series has been decomposed into $\{d^{(n)}\}$, $\{d'^{(n)}\}$, $\{D^{+(n)}\}$, and $\{D^{-(n)}\}$.

becomes $\sim m_e^2 M z_L^{2c_e+1}/(2c_e+1)m_B^2$ so that unnecessarily large m_B is required to reproduce a small neutrino mass. Further $c_e > \frac{1}{2}$ yields an additional exotic mode with a mass of $O(10\,\text{GeV})$. We adopt $c_e, c_\mu, c_\tau < -\frac{1}{2}$.

The Majorana field $\hat{\chi}$ is decomposed as

$$\hat{\chi} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \hat{\chi}^c = e^{i\delta_C} \begin{pmatrix} +\sigma^2 \eta^* \\ -\sigma^2 \xi^* \end{pmatrix} = \hat{\chi}. \tag{34}$$

The fields are expanded as

$$\begin{aligned}
\frac{1}{z^2}\begin{pmatrix} \nu \\ \nu' \end{pmatrix} &= \sqrt{k}\sum_{n=0}^{\infty}\left\{ \nu_L^{+(n)}(x)\begin{pmatrix} f_L^{\nu^{+(n)}}(z) \\ g_L^{\nu^{+(n)}}(z) \end{pmatrix} + \nu_R^{+(n)}(x)\begin{pmatrix} f_R^{\nu^{+(n)}}(z) \\ g_R^{\nu^{+(n)}}(z) \end{pmatrix} \right\} \\
&\quad + \sqrt{k}\sum_{n=1}^{\infty}\left\{ \nu_L^{-(n)}(x)\begin{pmatrix} f_L^{\nu^{-(n)}}(z) \\ g_L^{\nu^{-(n)}}(z) \end{pmatrix} + \nu_R^{-(n)}(x)\begin{pmatrix} f_R^{\nu^{-(n)}}(z) \\ g_R^{\nu^{-(n)}}(z) \end{pmatrix} \right\}, \\
\eta &= \sum_{n=0}^{\infty} \nu_L^{+(n)}(x)\, h^{\nu^{+(n)}} + \sum_{n=1}^{\infty} \nu_L^{-(n)}(x)\, h^{\nu^{-(n)}}, \\
\nu_R^{\pm(n)} &= \pm e^{i\delta_C}\sigma^2 \left(\nu_L^{\pm(n)}\right)^*.
\end{aligned} \tag{35}$$

Wave functions are normalized as

$$\int_1^{z_L} dz\left\{|f_L|^2 + |g_L|^2 + |f_R|^2 + |g_R|^2\right\} + |h|^2 = 1 \tag{36}$$

in each mode. The explicit forms of the wave functions are given in Eq. (B3).

3. *W*- and *Z*-couplings

The γ-, W-, and Z-couplings of the fermion fields are contained in the part of the action

$$\int d^4x \int_1^{z_L} \frac{dz}{k} \sum_J \bar{\check{\Psi}}^J \gamma^\mu (-i)\left(g_A A_\mu^{SO(5)} + g_B Q_X A_\mu^{U(1)_X}\right) \check{\Psi}^J \tag{37}$$

where $\check{\Psi}^J = z^{-2}\Psi^J$. By inserting the KK expansions of the gauge and fermion fields into Eq. (37), $\gamma^{(n)}$, $W^{(n)}$, and $Z^{(n)}$ couplings among the fermion KK modes are evaluated. The photon $\gamma = \gamma^{(0)}$ couplings are universal. They are diagonal in the KK space, and are given by $e\,Q_{\text{EM}} = g_w \sin\theta_W^0 (T_L^3 + T_R^3 + Q_X)$.

The $W = W^{(0)}$ couplings in the first generation of the quark multiplets are given by

$$\begin{aligned}
\frac{g_w}{\sqrt{2}} W_\mu \Bigg[&\sum_{n,m=0}^{\infty}\left\{ \hat{g}_{L,nm}^{Wud} \bar{u}_L^{(n)} \gamma^\mu d_L^{(m)} + \hat{g}_{R,nm}^{Wud} \bar{u}_R^{(n)} \gamma^\mu d_R^{(m)} \right\} \\
&+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{ \hat{g}_{L,nm}^{WuD} \bar{u}_L^{(n)} \gamma^\mu D_L^{(m)} + \hat{g}_{R,nm}^{WuD} \bar{u}_R^{(n)} \gamma^\mu D_R^{(m)} \right\} \Bigg]
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
\hat{g}_{L/R,nm}^{Wud} = \sqrt{kL}\int_1^{z_L} dz \Bigg\{ &h_{W^{(0)}}^L f_{L/R}^{u^{(n)}*} f_{L/R}^{d^{(m)}} + h_{W^{(0)}}^R g_{L/R}^{u^{(n)}*} g_{L/R}^{d^{(m)}} \\
&+ \frac{i}{\sqrt{2}} \hat{h}_{W^{(0)}} \left(f_{L/R}^{u^{(n)}*} g_{L/R}^{d^{(m)}} - g_{L/R}^{u^{(n)}*} f_{L/R}^{d^{(m)}} \right) \Bigg\}.
\end{aligned} \tag{39}$$

$\hat{g}^{WuD}_{L/R,nm}$ is obtained by replacing $d^{(m)}$ by $D^{(m)}$ in Eq. (39). The values in the SM correspond to $\hat{g}^{Wud}_{L,00} = 1$ and $\hat{g}^{Wud}_{R,00} = 0$. In the RS space off-diagonal components of $\hat{g}^{Wud}_{L/R,nm}$ are nonvanishing. For $\theta_H = 0.1$ and $m_{\rm KK} = 13$ TeV, for instance, the coupling matrices are given by

$$\hat{g}^{Wud}_L = \begin{pmatrix} 0.997645 & -0.024904 & 0.000020 & -0.002827 & 10^{-6} & \cdots \\ -0.024904 & 0.002498 & 0.028389 & 10^{-7} & 0.000510 & \\ 0.000020 & 0.028389 & 0.997618 & -0.024548 & 0.000022 & \\ -0.002827 & 10^{-7} & -0.024548 & 0.002498 & 0.027021 & \\ 10^{-6} & 0.000510 & 0.000022 & 0.027021 & 0.997620 & \\ \vdots & & & & & \ddots \end{pmatrix},$$

$$\hat{g}^{Wud}_R = \begin{pmatrix} 10^{-12} & 10^{-7} & 10^{-7} & 10^{-9} & 10^{-7} & \cdots \\ 10^{-8} & 0.002498 & 0.024145 & 10^{-8} & 10^{-6} & \\ 10^{-7} & 0.024145 & 0.997632 & -0.022564 & 0.000018 & \\ 10^{-10} & 10^{-8} & -0.022564 & 0.002498 & 0.025826 & \\ 10^{-8} & 10^{-6} & 0.000018 & 0.025826 & 0.997625 & \\ \vdots & & & & & \ddots \end{pmatrix} \tag{40}$$

where 10^{-7}, for instance, implies $O(10^{-7})$. Notice that the couplings for the KK excited states are nearly vector-like. However, they have very small axial-vector components; $\hat{g}^{Wud}_A = \frac{1}{2}(\hat{g}^{Wud}_R - \hat{g}^{Wud}_L)$ is $O(10^{-3})$ or less. As is shown below, those small numbers must be properly taken into account to establish the coupling sum rules. The $\hat{g}^{WuD}_{L/R,nm}$ couplings are very small; $\max |\hat{g}^{WuD}_{L/R,nm}| \sim 2 \times 10^{-6}$.

In the lepton sector there are two types of the neutrino towers, $\{\nu^{+(n)}\}$ (ν_{e1} series) and $\{\nu^{-(n)}\}$ (ν_{e2} series), both of which have W-couplings. In the first generation the couplings are given by

$$\frac{g_w}{\sqrt{2}} W_\mu \Bigg[\sum_{n,m=0}^{\infty} \Big\{ \hat{g}^{W\nu_{e1}e}_{L,nm}\, \bar{\nu}^{+(n)}_{eL} \gamma^\mu e^{(m)}_L + \hat{g}^{W\nu_{e1}e}_{R,nm}\, \bar{\nu}^{+(n)}_{eR} \gamma^\mu e^{(m)}_R \Big\} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \Big\{ \hat{g}^{W\nu_{e2}e}_{L,nm}\, \bar{\nu}^{-(n)}_{eL} \gamma^\mu e^{(m)}_L + \hat{g}^{W\nu_{e2}e}_{R,nm}\, \bar{\nu}^{-(n)}_{eR} \gamma^\mu e^{(m)}_R \Big\} \Bigg] \tag{41}$$

where

$$\hat{g}^{W\nu_{e1}e}_{L,nm} = \sqrt{kL} \int_1^{z_L} dz \Bigg\{ h^L_{W^{(0)}} f^{\nu_e^{+(n)}*}_{L/R} f^{e^{(m)}}_{L/R} + h^R_{W^{(0)}} g^{\nu_e^{+(n)}*}_{L/R} g^{e^{(m)}}_{L/R} + \frac{i}{\sqrt{2}} \hat{h}_{W^{(0)}} \Big(f^{\nu_e^{+(n)}*}_{L/R} g^{e^{(m)}}_{L/R} - g^{\nu_e^{+(n)}*}_{L/R} f^{e^{(m)}}_{L/R} \Big) \Bigg\},$$

$$\hat{g}^{W\nu_{e2}e}_{L,nm} = \sqrt{kL} \int_1^{z_L} dz \Bigg\{ h^L_{W^{(0)}} f^{\nu_e^{-(n)}*}_{L/R} f^{e^{(m)}}_{L/R} + h^R_{W^{(0)}} g^{\nu_e^{-(n)}*}_{L/R} g^{e^{(m)}}_{L/R} + \frac{i}{\sqrt{2}} \hat{h}_{W^{(0)}} \Big(f^{\nu_e^{-(n)}*}_{L/R} g^{e^{(m)}}_{L/R} - g^{\nu_e^{-(n)}*}_{L/R} f^{e^{(m)}}_{L/R} \Big) \Bigg\}. \tag{42}$$

The values in the SM correspond to $\hat{g}_{L,00}^{W\nu_{e1}e} = 1$ and $\hat{g}_{R,00}^{W\nu_{e1}e} = 0$. For $\theta_H = 0.1$, $m_{\rm KK} = 13$ TeV, and $M_e = 10^3$ TeV, for instance, the coupling matrices are given by

$$\hat{g}_L^{W\nu_{e1}e} = \begin{pmatrix} 0.997647 & -0.023607 & 0.000018 & -0.002964 & 10^{-7} & \cdots \\ -0.016689 & 0.001766 & 0.020087 & 10^{-8} & 0.000319 & \\ -0.000013 & -0.020092 & -0.705422 & 0.017130 & 0.000015 & \\ 0.002094 & 10^{-8} & 0.017115 & -0.001765 & -0.019154 & \\ 10^{-7} & -0.000318 & -0.000015 & -0.019170 & -0.705382 & \\ \vdots & & & & & \ddots \end{pmatrix},$$

$$\hat{g}_R^{W\nu_{e1}e} = \begin{pmatrix} 10^{-23} & 10^{-17} & 10^{-17} & 10^{-19} & 10^{-17} & \cdots \\ 10^{-9} & 0001766 & 0.016811 & 10^{-8} & -0.000095 & \\ 10^{-8} & -0.016814 & -0.705432 & 0.015656 & -0.000012 & \\ 10^{-10} & 10^{-8} & 0.015642 & -0.001765 & -0.018204 & \\ 10^{-9} & 0.000095 & -0.000012 & -0.018219 & -0.705385 & \\ \vdots & & & & & \ddots \end{pmatrix} \tag{43}$$

and

$$\hat{g}_L^{W\nu_{e2}e} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0.016688 & -0.001766 & -0.020086 & 10^{-8} & -0.000319 & \\ 0.000013 & 0.020092 & 0.705422 & -0.017130 & 0.000015 & \\ -0.002094 & 10^{-8} & -0.017107 & 0.001764 & 0.019145 & \\ 10^{-7} & 0.000319 & 0.000015 & 0.019171 & 0.705414 & \\ \vdots & & & & & \ddots \end{pmatrix},$$

$$\hat{g}_R^{W\nu_{e2}e} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 10^{-9} & -0001766 & -0.016810 & 10^{-8} & 0.000095 & \\ 10^{-8} & 0.016814 & 0.705432 & -0.015656 & 0.000012 & \\ 10^{-10} & 10^{-8} & -0.015635 & 0.001764 & 0.018196 & \\ 10^{-9} & -0.000095 & 0.000012 & 0.018220 & 0.705418 & \\ \vdots & & & & & \ddots \end{pmatrix}. \tag{44}$$

Here we have set $\hat{g}_{L/R,0n}^{W\nu_{e2}e} = 0$. It is seen that in the part of the KK excited states, $\hat{g}_{L,nm}^{W\nu_{e1}e} \sim \hat{g}_{R,nm}^{W\nu_{e1}e}$ and $\hat{g}_{L,nm}^{W\nu_{e2}e} \sim \hat{g}_{R,nm}^{W\nu_{e2}e}$ for $n, m \geq 1$, and $\hat{g}_{L/R,nm}^{W\nu_{e1}e} + \hat{g}_{L/R,nm}^{W\nu_{e2}e} \sim 0$ for $n \geq 1$, $m \geq 0$. Those couplings are almost vector-like. As in the (u, d) case, they have very small axial-vector components; $\hat{g}_A^{W\nu_{e1}e}$ and $\hat{g}_A^{W\nu_{e2}e}$ are $O(10^{-3})$ or less.

The W-couplings of the zero modes, namely the couplings of quarks and leptons in three generations, are summarized in Table 2. Except for (t, b) the W-couplings are universal to high accuracy; $\hat{g}_L^W \sim 0.997645 \equiv \hat{g}^{W,\rm GHU}$ and $\hat{g}_R^W \sim 0$. The observed lepton coupling should be identified as $g_w^{\rm obs} = g_w \, \hat{g}^{W,\rm GHU}$.

Table 2. The W-couplings of quarks and leptons in units of g_w for $\theta_H = 0.1$, $m_{\rm KK} = 13\,$TeV, and $M = 10^3\,$TeV.

	$\hat{g}_L^W$	$\hat{g}_R^W$
(ν_e, e)	0.997647	3×10^{-23}
(ν_μ, μ)	0.997644	3×10^{-21}
(ν_τ, τ)	0.997642	4×10^{-20}
(u, d)	0.997645	2×10^{-12}
(c, s)	0.997643	8×10^{-10}
(t, b)	0.997969	0.000011

The $Z = Z^{(0)}$ couplings are evaluated similarly. The couplings in the up-type quark sector are given in the form

$$\frac{g_w}{\cos\theta_W^0} Z_\mu \sum_{n,m=0}^{\infty} \left\{ \hat{g}^{Zu}_{L,nm} \bar{u}_L^{(n)} \gamma^\mu u_L^{(m)} + \hat{g}^{Zu}_{R,nm} \bar{u}_R^{(n)} \gamma^\mu u_R^{(m)} \right\}. \tag{45}$$

As is suggested from the structure of the wave functions in Eq. (22), it is convenient to decompose the Z-couplings into the $U(1)_{\rm EM}$ part and the rest. We define

$$h^Q_{Z^{(n)}}(z) = \sqrt{\frac{2}{r_{Z^{(n)}}}}\, C(z, \lambda_{Z^{(n)}}) \tag{46}$$

and write

$$\hat{g}^{Zu}_{L/R,nm} = \hat{g}^{Zu,su2}_{L/R,nm} - \sin^2\theta_W^0\, \hat{g}^{Zu,\rm EM}_{L/R,nm}. \tag{47}$$

Then

$$\begin{aligned}
\hat{g}^{Zu,su2}_{L/R,nm} &= T_u^3 \cos\theta_W^0 \sqrt{kL} \int_1^{z_L} dz \left\{ h^L_{Z^{(0)}} f^{u^{(n)}*}_{L/R} f^{u^{(m)}}_{L/R} + h^R_{Z^{(0)}} g^{u^{(n)}*}_{L/R} g^{u^{(m)}}_{L/R} \right. \\
&\qquad \left. + \frac{i}{\sqrt{2}} \hat{h}_{Z^{(0)}} \left(f^{u^{(n)}*}_{L/R} g^{u^{(m)}}_{L/R} - g^{u^{(n)}*}_{L/R} f^{u^{(m)}}_{L/R} \right) \right\}, \\
\hat{g}^{Zu,\rm EM}_{L/R,nm} &= Q_u \cos\theta_W^0 \sqrt{kL} \int_1^{z_L} dz\, h^Q_{Z^{(0)}} \left\{ f^{u^{(n)}*}_{L/R} f^{u^{(m)}}_{L/R} + g^{u^{(n)}*}_{L/R} g^{u^{(m)}}_{L/R} \right\}
\end{aligned} \tag{48}$$

where $T_u^3 = \frac{1}{2}$ and $Q_u = \frac{2}{3}$. With the normalization $r_{Z^{(0)}}$ in Eq. (22), $\hat{g}^{Zu,su2}_{L,00} = 0.498844$, $\hat{g}^{Zu,su2}_{R,00} = 5 \times 10^{-12}$, $\hat{g}^{Zu,\rm EM}_{L,00} = 0.666791$, and $\hat{g}^{Zu,\rm EM}_{R,00} = 0.666725$ for $\theta_H = 0.1$, $m_{\rm KK} = 13\,$TeV, and $\sin^2\theta_W^0 = 0.230634$. The Z-coupling matrices are given by

$$\hat{g}_L^{Zu} = \begin{pmatrix} 0.345059 & -0.012453 & 0.000009 & -0.001414 & 10^{-7} & \cdots \\ -0.012453 & -0.152531 & 0.014195 & -0.000004 & 0.000255 & \\ 0.000009 & 0.014195 & 0.345047 & -0.012275 & 0.000010 & \\ -0.001414 & -0.000004 & -0.012275 & -0.152531 & 0.013511 & \\ 10^{-7} & 0.000255 & 0.000010 & 0.013511 & 0.345048 & \\ \vdots & & & & & \ddots \end{pmatrix},$$

$$\hat{g}_R^{Zu} = \begin{pmatrix} -0.153769 & 0.000012 & 10^{-7} & -0.000011 & 10^{-7} & \cdots \\ 0.000012 & -0.152536 & 0.012073 & -0.000003 & 0.000001 & \\ 10^{-7} & 0.012073 & 0.345054 & -0.011282 & 0.000008 & \\ -0.000011 & -0.000003 & -0.011282 & -0.152532 & 0.012913 & \\ 10^{-7} & 0.000001 & 0.000008 & 0.012913 & 0.345050 & \\ \vdots & & & & & \ddots \end{pmatrix}. \tag{49}$$

The couplings in the space of KK excited states, namely $n, m \geq 1$ elements of $\hat{g}_{nm}^{Zu}$, are nearly vector-like. The axial-vector components are small; $\hat{g}_A^{Zu} = \frac{1}{2}(\hat{g}_R^{Zu} - \hat{g}_L^{Zu})$ is $O(10^{-3})$ or less.

In the down-type quark sector there are $\{d^{(n)}\}$ and $\{D^{(n)}\}$ series. The couplings are written as

$$\begin{aligned} \frac{g_w}{\cos\theta_W^0} Z_\mu \Bigg[& \sum_{n,m=0}^{\infty} \left\{ \hat{g}_{L,nm}^{Zdd} \bar{d}_L^{(n)} \gamma^\mu d_L^{(m)} + \hat{g}_{R,nm}^{Zdd} \bar{d}_R^{(n)} \gamma^\mu d_R^{(m)} \right\} \\ & + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \left\{ \hat{g}_{L,nm}^{ZdD} \bar{d}_L^{(n)} \gamma^\mu D_L^{(m)} + \hat{g}_{R,nm}^{ZdD} \bar{d}_R^{(n)} \gamma^\mu D_R^{(m)} \right\} \\ & + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \left\{ \hat{g}_{L,nm}^{ZDd} \bar{D}_L^{(n)} \gamma^\mu d_L^{(m)} + \hat{g}_{R,nm}^{ZDd} \bar{D}_R^{(n)} \gamma^\mu d_R^{(m)} \right\} \\ & + \sum_{n,m=1}^{\infty} \left\{ \hat{g}_{L,nm}^{ZDD} \bar{D}_L^{(n)} \gamma^\mu D_L^{(m)} + \hat{g}_{R,nm}^{ZDD} \bar{D}_R^{(n)} \gamma^\mu D_R^{(m)} \right\} \Bigg]. \end{aligned} \tag{50}$$

With the decomposition $\hat{g}_{L/R,nm}^{Zdd} = \hat{g}_{L/R,nm}^{Zdd,su2} - \sin^2\theta_W^0\, \hat{g}_{L/R,nm}^{Zdd,\mathrm{EM}}$ etc., the couplings are given by

$$\begin{aligned} \hat{g}_{L/R,nm}^{Zdd,su2} &= T_d^3 \cos\theta_W^0 \sqrt{kL} \int_1^{z_L} dz \left\{ h_{Z^{(0)}}^L f_{L/R}^{d^{(n)}*} f_{L/R}^{d^{(m)}} + h_{Z^{(0)}}^R g_{L/R}^{d^{(n)}*} g_{L/R}^{d^{(m)}} \right. \\ &\quad \left. + \frac{i}{\sqrt{2}} \hat{h}_{Z^{(0)}} \left(f_{L/R}^{d^{(n)}*} g_{L/R}^{d^{(m)}} - g_{L/R}^{d^{(n)}*} f_{L/R}^{d^{(m)}} \right) \right\}, \\ \hat{g}_{L/R,nm}^{Zdd,\mathrm{EM}} &= Q_d \cos\theta_W^0 \sqrt{kL} \int_1^{z_L} dz\, h_{Z^{(0)}}^Q \left(f_{L/R}^{d^{(n)}*} f_{L/R}^{d^{(m)}} + g_{L/R}^{d^{(n)}*} g_{L/R}^{d^{(m)}} \right. \\ &\quad \left. + h_{L/R}^{d^{(n)}*} h_{L/R}^{d^{(m)}} + k_{L/R}^{d^{(n)}*} k_{L/R}^{d^{(m)}} \right). \end{aligned} \tag{51}$$

Here $T_d^3 = -\frac{1}{2}$ and $Q_d = -\frac{1}{3}$. The expressions for the $\hat{g}^{ZdD}, g^{ZDd}, g^{ZDD}$ components are obtained by the replacement $d^{(n)} \to D^{(n)}$ and/or $d^{(m)} \to D^{(m)}$ in Eq. (51). Note that $\hat{g}^{Zdd,su2}_{L,00} = -0.498844$, $\hat{g}^{Zdd,su2}_{R,00} = -1 \times 10^{-13}$, $\hat{g}^{Zdd,\mathrm{EM}}_{L,00} = -0.333395$, and $\hat{g}^{Zdd,\mathrm{EM}}_{R,00} = -0.333372$ for $\theta_H = 0.1$, $m_{\mathrm{KK}} = 13$ TeV, and $\sin^2\theta_W^0 = 0.230634$.

The Z-coupling matrices are given by

$$
\hat{g}_L^{Zdd} = \begin{pmatrix}
-0.421952 & 0.012453 & -0.000011 & 0.001414 & 10^{-7} & \cdots \\
0.012453 & 0.075640 & -0.014195 & 0.000002 & -0.000255 & \\
-0.000011 & -0.014195 & -0.421937 & 0.012275 & -0.000012 & \\
0.001414 & 0.000002 & 0.012275 & 0.0756407 & -0.013511 & \\
10^{-7} & -0.000255 & -0.000012 & -0.013511 & -0.421938 & \\
\vdots & & & & & \ddots
\end{pmatrix},
$$

$$
\hat{g}_R^{Zdd} = \begin{pmatrix}
0.076887 & 10^{-8} & 10^{-8} & 10^{-10} & 10^{-8} & \cdots \\
10^{-8} & 0.075643 & -0.012073 & 0.000002 & -0.000001 & \\
10^{-8} & -0.012073 & -0.421945 & 0.011282 & -0.000010 & \\
10^{-10} & 0.000002 & 0.011282 & 0.075641 & -0.012913 & \\
10^{-8} & -0.000001 & -0.000010 & -0.012913 & -0.421940 & \\
\vdots & & & & & \ddots
\end{pmatrix}. \tag{52}
$$

As in the case of $\hat{g}^{Zu}$, the couplings in the space of the KK excited states are almost vector-like; $\hat{g}^{Zdd}_A = \frac{1}{2}(\hat{g}^{Zdd}_R - \hat{g}^{Zdd}_L)$ is $O(10^{-3})$ or less. The $su2$ components of $\{D^{(n)}\}$ series are very small; $|\hat{g}^{ZDD,su2}_{L/R,nm}|$ is $O(10^{-12})$ or less. Off-diagonal elements of $\hat{g}^{ZDD,\mathrm{EM}}_{L/R}$ are $O(10^{-6})$ or less. All diagonal elements are about -0.33339 and $|\hat{g}^{ZDD,\mathrm{EM}}_{L,nn} - \hat{g}^{ZDD,\mathrm{EM}}_{R,nn}| = O(10^{-6})$. Also $\hat{g}^{ZdD}_{L/R,nm} = (\hat{g}^{ZDd}_{L/R,mn})^* = O(10^{-6})$ or less.

The Z-couplings of charged lepton multiplets have the same structure as in the up-type quark sector. The couplings of the electron multiplet are

$$
\frac{g_w}{\cos\theta_W^0} Z_\mu \sum_{n,m=0}^{\infty} \left\{ \hat{g}^{Ze}_{L,nm} \bar{e}_L^{(n)} \gamma^\mu e_L^{(m)} + \hat{g}^{Ze}_{R,nm} \bar{e}_R^{(n)} \gamma^\mu e_R^{(m)} \right\}, \tag{53}
$$

where $\hat{g}^{Ze}_{L/R,nm}$ are given by the expressions obtained by replacing, in Eqs. (47) and (48), $u^{(n)}$ by $e^{(n)}$, T_u^3 by $T_e^3 = -\frac{1}{2}$, and Q_u by $Q_e = -1$. $\hat{g}^{Ze,su2}_{L,00} = -0.498845$, $\hat{g}^{Ze,su2}_{R,00} = -4 \times 10^{-15}$, $\hat{g}^{Ze,\mathrm{EM}}_{L,00} = -1.00019$, and $\hat{g}^{Ze,\mathrm{EM}}_{R,00} = -1.00009$ for $\theta_H = 0.1$, $m_{\mathrm{KK}} = 13$ TeV, and $\sin^2\theta_W^0 = 0.230634$. The

Z-coupling matrices are given by

$$\hat{g}_L^{Ze} = \begin{pmatrix} -0.268168 & 0.011804 & -0.000006 & 0.001482 & 10^{-7} & \cdots \\ 0.011804 & 0.229421 & -0.014207 & 0.000006 & -0.000256 & \\ -0.000006 & -0.014207 & -0.268158 & 0.012113 & -0.000008 & \\ 0.001482 & 0.000006 & 0.012113 & 0.229421 & -0.013556 & \\ 10^{-7} & -0.000256 & -0.000008 & -0.013556 & -0.268158 & \\ \vdots & & & & & \ddots \end{pmatrix},$$

$$\hat{g}_R^{Ze} = \begin{pmatrix} 0.230654 & 10^{-7} & 10^{-8} & 10^{-7} & 10^{-8} & \cdots \\ 10^{-7} & 0.229429 & -0.011890 & 0.000004 & 0.000067 & \\ 10^{-8} & -0.011890 & -0.268163 & 0.011071 & -0.000006 & \\ 10^{-7} & 0.000004 & 0.011071 & 0.229423 & -0.012884 & \\ 10^{-8} & 0.000067 & -0.000006 & -0.012884 & -0.268160 & \\ \vdots & & & & & \ddots \end{pmatrix}. \tag{54}$$

The axial-vector components in the space of KK excited states are small; $\hat{g}^{Ze}_{A,nm}$ $(n, m \geq 1)$ are $O(10^{-3})$ or less.

In the neutrino sector the couplings are given by

$$\begin{aligned} \frac{g_w}{\cos\theta_W^0} Z_\mu \Bigg[& \sum_{n,m=0}^{\infty} \left\{ \hat{g}^{Z\nu_{e11}}_{L,nm} \bar{\nu}^{+(n)}_{eL} \gamma^\mu \nu^{+(m)}_{eL} + \hat{g}^{Z\nu_{e11}}_{R,nm} \bar{\nu}^{+(n)}_{eR} \gamma^\mu \nu^{+(m)}_{eR} \right\} \\ & + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ \hat{g}^{Z\nu_{e12}}_{L,nm} \bar{\nu}^{+(n)}_{eL} \gamma^\mu \nu^{-(m)}_{eL} + \hat{g}^{Z\nu_{e12}}_{R,nm} \bar{\nu}^{+(n)}_{eR} \gamma^\mu \nu^{-(m)}_{eR} \right\} \\ & + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \hat{g}^{Z\nu_{e21}}_{L,nm} \bar{\nu}^{-(n)}_{eL} \gamma^\mu \nu^{+(m)}_{eL} + \hat{g}^{Z\nu_{e21}}_{R,nm} \bar{\nu}^{-(n)}_{eR} \gamma^\mu \nu^{+(m)}_{eR} \right\} \\ & + \sum_{n,m=1}^{\infty} \left\{ \hat{g}^{Z\nu_{e22}}_{L,nm} \bar{\nu}^{-(n)}_{eL} \gamma^\mu \nu^{-(m)}_{eL} + \hat{g}^{Z\nu_{e22}}_{R,nm} \bar{\nu}^{-(n)}_{eR} \gamma^\mu \nu^{-(m)}_{eR} \right\} \Bigg] \end{aligned} \tag{55}$$

where

$$\begin{aligned} \hat{g}^{Z\nu_{eab}}_{L/R,nm} = T^3_{\nu_e} \cos\theta_W^0 \sqrt{kL} \int_1^{z_L} dz \, \Bigg\{ & h^L_{Z^{(0)}} f^{\nu_e^{a(n)}*}_{L/R} f^{\nu_e^{b(m)}}_{L/R} + h^R_{Z^{(0)}} g^{\nu_e^{a(n)}*}_{L/R} g^{\nu_e^{b(m)}}_{L/R} \\ & + \frac{i}{\sqrt{2}} \hat{h}_{Z^{(0)}} \left(f^{\nu_e^{a(n)}*}_{L/R} g^{\nu_e^{b(m)}}_{L/R} - g^{\nu_e^{a(n)}*}_{L/R} f^{\nu_e^{b(m)}}_{L/R} \right) \Bigg\}. \end{aligned} \tag{56}$$

Here $T^3_{\nu_e} = \frac{1}{2}$ and we have denoted that $(\nu_e^{1(n)}, \nu_e^{2(n)}) = (\nu_e^{+(n)}, \nu_e^{-(n)})$. Note $\hat{g}^{Z\nu_{e21}}_{L/R} = (\hat{g}^{Z\nu_{e12}}_{L/R})^\dagger$. We set $\hat{g}^{Z\nu_{e12}}_{L/R,n0}, \hat{g}^{Z\nu_{e21}}_{L/R,0n}, \hat{g}^{Z\nu_{e22}}_{L/R,0n}, \hat{g}^{Z\nu_{e22}}_{L/R,n0} = 0$. The Z-coupling matrices are given by

$$
\hat{g}^{Z\nu_{e11}}_L = \begin{pmatrix}
0.498845 & -0.008345 & -0.000008 & 0.001047 & 10^{-7} & \cdots \\
-0.008345 & 0.000624 & -0.007102 & 10^{-8} & -0.000113 & \\
-0.000008 & -0.007102 & 0.249412 & -0.006051 & 0.000007 & \\
0.001047 & 10^{-8} & -0.006051 & 0.000624 & 0.006772 & \\
10^{-7} & -0.000113 & 0.000007 & 0.006772 & 0.249384 & \\
\vdots & & & & & \ddots
\end{pmatrix},
$$

$$
\hat{g}^{Z\nu_{e11}}_R = \begin{pmatrix}
10^{-32} & 10^{-18} & 10^{-17} & 10^{-19} & 10^{-18} & \cdots \\
10^{-18} & 0.000624 & -0.005944 & 10^{-8} & 0.000034 & \\
10^{-17} & -0.005944 & 0.249417 & -0.005531 & 0.000006 & \\
10^{-19} & 10^{-8} & -0.005531 & 0.000624 & 0.006436 & \\
10^{-18} & 0.000034 & 0.000006 & 0.006436 & 0.249386 & \\
\vdots & & & & & \ddots
\end{pmatrix},
$$

$$
\hat{g}^{Z\nu_{e22}}_L = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0.000624 & -0.007102 & 10^{-8} & -0.000113 & \\
0 & -0.007102 & 0.249412 & -0.006048 & 0.000007 & \\
0 & 10^{-8} & -0.006048 & 0.000623 & 0.006769 & \\
0 & -0.000113 & 0.000007 & 0.006769 & 0.249407 & \\
\vdots & & & & & \ddots
\end{pmatrix},
$$

$$
\hat{g}^{Z\nu_{e22}}_R = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0.000624 & -0.005943 & 10^{-8} & 0.000034 & \\
0 & -0.005943 & 0.249417 & -0.005528 & 0.000006 & \\
0 & 10^{-8} & -0.005528 & 0.000623 & 0.006433 & \\
0 & 0.000034 & 0.000006 & 0.006433 & 0.249409 & \\
\vdots & & & & & \ddots
\end{pmatrix},
$$

$$
\hat{g}^{Z\nu_{e12}}_L = \begin{pmatrix}
0 & 0.008344 & 0.000008 & -0.001047 & 10^{-7} & \cdots \\
0 & -0.000624 & 0.007102 & 10^{-8} & 0.000113 & \\
0 & 0.007102 & -0.249412 & 0.006048 & -0.000007 & \\
0 & 10^{-8} & 0.006051 & -0.000623 & -0.006772 & \\
0 & 0.000113 & -0.000007 & -0.006769 & -0.249395 & \\
\vdots & & & & & \ddots
\end{pmatrix},
$$

$$
\hat{g}^{Z\nu_{e12}}_R = \begin{pmatrix}
0 & 10^{-18} & 10^{-18} & 10^{-19} & 10^{-18} & \cdots \\
0 & -0.000624 & 0.005944 & 10^{-8} & -0.000034 & \\
0 & 0.005943 & -0.249417 & -0.005528 & -0.000006 & \\
0 & 10^{-8} & 0.005531 & -0.000623 & -0.006436 & \\
0 & -0.000034 & -0.000006 & -0.006433 & -0.249397 & \\
\vdots & & & & & \ddots
\end{pmatrix}. \qquad (57)
$$

Table 3. The Z-couplings of leptons and quarks in units of $g_w/\cos\theta_W^0$ for $\theta_H = 0.1$, $m_{\rm KK} = 13$ TeV, and $M = 10^3$ TeV. For reference the SM values $T_L^3 - \sin^2\theta_W^{\rm SM} Q_{\rm EM}$ with $\sin^2\theta_W^{\rm SM} = 0.2312$ are listed as well.

	$\hat{g}_L^Z$	$\hat{g}_R^Z$		$\hat{g}_L^Z$	$\hat{g}_R^Z$
ν_e	0.498845	2×10^{-32}	e	− 0.268168	0.230654
ν_μ	0.498843	1×10^{-32}	μ	− 0.268167	0.230654
ν_τ	0.498842	1×10^{-32}	τ	− 0.268166	0.229961
SM	0.5	0	SM	− 0.2688	0.2312
u	0.345059	− 0.153769	d	− 0.421952	0.076887
c	0.345058	− 0.153769	s	− 0.421950	0.076887
t	0.345390	− 0.153439	b	− 0.421945	0.076890
SM	0.3459	− 0.1541	SM	− 0.4229	0.0771

Notice that in the space of KK excited states $\hat{g}_{L,nm}^{Z\nu_{e11}} \sim \hat{g}_{L,nm}^{Z\nu_{e22}} \sim -\hat{g}_{L,nm}^{Z\nu_{e12}} \sim -\hat{g}_{L,nm}^{Z\nu_{e21}}$, $\hat{g}_{R,nm}^{Z\nu_{e11}} \sim \hat{g}_{R,nm}^{Z\nu_{e22}} \sim -\hat{g}_{R,nm}^{Z\nu_{e12}} \sim -\hat{g}_{R,nm}^{Z\nu_{e21}}$, and $|\hat{g}_{R,nm}^{Z\nu_{eab}} - \hat{g}_{L,nm}^{Z\nu_{eab}}|$ are $O(10^{-3})$ or less for $n, m \geq 1$.

The Z-couplings of the zero modes, namely those of leptons and quarks in three generations, are summarized in Table 3. The deviations from the SM values are tiny.

In observation the weak coupling constant is measured from the $W\nu_e e$ and $W\nu_\mu\mu$ couplings. Normalized by $\hat{g}_{L,00}^{W\nu_{e1}e}$, the Z-couplings in the first generation become

$$\frac{1}{\hat{g}_{L,00}^{W\nu_{e1}e}} \begin{pmatrix} \hat{g}_{L,00}^{Z\nu_{e11}} \\ \hat{g}_{R,00}^{Z\nu_{e11}} \\ \hat{g}_{L,00}^{Ze} \\ \hat{g}_{R,00}^{Ze} \end{pmatrix} = \begin{pmatrix} 0.500022 \\ 2 \times 10^{-32} \\ -0.268800 \\ 0.231198 \end{pmatrix},$$

$$\frac{1}{\hat{g}_{L,00}^{W\nu_{e1}e}} \begin{pmatrix} \hat{g}_{L,00}^{Zu} \\ \hat{g}_{R,00}^{Zu} \\ \hat{g}_{L,00}^{Zdd} \\ \hat{g}_{R,00}^{Zdd} \end{pmatrix} = \begin{pmatrix} 0.345873 \\ -0.154132 \\ -0.422947 \\ 0.077068 \end{pmatrix}. \tag{58}$$

The values in Eq. (58) are very close to those in the SM with $\sin^2\theta_W^{\rm SM} = 0.2312$. For (t, b) quarks

$$\frac{1}{\hat{g}_{L,00}^{W\nu_{e1}e}} \begin{pmatrix} \hat{g}_{L,00}^{Zt} \\ \hat{g}_{R,00}^{Zt} \\ \hat{g}_{L,00}^{Zbb} \\ \hat{g}_{R,00}^{Zbb} \end{pmatrix} = \begin{pmatrix} 0.346205 \\ -0.153801 \\ -0.422940 \\ 0.077071 \end{pmatrix}. \tag{59}$$

The deviations in the Zb couplings from the SM are very small, which should be contrasted to the situation in some models in the RS warped space formulated in the early days [32]. The Zt_L coupling is 0.09% larger than that in the SM, whereas the Zt_R coupling is 0.19% smaller than that in the SM.

As shown in Eqs. (47) and (48), the Z-coupling is decomposed into the $su2$ part and EM part. The $su2$ part consists of three components: the $h_{Z^{(0)}}^L$ [a], $h_{Z^{(0)}}^R$ [b], $\hat{h}_{Z^{(0)}}$ [c] parts,

$\hat{g}^{Z,su2}_{L/R} = \hat{g}^{Z,a}_{L/R} + \hat{g}^{Z,b}_{L/R} + \hat{g}^{Z,c}_{L/R}$. All of them are important. For (u, d) quarks, for instance,

$$\begin{array}{ccccc} & u_L & d_L & u_R & d_R \\ \hat{g}^{Z,a} : & 0.49884 & -0.49884 & 0.0012459 & -0.000026196 \\ \hat{g}^{Z,b} : & 10^{-23} & 10^{-26} & 0.0012459 & -0.000026196 \\ \hat{g}^{Z,c} : & 10^{-12} & 10^{-13} & -0.0024918 & 0.000052392 \\ \hat{g}^{Z,su2} : & 0.49884 & -0.49884 & 10^{-11} & 10^{-13} \end{array} . \tag{60}$$

It is seen that the $SO(5)$ structure is crucial to have consistent gauge couplings of quarks and leptons. Vanishingly small $\hat{g}^{Zu,su2}_R$ and $\hat{g}^{Zd,su2}_R$ are due to the cancellation among the a, b, and c components, which is possible in the $SO(5) \times U(1)_X$ gauge theory, but not in $SU(2)_L \times SU(2)_R \times U(1)$ gauge theory.

The WWZ-coupling is evaluated similarly. Triple couplings are written as

$$\begin{aligned} & -ig_w \cos\theta^0_W \sum_{n,m,\ell} \hat{g}_{W^{\dagger(n)}W^{(m)}Z^{(\ell)}} K[W^{\dagger(n)}, W^{(m)}, Z^{(\ell)}], \\ & K[A, B, C] = A^{\mu\nu} B_\mu C_\nu + B^{\mu\nu} C_\mu A_\nu + C^{\mu\nu} A_\mu B_\nu, \\ & A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ \text{ etc.} \end{aligned} \tag{61}$$

The SM value is $g^{\rm SM}_{WWZ} = g_w \cos\theta^{\rm SM}_W$. In the current model one finds

$$\begin{aligned} \hat{g}_{W^{\dagger(n)}W^{(m)}Z^{(\ell)}} = \frac{\sqrt{kL}}{2\cos\theta^0_W} \int_1^{z_L} \frac{dz}{z} \Big\{ & 2h^L_{W^{(n)}} h^L_{W^{(m)}} h^L_{Z^{(\ell)}} + 2h^R_{W^{(n)}} h^R_{W^{(m)}} h^R_{Z^{(\ell)}} \\ & + \left(h^L_{W^{(n)}} + h^R_{W^{(n)}}\right) \hat{h}_{W^{(m)}} \hat{h}_{Z^{(\ell)}} + \hat{h}_{W^{(n)}} \left(h^L_{W^{(m)}} + h^R_{W^{(m)}}\right) \hat{h}_{Z^{(\ell)}} \\ & + \hat{h}_{W^{(n)}} \hat{h}_{W^{(m)}} \left(h^L_{Z^{(\ell)}} + h^R_{Z^{(\ell)}}\right) \Big\}. \end{aligned} \tag{62}$$

For $\theta_H = 0.1$ and $m_{\rm KK} = 13\,$TeV

$$\begin{aligned} & \hat{g}_{W^{\dagger(0)}W^{(0)}Z^{(0)}} - 1 = 2.7 \times 10^{-7}, \\ & \begin{pmatrix} \hat{g}_{W^{\dagger(0)}W^{(0)}Z^{(1)}} \\ \hat{g}_{W^{\dagger(0)}W^{(0)}Z^{(2)}} \\ \hat{g}_{W^{\dagger(0)}W^{(0)}Z^{(3)}} \\ \hat{g}_{W^{\dagger(0)}W^{(0)}Z^{(4)}} \end{pmatrix} = \begin{pmatrix} -2 \times 10^{-4} \\ 3 \times 10^{-8} \\ -8 \times 10^{-6} \\ -3 \times 10^{-8} \end{pmatrix}. \end{aligned} \tag{63}$$

The deviation in the $W^\dagger WZ = W^{\dagger(0)} W^{(0)} Z^{(0)}$ coupling is extremely tiny.

4. Fermion 1-loop corrections

With the W- and Z-couplings obtained in the previous section we are going to evaluate oblique corrections in the GUT-inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU. Let gauge fields X_μ and Y_μ couple to fermions ψ_1 and ψ_2 by

$$X_\mu \bar{\psi}_2 \gamma^\mu \left(g^{X,V}_{21} - g^{X,A}_{21} \gamma^5\right) \psi_1 + Y_\mu \bar{\psi}_1 \gamma^\mu \left(g^{Y,V}_{12} - g^{Y,A}_{12} \gamma^5\right) \psi_2. \tag{64}$$

The vacuum polarization $\Pi^{\mu\nu}_{XY}(p)$ in which fermions ψ_1 and ψ_2 are running along the loop is given by

$$
\begin{aligned}
i\,\Pi^{\mu\nu}_{XY}(p) = (-1)\int \frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\,(-i\gamma^\mu)\left(g^{X,V}_{21} - g^{X,A}_{21}\gamma^5\right)\frac{i}{\not{q} - m_1 + i\epsilon} \\
\times\,(-i\gamma^\nu)\left(g^{Y,V}_{12} - g^{Y,A}_{12}\gamma^5\right)\frac{i}{\not{q} + \not{p} - m_2 + i\epsilon}.
\end{aligned}
\tag{65}
$$

In the dimensional regularization it becomes

$$
\begin{aligned}
\Pi^{\mu\nu}_{XY}(p) &= -\frac{i}{(4\pi)^{d/2}}\,\Gamma\left(2-\tfrac{1}{2}d\right)\int_0^1 dx\left(\frac{\mu^2}{\Delta}\right)^{2-d/2} \\
&\quad\times\Big[\left(g^{X,V}_{21}g^{Y,V}_{12} + g^{X,A}_{21}g^{Y,A}_{12}\right)\left\{\left(x(1-x)p^2 - \Delta\right)\eta^{\mu\nu} - 2x(1-x)p^\mu p^\nu\right\} \\
&\quad + \left(g^{X,V}_{21}g^{Y,V}_{12} - g^{X,A}_{21}g^{Y,A}_{12}\right)m_1 m_2\eta^{\mu\nu}\Big] \\
&\equiv \Pi_{XY}(p^2)\,\eta^{\mu\nu} - \Sigma_{XY}(p^2)\,p^\mu p^\nu, \\
\Delta &= \Delta(x;p^2,m_1,m_2) = -x(1-x)p^2 + (1-x)m_1^2 + x m_2^2.
\end{aligned}
\tag{66}
$$

Expanded around $d = 4$, $\Pi_{XY}(p)$ contains divergent pole terms:

$$
\begin{aligned}
\Pi_{XY}\left(p^2\right) &= \Pi^{\mathrm{div}}_{XY}\left(p^2\right) + \Pi^{\mathrm{finite}}_{XY}\left(p^2\right), \\
\Pi^{\mathrm{div}}_{XY}\left(p^2\right) &= \frac{1}{4\pi^2}\Big[\left(g^{X,V}_{21}g^{Y,V}_{12} + g^{X,A}_{21}g^{Y,A}_{12}\right)\left\{\tfrac{1}{3}p^2 - \tfrac{1}{2}\left(m_1^2 + m_2^2\right)\right\} \\
&\quad + \left(g^{X,V}_{21}g^{Y,V}_{12} - g^{X,A}_{21}g^{Y,A}_{12}\right)m_1 m_2\Big]\hat{E} \equiv K^{\mathrm{div}}\hat{E}, \\
\Pi^{\mathrm{finite}}_{XY}\left(p^2\right) &= \frac{1}{4\pi^2}\int_0^1 dx\,\ln\Delta\Big[\left(g^{X,V}_{21}g^{Y,V}_{12} + g^{X,A}_{21}g^{Y,A}_{12}\right)\left\{x(1-x)\,p^2 - \Delta\right\} \\
&\quad + \left(g^{X,V}_{21}g^{Y,V}_{12} - g^{X,A}_{21}g^{Y,A}_{12}\right)m_1 m_2\Big], \\
\hat{E} &= -\frac{2}{4-d} + \gamma_E - \ln 4\pi\mu^2.
\end{aligned}
\tag{67}
$$

In terms of

$$
\begin{aligned}
b_0(s,m_1,m_2) &= \int_0^1 dx\,\ln\Delta(x;s,m_1,m_2) = b_0(s,m_2,m_1), \\
b_1(s,m_1,m_2) &= \int_0^1 dx\,x\ln\Delta(x;s,m_1,m_2) = \int_0^1 dx\,(1-x)\ln\Delta(x;s,m_2,m_1), \\
b_2(s,m_1,m_2) &= \int_0^1 dx\,x(1-x)\ln\Delta(x;s,m_1,m_2) = b_2(s,m_2,m_1),
\end{aligned}
\tag{68}
$$

$\Pi^{\rm finite}_{XY}(p^2)$ is expressed as

$$\Pi^{\rm finite}_{XY}\left(p^2\right) = \frac{1}{4\pi^2}\bigg[\left(g^{X,V}_{21}g^{Y,V}_{12} + g^{X,A}_{21}g^{Y,A}_{12}\right)\Big\{2p^2\, b_2\left(p^2, m_1, m_2\right) - m_1^2\, b_1\left(p^2, m_2, m_1\right) - m_2^2\, b_1\left(p^2, m_1, m_2\right)\Big\} + \left(g^{X,V}_{21}g^{Y,V}_{12} - g^{X,A}_{21}g^{Y,A}_{12}\right) m_1 m_2\, b_0\left(p^2, m_1, m_2\right)\bigg]. \tag{69}$$

We note that

$$\Pi^{\rm finite}_{XY}(p^2, m_1, m_2) = \mu^2\, \Pi^{\rm finite}_{XY}\left(\frac{p^2}{\mu^2}, \frac{m_1}{\mu}, \frac{m_2}{\mu}\right) - K^{\rm div}\ln\mu^2 \tag{70}$$

where $K^{\rm div}$ is defined in Eq. (67). It will be seen to be convenient to take $\mu = m_{\rm KK}$ in Eq. (70) to evaluate finite parts of oblique corrections in the current model.

In Sect. 3 we have obtained W- and Z-couplings of fermions. We adopt the convention for vector and axial-vector couplings given by $g_V = \frac{1}{2}(g_R + g_L)$ and $g_A = \frac{1}{2}(g_R - g_L)$ so that $g_R g_R' + g_L g_L' = 2(g_V g_V' + g_A g_A')$. To simplify expressions we introduce

$$\begin{aligned}
&G[\, p^2; \hat{g}_V, \hat{g}_A; \hat{g}_{V'}, \hat{g}_{A'}; m_{1n}, m_{2\ell}\,] \\
&= \sum_{n,\ell=0}^{\infty}\bigg[\left(\hat{g}_{V,n\ell}\hat{g}_{V',\ell n} + \hat{g}_{A,n\ell}\hat{g}_{A',\ell n}\right)\Big\{\left(\tfrac{1}{3}p^2 - \tfrac{1}{2}\left(m_{1n}^2 + m_{2\ell}^2\right)\right)\hat{E} \\
&\quad + 2p^2\, b_2(p^2, m_{1n}, m_{2\ell}) - m_{1n}^2\, b_1(p^2, m_{2\ell}, m_{1n}) - m_{2\ell}^2\, b_1(p^2, m_{1n}, m_{2\ell})\Big\} \\
&\quad + \Big(\hat{g}_{V,n\ell}\hat{g}_{V',\ell n} - \hat{g}_{A,n\ell}\hat{g}_{A',\ell n}\Big) m_{1n} m_{2\ell}\left\{\hat{E} + b_0(p^2, m_{1n}, m_{2\ell})\right\}\bigg].
\end{aligned} \tag{71}$$

Then, for the WW vacuum polarization, contributions from the (u, d) multiplets are given by

$$\Pi^{ud}_{WW}(p^2) = \frac{N_C g_w^2}{8\pi^2}\bigg\{G\Big[p^2; \hat{g}_V^{Wud}, \hat{g}_A^{Wud}; \hat{g}_V^{W^\dagger ud}, \hat{g}_A^{W^\dagger ud}; m_{u^{(n)}}, m_{d^{(\ell)}}\Big] + G\Big[p^2; \hat{g}_V^{WuD}, \hat{g}_A^{WuD}; \hat{g}_V^{W^\dagger uD}, \hat{g}_A^{W^\dagger uD}; m_{u^{(n)}}, m_{D^{(\ell)}}\Big]\bigg\} \tag{72}$$

where $N_C = 3$. Note that $\hat{g}^{W^\dagger ud}_{V/A,\ell n} = (\hat{g}^{Wud}_{V/A,n\ell})^*$ etc. Contributions from the (ν_e, e) multiplets are

$$\Pi^{\nu_e e}_{WW}(p^2) = \frac{g_w^2}{8\pi^2}\sum_{a=1}^{2} G\Big[p^2; \hat{g}_V^{W\nu_{ea}e}, \hat{g}_A^{W\nu_{ea}e}; \hat{g}_V^{W^\dagger\nu_{ea}e}, \hat{g}_A^{W^\dagger\nu_{ea}e}; m_{\nu_{ea}^{(n)}}, m_{e^{(\ell)}}\Big]. \tag{73}$$

For the ZZ vacuum polarization, contributions from the (u, d) multiplets are

$$\begin{aligned}
\Pi^{ud}_{ZZ}(p^2) = \frac{N_C g_w^2}{4\pi^2\cos^2\theta_W^0}\Big\{&G\big[p^2; \hat{g}_V^{Zu}, \hat{g}_A^{Zu}; \hat{g}_V^{Zu}, \hat{g}_A^{Zu}; m_{u^{(n)}}, m_{u^{(\ell)}}\big] \\
&+ G\big[p^2; \hat{g}_V^{Zdd}, \hat{g}_A^{Zdd}; \hat{g}_V^{Zdd}, \hat{g}_A^{Zdd}; m_{d^{(n)}}, m_{d^{(\ell)}}\big] \\
&+ G\big[p^2; \hat{g}_V^{ZDD}, \hat{g}_A^{ZDD}; \hat{g}_V^{ZDD}, \hat{g}_A^{ZDD}; m_{D^{(n)}}, m_{D^{(\ell)}}\big] \\
&+ 2\, G\big[p^2; \hat{g}_V^{ZdD}, \hat{g}_A^{ZdD}; \hat{g}_V^{ZDd}, \hat{g}_A^{ZDd}; m_{d^{(n)}}, m_{D^{(\ell)}}\big]\Big\}.
\end{aligned} \tag{74}$$

Here we have set $\hat{g}^{ZdD}_{V/A,n0} = \hat{g}^{ZDd}_{V/A,0n} = \hat{g}^{ZDD}_{V/A,n0} = \hat{g}^{ZDD}_{V/A,0n} = 0$. Contributions from the (ν_e, e) multiplets are

$$\Pi^{\nu_e e}_{ZZ}(p^2) = \frac{g_w^2}{4\pi^2\cos^2\theta^0_W}\Big\{ G\left[p^2; \hat{g}^{Ze}_V, \hat{g}^{Ze}_A; \hat{g}^{Ze}_V, \hat{g}^{Ze}_A; m_{e^{(n)}}, m_{e^{(\ell)}}\right] + \sum_{a=1}^{2}\sum_{b=1}^{2} G\Big[p^2; \hat{g}^{Z\nu_{eab}}_V, \hat{g}^{Z\nu_{eab}}_A; \hat{g}^{Z\nu_{eba}}_V, \hat{g}^{Z\nu_{eba}}_A; m_{\nu^{(n)}_{ea}}, m_{\nu^{(\ell)}_{eb}}\Big]\Big\}. \tag{75}$$

The photon couplings are universal. They are diagonal and vector-like. Noting that $b_1(p^2, m, m) = \frac{1}{2} b_0(p^2, m, m)$, one finds for the $\gamma\gamma$ vacuum polarization that

$$\Pi^{ud}_{\gamma\gamma}(p^2) = \frac{N_C g_w^2 \sin^2\theta^0_W}{4\pi^2}\, p^2 \Big\{ \sum_{n=0}^{\infty} Q_u^2 \left(\tfrac{1}{3}\hat{E} + 2b_2(p^2, m_{u^{(n)}}, m_{u^{(n)}})\right) + \sum_{n=0}^{\infty} Q_d^2 \left(\tfrac{1}{3}\hat{E} + 2b_2(p^2, m_{d^{(n)}}, m_{d^{(n)}})\right) + \sum_{n=1}^{\infty} Q_D^2 \left(\tfrac{1}{3}\hat{E} + 2b_2(p^2, m_{D^{(n)}}, m_{D^{(n)}})\right) \Big\},$$

$$\Pi^{\nu_e e}_{\gamma\gamma}(p^2) = \frac{g_w^2 \sin^2\theta^0_W}{4\pi^2}\, p^2 \sum_{n=0}^{\infty} Q_e^2 \left(\tfrac{1}{3}\hat{E} + 2b_2(p^2, m_{e^{(n)}}, m_{e^{(n)}})\right) \tag{76}$$

where $Q_u = \frac{2}{3}$, $Q_d = Q_D = -\frac{1}{3}$, and $Q_e = -1$. For the $Z\gamma$ vacuum polarization one finds that

$$\Pi^{ud}_{Z\gamma}(p^2) = \frac{N_C g_w^2 \sin\theta^0_W}{4\pi^2\cos\theta^0_W}\, p^2 \Big\{ \sum_{n=0}^{\infty} Q_u \hat{g}^{Zu}_{V,nn} \left(\tfrac{1}{3}\hat{E} + 2b_2(p^2, m_{u^{(n)}}, m_{u^{(n)}})\right) + \sum_{n=0}^{\infty} Q_d \hat{g}^{Zdd}_{V,nn} \left(\tfrac{1}{3}\hat{E} + 2b_2(p^2, m_{d^{(n)}}, m_{d^{(n)}})\right) + \sum_{n=1}^{\infty} Q_D \hat{g}^{ZDD}_{V,nn} \left(\tfrac{1}{3}\hat{E} + 2b_2(p^2, m_{D^{(n)}}, m_{D^{(n)}})\right) \Big\},$$

$$\Pi^{\nu_e e}_{Z\gamma}(p^2) = \frac{g_w^2 \sin\theta^0_W}{4\pi^2\cos\theta^0_W}\, p^2 \sum_{n=0}^{\infty} Q_e \hat{g}^{Ze}_{V,nn} \left(\tfrac{1}{3}\hat{E} + 2b_2(p^2, m_{e^{(n)}}, m_{e^{(n)}})\right). \tag{77}$$

Note that $\Pi_{\gamma\gamma}(0) = \Pi_{Z\gamma}(0) = 0$ as a consequence of the Ward–Takahashi identity in $U(1)_{\rm EM}$.

Expressions for $\Pi(p^2)$ for the second and third generations are obtained similarly.

5. Coupling sum rules

Each $\Pi(p^2)$ in the previous section contains divergent terms proportional to $\hat{E}$. In the SM some of them are absorbed by renormalization constants, and specific combinations of the $\Pi(p^2)$'s, namely the S, T, and U combinations, remain finite [28–31]. In GHU all KK modes of fermions contribute to $\Pi(p^2)$, and their couplings $\hat{g}^W_{V/A}$ and $\hat{g}^Z_{V/A}$ are highly nontrivial. The couplings $\hat{g}^W_{V/A}$ and $\hat{g}^Z_{V/A}$ take the matrix form with nonvanishing off-diagonal elements. Further, even in the subspace of the KK excited states the axial vector couplings are nonvanishing.

In this section we show that there exist three identities among the W- and Z-coupling matrices in each fermion doublet-multiplet, which are associated with the divergent terms in $\Pi_{WW}(p^2)$, $\Pi_{ZZ}(p^2)$, and $\Pi_{Z\gamma}(p^2)$. We define the W^3 coupling matrix $\hat{g}_V^{W^3}$, say for the (u, d) multiplet, by

$$\begin{aligned}\hat{g}^{Zu}_{V,n\ell} &= \hat{g}^{W^3u}_{V,n\ell} - \sin^2\theta^0_W\, Q_u\, \delta_{n\ell},\\ \begin{pmatrix}\hat{g}^{Zdd}_{V,n\ell}\\ \hat{g}^{ZDD}_{V,n\ell}\end{pmatrix} &= \begin{pmatrix}\hat{g}^{W^3dd}_{V,n\ell}\\ \hat{g}^{W^3DD}_{V,n\ell}\end{pmatrix} - \sin^2\theta^0_W\, Q_d\, \delta_{n\ell}.\end{aligned} \tag{78}$$

We stress that $\hat{g}_V^{W^3u}$ and $\hat{g}_V^{W^3dd}$ slightly differ from $\hat{g}_V^{Zu,su2}$ and $\hat{g}_V^{Zdd,su2}$ as defined in Eqs. (47) and (51). Numerically all elements of $\hat{g}_V^{W^3DD}$, $\hat{g}_A^{ZDD}$, $\hat{g}_{V/A}^{ZdD}$, and $\hat{g}_{V/A}^{ZDd}$ are $O(10^{-6})$ or less. In the following we safely omit the contributions coming from the D modes in the expressions for the coupling sum rules.

We define

$$\begin{aligned}
A_0^{ud} &= \mathrm{Tr}\,\Big\{\hat{g}_V^{W^3u}\hat{g}_V^{W^3u} + \hat{g}_A^{Zu}\hat{g}_A^{Zu} + \hat{g}_V^{W^3dd}\hat{g}_V^{W^3dd} + \hat{g}_A^{Zdd}\hat{g}_A^{Zdd}\Big\},\\
A_1^{ud} &= \sum_{n,\ell}\Big\{\hat{g}^{W^3u}_{V,n\ell}\hat{g}^{W^3u}_{V,\ell n}(m_{u^{(n)}} - m_{u^{(\ell)}})^2 + \hat{g}^{Zu}_{A,n\ell}\hat{g}^{Zu}_{A,\ell n}(m_{u^{(n)}} + m_{u^{(\ell)}})^2\\
&\qquad + \hat{g}^{W^3dd}_{V,n\ell}\hat{g}^{W^3dd}_{V,\ell n}(m_{d^{(n)}} - m_{d^{(\ell)}})^2 + \hat{g}^{Zdd}_{A,n\ell}\hat{g}^{Zdd}_{A,\ell n}(m_{d^{(n)}} + m_{d^{(\ell)}})^2\Big\},\\
B^{ud} &= Q_u \mathrm{Tr}\,\hat{g}_V^{W^3u} + Q_d \mathrm{Tr}\,\hat{g}_V^{W^3dd},\\
C^{ud} &= Q_u^2 \mathrm{Tr}\, I + Q_d^2 \mathrm{Tr}\, I,\\
D_0^{ud} &= \mathrm{Tr}\,\Big\{\hat{g}_V^{Wud}\hat{g}_V^{W^\dagger ud} + \hat{g}_A^{Wud}\hat{g}_A^{W^\dagger ud}\Big\},\\
D_1^{ud} &= \sum_{n,\ell}\Big\{\hat{g}^{Wud}_{V,n\ell}\hat{g}^{W^\dagger ud}_{V,\ell n}(m_{u^{(n)}} - m_{d^{(\ell)}})^2 + \hat{g}^{Wud}_{A,n\ell}\hat{g}^{W^\dagger ud}_{A,\ell n}(m_{u^{(n)}} + m_{d^{(\ell)}})^2\Big\}.
\end{aligned} \tag{79}$$

Here Tr in $Q_u^2 \mathrm{Tr}\, I$ implies the trace over the $u^{(n)}$ states. Then the divergent parts of $\Pi^{ud}(p^2)$ are expressed as

$$\begin{aligned}
\Pi^{ud}_{ZZ}(p^2)^{\rm div} &= \frac{N_C g_w^2}{4\pi^2\cos^2\theta^0_W}\left\{\frac{1}{3}\Big(A_0^{ud} - 2\sin^2\theta^0_W B^{ud} + \sin^4\theta^0_W C^{ud}\Big)p^2 - \frac{1}{2}A_1^{ud}\right\}\hat{E},\\
\Pi^{ud}_{Z\gamma}(p^2)^{\rm div} &= \frac{N_C g_w^2 \sin\theta^0_W}{4\pi^2\cos\theta^0_W}\,\frac{1}{3}\Big(B^{ud} - \sin^2\theta^0_W C^{ud}\Big)p^2\,\hat{E},\\
\Pi^{ud}_{\gamma\gamma}(p^2)^{\rm div} &= \frac{N_C g_w^2\sin^2\theta^0_W}{4\pi^2}\,\frac{1}{3}C^{ud}p^2\,\hat{E},\\
\Pi^{ud}_{WW}(p^2)^{\rm div} &= \frac{N_C g_w^2}{8\pi^2}\left\{\frac{1}{3}D_0^{ud}p^2 - \frac{1}{2}D_1^{ud}\right\}\hat{E}.
\end{aligned} \tag{80}$$

For the (ν_e, e) doublet-multiplet

$$\hat{g}^{Ze}_{V,n\ell} = \hat{g}^{W^3e}_{V,n\ell} - \sin^2\theta^0_W\, Q_e\delta_{n\ell} \tag{81}$$

and we define

$$
\begin{aligned}
A_0^{\nu_e e} &= \mathrm{Tr}\left\{\hat{g}_V^{W^3 e}\hat{g}_V^{W^3 e} + \hat{g}_A^{Ze}\hat{g}_A^{Ze} + \sum_{a=1}^{2}\sum_{b=1}^{2}\left(\hat{g}_V^{Z\nu_{eab}}\hat{g}_V^{Z\nu_{eba}} + \hat{g}_A^{Z\nu_{eab}}\hat{g}_A^{Z\nu_{eba}}\right)\right\},\\
A_1^{\nu_e e} &= \sum_{n,\ell}\Bigg(\hat{g}_{V,n\ell}^{W^3 e}\hat{g}_{V,\ell n}^{W^3 e}(m_{e^{(n)}} - m_{e^{(\ell)}})^2 + \hat{g}_{A,n\ell}^{Ze}\hat{g}_{A,\ell n}^{Ze}(m_{e^{(n)}} + m_{e^{(\ell)}})^2\\
&\quad + \sum_{a=1}^{2}\sum_{b=1}^{2}\left\{\hat{g}_{V,n\ell}^{Z\nu_{eab}}\hat{g}_{V,\ell n}^{Z\nu_{eba}}(m_{\nu_{ea}^{(n)}} - m_{\nu_{eb}^{(\ell)}})^2 + \hat{g}_{A,n\ell}^{Z\nu_{eab}}\hat{g}_{A,\ell n}^{Z\nu_{eba}}(m_{\nu_{ea}^{(n)}} + m_{\nu_{eb}^{(\ell)}})^2\right\}\Bigg),\\
B^{\nu_e e} &= Q_e \mathrm{Tr}\,\hat{g}_V^{W^3 e},\\
C^{\nu_e e} &= Q_e^2 \mathrm{Tr}\, I,\\
D_0^{\nu_e e} &= \sum_{a=1}^{2}\mathrm{Tr}\left\{\hat{g}_V^{W\nu_{ea}e}\hat{g}_V^{W^\dagger\nu_{ea}e} + \hat{g}_A^{W\nu_{ea}e}\hat{g}_A^{W^\dagger\nu_{ea}e}\right\},\\
D_1^{\nu_e e} &= \sum_{a=1}^{2}\sum_{n,\ell}\left\{\hat{g}_{V,n\ell}^{W\nu_{ea}e}\hat{g}_{V,\ell n}^{W^\dagger\nu_{ea}e}(m_{\nu_{ea}^{(n)}} - m_{e^{(\ell)}})^2 + \hat{g}_{A,n\ell}^{W\nu_{ea}e}\hat{g}_{A,\ell n}^{W^\dagger\nu_{ea}e}(m_{\nu_{ea}^{(n)}} + m_{e^{(\ell)}})^2\right\}.
\end{aligned}
\tag{82}
$$

The divergent parts of $\Pi^{\nu_e e}(p^2)$ are given by the expressions in Eq. (80) where $N_C = 1$ and the superscript 'ud' is replaced by '$\nu_e e$'. Note that these coefficients depend on the fermion doublet; $A_0^{ud} \neq A_0^{\nu_e e}$ etc.

Although all of the coupling matrices $\hat{g}_{V/A}$ are rather nontrivial as shown in Sect. 3, there appear astonishing relations among A_0, A_1, B, D_0, and D_1. We are going to establish, by numerical evaluation from the coupling matrices, the following coupling sum rules

$$
\begin{cases} A_0^{ud} = h^{ud} B^{ud} \\ D_0^{ud} = 2A_0^{ud} \\ D_1^{ud} = 2A_1^{ud} \end{cases}, \quad
\begin{cases} A_0^{\nu_e e} = h^{\nu_e e} B^{\nu_e e} \\ D_0^{\nu_e e} = 2A_0^{\nu_e e} \\ D_1^{\nu_e e} = 2A_1^{\nu_e e} \end{cases}
\tag{83}
$$

to high accuracy, where

$$
h^{ud} = \hat{g}_{L,00}^{Zu,su2} - \hat{g}_{L,00}^{Zdd,su2}, \quad h^{\nu_e e} = \hat{g}_{L,00}^{Z\nu_{e11}} - \hat{g}_{L,00}^{Ze,su2}. \tag{84}
$$

Similar relations hold for the second and third generations. For the (t, b) doublet, we use $h^{tb} = -2\hat{g}_{L,00}^{Zbb,su2}$. Numerical values of $\hat{g}_{L,00}^{Z,su2}$ and $\hat{g}_{L,00}^{W}$ for $\theta_H = 0.1$, $m_{\mathrm{KK}} = 13$ TeV, and $M = 10^3$ TeV are summarized in Table 4. The factors h are close to, but not exactly 1.

In the SM $h = 1$, $A_0 = B = \frac{1}{2}D_0 = \frac{1}{4}$, $A_1^{ud} = \frac{1}{2}D_1^{ud} = \frac{1}{4}(m_u^2 + m_d^2)$, etc. so that the relations in Eq. (83) are satisfied for each doublet. In the current GHU model the relations are highly nontrivial. We have included contributions coming from the KK modes $n = 0$ to $n = 12$. The mass spectrum of the KK states and the 13-by-13 coupling matrices are determined with double

Table 4. The couplings $\hat{g}^{Z,su2}_{L,00}$ and $\hat{g}^{W}_{L,00}$ for $\theta_H = 0.1$, $m_{\rm KK} = 13$ TeV, and $M = 10^3$ TeV.

	$2\hat{g}^{Z\nu_{e11}}_{L,00}$	$-2\hat{g}^{Ze,su2}_{L,00}$	$\hat{g}^{W\nu_{e1}e}_{L,00}$
(ν_e, e)	0.997690	0.997691	0.997647
(ν_μ, μ)	0.997686	0.997687	0.997644
(ν_τ, τ)	0.997684	0.997684	0.997642
	$2\hat{g}^{Zu,su2}_{L,00}$	$-2\hat{g}^{Zdd,su2}_{L,00}$	$\hat{g}^{Wud}_{L,00}$
(u, d)	0.997688	0.997688	0.997645
(c, s)	0.997685	0.997685	0.997643
(t, b)	0.998344	0.997671	0.997969

Table 5. The coupling sum rules. The values of A_0, A_1, Δ_S, Δ_T, Δ_U, and h in Eqs. (83)–(85) are tabulated for each doublet for $\theta_H = 0.1$, $m_{\rm KK} = 13$ TeV, and $M = 10^3$ TeV. The numerical values are evaluated by including the contributions coming from the KK towers of fermions up to the $n = 12$ level.

	A_0	$A_1/m_Z{}^2$	Δ_S	Δ_T	Δ_U	h
(ν_e, e)	3.24204	44.1410	3.6×10^{-5}	5.3×10^{-5}	-3.4×10^{-7}	0.997690
(ν_μ, μ)	3.24214	43.5089	7.1×10^{-5}	5.3×10^{-5}	-3.4×10^{-7}	0.997687
(ν_τ, τ)	3.24219	43.1535	-1.5×10^{-5}	5.3×10^{-5}	-2.7×10^{-7}	0.997684
(u, d)	3.24210	43.7063	-4.0×10^{-6}	5.5×10^{-5}	-3.4×10^{-7}	0.997688
(c, s)	3.24217	43.2862	2.0×10^{-5}	5.3×10^{-5}	-3.4×10^{-7}	0.997685
(t, b)	3.24262	42.5613	1.0×10^{-4}	4.1×10^{-5}	-2.8×10^{-7}	0.997671

precision. To confirm the accuracy of the coupling sum rules we introduce

$$\Delta_S = \frac{A_0 - hB}{A_0},$$

$$\Delta_T = \frac{A_1 - \frac{1}{2}D_1}{A_1},$$

$$\Delta_U = \frac{A_0 - \frac{1}{2}D_0}{A_0}. \tag{85}$$

Obtained results for A_0, A_1, Δ_S, Δ_T, and Δ_U are summarized in Table 5.

It is seen that the coupling sum rules (83) are valid with 5- to 7-digit accuracy, at least numerically. In view of the nontrivial matrix structure of the gauge couplings the coupling sum rules (83) are highly nontrivial. The relations are expected as the consequences of the 5D gauge invariance in GHU. Although the values of A_0 and A_1 increase with n, Δ_S, Δ_T, and Δ_U remain small. For the (ν_e, e) multiplet, for instance, as n increases from 12 to 16, the variations are A_0: $3.24 \to 4.24$, $A_1/m_Z{}^2$: $44.1 \to 57.9$, Δ_S: $3.6 \times 10^{-5} \to 7.4 \times 10^{-5}$, Δ_T: $5.34 \times 10^{-5} \to 5.34 \times 10^{-5}$, Δ_U: $-3.44 \times 10^{-7} \to -3.41 \times 10^{-7}$.

In this paper we have considered the vacuum polarization tensors of the photon, W, and Z bosons only. Rigorous theoretical derivation of the coupling sum rules would require treating all the KK towers of the $SO(5) \times U(1)_X$ gauge bosons, which is beyond the scope of the current paper. It is expected that similar coupling sum rules hold even in each sector of the KK excited modes of the gauge fields.

One comment is in order about the appearance of the h factor in the relation $A_0 = hB$. The relation involves the Z- and $U(1)_{\rm EM}$ couplings. As emphasized in Sect. 3, around Eq. (60), the Z-couplings have the effective $SU(2)_{\rm eff} \times U(1)_{\rm EM}$ structure. In the SM the Z-couplings to quarks and leptons are given by $(g_w/\cos\theta_W^{\rm SM})(T_L^3 - \sin^2\theta_W^{\rm SM} Q_{\rm EM})$. In GHU the W- and Z-couplings to a doublet $\beta = [(u, d), (\nu_e, e), \cdots]$ are given approximately by

$$W : \frac{g_w \, \hat{g}_{L,00}^{W\beta}}{\sqrt{2}} \, \big(T_{\rm eff}^1 + i\, T_{\rm eff}^2\big)\,,$$

$$Z : \frac{g_w \, h^\beta}{\cos\theta_W^0} \left(T_{\rm eff}^3 - \frac{\sin^2\theta_W^0}{h^\beta} \, Q_{\rm EM} \right), \tag{86}$$

where left-handed (right-handed) quarks and leptons are $SU(2)_{\rm eff}$ doublets (singlets), and $h^\beta = \hat{g}_{L,00}^{Z\beta_u, su2} - \hat{g}_{L,00}^{Z\beta_d, su2}$ for $\beta = (\beta_u, \beta_d)$. The factor h^β is not equal to 1 in GHU even at the tree level. It is very close to $\hat{g}_{L,00}^{W\beta}$. The relevant quantity for the forward-backward asymmetry in $e^- e^+ \to f\bar{f}$ at the Z pole, for instance, is $\sin^2\theta_W^0/h^\beta$ which is about $\sin^2\theta_W^{\rm SM}$. The h factor in Eq. (86) effectively appears in the relation $A_0 = hB$, which affects the definition of the S parameter in GHU as discussed below.

6. Improved oblique parameters

The oblique parameters S, T, and U of Peskin–Takeuchi are useful to investigate new physics beyond the SM. These parameters are expressed in terms of the vacuum polarization tensors of W, Z, and photon. Certain combinations of those vacuum polarization tensors are finite, and are expected to represent important parts of the corrections to physical quantities.

In GHU some improvement is necessary. In the most general situation the S, T, and U parameters should be defined as certain combinations of the vacuum polarization tensors of all $SO(5) \times U(1)_X$ gauge fields including the KK excited modes. Only the combinations which are finite at the quantum level could serve as quantities measuring corrections to physical quantities. In this section we examine the finite corrections to the S, T, and U parameters associated with the vacuum polarization tensors of W, Z, and photon. We should remember that these quantities are not directly measured physical quantities. Directly measured physical quantities expressed in terms of four-fermi vertices, for instance, involve contributions coming from the KK modes of the gauge bosons in GHU.

In the previous section we have established three coupling sum rules to high accuracy. For each fermion doublet β the sum rules are

$$\begin{cases} A_0^\beta = h^\beta B^\beta & \text{for } S, \\ A_1^\beta = \frac{1}{2} D_1^\beta & \text{for } T, \\ A_0^\beta = \frac{1}{2} D_0^\beta & \text{for } U. \end{cases} \tag{87}$$

With these sum rules at hand we propose the following S, T, and U for each fermion doublet β at the 1-loop level:

$$
\begin{aligned}
\alpha_* S^\beta &= \frac{\sin^2 2\theta_W^0}{m_Z^2}\Big\{\Pi_{ZZ}^\beta\left(m_Z^2\right) - \Pi_{ZZ}^\beta(0) - \frac{\cos 2\theta_W^0 + h^\beta - 1}{\sin\theta_W^0\cos\theta_W^0}\Pi_{Z\gamma}^\beta\left(m_Z^2\right) \\
&\quad - \left(1 + \frac{h^\beta - 1}{\cos^2\theta_W^0}\right)\Pi_{\gamma\gamma}^\beta\left(m_Z^2\right)\Big\}, \\
\alpha_* T^\beta &= \frac{1}{m_Z^2\cos^2\theta_W^0}\Pi_{WW}^\beta(0) - \frac{1}{m_Z^2}\Pi_{ZZ}^\beta(0), \\
\alpha_* U^\beta &= \frac{1}{m_Z^2\cos^2\theta_W^0}\left\{\Pi_{WW}^\beta\left(m_W^2\right) - \Pi_{WW}^\beta(0)\right\} \\
&\quad - \frac{\cos^2\theta_W^0}{m_Z^2}\left\{\Pi_{ZZ}^\beta\left(m_Z^2\right) - \Pi_{ZZ}^\beta(0)\right\} \\
&\quad - \frac{\sin 2\theta_W^0}{m_Z^2}\Pi_{Z\gamma}^\beta\left(m_Z^2\right) - \frac{\sin^2\theta_W^0}{m_Z^2}\Pi_{\gamma\gamma}^\beta\left(m_Z^2\right)
\end{aligned}
\tag{88}
$$

where $\alpha_* = \alpha_{\rm EM}(m_Z^2)$. In GHU $m_Z\cos\theta_W^0 \neq m_W^{\rm tree}$. The terms proportional to $h^\beta - 1$ in $\alpha_* S^\beta$ represent the improvement from the standard expression for S. It is straightforward to confirm that S, T, U defined by Eq. (88) are finite as a consequence of the sum rules in Eq. (87). In the numerical evaluation of finite S^β, T^β, U^β by using the gauge coupling matrices of finite-dimensional rows and columns, one has to use the h factor defined by $h^\beta = A_0^\beta/B^\beta$, otherwise the result would be afflicted with the uncertainty associated with the divergence. Also notice that the weak mixing angle θ_W^0 entering in Eq. (88) is the angle defined in Eq. (13).

To explicitly express S^β, T^β, and U^β in terms of the gauge couplings and mass spectra, it is convenient to introduce

$$
\begin{aligned}
\check{b}_0(s, m_1, m_2) &= b_0(s, m_1, m_2) - b_0(0, m_1, m_2), \\
\check{b}_1(s, m_1, m_2) &= b_1(s, m_1, m_2) - b_1(0, m_1, m_2), \\
J_\pm(m_V, m_1, m_2) &= -b_2\left(m_V^2, m_1, m_2\right) \pm \frac{m_1 m_2}{2m_V^2}\check{b}_0\left(m_V^2, m_2, m_1\right) \\
&\quad + \frac{m_1^2}{2m_V^2}\check{b}_1\left(m_V^2, m_2, m_1\right) + \frac{m_2^2}{2m_V^2}\check{b}_1\left(m_V^2, m_1, m_2\right),
\end{aligned}
\tag{89}
$$

and

$$H[\hat{g}_V, \hat{g}_A; \hat{g}_{V'}, \hat{g}_{A'}; m_V, m_{1n}, m_{2\ell}]$$
$$= \sum_{n,\ell=0}^{\infty} \Big\{ \hat{g}_{V,n\ell}\hat{g}_{V',\ell n} J_-(m_V, m_{1n}, m_{2\ell}) + \hat{g}_{A,n\ell}\hat{g}_{A',\ell n} J_+(m_V, m_{1n}, m_{2\ell}) \Big\},$$
$$K[\hat{g}_V, \hat{g}_A; \hat{g}_{V'}, \hat{g}_{A'}; m_{1n}, m_{2\ell}]$$
$$= \sum_{n,\ell=0}^{\infty} \Big\{ \big(\hat{g}_{V,n\ell}\hat{g}_{V',\ell n} + \hat{g}_{A,n\ell}\hat{g}_{A',\ell n}\big)\big[m_{1n}^2 b_1(0, m_{2\ell}, m_{1n}) + m_{2\ell}^2 b_1(0, m_{1n}, m_{2\ell})\big]$$
$$- \big(\hat{g}_{V,n\ell}\hat{g}_{V',\ell n} - \hat{g}_{A,n\ell}\hat{g}_{A',\ell n}\big) m_{1n} m_{2\ell}\, b_0(0, m_{1n}, m_{2\ell}) \Big\}. \tag{90}$$

We note that $b_0(s, m, m) = 2b_1(s, m, m)$. Then $S^{(u,\,d)}$ is given by

$$\alpha_* S^{(u,d)} = -\frac{2N_C \sin^2\theta_W^0}{\pi^2} \Big\{ H\Big[\hat{g}_V^{W^3u}, \hat{g}_A^{Zu}; \hat{g}_V^{W^3u}, \hat{g}_A^{Zu}; m_Z, m_{u^{(n)}}, m_{u^{(\ell)}}\Big]$$
$$+ H\Big[\hat{g}_V^{W^3dd}, \hat{g}_A^{Zdd}; \hat{g}_V^{W^3dd}, \hat{g}_A^{Zdd}; m_Z, m_{d^{(n)}}, m_{d^{(\ell)}}\Big]$$
$$+ h^{(u,d)} \sum_{n=0}^{\infty} \Big(Q_u \hat{g}_{V,nn}^{W^3u} b_2\big(m_Z^2, m_{u^{(n)}}, m_{u^{(n)}}\big) + Q_d \hat{g}_{V,nn}^{W^3dd} b_2\big(m_Z^2, m_{d^{(n)}}, m_{d^{(n)}}\big) \Big) \Big\}. \tag{91}$$

$T^{(u,\,d)}$ is given by

$$\alpha_* T^{(u,d)} = \frac{N_C}{4\pi^2 \cos^2\theta_W^0 m_Z^2} \Big\{ -\frac{1}{2} K\Big[\hat{g}_V^{Wud}, \hat{g}_A^{Wud}; \hat{g}_V^{W^\dagger ud}, \hat{g}_A^{W^\dagger ud}; m_{u^{(n)}}, m_{d^{(\ell)}}\Big]$$
$$+ K\Big[\hat{g}_V^{W^3u}, \hat{g}_A^{Zu}; \hat{g}_V^{W^3u}, \hat{g}_A^{Zu}; m_{u^{(n)}}, m_{u^{(\ell)}}\Big]$$
$$+ K\Big[\hat{g}_V^{W^3dd}, \hat{g}_A^{Zdd}; \hat{g}_V^{W^3dd}, \hat{g}_A^{Zdd}; m_{d^{(n)}}, m_{d^{(\ell)}}\Big] \Big\}. \tag{92}$$

$U^{(u,\,d)}$ is given by

$$\alpha_* U^{(u,d)} = \frac{N_C}{2\pi^2} \Big\{ -\frac{1}{2} H\Big[\hat{g}_V^{Wud}, \hat{g}_A^{Wud}; \hat{g}_V^{W^\dagger ud}, \hat{g}_A^{W^\dagger ud}; m_W, m_{u^{(n)}}, m_{d^{(\ell)}}\Big]$$
$$+ H\Big[\hat{g}_V^{W^3u}, \hat{g}_A^{Zu}; \hat{g}_V^{W^3u}, \hat{g}_A^{Zu}; m_Z, m_{u^{(n)}}, m_{u^{(\ell)}}\Big]$$
$$+ H\Big[\hat{g}_V^{W^3dd}, \hat{g}_A^{Zdd}; \hat{g}_V^{W^3dd}, \hat{g}_A^{Zdd}; m_Z, m_{d^{(n)}}, m_{d^{(\ell)}}\Big] \Big\}. \tag{93}$$

For the lepton doublet (ν_e, e), $S^{(\nu_e,e)}$ is given by

$$\alpha_* S^{(\nu_e,e)} = -\frac{2\sin^2\theta_W^0}{\pi^2} \Big\{ H\Big[\hat{g}_V^{W^3e}, \hat{g}_A^{Ze}; \hat{g}_V^{W^3e}, \hat{g}_A^{Ze}; m_Z, m_{e^{(n)}}, m_{e^{(\ell)}}\Big]$$
$$+ \sum_{a=1}^{2}\sum_{b=1}^{2} H\Big[\hat{g}_V^{Z\nu_{eab}}, \hat{g}_A^{Z\nu_{eab}}; \hat{g}_V^{Z\nu_{eba}}, \hat{g}_A^{Z\nu_{eba}}; m_Z, m_{\nu_{ea}^{(n)}}, m_{\nu_{eb}^{(\ell)}}\Big]$$
$$+ h^{(\nu_e,e)} \sum_{n=0}^{\infty} Q_e \hat{g}_{V,nn}^{W^3e} b_2\big(m_Z^2, m_{e^{(n)}}, m_{e^{(n)}}\big) \Big\}. \tag{94}$$

Table 6. Corrections to S^β, T^β, U^β for $\theta_H = 0.1$ and $m_{\rm KK} = 13$ TeV. The numerical values are evaluated by including the contributions coming from the KK towers of fermions up to the $n = n_{\rm max} = 12$ level. The values in the neutrino sector are obtained by setting $m_{\nu_e} = m_{\nu_\mu} = m_{\nu_\tau} = 10^{-12}$ GeV and Majorana masses $M_e = M_\mu = M_\tau = 10^6$ GeV. In the last row the average increments per level, namely (total)/$n_{\rm max}$, are listed.

β	S^β	T^β	U^β
(ν_e, e)	0.0010	0.0126	3.7×10^{-6}
(ν_μ, μ)	0.0009	0.0122	3.7×10^{-6}
(ν_τ, τ)	0.0016	0.0129	3.7×10^{-6}
(u, d)	0.0028	0.0382	1.1×10^{-5}
(c, s)	0.0026	0.0360	1.1×10^{-5}
(t, b)	0.0013	0.0058	7.9×10^{-6}
total	0.010	0.12	0.00004
per level	8.4×10^{-4}	9.7×10^{-3}	3.4×10^{-6}

$T^{(\nu_e,e)}$ is given by

$$\alpha_* T^{(\nu_e,e)} = \frac{1}{4\pi^2 \cos^2\theta_W^0 m_Z^2}\bigg\{ -\frac{1}{2}\sum_{a=1}^{2} K\Big[\hat g_V^{W\nu_{ea}e}, \hat g_A^{W\nu_{ea}e}; \hat g_V^{W^\dagger\nu_{ea}e}, \hat g_A^{W^\dagger\nu_{ea}e}; m_{\nu_{ea}^{(n)}}, m_{e^{(\ell)}}\Big] + K\Big[\hat g_V^{W^3e}, \hat g_A^{Ze}; \hat g_V^{W^3e}, \hat g_A^{Ze}; m_{e^{(n)}}, m_{e^{(\ell)}}\Big] + \sum_{a=1}^{2}\sum_{b=1}^{2} K\Big[\hat g_V^{Z\nu_{eab}}, \hat g_A^{Z\nu_{eab}}; \hat g_V^{Z\nu_{eba}}, \hat g_A^{Z\nu_{eba}}; m_{\nu_{ea}^{(n)}}, m_{\nu_{eb}^{(\ell)}}\Big]\bigg\}. \tag{95}$$

$U^{(\nu_e,e)}$ is given by

$$\alpha_* U^{(\nu_e,e)} = \frac{1}{2\pi^2}\bigg\{ -\frac{1}{2}\sum_{a=1}^{2} H\Big[\hat g_V^{W\nu_{ea}e}, \hat g_A^{W\nu_{ea}e}; \hat g_V^{W^\dagger\nu_{ea}e}, \hat g_A^{W^\dagger\nu_{ea}e}; m_W, m_{\nu_{ea}^{(n)}}, m_{e^{(\ell)}}\Big] + H\Big[\hat g_V^{W^3e}, \hat g_A^{Ze}; \hat g_V^{W^3e}, \hat g_A^{Ze}; m_Z, m_{e^{(n)}}, m_{e^{(\ell)}}\Big] + \sum_{a=1}^{2}\sum_{b=1}^{2} H\Big[\hat g_V^{Z\nu_{eab}}, \hat g_A^{Z\nu_{eab}}; \hat g_V^{Z\nu_{eba}}, \hat g_A^{Z\nu_{eba}}; m_Z, m_{\nu_{ea}^{(n)}}, m_{\nu_{eb}^{(\ell)}}\Big]\bigg\}. \tag{96}$$

Formulas for the quark-lepton multiplets in the second and third generations are obtained similarly.

We have evaluated the improved S, T, U described above from the gauge coupling matrices determined in the space of the KK modes of $n = 0$ to $n_{\rm max} = 12$ levels. In the evaluation the usage of the identity (70) reduces numerical errors. In the combinations of the above S^β, T^β, U^β the sum of the $K^{\rm div}\ln\mu^2$ part in Eq. (70) vanishes thanks to the coupling sum rules in Eq. (87). In Table 6 we have tabulated the values of S^β, T^β, U^β beyond the SM contributions for $\theta_H = 0.1$, $m_{\rm KK} = 13$ TeV, and $M = 10^3$ TeV. The total values are $S \sim 0.01$, $T \sim 0.12$, and $U \sim 0.00004$ when the contributions of fermions up to the $n_{\rm max} = 12$ level are taken into account.

Unlike the case of the coupling sum rules, however, the parameters S, T, U evaluated in this manner seem to increase as $n_{\rm max}$ is increased. Let us denote these parameters as $S^{(0)}$, $T^{(0)}$, $U^{(0)}$ to stress that they are oblique parameters associated with the zero modes $W^{(0)}$, $Z^{(0)}$, and $\gamma^{(0)}$

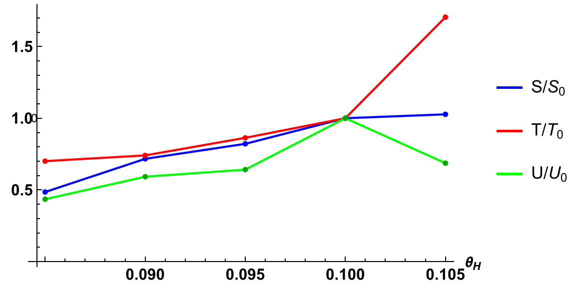

Fig. 1. The θ_H-dependence of the S, T, U parameters is plotted for $m_{\rm KK} = 13\,$TeV and $n_{\rm max} = 12$. S_0, T_0, and U_0 are the values of S, T, and U at $\theta_H = 0.1$, respectively.

of the gauge fields. The average increments in $S^{(0)}$, $T^{(0)}$, $U^{(0)}$ per level are small (as listed in the last row in Table 6), but do not vary very much in the range $12 \leq n_{\rm max} \leq 16$. This does not necessarily mean that oblique corrections to physical quantities become large in GHU. The current estimates of S, T, U from experimental data in the SM framework are $S^{\rm RPP} = -0.02 \pm 0.10$, $T^{\rm RPP} = 0.03 \pm 0.12$, and $U^{\rm RPP} = 0.01 \pm 0.11$ where the superscript RPP stands for Review of Particle Physics [38]. As emphasized at the beginning of this section, oblique corrections associated not only with $W^{(0)}$, $Z^{(0)}$, and $\gamma^{(0)}$, but also with the KK excited modes $W^{(n)}$, $Z^{(n)}$, and $\gamma^{(n)}$, become important for physical observable quantities in GHU. In particular, the couplings of left-handed quarks and leptons to $W^{(1)}$, $Z^{(1)}$, and $\gamma^{(1)}$ are large in the GUT-inspired GHU. To compare with $S^{\rm RPP}$, $T^{\rm RPP}$, and $U^{\rm RPP}$, one needs to include, in addition to $S^{(0)}$, $T^{(0)}$, and $U^{(0)}$, oblique corrections to the propagators of the KK gauge bosons. Contributions coming from internal fermions at the high KK levels equally affect the oblique corrections to the KK gauge boson propagators. To have definitive understanding of the contributions of KK fermions at the 1-loop level in GHU, it is necessary to directly evaluate observable quantities, which is left for future investigation.

So far we have presented the results for $\theta_H = 0.1$ with $m_{\rm KK} = 13\,$TeV. The θ_H-dependence of the oblique parameters is explored similarly. It should be noted that with $m_{\rm KK} = 13\,$TeV fixed, the value of θ_H can be lowered to $\theta_H^{\rm min} \sim 0.08$ to reproduce the mass of the top quark. To realize smaller values of θ_H, one needs to increase $m_{\rm KK}$. One expects that the oblique corrections should get smaller as θ_H gets smaller. Indeed this is the case. The θ_H-dependences of S, T, and U are depicted in the range $0.085 \leq \theta_H \leq 0.105$ in Fig. 1 for $m_{\rm KK} = 13\,$TeV and $n_{\rm max} = 12$. For larger values of θ_H the oblique corrections get larger. It is anticipated from the viewpoint that as θ_H varies from 0 to π, the gauge symmetry changes from $SU(2)_L \times U(1)_Y$ to $SU(2)_R \times U(1)_{Y'}$. The reliable numerical evaluation of the Z-couplings, particularly in the bottom quark tower, becomes harder for larger values of θ_H because of the singular behavior of the $h_{L/R}^{b^{(n)}}$ and $k_{L/R}^{b^{(n)}}$ components of the wave functions near the UV brane at $z = 1$.

7. Summary and discussions

In this paper we have examined the GUT-inspired $SO(5) \times U(1)_X \times SU(3)_C$ GHU model in the RS warped space. The W- and Z-couplings of quarks, leptons, and their KK excited modes take the matrix form in the KK space. These coupling matrices have nontrivial off-diagonal elements, and have both vector and axial-vector components. Nevertheless, these coupling matrices satisfy three sum rules (87). We have confirmed these coupling sum rules numerically from the evaluated W- and Z-coupling matrices. Rigorous derivation of the coupling sum rules would require the full treatment of the gauge bosons in the $SO(5) \times U(1)_X$ theory. It is noteworthy that the sum rules hold even in the subspace of the W, Z, and photon vacuum polarization tensors to very high accuracy. The appearance of the $h^\beta \neq 1$ factor in the relation $A_0^\beta = h^\beta B^\beta$ in Eq. (87) is anticipated from the vertex correction in the Z-couplings at the tree level as exhibited in the approximate formula in Eq. (86).

With the coupling sum rules at hand, one can evaluate the finite oblique corrections unambiguously. The corrections are evaluated by using the mass spectrum and gauge coupling matrices determined numerically. We have found for $\theta_H = 0.1$ and $m_{\rm KK} = 13$ TeV that $S \sim 0.01$, $T \sim 0.12$, and $U \sim 0.00004$ when the contributions of the fermion loops up to the $n_{\rm max} = 12$ level are taken into account. It was argued at a very early stage of the investigation [32] that there may arise a large correction to S in gauge theory in the RS space. We have found that the corrections in the GUT-inspired GHU are small by direct evaluation of 1-loop diagrams. We note that Yoon and Peskin have evaluated the oblique corrections in a different $SO(5) \times U(1)_X$ GHU model using a different method [14]. Their result also indicates small corrections for $m_{\rm KK} = 13$ TeV. However, to have definitive understanding of the contributions of KK fermions at the 1-loop level, it is necessary to evaluate observable quantities, by taking account of oblique corrections to the KK modes of the gauge fields.

The coupling sum rules presented in this paper are highly nontrivial. There must be some reason behind them, possibly originating from the 5D gauge invariance in the GHU scheme. Further investigation is necessary.

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Appendix A. Basis functions

We summarize the basis functions used for wave functions of gauge and fermion fields. For gauge fields we introduce

$$
\begin{aligned}
F_{\alpha,\beta}(u, v) &\equiv J_\alpha(u) Y_\beta(v) - Y_\alpha(u) J_\beta(v), \\
C(z; \lambda) &= \frac{\pi}{2} \lambda z z_L F_{1,0}(\lambda z, \lambda z_L), \\
S(z; \lambda) &= -\frac{\pi}{2} \lambda z F_{1,1}(\lambda z, \lambda z_L), \\
C'(z; \lambda) &= \frac{\pi}{2} \lambda^2 z z_L F_{0,0}(\lambda z, \lambda z_L), \\
S'(z; \lambda) &= -\frac{\pi}{2} \lambda^2 z F_{0,1}(\lambda z, \lambda z_L), \\
\hat{S}(z; \lambda) &= \frac{C(1; \lambda)}{S(1; \lambda)} S(z; \lambda),
\end{aligned}
\tag{A1}
$$

where $J_\alpha(u)$ and $Y_\alpha(u)$ are Bessel functions of the first and second kinds. They satisfy

$$-z\frac{d}{dz}\frac{1}{z}\frac{d}{dz}\begin{pmatrix} C \\ S \end{pmatrix} = \lambda^2 \begin{pmatrix} C \\ S \end{pmatrix}, \tag{A2}$$

with the BCs $C(z_L; \lambda) = z_L$, $C'(z_L; \lambda) = S(z_L; \lambda) = 0$, $S'(z_L; \lambda) = \lambda$, and $CS' - SC' = \lambda z$.

For fermion fields with a bulk mass parameter c, we define

$$\begin{aligned}
\begin{pmatrix} C_L \\ S_L \end{pmatrix}(z; \lambda, c) &= \pm\frac{\pi}{2}\lambda\sqrt{zz_L}F_{c+\frac{1}{2},c\mp\frac{1}{2}}(\lambda z, \lambda z_L), \\
\begin{pmatrix} C_R \\ S_R \end{pmatrix}(z; \lambda, c) &= \mp\frac{\pi}{2}\lambda\sqrt{zz_L}F_{c-\frac{1}{2},c\pm\frac{1}{2}}(\lambda z, \lambda z_L), \\
\begin{pmatrix} \hat{S}_L \\ \hat{C}_R \end{pmatrix}(z; \lambda, c) &= \frac{C_L(1; \lambda, c)}{S_L(1; \lambda, c)}\begin{pmatrix} S_L \\ C_R \end{pmatrix}(z; \lambda, c), \\
\begin{pmatrix} \hat{S}_R \\ \hat{C}_L \end{pmatrix}(z; \lambda, c) &= \frac{C_R(1; \lambda, c)}{S_R(1; \lambda, c)}\begin{pmatrix} S_R \\ C_L \end{pmatrix}(z; \lambda, c).
\end{aligned} \tag{A3}$$

These functions satisfy

$$\begin{aligned}
&D_+(c)\begin{pmatrix} C_L \\ S_L \end{pmatrix} = \lambda\begin{pmatrix} S_R \\ C_R \end{pmatrix}, \\
&D_-(c)\begin{pmatrix} C_R \\ S_R \end{pmatrix} = \lambda\begin{pmatrix} S_L \\ C_L \end{pmatrix}, \quad D_\pm(c) = \pm\frac{d}{dz} + \frac{c}{z},
\end{aligned} \tag{A4}$$

with the BCs $C_{R/L} = 1$, $S_{R/L} = 0$ at $z = z_L$, and $C_L C_R - S_L S_R = 1$. We also use

$$\begin{aligned}
\mathcal{C}_{L1}(z; \lambda, c, \tilde{m}) &= C_L(z; \lambda, c+\tilde{m}) + C_L(z; \lambda, c-\tilde{m}), \\
\mathcal{C}_{L2}(z; \lambda, c, \tilde{m}) &= S_L(z; \lambda, c+\tilde{m}) - S_L(z; \lambda, c-\tilde{m}), \\
\mathcal{S}_{L1}(z; \lambda, c, \tilde{m}) &= S_L(z; \lambda, c+\tilde{m}) + S_L(z; \lambda, c-\tilde{m}), \\
\mathcal{S}_{L2}(z; \lambda, c, \tilde{m}) &= C_L(z; \lambda, c+\tilde{m}) - C_L(z; \lambda, c-\tilde{m}), \\
\mathcal{C}_{R1}(z; \lambda, c, \tilde{m}) &= C_R(z; \lambda, c+\tilde{m}) + C_R(z; \lambda, c-\tilde{m}), \\
\mathcal{C}_{R2}(z; \lambda, c, \tilde{m}) &= S_R(z; \lambda, c+\tilde{m}) - S_R(z; \lambda, c-\tilde{m}), \\
\mathcal{S}_{R1}(z; \lambda, c, \tilde{m}) &= S_R(z; \lambda, c+\tilde{m}) + S_R(z; \lambda, c-\tilde{m}), \\
\mathcal{S}_{R2}(z; \lambda, c, \tilde{m}) &= C_R(z; \lambda, c+\tilde{m}) - C_R(z; \lambda, c-\tilde{m}).
\end{aligned} \tag{A5}$$

Appendix B. Wave functions

Wave functions of down-type quark and neutrino multiplets are given below.

B.1. Down-type quarks

The (d, d', D^+, D^-) fields are expanded as in Eq. (30). The wave functions are given by

$$\begin{pmatrix} f_L^{d^{(n)}}(z) \\ g_L^{d^{(n)}}(z) \\ h_L^{d^{(n)}}(z) \\ k_L^{d^{(n)}}(z) \end{pmatrix} = \frac{1}{\sqrt{r_{d^{(n)}}}} \begin{pmatrix} \alpha_{d^{(n)}} C_L(z; \lambda_{d^{(n)}}, c_u) \\ \beta_{d^{(n)}} S_L(z; \lambda_{d^{(n)}}, c_u) \\ a_{d^{(n)}} \mathcal{C}_{L2}(z; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d}) + b_{d^{(n)}} \mathcal{C}_{L1}(z; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d}) \\ a_{d^{(n)}} \mathcal{S}_{L1}(z; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d}) + b_{d^{(n)}} \mathcal{S}_{L2}(z; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d}) \end{pmatrix}$$

$$\begin{pmatrix} f_R^{d^{(n)}}(z) \\ g_R^{d^{(n)}}(z) \\ h_R^{d^{(n)}}(z) \\ k_R^{d^{(n)}}(z) \end{pmatrix} = \frac{1}{\sqrt{r_{d^{(n)}}}} \begin{pmatrix} \alpha_{d^{(n)}} S_R(z; \lambda_{d^{(n)}}, c_u) \\ \beta_{d^{(n)}} C_R(z; \lambda_{d^{(n)}}, c_u) \\ a_{d^{(n)}} \mathcal{S}_{R2}(z; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d}) + b_{d^{(n)}} \mathcal{S}_{R1}(z; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d}) \\ a_{d^{(n)}} \mathcal{C}_{R1}(z; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d}) + b_{d^{(n)}} \mathcal{C}_{R2}(z; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d}) \end{pmatrix} \tag{B1}$$

for the $d^{(n)}$ mode where

$$\begin{aligned} \beta_{d^{(n)}} &= -i \frac{\cos\frac{1}{2}\theta_H S_R(1, \lambda_{d^{(n)}}, c_u)}{\sin\frac{1}{2}\theta_H C_R(1, \lambda_{d^{(n)}}, c_u)} \alpha_{d^{(n)}}, \\ a_{d^{(n)}} &= i \frac{\mu_d S_R(1, \lambda_{d^{(n)}}, c_u)}{\sin\frac{1}{2}\theta_H} \frac{\mathcal{S}_{L2}(1; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d})}{\mathcal{F}_1(1; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d})} \alpha_{d^{(n)}}, \\ b_{d^{(n)}} &= -\frac{\mathcal{S}_{L1}(1; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d})}{\mathcal{S}_{L2}(1; \lambda_{d^{(n)}}, c_{D_d}, \tilde{m}_{D_d})} a_{d^{(n)}}, \\ \mathcal{F}_1 &= \mathcal{S}_{L1}\mathcal{S}_{R1} - \mathcal{S}_{L2}\mathcal{S}_{R2}. \end{aligned} \tag{B2}$$

For the $D^{(n)}$ mode the formulas are obtained by replacing $d^{(n)}$ by $D^{(n)}$ in Eqs. (B1) and (B2).

Except for the $d^{(0)}$ mode, namely d-quark, the $d^{(n)}$ $(n \geq 1)$ modes are mostly contained in (d, d') fields, whereas the $D^{(n)}$ $(n \geq 1)$ modes are mostly contained in (D^+, D^-) fields. In Table B1 the norm of each component ($N_f = \int_1^{z_L} dz\, |f|^2$ etc.) is tabulated. For comparison we list the norms of u and u' components of the $u^{(n)}$ modes in Table B2.

One can see that $(u_L^{(0)}, d_L^{(0)})$ is an $SU(2)_L$ doublet. On the other hand $u_R^{(0)}$ and $d_R^{(0)}$ are nearly $SU(2)_L$ singlets. Further $d_R^{(0)}$ has major components in the $D^\pm$ fields. Its $SU(2)_R$ portion is small. Although the W boson acquires a small $SU(2)_R$ portion at $\theta_H = 0.1$, its coupling to $(u_R^{(0)}, d_R^{(0)})$ is suppressed significantly.

B.2. Neutrinos

The $(\nu, \nu', \hat{\chi})$ fields are expanded as in Eq. (35). The wave functions are given by

$$\begin{pmatrix} f_L^{\nu^{\pm(n)}}(z) \\ g_L^{\nu^{\pm(n)}}(z) \\ f_R^{\nu^{\pm(n)}}(z) \\ g_R^{\nu^{\pm(n)}}(z) \end{pmatrix} = \frac{1}{\sqrt{r_{\nu^{\pm(n)}}}} \begin{pmatrix} \sin\frac{1}{2}\theta_H C_L(z; \lambda_{\nu^{\pm(n)}}, c_e)/S_R(1; \lambda_{\nu^{\pm(n)}}, c_e) \\ -i\cos\frac{1}{2}\theta_H S_L(z; \lambda_{\nu^{\pm(n)}}, c_e)/C_R(1; \lambda_{\nu^{\pm(n)}}, c_e) \\ \sin\frac{1}{2}\theta_H S_R(z; \lambda_{\nu^{\pm(n)}}, c_e)/S_R(1; \lambda_{\nu^{\pm(n)}}, c_e) \\ -i\cos\frac{1}{2}\theta_H C_L(z; \lambda_{\nu^{\pm(n)}}, c_e)/C_R(1; \lambda_{\nu^{\pm(n)}}, c_e) \end{pmatrix},$$

$$h^{\nu^{\pm(n)}} = \frac{1}{\sqrt{r_{\nu^{\pm(n)}}}} \frac{-im_B}{k\lambda_{\nu^{\pm(n)}} \mp M}. \tag{B3}$$

The normalization factor $r_{\nu^{\pm(n)}}$ is determined by Eq. (36).

Table B1. The norm of each component for the $d^{(n)}$ and $D^{(n)}$ modes. $N_f = \int_1^{z_L} dz\,|f|^2$ etc.

	$N_f\,(d)$	$N_g\,(d')$	$N_h\,(D^+)$	$N_k\,(D^-)$
$d_L^{(0)}$	1.	1×10^{-23}	3×10^{-14}	2×10^{-14}
$d_L^{(1)}$	2×10^{-22}	1.	1×10^{-10}	1×10^{-11}
$d_L^{(2)}$	1.	7×10^{-22}	1×10^{-12}	1×10^{-12}
$d_L^{(3)}$	8×10^{-22}	1.	3×10^{-10}	2×10^{-10}
$d_L^{(4)}$	1.	2×10^{-21}	2×10^{-12}	1×10^{-12}
$d_R^{(0)}$	5×10^{-5}	0.021	0.387	0.592
$d_R^{(1)}$	1×10^{-14}	1.	2×10^{-10}	1×10^{-10}
$d_R^{(2)}$	1.	3×10^{-14}	2×10^{-12}	8×10^{-13}
$d_R^{(3)}$	2×10^{-14}	1.	3×10^{-10}	4×10^{-10}
$d_R^{(4)}$	1.	3×10^{-14}	3×10^{-12}	8×10^{-13}
$D_L^{(1)}$	9×10^{-14}	8×10^{-11}	0.697	0.303
$D_L^{(2)}$	2×10^{-12}	7×10^{-11}	0.633	0.367
$D_L^{(3)}$	2×10^{-13}	3×10^{-10}	0.284	0.716
$D_L^{(4)}$	3×10^{-12}	1×10^{-10}	0.781	0.219
$D_R^{(1)}$	2×10^{-5}	7×10^{-3}	0.125	0.868
$D_R^{(2)}$	7×10^{-6}	3×10^{-3}	0.952	0.045
$D_R^{(3)}$	6×10^{-6}	2×10^{-3}	0.034	0.964
$D_R^{(4)}$	5×10^{-6}	2×10^{-3}	0.973	0.025

Table B2. The norm of the $u^{(n)}$ modes. $N_f = \int_1^{z_L} dz\,|f|^2$ etc.

	$N_f\,(u)$	$N_g\,(u')$
$u_L^{(0)}$	1.	2×10^{-20}
$u_L^{(1)}$	2×10^{-19}	1.
$u_L^{(2)}$	1.	3×10^{-19}
$u_L^{(3)}$	6×10^{-19}	1.
$u_L^{(4)}$	1.	8×10^{-19}
	$N_f\,(u)$	$N_g\,(u')$
$u_R^{(0)}$	0.002	0.998
$u_R^{(1)}$	2×10^{-11}	1.
$u_R^{(2)}$	1.	1×10^{-11}
$u_R^{(3)}$	1×10^{-11}	1.
$u_R^{(4)}$	1.	1×10^{-11}

The $\nu^{+(0)}$ mode is nearly left-handed, saturated with ν_L. $\nu^{\pm(n)}$ $(n \geq 1)$ modes are almost vector-like. $\nu^{\pm(2\ell-1)}$ $(\ell \geq 1)$ modes are saturated by ν_L and ν_R, whereas $\nu^{\pm(2\ell)}$ $(\ell \geq 1)$ modes are saturated by ν'_L and ν'_R. As an example we take $m_{\nu_e} = 10^{-3}$ eV and $M_e = 10^6$ GeV, which gives $m_{Be} = 4.8 \times 10^5$ GeV. The norm of each component is tabulated in Table B3.

Table B3. The norm of the $\nu^{\pm(n)}$ modes. $N_f = \int_1^{z_L} dz\,|f|^2$ etc.

	$N_{f_L}(\nu)$	$N_{g_L}(\nu')$	$N_{f_R}(\nu)$	$N_{g_R}(\nu')$	N_h
$\nu^{+(0)}$	1.	6×10^{-61}	1×10^{-20}	4×10^{-18}	1×10^{-18}
$\nu^{+(1)}$	1×10^{-28}	0.5	1×10^{-17}	0.5	1×10^{-15}
$\nu^{+(2)}$	0.5	9×10^{-28}	0.5	5×10^{-17}	1×10^{-17}
$\nu^{+(3)}$	2×10^{-27}	0.5	4×10^{-17}	0.5	4×10^{-15}
$\nu^{+(4)}$	0.5	7×10^{-26}	0.5	1×10^{-15}	3×10^{-16}
$\nu^{-(1)}$	2×10^{-28}	0.5	2×10^{-17}	0.5	2×10^{-15}
$\nu^{-(2)}$	0.5	1×10^{-27}	0.5	5×10^{-17}	1×10^{-17}
$\nu^{-(3)}$	3×10^{-27}	0.5	8×10^{-17}	0.5	7×10^{-15}
$\nu^{-(4)}$	0.5	2×10^{-26}	0.5	3×10^{-16}	7×10^{-17}

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