SUPERSYMMETRIC QUANTUM MECHANICS
and
CRITICAL POTENTIALS*

LUIS J. BOYA
Departamento de Fisica Teórica, Facultad de Ciencias
Universidad de Zaragoza, 50009 Zaragoza, Spain

ABSTRACT

Various examples of supersymmetric quantum mechanical problems are presented, for which the scattering amplitudes behave in a non-generic way. Critical potentials include δ-potentials, Pöschl-Teller potentials and intermediate range potentials falling off like $|x|^{-2}$. Conditions for the appearance of those critical potentials are established.

I. INTRODUCTION

Supersymmetric quantum mechanics (Susy Q.M.) originally set-up by Witten [1], is well established by now [2], [3]. In the one-dimensional case, the only one we shall consider in this paper, it affords a pairing of states of the same energy between two hamiltonians, with the possible exception of the ground state, which might have no partner. While most authors consider mainly the bound state problem, interesting results in Susy Q.M. are also obtained for the scattering states [4].

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In this paper we review some scattering problems in one-dimensional Susy Q.M., pointing out the non-generic character of the scattering amplitude for the chosen examples. Our selection includes a class of intermediate range potentials for which the phase shift behaviour exhibits pecularities.

The organization of this paper is as follows: in §2 we review briefly Susy Q.M.; in §3 we set the formalism of one-dimensional scattering in order to establish the generic low-energy and high-energy behaviour of the (only two) scattering amplitudes. Examples are treated thereafter, including the delta potentials (§4) and the Pöschl-Teller potential (§5); some slowly-falling potentials (decaying as $|x|^{-2}$ at infinity) are worked out in §6. In §7 we establish some general conditions for the existence of intermediate range critical potentials, and in §8 we put forward our conclusions.

II. SUPERSYMMETRIC QUANTUM MECHANICS

Supersymmetric quantum mechanics deals essentially with two hermitian anticommuting operators, Q and C, with known squares, namely [5]

$$\{Q,C\} = 0 \tag{II.1}$$

$$C^2 = I , \qquad Q^2 = H \tag{II.2}$$

where H is the hamiltonian of the system; Q is called the supercharge. (II.1) implies that the spectrum of Q is symmetrical around zero, namely if λ is in the spectrum of Q so is $-\lambda$; this is the crucial property of any supersymmetric theory, [5].

One of the simplest realizations of the supercharge is to write

$$Q = \sigma_2 p + \sigma_1 W(x) \tag{II.3}$$

where σ_i are the Pauli matrices. The real function W(x) is called the superpotential, and x labels a single space dimension. (II.3) gives rise to the most general one-dimensional Susy Q.M. problem [2], [3]. As our operator C we can take just σ_3. The hamiltonian is

$$H = Q^2 = p^2 + W^2(x) - \sigma_3 W'(x) \tag{II.4}$$

which splits in two unidimensional hamiltonians $H_\pm = p^2 + W^2 \mp W'$ making up a Susy pair.

If the eigenvalue equations are

$$H_+ u_k(x) = \varepsilon_k^+ u_k(x) \ , \quad H_- v_k(x) = \varepsilon_k^- v_k(x) \tag{II.5}$$

we have positive semidefinite energies and

$$\varepsilon_k^+ = \varepsilon_k^- = \varepsilon_k \qquad \text{for} \qquad \varepsilon_k^\pm > 0 \quad \text{strictly} \tag{II.6}$$

but the zero-energy eigenstates need not exist nor be paired. We choose the u's and v's eigenvectors of C also, namely $Cu_k = +u_k$, $Cv_k = -v_k$, so that Q interchanges them; from the explicit form (II.3) we obtain

$$\left(-\frac{d}{dx} + W(x)\right) u_k(x) = \sqrt{\varepsilon_k}\ v_k(x) \tag{II.7a}$$

$$\left(+\frac{d}{dx} + W(x)\right) v_k(x) = \sqrt{\varepsilon_k}\ u_k(x) \tag{II.7b}$$

as relation between the wavefunctions.

Both bound states and scattering states are paired by Susy. In particular, as (II.7) are local equations, the asymptotic values of the superpotential $W(\pm\infty)$ would relate the phase shifts for $u_k(x)$ and $v_k(x)$.

For example if we deal with even potentials, which implies odd superpotential, $W(x) = -W(-x)$, and describe the scattering by the even and odd phase shifts $\delta_{\pm}(k)$ (see §III for details), we have the asymptotic behaviour

$$u_k(x) \xrightarrow{x>>} N_{1+} \cos(k_x + \delta_{1+}) \quad \text{or} \quad N_{1-} \sin(k_x + \delta_{1-}) \tag{II.8a}$$

$$v_k(x) \xrightarrow{x>>} N_{2+} \cos(k_x + \delta_{2+}) \quad \text{or} \quad N_{2-} \sin(k_x + \delta_{2-}) \tag{III.8b}$$

that is, the relation between the phase shifts are, due to (II.7),

$$\delta_{2-}(k) = \delta_{1+}(k) - \alpha \tag{II.9a}$$

$$\delta_{2+}(k) = \delta_{1-}(k) - \alpha \tag{II.9b}$$

$$tg_{\alpha} = W(+\infty)/k \quad . \tag{II.10}$$

We shall make ample use of these relations later.

III. GENERIC BEHAVIOUR OF UNIDIMENSIONAL SCATTERING

We consider the unidimensional Schrödinger equation

$$\psi''(x) + k^2\psi(x) = V(x)\psi(x) \tag{III.1}$$

for a decaying potential $V(x) \to 0$ for $|x|$ large. The normal scattering wave function is

$$\psi(x) \xrightarrow{|x|>>} e^{ikx} + f_{\varepsilon}(k)e^{ik|x|}, \quad \varepsilon = \rightleftharpoons \tag{III.2}$$

where the scattering amplitudes are $f_{\rightarrow}(k) = f_{forward}$, and $f_{\leftarrow}(k) = f_{backward}$. If the incident wave comes from the right, we have zurdo scattering with wavefunction

$$\psi(x) \xrightarrow{|x|>>} e^{-ikx} + \check{f}_{\epsilon}(k)e^{ik|x|} \quad . \tag{III.2a}$$

The S-matrix in the in-out representation is

$$S(k) = \begin{pmatrix} 1 + f_{\rightarrow}(k) & \check{f}_{\leftarrow}(k) \\ f_{\leftarrow}(k) & 1 + \check{f}_{\rightarrow}(k) \end{pmatrix} . \tag{III.3}$$

For a local potential $V = V(x)$ time reversal invariance is automatic and implies $f_{\rightarrow}(k) = \check{f}_{\rightarrow}(k)$; if we restrict ourselves for simplicity to even potentials, $V(x) = +V(-x)$, we also have $f_{\leftarrow}(k) = \check{f}_{\leftarrow}(k)$. Unitarity of S limits furthermore the amplitudes to two real numbers, namely we have the conditions

$$\sigma(k) \equiv |f_{\rightarrow}(k)|^2 + |f_{\leftarrow}(k)|^2 = -2\text{Re } f_{\rightarrow}(k) \tag{III.4}$$

(which is the unidimensional optical theorem), and

$$\text{Re}(1 + f_{\rightarrow}(k))f^{*}_{\leftarrow}(k) = 0 \qquad \text{(phase relation)} . \tag{III.5}$$

The total scattering coefficient $\sigma(k)$ is dimensionless and bounded by 4, $0 \leq \sigma(k) \leq 4$. See [6], [7].

We can also introduce parity waves (the orthogonal group in one dimension is just parity, $O(1) = Z_2$), with even and odd phase shifts $\delta_{\pm}(k)$ [8]. The relations are

$$f_{\rightarrow}(k) = f_{+}(k) + f_{-}(k) , \qquad f_{\leftarrow}(k) = f_{+}(k) - f_{-}(k) \tag{III.6}$$

$$f_{\pm}(k) = ie^{i\delta\pm(k)} \sin \delta_{\pm}(k) \tag{III.7}$$

$$\sigma = 2(\sin^2 \delta_{+} + \sin^2 \delta_{-}) . \tag{III.8}$$

To study the dependence of the scattering amplitudes on the energy k^2, we consider separately the low- and the high-energy regions.

1) Low energy region. The odd wave phase shift $\delta_-(k)$ is the same as the s-wave phase shift $\delta_0(k)$ of the corresponding three dimensional problem, for which we know the effective range approximation is a good one; consequently

$$k \cot \delta_-(k) = c + dk^2 + O(k^4) \ , \quad (\text{low } k) \ ; \qquad \text{(III.9b)}$$

but we can derive an analogous formula for $\delta_+(k)$ by just changing the boundary conditions in the classical derivation of (III.9b) due to Bethe [9]. We obtain in this way

$$k \operatorname{tg} \delta_+(k) = a + bk^2 + O(k^4) \ , \quad (\text{low } k) \ . \qquad \text{(III.9a)}$$

In (III.9) a, b, c and d are constants. The generic behaviour would be characterized by normal values of the 0^{th} order terms:

$$0 \neq (a,c) \neq \infty \ .$$

In particular, generically

$$\delta_+(k = 0) = \pi/2 \qquad \text{mod } \pi \qquad \text{(III.11+)}$$

$$\delta_-(k = 0) = 0 \qquad \text{mod } \pi \qquad \text{(III.11-)}$$

The mod π ambiguity is of course resolved by the convention $\delta_\pm(k \to \infty) = 0$, and then the zero energy limit of $\delta_\pm$ just counts the number of even or odd bound states; still in the generic situation

$$\delta_+(0) = (n_+ - \tfrac{1}{2})\pi \qquad \text{(III.12+)}$$

$$\delta_-(0) = n_-\pi \qquad \text{(III.12-)}$$

where n_+ (n_-) is the number of even (odd) bound states, which is Levinson's theorem in one dimension.

2) High energy region. It can be easily deduced from the integral equation for scattering [10]

$$\psi(x) = e^{ikx} + \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{ik|x-x'|}\, V(x')\psi(x')dx' \qquad \text{(III.13)}$$

through the Born approximation, which gives

$$f_{\rightarrow}(k) \xrightarrow{k>>} f^B_{\rightarrow}(k) = \frac{1}{2ik}\int_{-\infty}^{+\infty} V(x)dx = \frac{\text{cons.}}{k} \qquad \text{(III.14}\rightarrow\text{)}$$

$$f_{\leftarrow}(k) \xrightarrow{k>>} f^B_{\leftarrow}(k) = \frac{1}{2ik}\int_{-\infty}^{+\infty} V(x)e^{2ikx}\, dx = o\left(\frac{1}{k}\right) , \qquad \text{(III.14}\leftarrow\text{)}$$

so the rapidly oscillating term exp(2ikx) damps $f_{\leftarrow}(k)$ with respect to $f_{\rightarrow}(k)$ at large k.

So we have the following generic behaviour of the amplitudes in one dimensional scattering:

a) isotropy at low k , (III.15.0)

because $\delta_-(k) \sim 0$ means $f_{\leftarrow} \simeq f_{\rightarrow}$, see (III.6); this is the same as in three dimensional scattering, which is dominated by the s-wave at low k.

b) transparency at high k , (III.15.∞)

because $f_{\leftarrow}(k \rightarrow \infty) \sim 0$, the high-energy behaviour is dominated by the forward amplitude; this is also true in three dimensions ("forward diffraction peak"; see e.g. [11]).

Without being too precise, we shall call <u>critical</u> a potential whose scattering amplutides do not behave like (III.15). We next present some examples of supersymmetric critical potentials.

IV. THE DELTA POTENTIAL

To illustrate a simple case of critical potential, let us look at the well-known case of the delta potential in one dimension

$$V(x) = g\delta(x) \ . \tag{IV.1}$$

The Susy aspect of the delta potential has been established already [3]; namely

$$W(x) = (g/2)\ \mathrm{sign}(x) \tag{IV.2}$$

is the appropriate superpotential. Supersymmetry just interchanges g with -g. Besides the bound state wavefunction, solution of $(-d/dx + W)u_0 = 0$:

$$u_0(x) = \sqrt{-g/2}\, \exp(+g|x|/2) \tag{IV.3}$$

(valid for $g < 0$ only), there are scattering states, which might be obtained by solving directly the spectral equation for $Q = \sigma_2 p + \sigma_1 W$ of (II.3), namely

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (g\ \mathrm{sign}(x))/2 & \lambda \\ -\lambda & -(g\ \mathrm{sign}(x))/2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{IV.4}$$

with $\lambda^2 = k^2 + g^2/4$; the scattering amplitudes are [3]

$$f_{\rightarrow}(k) = f_{\leftarrow}(k) = \frac{g}{2ik - g} \tag{IV.5}$$

as expected, cfr. Lapidus [8], [12].

So we see that the scattering is <u>isotropic for all k</u>, and in this sense the δ-potential is critical: the expected low energy isotropy persists at all energies. For the phase shifts we obtain

$$\mathrm{tg}\ \delta_+(k) = -g/2k\ , \qquad \delta_-(k) = 0\ . \tag{IV.6}$$

The effective range approximation is exact for $\delta_+(k)$, with quadratic coefficient ("effective range," cfr. (III.9a)) equal to zero; the odd phase shift is zero; also the scattering amplitude is given in full by the pole of the bound state, at $k_b = -ig/2$.

Of course, the critical nature of the delta potential is easy to understant: its "zero" range means that the isotropy condition is always valid (i.e., for all k).

The relations (II.9) are of course satisfied, but in a peculiar way: as the odd phase is zero, we obtain just $\delta_{1+} = -\delta_{2+} = -\mathrm{arctg}(g/2k)$, and the exactness of the effective range formula can be understood: it is a consequence of zero range potential ($\delta_- = 0$) and supersymmetry $\delta_{1+} - \delta_{2-} = -W(\infty)/k$.

V. THE PÖSCHL-TELLER POTENTIAL

The potential

$$V(x) = g\ \mathrm{sech}^2\ x \tag{V.1}$$

is a particular case of the so-called Bargmann potentials [13]. Solutions for arbitrary g can be found easily [14]. The critical case occurs for $g = -\ell(\ell + 1)$, $\ell = 0,1,2,\cdots$; in this case the solution can be obtained completely by algebraic methods. The superpotential is

$$W(x) = -\ell \tanh x \tag{V.2}$$

and connects g_ℓ with $g_{\ell-1}$ as Susy pairs, see [15] and also [3]; for $\ell = 0$ we connect therefore the free equation $u''(x) + k^2 u(x) = 0$ with the equation $v''(x) + k^2 v(x) = -2\mathrm{sech}^2(x) \cdot v(x)$, hence the partner of the incoming wave solution $u_k(x) = \exp(ikx)$ gives the exact scattering wave for $v(x)$; and by iteration the solution for arbitrary ℓ is obtained; the result is [3]

$$f_{\leftarrow}(k) = 0 \qquad \text{(reflectionless, transparent potential)} \tag{V.3←}$$

$$1 + f_{\rightarrow}(k) = \frac{ik - 1}{ik + 1}\,\frac{ik - 2}{ik + 2} \cdots \frac{ik - \ell}{ik + \ell}\,, \tag{V.3→}$$

i.e. we obtain a beautiful factorizable S-matrix of transparent type.

As the potential is attractive and even, there are also bound states (for $\ell > 0$) [3] at energies

$$\varepsilon(m) = -m^2\,; \qquad m = 1,2,3,\cdots,\ell\ . \tag{V.4}$$

So the $-\ell(\ell + 1)\mathrm{sech}^2(x)$ potential is critical in a way complementary to the delta potential: the latter is isotropic ($\delta_-(k) = 0$), the former transparent ($f_{\leftarrow}(k) = 0$).

The P-T potential for $g_\ell = -\ell(\ell + 1)$ is critical because g_ℓ is critical: namely we have a zero energy resonance each time g crosses the value $-\ell(\ell + 1)$. This produces a shift of $\pi/2$ in the corresponding δ's, and makes $\delta_+(0) = \delta_-(0)$ possible (this is a well-known phenomenon, see e.g. Newton [16]); then Susy maintains the equality to any k, and $\delta_+(k) = \delta_-(k)$ of course implies $f_{\leftarrow}(k) = 0$ (transparency condition), cfr. (III.6,7). For example for $\ell = 1$ we just have a zero energy resonance

[14] of odd parity, because the unique true bound state is of course even, at $\varepsilon = -1$; therefore $\delta_-(0) = +\pi/2$ instead of being zero, so $\delta_-(0) = \delta_+(0)$. This causes $f_+(0) = -2$, and $\sigma(0) = 4$, a typical resonance maximum, cfr. (IV.4).

The anomalous behaviour alternates: for $\ell = 2$ we get $\delta_+(0) = \pi$ instead of $\pi/2$ because the resonance is now in even wave, so $\sigma(0) = 0$ although it can again be considered as a resonant behaviour, the generic value would be $\sigma(0) = |-1|^2 + |-1|^2 = 2$; etc.

The scattering amplitude is again given in full by a rational function with poles at the bound states: the scattering is just additive in the angles $\phi(m) = \text{arctg}\, k/m$, $m = 1,\cdots,\ell$, that is to say

$$1 + f_+(k) = \exp[-2i \sum_{m=1}^{\ell} \text{arctg}(k/m)] \tag{V.5}$$

Finally, as it can be also deduced from (II.9,10), the effective range approximation is exact for the <u>first regular</u> phase shift, namely $\delta_+(\delta_-)$ for $\ell = 1(2)$, to wit

$$k \,\text{tg}\, \delta_+(k) = 1 \quad (\ell = 1)\ , \quad k \cot \delta_-(k) = \frac{2}{3} + \frac{1}{3} k^2 \ (\ell = 2) \tag{V.6}$$

For $\ell \geq 3$ the effective range approximation is not exact.

The form (V.3) of the scattering amplitude is equivalent to a product of two gamma functions divided by the conjugates (note that $|1 + f_+| = 1$ as $f_- = 0$); in fact, the solution for non-critical g can be also expressed in this way, see [14]. This is reminiscent of the Veneziano amplitude, or of the factorizable S-matrix in some soluble two dimensional models; in fact, a relation between the $\text{sech}^2\, x$ potential and the sine-Gordon equation has already been made, see [17].*

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VI. INTERMEDIATE RANGE POTENTIALS

We studied in [18] the partner potentials

$$V_+(x) = \frac{2}{1 + x^2} , \qquad V_-(x) = \frac{6x^2 - 2}{(1 + x^2)^2} \tag{VI.1}$$

coming from the superpotential $W(x) = 2x/(1 + x^2)$. As here $W(\infty) = 0$, the general equations (II.9) give

$$\delta_{2+}(k) = \delta_{1-}(k) , \qquad \delta_{2-}(k) = \delta_{1+}(k) \tag{VI.2}$$

which imply, of course, that all δ's cannot be generic. To calculate the δ's one has to resort to numerical integration.

Here the culprit of non-genericity is the long range character of the interaction, which introduces extra $\pi/2$ factors. For example $V_+(x) \xrightarrow{x>>} 2/x^2 = \ell(\ell + 1)/x^2$ for $\ell = 1$, which introduces a value of $-\pi/2$ at $k = 0$ in both δ_{1+} and δ_{1-} because of the centrifugal term; hence <u>both</u> δ_1's are anomalous; as for $V_-(x)$, the long range behaviour is $\propto 6/x^2 = \ell(\ell + 1)/x^2$ for $\ell = 2$, and this introduces an extra $-\pi$ value at $k = 0$ in both $\delta_{2\pm}$, so they are anomalous, too. This makes (VI.2) possible for $k = 0$, and again Susy takes care of the equality for arbitrary k. A detailed discussion is in [18].

It could happen that only one of the partner members is anomalous, for example, starting from the function

$$u_0(x) = \frac{1}{\sqrt{1 + x^2}} \tag{VI.3}$$

as the (unnormalized) Susy-exact ground state, we obtain from the equation $u_0(x) = \exp - \int w\, dx$ the superpotential $W(x)$ and the ordinary potentials $V_\pm = W^2 \pm W'$, namely

$$W(x) = \frac{x}{1 + x^2} \qquad \text{(VI.4)}$$

$$V_+(x) = \frac{1}{(1 + x^2)^2}\ , \qquad V_-(x) = \frac{2x^2 - 1}{(1 + x^2)^2}\ . \qquad \text{(VI.4')}$$

$V_+(x)$ represents a repulsive barrier, and decreases like $|x|^{-4}$ for large $|x|$; hence is not critical in any sense; in particular $\delta_{1+}(0) = -\pi/2$ (see III.12+) and $\delta_{1-}(0) = 0$. Now however $V_-(x)$ represents a craterlike potential which supports a zero energy true even bound state, not just a zero energy resonance, namely the state (VI-3) we started from! Also $V_-(x)$ decreases like $2|x|^{-2}$ for large $|x|$; hence its phase shifts acquire both an extra $-\pi/2$ at $k = 0$, but $\delta_{2+}(0)$ also has a $+\pi/2$ value from the bound state, (III.12+); therefore

$$\delta_{2+}(0) = -\frac{\pi}{2} + \frac{\pi}{2} = 0 = \delta_{1-}(0)\ , \quad \delta_{2-}(0) = -\pi/2 = \delta_{1+}(0) \qquad \text{(VI.5)}$$

in agreement with (II.9) for $W(\infty) = 0$, which is the case.

The wave function (VI.3) is inspired in some vortex functions [19].

when $W(x) = $ cons., and then $\delta_- = 0$. For the $-\ell(\ell + 1)\text{sech}^2\, x$ case, both δ_{1+} and δ_{2-} are either normal or anomalous, so the difference is still $\pi/2$ mod π at $k = 0$; the same is true for δ_{1-} and δ_{2+}.

VIII. CONCLUSIONS

We have made a fairly detailed study of some Susy potentials; most of them are critical, that is to say, their phase shift behave non generically. For the three cases: zero range or delta potential, transparent $\text{sech}^2\, x$ potential and intermediate range or $V(x) \propto |x|^{-2}$, we think we understand the critical behaviour: in particular a zero energy resonance must be counted as a "half-bound" [16] state, and a centrifugal term $r(r+1)/x^2$ contributes an additional $-r\pi/2$ to the phase shifts [18].

As another critical potential we should consider the Coulomb potential in one dimension, with superpotential $W(x) = \gamma - |x|^{-1}$; we propose to study this in a future paper.

VII. GENERALIZATIONS

We consider now some generalizations of (VI.3); e.g. if the wave function of the ground state $u_0(x)$ decreases like a power $-r$ at $|x|$ large, $W(x) = -u'(x)/u(x)$ decreases like $|x|^{-1}$, therefore $W(\infty) = 0$, and either of $V_\pm = W^2 \pm W'$ (or both) are critical. In fact both are critical, except when $u_0(x) \to |x|^{-1}$, because

$$u_0(x) \xrightarrow{|x|>>} |x|^{-r} \Rightarrow V_\pm(x) \xrightarrow{|x|>>} \frac{r}{x^2}(r \mp 1) \qquad \text{(VII.1)}$$

which implies also a connection between the number of bound states; namely, because

$$\delta_{1+}(0) = (n^{(1} - \tfrac{1}{2})\pi - (r-1)\pi/2\ , \quad \delta_{1-}(0) = n_-^{(1}\pi - (r-1)\frac{\pi}{2} \qquad \text{(VII.2.1)}$$

$$\delta_{2+}(0) = (n_+^{(2} - \tfrac{1}{2})\pi - r\pi/2\ , \quad \delta_{2-}(0) = (n_-^{(2)})\pi - r\pi/2 \qquad \text{(VII.2.2)}$$

we obtain, from (II.9), $\delta_{1+}(k) = \delta_{2-}(k)$, $\delta_{2+}(k) = \delta_{1-}(k)$, or

$$n_+^{(1} = n_-^{(2}\ , \qquad n_+^{(2} = n_-^{(1} + 1 \qquad \text{(VII.3)}$$

which holds independently of r.

For $W(\infty) \neq 0$ we do not obtain, in general, critical potentials, for (II.9) gives

$$\delta_{2-}(0) - \delta_{1+}(0) = \pm\pi/2 \quad \text{mod } \pi \tag{VII.4a}$$

$$\delta_{2+}(0) - \delta_{1-}(0) = \pm\pi/2 \quad \text{mod } \pi \tag{VII.4b}$$

which is to be expected, as the even phase shifts start at $\pi/2$, the odd at 0 (both mod π). The exceptional case of the delta potential obtains

BIBLIOGRAPHY

[1] WITTEN, E.- Nucl. Phys. B188, 513 (1981).

[2] For elementary reviews see BLOCKEY, C.A. and STEDMAN, G.E.- Eur. J. Phys. 6, 218 (1985) or HAYMAKER, R.W. and RAU, A.R.P.- Ann. J. Phys. 54, 928 (1986). See also GENDENSTEIN, L.E. and KRIVE, J.V.- Sov. Phys. (Uspekhi) 28, 645 (1985).

[3] BOYA, L.J.- Eur. J. Phys (to appear, 1988).

[4] A rigorous treatment of the continuous spectrum in Susy Q.M. from the mathematical standpoint can be consulted in BOLLÉ, D., GESZTESY, F., GROSSE, H., SCHWEIGER, W. and SIMON, B.- J. Math. Phys. 28, 1512 (1987).

[5] This is the modern introduction to Susy Q.M.; for a similar treatment see JAFFE, A. et al.,- Ann. Phys. 178, 313 (1987).

[6] FADDEEV, L.D.- Am. Math. Soc. Trasl. 2, 139 (1965).

[7] CHADAN, K. and SABATIER, P.C.- "Inverse Problems in Quantum Scattering Theory," Springer (Berlin) 1977, Ch. XVII.

[8] LAPIDUS, I.R.- Am. J. Phys. 37, 931 (1969), see also EBERLY, H.- Am. J. Phys. 33, 771 (1965).

[9] BETHE, H.A.- Phys. Rev. 76, 38 (1949); BETHE, H.A. and MORRISON, P.- "Elementary Nuclear Theory" (2nd ed.), J. Wiley (N.Y.) 1956, p. 55.

[10] JAMES, P.B.- Am. J. Phys. 38, 1319 (1970).

[11] RODBERG, L.S. and THALER, R.M.- "The Quantum Theory of Scattering," Academic Press (N.Y.) 1967, p. 61.

[12] LAPIDUS, I.R.- Am. J. Phys. 37, 1064 (1969), see also [8].

[13] BARGMANN, V.- Rev. Mod. Phys. 21, 488 (1949); see also the original work, PÖSCHL, L.G. and TELLER, E.- Zeit. f. Phys. 83, 143 (1933), where a more general potential is considered.

[14] LANDAU, L.D. and LIFSHITZ, E.M.- "Quantum Mechanics," Pergamon (London) 1977, p. 69.

[15] DODD, R.K., EILBECK, J.C., GIBBON, J.D. and MORRIS, H.C.- "Solitons and Nonlinear Wave Equations," Academic Press (N.Y.) 1982, p. 78.

[16] NEWTON, R.- "Scattering Theory of Waves and Particles" (2nd ed.), Springer (Berlin), 1986, p. 311.

[17] KOREPIN, V.E.- Theor. Math. Phys. 34, 3 (1978).

[18] BOYA, L.J., KMIECIK, M. and BOHM, A.- Phys. Rev. D35, 1255 (1987).

[19] NIELSEN, and SCHRÖR, B.- Phys. Lett. B66, 475 (1977).