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On curvature bounds in Lorentzian length spaces

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Abstract

We introduce several new notions of (sectional) curvature bounds for Lorentzian pre-length spaces: On the one hand, we provide convexity/concavity conditions for the (modified) time separation function, and, on the other hand, we study four-point conditions, which are suitable also for the non-intrinsic setting. Via these concepts, we are able to establish (under mild assumptions) the equivalence of all previously known formulations of curvature bounds. In particular, we obtain the equivalence of causal and timelike curvature bounds as introduced by Kunzinger and Sämann.

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Contents

1 INTRODUCTION

The theory of Lorentzian length spaces aims to give a synthetic description of Lorentzian geometry. Inspired by the transformative effect, its metric predecessor (cf., e.g. [5, 11, 12]) has had on the field of Riemannian geometry, after its introduction in [24] the theory has quickly branched out from Lorentzian Alexandrov geometry (e.g. [4, 8, 9]) into a variety of fields, in particular into optimal transport and metric measure geometry (e.g. [10, 15, 27]), causality theory (e.g. [1, 14, 25]), and general relativity (e.g. [19, 26, 28]).

During the initial development of the theory, one of the main goals was to establish a synthetic version of sectional curvature bounds via triangle comparison, a characterisation which in the smooth setting is known even for semi-Riemannian manifolds due to [3]. These descriptions of curvature bounds are also a topic of substantial interest in Alexandrov geometry. Indeed, for metric spaces, there is an abundance of different formulations for (sectional) curvature bounds, cf. [5, 11, 12].

Some of these have been added to the Lorentzian repertoire as well, such as, for example, the so-called monotonicity condition. In fact, both [9] and [6], more or less simultaneously, introduced the concept of hyperbolic angles into the synthetic Lorentzian theory, and gave a formulation of timelike curvature bounds expressed via the monotonic behaviour of angles. In [6], there is also a formulation using angles directly in relation to their comparison angles, but only for lower curvature bounds, and in [9], angle comparison is only obtained as an implication of ordinary curvature bounds and not vice versa.

As is evident from Alexandrov geometry, having a wide array of different characterisations at one's disposal is vital to producing a rich and flourishing theory of synthetic geometry. In this work, we collect all the currently known approaches to (sectional) curvature bounds in Lorentzian pre-length spaces, add several new ones and prove equivalence of all these notions under suitable assumptions. The precise interdependence of all of these concepts will be established in Theorem 5.1.

2 PRELIMINARIES

Since the theory of Lorentzian pre-length spaces is by now quite well established, we are not going to repeat the basic definitions, instead referring to [24] for the fundamentals, and to [6, 9] for more in-depth discussions of angles and some of the curvature bounds discussed here.

Remark 2.1 (Notations and conventions). For the sake of readability and consistency in notation, we collect some generalities here. Let X denote a Lorentzian pre-length space.

(i) Unless explicitly stated otherwise, K is assumed to be any real number. It symbolises the curvature of the model space, and hence, it is only explicitly mentioned when necessary.
(ii) By a *distance realiser*, we mean a curve that attains the τ-distance between its endpoints, that is, if γ is a causal curve from x to y, then γ is a distance realiser if $\tau(x, y) = L_\tau(\gamma)$. Instead of labelling the curve, we may also use the notation $[x, y]$ for a distance realiser from x to y.

(iii) By a *timelike triangle* in X, we mean a collection of three timelike related points $x \ll y \ll z, \tau(x,z) < \infty$, and three distance realisers pairwise joining them. By an *admissible causal triangle*, we mean a collection of three points $x \leqslant y \ll z$ or $x \ll y \leqslant z, \tau(x,z) < \infty$, together with distance realisers between timelike-related points. In other words, one of the two short sides is allowed to be null, and if it is, it need not be realised by a curve. When the context allows for it, we may refer to either of these as just a triangle, and both are denoted by $\Delta(x,y,z)$.

(iv) Given a triangle $\Delta(x,y,z)$ in X, a *comparison triangle* is a triangle in $\mathbb{L}^2(K)$ with the same side lengths. The existence of comparison triangles is established in the so-called Realisability lemma, see [3, Lemma 2.1].

(v) We say that a triangle or hinge (cf. Definition 2.5 below) in X *satisfies size bounds for* K if there exists a (unique) comparison configuration in $\mathbb{L}^2(K)$, the Lorentzian model space of constant curvature K. Throughout the paper, we assume any such configuration to satisfy size bounds. As will become evident later on, this is precisely the case when the largest τ-value in this configuration is less than D_K.

(vi) To increase readability, we are going to mark points arising in comparison triangles with a bar, in comparison hinges with a tilde and in four-point comparison configurations (cf. Definition 4.4 below) with a hat. That is, if $x \in X$, then the corresponding point in a comparison triangle will be denoted by $\bar{x}$, the corresponding point in a comparison hinge will be denoted by $\tilde{x}$ and the corresponding point in a four-point comparison configuration will be denoted by $\hat{x}$. To ease the notational burden we will, however, not mark the time separation function in the comparison spaces by $\bar{\tau}$. Instead, we will always just write τ since the marking of the arguments will clearly indicate when the time separation function in the model space is considered.

(vii) We will use the slightly updated version of curvature bounds introduced in [7]. In order for these conditions to be non-void, that is, trivially satisfied by pathological spaces where, for example, $\tau \equiv \infty$, we will always assume X to be chronological. This is not a substantial restriction since any space that satisfies some timelike curvature bound in the original formulation in [24, Definition 4.7] is chronological by definition, as there τ was supposed to be finite on comparison neighbourhoods.

(viii) The notation regarding hyperbolic angles (which are defined as limits) will use curves, while angles in the comparison spaces will use the endpoints of the corresponding curves (as the model spaces are uniquely geodesic, this is consistent).

(ix) Throughout this paper, we introduce several formulations of curvature bounds, all of which are built on the notion of comparison neighbourhoods. We will adhere to the following terminology: a Lorentzian pre-length space is said to have *curvature bounded below (resp. above) by* K in any of these senses if it is covered by the corresponding ($\geqslant K$)-(resp. ($\leqslant K$)-)comparison neighbourhoods. Moreover, we say that X has *curvature globally bounded below (resp. above) by* K in any of these senses if X is a corresponding ($\geqslant K$)-(resp. ($\leqslant K$)-)comparison neighbourhood. In general, implications on comparison neighbourhoods without further assumptions on the neighbourhoods yield implications on curvature bounds for X, so that we will not highlight this in every statement. However, if this only holds under additional assumptions, we will explicitly formulate the implication for curvature bounds on X.

For completeness, we briefly repeat the most important definitions surrounding hyperbolic angles, following [9].

Definition 2.2 (K-comparison angles and sign). Let X be a Lorentzian pre-length space, $\Delta(x,y,z)$ an admissible causal triangle in X, and $\Delta(\bar{x},\bar{y},\bar{z})$ a comparison triangle in $\mathbb{L}^2(K)$ for $\Delta(x,y,z)$. Assume that, say, x is adjacent to two timelike sides (the following definition clearly works for any other vertex who is between two timelike sides).

(i) The *K-comparison angle* at x is defined as the ordinary hyperbolic angle at $\bar{x}$ between $\bar{y}$ and $\bar{z}$:

$$\tilde{\measuredangle}_x^K(y,z) := \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y},\bar{z}) = \operatorname{arcosh}(|\langle\gamma'_{\bar{x}\bar{y}}(0),\gamma'_{\bar{x}\bar{z}}(0)\rangle|)\,, \tag{1}$$

where we assume the mentioned geodesics to be unit speed parametrised.

(ii) The *sign σ* of a K-comparison angle is the sign of the corresponding inner product (in the $-,+,\ldots,+$ convention). That is, in this notation, the sign is -1 if the angle is measured at x or z and 1 if the angle is measured at y.

(iii) The *signed K-comparison angle* is defined as $\tilde{\measuredangle}_x^{K,S}(y,z) := \sigma\tilde{\measuredangle}_x^K(y,z)$.

Definition 2.3 (Angles). Let X be a Lorentzian pre-length space and let α and β be two timelike curves of arbitrary time orientation emanating from $\alpha(0) = \beta(0) =: x$.

(i) The *angle* between α and β, if it exists, is defined as

$$\measuredangle_x(\alpha,\beta) := \lim_{s,t\to 0} \tilde{\measuredangle}_x^0(\alpha(s),\beta(t))\,, \tag{2}$$

where the limit only takes values of s and t into account for which the triple $(x,\alpha(s),\beta(t))$ (or some permutation thereof) forms an admissible causal triangle.[†]

(ii) The *sign σ* of an angle is -1 if α and β have the same time orientation and 1 otherwise. The *signed angle* is defined as $\measuredangle_x^S(\alpha,\beta) := \sigma\measuredangle_x(\alpha,\beta)$.

Remark 2.4. Note that one could also look at angles defined using any K instead of 0 in (2). However, the limit is the same regardless of the model space in which it is considered due to [9, Proposition 2.14]. Although this reference uses strong causality to ensure size bounds, this is not actually necessary, the condition from Definition 3.1(i) is sufficient: As τ is continuous near (p,p) (since $\tau(p,p) = 0 < D_K$) and $\tau^{-1}([0,D_K))$ is open, for two curves α,β emanating from p, $(\alpha(t),\beta(t))$ will initially stay in $\tau^{-1}([0,D_K))$. This is why we will usually drop the superscript K in the comparison angle and just write $\tilde{\measuredangle}_x(y,z)$. Similarly, we will write $\measuredangle_{\bar{x}}(\bar{y},\bar{z})$ instead of $\measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y},\bar{z})$ for angles between points in $\mathbb{L}^2(K)$ (which do not necessarily arise as comparison angles).

Definition 2.5 (Hinges and comparison hinges). Let X be a Lorentzian pre-length space. Let $\alpha : [0,a] \to X$ and $\beta : [0,b] \to X$ be two timelike distance-realisers emanating from the same point $x := \alpha(0) = \beta(0)$. Then, we say that α and β and their associated angle form a *hinge*[‡] *at* x, and denote it by (α,β). In particular, if the angle $\measuredangle_x(\alpha,\beta)$ is finite, then by a *comparison hinge* in

[†] As admissible causal triangles arise as limits of timelike triangles, one can also restrict this to timelike triangles.

[‡] In [9, Definition 2.11], such a constellation is only called a hinge if the angle exists, meaning that the lim sup is a limit and is finite. Here, we drop these restrictions, see Definition 3.14(iv) below.

$\mathbb{L}^2(K)$, denoted[†] by $(\tilde{\alpha}, \tilde{\beta})$, we mean a constellation $\tilde{x}$ together with two distance-realisers $\tilde{\alpha}$ and $\tilde{\beta}$ emanating from that point, such that they have the same length (and time orientation) as α and β, respectively, and such that the angle between them is the same, that is, $\measuredangle_x^S(\alpha, \beta) = \measuredangle_{\tilde{x}}^S(\tilde{\alpha}(a), \tilde{\beta}(b))$.

Another concept we shall require is the so-called finite diameter of a Lorentzian pre-length space. Introduced in [7], this number essentially bounds the length of geodesics for which it is possible to implement comparison methods.

Definition 2.6 (Finite diameter). Let X be a Lorentzian pre-length space.

(i) The *finite diameter* of X is

$$\mathrm{diam}_{\mathrm{fin}} = \sup(\{\tau(x, y) \mid x \ll y\} \setminus \{\infty\}), \tag{3}$$

that is, the supremum of all values τ takes except ∞.

(ii) By D_K we denote the finite diameter of $\mathbb{L}^2(K)$. In particular,

$$D_K = \mathrm{diam}_{\mathrm{fin}}(\mathbb{L}^2(K)) = \begin{cases} \infty, & \text{if } K \geqslant 0, \\ \frac{\pi}{\sqrt{-K}}, & \text{if } K < 0. \end{cases} \tag{4}$$

Note that the formula of D_K is pleasantly similar to the diameter of the Riemannian model spaces in metric geometry. Finally, we introduce the concept of being (locally) r-geodesic, another notion very similar to its metric counterpart.

Definition 2.7 (r-geodesic). Let X be a Lorentzian pre-length space and let $0 < r \leqslant \infty$.

(i) A subset $U \subseteq X$ is called r-geodesic if for all $p \ll q$ in U with $\tau(p, q) < r$, there exists a distance realiser in U connecting them. In particular, if U is ∞-geodesic, then this corresponds to the original definition of being geodesic, cf. [24, Definition 3.27], without the assumption of the existence of null realisers.

(ii) If one additionally assumes the existence of null realisers, we call U *causally r-geodesic*.

(iii) X is called *locally (causally) r-geodesic* if every point has a neighbourhood which is (causally) r-geodesic.

3 | CURVATURE COMPARISON FOR LORENTZIAN PRE-LENGTH SPACES

In this section, we recall and amend the currently known characterisations of timelike curvature bounds. In particular, for those cases where, as of yet, only implications between certain notions are available, we show full equivalence. Timelike curvature bounds in the setting of Lorentzian pre-length spaces were first introduced in [24, Definition 4.7] and slightly updated in [7, Definition 2.7].

[†] As the sides of a comparison hinge are unique, we may also denote a comparison hinge by its endpoints (and the vertex where the angle is measured), that is, $(\tilde{\alpha}(a), \tilde{x}, \tilde{\beta}(b))$.

Definition 3.1 (Curvature bounds by timelike triangle comparison). Let X be a Lorentzian pre-length space. An open subset U is called a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood) in the sense of *timelike triangle comparison* if:

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.
(ii) U is D_K-geodesic.
(iii) Let $\Delta(x, y, z)$ be a timelike triangle in U, with p, q two points on the sides of $\Delta(x, y, z)$. Let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle in $\mathbb{L}^2(K)$ for $\Delta(x, y, z)$ and $\bar{p}, \bar{q}$ comparison points for p and q, respectively. Then,

$$\tau(p, q) \leqslant \tau(\bar{p}, \bar{q}) \quad (\text{resp. } \tau(p, q) \geqslant \tau(\bar{p}, \bar{q})). \tag{5}$$

Note that within a $(\geqslant K)$-comparison neighbourhood, $p \ll q$ implies $\bar{p} \ll \bar{q}$, and within a $(\leqslant K)$-comparison neighbourhood, $\bar{p} \ll \bar{q}$ implies $p \ll q$.

As a first different formulation, we mention one-sided triangle comparison. This was originally introduced in [6] for lower curvature bounds only, but can easily be adapted to work for upper curvature bounds as well.

Definition 3.2 (Curvature bounds by one-sided timelike triangle comparison). Let X be a Lorentzian pre-length space. An open subset U is called a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood) in the sense of *one-sided timelike triangle comparison* if:

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.
(ii) U is D_K-geodesic.
(iii) Let $\Delta(x, y, z)$ be a timelike triangle in U. Let p be a point on one side of the triangle and denote by $v \in \{x, y, z\}$ the vertex opposite of p. Let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle in $\mathbb{L}^2(K)$ for $\Delta(x, y, z)$ and let $\bar{p}$ be a comparison point for p. Then[†]

$$\tau(p, v) \leqslant \tau(\bar{p}, \bar{v}) \text{ and } \tau(v, p) \leqslant \tau(\bar{v}, \bar{p}),$$

$$(\text{resp. } \tau(p, v) \geqslant \tau(\bar{p}, \bar{v}) \text{ and } \tau(v, p) \geqslant \tau(\bar{v}, \bar{p})).$$

Proposition 3.3 (One-sided triangle comparison). *Let U be an open subset in a Lorentzian pre-length space X. Then, U is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of timelike triangle comparison if and only if it is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of one-sided timelike triangle comparison.*

Proof. Concerning the non-trivial direction, the case for $(\leqslant K)$-comparison neighbourhoods is shown in [6, Proposition 4], so we only need to consider upper curvature bounds. Moreover, if, say, $p \in [x, y]$ and $q \in [y, z]$, then the proof of [6, Proposition 4] works for $(\geqslant K)$-comparison neighbourhoods in complete analogy, by reversing the inequalities between τ-values, as well as between the nonnormalised angles established in [6, Lemmas 1 & 2].

The other case of one point being on the longest side is where the cases of lower and upper curvature bounds differ (and this is, in fact, also the reason why we, in contrast to [6], require the τ-inequalities to hold even if there is no timelike relation between a point and the opposing vertex).

[†] Note that as X is supposed to be chronological, at most one of the τ-values in X is positive in each case.

Let, say, $p \in [x, y]$ and $q \in [x, z]$. If $\tau(\bar{p}, \bar{q}) = 0$, then (5) is trivially satisfied. So, assume $\bar{p} \ll \bar{q}$. Let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle for $\Delta(x, y, z)$ and let $\Delta(\bar{x}', \bar{p}', \bar{z}')$ be a comparison triangle for the subtriangle $\Delta(x, p, z)$. Since U is a $(\geqslant K)$-comparison neighbourhood in the sense of one-sided timelike triangle comparison, we infer $\tau(p, z) \geqslant \tau(\bar{p}, \bar{z})$, hence also $\tau(\bar{p}', \bar{z}') = \tau(p, z) \geqslant \tau(\bar{p}, \bar{z})$. The triangles $\Delta(\bar{x}', \bar{p}', \bar{z}')$ and $\Delta(\bar{x}, \bar{p}, \bar{z})$ have two sides of equal length and an inequality between the lengths of their third sides, so we obtain $\measuredangle_{\bar{x}}(\bar{p}, \bar{z}) \geqslant \measuredangle_{\bar{x}'}(\bar{p}', \bar{z}')$ by law of cosines monotonicity, cf. [9, Remark 2.5]. Clearly, this further yields

$$\measuredangle_{\bar{x}}(\bar{p}, \bar{q}) = \measuredangle_{\bar{x}}(\bar{p}, \bar{z}) \geqslant \measuredangle_{\bar{x}'}(\bar{p}', \bar{z}') = \measuredangle_{\bar{x}'}(\bar{p}', \bar{q}'). \tag{6}$$

Applying law of cosines monotonicity once more to the subtriangles $\Delta(\bar{x}, \bar{p}, \bar{q})$ and $\Delta(\bar{x}', \bar{p}', \bar{q}')$ of $\Delta(\bar{x}, \bar{p}, \bar{z})$ and $\Delta(\bar{x}', \bar{p}', \bar{z}')$, respectively, we obtain $\tau(\bar{p}, \bar{q}) \leqslant \tau(\bar{p}', \bar{q}')$. By one-sided comparison in the subtriangle $\Delta(x, p, z)$ of $\Delta(x, y, z)$, we get $\tau(p, q) \geqslant \tau(\bar{p}', \bar{q}')$, hence the desired inequality of $\tau(p, q) \geqslant \tau(\bar{p}, \bar{q})$ follows. Assuming the opposite timelike relation of $\bar{q} \ll \bar{p}$, the proof works out just the same. □

Throughout this work, the property of a Lorentzian pre-length space being *locally causally closed*, cf. [24, Definition 3.4], is used several times. This should be compared with the slightly weaker notion of being *locally weakly causally closed*, which was introduced in [1, Definition 2.19] to better resemble the smooth setting below the level of strong causality. It turns out that these two notions are equivalent under the assumption of strong causality, see [1, Proposition 2.21]. In a similar spirit, the following lemma shows in what way one could also work with this latter definition in the present paper.

Lemma 3.4 (Causally closed comparison neighbourhoods). *Let X be a locally weakly causally closed and strongly causal Lorentzian pre-length space which has curvature bounded below (resp. above) by K in the sense of timelike triangle comparison. Then, each point has a comparison neighbourhood which is causally closed. This will also work in any of the following senses, except the four-point condition.*

Proof. Let $x \in X$, and let U_1 be a weakly causally closed neighbourhood. Let U_2 be a curvature comparison neighbourhood. According to (i) in Definition 3.1, we have that $(U_2 \times U_2) \cap \tau^{-1}([0, D_K))$ is open, so we find U_3 with $x \in U_3 \subseteq U_2$ and $U_3 \times U_3 \subseteq \tau^{-1}([0, D_K))$. Then, by strong causality, there exists a causally convex set $V = \cap_{i=1}^{n} I(p_i, q_i)$ with $x \in V \subseteq U_1 \cap U_3$. We claim that V is a causally closed comparison neighbourhood. The properties (i) and (iii) follow as we are just restricting. For (ii), note that distance realisers in U_2 with endpoints in V are automatically contained in V by causal convexity. To see causal closure, let $x_n \leqslant y_n$ in V with $x_n \to x$ and $y_n \to y$. As $x_n, y_n \in U_3$, we have $\tau(x_n, y_n) < D_K$, so as U_2 is D_K-geodesic, there is a causal distance realiser γ_n connecting x_n and y_n. In particular, there is a causal curve joining them contained in V, which in the terminology of [1] is denoted by $x_n \leqslant_V y_n$, from which it follows by the weak causal closure that $x \leqslant_V y$, that is, x and y are also joined by a causal curve in V. Thus, we obtain $x \leqslant y$ and hence V is causally closed. □

The following three characterisations were all to some extent developed in [9] and use the notion of being *locally strictly timelike geodesically connected*, see [9, Definition 1.12]. It turns out that this is equivalent to the seemingly more natural property of being regular, which is why we will use this formulation instead (see Lemma 3.7 below). The following concept of regularity for

Lorentzian pre-length spaces was introduced in [7, Definition 2.4] (and has to be compared to the notion of regularity of Lorentzian length spaces, cf. [24, Definition 3.22]). The two concepts are equivalent under the assumption of strong causality, see Lemma 3.6 below.

Definition 3.5 (Regularity). A Lorentzian pre-length space X is called *regular* if every distance realiser between timelike related points is timelike, that is, it cannot contain a null segment.

Lemma 3.6 (Regularly localisable vs. regular and localisable). *A strongly causal Lorentzian pre-length space is regularly localisable if and only if it is regular and localisable.*

Proof. A regularly localisable Lorentzian pre-length space is automatically regular in the sense of Definition 3.5. By [19, Lemma 4.3], we can choose localisable neighbourhoods Ω such that the local time separation function agrees with τ. Thus, also a distance realiser w.r.t. the local time separation function is a global distance realiser, and has to stay timelike. □

Lemma 3.7 (Regularity and local strictly timelike geodesic connectedness). *A D_K-geodesic Lorentzian pre-length space X such that $t \mapsto L(\gamma|_{[0,t]})$ is continuous*† *for every distance realiser $\gamma : [0, b] \to X$ is regular if and only if it is locally strictly timelike geodesically connected.*

Proof. If X is regular, any distance realiser between timelike-related points is timelike by definition.

If X is locally strictly timelike geodesically connected, let $x \ll y$ and $\gamma : [0, b] \to X$ a causal distance realiser connecting x to y. The function $f(t) =: L(\gamma|_{[0,t]}) = \tau(x, \gamma(t))$ is continuous and monotonically increasing and, as γ is a distance realiser, we have $\tau(\gamma(s), \gamma(t)) = f(t) - f(s)$ for $s \leqslant t$. If γ is not timelike, we have $s < t$ such that $\tau(\gamma(s), \gamma(t)) = 0$, so f is constant on $[s, t]$, but $f(0) < f(b)$. W.l.o.g. assume $f(0) < f(s)$ and let s be minimal such that f is constant on $[s, t]$. Then, we have $f(s - \varepsilon) < f(s) = f(s + \varepsilon)$. Thus, at $p = \gamma(s)$, we have a distance realiser $\gamma|_{[s-\varepsilon,s+\varepsilon]}$ with $\gamma(s - \varepsilon) \ll \gamma(s + \varepsilon)$, but $\gamma(s) \not\ll \gamma(s + \varepsilon)$, contradicting the assumption that X is strictly timelike geodesically connected in a neighbourhood of p. □

Next, we turn to the so-called monotonicity condition. This equivalent formulation was introduced in [9, Definition 4.9] and updated in [7, Definition 2.15]. Intuitively, it says that signed comparison angles cannot increase (decrease) when approaching the vertex. Note that at first glance, this seems opposite to (iii) in the following definition, but this apparent contradiction is resolved by noting that increasing the inputs in θ causes one to move away from the vertex.

Definition 3.8 (Curvature bounds by monotonicity comparison). Let X be a regular Lorentzian pre-length space. An open subset U is called a ($\geqslant K$)- (resp. ($\leqslant K$)-)comparison neighbourhood in the sense of *monotonicity comparison* if:

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.
(ii) U is D_K-geodesic.
(iii) let $\alpha : [0, a] \to U, \beta : [0, b] \to U$ be timelike distance realisers such that $x := \alpha(0) = \beta(0)$ and such that $L(\alpha), L(\beta), \tau(\alpha(a), \beta(b)), \tau(\beta(b), \alpha(a)) < D_K$. Let $\theta : D \to [0, \infty)$ be defined

† This can, for example, be achieved by assuming that any point has a neighbourhood U such that $\tau|_{U \times U}$ is continuous, which is the case in any space with curvature bounds, see [24, Lemma 3.33] (whose proof also works with τ being merely continuous in this sense).

by $\theta(s,t) := \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(s),\beta(t))$ ($D \subseteq (0,a] \times (0,b]$ is the set where this is defined, that is, the set of points where there is some causal relation between $\alpha(s)$ and $\beta(t)$ and the comparison triangle exists). Then, θ is monotonically increasing (resp. decreasing).

The equivalence between monotonicity comparison and triangle comparison has already been established in [9].

Proposition 3.9 (Equivalence of triangle and monotonicity comparison). *Let $U \subseteq X$ be an open subset in a regular Lorentzian pre-length space X. Then, U is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of timelike triangle comparison if and only if it is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of monotonicity comparison.*

Proof. See [9, Theorem 4.13]. □

Remark 3.10 (One-sided monotonicity comparison). As it turns out, similar to the case of triangle comparison, there is also a one-sided version of monotonicity comparison. This means that we leave the parameter of one of the curves fixed. Clearly, ordinary monotonicity comparison implies one-sided monotonicity comparison. Conversely, one simply varies the parameters one after the other (taking care to make the points stay timelike related), to get from one-sided monotonicity comparison to the original formulation. As will be seen below, the one-sided version is more convenient to work with.

The next formulation of curvature bounds is closely related to monotonicity comparison. Instead of talking about monotonic behaviour of comparison angles when going along the sides of a hinge, we now require an inequality between hyperbolic angles and comparison angles. A formulation of curvature bounds using angles was first introduced in [6]. However, this uses a slightly different definition for angles. The following definition is better suited for our setting. Note that as in [6], for the case of upper curvature bounds, one needs to explicitly assume one case of the triangle inequality for angles.

Definition 3.11 (Curvature bounds by angle comparison). Let X be a regular Lorentzian pre-length space. An open subset U is called a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of *angle comparison* if:

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.
(ii) U is D_K-geodesic.
(iii) Let $\alpha : [0,a] \to U$, $\beta : [0,b] \to U$ be distance realisers such that $L(\alpha), L(\beta), \tau(\alpha(a),\beta(b))$, $\tau(\beta(b),\alpha(a)) < D_K$ and such that $x := \alpha(0) = \beta(0)$ and $\alpha(a), \beta(b)$ are causally related. Then,

$$\measuredangle_x^{\mathrm{S}}(\alpha,\beta) \leqslant \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(a),\beta(b)) \quad (\text{resp. } \measuredangle_x^{\mathrm{S}}(\alpha,\beta) \geqslant \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(a),\beta(b)))\,. \tag{7}$$

(vi) For $(\geqslant K)$-comparison neighbourhoods only: let $\alpha, \beta, \gamma : [0,\varepsilon) \to U$ be distance realisers, all emanating from the same point $x := \alpha(0) = \beta(0) = \gamma(0)$. Suppose that α and γ have the same time orientation and β has the opposite time orientation. Then, we have the following special case of the triangle inequality of angles:

$$\measuredangle_x(\alpha,\gamma) \leqslant \measuredangle_x(\alpha,\beta) + \measuredangle_x(\beta,\gamma)\,. \tag{8}$$

Note that for a Lorentzian pre-length space X with curvature bounded below, in point (*iv*), one can also take the curves as maps into X, since angles only depend on the initial segments of the curves anyways.

Remark 3.12 (Only considering timelike triangles). While Definition 3.11 requires all admissible causal triangles to satisfy the angle condition at each vertex where an angle is defined, it is often more convenient to work only with timelike triangles. However, it becomes clear, when using our new vocabulary, that only requiring the angle condition to hold at each vertex of every timelike triangle is an equivalent constraint.

Consider, for example, an admissible causal triangle $\Delta(x, y, z)$ where $x \ll y \leqslant z$, with $\tau(y, z) = 0$ with a failing angle condition. (The case $x \leqslant y \ll z$ is similar.) We show that moving y slightly can create a timelike triangle with a failing angle condition. Let $\alpha : [0, a] \to X$ be a distance realiser from x to y and $\beta : [0, b] \to X$ a distance realiser from x to z. By regularity, the side $[y, z]$ contains no timelike segments, so the only angle which is defined in $\Delta(x, y, z)$ is at x. It follows that, if an angle condition fails, it necessarily does so at x. As $s \nearrow a$, the triangle $\Delta(x, \alpha(s), z)$ is timelike and converges to the original $\Delta(x, y, z)$. The signed angle $\measuredangle_x^{\mathrm{S}}(\alpha, \beta)$ is not dependent on the endpoint of α, while the signed comparison angles $\tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(s), \beta(b))$ vary continuously with s. The failure of the angle condition at x is an open condition, and so, for s sufficiently close to a, the timelike triangle $\Delta(x, \alpha(s), z)$ also has a failing angle condition at x.

Hence, the existence of an admissible causal triangle with failing angle condition implies the existence of a timelike triangle (of comparable size) with failing angle condition. The contrapositive then tells us that a space has a curvature bound with respect to angle comparison in admissible causal triangles if it does so with respect to angle comparison in timelike triangles. The converse implication is tautological and the two notions are therefore equivalent. In particular, we refrain from introducing 'causal angle comparison', since it would anyways be automatically equivalent to Definition 3.11.

We now show that monotonicity comparison implies angle comparison. As [6] uses a different convention and the proof is elementary, we give it anew. We intend to form an implication circle, so the converse implication is proven later. In that proof, a technical detail requires us to assume the triangle inequality of angles as displayed in (8), which was achieved in [9, Theorem 4.5(ii)] using *geodesic prolongation*, cf. [9, Definition 4.2].

As this is a rather strong property, however, we believe that it is, in fact, more natural to directly impose this condition, whenever necessary. We will do this by saying that X satisfies (8).

Proposition 3.13 (Monotonicity comparison implies angle comparison). *Let $U \subseteq X$ be an open subset in a regular Lorentzian pre-length space. In the case of lower curvature bounds, additionally assume that X satisfies (8). Then, if U is a ($\geqslant K$)- (resp. ($\leqslant K$)-)comparison neighbourhood in the sense of monotonicity comparison, it is also a ($\geqslant K$)- (resp. ($\leqslant K$)-)comparison neighbourhood in the sense of angle comparison.*

Proof. (i) and (ii) are the same in both definitions, and (iv) in the case of lower curvature bounds is assumed directly. So, the only point to check is (iii) in Definition 3.11. To this end, given distance realisers α and β in U as in Definition 3.11(iii), we have by definition

$$\measuredangle_x^{\mathrm{S}}(\alpha, \beta) = \lim_{s,t \to 0} \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(s), \beta(t)) = \lim_{s,t \to 0} \theta(s, t)\,. \tag{9}$$

As θ is monotonous by assumption, the desired inequality holds in the limit as well. $\square$

Finally, we turn to hinge comparison. This uses the construction of hinges and comparison hinges, and the distinguishing inequality pertains to the opposite side of the angle that forms the hinge. As will be seen below, this is closely related to angle comparison via the law of cosines. To establish a proper equivalence between hinge comparison and angle comparison, however, we need to additionally assume the same case of the triangle inequality of angles as in Definition 3.11.

Definition 3.14 (Curvature bounds by hinge comparison). Let X be a regular Lorentzian pre-length space. An open subset U is called a ($\geqslant K$)- (resp. ($\leqslant K$)-)comparison neighbourhood in the sense of *hinge comparison* if:

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.
(ii) U is D_K-geodesic.
(iii) Let $\alpha : [0, a] \to U$, $\beta : [0, b] \to U$ be distance realisers emanating from the same point $x = \alpha(0) = \beta(0)$ such that $L(\alpha), L(\beta), \tau(\alpha(a), \beta(b)), \tau(\beta(b), \alpha(a)) < D_K$ and such that the angle $\measuredangle_x(\alpha, \beta)$ is finite. Let $(\tilde{\alpha}, \tilde{\beta})$ form a comparison hinge for (α, β) in $\mathbb{L}^2(K)$. Then

$$\tau(\alpha(a), \beta(b)) \geqslant \tau(\tilde{\alpha}(a), \tilde{\beta}(b)) \quad (\text{resp. } \tau(\alpha(a), \beta(b)) \leqslant \tau(\tilde{\alpha}(a), \tilde{\beta}(b)))\,. \tag{10}$$

(iv) Let α, β be as in (iii), without the restriction of finite angle. For ($\geqslant K$)-comparison neighbourhoods, we assume that if α, β point in different time directions, the angle can never be infinite, and for ($\leqslant K$)-comparison neighbourhoods, we assume that if α, β point in the same time directions, the angle can never be infinite.[†]
(v) For ($\geqslant K$)-comparison neighbourhoods only: let $\alpha, \beta, \gamma : [0, \varepsilon) \to U$ be distance realisers all emanating from the same point $x := \alpha(0) = \beta(0) = \gamma(0)$. Suppose that α and γ have the same time-orientation and β has the opposite time orientation. Then, we have the following special case of the triangle inequality of angles:

$$\measuredangle_x(\alpha, \gamma) \leqslant \measuredangle_x(\alpha, \beta) + \measuredangle_x(\beta, \gamma)\,. \tag{11}$$

Proposition 3.15 (Equivalence of angle and hinge comparison). *Let U be an open subset in a regular Lorentzian pre-length space X. Then, U is a ($\geqslant K$)- (resp. ($\leqslant K$)-)comparison neighbourhood in the sense of angle comparison if and only if it is a ($\geqslant K$)- (resp. ($\leqslant K$)-)comparison neighbourhood in the sense of hinge comparison.*

Proof. Definition 3.11(iv) and Definition 3.14(v) as well as (i) and (ii) in both formulations are the same. Thus, only the case of (iii) in both conditions as well as 3.14(iv) are of interest. Concerning 3.14(iv), note that angle comparison for, say, lower curvature bounds, yields

$$\measuredangle_x^{\mathrm{S}}(\alpha, \beta) \leqslant \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(a), \beta(b)) \tag{12}$$

for any two distance realisers as in Definition 3.11(iii). Clearly, any comparison angle is finite by definition (it is a hyperbolic angle in the model spaces between timelike distance realisers). If $\sigma = 1$, that is, if the two curves have different time orientation, then this becomes an inequality for non-signed (comparison) angles, and hence, $\measuredangle_x(\alpha, \beta) < \infty$ follows. For upper curvature bounds, the inequality on signed angles is reversed, which is why we get the implication for finite $\measuredangle_x(\alpha, \beta)$ for curves with the same time orientation (causing the inequality to reverse once again).

[†] This can be viewed as the limit of (iii) as $\measuredangle_x(\alpha, \beta) \to \infty$ and agrees with [9, Lemma 4.10]. The rationale behind (iv) is to avoid the case of curvature bounds being trivially satisfied when angles are infinite.

For (iii), we start out by noting that hinges and triangles are closely related concepts. Indeed, given any hinge (α, β) emanating from x such that the endpoints $\alpha(a)$ and $\beta(b)$ of the curves are causally related, we can form a triangle $\Delta(x, \alpha(a), \beta(b))$ (the order of the points might change depending on the time orientation of the curves, and the side opposite of x might be null, but this is not important for our arguments). Conversely, any timelike triangle $\Delta(x, y, z)$ gives a hinge at x (in fact, at any of the three points), by using the two sides adjacent to x.

Say (α, β) is a hinge at x with finite angle, both curves are future-directed, and $\alpha(a) \leqslant \beta(b)$. Consider the comparison triangle $\Delta(\bar{x}, \bar{\alpha}(a), \bar{\beta}(b))$ (cf. the realisabilty lemma [3, Lemma 2.1]) and the comparison hinge $(\tilde{\alpha}(a), \tilde{x}, \tilde{\beta}(b))$. By construction, we have $\measuredangle_x(\alpha, \beta) = \measuredangle_{\tilde{x}}(\tilde{\alpha}(a), \tilde{\beta}(b))$ and $\tilde{\measuredangle}_x(\alpha(a), \beta(b)) = \measuredangle_{\bar{x}}(\bar{\alpha}(a), \bar{\beta}(b))$. Moreover, the comparison hinge $(\tilde{\alpha}(a), \tilde{x}, \tilde{\beta}(b))$ can be viewed as a geodesic triangle, although the side connecting $\tilde{\alpha}(a)$ and $\tilde{\beta}(b))$ might not be causal. In any case, the sides adjacent to $\bar{x}$ and $\tilde{x}$ have the same lengths, so we have $\measuredangle_x(\alpha, \beta) \leqslant \tilde{\measuredangle}_x(\alpha(a), \beta(b))$ if and only if $\tau(\alpha(a), \beta(b)) \geqslant \tau(\tilde{\alpha}(a), \tilde{\beta}(b))$ due[†] to the hinge lemma [3, Lemma 2.2]. The case of α and β having different time orientation (or both being past-directed) is completely analogous.

Finally, we need to touch on a small technicality about causal relations. While angle comparison talks about curves where the endpoints are causally related, this is not the case for hinge comparison, meaning that one needs to conclude from angle comparison the fact that hinge comparison is also valid in configurations where the endpoints are not causally related. Clearly, this is only possible if the curves have the same time orientation, say both are future directed. Moreover, for upper curvature bounds, the inequality in hinge comparison reads $\tau(\alpha(a), \beta(b)) \leqslant \tau(\tilde{\alpha}(a), \tilde{\beta}(b))$, which is trivially satisfied if $\alpha(a)$ and $\beta(b)$ are not causally related. So, assume that we are in the case of lower curvature bounds and let (α, β) be a hinge with both realisers future-directed and assume that there is no causal relation between $\alpha(a)$ and $\beta(b)$. We essentially need to show that there is no timelike relation between $\bar{\alpha}(a)$ and $\bar{\beta}(b)$. Note that contrary to comparison triangles, comparison hinges have the useful property that sub-comparison hinges 'live inside' the original one. In other words, if $(\tilde{\alpha}, \tilde{\beta})$ is a comparison hinge for (α, β), then $(\tilde{\alpha}|_{[0,s]}, \tilde{\beta})$ is a comparison hinge for $(\alpha|_{[0,s]}, \beta)$. Clearly, $\alpha(\delta) \ll \beta(b)$ for small enough $\delta > 0$. Thus, together with $\tau(\alpha(a), \beta(b)) = 0$ and the mean value theorem (τ is continuous in a comparison neighbourhood), we infer that for each $\varepsilon > 0$ small enough, there is a parameter s' such that $\tau(\alpha(s'), \beta(b)) = \varepsilon$. Thus, we infer $\varepsilon = \tau(\alpha(s'), \beta(b)) \geqslant \tau(\tilde{\alpha}(s'), \tilde{\beta}(b))$. Since $\tau(\tilde{\alpha}(s), \tilde{\beta}(b))$ is clearly monotonically decreasing in s as well and $\varepsilon > 0$ was arbitrary, we arrive at $\tau(\tilde{\alpha}(a), \tilde{\beta}(b)) = 0$, as claimed. □

4 | NEW CHARACTERISATIONS OF CURVATURE BOUNDS

In this chapter, we introduce several characterisations of curvature bounds which are new in the Lorentzian context.

4.1 | Timelike and causal curvature bounds

Before we go on to introduce new formulations of curvature bounds, however, we want to briefly touch on the interplay between causal and timelike curvature bounds. Causal curvature bounds were also introduced in [24, Definition 4.14]. In (ii) of that definition, it was required that

[†] We prefer to use this result instead of our version of the law of cosines, as the triangle $\Delta(\tilde{x}, \tilde{\alpha}(a), \tilde{\beta}(b))$ has a possibly spacelike side.

comparison neighbourhoods be causally geodesic. However, since the defining inequalities on τ are only required between (comparison) points on timelike sides of an admissible causal triangle, the existence of null realisers is not necessary. For this reason, in the definition of causal curvature bounds we give below, we require comparison neighbourhoods to be merely D_K-geodesic (instead of causally D_K-geodesic), just as in the other formulations of curvature bounds. Note that with this modification, all results about causal curvature bounds that have been obtained in the literature so far retain their validity. Most importantly, with this reformulation, we are able to show that causal and timelike curvature bounds are, in fact, equivalent.

Definition 4.1 ((Strict) causal curvature bounds by triangle comparison). Let X be a Lorentzian pre-length space. An open subset U is called a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of *causal triangle comparison* if:

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.
(ii) U is D_K-geodesic.
(iii) Let $\Delta(x, y, z)$ be an admissible causal triangle in U, with p, q two points on the timelike sides of $\Delta(x, y, z)$. Let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle in $\mathbb{L}^2(K)$ for $\Delta(x, y, z)$ and $\bar{p}, \bar{q}$ comparison points for p and q, respectively. Then,

$$\tau(p, q) \leqslant \tau(\bar{p}, \bar{q}) \quad (\text{resp. } \tau(p, q) \geqslant \tau(\bar{p}, \bar{q})). \tag{13}$$

If in (iii), we additionally have

$$p \leqslant q \Rightarrow \bar{p} \leqslant \bar{q} \quad (\text{resp. } p \leqslant q \Leftarrow \bar{p} \leqslant \bar{q}), \tag{14}$$

then U is called a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of *strict causal triangle comparison*.

Theorem 4.2 (Timelike and (strict) causal curvature bounds). *Let X be a Lorentzian pre-length space.*

(i) *Let $U \subseteq X$ be open. Then, U is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of timelike triangle comparison if and only if it is one in the sense of causal triangle comparison.*
(ii) *Let $U \subseteq X$ be open. Then, U is a $(\geqslant K)$-comparison neighbourhood in the sense of causal triangle comparison if and only if it is one in the sense of strict causal triangle comparison. If U is locally causally closed, the analogous statement about U being a $(\leqslant K)$-comparison neighbourhood holds as well.*
(iii) *X has curvature bounded below (resp. above) by K in the sense of timelike triangle comparison if and only if it has the same bound in the sense of causal triangle comparison.*
(iv) *X has curvature bounded below by K in the sense of causal triangle comparison if and only if it has the same bound in the sense of strict causal triangle comparison. If X is strongly causal and locally causally closed, the analogous statement about X having curvature bounded above by K holds as well.*

Proof.

(i) For the non-trivial direction of the claim, suppose that U is a comparison neighbourhood in the sense of timelike triangle comparison, and let $\Delta(x, y, z)$ be a causal triangle in U satisfying size bounds, and with $\tau(y,z)=0$. Let p and q be points on the timelike sides of the triangle. Furthermore, let $\Delta(\bar{x},\bar{y},\bar{z})$ be a comparison triangle for $\Delta(x,y,z)$ and denote by $\bar{p}$ and $\bar{q}$ the comparison points for p and q in that triangle, respectively. We distinguish the following cases:

(1) $p,q\neq y$: Let $p\in[x,y]$, $q\in[x,z]$, and let $\alpha:[0,1]\to U$ be a geodesic realising $[x,y]$, so $y=\alpha(1)$ and, say, $p=\alpha(t_0)$ for $t_0\in(0,1)$. Set $y_t:=\alpha(t)$, then for any $t\in(t_0,1)$, $\Delta(x,y_t,z)$ is a timelike triangle containing p and q. Let $\Delta(\bar{x},\bar{y}_t,\bar{z})$ be a comparison triangle for $\Delta(x,y_t,z)$ and denote by $\bar{p}_t$ and $\bar{q}_t$ the comparison points for p and q therein. Then, by timelike triangle comparison for $\Delta(x,y_t,z)$, we have $\tau(p,q)\leqslant\tau(\bar{p}_t,\bar{q}_t)$ (resp. $\tau(p,q)\geqslant\tau(\bar{p}_t,\bar{q}_t)$). We now argue that $\bar{y}_t\to\bar{y}$. Clearly, $y_t\to y$ and hence, since τ is continuous, $\tau(\bar{x},\bar{y}_t)=\tau(x,y_t)\to\tau(x,y)=\tau(\bar{x},\bar{y})$ and $\tau(\bar{y}_t,\bar{z})=\tau(y_t,z)\to\tau(y,z)=\tau(\bar{y},\bar{z})$. Fixing the segment $[\bar{x},\bar{z}]$ in its place and assuming that it is vertical (after applying a suitable Lorentz transformation), we see that $\bar{y}_t$ arises as the unique (up to reflection on $[\bar{x},\bar{z}]$) point of intersection of hyperbolas with centres $\bar{x}$ and $\bar{z}$, respectively. Since τ is continuous, these hyperbolas transform continuously in t, with the one centred at $\bar{z}$ degenerating into two line segments as $t\to 1$. This shows $\bar{y}_t\to\bar{y}$, which immediately implies $\bar{p}_t\to\bar{p}$ and $\bar{q}_t\to\bar{q}$, and so, $\tau(\bar{p}_t,\bar{q}_t)\to\tau(\bar{p},\bar{q})$. Thus, we get $\tau(p,q)\leqslant\tau(\bar{p},\bar{q})$ (resp. $\tau(p,q)\geqslant\tau(\bar{p},\bar{q})$), as claimed. The case of $p\in[x,z]$, $q\in[x,y]$ is analogous.

(2) $q=y$, $p\in[x,z]$: Again, let $[x,y]$ be realised by the geodesic $\alpha:[0,1]\to U$ and set $q_t:=\alpha(t)$. By case (1), $\tau(p,q_t)\leqslant\tau(\bar{p},\bar{q}_t)$ (resp. $\tau(p,q_t)\geqslant\tau(\bar{p},\bar{q}_t)$), where $\bar{q}_t$ is the comparison point to q in the triangle $\Delta(\bar{x},\bar{y},\bar{z})$. Letting $t\nearrow 1$, we obtain $\tau(p,q)\leqslant\tau(\bar{p},\bar{q})$ (resp. $\tau(p,q)\geqslant\tau(\bar{p},\bar{q})$) also in this case.

(3) $p=y$: Then, $0=\tau(p,q)=\tau(\bar{p},\bar{q})$.

(ii) Let U as in (ii) be a comparison neighbourhood in the sense of causal triangle comparison. Let $\Delta(x,y,z)$ be a causal triangle satisfying size bounds, and let p,q each be either one of x,y,z or lie on a timelike side of $\Delta(x,y,z)$. Let $\Delta(\bar{x},\bar{y},\bar{z})$ be a comparison triangle and $\bar{p},\bar{q}$ comparison points. Note that if p,q are both vertices or lie on the same side, the required inequalities are always satisfied. So, we may suppose that p lies in the interior of the side $[x,z]$, say $p=\alpha(t_0)$, $t_0\in(0,1)$, where the geodesic $\alpha:[0,1]\to U$ realises $[x,z]$. Let $p_t:=\alpha(t)$ for $t<t_0$. Then, $p_t\ll p$ and $p_t\to p$ as $t\nearrow t_0$, and similarly for the comparison points in $\Delta(\bar{x},\bar{y},\bar{z})$, we have $\bar{p}_t\to\bar{p}$ as $t\nearrow t_0$. From this and the reverse triangle inequality for τ, it follows that $p\leqslant q\Rightarrow\tau(p_t,q)>0$ for all $t<t_0$. If U is causally closed, the converse implication holds as well. Moreover, also, in $\mathbb{L}^2(K)$, we have $\tau(\bar{p}_t,\bar{q})>0$ for all $t<t_0$ if and only if $\bar{p}\leqslant\bar{q}$. Since $\tau(p_t,q)\leqslant\tau(\bar{p}_t,\bar{q})$ (resp. $\tau(p_t,q)\geqslant\tau(\bar{p}_t,\bar{q})$), this verifies (14) and thereby shows that U is also a comparison neighbourhood in the sense of strict causal triangle comparison.

(iii) This is immediate from (i).

(iv) Recalling Lemma 3.4, this is a direct consequence of (ii). □

Remark 4.3 (One-sided versions of (strict) causal triangle comparison). In analogy to Definition 3.2, one can also introduce one-sided versions of (strict) causal triangle comparison by requiring one of p,q to be a vertex of the triangle. The implication from (strict) causal triangle comparison to (strict) causal one-sided triangle comparison is obvious. The implications from strict causal one-sided triangle comparison to causal one-sided triangle comparison and further to timelike one-sided triangle comparison are similiarly obvious. Thus, under the assumptions of Theorem 4.2, all of these notions are equivalent.

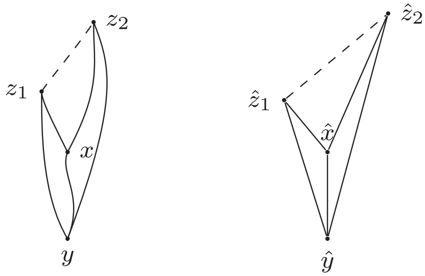

FIGURE 1 A timelike four-point configuration in X and a corresponding comparison configuration.

4.2 | The four-point condition

In Alexandrov geometry, the four-point condition is a convenient reformulation used in both upper and lower curvature problems. Notably, it is used in a version of Toponogov's theorem, cf. [13]. Its biggest advantage is that it does not require the existence of distance realisers, that is, it also works in a non-intrinsic setting. It is somewhat unique in the sense that, as the name suggests, it uses four points in contrast to essentially all previous formulations, which used three points (forming a hinge or a triangle), but at the same time, the formulation is still fundamentally geometric in nature, so to say, in contrast to the convexity/concavity condition on τ, which seems more analytical. The four-point condition is a bit more natural for the curvature bounded below case, which is why we give the definitions separately. As in the metric version, the four-point condition can be expressed both via distance and angle inequalities.

Before giving the definition, it will be convenient to lay out some notational conventions.

Definition 4.4 (Four-point configurations). Let X be a Lorentzian pre-length space.

(i) By a *timelike future four-point configuration*, we mean a tuple of four points in X, usually denoted by (y, x, z_1, z_2), satisfying the relations $y \ll x \ll z_1$ and $x \ll z_2$. It is called *endpoint-causal* if $z_1 \leqslant z_2$.

(ii) By a *causal future four-point configuration*, we mean a tuple of four points in X, usually denoted by (y, x, z_1, z_2), satisfying the relations $y \ll x \leqslant z_1$ and $x \leqslant z_2$. It is called *endpoint-causal* if $z_1 \leqslant z_2$.

(iii) Given a timelike (resp. causal) future four-point configuration (y, x, z_1, z_2) in X, by a *four-point comparison configuration* in $\mathbb{L}^2(K)$, we mean a tuple of four points $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$ such that $\tau(y, x) = \tau(\hat{y}, \hat{x})$, $\tau(y, z_i) = \tau(\hat{y}, \hat{z}_i)$ and $\tau(x, z_i) = \tau(\hat{x}, \hat{z}_i)$, $i = 1, 2$, and such that $\hat{z}_1$ and $\hat{z}_2$ lie on opposite sides of the line through $\hat{y}$ and $\hat{x}$, see Figure 1.

(iv) A timelike (resp. causal) future four-point configuration in X is called *left (resp. right) straight* if $\tau(y, z_1) = \tau(y, x) + \tau(x, z_1)$ (resp. $\tau(y, z_2) = \tau(y, x) + \tau(x, z_2)$), that is, y, x and z_1 (resp. z_2) lie on a distance realiser, if it exists. Note that the four-point comparison configuration (if it exists) of a four-point configuration is straight if and only if the original four-point configuration is straight, see Figure 2.

(v) There are past versions of all of the aforementioned concepts, which result from reversing all causality relations in the obvious way. The resulting tuple will then be denoted by (z_2, z_1, x, y).

When proving statements where some formulation of curvature bounds implies a curvature bound expressed via four-point configurations, we will only explicitly show how to

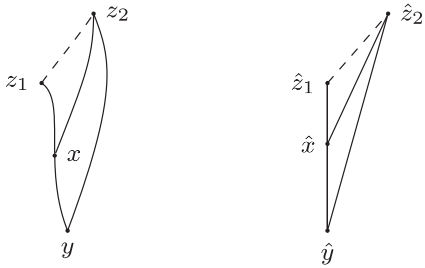

FIGURE 2 A (left) straight timelike four-point configuration in X and a corresponding comparison configuration.

obtain the desired inequality for a future configuration. The case of a past configuration always follows symmetrically. Since the list of decorating adjectives for four-point configurations is already quite long, we decided to omit the word 'future' when dealing with future four-point configurations (which, in any case, are also clearly identified by the order of points in the above notation).

Intuitively, these four-point configurations could be thought of as two admissible causal (or even timelike) triangles $\Delta(y, x, z_i)$ that share the side $[y, x]$, but technically, one has to be careful with this as the points in X might not form a triangle if there are no geodesics joining the points.

Definition 4.5 (Size bounds for four-point configurations). Similar to the corresponding terminology for triangles and hinges, a four-point configuration (y, x, z_1, z_2) in a Lorentzian pre-length space X is said to *satisfy size bounds for K* if there exists a four-point comparison configuration in $\mathbb{L}^2(K)$. Evidently, this is the case precisely if $\tau(y, z_1) < D_K$ and $\tau(y, z_2) < D_K$. Note that the four-point comparison configuration is unique up to isometry of $\mathbb{L}^2(K)$. Throughout this work, we will assume that all mentioned four-point configurations satisfy size bounds.

It turns out that lower and upper curvature bounds in the sense of any four-point condition have to be formulated quite differently, which is why we introduce them separately. We will go into more detail below.

Definition 4.6 (Lower curvature bounds by timelike (resp. causal) four-point condition). Let X be a Lorentzian pre-length space. An open subset U of X is called a $(\geqslant K)$-comparison neighbourhood in the sense of the *timelike (resp. causal) four-point condition* if:

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.
(ii) Let (y, x, z_1, z_2) be a timelike (resp. causal) and endpoint-causal four-point configuration in U. Let $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$ be a four-point comparison configuration in $\mathbb{L}^2(K)$. Then,

$$\tau(z_1, z_2) \geqslant \tau(\hat{z}_1, \hat{z}_2) . \tag{15}$$

In addition, for any timelike (resp. causal) and endpoint-causal past four-point configuration (z_2, z_1, x, y) and a comparison configuration $(\hat{z}_2, \hat{z}_1, \hat{x}, \hat{y})$, we require

$$\tau(z_2, z_1) \geqslant \tau(\hat{z}_2, \hat{z}_1) . \tag{16}$$

In the spirit of the equivalence between timelike and causal curvature bounds established in Theorem 4.2, we also give a more general version of the four-point condition. This is a priori a stricter property as it says more about a greater number of configurations. Their equivalence (under some mild assumptions) will be demonstrated below. Unsurprisingly, it is very convenient to have different equivalent formulations of the same property at hand. In particular, we expect the strict causal triangle comparison and the strict causal four-point condition to be especially useful for the slightly adapted setting of so-called *Lorentzian metric spaces*, cf. [26], where the time separation function τ is replaced by a function ℓ that additionally encodes the causal relation. These conditions can also be more concisely formulated in terms of ℓ.

Definition 4.7 (Lower curvature bounds by strict causal four-point condition). Let X be a Lorentzian pre-length space. An open subset U in X is called a $(\geqslant K)$-comparison neighbourhood in the sense of the *strict causal four-point condition* if it is a $(\geqslant K)$-comparison neighbourhood in the sense of the causal four-point condition, where condition (ii) in Definition 4.6 is strengthened to:

(ii') Let (y, x, z_1, z_2) be a causal four-point configuration in U (not necessarily endpoint-causal). Let $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$ be a four-point comparison configuration in $\mathbb{L}^2(K)$. Then,

$$\tau(z_1, z_2) \geqslant \tau(\hat{z}_1, \hat{z}_2) \text{ and} \tag{17}$$

$$\hat{z}_1 \leqslant \hat{z}_2 \Rightarrow z_1 \leqslant z_2 \,. \tag{18}$$

In addition, for any causal past four-point configuration (z_2, z_1, x, y) and a comparison configuration $(\hat{z}_2, \hat{z}_1, \hat{x}, \hat{y})$, we require

$$\tau(z_2, z_1) \geqslant \tau(\hat{z}_2, \hat{z}_1) \text{ and} \tag{19}$$

$$\hat{z}_2 \leqslant \hat{z}_1 \Rightarrow z_2 \leqslant z_1 \,. \tag{20}$$

Similar to the metric case, the timelike four-point condition cannot only be described via a distance estimate but also via the behaviour of (comparison) angles, cf. [13, Definition 2.3] and [11, Definition II.1.10].

Lemma 4.8 (Angle version of the timelike four-point condition for lower curvature bounds). *Let U be an open subset in a Lorentzian pre-length space X which satisfies Definition 4.6(i). Let (y, x, z_1, z_2) be a timelike and endpoint-causal four-point configuration in U and let $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$ be a comparison configuration in $\mathbb{L}^2(K)$. Then, $\tau(z_1, z_2) \geqslant \tau(\hat{z}_1, \hat{z}_2)$ if and only if*

$$\tilde{\measuredangle}_x(z_1, z_2) \leqslant \tilde{\measuredangle}_x(z_1, y) + \tilde{\measuredangle}_x(y, z_2) \,. \tag{21}$$

In addition, if (z_2, z_1, x, y) is a timelike past and endpoint-causal four-point configuration and $(\hat{z}_2, \hat{z}_1, \hat{x}, \hat{y})$ a comparison configuration, then $\tau(z_2, z_1) \geqslant \tau(\hat{z}_2, \hat{z}_1)$ if and only if (21) is satisfied.

Proof. It will suffice to only consider the future case. Let $\Delta(\bar{x}, \bar{z}_1, \bar{z}_2)$ be a comparison triangle for (the possibly merely causal) triangle $\Delta(x, z_1, z_2)$, then by definition, $\tilde{\measuredangle}_x(z_1, z_2) = \measuredangle_{\bar{x}}(\bar{z}_1, \bar{z}_2)$. For the hyperbolic angles in the comparison configuration $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$, we have

$$\measuredangle_{\hat{x}}(\hat{z}_1, \hat{z}_2) = \measuredangle_{\hat{x}}(\hat{y}, \hat{z}_1) + \measuredangle_{\hat{x}}(\hat{y}, \hat{z}_2) = \tilde{\measuredangle}_x(z_1, y) + \tilde{\measuredangle}_x(y, z_2), \tag{22}$$

where the first equality is due to the triangle equality of angles in (two-dimensional) spacetimes. Note that $\tau(\bar{x}, \bar{z}_1) = \tau(x, z_1) = \tau(\hat{x}, \hat{z}_1)$ and $\tau(\bar{x}, \bar{z}_2) = \tau(x, z_2) = \tau(\hat{x}, \hat{z}_2)$, that is, the sides adjacent to the angles at $\bar{x}$ and $\hat{x}$ have the same lengths. Thus, we can use the Hinge Lemma, cf. [3, Lemma 2.2], to read off the desired equivalence directly. □

Proposition 4.9 (Angle comparison implies timelike four-point condition for lower curvature bounds). *Let U be an open subset in a regular Lorentzian pre-length space X. If U is a $(\geqslant K)$-comparison neighbourhood in the sense of angle comparison, then it is a $(\geqslant K)$-comparison neighbourhood in the sense of the timelike four-point condition.*

Proof. Let U be a $(\geqslant K)$-comparison neighbourhood in the sense of angle comparison and let (y, x, z_1, z_2) be a timelike and endpoint-causal four-point configuration in U. Take distance realisers α from x to z_1, β from x to y, γ from x to z_2, which we know exist by Definition 3.11(ii). We obtain the following inequalities for angles:

$$\tilde{\measuredangle}_x(z_1, z_2) \leqslant \measuredangle_x(\alpha, \gamma) \leqslant \measuredangle_x(\alpha, \beta) + \measuredangle_x(\beta, \gamma) \leqslant \tilde{\measuredangle}_x(z_1, y) + \tilde{\measuredangle}_x(y, z_2), \tag{23}$$

where we used (iii) (with corresponding signs) and (iv) in Definition 3.11. The claim therefore follows from Lemma 4.8. The same inequality can be obtained for a past four-point configuration in complete analogy. □

When introducing a four-point condition for timelike curvature bounded above, we run into the following problem: in the above proof, the inequalities from angle comparison and hinge comparison reverse, but the triangle inequality of angles does not. However, there is a way around this, as equality in the triangle inequality of angles is enough to obtain inequalities in the opposite direction. This is achieved by restricting to straight four-point configurations. We also briefly want to justify (ii) in the following definition. In essence, the existence of τ-midpoints is assumed[†] in order to ensure that the definition does turn into a void statement when its assumptions cannot be met. Indeed, there exist exotic spaces without distance realisers, where there simply exist too few (or none at all) straight four-point configurations, in which case the curvature bound might be trivially satisfied. As an example for such a space, consider a locally finite random selection of points in the Minkowski plane, equipped with the restrictions of the causal relation and time separation function from ambient space. Then, almost surely no three points lie on a line.

Definition 4.10 (Upper curvature bounds by timelike (resp. causal) four-point condition). Let X be a Lorentzian pre-length space. An open subset U is called a $(\leqslant K)$-comparison neighbourhood in the sense of the *timelike (resp. causal) four-point condition* if:

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.

[†] The existence of τ-midpoints is comparatively strong, but it is easy to formulate and we are mostly working in an intrinsic setting anyways (where the existence of such points is not automatic). Technically, requiring a weaker condition, like the existence of distance realisers that are partially defined on a dense subset of some interval suffices.

(ii) For all $x \ll z$ in U with $\tau(x,z) < D_K$, there exists a τ-midpoint in U, that is, a point $y \in U$ such that $\tau(x,y) = \tau(y,z) = \frac{1}{2}\tau(x,z)$.

(iii) Let (y, x, z_1, z_2) be a straight timelike (resp. causal) and endpoint-causal four-point configuration in U. Let $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$ be a straight four-point comparison configuration in $\mathbb{L}^2(K)$. Then,

$$\tau(z_1, z_2) \leqslant \tau(\hat{z}_1, \hat{z}_2)\,. \tag{24}$$

In addition, for any straight timelike (resp. causal) and endpoint-causal past four-point configuration (z_2, z_1, x, y) and a comparison configuration $(\hat{z}_2, \hat{z}_1, \hat{x}, \hat{y})$, we require

$$\tau(z_2, z_1) \leqslant \tau(\hat{z}_2, \hat{z}_1)\,. \tag{25}$$

Definition 4.11 (Upper curvature bounds by strict causal four-point condition). Let X be a Lorentzian pre-length space. An open subset U of X is called a $(\leqslant K)$-comparison neighbourhood in the sense of the *strict causal four-point condition* if it is a $(\leqslant K)$-comparison neighbourhood in the sense of the causal four-point condition, where under the assumptions of condition (iii), we additionally require

$$z_1 \leqslant z_2 \Rightarrow \hat{z}_1 \leqslant \hat{z}_2\,, \tag{26}$$

and for past configurations, we additionally require

$$z_2 \leqslant z_1 \Rightarrow \hat{z}_2 \leqslant \hat{z}_1\,. \tag{27}$$

Remark 4.12 (Endpoint causality in the strict four-point condition). Note that for upper curvature bounds, a more general formulation allowing for non-endpoint-causal four-point configurations is superfluous. Indeed, whenever z_1 and z_2 are not causally related, both the inequality $\tau(z_1, z_2) \leqslant \tau(\hat{z}_1, \hat{z}_2)$ and the implication $z_1 \leqslant z_2 \Rightarrow \hat{z}_1 \leqslant \hat{z}_2$ are trivially satisfied. It therefore essentially only makes sense to consider endpoint-causal four-point configurations.

Remark 4.13 (Relevant constellations of causal four-point configurations). Here, we show that it is not necessary to look at causal four-point configurations (y, x, z_1, z_2) where $x \leqslant z_2$ are null related, where for curvature bounded below in the sense of the strict causal four-point condition, one additionally needs that the space (or the comparison neighbourhood) is regular to conclude this. However, the latter is not an actual restriction since the only statement involving the strict causal four-point condition, Proposition 4.19, assumes this anyways.

To begin with, we cannot have $z_1 \ll z_2$ as otherwise $x \leqslant z_1 \ll z_2$ would yield a timelike relation $x \ll z_2$, and the same works for $\hat{z}_1 \not\ll \hat{z}_2$. In particular, the τ-inequality in any four-point condition is trivially satisfied. Moreover, $\hat{z}_1 \not\leqslant \hat{z}_2$ unless the four-point situation is left-straight and x is null before z_1 (as $\hat{z}_1$ and $\hat{z}_2$ are on opposite sides of the line extending $[\hat{y}, \hat{x}]$). Under the previously mentioned assumption of regularity, $x = z_1$ follows in this case (as y, x, z_1 are collinear with $y \ll x$ and x, z_1 null related), and therefore, lower curvature bounds in the sense of the strict causal four-point condition automatically hold.

For upper curvature bounds in the sense of the strict causal four-point condition, assume that (y, x, z_1, z_2) is straight with $x \leqslant z_2$ null related. Since we need to consider endpoint-causal configurations by Remark 4.12, it must be the case that $x \leqslant z_1$ are null related as well, otherwise $x \ll z_1 \leqslant z_2$ would yield $x \ll z_2$, a contradiction to them being null related.

If (y, x, z_1, z_2) is left straight, we have $\tau(y, z_1) = \tau(y, x) + \tau(x, z_1) = \tau(y, x)$. In particular, the comparison points $\hat{y}, \hat{x}, \hat{z}_1$ lie on a line, and as $\tau(\hat{x}, \hat{z}_1) = 0$, we conclude $\hat{z}_1 = \hat{x} \leqslant \hat{z}_2$. Thus, $\hat{z}_1 = \hat{x} \leqslant \hat{z}_2$. On the other hand, if it is right straight, then $\tau(y, z_2) = \tau(y, x) + \tau(x, z_2) = \tau(y, x)$. Further, $\tau(y, z_2) \geqslant \tau(y, z_1) + \tau(z_1, z_2) = \tau(y, z_1)$ and also $\tau(y, z_1) \geqslant \tau(y, x) + \tau(x, z_1) = \tau(y, x)$, so $\tau(y, x) = \tau(y, z_1) = \tau(y, z_2)$. In particular, this configuration is also left straight, making both $\hat{y}, \hat{x}, \hat{z}_1$ and $\hat{y}, \hat{x}, \hat{z}_2$ lie on a line, which forces $\hat{x} = \hat{z}_1 = \hat{z}_2$. This shows that (26) is satisfied.

Finally, if $x = z_1$ or $x = z_2$, any τ-inequality and implication of causal relation is trivially satisfied.

Altogether (assuming that the space is regular in the case of the strict causal four-point condition for curvature bounded below), we can always assume that all four points are distinct and $x \ll z_2$.

In order to show that angle comparison implies timelike four-point comparison in the case of upper curvature bounds, we require the following auxiliary result. It is, in fact, a variant of [9, Theorem 4.5(i)], where we do not rely on the fact that one of the angles exists, cf. [9, Lemma 4.10].

Lemma 4.14 (Triangle inequality of angles, special case). *Let X be a Lorentzian pre-length space with curvature bounded above by K in the sense of timelike triangle comparison. Let α and β be future-directed distance realisers, and let γ be a past-directed distance realiser, all emanating from the same point p, such that the concatenation of γ and β again is a distance realiser. Then,*

$$\measuredangle_p(\alpha, \gamma) \leqslant \measuredangle_p(\alpha, \beta)\,. \tag{28}$$

Since $\measuredangle_p(\beta, \gamma) = 0$, cf. [9, Lemma 3.4], this amounts to the following triangle inequality of angles:

$$\measuredangle_p(\alpha, \gamma) \leqslant \measuredangle_p(\alpha, \beta) + \measuredangle_p(\beta, \gamma)\,. \tag{29}$$

Proof. Choose any parameters r, s, t such that, say, $x = \gamma(r), y = \beta(s)$ and $z = \alpha(t)$ form a timelike triangle $\Delta(x, y, z)$ (the direction of the timelike relation between the points on α and β is not important). Consider the two subtriangles $\Delta(x, p, z)$ and $\Delta(p, y, z)$ and consider a comparison configuration consisting of $\Delta(\bar{x}, \bar{p}, \bar{z})$ and $\Delta(\bar{p}, \bar{y}, \bar{z})$ (such that they share the common side between $\bar{p}$ and $\bar{z}$). Due to upper curvature bounds and the Alexandrov Lemma, cf. [8, Proposition 2.42], this is a concave configuration, that is,

$$\tilde{\measuredangle}_p(\gamma(r), \alpha(t)) = \measuredangle_{\bar{p}}(\bar{x}, \bar{z}) \leqslant \measuredangle_{\bar{p}}(\bar{y}, \bar{z}) = \tilde{\measuredangle}_p(\beta(s), \alpha(t))\,. \tag{30}$$

The desired inequality then follows from the definition of angles as limits of comparison angles. □

Proposition 4.15 (Angle comparison implies timelike four-point condition for upper curvature bounds). *Let U be an open subset of a regular Lorentzian pre-length space X. If U is a $(\leqslant K)$-comparison neighbourhood in the sense of angle comparison, then it is also a $(\leqslant K)$-comparison neighbourhood in the sense of the timelike four-point condition.*

Proof. Properties (i) and (ii) in Definition 4.10 follow directly from (i) and (ii) in Definition 3.11.

So, let (y, x, z_1, z_2) in U be a straight timelike and endpoint-causal four-point configuration. Take distance realisers (which exist by our assumptions in Definition 3.11) α from x to z_1, β from x to y and γ from x to z_2. Note that for, say, a left straight configuration, α and β fit together to a distance realiser from y through x to z_1 (the right straight case works analogously, with β, γ fitting together). In particular, $\measuredangle_x(\alpha, \beta) = 0$ (by [9, Lemma 3.4]). Let $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$ be a (straight) comparison configuration for (y, x, z_1, z_2). In particular, $\Delta(\hat{y}, \hat{x}, \hat{z}_2)$ is a comparison triangle for $\Delta(y, x, z_2)$. Similar to the lower curvature bounds case, we obtain the following inequality for angles:

$$\begin{aligned} \measuredangle_x(\alpha, \gamma) = \underbrace{\measuredangle_x(\beta, \alpha)}_{=0} + \measuredangle_x(\alpha, \gamma) \geqslant \measuredangle_x(\beta, \gamma) \\ \geqslant \tilde{\measuredangle}_x(y, z_2) = \measuredangle_{\hat{x}}(\hat{y}, \hat{z}_2) = \measuredangle_{\hat{x}}(\hat{z}_1, \hat{z}_2), \end{aligned} \tag{31}$$

where we used Lemma 4.14, Definition 3.11(iii) (with the sign of the angles already taken into account), and the triangle equality for angles in $\mathbb{L}^2(K)$.

Let $(\tilde{x}, \tilde{z}_1, \tilde{z}_2)$ form a comparison hinge for (α, γ) in $\mathbb{L}^2(K)$. Then, hinge comparison, cf. Definition 3.14 and Proposition 3.15 yield

$$\tau(z_1, z_2) \leqslant \tau(\tilde{z}_1, \tilde{z}_2). \tag{32}$$

The comparison hinge $(\tilde{x}, \tilde{z}_1, \tilde{z}_2)$ and the triangle $\Delta(\hat{x}, \hat{z}_1, \hat{z}_2)$ have two sides of equal length, and $\measuredangle_{\hat{x}}(\hat{z}_1, \hat{z}_2) \leqslant \measuredangle_x(\alpha, \gamma) = \measuredangle_{\tilde{x}}(\tilde{z}_1, \tilde{z}_2)$ by (31). Thus, law of cosines monotonicity (cf. [9, Remark 2.5]) implies $\tau(\tilde{z}_1, \tilde{z}_2) \leqslant \tau(\hat{z}_1, \hat{z}_2)$, which together with (32) gives the desired inequality $\tau(z_1, z_2) \leqslant \tau(\hat{z}_1, \hat{z}_2)$. The case of a past four-point configuration follows analogously. □

Proposition 4.16 (Angle version of the timelike four-point condition for upper curvature bounds). *Let U be an open subset in a Lorentzian pre-length space X which satisfies Definition 4.6(i). Let (y, x, z_1, z_2) be a straight timelike and endpoint-causal four-point configuration in U and let $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$ be a comparison configuration in $\mathbb{L}^2(K)$. Then, $\tau(z_1, z_2) \leqslant \tau(\hat{z}_1, \hat{z}_2)$ if and only if*

$$\tilde{\measuredangle}_x(z_1, z_2) \geqslant \tilde{\measuredangle}_x(z_1, y) + \tilde{\measuredangle}_x(y, z_2). \tag{33}$$

In addition, if (z_2, z_1, x, y) is a timelike and endpoint-causal past four-point configuration and $(\hat{z}_2, \hat{z}_1, \hat{x}, \hat{y})$ a comparison configuration, then $\tau(z_2, z_1) \leqslant \tau(\hat{z}_2, \hat{z}_1)$ if and only if (33) is satisfied.

Proof. The proof is completely analogous to the lower curvature bounds version, see Lemma 4.8. □

Note that in (33), one of the angles on the right-hand side is zero, depending on whether one deals with a left straight or a right straight configuration.

Proposition 4.17 (Timelike vs. causal four-point condition). *Let X be a Lorentzian pre-length space, and let $U \subseteq X$ be open, regular and D_K-geodesic. Then, U is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of the timelike four-point condition if and only if U is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of the causal four-point condition.*

In particular, if X is strongly causal, locally D_K-geodesic and regular, then it has curvature bounded below (resp. above) by K in the sense of the timelike four-point condition if and only if it has the same bound in the sense of the causal four-point condition.

Proof. The direction from causal to timelike is clear, as any (straight) timelike four-point configuration is also a (straight) causal four-point configuration.

For the converse direction, let (y, x, z_1, z_2) be a causal and endpoint-causal four-point configuration. Let $\alpha : [0,1] \to X$ be the timelike distance realiser from y to x. Set $x^t := \alpha(t)$, then for all $t < 1$, the four-point configuration (y, x^t, z_1, z_2) is timelike and endpoint-causal, and straight if (y, x, z_1, z_2) was straight. Note that by continuity of τ, we can choose the four-point comparison configuration $(\hat{y}^t, \hat{x}^t, \hat{z}_1^t, \hat{z}_2^t)$ of (y, x^t, z_1, z_2) such that each of the points converges to the corresponding point in the four-point comparison situation $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$ of (y, x, z_1, z_2). In particular, we have $\tau(\hat{z}_1^t, \hat{z}_2^t) \to \tau(\hat{z}_1, \hat{z}_2)$, and $\tau(z_1, z_2)$ remains independent of t. For lower curvature bounds, we know $\tau(z_1, z_2) \geqslant \tau(\hat{z}_1^t, \hat{z}_2^t)$, thus we also have $\tau(z_1, z_2) \geqslant \tau(\hat{z}_1, \hat{z}_2)$. In the case of upper curvature bounds, we get corresponding inequalities in the other direction. The case of a past four-point configuration follows analogously.

Finally, note that the additional assumptions in the second part of the claim are required since comparison neighbourhoods in the sense of any four-point condition need not be D_K-geodesic. Concerning the non-trivial direction, let $x \in X$ and suppose that U is a comparison neighbourhood of x in the sense of the timelike four-point condition. Then, we find a neighbourhood V of x which is D_K-geodesic. Any intersection of timelike diamonds inside $U \cap V$ is, due to causal convexity, easily seen to be a regular and D_K-geodesic comparison neighbourhood, hence the first statement of the proposition applies. □

Next, we show that curvature bounds in the sense of the causal four-point condition imply curvature bounds in the sense of monotonicity comparison.

Proposition 4.18 (Causal four-point condition implies monotonicity comparison). *Let X be a Lorentzian pre-length space and let $U \subseteq X$ be open, regular and D_K-geodesic. If U is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of the causal four-point condition, then U is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of monotonicity comparison.*

In particular, if X is strongly causal, regular and locally D_K-geodesic, and X has curvature bounded below (resp. above) by K in the sense of the causal four-point condition, then it also has the same bound in the sense of monotonicity comparison.

Proof. We only demonstrate the case of lower curvature bounds, the upper curvature bounds case is entirely analogous. Let U be as in the statement. The first two conditions in Definition 3.8 are satisfied by assumption. For the third condition, let $\alpha : [0, a] \to X, \beta : [0, b] \to X$ be a hinge with $\alpha(0) = \beta(0)$.

There are two cases to consider, one where the two curves have the same time-orientation (say both future-directed), and one where they have different time orientation (say α is future-directed and β is past-directed).

First, we consider the case of α and β being future-directed. We need to show that the partial function $\theta(s, t) = \tilde{\measuredangle}_y^{K,S}(\alpha(s), \beta(t))$ is monotonically increasing. By Remark 3.10, it suffices to establish one-sided monotonicity. The future-directed case technically breaks down into three subcases, depending on the relations between the points on the curves (see Figure 3 for a rough sketch of the in total four subcases). Let $s_+ > s_- > 0$ and $t > 0$ be such that $\alpha(s_+) \leqslant \beta(t)$ (the case of $\alpha(s_-) \leqslant \beta(t) \leqslant \alpha(s_+)$ follows analogously). These correspond to the cases (i) and (ii) in Figure 3, respectively. Note that in case (ii), we might deal with a causal four-point configuration if $\alpha(s_-) \leqslant \beta(t)$ are null related. Set $y := \alpha(0) = \beta(0), x := \alpha(s_-), z_1 := \alpha(s_+)$ and $z_2 := \beta(t)$. We need to show that $\theta(s_-, t) \leqslant \theta(s_+, t)$, that is, $\tilde{\measuredangle}_y(x, z_2) \geqslant \tilde{\measuredangle}_y(z_1, z_2)$ (recall that θ is defined

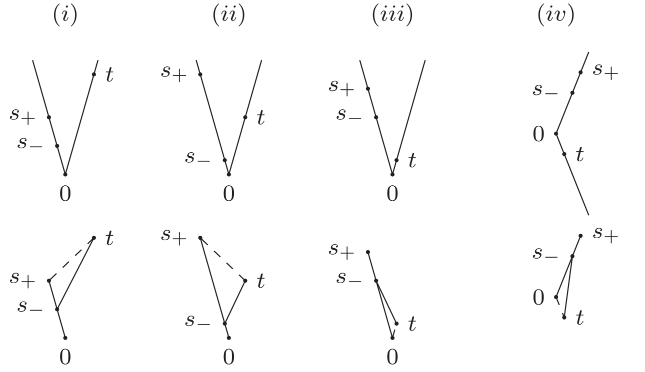

FIGURE 3 The four possible subcases of endpoint-causal straight four-point configurations which arise from a hinge.

using signed angles). By construction, (y, x, z_1, z_2) forms a left straight timelike and endpoint-causal four-point configuration. Let $(\hat{y}, \hat{x}, \hat{z}_1, \hat{z}_2)$ be a comparison configuration, then $\Delta(\hat{y}, \hat{x}, \hat{z}_2)$ is a comparison triangle for $\Delta(y, x, z_2)$. Let $\Delta(\bar{y}, \bar{z}_1, \bar{z}_2)$ be a comparison triangle for (the possibly merely admissible causal triangle) $\Delta(y, z_1, z_2)$ and let $\bar{x}$ be the comparison point for x in $\Delta(\bar{y}, \bar{z}_1, \bar{z}_2)$. We have $\tilde{\measuredangle}_y(x, z_2) = \measuredangle_{\hat{y}}(\hat{x}, \hat{z}_2) = \measuredangle_{\hat{y}}(\hat{z}_1, \hat{z}_2)$ and $\tilde{\measuredangle}_y(z_1, z_2) = \measuredangle_{\bar{y}}(\bar{z}_1, \bar{z}_2)$, so the desired inequality reads $\measuredangle_{\hat{y}}(\hat{z}_1, \hat{z}_2) \geqslant \measuredangle_{\bar{y}}(\bar{z}_1, \bar{z}_2)$. The two triangles $\Delta(\bar{y}, \bar{z}_1, \bar{z}_2)$ and $\Delta(\hat{y}, \hat{z}_1, \hat{z}_2)$ have two sides of equal length, and by four-point comparison, we know $\tau(\bar{z}_1, \bar{z}_2) = \tau(z_1, z_2) \geqslant \tau(\hat{z}_1, \hat{z}_2)$. Thus, $\tilde{\measuredangle}_y(x, z_2) = \measuredangle_{\hat{y}}(\hat{z}_1, \hat{z}_2) \geqslant \measuredangle_{\bar{y}}(\bar{z}_1, \bar{z}_2) = \tilde{\measuredangle}_y(z_1, z_2)$ follows by law of cosines monotonocity, cf. [9, Remark 2.5]).

For the remaining subcase of the future-directed case, let s_-, s_+ and t be such that $\beta(t) \leqslant \alpha(s_-) \ll \alpha(s_+)$, see (iii) in Figure 3 (note that also here one might deal with a causal four-point configuration if $\beta(t) \leqslant \alpha(s_-)$ are null related. Set $z_2 := \alpha(0) = \beta(0), z_1 := \beta(t), x := \alpha(s_-)$ and $y := \alpha(s_+)$ and consider the resulting left straight timelike and endpoint-causal past four-point configuration (z_2, z_1, x, y).

Construct a comparison configuration $(\hat{z}_2, \hat{z}_1, \hat{x}, \hat{y})$ as well as a comparison triangle $\Delta(\bar{z}_2, \bar{z}_1, \bar{y})$ for the triangle $\Delta(z_2, z_1, y)$. The triangles $\Delta(\bar{z}_2, \bar{z}_1, \bar{y})$ and $\Delta(\hat{z}_2, \hat{z}_1, \hat{y})$ have two sides of equal length, and by four-point comparison, we know $\tau(z_2, z_1) = \tau(\bar{z}_2, \bar{z}_1) \geqslant \tau(\hat{z}_2, \hat{z}_1)$. Thus, by law of cosines, we obtain $\measuredangle_{\bar{y}}(\bar{z}_2, \bar{z}_1) \leqslant \measuredangle_{\hat{y}}(\hat{z}_2, \hat{z}_1)$. Let $\bar{x}$ be a comparison point for x in $\Delta(\bar{z}_2, \bar{z}_1, \bar{y})$ and consider the subtriangles $\Delta(\bar{z}_1, \bar{x}, \bar{y})$ and $\Delta(\hat{z}_1, \hat{x}, \hat{y})$ of $\Delta(\bar{z}_2, \bar{z}_1, y)$ and $\Delta(\hat{z}_2, \hat{z}_1, \hat{y})$, respectively. They have two sides of equal length, and the angles at $\bar{y}$ (resp. $\hat{y}$) agree with the ones in the original triangles, that is, $\measuredangle_{\bar{y}}(\bar{x}, \bar{z}_1) = \measuredangle_{\bar{y}}(\bar{z}_2, \bar{z}_1) \leqslant \measuredangle_{\hat{y}}(\hat{z}_2, \hat{z}_1) = \measuredangle_{\hat{y}}(\hat{x}, \hat{z}_1)$. Thus, we get $\tau(\bar{x}, \bar{z}_1) \geqslant \tau(\hat{x}, \hat{z}_1) = \tau(x, z_1)$. Finally, we can relate a comparison triangle $\Delta(\bar{z}_2', \bar{z}_1', \bar{x}')$ for $\Delta(z_2, z_1, x)$ to the subtriangle $\Delta(\bar{z}_2, \bar{z}_1, \bar{x})$ of $\Delta(\bar{z}_2, \bar{z}_1, \bar{y})$. They have two sides of equal length, and from the above arguments, we know $\tau(\bar{x}, \bar{z}_1) \leqslant \tau(x, z_1) = \tau(\bar{x}', \bar{z}_1')$. Hence, the desired inequality $\measuredangle_{\bar{z}_2}(\bar{x}, \bar{z}_1) \geqslant \measuredangle_{\bar{z}_2'}(\bar{y}', \bar{z}_1')$ follows.

At last, consider the case of α being future-directed and β being past-directed, see (iv) in Figure 3. In this case, for any choice of parameters, $\alpha(s_-), \alpha(s_+)$ and $\beta(t)$ (together with the origin) yield a straight timelike and endpoint-causal past four-point configuration. Labelling the points as in the case (iii) depicted in Figure 3, we observe that we are actually in the same situation, with the only difference being that the causal relation between z_1 and z_2 is reversed. Regardless, the arguments are completely analogous.

The second part of the statement follows just as in Proposition 4.17. □

Proposition 4.19 (Causal vs. strict causal four-point condition). *Let X be a Lorentzian pre-length space and let $U \subseteq X$ be open, D_K-geodesic and regular. In the case of lower curvature bounds, assume in addition that U is causally closed. Then, U is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of the causal four-point condition if and only if it is a $(\geqslant K)$- (resp. $(\leqslant K)$-)comparison neighbourhood in the sense of the strict causal four-point condition.*

In particular, if X is strongly causal, regular and locally D_K-geodesic, then X has curvature bounded below (resp. above) by K in the sense of the causal four-point condition, if and only if it has the same bound in the sense of the strict causal four-point condition.

Proof. In both cases of implications, one implication is obvious from the definitions, so we only need to show that the causal four-point condition implies the strict causal four-point condition. For $(\leqslant K)$-comparison neighbourhoods, let first (y, x, z_1, z_2) be a left straight causal and endpoint-causal four-point configuration. Then, $x \ll z_1$ since $y \ll x$ and U is regular. Let $\alpha : [0,1] \to X$ be the timelike distance realiser connecting x to z_1, and set $z_1^t := \alpha(t)$, then $z_1^t \ll z_2$ for all $t < 1$. Consider the straight timelike four-point configuration (y, x, z_1^t, z_2) and a comparison configuration $(\hat{y}, \hat{x}, \hat{z}_1^t, \hat{z}_2)$. Then, we have $0 < \tau(z_1^t, z_2) \leqslant \tau(\hat{z}_1^t, \hat{z}_2)$. Note that $(\hat{y}, \hat{x}, \hat{z}_1^t, \hat{z}_2)$ can be chosen so that it converges to a comparison configuration for (y, x, z_1, z_2) as $t \nearrow 1$, that is, $\hat{z}_1^t \to \hat{z}_1$. Thus, in the limit, we get $\hat{z}_1 \leqslant \hat{z}_2$, as required.

If (y, x, z_1, z_2) is instead a right straight causal and endpoint-causal four-point configuration, let β be the timelike distance realiser connecting y to z_1 and γ be the timelike distance realiser connecting y to x. Then, for all $t < 1$, there is an $s < 1$ such that $\gamma(t) \ll \beta(s)$, and we can make this choice s_t continuously and such that $\lim_{t \nearrow 1} s_t = 1$. Set $x^t = \gamma(t)$ and $z_1^t = \beta(s_t)$. Consider the right straight timelike and endpoint-causal four-point configuration (y, x^t, z_1^t, z_2), then we have $\tau(z_1^t, z_2) \geqslant \tau(z_1^t, z_1) > 0$. For a comparison configuration $(\hat{y}, \hat{x}^t, \hat{z}_1^t, \hat{z}_2)$, we have $0 < \tau(z_1^t, z_2) \leqslant \tau(\hat{z}_1^t, \hat{z}_2)$ by the causal four-point condition. In particular, $\hat{z}_1^t \leqslant \hat{z}_2$ for all $t \in [0, 1)$. Note that $(\hat{y}, \hat{x}^t, \hat{z}_1^t, \hat{z}_2)$ can be chosen so that it converges to a comparison configuration for (y, x, z_1, z_2) as $t \nearrow 1$, so in the limit, we get $\hat{z}_1 \leqslant \hat{z}_2$, as required.

Now we have to look at whether $(\geqslant K)$-comparison neighbourhoods in the sense of the causal four-point condition are also such in the sense of the strict causal four-point condition. Let (y, x, z_1, z_2) be a causal four-point configuration which is not necessarily endpoint-causal). For now, consider the case where $x \ll z_1$ is timelike. First, we look at the inequality between the τ's. Let α be the timelike distance realiser from x to z_1, set $z_1^t := \alpha(t)$ and consider the causal four-point configuration (y, x, z_1^t, z_2). This yields a four-point comparison configuration $(\hat{y}, \hat{x}, \hat{z}_1^t, \hat{z_2})$ (note that this can be chosen in such a way that only $\hat{z}_1^t$ depends on t).

By Proposition 4.18, we know that U is also a comparison neighbourhood in the sense of monotonicity comparison. Thus, we have that $\tilde{\measuredangle}_x(y, \alpha(t)) = \measuredangle_{\hat{x}}(\hat{y}, \hat{z}_1^t)$ is increasing in t. By the triangle equality of angles in $\mathbb{L}^2(K)$, we know $\measuredangle_{\hat{x}}(\hat{z}_1^t, \hat{z}_2) = \measuredangle_{\hat{x}}(\hat{y}, \hat{z}_1^t) + \measuredangle_{\hat{x}}(\hat{y}, \hat{z}_2)$, hence also $\measuredangle_{\hat{x}}(\hat{z}_1^t, \hat{z}_2)$ is increasing in t. Now we claim that $\tau(\hat{z}_1^t, \hat{z}_2)$ is strictly monotonically decreasing in t whenever $\hat{z}_1^t \leqslant \hat{z}_2$. Let $s < t$. Let $\hat{z}_1^{st}$ be the point on the side $[\hat{x}, \hat{z}_1^t]$ such that $\tau(\hat{x}, \hat{z}_1^{st}) = \tau(x, \alpha(s))$. Note that $\hat{z}_1^{st}$ is a comparison point on the side of a triangle, while $\hat{z}_1^s$ is a vertex of a comparison triangle. Then we observe:

- $\tau(\hat{z}_1^{st}, \hat{z}_2) > \tau(\hat{z}_1^t, \hat{z}_2)$ by reverse triangle inequality ($\hat{z}_1^{st} \ll \hat{z}_1^t$),
- $\tau(\hat{z}_1^s, \hat{z}_2) > \tau(\hat{z}_1^{st}, \hat{z}_2)$ by law of cosines monotonicity: These can each be completed to a triangle with $\hat{x}$. The other side lengths corresponding to each other agree and we know an inequality between the angles at $\hat{x}$.

In particular, we can look at the functions $f(t) = \tau(z_1^t, z_2)$ and $g(t) = \tau(\hat{z}_1^t, \hat{z}_2)$. We have just proven that g is strictly monotonically decreasing whenever $\hat{z}_1^t \leqslant \hat{z}_2$. The reverse triangle inequality proves that f is as well whenever $z_1^t \leqslant z_2$. By Remark 4.12, both causal relations are certainly satisfied for small t. Ultimately, we have to show that $\tau(z_1, z_2) \geqslant \tau(\hat{z}_1, \hat{z}_2)$, that is, $f(1) \geqslant g(1)$. We even show that $f(t) \geqslant g(t)$ for all $t \in (0, 1]$. If t is such that $f(t) > 0$, we know $z_1^t \ll z_2$, so we can apply the causal four-point condition to the timelike and endpoint-causal four-point configuration (y, x, z_1^t, z_2) and a corresponding comparison configuration to get that $f(t) \geqslant g(t)$. If $f(t) = 0$, we have to show $g(t) = 0$ as well. Let us now indirectly assume that there is t_1 such that $0 = f(t_1) < g(t_1) = \tau(\hat{z}_1^t, \hat{z}_2)$. By the reverse triangle inequality, we then also have $\tau(\hat{x}, \hat{z}_2) \geqslant \tau(\hat{x}, \hat{z}_1^{t_1}) + \tau(\hat{z}_1^{t_1}, \hat{z}_2) > 0$. Again, by Remark 4.13, we further have that $f(0) = \tau(x, z_2) = \tau(\hat{x}, \hat{z}_2) = g(0) > 0$, and thus, for small enough t also $f(t) > 0$ by continuity of τ. In particular, there exists $t_0 \in (0, t_1)$ such that $f(t_0) > 0$. By the above argument, we gather $f(t_0) \geqslant g(t_0)$. As f is continuous, there is a $t^* \leqslant t_1$ such that $f(t^*) = \min\left(\frac{g(t_1)}{2}, f(t_0)\right)$. Then, we have $0 < f(t^*) < g(t_1) \leqslant g(t^*)$ in contradiction to the causal four-point condition.

As to the implication of the causal relations, recall that we are still in the case of $x \ll z_1$ and suppose towards a contradiction that $z_1 \not\leqslant z_2$ but $\hat{z}_1 \leqslant \hat{z}_2$. Since U is a causally closed neighbourhood, we infer that $\not\leqslant$ is open, that is, $f(t) = 0$ for t close enough to 1. However, since $\hat{z}_1 \leqslant \hat{z}_2$, it follows that $\hat{z}_1^t \ll \hat{z}_2$ for all $t \in [0, 1)$, which, in turn, gives $f(t) < g(t)$, a contradiction to the paragraph above.

Finally, consider the case of $x \not\ll z_1$ being null related. We follow the same proof as above, but have to replace all the arguments leading to g being strictly monotonically decreasing. In this case, α is a null curve from x to z_1. In particular, we have $z_1^s \leqslant z_1^t$ for $s < t$ and hence $\tau(y, z_1^t) \geqslant \tau(y, z_1^s) + \tau(z_1^s, z_1^t)$ by the reverse triangle inequality. Moreover, the second term on the right-hand side is zero (since α is null), which is why the reverse triangle inequality must be strict since otherwise regularity of U would be violated. Thus, we conclude $\tau(y, z_1^t) > \tau(y, z_1^s)$. Arrange the four-point comparison configurations $(\hat{y}, \hat{x}, \hat{z}_1^s, \hat{z}_2)$ and $(\hat{y}, \hat{x}, \hat{z}_1^t, \hat{z}_2)$ such that they share the triangle $\Delta(\hat{y}, \hat{x}, \hat{z}_2)$. Note that $\hat{x} \leqslant \hat{z}_1^s$ and $\hat{x} \leqslant \hat{z}_1^t$ are null related and point towards the left (by convention), hence $\hat{x}, \hat{z}_1^s$ and $\hat{z}_1^t$ all lie on a null geodesic. In particular, $\hat{z}_1^s$ and $\hat{z}_1^t$ are causally related, and since $\tau(\hat{y}, \hat{z}_1^s) < \tau(\hat{y}, \hat{z}_1^t)$, we have $\hat{z}_1^s \leqslant \hat{z}_1^t$. By the reverse triangle inequality and the fact that $\hat{z}_1^s, \hat{z}_1^t$ and $\hat{z}_2$ do not all lie on a single distance realiser, we have $\tau(\hat{z}_1^s, \hat{z}_2) > \tau(\hat{z}_1^t, \hat{z}_2)$ for all $s < t$ whenever $\hat{z}_1^t \leqslant \hat{z}_2$, that is, g is monotonically decreasing, and indeed strictly so on $\{t \mid \hat{z}_1^t \leqslant \hat{z}_2\} \subseteq [0, 1]$. The rest of the proof works as in the case of $x \ll z_1$.

The case of a past four-point configuration follows analogously.

The final claim of the proposition follows just as in Proposition 4.17. □

4.3 | Convexity and concavity of τ

Similar to the metric setting (cf., e.g. [5]), a characterisation of curvature bounds via convexity or concavity properties of modified distance functions relies crucially on the analytic properties of solutions to the differential equation

$$f'' - Kf = \lambda\,, \tag{34}$$

its homogeneous variant

$$f'' - Kf = 0\,, \tag{35}$$

as well as the corresponding differential inequalities[†] $f'' - Kf \geqslant \lambda$ (resp. $\leqslant \lambda$). We therefore begin this section by deriving some essentials of the solution theory for (34). In the geometric applications we are interested in, the function f will typically only be continuous. For such functions, the standard solution concept is the distributional one (although for our purposes, the most useful concept is 'in the sense of Jensen', cf. Definition 4.21 below). Due to the hypoellipticity of any constant coefficient ordinary differential operator, this makes no difference in the case of equality in (34), whose general solution is given explicitly by

$$f(t) = \begin{cases} \alpha \cos(\sqrt{K}t) + \beta \sin(\sqrt{K}t) + \frac{\lambda}{K} & (K < 0) \\ \lambda \frac{t^2}{2} + \alpha t + \beta & (K = 0) \\ \alpha \cosh(\sqrt{|K|}t) + \beta \sinh\left(\sqrt{|K|}t\right) + \frac{\lambda}{K} & (K > 0), \end{cases} \tag{36}$$

where $\alpha, \beta \in \mathbb{R}$. In the inequality case, f being a solution in the sense of distributions to $f'' - Kf \geqslant \lambda$ (resp. $\leqslant \lambda$), or a distributional *subsolution* (resp. *supersolution*), means that, for any smooth non-negative test function φ with compact support in I, we have

$$\int_I f(t)\varphi''(t) - Kf(t)\varphi(t) - \lambda\varphi(t)\,dt \geqslant 0 \quad (\text{resp. } \leqslant 0). \tag{37}$$

Such functions automatically are of higher regularity.

Proposition 4.20 (Almost-convexity for distributional subsolutions). *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a continuous function which is a distributional subsolution (resp. supersolution) to (34) and let $t_0 \in I$. Then there is a $c > 0$ (resp. $c < 0$) such that $f(t) + ct^2$ is a convex (resp. concave) function near t_0. In particular, f is locally Lipschitz and possesses one-sided derivatives at every point of I. For these, we have $f'(t^-) \leqslant f'(t^+)$ (resp. $f'(t^-) \geqslant f'(t^+)$) at each $t \in I$.*

Proof. Let f be a distributional subsolution. By assumption, for any c, we have

$$(f(t) + ct^2)'' \geqslant Kf(t) + \lambda + 2c,$$

and the right-hand side can be made non-negative near t_0 for $c > 0$ sufficiently big since f is continuous, hence locally bounded. It follows that $f(t) + ct^2$ is a convex distribution, hence a convex function near t_0 (cf. [20, Theorem 4.1.6]). The remaining claims follow from well-known properties of convex functions (cf. [21, Corollary 1.1.6]). The supersolution case follows from the subsolution one by considering $-f$. □

For the modified distance function, we are going to study below, the following alternative solution concept will be relevant.

Definition 4.21. A continuous function $f : I \to \mathbb{R}$ (I an interval) is called a solution to $f'' - Kf \geqslant \lambda$ in the sense of Jensen[‡] (ITSJ) if the following holds: If $t_1, t_2 \in I$, $t_1 < t_2$, $|t_1 - t_2| < D_K$ and g is the unique solution to (34) with $g(t_i) = f(t_i)$ ($i = 1, 2$), then $f(t) \leqslant g(t)$ for all $t \in [t_1, t_2]$.

[†] This is a convexity (resp. concavity) condition on f, cf. [2].

[‡] In [5], this property is called Jensen's inequality, which motivates our terminology here.

Also here we speak of subsolutions† ITSJ, and the supersolution case is defined analogously with the inequalities reversed. As in the distributional case, ITSJ solutions enjoy additional regularity properties:

Proposition 4.22 (Almost-convexity for Jensen subsolutions). *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a continuous function which is a subsolution (resp. supersolution) to (34) in the sense of Jensen and let $t_0 \in I$. Then there is a $c > 0$ (resp. $c < 0$) such that $f(t) + ct^2$ is a convex (resp. concave) function near t_0. In particular, f is locally Lipschitz and possesses one-sided derivatives at every point of I. For these, we have $f'(t^-) \leqslant f'(t^+)$ at each $t \in I$.*

Proof. To show convexity of $f(t) + ct^2$, we need to establish that, for any $\lambda \in (0,1)$ and $t_1 < t_2$ near t_0, we have

$$\begin{aligned} &\lambda(f(t_1) + ct_1^2) + (1-\lambda)(f(t_2) + ct_2^2) \\ &\qquad \geqslant f(\lambda t_1 + (1-\lambda)t_2) + c(\lambda t_1 + (1-\lambda)t_2)^2. \end{aligned} \tag{38}$$

For $|t_1 - t_2|$ small, we can pick $g = g_{t_1,t_2}$ as in Definition 4.21 with $g(t_i) = f(t_i)$ $(i = 1,2)$. Now picking $c = c(t_1,t_2) > 0$ such that $g(t) + ct^2$ becomes convex near t_0 and inserting in (38), we obtain

$$\begin{aligned} &\lambda(f(t_1) + ct_1^2) + (1-\lambda)(f(t_2) + ct_2^2) \\ &\qquad = \lambda(g(t_1) + ct_1^2) + (1-\lambda)(g(t_2) + ct_2^2) \\ &\qquad \geqslant g(\lambda t_1 + (1-\lambda)t_2) + c(\lambda t_1 + (1-\lambda)t_2)^2 \\ &\qquad \geqslant f(\lambda t_1 + (1-\lambda)t_2) + c(\lambda t_1 + (1-\lambda)t_2)^2. \end{aligned}$$

Hence, the claim will follow once we are able to show that $c(t_1,t_2)$ can be chosen to remain uniformly bounded for $t_1 < t_2$ sufficiently near to t_0. Equivalently, we require a uniform lower bound on g'' in a fixed small neighbourhood of t_0. Now since g is a solution to (34), g'' is a solution to the corresponding homogeneous equation (35). From the explicit formulae (36), the claim for $K = 0$ follows immediately. So, suppose first that $K > 0$. If $g''|_{[t_1,t_2]}$ has its minimum at t_1 or t_2 (e.g. if it is monotonous), then employing (35) for g together with the fact that $g(t_i) = f(t_i)$ $(i = 1,2)$ allows us to conclude local uniform boundedness of g'' from that of f. Otherwise, $|g''|$ must attain a local maximum in $[t_1,t_2]$, say at $\hat{t}$. Then $|g''| \geqslant \frac{1}{2}|g''(\hat{t})|$ in some ball $B_r(\hat{t})$ around $\hat{t}$. Now (35) for g'' together with (36) shows that $g''(t) = A\cos(\sqrt{K}(t-\hat{t}))$ for suitable amplitude A, both depending on t_1, t_2. Since, however, the frequency $\sqrt{K}$ depends only on K, the same is true for the radius $r = r(K)$. Hence, if we restrict to a ball of radius $r(K)$ around t_0, then t_i will both lie in $B_r(\hat{t})$, so that (using (34) for g) we have

$$|g''(\hat{t})| < 2|g''(t_1)| \leqslant 2K|g(t_1)| + \lambda = 2K|f(t_1)| + \lambda,$$

again allowing us to infer local boundedness of g'' around t_0 from that of f. In the case $K > 0$, again by (35) and (36), we can write g'' in one of the forms $C\sinh(t+\theta)$, $C\cosh(t+\theta)$, Ce^t, or

† The defining inequality in Definition 4.21 is the reason we chose the names sub- and supersolution as we did.

Ce^{-t}. Since none of these functions has an interior minimum on any finite interval, the claim follows as in the case of a boundary minimum above. □

We are going to need a comparison result on the homogeneous version of (34), which can be found in [17, p. 23], cf. also [22, Theorem 5.1.1].

Lemma 4.23. *Let $\psi : [0, L] \to \mathbb{R}$ be a smooth solution to $\psi'' - K\psi \geqslant 0$, $\psi(0) = 0$, $\psi(L) = 0$ and assume that $L < D_K$. Then $\psi(t) \leqslant 0$ for all $t \in [0, L]$.*

It is then immediate that the same conclusion holds if $\psi(0), \psi(L)$ are supposed to be $\leqslant 0$ (cf. Corollary 4.29 below for a strengthening of this result).

The following result is a slight generalisation of [5, Theorem 3.14] (stated without proof there). Since we will repeatedly rely on arguments required for establishing it, we give a complete proof.

Theorem 4.24. *Let $I = (a, b) \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a continuous function. The following are equivalent.*

(i) *f is a solution to $f'' - Kf \geqslant \lambda$ in the distributional sense.*
(ii) *f is locally Lipschitz and is a solution to $f'' - Kf \geqslant \lambda$ in the support sense (ITSS), that is, for all $t_0 \in I$ there is a solution $g : I \to \mathbb{R}$ of (34) with $g(t_0) = f(t_0)$ and $f \geqslant g$ on $[t_0 - D_K, t_0 + D_K] \cap I$.*†
(iii) *f is a solution to $f'' - Kf \geqslant \lambda$ in the sense of Jensen.*

The corresponding statement with all inequalities reversed in (i)–(iii) holds as well.

Proof. Due to Propositions 4.20 and 4.22, we may assume f to be locally Lipschitz throughout.

(i)⇒(iii): Let $\varphi \in C_c^\infty((-1,1))$, $\varphi \geqslant 0$, $\int \varphi(t)\,dt = 1$ and set $\varphi_\varepsilon(t) := \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)$. Finally, let $f_\varepsilon := f * \varphi_\varepsilon$. Then, f_ε is smooth on $I_\varepsilon := (a + \varepsilon, b - \varepsilon)$ and satisfies $f_\varepsilon'' - Kf_\varepsilon \geqslant \lambda$ on its domain. Let $t_1 < t_2 \in I$, $|t_1 - t_2| < D_K$ and let $\varepsilon > 0$ be so small that $[t_1, t_2] \subseteq I_\varepsilon$. Let g_ε be the unique solution to (34) with $g_\varepsilon(t_i) = f_\varepsilon(t_i)$ $(i = 1, 2)$. Then, $\psi_\varepsilon := f_\varepsilon - g_\varepsilon$ is smooth, $\psi_\varepsilon'' - K\psi_\varepsilon \geqslant 0$, and $\psi_\varepsilon(t_i) = 0$ for $i = 1, 2$. By Lemma 4.23, then, $\psi_\varepsilon \leqslant 0$ on $[t_1, t_2]$. Letting $\varepsilon \to 0$ implies that $f \leqslant g$ on $[t_1, t_2]$ (cf. [23, Theorem 11]).

(iii)⇒(ii): We first prove that, with t_1, t_2 and g as in Definition 4.21, that is, as in (iii), we also have that $f(t) \geqslant g(t)$ for all $t \in ((t_2 - D_K, t_1] \cup [t_2, t_1 + D_K]) \cap I$. We show this for any fixed $t_0 \in (t_2 - D_K, t_1] \cap I$, the other case being analogous. Denote by h the solution to (34) with $h(t_0) = f(t_0)$ and $h(t_2) = f(t_2)$. By (iii), $h \geqslant f$ on $[t_0, t_2]$ and we claim that $h \geqslant g$ on $[t_0, t_1]$: Supposing, to the contrary, the existence of some $\bar{t} \in [t_0, t_1]$ with $h(\bar{t}) < g(\bar{t})$, then since $h(t_1) \geqslant g(t_1)$, there would also have to be some $\hat{t} \in [t_0, t_1]$ with $g(\hat{t}) = h(\hat{t})$. But then since also $g(t_2) = f(t_2) = h(t_2)$, $g = h$ everywhere by unique solvability of the boundary value problem for (34), giving a contradiction. Consequently, $g(t_0) \leqslant h(t_0) = f(t_0)$, as claimed.

Now for given $t \in I$, consider two sequences $s_n \nearrow t$, $t_n \searrow t$, and let g_n be the unique solution to (34) with $g_n(s_n) = f(s_n)$ and $g_n(t_n) = f(t_n)$. Since f is locally Lipschitz, it follows from [18, Lemma 3] that both g_n and g_n' remain uniformly bounded in some fixed neighbourhood of t for n large. In particular, there is a subsequence of

† Note that in [5, Theorem 3.14 (b)], the inequality sign has to be reversed.

$(g_n(t), g_n'(t))$ that converges, so w.l.o.g. the whole sequence does. By continuous dependence on initial data, g_n therefore converges to the solution g with initial data $g(t) = \lim_n g_n(t)$ and $g'(t) = \lim_n g_n'(t)$.

For $t - D_K < s < t$ (the other case being analogous), we have for large enough n that $t_n^+ - D_K < s < t_n^- < t$. For such n, the above considerations show that $f(s) \geqslant g_n(s)$. Taking the limit as $n \to \infty$, we get $f(s) \geqslant g(s)$.

(ii)⇒(i): Assume first that f is smooth and let $t_0 \in I$ and g as in (ii). Then

$$f''(t_0) = \lim_{h\to 0} \frac{f(t_0+h) - 2f(t_0) + f(t_0-h)}{h^2} \tag{39}$$

$$\geqslant \lim_{h\to 0} \frac{g(t_0+h) - 2g(t_0) + g(t_0-h)}{h^2} \tag{40}$$

$$= g''(t_0) = \lambda + Kg(t_0) = \lambda + Kf(t_0) \tag{41}$$

gives the claim in this case. Next, suppose that f is only locally Lipschitz and let φ_ε be as above. For $s \in I$, let g_s be the solution from (ii) with $t_0 = s$. If f is differentiable at s, then (ii) implies that $g_s'(s) = f'(s)$. For such values of s, therefore, g_s is uniquely determined and can indeed be calculated from $f(s)$ and $f'(s)$ according to (36). Since $f' \in L^\infty_{loc}(I)$, it follows that $(s,t) \mapsto g_s(t) \in L^\infty(I \times I)$. Now fix $t_0 \in I$ and set (for t near t_0 and ε small)

$$h_\varepsilon(t) := \int g_{t_0-s}(t-s)\varphi_\varepsilon(s)\,ds = \int g_{t_0-s}(s)\varphi_\varepsilon(t-s)\,ds.$$

Then h_ε is a smooth solution to $h_\varepsilon'' - Kh_\varepsilon = \lambda$. Moreover,

$$f_\varepsilon(t_0) = \int f(t_0-s)\varphi_\varepsilon(s)\,ds = \int g_{t_0-s}(t_0-s)\varphi_\varepsilon(s)\,ds = h_\varepsilon(t_0)$$

and $f_\varepsilon(t) \geqslant h_\varepsilon(t)$ near t_0 (for ε small). From the smooth case, it then follows that $f_\varepsilon'' - Kf_\varepsilon \geqslant \lambda$ near t_0, so $\varepsilon \to 0$ gives the claim. □

We record the following consequence of the proof of (iii)⇒(ii) in Theorem 4.24.

Corollary 4.25 (Outer Jensen). *Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ locally Lipschitz. Then f is a subsolution in the sense of Jensen if and only if it satisfies the following condition: for all $t_1 < t_2 \in I$, the (unique) solution $g : I \to \mathbb{R}$ of (34) with $g(t_i) = f(t_i)$ satisfies $f \geqslant g$ on $(I \setminus (t_1, t_2)) \cap (t_2 - D_K, t_1 + D_K)$.*

The following solutions of (34) will be of particular interest to us.

Definition 4.26. The *modified distance* function $\mathrm{md}^K : [0, D_K) \to \mathbb{R}$ in the model space $\mathbb{L}^2(K)$ is the solution of the initial value problem

$$\begin{cases} (\mathrm{md}^K)'' - K\,\mathrm{md}^K = 1 \\ \mathrm{md}^K(0) = 0 \\ (\mathrm{md}^K)'(0) = 0. \end{cases} \tag{42}$$

The *modified sine* function $\mathrm{sn}^K = (\mathrm{md}^K)'$ and the *modified cosine* function $\mathrm{cn}^K = (\mathrm{sn}^K)'$ are the solutions of the following initial value problems for the corresponding homogeneous equation:

$$\begin{cases} (\mathrm{sn}^K)'' - K\,\mathrm{sn}^K = 0 \\ \mathrm{sn}^K(0) = 0 \\ (\mathrm{sn}^K)'(0) = 1 \end{cases} \tag{43}$$

and

$$\begin{cases} (\mathrm{cn}^K)'' - K\,\mathrm{cn}^K = 0 \\ \mathrm{cn}^K(0) = 1 \\ (\mathrm{cn}^K)'(0) = 0 \end{cases} \tag{44}$$

Explicitly:

	$\mathbf{md}^K(t)$	$\mathbf{sn}^K(t)$	$\mathbf{cn}^K(t)$
$K = 0$	$\frac{t^2}{2}$	t	1
$K = 1$	$\cosh(t) - 1$	$\sinh(t)$	$\cosh(t)$
$K = -1$	$1 - \cos(t)$	$\sin(t)$	$\cos(t)$

The role of the modified distance is the following, compare [5, Chapter 1, 1.1(a)].

Lemma 4.27 (md^K and geodesics in model spaces). *Let $p \in \mathbb{L}^2(K)$ and $\gamma : \mathbb{R} \to \mathbb{L}^2(K)$ be a τ-unit speed geodesic. Then, the partial function*

$$f(t) = \begin{cases} \mathrm{md}^K(\tau(p, \gamma(t))) & p \leqslant \gamma(t) \\ \mathrm{md}^K(\tau(\gamma(t), p)) & \gamma(t) \leqslant p \end{cases} \tag{45}$$

satisfies:

$$f'' - Kf = 1\,, \tag{46}$$

more precisely, $f = \mathrm{md}^K + C\,\mathrm{sn}^K + D\,\mathrm{cn}^K$ where this is positive, and f is not defined where the right-hand side is negative, with $C = f'(0)$ and $D = f(0)$. In particular, it is extensible to a solution on $\mathbb{R}$.

Proof. By rescaling, it suffices to consider the cases $K = -1, 0, +1$. For $K = 0$, after applying a suitable Lorentz transformation, we can assume that $p = (t_p, x_p)$ and $\gamma(t) = (t, 0)$. It then follows that $f(t) = \frac{t^2}{2} - t_p t + \frac{t_p^2 - x_p^2}{2} = \mathrm{md}^0(t) + f'(0)\,\mathrm{sn}^0(t) + f(0)\,\mathrm{cn}^0(t)$.

For the case $K = 1$, again by applying a suitable† Lorentz transformation, we may assume that $p = (t_p, x_p, y_p)$ and $\gamma(t) = (\sinh(t), 0, \cosh(t))$. Now for any causally related points v, w in de Sitter space $\mathbb{L}^2(1)$, the time separation function is given explicitly by $\tau(v, w) = \mathrm{arcosh}\langle v, w\rangle$

† Here, we view de Sitter space as the set $\{(t, x, y) \mid -t^2 + x^2 + y^2 = +1\}$ embedded in $\mathbb{R}^3_1$.

(cf., e.g. [16, (2.7)]). Consequently, for $p \leqslant \gamma(t)$, $\tau(p,\gamma(t)) = \operatorname{arcosh}(-t_p \sinh t + y_p \cosh t)$, and so,

$$f(t) = \mathrm{md}^1(\tau(p,\gamma(t))) = -t_p \sinh t + y_p \cosh t - 1$$
$$= \mathrm{md}^1(t) + f'(0)\,\mathrm{sn}^1(t) + f(0)\,\mathrm{cn}^1(t).$$

Finally, for the anti-de Sitter† case $K = -1$, we can w.l.o.g. assume $\gamma(t) = (\cos(t), \sin(t), 0)$, and we set $p = (s_p, t_p, x_p)$. Here, for causally related points v, w, we have $\tau(v,w) = \arccos(-\langle v, w\rangle)$ (using [16, (2.7)], together with [29, Lem. 4.24]). Thus, for $p \leqslant \gamma(t)$, we have

$$f(t) = \mathrm{md}^{-1}(\tau(p,\gamma(t))) = 1 + \langle p, \gamma(t)\rangle = 1 - s_p \cos t - t_p \sin t$$
$$= \mathrm{md}^{-1}(t) + f'(0)\,\mathrm{sn}^{-1}(t) + f(0)\,\mathrm{cn}^{-1}(t). \qquad \square$$

The definition of f in (45) requires a distinction in cases depending on the causal relation of p and $\gamma(t)$. This is also the main difference with respect to the metric machinery: if there is no causal relation between p and $\gamma(t)$, then f cannot give any information. This is why the next few results build up the theory in order to extend some known results from metric geometry to domains that consist of two intervals.

Lemma 4.28. *Let $f_i : (a,b) \to \mathbb{R}$, $i = 1,2$, be two solutions of (34) and let $t_1, t_2 \in I$ with $t_1 < t_2$. Then, if $f_1(t_1) = f_2(t_1)$ and $f_1(t_2) < f_2(t_2)$, we have $f_1 < f_2$ on $(t_1, \min(b, t_1 + D_K))$ and $f_1 > f_2$ on $(\max(a, t_1 - D_K), t_1)$. In particular, if $f_1(t_1) > f_2(t_1)$ and $f_1(t_2) < f_2(t_2)$, we have $f_1 < f_2$ on $(t_2, \min(b, t_1 + D_K))$ and $f_1 > f_2$ on $(\max(a, t_2 - D_K), t_1)$.*

Proof. Assume w.l.o.g. that $t_1 = 0$ and let $f := f_2 - f_1$. Then, f is a solution of the homogeneous problem (35), hence it is of the form $f = a\,\mathrm{cn}^K + b\,\mathrm{sn}^K$. As $f(0) = 0$, we have $f = b\,\mathrm{sn}^K$ and $b = \frac{f(t_2)}{\mathrm{sn}^K(t_2)} > 0$. By the explicit formula, $\mathrm{sn}^K(t) > 0$ for $t \in (0, D_K)$ and $\mathrm{sn}^K(t) < 0$ for $t \in (-D_K, 0)$. For f_1 and f_2 and general t_1, this means that $f_1 < f_2$ on $(t_1, \min(b, t_1 + D_K))$ and $f_1 > f_2$ on $(\max(a, t_1 - D_K), t_1)$.

For the 'in particular' statement, note that f has a zero between t_1 and t_2, so we can apply the main part of the lemma. $\square$

As a consequence, we obtain the following strengthening of Lemma 4.23.

Corollary 4.29. *Let the continuous function $\psi : [0,L] \to \mathbb{R}$ be a solution to $\psi'' - K\psi \geqslant 0$ in the sense of Jensen and assume that $\psi(0) \leqslant 0$ and $\psi(L) \leqslant 0$, with at least one of these inequalities strict. Then $\psi(t) < 0$ for all $t \in (0,L)$.*

Proof. Let φ_1 be the unique solution to (35) with $\varphi_1(0) = \psi(0)$ and $\varphi_1(L) = 0$, and φ_2 the one with $\varphi_2(0) = 0$ and $\varphi_2(L) = \psi(L)$. Then, $\varphi := \varphi_1 + \varphi_2$ is the unique solution to (35) with the same boundary conditions as ψ, so $\psi \leqslant \varphi$ on $[0,L]$. This proves the claim since at least one of φ_1 and φ_2 is strictly negative on $(0,L)$ by Lemma 4.28. $\square$

In the following results, we will require a Jensen-type solution concept that is applicable to domains more general than intervals.

† Here, we view anti-de Sitter space as the set $\{(s,t,x) \mid -s^2 - t^2 + x^2 = -1\}$ embedded in $\mathbb{R}^3_2$.

Definition 4.30. Let $U \subseteq \mathbb{R}$ be any subset. A continuous function $f : U \to \mathbb{R}$ is said to satisfy the *Jensen subsolution (resp. supersolution) inequality* for the parameters (t_1, t_2, t_3) with $t_1 < t_2 < t_3$ and $|t_1 - t_3| < D_K$ if $f(t_2) \leqslant g(t_2)$ (resp. $f(t_2) \geqslant g(t_2)$), where g is the unique solution to (34) with $g(t_i) = f(t_i)$ for $i = 1, 3$.

Remark 4.31 (Reformulation of subsolutions (resp. supersolutions)). Note that f is a subsolution (resp. supersolution) in the sense of Jensen if and only if it satisfies the Jensen subsolution (resp. supersolution) inequality for all parameters (t_1, t_2, t_3) with $t_1 < t_2 < t_3$ and $|t_1 - t_3| < D_K$.

Proposition 4.32 (Splitting Jensen). *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a continuous function. Let $t_1 < t_2 < t_3 < t_4$ and $|t_1 - t_4| < D_K$. If f satisfies the Jensen subsolution (supersolution) inequality for the parameters (t_1, t_2, t_3) and for (t_2, t_3, t_4), it also satisfies it for (t_1, t_2, t_4) and (t_1, t_3, t_4).*

Proof. Again, it will suffice to prove the subsolution case. Let g_{13}, g_{24} and g_{14} be the solutions to (34) with $g_{13}(t_1) = f(t_1)$, $g_{13}(t_3) = f(t_3)$, $g_{24}(t_2) = f(t_2)$ and $g_{24}(t_4) = f(t_4)$, $g_{14}(t_1) = f(t_1)$ and $g_{14}(t_4) = f(t_4)$. Then, our assumption is that $f(t_2) \leqslant g_{13}(t_2)$, as well as $f(t_3) \leqslant g_{24}(t_3)$. We then have to show, w.l.o.g., that $f(t_2) \leqslant g_{14}(t_2)$.

As $g_{13}(t_2) \geqslant g_{24}(t_2)$ and $g_{13}(t_3) \leqslant g_{24}(t_3)$, we can use Lemma 4.28 to conclude that $g_{24}(t_1) \leqslant f(t_1)$. Now set $\psi := g_{24} - g_{14}$. Then $\psi(t_1) \leqslant 0$ and $\psi(t_4) = 0$, so Lemma 4.23 implies that $\psi \leqslant 0$ on $[t_1, t_4]$. In particular, $f(t_2) = g_{24}(t_2) \leqslant g_{14}(t_2)$. □

Note that the above result could, in fact, be formulated entirely in terms of properties of g_{13}, g_{24} and g_{14}, without recourse to the function f. However, for the applications we have in mind, the formulation we chose is more appropriate.

As in the case of the model spaces (see Lemma 4.27), we will be interested in when a function f defined on two disjoint intervals is a subsolution of (34) when restricted to the set where it is non-negative. For our intended purposes (cf. Definition 4.35 below), these intervals will be closed. In this case, we note the following immediate but helpful consequence of Definition 4.21:

Remark 4.33. If f is a continuous function on an interval $[a, b]$, then f is a subsolution (resp. supersolution) to (34) ITSJ on $[a, b]$ if and only if it is one on (a, b). It follows from this, together with Theorem 4.24, that in the following result, the Jensen solution concept on intervals can also be expressed in terms of the equivalent notions given there.

Proposition 4.34. *Let $I = [a, b] \cup [c, d]$, $a < b < c < d$, and $f : I \to \mathbb{R}$ continuous with $f(b) = f(c) = 0$. Then the following are equivalent.*

(i) *f is a subsolution ITSJ of (34) in the following sense: For all $t_1, t_2, t_3 \in [a, b] \cup [c, d]$ with $t_1 < t_2 < t_3$ and $|t_1 - t_3| < D_K$, f satisfies the Jensen subsolution inequality.*
(ii) *f is extensible as a subsolution of (34) ITSJ on $[a, d]$.*
(iii) *f is a subsolution of (34) ITSJ on both parts of its domain, and for the solution g of (34) with $g(b) = g(c) = 0$, we have $f'(b^-) \leqslant g'(b)$ and $f'(c^+) \geqslant g'(c)$.*

An analogous equivalence holds for supersolutions.

Proof.

(ii)⇒(i) is clear.

(i)⇒(iii): The proof of Corollary 4.25 still works in the present setup and shows that for g as in (iii), we have $g \leqslant f$ on $[\max(a, c - D_K), b]$ (as well as on $[c, \min(d, a + D_K)]$). Therefore,

$$f'(b^-) = \lim_{t \nearrow b} \frac{f(b) - f(t)}{b - t} \leqslant \lim_{t \nearrow b} \frac{g(b) - g(t)}{b - t} = g'(b).$$

The inequality $f'(c_+) \geqslant g'(c)$ can be established in a similar way.

(iii)⇒(ii): Taking g as in (iii), we extend f to $[a, d]$ by setting $f(t) := g(t)$ for $t \in (b, c)$. We now verify the Jensen condition for this extended function. By Proposition 4.32, we only need to check three cases: $|t_1 - t_3| < D_K$ and either $t_1 < t_2 < t_3$ are all contained in either $[a, b]$ or $[b, c]$ or $[c, d]$ (where it is automatically satisfied), or (t_1, b, t_3) with $t_1 \in (a, b)$ and $t_3 \in (b, c)$, or, finally, (t_1, c, t_3) with $t_1 \in (b, c)$ and $t_3 \in (c, d)$. By symmetry, we only need to treat (t_1, b, t_3). Thus, let h be the unique solution to (34) with $h(t_1) = f(t_1)$ and $h(t_3) = f(t_3)$. The claim then is that $h(b) \geqslant f(b)$.

If we can show that $f \geqslant g$ on $[t_1, b]$ we will be done: Indeed, then $h(t_1) = f(t_1) \geqslant g(t_1)$ and $h(t_3) = g(t_3)$, so Lemma 4.23 implies $h(b) \geqslant g(b) = f(b)$. So, we are left with proving that $k := f - g$, which is a subsolution ITSJ of the homogeneous equation (35), is non-negative on $[t_1, b]$.

We now construct a supporting solution for k at b: Let $s_n \in (t_1, b)$, $s_n \nearrow b$ and denote by l_n the unique solution to (35) with $l_n(s_n) = k(s_n)$ and $l_n(b) = k(b)$. As in the proof of Theorem 4.24, (iii)⇒(ii), it then follows that (up to picking a subsequence) l_n converges (in C^2) to the solution l of the initial value problem to (35) with

$$l(b) = \lim_n l_n(b) = k(b),$$

$$l'(b) = \lim_n l_n'(b).$$

By the mean value theorem, $\frac{l_n(b) - l_n(s_n)}{b - s_n} = l_n'(\tilde{s}_n) \to l'(b)$ (where $\tilde{s}_n \in (s_n, b)$), which by construction implies that $l'(b) = k'(b^-)$. As in the proof of Theorem 4.24, (iii)⇒(ii), it follows that l is a supporting function for k at b, that is, $l \leqslant k$ on $[t_1, b]$. Furthermore, we have $l(b) = k(b) = 0$ and $l'(b) = k'(b^-) \leqslant 0$ by (iii), so (36) implies that $l \geqslant 0$ on $[t_1, b]$. Thus, finally, $k \geqslant l \geqslant 0$, so $f \geqslant g$ on $[t_1, b]$, as claimed. □

We now have the necessary tools at hand to introduce a characterisation of curvature bounds via a convexity/concavity property of τ, see [5, Theorems 8.23 & 9.25].

Definition 4.35 (Curvature bounds by convexity/concavity of τ). Let X be a regular Lorentzian pre-length space. An open subset U is called a ($\geqslant K$)- (resp. ($\leqslant K$)-)comparison neighbourhood in the sense of the *τ-convexity (resp. τ-concavity) condition* if:

(i) τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.
(ii) U is D_K-geodesic.

(iii) Let $p \in U$ and let $\gamma : [a,d] \to U$ be a timelike τ-arclength parametrised distance realiser† with $\tau(p,\gamma(d)) < D_K$, $\tau(\gamma(a),p) < D_K$ and $\tau(\gamma(a),\gamma(d)) = d - a < D_K$. We define the partial function on $[a,d]$

$$f(t) = \begin{cases} \mathrm{md}^K(\tau(p,\gamma(t))), \text{ if } p \leqslant \gamma(t) \\ \mathrm{md}^K(\tau(\gamma(t),p)), \text{ if } \gamma(t) \leqslant p \end{cases} \tag{47}$$

(compare this with (45), but note that here τ denotes the time separation in X). If f is not defined on a closed subset, extend it by setting it equal to 0 on the boundary of its domain. We require

$$f'' - Kf \geqslant 1 \quad (\text{resp. } f'' - Kf \leqslant 1)\,, \tag{48}$$

that is, f is a subsolution (resp. supersolution) in the sense of any of the equivalent formulations established in Proposition 4.34 and Remark 4.33.

Remark 4.36 (Domain of f). Define $b = \sup\{t \in [a,d] : \gamma(t) \leqslant p\}$, $c = \inf\{t \in [a,d] : p \leqslant \gamma(t)\}$ (if the respective set is nonemtpy). Then the function f defined in (47) has the domain: $[a,b] \cup [c,d]$ if both b, c are defined, $[a,b]$ if c is not defined, $[c,d]$ if b is not defined, and $\emptyset$ if neither are defined. Without the extension of f and if U is not causally closed, the points b, c may be missing from these sets.

Proposition 4.37 (Triangle comparison and τ- convexity (resp. concavity) condition are equivalent). *Let U be an open subset in a regular Lorentzian pre-length space X. Then, U is a $(\geqslant K)$-(resp. $(\leqslant K)$-)comparison neighbourhood in the sense of one-sided timelike triangle comparison if and only if it is a $(\geqslant K)$ (resp. $(\leqslant K)$) -comparison neighbourhood in the sense of the τ-convexity (resp. τ-concavity) condition.*

Proof. Since X is assumed to be regular, the first two conditions in Definitions 3.1 and 4.35 agree. It is left to check the third condition. We will do this for lower curvature bounds and mention where the case of upper curvature bounds is not analogous.

Below, when considering the convexity condition, we take a point $p \in U$ and a timelike distance realiser γ in τ-unit speed parametrisation. Define the partial function f as required, without the extension (cf. (47)). Note that the extension of f is still continuous and, by a limit argument, the extended f is a subsolution (resp. supersolution) ITSJ if and only if f before the extension was a subsolution (resp. supersolution) ITSJ. We then want to check that f is a subsolution of (34) ITSJ with $\lambda = 1$ by showing that it satisfies the Jensen subsolution inequality for any parameters $t_1 < t_2 < t_3$ with $|t_1 - t_3| < D_K$, cf. Remark 4.31. Take $t_1 < t_2 < t_3$ in the domain of γ with $|t_1 - t_3| < D_K$. We set $x = \gamma(t_1)$, $q = \gamma(t_2)$ and $y = \gamma(t_3)$, and assume that x and y are causally related to p.

On the other hand, when considering one-sided triangle comparison, we will let $\Delta(x,p,y)$ form a timelike triangle (in any possible permutation), and let q be a point on the side $[x,y]$ realised by γ in τ-unit speed. Let $x = \gamma(t_1)$, $q = \gamma(t_2)$ and $y = \gamma(t_3)$. Note that choosing q in the interior of $[x,y]$ is not a real restriction since otherwise we would have trivial equality of the τ-lengths in X and $\mathbb{L}^2(K)$.

† Any timelike distance realiser can be parametrised by τ-arclength, but this parametrisation need not be Lipschitz, cf. [24, Corollary 3.35].

Now we are ready to establish both directions simultaneously.

Case 1: First assume that the parameters are suitable for both timelike triangle comparison as well the Jensen inequality. More precisely, assume that x, y are timelike related to p and $p \leqslant q$ (in which direction the points are related is not important). In particular, t_1, t_2, t_3 all lie inside the domain of f. Then, we can consider both the one-sided triangle comparison for q in $\Delta(p, x, y)$ and the Jensen inequality for the parameters $t_1 < t_2 < t_3$, and we claim that they are equivalent.

For both triangle comparison and Jensen inequality, we take a comparison situation $\Delta(\bar{p}, \bar{x}, \bar{y})$ and a comparison point $\bar{q}$. In the present case 1, we additionally assume $\bar{p} \leqslant \bar{q}$. Define $\bar{\gamma}$ as the side $[\bar{x}, \bar{y}]$ in τ-unit speed parametrisation such that $\bar{\gamma}(t_1) = \bar{x}$, then $\bar{\gamma}(t_2) = \bar{q}$ and $\bar{\gamma}(t_3) = \bar{y}$.

We set

$$\bar{f}(t) := \begin{cases} \mathrm{md}^K(\tau(\bar{p}, \bar{\gamma}(t))), & \text{if } \bar{p} \leqslant \bar{\gamma}(t)\,, \\ \mathrm{md}^K(\tau(\bar{\gamma}(t), \bar{p})), & \text{if } \bar{\gamma}(t) \leqslant \bar{p}\,. \end{cases}$$

By Lemma 4.27, $\bar{f}$ is a solution of (46) and is given by $\mathrm{md}^K + B\,\mathrm{cn}^K + C\,\mathrm{sn}^K$ where both are defined, which is precisely where the latter is non-negative. Extend $\bar{f}$ to g, given by that formula. We have that $f(t_1) = \bar{f}(t_1)$ and $f(t_3) = \bar{f}(t_3)$, so g is the solution of (46) with $f(t_1) = g(t_1)$ and $f(t_3) = g(t_3)$.

Now one-sided triangle comparison from below precisely says that $\tau(p, q) \leqslant \tau(\bar{p}, \bar{q})$. The case of $\tau(q, p) = 0 \leqslant \tau(\bar{q}, \bar{p}) = 0$ is automatic: indeed, by push-up and chronology, $p \leqslant q$ implies $q \not\ll p$ (and similarly in the model space). As md^K is strictly monotonically increasing on $[0, D_K]$, this is equivalent to

$$f(t_2) = \mathrm{md}^K(\tau(p, q)) \leqslant \mathrm{md}^K(\tau(\bar{p}, \bar{q})) = \bar{f}(t_2)\,, \tag{49}$$

which is the Jensen subsolution inequality for $t_1 < t_2 < t_3$. The curvature bounded above case follows analogously, concluding Case 1.

For all of the remaining cases, note that the logic is more subtle: we aim to prove that all Jensen inequalities imply the desired curvature bound inequality, and that all curvature bound inequalities imply the desired Jensen inequality.

Before we continue, note that we can restrict to admissible causal triangles in the Jensen inequality. Indeed, it cannot be that both x, y are not timelike related to p: as $t_1 < t_3$, this could only be the case if we have $x \leqslant p \leqslant y$ all null related, but then p, q are causally unrelated, contrary to our assumption in the present case. In all of the following cases, we will therefore assume that x, y and p form an admissible causal triangle.

Case 2: We can extend Case 1 to admissible causal triangles. Thus, let x, y, q be causally related to p and not both x and y timelike related to p as well as $\bar{p}$ and $\bar{q}$ causally related (in the same direction as p and q), then timelike triangle comparison does not make sense immediately. If only one of x, y is not timelike related, by varying p and q, one can reduce this to Case 1 by a limiting procedure as in the second paragraph in Proposition 4.19.

We have now dealt with all cases where, up to symmetry, $p \leqslant q$ and $\bar{p} \leqslant \bar{q}$.

Furthermore, we have to consider curvature bounds above and below separately.

Case 3: Let now $p \leqslant q$ and $\bar{p}, \bar{q}$ be causally unrelated. Then, we automatically have $0 \leqslant f(t_2)$ and $0 > g(t_2)$. In the case of curvature bounds below, we use the main argument in the equivalence of strict causal curvature bounds (Theorem 4.2 (ii)). As the implication $p \leqslant q \Rightarrow \bar{p} \leqslant \bar{q}$ does not hold, this contradicts strict causal triangle comparison, which we know to be equivalent to one-sided timelike triangle comparison by Proposition 3.3 and Theorem 4.2, thus also some curvature bound inequality for τ fails. For the Jensen subsolution inequality, $f(t_2) \leqslant g(t_2)$ is violated, making these match.

Triangle comparison above is automatically satisfied. For the Jensen supersolution inequality, $f(t_2) \geqslant g(t_2)$ is automatically satisfied, making these match as well.

Case 4: Let now p, q be causally unrelated and $\bar{p} \leqslant \bar{q}$. In any curvature bound, $f(t_2)$ is not defined, so this does not correspond to a Jensen subsolution (resp. supersolution) inequality. Note that triangle comparison from below is automatically satisfied. Triangle comparison from above does not hold if and only if $\bar{p} \ll \bar{q}$, so we restrict to that case, seeking a contradiction. As $\bar{p} \leqslant \bar{q} \ll \bar{y}$, we also infer that $p \ll y$. Consider $q_t = \gamma(t)$ and $\bar{q}_t = \bar{\gamma}(t)$ for $t > t_2$, then $\bar{p} \ll \bar{q}_t$. Note that for large enough $t < t_3$, we do have $p \leqslant q_t$. Set t' to be the infimum of these t, then

$$\lim_{t \searrow t'} \tau(p, q_t) = 0 < \tau(\bar{p}, \bar{q}) \leqslant \tau(\bar{p}, \bar{q}_{t'}) = \lim_{t \searrow t'} \tau(\bar{p}, \bar{q}_t), \tag{50}$$

where both limits exist since their arguments are monotonically increasing in t. If $\lim_{t \searrow t'} \tau(p, q_t)$ were positive, we would have $p \ll q_t'$, contradicting the fact that t' was an infimum. This, in turn, contradicts the Jensen supersolution inequality for t close enough to t', concluding this case.

Case 5: Let now p, q be causally unrelated and $\bar{p}, \bar{q}$ causally unrelated too. Then $f(t_2)$ is undefined, so this does not correspond to a Jensen subsolution (resp. supersolution) inequality. For triangle comparison, $\tau(p, q) = \tau(\bar{p}, \bar{q})$, so this instance of triangle comparison below (resp. above) is automatically satisfied.

Case 6: Let $p \leqslant q$ and $\bar{p} \geqslant \bar{q}$, that is, the two sides of the Jensen inequality are in different settings with respect to our case distinction. As in Case 4, we know that $p \leqslant q \ll y$ and $\bar{x} \ll \bar{q} \leqslant \bar{p}$, so $x \ll p \ll y$. We claim that we can find a parameter $\tilde{t}_2$ suitable for Case 1 violating the required triangle comparison, thus Case 1 shows that the corresponding Jensen inequality does not hold. In other words, under the assumptions of curvature bounds (in the sense of triangle comparison or convexity/concavity), this case cannot occur. Set $q_t = \gamma(t)$, then the corresponding point is $\bar{q}_t = \bar{\gamma}(t)$. For curvature bounds below, we define the functions $a(t) = \tau(p, \gamma(t))$ and $\bar{a}(t) = \tau(\bar{p}, \bar{\gamma}(t))$. Then, we know that both a and $\bar{a}$ are increasing, they both attain 0 and some positive value, and for $\varepsilon > 0$ small enough, we have that $a(t_2 + \varepsilon) > 0$ but $\bar{a}(t_2 + \varepsilon) = 0$ (if the second was positive for all $\varepsilon > 0$, we would have $\bar{p} \leqslant \bar{q}$ as well, so we can apply Case 1 directly). Let $\varepsilon' := \sup\{\varepsilon \mid \bar{a}(t_2 + \varepsilon) = 0\}$. Then, $a(t_2 + \varepsilon') > 0, \bar{a}(t_2 + \varepsilon') = 0$. This allows us to find a value $\tilde{t}_2$ such that $a(\tilde{t}_2) > \bar{a}(\tilde{t}_2) > 0$. This violates triangle comparison from below and makes $(t_1, \tilde{t}_2, t_3)$ parameters suitable for Case 1, thus also the Jensen subsolution inequality fails.

For curvature bounds above, the argument is analogous, using the functions $a(t) = \tau(\gamma(t), p)$ and $\bar{a}(t) = \tau(\bar{\gamma}(t), \bar{p})$ instead.

Case 7: The case of $p \geqslant q$ as well as the case of p and q being causally unrelated and $\bar{p} \geqslant \bar{q}$ are symmetric with respect to time orientation. □

5 | EQUIVALENCES AMONG CURVATURE BOUNDS

Here, we collect the various interdependencies between synthetic sectional curvature bounds that have been established in the previous sections into the following main result.

Theorem 5.1 (Equivalent notions of curvature bounds for Lorentzian pre-length spaces). *Let X be a chronological*[†] *Lorentzian pre-length space. Recall that X may have curvature bounded below (resp. above) by K in any of the following senses:*

(i) *timelike triangle comparison,*
(ii) *one-sided timelike triangle comparison,*
(iii) *causal triangle comparison,*
(iv) *one-sided causal triangle comparison,*
(v) *strict causal triangle comparison,*
(vi) *one-sided strict causal triangle comparison,*
(vii) *monotonicity comparison,*
(viii) *one-sided monotonicity comparison,*
(ix) *angle comparison,*
(x) *hinge comparison,*
(xi) *timelike four-point condition,*
(xii) *angle version of timelike four-point condition,*
(xiii) *causal four-point condition,*
(xiv) *strict causal four-point condition,*
(xv) *τ-convexity (resp. τ-concavity) condition.*

In general, the following relations between these curvature bounds hold:

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iv) \Leftrightarrow (iii) \Leftrightarrow (v) \Leftrightarrow (vi), \text{ and } (xi) \Leftrightarrow (xii),$$

where in the case of upper curvature bounds for $(iii) \Leftrightarrow (v) \Leftrightarrow (vi)$, one needs to additionally assume that X is strongly causal and locally causally closed.

If X is regular, we additionally have:

$$(xv) \Leftrightarrow (i) \Leftrightarrow (viii) \Leftrightarrow (vii) \Rightarrow (ix) \Leftrightarrow (x) \Rightarrow (xi),$$

where in the case of lower curvature bounds for $(vii) \Rightarrow (ix)$, one needs to additionally assume that X satisfies (8).

Finally, if X is strongly causal, regular and locally D_K-geodesic, and in the case of upper curvature bounds additionally is locally causally closed and in the case of lower curvature bounds satisfies (8), then all aforementioned notions of curvature bounds are equivalent. All relations between the curvature bounds are depicted Figure 4.

6 | IMPLICATIONS OF CURVATURE BOUNDS

In this final section, we prove two implications of upper curvature bounds in the spirit of [11, Proposition II.2.2, Exercise II.2.3]. On the one hand, we infer that τ is bi-concave for curvature bounded above by 0, and on the other hand, we get that $\tau(p, \cdot)$ (resp. $\tau(\cdot, p)$) is concave for arbitrary upper curvature bounds.

[†] We assume all spaces to be chronological from the onset, but wanted to emphasise this in the main theorem.

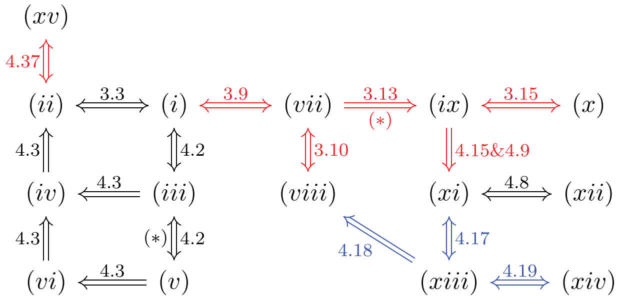

FIGURE 4 All relations between different formulations of curvature bounds for Lorentzian pre-length space. Black arrows are always valid, red arrows require X to be regular and blue arrows require X to be strongly causal, regular and locally D_K-geodesic. The two instances of additional assumptions for one direction of curvature bounds are marked by $(*)$ in the figure.

6.1 | Bi-concavity of τ

The bi-concavity of τ for spaces with curvature bounded above by 0 essentially follows by the intercept theorem in the Minkowski plane.

Proposition 6.1 (Bi-concavity of τ). *Let X be a Lorentzian pre-length space with curvature bounded above by 0 (in the sense of strict causal triangle comparison). Let U be a comparison neighbourhood in X. Then $\tau|_{(U\times U)}$ is 'timelike bi-concave', that is, for any two constant speed parametrised timelike distance realisers $\alpha, \beta : [0,1] \to X$ with the same time orientation such that $\alpha(1) \leqslant \beta(1)$ and $\alpha(0) \leqslant \beta(0)$, we have*

$$\tau(\alpha(t), \beta(t)) \geqslant t\tau(\alpha(1), \beta(1)) + (1-t)\tau(\alpha(0), \beta(0)) \,. \tag{51}$$

Proof. Say without loss of generality that both curves are future-directed. Let us first assume that $\alpha(0) = \beta(0)$. Let $\Delta(\bar\alpha(0), \bar\alpha(1), \bar\beta(1))$ be a comparison triangle for $\Delta(\alpha(0), \alpha(1), \beta(1))$. The intercept theorem yields the elementary equality $\tau(\bar\alpha(t), \bar\beta(t)) = t\tau(\bar\alpha(1), \bar\beta(1)) = t\tau(\alpha(1), \beta(1))$. By curvature bounds from above, we infer $\tau(\alpha(t), \beta(t)) \geqslant \tau(\bar\alpha(t), \bar\beta(t))$, and hence $\tau(\alpha(t), \beta(t)) \geqslant t\tau(\alpha(1), \beta(1))$, which already shows (51), as $\alpha(0) = \beta(0)$ and $\alpha(t) \leqslant \beta(t)$.

Now assume $\alpha(0) \leqslant \beta(0)$. Then, from $\alpha(0) \leqslant \beta(0) \ll \beta(1)$, we infer $\alpha(0) \ll \beta(1)$ by the transitivity of $\ll$. Let $\gamma : [0,1] \to X$ be a constant speed parametrised timelike distance realiser from $\alpha(0)$ to $\beta(1)$, that is, $\gamma(0) = \alpha(0)$ and $\gamma(1) = \beta(1)$. Then, $\Delta(\gamma(0), \beta(0), \gamma(1))$ and $\Delta(\gamma(0), \alpha(1), \gamma(1))$ form two triangles which fit into the special case above (where the first configuration has a reversed time orientation). Thus, we infer $\tau(\alpha(t), \gamma(t)) \geqslant t\tau(\alpha(1), \gamma(1))$ and $\tau(\gamma(t), \beta(t)) \geqslant (1-t)\tau(\gamma(0), \beta(0))$. In particular, the intercept theorem also yields that the causal relation in the Minkowski plane is preserved, that is, we have $\bar\alpha(t) \leqslant \bar\gamma(t) \leqslant \bar\beta(t)$ for all $t \in [0,1]$ and hence $\alpha(t) \leqslant \gamma(t) \leqslant \beta(t)$ by strict causal curvature bounds. Then via the reverse triangle inequality obtain

$$\begin{aligned}\tau(\alpha(t), \beta(t)) &\geqslant \tau(\alpha(t), \gamma(t)) + \tau(\gamma(t), \beta(t)) \\ &\geqslant t\tau(\alpha(1), \gamma(1)) + (1-t)\tau(\gamma(0), \beta(0)) \\ &= t\tau(\alpha(1), \beta(1)) + (1-t)\tau(\alpha(0), \beta(0)) \,.\end{aligned}$$

□

6.2 | Concavity of τ

Corollary 6.2 (Concavity of τ). *Let X be a Lorentzian pre-length space with curvature bounded above by K in the sense of strict causal triangle comparison. Let U be a comparison neighbourhood in X. Then τ is concave on $(U \times U) \cap \tau^{-1}([0, \frac{D_K}{2}))$, that is, for all $p \in U$ and all timelike geodesics γ contained in either $I^+(p) \cap U$ or $I^-(p) \cap U$ that have τ-distance less than $\frac{D_K}{2}$ to p, we have*

$$\tau(p, \gamma(t)) \geqslant t\tau(p, \gamma(1)) + (1-t)\tau(p, \gamma(0)) \tag{52}$$

or

$$\tau(\gamma(t), p) \geqslant t\tau(\gamma(1), p) + (1-t)\tau(\gamma(0), p) \tag{53}$$

for all $t \in [0, 1]$.

Proof. An elementary calculation yields that if τ is bi-concave in the sense of Proposition 6.1, then it is also concave in the sense of (52) and (53). In particular, the current proposition is valid for $K = 0$. Moreover, any space with timelike curvature bounded above by $K > 0$ also has timelike curvature bounded above by 0, which can easily be seen from [4, Lemma 6.1].

Thus, it is only left to consider the case of $K < 0$, which up to scaling we can reduce to $K = -1$. In this case, we claim that the time separation on $\mathbb{L}^2(-1)$, which in the remainder of this proof shall be denoted by $\bar{\tau}$, is concave. Assuming the claim, we then consider a triangle $\Delta(p, \gamma(0), \gamma(1))$ and the corresponding comparison triangle and compute

$$\begin{aligned}\tau(p, \gamma(t)) \geqslant \bar{\tau}(\bar{p}, \bar{\gamma}(t)) &\geqslant t\bar{\tau}(\bar{p}, \bar{\gamma}(1)) + (1-t)\bar{\tau}(\bar{p}, \bar{\gamma}(0)) \\ &= t\tau(p, \gamma(1)) + (1-t)\tau(p, \gamma(0)) \,.\end{aligned}$$

Showing that $\bar{\tau}$ is concave is an elementary calculation. Indeed, after applying a suitable Lorentz transformation and parameter shift, we can assume $p = (\cosh(\omega), 0, \sinh(\omega))$ and $\gamma(t) = (\cos(t), \sin(t), 0)$, so that

$$f(t) := \bar{\tau}(p, \gamma(t)) = \arccos(\cosh(\omega)\cos(t)) \,, \tag{54}$$

which is defined on $\left(-\pi, -\arccos\left(\frac{1}{\cosh(\omega)}\right)\right] \cup [\arccos\left(\frac{1}{\cosh(\omega)}\right), \pi)$. Then, we get

$$f''(t) = -\frac{\cosh(\omega)\cos(t)(\cosh(\omega)^2 - 1)}{\sqrt{1 - \cosh(\omega)^2\cos(t)^2}^{\,3}} \,. \tag{55}$$

Clearly, both the numerator and the denominator are positive (for $t \leqslant \frac{\pi}{2}$ and whenever defined), so we have $f''(t) \leqslant 0$, showing that $\bar{\tau}(\bar{p}, \bar{\gamma}(t))$ is concave in t on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (where it is defined). □

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