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VARIOUS MATTER MODELS IN GENERALISED BIANCHI SPACE-TIMES

Solutions of General Relativity admitting homotheties of the Bianchi classification

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Nikolaos Th. Chatzarakis

"Cosmologists are often in error, but never in doubt."

Lev Davidovich Landau

"Time present and time past
Are both perhaps present in time future,
And time future contained in time past.
If all time is eternally present
All time is unredeemable"

T.S. Eliot, *The Four Quartets*

Summary of Methods and Results

This thesis concerns the proof that a specific action of (three-dimensional and real) Bianchi groups on a (four-dimensional) space-time is (i) possible to be formulated as a particular foliation that covers the manifold of the space-time, and (ii) consists a solution to the Einstein field equations of General Relativity under several different matter models - namely, the vacuum, the pseudo-vacuum (a scalar field), the electro-vacuum (a free electromagnetic field), and the perfect fluid (in the classical macroscopic formulation).

There are three fundamental methods utilised:

(1) In the case of the fundamental theorem, presented in Chapter 2, the methods of algebra and differential geometry are used to prove the theorem.
(2) In the case of the actual construction of the space-time, in Chapter 3, everything appears as a simple exercise on differential geometry - the main feature of it used being the definitions of connection and curvature.
(3) Finally, when proving the existence and uniqueness of the Einstein system, the whole result is based on the Picard-Lindelöf theorem, that is based on real analysis and fixed point theorem (like Banach).

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Abstract

In the standard treatment, a group acts by isometries on a space-time, imposing its generators as the Killing vectors of this space-time, hence the corresponding solution to Einstein's field equations bears these physical symmetries. This treatment has produced a number of results with particular interest; examples of this are the spatially homogeneous cosmological models, some inhomogeneous cosmological models, but also several solutions of gravitational radiation. However, most of these solutions are carried out in vacuum or with the utilization of a perfect fluid as source. In the cases where either classical macroscopic (Euler) matter, or kinetic microscopic (Vlasov or Boltzmann) matter is used, the analysis is usually carried out with respect to an orthogonal slicing of space-time, which is further restrictive on the freedom of the action of the Bianchi group on the space-time.

In this work, we attempt to generalise these works by assuming that the Bianchi group acts by homotheties and the quotient is any one-dimensional submanifold invariant to the action of the group. We propose that such a space-time can be constructed, given the action is free and regular, *i.e.*, that the orbits of the group are three-dimensional submanifolds of the space-time. Moreover, we propose that the transversal vector field (1) commutes with the Bianchi group generators, and (2) is tangent to a geodesic at any point of the homogeneous hypersurface. Consequently, such a space-time may indeed be a solution of the Einstein equations.

We specify this even further, by specifying the matter fields that act as the source of the Einstein equations. Initially, we prove that vacuum solutions of this set-up exist. Second, we assume free scalar fields as the source, thus pairing the Einstein equations with the Klein-Gordon one (the Einstein-scalar field system); once again, this system is also integrable under the condition that the scalar fields propagate along the orbits of the group (that is, they inherit the homotheties of the group). Following, we assume free electromagnetic fields as the source of the Einstein equations, which are now paired by the Maxwell equations (the Einstein-Maxwell system); such a system is also integrable under the condition that the electromagnetic fields inherit the homotheties of the space-time (*i.e.*, that the electromagnetic waves propagate along the orbits of the group). Interestingly, in both cases, these conditions can be proved. Finally, we consider the case of perfect fluids; in this case, the Einstein equations are combined with the Euler equations of fluid dynamics (the Einstein-Euler system) and are similarly integrable; the last case is particularly interesting, since, apart from the usual condition that the fluid inherits the symmetries of the space-time, it poses restrictions in the equation of state of the fluid.

All three cases are followed by realistic examples, some of which can be found in the literature. Interestingly, the particular space-times (where the Bianchi groups act freely and regularly by homotheties) reveal certain peculiarities that are not present in the usually considered (spatially or space-time) homogeneous space-times. Moreover, we can prove that solutions with such peculiarities are not unique; given initial conditions sufficiently close to them, other similar solutions (with the same peculiarities) can be found.

CHAPTER 1

Introduction

1. The Purpose of this Thesis

A space-time is called *homogeneous* if it is a non-empty manifold on which a Lie group of symmetries acts transitively, so that the the elements of this group are identified as symmetries for the space-time. These symmetries are defined as (smooth) vector fields whose local flow diffeomorphisms preserve some property of the space-time and are classified according to the type of this preservation [**1**]. If the group acts by isometries (whereas the metric is preserved along the local flow of the generators), then its generators are identified as Killing vectors of the entire space-time; if the group acts by homotheties (whereas the metric is preserved up to a constant factor along the local flow of the generators), then its generators are homothetic vectors of the space-time; finally, if the group acts by conformal symmetries (whereas the metric is preserved up to a conformal factor, depending on the local coordinates, along the local flow of the generators), its generators are conformal motions of the space-time [**2**]. When a space-time occurs as a solution to Einstein's field equations of General Relativity, then the symmetry group is essential to divulge the symmetries of the matter fields and the physical meaning of the space-time. Consequently, the Killing, homothetic, or conformal vector fields and their algebra contain and represent the physical meaning of the specific space-times.

This category becomes particularly important if the group acts transitively on a 3-d hypersurface of the space-time alone, allowing for an invariant vector field. This case concerns the 3-dimensional Lie groups of symmetries, classified by Bianchi in eleven types, six of which (namely I, II, VI_0, VII_0, $VIII$ and IX) are unimodular and belong to Class A, and the five remaining (namely III, IV, V, VI_h, VII_h) are solvable but not unimodular and belong to Class B; it is important that those belonging to Class B contain a 2-dimensional real Lie subgroup. These groups can act (usually by isometries) on either a spacelike hypersurface, $\mathcal{S}_3$, a timelike hypersurface $\mathcal{T}_3$, or a light-like hypersurface, $\mathcal{N}_3$, deeming the space-time *hypersurface homogeneous.* Essentially, the geodesics defined on the 3-dimensional sub-space are adjoint representations of the particular group orbits. Of course, the orbits of the group can be of dimension less than 3, resulting to surface homogeneous, curve homogeneous, *etc.* space-times.

Such (hypersurface) homogeneous space-times are of particular interest in astrophysical and cosmological context, as they may present realistic models -

eg. the flat Friedmann-Lemaître-Robertson-Walker cosmology is a particular subcase of Bianchi I acting on $\mathcal{S}_3$, while the Schwarzschild solution is a case of (or rather contains) Bianchi IX acting on $\mathcal{T}_3$. Petrov classified all vacuum and Λ-vacuum solutions of General Relativity that admit a Bianchi group action with any type of orbit and provided the canonical form of the metric for each [**3**]. However, it was Taub who originally applied such techniques to physically meaningful space-times [**4**]. Specifically, Taub assumed that the Bianchi classes of the real 3-dimensional Lie groups can act transitively on $\mathcal{S}_3$ with a timelike vector being invariant to this action; hence the spatial symmetries of such a space-time are entirely contained in one of the eleven Bianchi groups and its spatial geodesics are purely determined by this group orbits. Constraining these spatial hypersurfaces orthogonally to the timelike coordinate and assuming a perfect fluid as a source to Einstein's equations, these space-times have been widely used to describe spatially homogeneous cosmological models, as examined thoroughly by Ellis and Collins [**5**] *inter alia*, while Ryan and Shepley covered the fundamental results of these early investigations in [**8**]. These spatially homogeneous cosmological models are called Bianchi models or *homogeneous of dimension 3* models.[1] Among many, Collins and Stewart [**6**] and Wainwright and Hewitt [**7**] studied the evolution, constraints and stability of such models in the context of dynamical systems; some of these studies also focused on the viability of the models and their comparison to observational evidence. The fundamental results can be found in [**9**].

Assuming a different slicing of the space-time, so that a Bianchi group does not act transitively on the spatial hypersurfaces; in this case, the orthogonal slicing over the temporal coordinate is not necessary, hence the temporal coordinate can be contained in the geodesics that represent the symmetry group orbits. In this sense, the spatial homogeneity of the model is not ensured, but generalized space-time homogeneity can be considered. Ryan and Shepley gave insights towards space-time homogeneouns models in the context of General Relativity [**8**], with the Gödel space-time as a fundamental example [**10**]. Several studies attempted to associate such slices with inhomogeneous cosmological models; the usual treatment considered a $2-d$ homogeneous symmetry over the space-time, so that the temporal coordinate to be distinguished, hence this model became known as *homogeneous of dimension 2* cosmologies.[2] Szafron gave a first account of such solutions in [**11**], while Collins and Szafron utilized the concept of intrinsic symmetries of the space-time to consider a whole class of inhomogeneous cosmologies that follow the Bianchi classification [**12**]; Carmeli and Charach, Wainwright, and Rainer and Schmidt *inter alia* gave similar account in [**13, 14, 15**], concerned mainly with the irrotational models, while Krasiński focused in the rotational models

[1] In cosmology, it is usual to call these models G_3, though this is not to be confused with the Galilean group.

[2] In cosmology, it is usual to call these models G_2, though this is not to be confused with the the automorphism group of the octonion algebra.

obeying the same symmetries [**16**]. Van Helst *et al.* considered an holistic approach founded on dynamical systems that deals with homogeneous of dimension 2 cosmologies [**17**]; a thorough examination of all these cases can be found in [**18**]. We should notice again that perfect fluid was utilized as a source.

The choice of the orthogonal $1+3$ slicing is useful and easily associated with physical properties of the space-time, however it is generally restrictive. The choice of 3-d homogeneous models in the spatially homogeneous case is further restricted to 2-d homogeneous models in the inhomogeneous case, since time must be distinguished -for a general treatment of the case, see [**19**]. In general, if the Bianchi groups are allowed to transitively act in any 3-dimensional sub-space of a space-time, inhomogeneous models with symmetry of dimension 3 might also occur. In this case, the usual $1+3$ slicing of the space-time is not possible, as time cannot be *à priori* distinguished. Wainwright and subsequent studies of the 2-d homogeneous models admitted a $1+1+2$ slicing, while Nilsson and Uggla attempted to generalize this formalism in the case of a $1+3$ slicing over any vector, independent of its signature [**20**]; Harness also has probed towards this direction [**21**]. In general, it is possible to slice the space-time along any chosen coordinate (timelike, spacelike or null) and apply the very same techniques.

Most of these models were considered with a fluid as source, utilizing the standard hydrodynamic description. Knowing that actual matter may deviate from this treatment, due to its underlying statistical character, we may choose to describe it in the context of Vlasov distribution of particle, hence departing from the Einstein-fluid system towards the Einstein-Vlasov system -ref. [**22**] by Andreasson may provide the fundamentals to the treatment of such a system. Several studies have been conducting on 3-d spatially homogeneous cosmological models, obeying the symmetries of a particular Bianchi group and generated by the Einstein-Vlasov system, including [**23, 24, 25**]; the methodology and main results are summarized by Rendall in [**26**], while Ringström presented a thorough examination of the case of maximum symmetry -the FLRW universe- in [**27**]. However, little work has been done so far in 2-d homogeneous cosmological models -except perhaps for [**28**]. This gap in the literature can be bridged by extending the afore-mentioned works.

What is attempted in this work is the study of the Einstein system in the case of a Bianchi group acting freely without any restriction on the nature of the homogeneous hypersurfaces. In order to achieve this, we must first drop the assumption of a transitive action, since this allows for a wide variety of orbits. As we mentioned, a Bianchi group acting transitively may have orbits of dimension less than three; seeking for the entire hypersurface to be homogeneous, we need the group to act freely and regularly.

Another assumption that is dropped is that of orthogonal slicing. In the studies mentioned, the quotient of the group is a unique transversal vector field in the space-time, orthogonal to the homogeneous hypersurface. In the case of homogeneous cosmological models, this vector field can coincide with the coordinate time; for inhomogeneous cosmological models, it can coincide with either the coordinate time, or with one of the spatial coordinates depending on the signature of the homogeneous hypersurfaces. In our case, the orthogonality is dropped, hence the uniqueness of the transversal is also dropped. As a result, the transversal vector field of the group will be any of all vectors that are invariant under the action of the group.

Furthermore, seeking to allow for a completely free and regular action, we are also bound to drop the assumption of it acting by isometries. The case of the Bianchi group acting by homotheties and conformal symmetries has been studied by Steele in the case of vacuum [**31**], and by Kramer and Carot in the case of perfect fluids [**32**]. Attempting to extend it to the case of collisionless matter, we acknowledge that the Vlasov equation for massive particles is invariant under isometries alone; it is the case of massless particles (radiation) that allows the Vlasov equation to be invariant for homotheties - which is the case we are interested at.

As for the freedom of the nature of orbits on the homogeneous manifold, there are two reasons to consider. If the group orbits are spacelike, then we arrive to an extension of the usual 3-d spatially homogeneous cosmologies; if the group orbits are timelike, then we arrive to an extension of the 2-d homogeneous cosmologies, or even to some cases of inhomogeneous cosmologies - proposing that the Bianchi group can be divided to a subgroup of dimensnion 2, which will act on 2-dimensional spatial surfaces. Hence, the case of non-null orbits combines a large case of universes dominated by collisionless massless particles. Though, the case of lightlike orbits is somehow vague, Kramer *et al.* [**29**] and Hall [**1**] concluded that motions along null hypersurfaces denotes space-times that are solutions of General Relativity describing plane waves (at least locally, where the null homothetic vector exists). Consequently, allowing for null group orbits in the space-time, we consider gravitational waves coming through collisionless matter. As in the case of homogeneous cosmologies, these gravitational waves are not only plane, but may have up to eleven symmetry groups - *eg.* the action of Bianchi I would result to plane gravitational waves, the action of Bianchi III would result to cylindrical gravitation waves, and the action of Bianchi IX would result to spherical gravitational waves.

Thus, the work can be summarized in the proof of a theorem, as follows

THEOREM (Construction of a Homogeneous Space-time). *Assume there is a four-dimensional space-time $\mathcal{V}_4$ and a three-dimensional group $\mathcal{G}$, such that*

(1) *the group $\mathcal{G}$ acts freely and regularly on the space-time $\mathcal{V}_3$, admitting 3-dimensional orbits on it, and*
(2) *the group $\mathcal{G}$ acts by homotheties on the space-time $\mathcal{V}_4$, hence the generators of the group are homothetic vector fields of the space-time.*

For any such space-time, there exists some vector field $\vec{\zeta}$ in the neighbourhood of the generators of $\mathcal{G}$ that satisfies the following properties:

(1) *it is invariant under the action of the group, i.e., it commutes with the generators of the group;*

(2) *it is tangent to a geodesic at any point of the space-time; and*
(3) *is null.*

Then, a coordinate patch can be constructed that covers locally the space-time, based on the transversal $\vec{\zeta}$ *and the generators of the group.*

In the proof of the theorem, we proceed on the following stages: First, we prove that $\vec{\zeta}$ commutes with the generators, as a consequence of the free and regular action of the group. Second, we prove that $\vec{\zeta}$ is tangent to a geodesic by using its commutation with the generators and the action of the group by homotheties. Third, we specify $\vec{\zeta}$ to be null so as to ensure a convenient and generic normalisation condition. Finally, we construct a coordinate chart for the space-time using the affine parameter of the geodesic on which $\vec{\zeta}$ is tangent, and the canonical coordinates of the group; thus, $\vec{z}$ and the generators span a 4-dimensional vector basis. This is a universal result, as the orientation of the generators of the group, as well as the signature of the homogeneous hypersurface are unspecified, while any vector transversal to the group (that is geodesic and null) can play the role of $\vec{\zeta}$ at any point of the space-time.

The thesis is organized as follows: In Chapter 2, we present the main notions of the group acting freely and regularly on the space-time and the fundamental relations of vectors on this space-time. Following, the construction of the space-time is schematized, hence the three parts of the theorem are proved. Finally, we present the two possible forms of the metric as well as the Einstein field equations, the conservation laws for the matter fields and the four possible forms of these matter fields - namely, the scalar fields, the electromagnetic fields, the classical macroscopic fluid, and the Vlasov (collisionless) matter.

In Chapter 3, we deal with the parametrisation of the space-time under the coordinate chart specified in the previous chapter. We use the conditions of the theorem to specify the structure of the metric - and we explain why other structures are not discussed. We compute the Levi-Civita connections, the curvature tensors and the energy-momentum equation in the most generic form. What we aim to prove is that a non-null transversal is restrictive and yields trivial or well-known results; thus, a null transversal must be operationalised. In this case, the third part of the theorem is easily proved.

In Chapter 4, we consider the case of vacuum solution (the Einstein system). In this case, the Einstein tensor is set equal to zero, due to the absence of any matter fields. Assuming a set of coordinates adapted to the transversal and the group, the derivatives of the metric along the transversal are reduced to simple partial derivatives, while the derivatives along the group are given by the group. Hence, the Einstein equations reduce to ordinary differential equations. Following, the existence and uniqueness of solutions can be proved given a set of initial data (on an orbit of the Bianchi group) and a fixed interval of the independent variable (along the transversal collineation);[3]. these

[3]A comment must be made here about the use of the word "collineation" throughout this thesis. Usually, a vector field is termed collineation (with respect to something) when the

are proved by means of a fixed-point argument and the Picard-Lindelöf theorem. We also attempt to extend the the fixed interval of the independent variable so as to cover the entirety of the space-time. Finally, examples of such solutions are presented, such as the Minkowski space-time.

In Chapter 5, we extend our research to the $\Lambda-$vacuum solution (the Einstein system with a cosmological constant). In this case, the Einstein tensor is not zero, but proportional to the metric tensor; nevertheless, the rest of the set-up remains the same. The Einstein equations are reduced to ordinary differential equations and the existence and uniqueness of their solutions can be proved by means of the same fixed-point argument and the Picard-Lindelöf theorem. Examples of this case, *e.g.*, the de-Sitter space-time, are provided as well.

In Chapter 6, we turn our emphasis on electro-vacuum solutions (the Eintein-Maxwell system), where the presence of free (charge-less) electromagnetic fields in the space-time is allowed. In this case, the Einstein tensor is equal to the Maxwell energy-momentum tensor, which is given with respect to the Faraday electromagnetic tensor. By proving that the electromagnetic fields inherit the symmetries of the space-time, we show that the derivatives of the Faraday tensor along the transversal reduce to simple partial derivatives, while its derivatives along the group are determined by relations similar to those for the metric; hence, the Maxwell equations reduce to a set of ordinary differential equations with respect to the same independent variable as the metric and a set of constraints. Thus, the complete Einstein-Maxwell system is an ordinary differential system, which can be solved in the same manner as the Einstein system in Chapters III and IV. Examples of space-times with free electromagetic fields are given in the end.

Chapter 7 introduces matter fields in the context of a perfect fluid (the Einstein-Euler system). Now, the Einstein tensor is equal to the energy-momentum tensor of a perfect fluid, determined by the latter's energy density and pressure -two scalar quantities- and the velocity of the fluid. What we prove here is that the matter-energy density and the isotropic pressure of the fluid, as well as the observer's velocity vector inherit the symmetries of the space-time; consequently, their derivatives along the quotient are reduced to simple partial derivatives along the chosen parameter, and their derivatives along the group are determined by simple relation similar to those for the metric. In this case, both the Einstein and the Euler equations are simply ordinary differential equations. Their solution over a given interval of the independent variable can follow using a fixed-point argument and the Picard-Lindelöf theorem; an extension is possible trivially. Finally, a number of

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"movement of a specific quantity along it can be described as an isomorphism of this vector field (*e.g.*, a "curvature collineation" is one that preserves the Riemann tensor, a "matter collineation" one that preserves the stress-energy-momentum tensor, *etc.*). In our case, this vector field does preserve the group elements, as it commutes with them. Therefore, we will use the term in the following meaning: any vector field in the quotient of the group that preserves the group action will be termed a "transversal collineation".

known perfect fluid solutions that yield homothetic vectors are considered as possible examples of this case.

Chapter 8 concludes the thesis, summarizing the results and suggesting possible extensions of the work for the future. The main goal is to consider the Einstein-Vlasov and the Einstein-Boltzman systems under the same set-up.

2. A Notice on Notions and Notation

Before we proceed, it is useful to make a short comment on the notions and notation used so far, that will be used in the dissertation as follows.

The main geometric features employed in relativistic gravity and cosmology, hence in the modified theories of gravity and cosmology as well, are the tensors. Tensors are geometric objects that map other objects to itself in a multi-linear manner; given a metric affine manifold with a coordinate basis, tensors can be expressed as multidimensional arrays, whose elements correspond to a mapping on the specific basis. Such an object is expressed with indices in the form

$$T^{\alpha_1\alpha_2\ldots\alpha_n}{}_{\kappa_1\kappa_2\ldots\kappa_m}\,,$$

where $n+m$ is the rank of the tensor. The simplest form of tensors are the scalars (0-rank) and the vectors (1-rank).

The full definition of tensors is given by means of a coordinate basis change, since these objects remain unaffected, or rather invariant in such changes, proposing that the manifold on which they are defined in affine. As a result, given an "old" coordinate basis, $\{x^\mu\}$, and a "new" one, $\{\tilde{x}^\mu\}$, along with the (reversible) transformation rules, $\tilde{x}^\mu = \tilde{x}^\mu\left(x^\nu\right)$, any tensor follows the following transformation rule,

$$\tilde{T}^{\alpha_1\alpha_2\ldots}{}_{\kappa_1\kappa_2\ldots} = \frac{\partial\tilde{x}^{\alpha_1}}{\partial x^{\beta_1}}\frac{\partial\tilde{x}^{\alpha_2}}{\partial x^{\beta_2}}\ldots\frac{\partial x^{\lambda_1}}{\partial\tilde{x}^{\kappa_1}}\frac{\partial x^{\lambda_1}}{\partial\tilde{x}^{\kappa_1}}\ldots T^{\beta_1\beta_2\ldots}{}_{\lambda_1\lambda_2\ldots}\,.$$

Any other multidimensional array of arithmetics that does not follow this transformation rule during a change of the coordinate basis, is not considered an invariant of the manifold and, thus, is not a tensor.

The indices of the tensors can be upper or lower, depending on whether they correspond to the tangent or the cotangent space defined by the coordinate basis on the manifold. Upper indices correspond to the tangent space, that is defined by means of the coordinate curves tangent on the unit vectors; lower incides correspond to the cotangent space, that is defined by means of the coordinate surfaces vertical to the unit vectors. Greek letters will be used for the indices of tensors that are defined on 4-d pseudo-Riemannian or Einstein manifolds (that have non-degenerate metric and curvature proportional to it), used in General Relativity, where $\alpha = 0$ denotes the temporal components and $\alpha = 1, 2, 3$ denote the spatial components; latin letters will be used for the indices of tensors defined on 3-d Riemannian manifolds (that have positively defined metric and curvature), where $i = 1, 2, 3$ correspond to spatial components only.

Since the analysis is conducted on curved differentiable manifolds, many forms of differentiation shall appear. Partial derivative with respect to the coordinate basis shall be denoted as

$$\frac{\partial U}{\partial x^\mu} = \partial_\mu U\,.$$

Thus, the covariant derivative is defined as the derivative of a tensor along the tangent curves of the manifold; it is an extension of the partial derivative, equal to it when scalars are considered and diverging from it when higher-order tensors are differentiated. The divergence results from the affine connection of the manifold, measuring the latter's divergence from a flat Euclidean space, and is as large as the order of the differentiated tensor. More specifically, given a scalar, Φ, we have

$$\nabla_\mu \Phi = \partial_\mu \Phi \,;$$

given a vector, V^μ in the tangent space and V_μ in the cotangent space, we have

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda \quad \text{and} \quad \nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda{}_{\mu\nu} V_\lambda \,;$$

finally, given a 2-rank tensor, $T^{\mu\nu}$, or $T_{\mu\nu}$, or $T^\mu{}_\nu$, we have

$$\begin{aligned} \nabla_\mu T^{\rho\sigma} &= \partial_\mu T^{\rho\sigma} + \Gamma^\rho{}_{\mu\lambda} T^{\lambda\sigma} + \Gamma^\sigma{}_{\mu\lambda} T^{\rho\lambda} \,, \\ \nabla_\mu T_{\rho\sigma} &= \partial_\mu T_{\rho\sigma} - \Gamma^\lambda{}_{\mu\rho} T_{\lambda\sigma} - \Gamma^\lambda{}_{\mu\sigma} T_{\rho\lambda} \ \text{ and} \\ \nabla_\mu T^\rho{}_\sigma &= \partial_\mu T^\rho{}_\sigma + \Gamma^\rho{}_{\mu\lambda} T^\lambda{}_\sigma - \Gamma^\lambda{}_{\mu\sigma} T^\rho{}_\lambda \,, \end{aligned}$$

and so on. $\Gamma^\alpha{}_{\beta\gamma}$ is the affine connection of the manifold, a non-tensor object, as it is not invariant in the manifold (it does not transform accordingly when the coordinates change).

Moreover, throughout this work, Greek letters shall be used for indices on the 4-dimensional space-time, while Latin letters for the indices on the 3-dimensional homogeneous sub-space. Following Gourgoulhon, $\{\alpha, \beta, \gamma, ...\}$ and $\{a, b, c, ...\}$ shall be used as free indices, while $\{\kappa, \lambda, \mu, \nu, ...\}$ and $\{i, j, k, l, ...\}$ as contracting indices [**33**]; this could be is used in accordance to some parts of the literature to ease the immediate identification of the tensor rank of each equation. Round brackets among the indices denote symmetry, angle brackets denote trace-free symmetry and square brackets denote skew-symmetry.

Finally, to distinguish between forms and vectors, we shall employ distinct symbols: bold for the former, and an arrow-above for the latter. As a result,

$$\vec{u} = u^i \frac{\partial}{\partial x^i}$$

is a vector spanned in the tangent space, $T_p\mathcal{M}$, while

$$\mathbf{u} = u_i \mathrm{d}x^i$$

is the corresponding 1-form spanned in the dual or cotangent space, $T_p^*\mathcal{M}$ (at some point p). This is done as to avoid the confusion of using indices to denote vectors that belong in a certain algebra, rather than coordinates (which is the convention). We should note that the metric and other famous rank 2 tensors (*e.g.*, the Faraday tensor, the stress-energy-momentum tensor) are usually written as 2−forms, so they will follow the same writing.

CHAPTER 2

The Construction of the Space-Time

1. Introduction

In this chapter, we discuss the construction of the space-time under the particular conditions of a Bianchi group acting freely and regularly on it. The chapter begins with a brief review on groups and their action on manifolds, specifically about the classification of Bianchi groups.

The first part of the chapter leads to a theorem describing the construction of such a space-time, such that the foliation does not occur under an orthogonal (time-like or space-like) slicing, but following a null collineation that is geodesic and invariant under the action of the group. The second part discusses the the elements of the space-time, *i.e.* the connection, the curvature elements and the matter models.

2. Manifolds and Groups

Consider a space-time $(\mathcal{V}_4, \boldsymbol{g})$, where $\mathcal{V}_4$ is a (non-empty) pseudo-Riemannian manifold and $\boldsymbol{g}$ the (definite) metric imposed to it, with signature $(-,+,+,+)$; $\mathrm{T}\mathcal{V}_4(q)$ is the tangent bundle of $\mathcal{V}$ on a point $q \in \mathcal{V}_4$. Consider $\mathcal{G}$ a 3-dimensional group of symmetries with algebra $\mathfrak{G}$, that acts freely and regularly on $(\mathcal{V}_4, \boldsymbol{g})$; hence a 3-dimensional submanifold $\mathcal{M} \subset \mathcal{V}_4$ exists, whose symmetry group is $\mathcal{G}$. It is important to explain what free and regular mean, since we use the definitions set out by Olver [**34**]. For the former we have:

Definition 2.1 (Free Action). *The action of a group $\mathcal{G}$ on a manifold $\mathcal{V}$ is free if the isotropy subgroup of any individual point $z_0 \in \mathcal{V}$ is trivial.*

This definition is similar to the usual definition of a free action (*e.g.* in [**35**]), since it implies that the action contains no fixed points - *i.e.*, $\mathfrak{g} \cdot x \neq x$ for $\mathfrak{g} \in \mathcal{G}$. Now, for the latter, we have:

Definition 2.2 (Regular Action). *The action of a group $\mathcal{G}$ on a manifold $\mathcal{V}$ is regular if*

- *all orbits of the group have the same dimension, and*
- *each point $z \in \mathcal{V}$ has a system of arbitrarily small neighbourhoods $U(z)$ such that each orbit of the group $O(z')$ for $z' \in U(z)$ intersects these neighbourhoods in a pathwise connected subset.*

This definition is not identical to the usual definition of regular action, since it does not contain transitivity by definition.[1]

It follows from the "Homogeneous Space Construction Theorem" that $\mathcal{M}$ is a homogeneous manifold represented as the left coset space $\mathcal{M} = \mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ a closed subgroup of $\mathcal{G}$, whose dimension is given as

$$\dim\mathcal{H} = \dim\mathcal{G} - \dim\mathcal{M} \ .$$

Of course, if $\mathcal{H}$ is a discrete closed subgroup of $\mathcal{G}$, then $\dim\mathcal{H} = 0$; hence, $\dim\mathcal{G} = \dim\mathcal{M} = 3$ [**36**].

Given a point h_1 on this hypersurface is "moved" along an orbit on this hypersurface according to $h_2 = \mathfrak{g} \cdot h_1$, where $\mathfrak{g} \in \mathcal{G}$. If $L_{\mathfrak{g}} : \mathcal{G} \to \mathcal{G}$ the diffeomorphism on the group, or equivalently $L_{\mathfrak{g}} : \mathcal{M} \to \mathcal{M}$ the diffeomorphism on the 3-dimensional sub-manifold, such that

$$h_2 = L_{\mathfrak{g}} h_1 = \mathfrak{g} \cdot h_1 \ , \tag{1}$$

is unique for any $\mathfrak{g}, h_1, h_2 \in \mathcal{M}$ and has the following properties

$$\big(L_{\mathfrak{g}}\big)^{-1} = L_{\mathfrak{g}^{-1}} \quad \text{and} \ \ L_{\mathfrak{g}} \circ L_h = L_{\mathfrak{g}\cdot h} \ . \tag{2}$$

Then, the derivative of the diffeomorphism "moves" vectors, $\vec{v}$, from the tangent space of $\mathcal{M}$ on some point $h_1 \in \mathcal{M}$ to the tangent space of $\mathcal{M}$ on some point $h_2 \in \mathcal{M}$ as

(3)

$$\mathrm{d}L_{\mathfrak{g}} : \mathrm{T}\mathcal{M}\left(h_1\right) \to \mathrm{T}\mathcal{M}\left(h_2\right) : \vec{v}_{h_2} = \vec{v}_{\mathfrak{g}\cdot h_1} = \mathrm{d}L_{\mathfrak{g}}\left(\vec{v}_{h_1}\right) = \big(v^i\big)_{h_1} \frac{\partial \left(L^j\right)_{\mathfrak{g}}}{\partial x^i}\bigg|_{h_1} \partial_j \ .$$

In this, we consider that $\big(L^j\big)_{\mathfrak{g}}(x)$ are the components of the left translation $L_{\mathfrak{g}}(h) = \mathfrak{g} \cdot h$ in some local coordinates $\{x^i\}$; vectors like $\vec{v}$ that fulfill these identities are known as left invariant vector fields and have the following properties:

$$\begin{array}{rl} (i) & \mathrm{d}L_{\mathfrak{g}}\left(\vec{v}_h\right) = \mathrm{d}L_{\mathfrak{g}}\left(\mathrm{d}L_h(\vec{v}_0)\right) = \big(\mathrm{d}L_{\mathfrak{g}} \circ \mathrm{d}L_h\big)(\vec{v}_0) = \mathrm{d}L_{\mathfrak{g}\cdot h}\left(\vec{v}_0\right) = \vec{v}_{\mathfrak{g}\cdot h} \\ (ii) & \vec{v}_{\mathfrak{g}} = \mathrm{d}L_{\mathfrak{g}}\left(\vec{v}_0\right) \to \vec{v}_0 = \mathrm{d}L_{\mathfrak{g}^{-1}} \\ (iii) & \mathrm{d}L_{\mathfrak{g}}\big([\vec{v}_h, \vec{v}_f]\big) = \big[\mathrm{d}L_{\mathfrak{g}}\left(\vec{v}_h\right), \mathrm{d}L_{\mathfrak{g}}\left(\vec{v}_f\right)\big] = [\vec{v}_{\mathfrak{g}\cdot h}, \vec{v}_{\mathfrak{g}\cdot f}] \end{array} \tag{4}$$

Let us first consider the group. Without loss of generality, we can assume that the group referred to is a Bianchi group; that is, a three-dimensional

[1]Transitive actions are usually defined as those for which, for any two points $x\,, y \in \mathcal{M}$ there is some $\mathfrak{g} \in \mathcal{G}$ such that $\mathfrak{g} \cdot x = y$ [**35**]. Following this idea, regular actions are defined as simply transitive, that is, as actions that are both free and transitive; hence, regular actions, in this fashion, are those for which there is a unique $\mathfrak{g} \in \mathcal{G}$ such that any two different points $x\,, y \in \mathcal{G}$ are uniquely related as $\mathfrak{g} \cdot x = y$.

To give an example of why the two definitions are not alike, let us take a direct sum of a one-dimensional and a two-dimensional Lie group that act simply transitively (regularly in the usual definition for ordinary groups) on the direct sum of a straight and a plane, respectively. Then, we have orbits of unequal dimensions; so this action is not regular (in the definition given by Olver).

real group whose non-isomorphic structure is determined by a closed set of commutators. Any such group belongs to a classification of nine, the importance of which lies in a complete accounting of the symmetry groups of three-dimensional real spaces [**37**]. All Bianchi groups except for types $VIII$ and IX can be constructed as a semidirect product of $\mathbb{R}^2$ and $\mathbb{R}$, with $\mathbb{R}$ acting on $\mathbb{R}^2$ by some 2×2 matrix A; as for types $VIII$ and IX, they are directly related to the special linear and the special orthogonal groups respectively. Notably, the class is divided to two sub-classes: Class A refers to those that are unimodular and Class B to those that are non-unimodular.

Let $\vec{\xi_a}$ be the generators of a Bianchi group, which act by homotheties on the 3-dimensional sub-manifold $\mathcal{M}$, hence

(5) $$[\vec{\xi_a}, \vec{\xi_b}] = \nabla_{\vec{\xi_a}} \vec{\xi_b} - \nabla_{\vec{\xi_b}} \vec{\xi_a} = C^m{}_{ab} \vec{\xi_m}\,,$$

where $C^c{}_{ab}$ the structure constants of the Bianchi group. The generators of the Bianchi group are also related by means of the Jacobi identity,

(6) $$[\vec{\xi_a}, [\vec{\omega_b}, \vec{\xi_c}]] + [\vec{\xi_b}, [\vec{\xi_c}, \vec{\xi_a}]] + [\vec{\xi_c}, [\vec{\xi_a}, \vec{\xi_b}]] = 0\,,$$

which can also be written with respect to the structure constants, as

(7) $$C^d{}_{am} C^m{}_{bc} + C^d{}_{bm} C^m{}_{ca} + C^d{}_{cm} C^m{}_{ab} = 0\,.$$

For a detailed account of the non-isomorphic structure and the automorphisms of the Bianchi groups, see Tables 1 and 2.

It is easy to see that the form of the matrix A is sufficient to determine the particular Bianchi group:

- Bianchi I corresponds to any simply connected group whose centre is $\mathbb{R}^3$ and outer automorphism the three-dimensional general linear group. The matrix is zero ($A = 0$).
- Bianchi II corresponds to any simply connected group whose centre is $\mathbb{R}$ and outer automorphism the two-dimensional general linear group. The matrix is nilpotent, but not zero. The corresponding algebra is the Heisenberg algebra.
- Bianchi III corresponds to any simply connected group with centre $\mathbb{R}$ and outer automorphism the group of non-zero real numbers. The matrix A has one zero and one non-zero eigenvalue. The corresponding algebra is solvable and non-unimodular.
- Bianchi IV corresponds to any simply connected group with trivial centre and outer automorphism the product of the reals and a group of order 2. The matrix A has two equal non-zero eigenvalues, but it is not diagonisable.
- Bianchi V corresponds to any simply connected group with trivial centre and outer automorphism group the elements of the two-dimensional general linear group with determinant equal to 1 or -1. The matrix A has two eigenvalues and it is diagonisable.

TABLE 1. Properties of the Class A Bianchi Groups

Bianchi Group	Non-isomorphic Structure	Automorphism Group	Lie Algebra	Dimension of Automorphisms	Dimension of Isometries
I	$[\vec{\xi}_1,\vec{\xi}_2]=0$ $[\vec{\xi}_2,\vec{\xi}_3]=0$ $[\vec{\xi}_3,\vec{\xi}_1]=0$	$\mathcal{GL}(3,\mathbb{R})$	$\mathfrak{U}(1)^3$	9	6
II	$[\vec{\xi}_1,\vec{\xi}_2]=0$ $[\vec{\xi}_2,\vec{\xi}_3]=0$ $[\vec{\xi}_3,\vec{\xi}_1]=-\vec{\xi}_2$	$\begin{pmatrix}\det\lvert A\rvert & 0\\ \vec{v}^T & A\end{pmatrix}$ where $A\in\mathcal{GL}(2,\mathbb{R})$ and $\vec{v}\in\mathbb{R}^2$	$\mathfrak{Heis}_3$	6	4
VI_0	$[\vec{\xi}_1,\vec{\xi}_2]=0$ $[\vec{\xi}_2,\vec{\xi}_3]=0$ $[\vec{\xi}_3,\vec{\xi}_1]=-\vec{\xi}_1$	$\begin{pmatrix}c & -d & 0\\ d & c & 0\\ \vec{v}^T & & 1\end{pmatrix}$ where $c,d\in\mathbb{R}$, such that c or $d\neq 0$ and $\vec{v}\in\mathbb{R}^2$	$\mathfrak{iso}(1,1)$	4	3
VII_0	$[\vec{\xi}_1,\vec{\xi}_2]=0$ $[\vec{\xi}_2,\vec{\xi}_3]=\vec{\xi}_1$ $[\vec{\xi}_3,\vec{\xi}_1]=\vec{\xi}_2$	$\begin{pmatrix}-c & d & 0\\ d & c & 0\\ \vec{v}^T & & 1\end{pmatrix}$ where $c,d\in\mathbb{R}$, such that c or $d\neq 0$ and $\vec{v}\in\mathbb{R}^2$	$\mathfrak{iso}(2)$	4	3
$VIII$	$[\vec{\xi}_1,\vec{\xi}_2]=\vec{\xi}_1$ $[\vec{\xi}_2,\vec{\xi}_3]=\vec{\xi}_3$ $[\vec{\xi}_3,\vec{\xi}_1]=-2\vec{\xi}_2$	$\mathcal{SO}(2,\mathbb{R})$	$\mathfrak{so}(2,1)$	3	3, 4
IX	$[\vec{\xi}_1,\vec{\xi}_2]=\vec{\xi}_3$ $[\vec{\xi}_2,\vec{\xi}_3]=\vec{\xi}_1$ $[\vec{\xi}_3,\vec{\xi}_1]=\vec{\xi}_2$	$\mathcal{SO}(3)$	$\mathfrak{so}(3)$	3	3, 4

- Bianchi VI_0 corresponds to any simply connected group with trivial centre and outer automorphism the product of the positive real numbers with the dihedral group of order 8. The matrix A has non-zero distinct real eigenvalues with zero sum. The corresponding Lie algebra of the two-dimensional Poincaré group, *i.e.* the group of isometries of two-dimensional Minkowski space.
- Bianchi VI_h corresponds to any simply connected group with trivial center and outer automorphism group a product of the non-zero real numbers and a group of order 2. The matrix A has non-zero distinct real eigenvalues with non-zero sum. The corresponding algebra is solvable and non-unimodular.
- Bianchi VII_0 corresponds to any simply connected group with trivial centre and outer automorphism a product of the non-zero real numbers and a group of order 2. The matrix A has non-zero imaginary eigenvalues.

- Bianchi VII_h corresponds to any simply connected group with trivial centre and outer automorphism group the non-zero real numbers. The matrix A has strictly complex (non-real and non-imaginary) eigenvalues.
- Bianch $VIII$ corresponds to any simply connected group with centre $\mathbb{Z}$ and its outer automorphism group has order 2. The corresponding algebra is that of a two-dimensional special linear group of traceless 2×2 matrices, which is simple and unimodular.
- Bianchi IX corresponds to any simply connected group with centre of order 2 and trivial outer automorphism group. This is equivalent to a spin group. The corresponding Lie algebra is that of a two-dimensional special orthogonal group, which is simple and unimodular.

Table 2. Properties of the Class B Bianchi Groups

Bianchi Group	Non-isomorphic Structure	Automorphism Group	Lie Algebra	Dimension of Automrphisms	Dimension of Isometries
III	$[\vec{\xi}_1, \vec{\xi}_2] = 0$ $[\vec{\xi}_2, \vec{\xi}_3] = 0$ $[\vec{\xi}_3, \vec{\xi}_1] = -\vec{\xi}_1$	$\begin{pmatrix} 1 & \vec{v} & \\ 0 & c & d \\ 0 & d & c \end{pmatrix}$ where $c, d \in \mathbb{R}$, such that c or $d \neq 0$ and $\vec{v} \in \mathbb{R}^2$	$\mathfrak{R} \times \mathfrak{l}_2$	4	4
IV	$[\vec{\xi}_1, \vec{\xi}_2] = 0$ $[\vec{\xi}_2, \vec{\xi}_3] = \vec{\xi}_2$ $[\vec{\xi}_3, \vec{\xi}_1] = -\vec{\xi}_1 - \vec{\xi}_2$	$\begin{pmatrix} 1 & \vec{v} & \\ 0 & c & d \\ 0 & 0 & c \end{pmatrix}$ where $c, d \in \mathbb{R}$, such that c or $d \neq 0$ and $\vec{v} \in \mathbb{R}^2$	$\mathfrak{R} \times \mathfrak{l}_2$	4	3
V	$[\vec{\xi}_1, \vec{\xi}_2] = 0$ $[\vec{\xi}_2, \vec{\xi}_3] = \vec{\xi}_2$ $[\vec{\xi}_3, \vec{\xi}_1] = -\vec{\xi}_1$	$\begin{pmatrix} 1 & \vec{v} \\ 0 & A \end{pmatrix}$ where $A \in \mathcal{GL}(2, \mathbb{R})$ and $\vec{v} \in \mathbb{R}^2$	$\mathfrak{R} \times \mathfrak{gl}(2)$	6	6
VI_h	$[\vec{\xi}_1, \vec{\xi}_2] = 0$ $[\vec{\xi}_2, \vec{\xi}_3] = h\vec{\xi}_2$ $[\vec{\xi}_3, \vec{\xi}_1] = -\vec{\xi}_1$	$\begin{pmatrix} 1 & \vec{v} & \\ 0 & c & d \\ 0 & d & c \end{pmatrix}$ where $c, d \in \mathbb{R}$, such that c or $d \neq 0$ and $\vec{v} \in \mathbb{R}^2$	$\mathfrak{R} \rtimes \mathfrak{R}^2$	4	3
VII_h	$[\vec{\xi}_1, \vec{\xi}_2] = 0$ $[\vec{\xi}_2, \vec{\xi}_3] = \vec{\xi}_1 + h\vec{\xi}_2$ $[\vec{\xi}_3, \vec{\xi}_1] = \vec{\xi}_2 - h\vec{\xi}_1$	$\begin{pmatrix} 1 & \vec{v} & \\ 0 & c & d \\ 0 & -d & c \end{pmatrix}$ where $c, d \in \mathbb{R}$, such that c or $d \neq 0$ and $\vec{v} \in \mathbb{R}^2$	$\mathfrak{R} \rtimes \mathfrak{R}^2$	4	3, 6

Also, from the contractions of the Jacobi identity, we easily prove that the Killing form,

(8) $$C^n{}_{ma} C^m{}_{nb} = C^n{}_{mb} C^m{}_{na} \,,$$

is a symmetric rank-2 tensor, and thus the following contraction between structure constants is also zero,

(9) $$C^n{}_{ab} C^m{}_{mn} = 0 \,.$$

These ensure the prerequisites of our theorem for the construction of a space-time.

Let us now consider the action of the group on the space-time. In general, there are three manners a group can act on a manifold:

(1) A group acts by *isometries* when the Lie derivative of the metric along the generators of the group vanishes

(10) $$\mathcal{L}_{\vec{\xi}_a} \boldsymbol{g} = 0 \,,$$

in which case, moving along the orbits of the group, distance are preserved. Examples of such actions include the usual 'physical symmetries' observed in many well-known solutions of General Relativity, such as translations, rotations and Lorentzian boosts. Generators of such a group are known as *Killing vector fields* of the manifold.[2]

Given two vectors, $\vec{x}$ and $\vec{y}$, and a metric, $\boldsymbol{g}$, the definition of an isometry leads to the Killing equation

(11) $$\boldsymbol{g}\big(\nabla_{\vec{x}} \vec{\xi}_a, \vec{y}\big) + \boldsymbol{g}\big(\vec{x}, \nabla_{\vec{\xi}_a} \vec{y}\big) = 0 \,.$$

(2) A group acts by *homotheties* when the Lie derivative of the metric along the generators of the group is

(12) $$\mathcal{L}_{\vec{\xi}_a} \boldsymbol{g} = \phi_a \boldsymbol{g} \,,$$

where $\phi_a \in \mathbb{R}$ are scalar constants, dependent only on the particular generator and g the 4-dimensional metric. In this case, moving along the orbits of the group, distances grow (or shrink) proportionately;

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[2]For example, in Minkowski space-time, in the standard coordinates,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \,,$$

the Killing vector fields correspond to the translations with respect to space and time,

$$\vec{\xi}_0 = \frac{\partial}{\partial t} \;,\quad \vec{\xi}_1 = \frac{\partial}{\partial x_1} \;,\quad \vec{\xi}_2 = \frac{\partial}{\partial x_2} \quad \text{and} \quad \vec{\xi}_3 = \frac{\partial}{\partial x_3} \,,$$

the rotations about all three spatial axes,

$$\vec{\xi}_4 = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \;,\quad \vec{\xi}_5 = -x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} \quad \text{and} \quad \vec{\xi}_6 = -x_1 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_1} \,,$$

and the Lorentzian boost along all three spatial axes,

$$\vec{\xi}_7 = x_1 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_1} \;,\quad \vec{\xi}_8 = x_2 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_2} \quad \text{and} \quad \vec{\xi}_9 = x_3 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_3} \,.$$

that is, an action by homotheties implies uniform scaling of 'objects' moving along the orbits of the group. Generators of such a group are known as *homothetic vectors* or *homothetic collineations*.[3]

Given two vectors, $\vec{x}$ and $\vec{y}$, and a metric, g, the definition of a homothety leads to the homothetic Killing equation

(13) $$\boldsymbol{g}\big(\nabla_{\vec{x}}\vec{\xi}_a, \vec{y}\big) + \boldsymbol{g}\big(\vec{x}, \nabla_{\vec{\xi}_a}\vec{y}\big) = 2\phi_a \boldsymbol{g}\big(\vec{x}, \vec{y}\big)\,.$$

(3) A group acts by *conformal symmetries* when the Lie derivative of the metric along the generators of the group is

(14) $$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{g} = \phi_a(x)\boldsymbol{g}\,,$$

where $\phi_a(x) \in \mathbb{R}$ are scalar functions, dependent both on the particular generator and g, and on the point x of the manifold. In this case, moving along orbits of the group, angles are preserved but distances are not. Generators of such a group are known as *conformal Killing vectors* or *conformal collineations*.[4]

Given two vectors, $\vec{x}$ and $\vec{y}$, and a metric, g, the definition of a homothety leads to the homothetic Killing equation

(15) $$\boldsymbol{g}\big(\nabla_{\vec{x}}\vec{\xi}_a, \vec{y}\big) + \boldsymbol{g}\big(\vec{x}, \nabla_{\vec{\xi}_a}\vec{y}\big) = \frac{2}{n}\boldsymbol{g}\big(\vec{x}, \vec{y}\big)\mathrm{div}_{\boldsymbol{g}}\left(\vec{\xi}_a\right)\,.$$

These groups can act on either a spacelike hypersurface, $\mathcal{S}_3$, a timelike hypersurface $\mathcal{T}_3$, or a light-like hypersurface, $\mathcal{N}_3$, deeming the space-time $\mathcal{V}$ *hypersurface homogeneous* (Stephani *et al.* 2003). As a result, the orbits

[3]For example, in the space-time defined by the metric [**44**]

$$\mathrm{d}s^2 = -e^{\frac{t}{r}}\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega\,,$$

there are three homothetic vectors: a scaling along the radial direction

$$\vec{\xi}_1 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}\,,$$

and three rotations, along angles θ and ϕ

$$\vec{\xi}_2 = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\;\;,\;\;\vec{\xi}_3 = -\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\quad\text{and}\quad\xi_4 = \frac{\partial}{\partial\phi}\,.$$

[4]For example, the space-time where the Bianchi III group acts by isometries,

$$\mathrm{d}s^2 = A^2(t)\left(e^{m\lambda t}\big(-\mathrm{d}t^2 + \mathrm{d}x_1^2\big) + e^{m(\lambda-1)t}\mathrm{d}x_2^2 + e^{-2x_1}\mathrm{d}x_3^2\right)\,,$$

admits one proper conformal symmetry

$$\vec{\xi}_1 = \frac{2}{m}\frac{\partial}{\partial t} + x_2\frac{\partial}{\partial x_2} + \lambda x_3\frac{\partial}{\partial x_3}\,,$$

for given parameters $m\,,\lambda \in \mathbb{R}$ and arbitrary function $A(t)$.
Similarly, the space-time where the Bianchi V group acts by isometries,

$$\mathrm{d}s^2 = A^2(t)\left(e^{m\lambda t}\big(-\mathrm{d}t^2 + \mathrm{d}x_1^2\big) + e^{2x_1}\left(e^{m(\lambda-1)t}\mathrm{d}x_2^2 + \mathrm{d}x_3^2\right)\right)\,,$$

admits one proper conformal symmetry

$$\vec{\xi}_1 = \frac{2}{m}\frac{\partial}{\partial t} + x_2\frac{\partial}{\partial x_2} + \lambda x_3\frac{\partial}{\partial x_3}\,,$$

for given parameters $m\,,\lambda \in \mathbb{R}$ and arbitrary function $A(t)$ [**45**].

of the group are identified with the homogeneous three-dimensional hypersurfaces (the *foliation*) of the space-time; as for the geodesics on the three-dimensional hpersurfaces are adjoint representations of the particular group orbits - of dimension 3 or less.

In the case considered, we will assume that the group acts by homotheties. As a result,

$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{g} = \phi_a \boldsymbol{g} \,. \tag{16}$$

We should notice that the multiplication of the homothety constants to the structure constants sums to zero,

$$\phi_i C^i{}_{ab} = 0 \,. \tag{17}$$

2.1. Preliminary relations between vectors. In order to proceed, we will need to calculate the covariant and the Lie derivatives of arbitrary vectors along arbitrary directions. In particular, we can calculate inner products of vector fields with covariant derivatives of vector fields. The first part is just Lie derivatives that do not need the specification of the space-time connection. Consider three vector fields, $\vec{v}$, $\vec{u}$ and $\vec{w}$, defined on the space-time. Then, the Lie derivatives of their inner products are

(18)
$$\begin{aligned}
&\vec{w}\big(\boldsymbol{g}(\vec{v},\vec{u})\big) = \mathcal{L}_{\vec{w}}\big(\boldsymbol{g}(\vec{v},\vec{u})\big) = \big(\mathcal{L}_{\vec{w}}g\big)(\vec{v},\vec{u}) + \boldsymbol{g}\big([\vec{w},\vec{v}],\vec{u}\big) - \boldsymbol{g}\big([\vec{u},\vec{w}],\vec{v}\big) \ , \\
&\vec{v}\big(\boldsymbol{g}(\vec{u},\vec{w})\big) = \mathcal{L}_{\vec{v}}\big(\boldsymbol{g}(\vec{u},\vec{w})\big) = \big(\mathcal{L}_{\vec{v}}\boldsymbol{g}\big)(\vec{u},\vec{w}) + \boldsymbol{g}\big([\vec{v},\vec{u}],\vec{w}\big) - \boldsymbol{g}\big([\vec{w},\vec{v}],\vec{u}\big) \ \text{and} \\
&\vec{u}\big(\boldsymbol{g}(\vec{w},\vec{v})\big) = \mathcal{L}_{\vec{u}}\big(\boldsymbol{g}(\vec{w},\vec{v})\big) = \big(\mathcal{L}_{\vec{u}}\boldsymbol{g}\big)(\vec{w},\vec{v}) + \boldsymbol{g}\big([\vec{u},\vec{w}],\vec{v}\big) - \boldsymbol{g}\big([\vec{v},\vec{u}],\vec{w}\big)
\end{aligned}$$

The corresponding relations for the covariant derivatives follow

(19)
$$\begin{aligned}
&\vec{w}\big(\boldsymbol{g}(\vec{v},\vec{u})\big) = \nabla_{\vec{w}}\big(\boldsymbol{g}(\vec{v},\vec{u})\big) = \boldsymbol{g}\big(\nabla_{\vec{w}}\vec{v},\vec{u}\big) + \boldsymbol{g}\big(\nabla_{\vec{u}}\vec{w},\vec{v}\big) - \boldsymbol{g}\big([\vec{u},\vec{w}],\vec{v}\big) \ , \\
&\vec{v}\big(\boldsymbol{g}(\vec{u},\vec{w})\big) = \nabla_{\vec{v}}\big(\boldsymbol{g}(\vec{u},\vec{w})\big) = \boldsymbol{g}\big(\nabla_{\vec{v}}\vec{u},\vec{w}\big) + \boldsymbol{g}\big(\nabla_{\vec{w}}\vec{v},\vec{u}\big) - \boldsymbol{g}\big([\vec{w},\vec{v}],\vec{u}\big) \ \text{and} \\
&\vec{u}\big(\boldsymbol{g}(\vec{w},\vec{v})\big) = \nabla_{\vec{u}}\big(\boldsymbol{g}(\vec{w},\vec{v})\big) = \boldsymbol{g}\big(\nabla_{\vec{u}}\vec{w},\vec{v}\big) + \boldsymbol{g}\big(\nabla_{\vec{v}}\vec{u},\vec{w}\big) - \boldsymbol{g}\big([\vec{v},\vec{u}],\vec{w}\big) \,.
\end{aligned}$$

This leads to formulae which are more or less the coordinate free versions of the familiar formulae for the Christoffel symbols of the first kind, with all indices lower. There are commutator terms because the vector fields aren't partial derivatives with respect to a system of coordinates and the derivatives of the metric are Lie derivatives. First of all, for any three vectors $\vec{x}_\alpha\,, \vec{x}_\beta\,, \vec{x}_\gamma \in T\mathcal{V}_4$, we have

$$\begin{aligned}
&\boldsymbol{g}\big(\nabla_{\vec{x}_\alpha}\vec{x}_\beta,\vec{x}_\gamma\big) + \boldsymbol{g}\big(\nabla_{\vec{x}_\gamma}\vec{x}_\alpha,\vec{x}_\beta\big) = \big(\mathcal{L}_{\vec{x}_\alpha}g\big)(\vec{x}_\beta,\vec{x}_\gamma) + \boldsymbol{g}\big([\vec{x}_\alpha,\vec{x}_\beta],\vec{x}_\gamma\big) \ , \\
&\boldsymbol{g}\big(\nabla_{\vec{x}_\beta}\vec{x}_\gamma,\vec{x}_\alpha\big) + \boldsymbol{g}\big(\nabla_{\vec{x}_\alpha}\vec{x}_\beta,\vec{x}_\gamma\big) = \big(\mathcal{L}_{\vec{x}_\beta}\boldsymbol{g}\big)(\vec{x}_\gamma,\vec{x}_\alpha) + \boldsymbol{g}\big([\vec{x}_\beta,\vec{x}_\gamma],\vec{x}_\alpha\big) \ \text{and} \\
&\boldsymbol{g}\big(\nabla_{\vec{x}_\gamma}\vec{x}_\alpha,\vec{x}_\beta\big) + \boldsymbol{g}\big(\nabla_{\vec{x}_\beta}\vec{x}_\gamma,\vec{x}_\alpha\big) = \big(\mathcal{L}_{\vec{x}_\gamma}\boldsymbol{g}\big)(\vec{x}_\alpha,\vec{x}_\beta) + \boldsymbol{g}\big([\vec{x}_\gamma,\vec{x}_\alpha],\vec{x}_\beta\big) \,.
\end{aligned}$$

Then, the Christoffel symbols of the first kind for generic directions can be given as

(20)

$$\boldsymbol{g}\big(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\alpha\big)=\Gamma_{\alpha\beta\gamma}=\frac{1}{2}\Big(\big(\mathcal{L}_{\vec{x}_\gamma}\boldsymbol{g}\big)(\vec{x}_\beta,\vec{x}_\alpha)+\big(\mathcal{L}_{\vec{x}_\beta}\boldsymbol{g}\big)(\vec{x}_\alpha,\vec{x}_\gamma)-\big(\mathcal{L}_{\vec{x}_\alpha}\boldsymbol{g}\big)(\vec{x}_\beta,\vec{x}_\gamma)\Big)$$
$$+\frac{1}{2}\Big(\boldsymbol{g}\big([\vec{x}_\gamma,\vec{x}_\beta],\vec{x}_\alpha\big)+\boldsymbol{g}\big([\vec{x}_\beta,\vec{x}_\alpha],\vec{x}_\gamma\big)-\boldsymbol{g}\big([\vec{x}_\alpha,\vec{x}_\gamma],\vec{x}_\beta\big)\Big)\,.$$

These Christoffel symbols are not symmetric with respect to the second and third indices, as is the usual case; the symmetry is broken due to the skew-symmetric term $\frac{1}{2}\Big(\boldsymbol{g}\big([\vec{x}_\gamma,\vec{x}_\beta],\vec{x}_\alpha\big)$, and can easily be restored if we assume that vectros $\vec{x}_\beta$ and $\vec{x}_\gamma$ commute. The Christoffel symbols of the second kind are easily derived from these by raising the first index as

(21)

$$\Gamma^\alpha{}_{\beta\gamma}=g^{-1}(\mathbf{x}^\alpha,\mathbf{x}^\mu)\Gamma_{\mu\beta\gamma}$$
$$=\frac{1}{2}g^{-1}(\mathbf{x}^\alpha,\mathbf{x}^\mu)\Big(\big(\mathcal{L}_{\vec{x}_\gamma}\boldsymbol{g}\big)(\vec{x}_\beta,\vec{x}_\mu)+\big(\mathcal{L}_{\vec{x}_\beta}g\big)(\vec{x}_\mu,\vec{x}_\gamma)-\big(\mathcal{L}_{\vec{x}_\mu}\boldsymbol{g}\big)(\vec{x}_\gamma,\vec{x}_\beta)\Big)$$
$$+\frac{1}{2}g^{-1}(\mathbf{x}^\alpha,\mathbf{x}^\mu)\Big(\boldsymbol{g}\big([\vec{x}_\gamma,\vec{x}_\beta],\vec{x}_\mu\big)+\boldsymbol{g}\big([\vec{x}_\beta,\vec{x}_\mu],\vec{x}_\gamma\big)-\boldsymbol{g}\big([\vec{x}_\mu,\vec{x}_\gamma],\vec{x}_\beta\big)\Big)\,,$$

where $\mathbf{x}^\alpha$, $\mathbf{x}^\beta$ are the 1-forms corresponding to $\vec{u}$ and $\vec{y}$ - that is the 'index-lowered' 1-forms

$$\mathbf{x}=\mathbf{g}\big(\vec{x},\cdot\big)\,,$$

and conversely

$$\vec{x}=g^{-1}\big(\mathbf{x},\cdot\big)\,.$$

It is worth pointing out that vector $\vec{x}$ and 1-form $\mathbf{x}$ are not necessarily dual, since the assumption of a tetrad basis $\big\{\vec{x}_\alpha\big\}_{\alpha=0,1,2,3}$ has not been made. If we allow this assumption, *i.e*, that $\big\{\vec{x}_\alpha\big\}_{\alpha=0,1,2,3}$ is a basis of the tangent space $T\mathcal{V}_4$, then $\big\{\mathbf{x}^\alpha\big\}_{\alpha=0,1,2,3}$ is a basis of the dual space $T^*\mathcal{V}_4$, such that

$$\mathbf{x}^\alpha\big(\vec{x}_\beta\big)=\delta^\alpha{}_\beta\,.$$

However, this assumption is not necessary, in the sense that these relations can hold for any number of vectors $\vec{x}$.[5]

Obviously, the covariant derivatives of any vector field $\vec{x}_\alpha$ with respect to any other $\vec{x}_\beta$ (even the same) can be given as usual, utilizing the Christoffel symbols of the second kind as,

$$\nabla_{\vec{x}_\beta}\vec{x}_\alpha=\Gamma^\mu{}_{\alpha\beta}\vec{x}_\mu\,, \tag{22}$$

where the Einstein notation is being used (*i.e.*, this relation assumes a summation over all μ's).

Another important relation, before we turn to Einstein's equations, is mixing the Lie and the covariant derivative of vectors. Let us assume that in

[5]This means that $\mathbf{x}^\alpha$ is not uniquely defined by $\vec{x}_\alpha$.

Eq. (18), one of the vectors multiplied is the covariant derivative of two other vectors; hence, if

$$\mathcal{L}_{\vec{x}_\alpha}\big(\boldsymbol{g}(\vec{v},\vec{x}_\beta)\big) = \big(\mathcal{L}_{\vec{x}_\alpha}\boldsymbol{g}\big)(\vec{v},\vec{x}_\beta) + \boldsymbol{g}\big([\vec{x}_\alpha,\vec{v}],\vec{x}_\beta\big) - \boldsymbol{g}\big([\vec{x}_\beta,\vec{x}_\alpha],\vec{v}\big)$$

then $\vec{v} = \nabla_{\vec{x}_\delta}\vec{x}_\gamma$, so that

(23)

$$\mathcal{L}_{\vec{x}_\alpha}\big(\boldsymbol{g}(\nabla_{\vec{x}_\delta}\vec{x}_\gamma,\vec{x}_\beta)\big) = \big(\mathcal{L}_{\vec{x}_\alpha}\boldsymbol{g}\big)(\nabla_{\vec{x}_\delta}\vec{x}_\gamma,\vec{x}_\beta)+\boldsymbol{g}\big([\vec{x}_\alpha,\nabla_{\vec{x}_\delta}\vec{x}_\gamma],\vec{x}_\beta\big)-\boldsymbol{g}\big([\vec{x}_\beta,\vec{x}_\alpha],\nabla_{\vec{x}_\delta}\vec{x}_\gamma\big)\,.$$

Similarly, from Eq. (20), we can compute the inner product of a covariant derivative and a Lie derivative as

(24)

$$\begin{aligned}2\boldsymbol{g}\big(\nabla_{\vec{x}_\delta}\vec{x}_\gamma,[\vec{x}_\alpha,\vec{x}_\beta]\big) =&\big(\mathcal{L}_{\vec{x}_\gamma}\boldsymbol{g}\big)\big([\vec{x}_\alpha,\vec{x}_\beta],\vec{x}_\delta\big) + \big(\mathcal{L}_{\vec{x}_\delta}\boldsymbol{g}\big)\big([\vec{x}_\alpha,\vec{x}_\beta],\vec{x}_\gamma\big) - \big(\mathcal{L}_{[\vec{x}_\alpha,\vec{x}_\beta]}\boldsymbol{g}\big)\big(\vec{x}_\delta,\vec{x}_\gamma\big)\\ &+\boldsymbol{g}\big(\big[\vec{x}_\gamma,[\vec{x}_\alpha,\vec{x}_\beta]\big],\vec{x}_\delta\big)+\boldsymbol{g}\big(\big[\vec{x}_\delta,[\vec{x}_\alpha,\vec{x}_\beta]\big],\vec{x}_\gamma\big)-\boldsymbol{g}\big([\vec{x}_\alpha,\vec{x}_\beta],[\vec{x}_\gamma,\vec{x}_\delta]\big)\,.\end{aligned}$$

So, from Eqs. (23) and (24), we may get

(25)

$$\begin{aligned}2\boldsymbol{g}\big(\mathcal{L}_{\vec{x}_\alpha}\nabla_{\vec{x}_\delta}\vec{x}_\gamma,\vec{x}_\beta\big) =&2\mathcal{L}_{\vec{x}_\alpha}\,\big(\boldsymbol{g}\big(\nabla_{\vec{x}_\delta}\vec{x}_\gamma,\vec{x}_\beta\big)\big) - 2\big(\mathcal{L}_{\vec{x}_\alpha}\boldsymbol{g}\big)\big(\nabla_{\vec{x}_\delta}\vec{y}_\gamma,\vec{x}_\beta\big)\\ &-\big(\mathcal{L}_{\vec{x}_\gamma}\boldsymbol{g}\big)\big([\vec{x}_\alpha,\vec{x}_\beta],\vec{x}_\delta\big)-\big(\mathcal{L}_{\vec{x}_\delta}\boldsymbol{g}\big)\big([\vec{x}_\alpha,\vec{x}_\beta],\vec{x}_\gamma\big)+\big(\mathcal{L}_{[\vec{x}_\alpha,\vec{x}_\beta]}\boldsymbol{g}\big)\big(\vec{x}_\gamma,\vec{x}_\delta\big)\\ &-\boldsymbol{g}\big(\big[\vec{x}_\gamma,[\vec{x}_\alpha,\vec{x}_\beta],\vec{x}_\delta\big)-\boldsymbol{g}\big(\big[\vec{x}_\delta,[\vec{x}_\alpha,\vec{x}_\beta],\vec{x}_\gamma\big)+\boldsymbol{g}\big([\vec{x}_\gamma,\vec{x}_\delta],[\vec{x}_\alpha,\vec{x}_\beta]\big)\,,\end{aligned}$$

where $\big(\mathcal{L}_{[\vec{x}_\alpha,\vec{x}_\beta]}\boldsymbol{g}\big)\big(\vec{x}_\gamma,\vec{x}_\delta\big) = \big(\mathcal{L}_{\vec{x}_\alpha}\mathcal{L}_{\vec{x}_\beta}\boldsymbol{g}\big)(\vec{x}_\gamma,\vec{x}_\delta) - \big(\mathcal{L}_{\vec{x}_\beta}\mathcal{L}_{\vec{x}_\alpha}\boldsymbol{g}\big)(\vec{x}_\gamma,\vec{x}_\delta)$.

The Lie derivative of the connection (Eq. (20)) along $\vec{w}$ is

(26)

$$\begin{aligned}2\mathcal{L}_{\vec{x}_\alpha}\,\big(\boldsymbol{g}\big(\nabla_{\vec{x}_\delta}\vec{x}_\gamma,\vec{x}_\beta\big)\big) =&\big(\mathcal{L}_{\vec{x}_\alpha}\mathcal{L}_{\vec{x}_\beta}\boldsymbol{g}\big)(\vec{x}_\gamma,\vec{x}_\delta)-\big(\mathcal{L}_{\vec{x}_\beta}\mathcal{L}_{\vec{x}_\gamma}\boldsymbol{g}\big)(\vec{x}_\gamma,\vec{x}_\gamma)+\big(\mathcal{L}_{\vec{x}_\alpha}\mathcal{L}_{\vec{x}_\delta}\boldsymbol{g}\big)(\vec{x}_\beta,\vec{x}_\gamma)\\ &+\big(\mathcal{L}_{\vec{x}_\gamma}\boldsymbol{g}\big)\big([\vec{x}_\alpha,\vec{x}_\beta],\vec{x}_\delta\big)+\big(\mathcal{L}_{\vec{x}_\gamma}\boldsymbol{g}\big)\big(\vec{x}_\beta,[\vec{x}_\alpha,\vec{x}_\delta]\big)\\ &-\big(\mathcal{L}_{\vec{x}_\beta}\boldsymbol{g}\big)\big([\vec{x}_\alpha,\vec{z}_\gamma],\vec{x}_\delta\big)-\big(\mathcal{L}_{\vec{x}_\beta}\boldsymbol{g}\big)\big(\vec{x}_\gamma,[\vec{x}_\alpha,\vec{x}_\delta]\big)\\ &+\big(\mathcal{L}_{\vec{x}_\delta}\boldsymbol{g}\big)\big([\vec{x}_\alpha,\vec{x}_\gamma],\vec{x}_\beta\big)+\big(\mathcal{L}_{\vec{x}_\delta}\boldsymbol{g}\big)\big(\vec{x}_\gamma,[\vec{x}_\alpha,\vec{x}_\beta]\big)\\ &+\boldsymbol{g}\big(\big[[\vec{x}_\gamma,\vec{x}_\beta],\vec{x}_\alpha\big],\vec{x}_\delta\big)+\boldsymbol{g}\big([\vec{x}_\gamma,\vec{x}_\beta],[\vec{x}_\alpha,\vec{x}_\delta]\big)\\ &-\boldsymbol{g}\big(\big[[\vec{x}_\gamma,\vec{x}_\delta],\vec{x}_\alpha\big],\vec{x}_\beta\big)-\boldsymbol{g}\big([\vec{x}_\gamma,\vec{x}_\delta],[\vec{x}_\alpha,\vec{x}_\beta]\big)\\ &+\boldsymbol{g}\big(\big[[\vec{x}_\beta,\vec{x}_\delta],\vec{x}_\alpha\big],\vec{x}_\gamma\big)+\boldsymbol{g}\big([\vec{x}_\beta,\vec{x}_\delta],[\vec{x}_\alpha,\vec{x}_\gamma]\big)\quad.\end{aligned}$$

Substituting Eq. (36) to Eq. (35), we obtain a relation that decomposes the Lie derivative of the covariant derivative of a vector along another vector to a series of Lie derivatives,

(27)

$$\begin{aligned}2\boldsymbol{g}\big(\mathcal{L}_{\vec{x}_\alpha}\nabla_{\vec{x}_\delta}\vec{x}_\gamma,\vec{x}_\beta\big) =&-2\big(\mathcal{L}_{\vec{x}_\alpha}\boldsymbol{g}\big)\big(\nabla_{\vec{x}_\delta}\vec{x}_\gamma,\vec{x}_\beta\big)\\ &+\big(\mathcal{L}_{\vec{x}_\alpha}\mathcal{L}_{\vec{x}_\gamma}\boldsymbol{g}\big)\big(\vec{x}_\beta,\vec{x}_\delta\big)+\big(\mathcal{L}_{\vec{x}_\alpha}\mathcal{L}_{\vec{x}_\delta}\boldsymbol{g}\big)\big(\vec{x}_\beta,\vec{x}_\gamma\big)-\big(\mathcal{L}_{\vec{x}_\beta}\mathcal{L}_{\vec{x}_\alpha}\boldsymbol{g}\big)\big(\vec{x}_\gamma,\vec{x}_\delta\big)\\ &+\big(\mathcal{L}_{\vec{x}_\delta}\boldsymbol{g}\big)\big(\vec{x}_\beta,[\vec{x}_\alpha,\vec{x}_\delta]\big)+\big(\mathcal{L}_{\vec{x}_\delta}\boldsymbol{g}\big)\big(\vec{x}_\beta,[\vec{x}_\alpha,\vec{x}_\gamma]\big)\\ &-\big(\mathcal{L}_{\vec{x}_\beta}\boldsymbol{g}\big)\big(\vec{x}_\delta,[\vec{x}_\alpha,\vec{x}_\gamma]\big)-\big(\mathcal{L}_{\vec{x}_\beta}\boldsymbol{g}\big)\big(\vec{x}_\gamma,[\vec{x}_\alpha,\vec{x}_\delta]\big)\\ &+\boldsymbol{g}\big(\big[[\vec{x}_\alpha,\vec{x}_\gamma],\vec{x}_\beta\big],\vec{x}_\delta\big)+\boldsymbol{g}\big(\big[[\vec{x}_\alpha,\vec{x}_\delta],\vec{x}_\beta\big],\vec{x}_\gamma\big)+\boldsymbol{g}\big(\big[[\vec{x}_\gamma,\vec{x}_\delta],\vec{x}_\alpha\big],\vec{x}_\beta\big)\\ &+\boldsymbol{g}\big([\vec{x}_\gamma,\vec{x}_\beta],[\vec{x}_\alpha,\vec{x}_\delta]\big)+\boldsymbol{g}\big([\vec{x}_\beta,\vec{x}_\delta],[\vec{x}_\alpha,\vec{x}_\gamma]\big)\,.\end{aligned}$$

This relation can be used to examine whether these two vectors are orthogonal or not; if they are, the above relation should be zero.

2.2. Riemann and Ricci curvature in a generalised frame. Curvature of (pseudo-)Riemannian manifolds is defined as the difference between the original and the final state of the parallel transport of a vector along a closed loop. This curvature is measured by the Riemann-Christoffel tensor, that is defined as

$$\boldsymbol{R}\left(\vec{v},\vec{u}\right)\vec{w} = \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} - \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{[\vec{v},\vec{u}]}\vec{w}\,, \tag{28}$$

for any vector fields $\vec{v}$, $\vec{u}$ and $\vec{w}$ on the manifold. Assuming a tetrad basis $\{\vec{x}_\alpha\}_{\alpha=0,1,2,3}$, this would be written as

$$R^{\alpha}{}_{\beta\gamma\delta}x_\alpha = \boldsymbol{R}\left(x_\gamma,x_\delta\right)x_\beta$$

Obviously, the last two indices of the Riemann tensor (γ and δ) are antisymmetric, regardless of the frame on which it is written.

Given any vectors $\vec{x}^\alpha$, $\vec{x}^\beta$, $\vec{x}^\gamma$ (not necessarily a tetrad basis), we have

$$\boldsymbol{R}\big(\vec{x}_\gamma,\vec{x}_\delta\big)\vec{x}_\alpha = \nabla_{\vec{x}_\gamma}\nabla_{\vec{x}_\delta}\vec{x}_\alpha - \nabla_{\vec{x}_\delta}\nabla_{\vec{x}_\gamma}\vec{x}_\alpha - \nabla_{[\vec{x}_\gamma,\vec{x}_\delta]}\vec{x}_\alpha\,, \tag{29}$$

where the covariant derivatives are given as

$$\nabla_{\vec{x}_\gamma}\vec{x}_\alpha = \Gamma^{\mu}{}_{\alpha\gamma}\vec{x}_\mu \tag{30}$$

and, subsequently, the second-order covariant derivatives are give as

(31)
$$\begin{aligned}\nabla_{\vec{x}_\gamma}\nabla_{\vec{x}_\delta}\vec{x}_\alpha =&\nabla_{\vec{x}_\gamma}\left(\Gamma^{\mu}{}_{\alpha\delta}\vec{x}_\mu\right) = \\ =&\big(\mathcal{L}_{\vec{x}_\gamma}\Gamma^{\mu}{}_{\alpha\delta}\big)\vec{x}_\mu + \Gamma^{mu}{}_{\alpha\delta}\nabla_{\vec{x}_\gamma}\vec{x}_\mu = \big(\mathcal{L}_{\vec{x}_\gamma}\Gamma^{\mu}{}_{\alpha\delta}\big)\vec{x}_\mu + \Gamma^{\mu}{}_{\alpha\delta}\Gamma^{\nu}{}_{\gamma\mu}\vec{x}_\nu\end{aligned}$$

We note again that the Christoffel symbols are matrices, not tensors; hence, they must be differentiated accordingly. As a result, the Riemann tensor can be written as

(32)
$$\boldsymbol{R}\big(\vec{x}_\gamma,\vec{x}_\delta\big)\vec{x}_\alpha = \big(\mathcal{L}_{\vec{x}_\gamma}\Gamma^{\mu}{}_{\alpha\delta}\big)\vec{x}_\mu - \big(\mathcal{L}_{\vec{x}_\delta}\Gamma^{\mu}{}_{\alpha\gamma}\big)\vec{x}_\mu + \Gamma^{\mu}{}_{\alpha\delta}\Gamma^{\nu}{}_{\gamma\mu}\vec{x}_\nu - \Gamma^{\mu}{}_{\alpha\gamma}\Gamma^{\nu}{}_{\delta\mu}\vec{x}_\nu - C^{\mu}{}_{\gamma\delta}\Gamma^{\nu}{}_{\alpha\mu}\vec{x}_\nu\,.$$

The only problem is to specify the Lie derivatives of the Christoffel symbols of the second kind. That is

$$\begin{aligned}\mathcal{L}_{\vec{x}_\delta}\Gamma^{\alpha}{}_{\beta\gamma} =&\mathcal{L}_{\vec{x}_\delta}\big(g^{-1}\left(\mathbf{x}^\alpha,\mathbf{x}^\mu\right)g\left(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\right)\big) = \\ =&\mathcal{L}_{\vec{x}_\delta}\big(g^{-1}\left(\mathbf{x}^\alpha,\mathbf{x}_\mu\right)\big)g\left(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\right) + g^{-1}\left(\mathbf{x}^\alpha,\mathbf{x}^\mu\right)\mathcal{L}_{\vec{x}_\delta}\big(g\left(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\right)\big) \\ =&-g^{-1}\big(\mathbf{x}^\alpha,\mathbf{x}^\kappa\big)g^{-1}\big(\mu,\lambda\big)\mathcal{L}_{\vec{x}_\delta}\big(g(\vec{x}_\kappa,\vec{x}_\lambda)\big)g\left(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\right) \\ &+g^{-1}\big(\mathbf{x}^\alpha,\mathbf{x}^\mu\big)\mathcal{L}_{\vec{x}_\delta}\big(g\left(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\right)\big)\,.\end{aligned}$$

To calculate this we require the Lie derivative of an inner product of two vectors ($\mathcal{L}_{\vec{x}_\delta}\big(g(\vec{x}_\kappa,\vec{x}_\lambda)\big)$), the inner product of a covariant derivative with a vector ($g\left(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\right)$), and the Lie derivative of the inner product of a covariant derivative with a vector ($\mathcal{L}_{\vec{x}_\delta}\big(g\left(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\right)\big)$). The first two are known, from Eqs. (18) and (20); the third is easy to compute if we substitute one of the vectors multiplied to Eq. (18) with the covariant derivative, so that

(33)
$$\mathcal{L}_{\vec{x}_\delta}\big(g(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu)\big) = \big(\mathcal{L}_{\vec{x}_\delta}g\big)(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu)+g\big([\vec{x}_\delta,\nabla_{\vec{x}_\gamma}\vec{x}_\beta],\vec{x}_\mu\big)-g\big([\vec{x}_\mu,\vec{x}_\delta],\nabla_{\vec{x}_\gamma}\vec{x}_\beta\big)\,.$$

From Eq. (20), we can compute the inner product of a covariant derivative and a Lie derivative as

(34)
$$\begin{aligned}2g\big(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,[\vec{x}_\mu,\vec{x}_\delta]\big) =&\big(\mathcal{L}_{\vec{x}_\beta}g\big)\big([\vec{x}_\mu,\vec{x}_\delta],\vec{x}_\gamma\big)+\big(\mathcal{L}_{\vec{x}_\gamma}g\big)\big([\vec{x}_\mu,\vec{x}_\delta],\vec{x}_\beta\big)-\big(\mathcal{L}_{[\vec{x}_\mu,\vec{x}_\delta]}g\big)\big(\vec{x}_\beta,\vec{x}_\gamma\big)\\ &+g\big([\vec{x}_\beta,[\vec{x}_\mu,\vec{x}_\delta]],\vec{x}_\gamma\big)+g\big([\vec{x}_\gamma,[\vec{x}_\mu,\vec{x}_\delta]],\vec{x}_\beta\big)-g\big([\vec{x}_\mu,\vec{x}_\delta],[\vec{x}_\beta,\vec{x}_\gamma]\big)\,.\end{aligned}$$

So, from Eqs. (23) and (24), we may get

(35)
$$\begin{aligned}2g\big(\mathcal{L}_{\vec{x}_\delta}\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\big) =&2\mathcal{L}_{\vec{x}_\delta}\,\big(g\big(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\big)\big)-2\big(\mathcal{L}_{\vec{x}_\delta}g\big)\big(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\big)\\ &-\big(\mathcal{L}_{\vec{x}_\beta}g\big)\big([\vec{x}_\delta,\vec{x}_\mu],\vec{x}_\gamma\big)-\big(\mathcal{L}_{\vec{x}_\gamma}g\big)\big([\vec{x}_\delta,\vec{x}_\mu],\vec{x}_\beta\big)+\big(\mathcal{L}_{[\vec{x}_\delta,\vec{x}_\mu]}g\big)\big(\vec{x}_\beta,\vec{x}_\gamma\big)\\ &-g\big([\vec{x}_\beta,[\vec{x}_\delta,\vec{x}_\mu],\vec{x}_\gamma\big)-g\big([\vec{x}_\gamma,[\vec{x}_\delta,\vec{x}_\mu],\vec{x}_\beta\big)+g\big([\vec{x}_\gamma,\vec{x}_\beta],[\vec{x}_\delta,\vec{x}_\mu]\big)\,,\end{aligned}$$

where $\big(\mathcal{L}_{[\vec{x}_\delta,\vec{x}_\mu]}g\big)\big(\vec{x}_\beta,\vec{x}_\gamma\big)=\big(\mathcal{L}_{\vec{x}_\delta}\mathcal{L}_{\vec{x}_\mu}g\big)(\vec{x}_\beta,\vec{x}_\gamma)-\big(\mathcal{L}_{\vec{x}_\mu}\mathcal{L}_{\vec{x}_\delta}g\big)(\vec{x}_\beta,\vec{x}_\gamma)$. Eventually, the Lie derivative of the connection (Eq. (20)) along $\vec{x}_\delta$ is

(36)
$$\begin{aligned}2\mathcal{L}_{\vec{x}_\delta}\,\big(g\big(\nabla_{\vec{x}_\gamma}\vec{x}_\beta,\vec{x}_\mu\big)\big) =&\big(\mathcal{L}_{\vec{x}_\delta}\mathcal{L}_{\vec{x}_\mu}g\big)(\vec{x}_\gamma,\vec{x}_\beta)-\big(\mathcal{L}_{\vec{x}_\mu}\mathcal{L}_{\vec{x}_\delta}g\big)(\vec{x}_\gamma,\vec{x}_\beta)+\big(\mathcal{L}_{\vec{x}_\delta}\mathcal{L}_{\vec{x}_\gamma}g\big)(\vec{x}_\mu,\vec{x}_\beta)\\ &+\big(\mathcal{L}_{\vec{x}_\beta}g\big)\big([\vec{x}_\delta,\vec{x}_\mu],\vec{x}_\gamma\big)+\big(\mathcal{L}_{\vec{x}_\beta}g\big)\big(\vec{x}_\mu,[\vec{x}_\delta,\vec{x}_\gamma]\big)\\ &-\big(\mathcal{L}_{\vec{x}_\mu}g\big)\big([\vec{x}_\delta,\vec{x}_\beta],\vec{x}_\gamma\big)-\big(\mathcal{L}_{\vec{x}_\mu}g\big)\big(\vec{x}_\beta,[\vec{x}_\delta,\vec{x}_\gamma]\big)\\ &+\big(\mathcal{L}_{\vec{x}_\gamma}g\big)\big([\vec{x}_\delta,\vec{x}_\beta],\vec{x}_\mu\big)+\big(\mathcal{L}_{\vec{x}_\gamma}g\big)\big(\vec{x}_\beta,[\vec{z}_\delta,\vec{x}_\mu]\big)\\ &+g\big([[\vec{x}_\beta,\vec{x}_\mu],\vec{x}_\delta],\vec{x}_\gamma\big)+g\big([\vec{x}_\beta,\vec{x}_\mu],[\vec{x}_\delta,\vec{x}_\gamma]\big)\\ &-g\big([[\vec{x}_\gamma,\vec{x}_\beta],\vec{x}_\delta],\vec{x}_\mu\big)-g\big([\vec{x}_\gamma,\vec{x}_\beta],[\vec{x}_\delta,\vec{x}_\mu]\big)\\ &+g\big([[\vec{x}_\mu,\vec{x}_\beta],\vec{x}_\delta],\vec{x}_\gamma\big)+g\big([\vec{x}_\mu,\vec{x}_\beta],[\vec{x}_\delta,\vec{x}_\gamma]\big)\,.\end{aligned}$$

Consequently, the Lie derivative of the Christoffel symbol is

(37)

$$\begin{aligned}\mathcal{L}_{\vec{x}_\delta}\Gamma^{\alpha}{}_{\beta\gamma} =&\frac{1}{2}g^{-1}\big(\mathbf{x}^\alpha,\mathbf{x}^\kappa\big)g^{-1}\big(\mathbf{x}^\mu,\mathbf{x}^\lambda\big)\Big(\left(\mathcal{L}_{\vec{x}_\delta}g\right)(\vec{x}_\kappa,\vec{x}_\lambda)\\
&\qquad\Big(\left(\mathcal{L}_{\vec{x}_\gamma}g\right)(\vec{x}_\beta,\vec{x}_\mu)+\left(\mathcal{L}_{\vec{x}_\beta}g\right)(\vec{x}_\mu,\vec{x}_\gamma)-\left(\mathcal{L}_{\vec{x}_\mu}g\right)(\vec{x}_\beta,\vec{x}_\gamma)\\
&\qquad\quad+g\left([\vec{x}_\delta,\vec{x}_\kappa],\vec{x}_\lambda\right)-g\left([\vec{x}_\lambda,\vec{x}_\delta],\vec{x}_\kappa\right)\Big)\\
&\qquad\quad-g\left([\vec{x}_\beta,\vec{x}_\gamma],\vec{x}_\mu\right)+g\left([\vec{x}_\beta,\vec{x}_\mu],\vec{x}_\gamma\right)-g\left([\vec{x}_\mu,\vec{x}_\gamma],\vec{x}_\beta\right)\Big)\\
&+\frac{1}{2}g^{-1}\big(\mathbf{x}^\alpha,\mathbf{x}^\mu\big)\Big(\left(\mathcal{L}_{\vec{x}_\delta}\mathcal{L}_{\vec{x}_\mu}g\right)(\vec{x}_\beta,\vec{x}_\gamma)\\
&\qquad-\left(\mathcal{L}_{\vec{x}_\mu}\mathcal{L}_{\vec{x}_\delta}g\right)(\vec{x}_\beta,\vec{x}_\gamma)+\left(\mathcal{L}_{\vec{x}_\delta}\mathcal{L}_{\vec{x}_\gamma}g\right)(\vec{x}_\beta,\vec{x}_\mu)\\
&\qquad+\left(\mathcal{L}_{\vec{x}_\beta}g\right)\left([\vec{x}_\delta,\vec{x}_\mu],\vec{x}_\gamma\right)+\left(\mathcal{L}_{\vec{x}_\beta}g\right)\left([\vec{x}_\delta,\vec{x}_\gamma],\vec{x}_\mu\right)\\
&\qquad-\left(\mathcal{L}_{\vec{x}_\mu}g\right)\left([\vec{x}_\delta,\vec{x}_\beta],\vec{x}_\gamma\right)-\left(\mathcal{L}_{\vec{x}_\mu}g\right)\left([\vec{x}_\delta,\vec{x}_\gamma],\vec{x}\right)\\
&\qquad+\left(\mathcal{L}_{\vec{x}_\gamma}g\right)\left([\vec{x}_\delta,\vec{x}_\beta],\vec{x}_\mu\right)+\left(\mathcal{L}_{\vec{x}_\gamma}g\right)\left([\vec{x}_\delta,\vec{x}_\mu],\vec{x}_\beta\right)\\
&\qquad+g\left(\big[[\vec{x}_\beta,\vec{x}_\mu],\vec{x}_\delta\big],\vec{x}_\gamma\right)+g\left([\vec{x}_\beta,\vec{x}_\mu],[\vec{x}_\delta,\vec{x}_\gamma]\right)\\
&\qquad-g\left(\big[[\vec{x}_\beta,\vec{x}_\gamma],\vec{x}_\delta\big],\vec{x}_\mu\right)-g\left([\vec{x}_\beta,\vec{x}_\mu],[\vec{x}_\delta,\vec{x}_\mu]\right)\\
&\qquad+g\left(\big[[\vec{x}_\mu,\vec{x}_\gamma],\vec{x}_\delta\big],\vec{x}_\beta\right)+g\left([\vec{x}_\mu,\vec{x}_\gamma],[\vec{x}_\delta,\vec{x}_\beta]\right)\Big)\,.\end{aligned}$$

As a result, the Riemann-Christoffel tensor is written as

(38)

$$\begin{aligned}R^{\alpha}{}_{\delta\beta\gamma} =&\frac{1}{2}g^{\alpha\mu}\Big((\mathcal{L}_{\vec{x}_\beta}\mathcal{L}_{\vec{x}_\mu}g)_{\gamma\delta}-(\mathcal{L}_{\vec{x}_\mu}\mathcal{L}_{\vec{x}_\beta}g)_{\gamma\delta}+(\mathcal{L}_{\vec{x}_\beta}\mathcal{L}_{\vec{x}_\delta}g)_{\gamma\mu}\\
&-(\mathcal{L}_{\vec{x}_\gamma}\mathcal{L}_{\vec{x}_\mu}g)_{\beta\delta}+(\mathcal{L}_{\vec{x}_\mu}\mathcal{L}_{\vec{x}_\gamma}g)_{\beta\delta}-(\mathcal{L}_{\vec{x}_\gamma}\mathcal{L}_{\vec{x}_\delta}g)_{\beta\mu}\\
&+C^{\nu}{}_{\beta\mu}(\mathcal{L}_{\vec{x}_\gamma}g)_{\delta\nu}+C^{\nu}{}_{\beta\delta}(\mathcal{L}_{\vec{x}_\gamma}g)_{\mu\nu}-C^{\nu}{}_{\beta\gamma}(\mathcal{L}_{\vec{x}_\mu}g)_{\nu\delta}-C^{\nu}{}_{\beta\delta}(\mathcal{L}_{\vec{x}_\mu}g)_{\gamma\nu}\\
&+C^{\nu}{}_{\beta\gamma}(\mathcal{L}_{\vec{x}_\delta}g)_{\mu\nu}+C^{\nu}{}_{\beta\mu}(\mathcal{L}_{\vec{x}_\delta}g)_{\gamma\nu}-C^{\nu}{}_{\gamma\mu}(\mathcal{L}_{\vec{x}_\beta}g)_{\delta\nu}-C^{\nu}{}_{\gamma\delta}(\mathcal{L}_{\vec{x}_\beta}g)_{\mu\nu}\\
&+C^{\nu}{}_{\gamma\beta}(\mathcal{L}_{\vec{x}_\mu}g)_{\delta\nu}+C^{\nu}{}_{\gamma\delta}(\mathcal{L}_{\vec{x}_\mu}g)_{\beta\nu}-C^{\nu}{}_{\gamma\beta}(\mathcal{L}_{\vec{x}_\delta}g)_{\mu\nu}-C^{\nu}{}_{\gamma\mu}(\mathcal{L}_{\vec{x}_\delta}g)_{\beta\nu}\\
&+C^{\kappa}{}_{\gamma\mu}C^{\lambda}{}_{\kappa\beta}g_{\delta\lambda}-C^{\kappa}{}_{\gamma\delta}C^{\lambda}{}_{\kappa\beta}g_{\mu\lambda}+C^{\kappa}{}_{\mu\delta}C^{\lambda}{}_{\kappa\beta}g_{\gamma\lambda}\\
&-C^{\kappa}{}_{\beta\mu}C^{\lambda}{}_{\kappa\gamma}g_{\lambda\delta}+C^{\kappa}{}_{\beta\delta}C^{\lambda}{}_{\kappa\gamma}g_{\lambda\mu}-C^{\kappa}{}_{\mu\delta}C^{\lambda}{}_{\kappa\gamma}g_{\beta\lambda}\Big)\\
&+\frac{1}{2}g^{\alpha\kappa}g^{\mu\lambda}\Big((\mathcal{L}_{\vec{x}_\gamma}g)_{\delta\kappa}+(\mathcal{L}_{\vec{x}_\delta}g)_{\gamma\kappa}-(\mathcal{L}_{\vec{x}_\kappa}g)_{\gamma\delta}+g_{\kappa\rho}C^{\rho}{}_{\gamma\delta}+g_{\gamma\rho}C^{\rho}{}_{\delta\kappa}+g_{\delta\rho}C^{\rho}{}_{\gamma\kappa}\Big)\\
&\quad\Big((\mathcal{L}_{\vec{x}_\mu}g)_{\beta\lambda}+(\mathcal{L}_{\vec{x}_\beta}g)_{\mu\lambda}-(\mathcal{L}_{\vec{x}_\lambda}g)_{\beta\mu}+g_{\lambda\sigma}C^{\sigma}{}_{\beta\mu}+g_{\mu\sigma}C^{\sigma}{}_{\beta\lambda}+g_{\beta\sigma}C^{\sigma}{}_{\mu\lambda}\Big)\\
&-\frac{1}{2}g^{\alpha\kappa}g^{\mu\lambda}\Big((\mathcal{L}_{\vec{x}_\beta}g)_{\delta\kappa}+(\mathcal{L}_{\vec{x}_\delta}g)_{\beta\kappa}-(\mathcal{L}_{\vec{x}_\kappa}g)_{\beta\delta}+g_{\kappa\rho}C^{\rho}{}_{\beta\delta}+g_{\beta\rho}C^{\rho}{}_{\delta\kappa}\Big)+g_{\delta\rho}C^{\rho}{}_{\beta\kappa}\\
&\quad\Big((\mathcal{L}_{\vec{x}_\mu}g)_{\gamma\lambda}+(\mathcal{L}_{\vec{x}_\gamma}g)_{\mu\lambda}-(\mathcal{L}_{\vec{x}_\lambda}g)_{\gamma\mu}+g_{\kappa\sigma}C^{\sigma}{}_{\gamma\mu}+g_{\mu\sigma}C^{\sigma}{}_{\gamma\lambda}+g_{\gamma\sigma}C^{\sigma}{}_{\mu\lambda}\Big)\\
&+\frac{1}{2}g^{\alpha\mu}\Big((\mathcal{L}_{\vec{x}_\delta}g)_{\mu\nu}+(\mathcal{L}_{\vec{x}_\nu}g)_{\delta\mu}-(\mathcal{L}_{\vec{x}_\mu}g)_{\delta\nu}-g_{\mu\lambda}C^{\lambda}{}_{\delta\nu}-g_{\delta\lambda}C^{\lambda}{}_{\mu\nu}+g_{\nu\lambda}C^{\lambda}{}_{\delta\mu}\Big)C^{\nu}{}_{\beta\gamma}\end{aligned}$$

Of course, the traces of the Riemann tensor are said to represent the local features of Riemannian curvature and are defined as

$$R_{\alpha\beta} = R^{\lambda}{}_{\alpha\lambda\beta} \quad \text{and} \tag{39}$$

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\lambda}{}_{\mu\lambda\nu} \,. \tag{40}$$

Given the relation, for the Riemann-Christoffel tensor, it is easy to express the Ricci tensor with respect to the derivatives of the metric and the commutators as

(41)

$$\begin{aligned}
R_{\alpha\beta} =& \frac{1}{2} g^{\kappa\lambda} \Big((\mathcal{L}_{\vec{x}_\kappa} \mathcal{L}_{\vec{x}_\lambda} g)_{\alpha\beta} - (\mathcal{L}_{\vec{x}_\kappa} \mathcal{L}_{\vec{x}_\lambda} g)_{\alpha\beta} + (\mathcal{L}_{\vec{x}_\kappa} \mathcal{L}_{\vec{x}_\alpha} g)_{\beta\lambda} \\
&- (\mathcal{L}_{\vec{x}_\beta} \mathcal{L}_{\vec{x}_\lambda} g)_{\alpha\kappa} + (\mathcal{L}_{\vec{x}_\lambda} \mathcal{L}_{\vec{x}_\beta} g)_{\alpha\kappa} - (\mathcal{L}_{\vec{x}_\beta} \mathcal{L}_{\vec{x}_\alpha} g)_{\kappa\lambda} \\
&+ C^{\mu}{}_{\alpha\lambda} (\mathcal{L}_{\vec{x}_\beta} g)_{\mu\kappa} + C^{\mu}{}_{\alpha\kappa} (\mathcal{L}_{\vec{x}_\beta} g)_{\mu\lambda} \\
&- C^{\mu}{}_{\alpha\beta} (\mathcal{L}_{\vec{x}_\lambda} g)_{\kappa\mu} - C^{\mu}{}_{\alpha\kappa} (\mathcal{L}_{\vec{x}_\lambda} g)_{\beta\mu} \\
&+ C^{\mu}{}_{\kappa\alpha} (\mathcal{L}_{\vec{x}_\beta} g)_{\lambda\mu} + C^{\mu}{}_{\kappa\alpha} (\mathcal{L}_{\vec{x}_\lambda} g)_{\beta\mu} \\
&+ C^{\mu}{}_{\beta\kappa} (\mathcal{L}_{\vec{x}_\lambda} g)_{\alpha\mu} - C^{\mu}{}_{\kappa\beta} (\mathcal{L}_{\vec{x}_\alpha} g)_{\lambda\mu} \\
&+ C^{\mu}{}_{\beta\lambda} C^{\nu}{}_{\mu\kappa} g_{\alpha\nu} - C^{\mu}{}_{\alpha\lambda} C^{\nu}{}_{\mu\kappa} g_{\beta\nu} + C^{\mu}{}_{\alpha\beta} C^{\nu}{}_{\mu\kappa} g_{\lambda\nu} \Big) \\
&+ \frac{1}{2} g^{\kappa\rho} g^{\lambda\sigma} \Big((\mathcal{L}_{\vec{x}_\alpha} g)_{\beta\rho} + (\mathcal{L}_{\vec{x}_\beta} g)_{\alpha\rho} - (\mathcal{L}_{\vec{x}_\rho} g)_{\alpha\beta} - g_{\rho\mu} C^{\mu}{}_{\alpha\beta} + g_{\alpha\mu} C^{\mu}{}_{\beta\rho} + g_{\beta\mu} C^{\mu}{}_{\alpha\rho} \Big) \\
&\Big((\mathcal{L}_{\vec{x}_\lambda} g)_{\kappa\sigma} + (\mathcal{L}_{\vec{x}_\kappa} g)_{\lambda\sigma} - (\mathcal{L}_{\vec{x}_\sigma} g)_{\kappa\lambda} + g_{\sigma\nu} C^{\nu}{}_{\kappa\lambda} + g_{\lambda\nu} C^{\nu}{}_{\kappa\sigma} + g_{\kappa\nu} C^{mu}{}_{\lambda\sigma} \Big) \\
&- \frac{1}{2} g^{\kappa\rho} g^{\lambda\sigma} \Big((\mathcal{L}_{\vec{x}_\alpha} g)_{\kappa\rho} + (\mathcal{L}_{\vec{x}_\kappa} g)_{\alpha\rho} - (\mathcal{L}_{\vec{x}_\rho} g)_{\alpha\kappa} - g_{\rho\mu} C^{\mu}{}_{\alpha\kappa} + g_{\alpha\mu} C^{\mu}{}_{\kappa\rho} + g_{\kappa\mu} C^{k}{}_{\alpha\rho} \Big) \\
&(\mathcal{L}_{\vec{x}_\beta} g)_{\lambda\sigma} + \Big((\mathcal{L}_{\vec{x}_\lambda} g)_{\beta\sigma} - (\mathcal{L}_{\vec{x}_\sigma} g)_{\beta\lambda} + g_{\sigma\nu} C^{\nu}{}_{\beta\lambda} + g_{\lambda\nu} C^{\nu}{}_{\beta\sigma} + g_{\beta\nu} C^{\nu}{}_{\lambda\sigma} \Big) \\
&- \frac{1}{2} g^{\kappa\lambda} \Big((\mathcal{L}_{\vec{x}_\alpha} g)_{\kappa\mu} + (\mathcal{L}_{\vec{x}_\mu} g)_{\alpha\kappa} - (\mathcal{L}_{\vec{x}_\kappa} g)_{\alpha\mu} - g_{\kappa\nu} C^{\nu}{}_{\alpha\mu} + g_{\alpha\nu} C^{\nu}{}_{\mu\kappa} + g_{\mu\nu} C^{\nu}{}_{\alpha\lambda} \Big) C^{\mu}{}_{\beta\lambda} \,,
\end{aligned}$$

and the Ricci scalar as

(42)

$$
\begin{aligned}
R =& \frac{1}{2} g^{\kappa\lambda} g^{\mu\nu} \Big((\mathcal{L}_{\vec{x}_\mu}\mathcal{L}_{\vec{x}_\nu} g)_{\kappa\lambda} - (\mathcal{L}_{\vec{x}_\nu}\mathcal{L}_{\vec{x}_\mu} g)_{\kappa\lambda} + (\mathcal{L}_{\vec{x}_\mu}\mathcal{L}_{\vec{x}_\kappa} g)_{\lambda\nu} \\
&\quad - (\mathcal{L}_{\vec{x}_\lambda}\mathcal{L}_{\vec{x}_\nu} g)_{\kappa\mu} + (\mathcal{L}_{\vec{x}_\nu}\mathcal{L}_{\vec{x}_\lambda} g)_{\kappa\mu} - (\mathcal{L}_{\vec{x}_\lambda}\mathcal{L}_{\vec{x}_\kappa} g)_{\mu\nu} \\
&\quad + C^{\rho}{}_{\kappa\nu} (\mathcal{L}_{\vec{x}_\lambda} g)_{\rho\mu} + C^{\rho}{}_{\kappa\mu} (\mathcal{L}_{\vec{x}_\lambda} g)_{\rho\nu} - C^{\rho}{}_{\kappa\lambda} (\mathcal{L}_{\vec{x}_\nu} g)_{\rho\mu} - C^{\rho}{}_{\kappa\mu} (\mathcal{L}_{\vec{x}_\nu} g)_{\rho\lambda} \\
&\quad + C^{\rho}{}_{\mu\kappa} (\mathcal{L}_{\vec{x}_\lambda} g)_{\rho\nu} + C^{\rho}{}_{\mu\kappa} (\mathcal{L}_{\vec{\nu}} g)_{\rho\lambda} + C^{\rho}{}_{\mu\lambda} (\mathcal{L}_{\vec{x}_\nu} g)_{\rho\kappa} - C^{\rho}{}_{\mu\lambda} (\mathcal{L}_{\vec{x}_\kappa} g)_{\rho\nu} \\
&\quad + C^{\rho}{}_{\lambda\nu} C^{\sigma}{}_{\rho\mu} g_{\kappa\sigma} - C^{\rho}{}_{\lambda\nu} C^{\sigma}{}_{\rho\mu} g_{\lambda\sigma} + C^{\rho}{}_{\kappa\lambda} C^{\sigma}{}_{\rho\mu} g_{\nu\sigma} \\
&+ \frac{1}{2} g^{\kappa\lambda} g^{\mu\pi} g^{\nu\sigma} \\
&\quad \Big((\mathcal{L}_{\vec{x}_\kappa} g)_{\lambda\pi} + (\mathcal{L}_{\vec{x}_\lambda} g)_{\kappa\pi} - (\mathcal{L}_{\vec{x}_\pi} g)_{\kappa\lambda} - g_{\pi\sigma} C^{\sigma}{}_{\kappa\lambda} + g_{\kappa\sigma} C^{\sigma}{}_{\lambda\pi} + g_{\lambda\sigma} C^{\sigma}{}_{\kappa\pi} \Big) \\
&\quad \Big((\mathcal{L}_{\vec{x}_\mu} g)_{\nu\rho} + (\mathcal{L}_{\vec{x}_\nu} g)_{\nu\rho} - (\mathcal{L}_{\vec{x}_\rho} g)_{wu} + g_{\rho\tau} C^{\tau}{}_{\mu\nu} + g_{\mu\tau} C^{\tau}{}_{\nu\rho} + g_{\mu\tau} C^{\tau}{}_{\mu\rho} \Big) \\
&- \frac{1}{2} g^{\kappa\lambda} g^{\mu\nu} g^{\pi\rho} \\
&\quad \Big((\mathcal{L}_{\vec{x}_\kappa} g)_{\mu\nu} + (\mathcal{L}_{\vec{x}_\mu} g)_{\kappa\nu} - (\mathcal{L}_{\vec{x}_\nu} g)_{\kappa\mu} - g_{\mu\sigma} C^{\sigma}{}_{\kappa\nu} + g_{\nu\sigma} C^{\sigma}{}_{\kappa\mu} \Big) \\
&\quad \Big((\mathcal{L}_{\vec{x}_\lambda} g)_{\pi\rho} + (\mathcal{L}_{\vec{x}_\pi} g)_{\lambda\rho} - (\mathcal{L}_{\vec{x}_\rho} g)_{\lambda\pi} + g_{\rho\tau} C^{\tau}{}_{\lambda\pi} + g_{\pi\tau} C^{\tau}{}_{\lambda\rho} \Big) \\
&- \frac{1}{2} g^{\kappa\lambda} g^{\mu\nu} \Big((\mathcal{L}_{\vec{x}_\kappa} g)_{\mu\rho} + (\mathcal{L}_{\vec{x}_\rho} g)_{\kappa\mu} - (\mathcal{L}_{\vec{x}_\mu} g)_{\kappa\rho} \\
&\quad + g_{\mu\sigma} C^{\sigma}{}_{\rho\kappa} + g_{\kappa\sigma} C^{\sigma}{}_{\rho\mu} - g_{\rho\sigma} C^{\sigma}{}_{\mu\kappa} \Big) C^{\rho}{}_{\lambda\nu} \,.
\end{aligned}
$$

As for the long-range effects of curvature, they are represented by the traceless and conformal Weyl tensor,

(43)

$$
W_{\alpha\beta\gamma\delta} = g_{\alpha\mu} R^{\mu}{}_{\beta\gamma\delta} - \frac{1}{2}\big(R_{\alpha\delta} g_{\beta\gamma} + R_{\beta\gamma} g_{\alpha\delta} - R_{\alpha\gamma} g_{\beta\delta} - R_{\beta\delta} g_{alpha\gamma}\big) + \frac{1}{6} R \big(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\big) \,.
$$

We should note that the Ricci tensor should be symmetric in its indices, proposing that the frame on which it is written is an orthonormal one. As for the Weyl tensor, it retains the symmetries of the Riemann tensor.

2.3. The construction of the space-time. This study focuses on space-times where a Bianchi group acts freely without any restriction on the nature of the homogeneous hypersurfaces. This comes at odds with previous examples of Bianchi group actions on space-times where the group acted transitively by isometries leading to an orthogonal slicing of the space-time. In our case, the following usual assumptions are dropped:

(1) **First assumption dropped: Transitive Action.**
The Bianchi group acts *freely* and *regularly* (in the definition of Olver [**34**]). This implies that the orbits of the Bianchi group are strictly of dimension three.

(2) **Second assumption dropped: Isometries.**
The Bianchi group acts by homotheties.
(3) **Third assumption dropped: Orthogonal Slicing.**
The quotient of the group is not a unique neither an orthogonal transversal vector field in the space-time, but may be any of all vectors that are invariant under the action of the group.

THEOREM 2.1 (Construction of a Homogeneous Space-time). *Assume there is a four-dimensional space-time $\mathcal{V}_4$ and a three-dimensional group $\mathcal{G}$, such that*

(1) *the group $\mathcal{G}$ acts freely and regularly on the space-time $\mathcal{V}_3$, admitting 3-dimensional orbits on it, and*
(2) *the group $\mathcal{G}$ acts by homotheties on the space-time $\mathcal{V}_4$, hence the generators of the group are homothetic vector fields of the space-time.*

For any such space-time, there exists some vector field $\vec{\zeta}$ in the neighbourhood of the generators of $\mathcal{G}$ that satisfies the following properties:

(1) *it is invariant under the action of the group, i.e., it commutes with the generators of the group;*
(2) *it is tangent to a geodesic at any point of the space-time; and*
(3) *is null.*

Then, a coordinate patch can be constructed that covers locally the space-time, based on the transversal $\vec{\zeta}$ and the generators of the group.

The scope of the theorem is to prove that the construction of the space-time is possible by choosing any such vector field at any point of the homogeneous hypersurfaces of the space-time.

PROOF. The proof will follow the following order: a point P is arbitrarily chosen on the space-time $\mathcal{V}_4$, and on it a vector $\vec{\zeta}$ that has the properties described in the theorem. The vector is subsequently "moved" along the group and the quotient of its action; the order of the "movement" does not matter (whether the vector is moved first along the group and then along the quotient or *vice versa*). The properties of the vector $\vec{\zeta}$ are matched to this "movement" in such a way that a certain map J is proved to exist, from the direct sum $\left[\mathbb{R} \times \mathcal{G}\right]$ to the space-time; this map serves as the coordinate patch that can locally cover $\mathcal{V}_4$.

Before we proceed: Let $\vec{\zeta}$ be some vector field in the quotient of the group action. Then, let C_ζ be the family of curves tangent to some (or all possible) $\vec{\zeta}$; this family of curves is parametrised by means of an affine parameter, z, which takes values in $\mathbb{R}$. As a result, the direction defined by them is spanned by the vector $\vec{v} = \dfrac{\partial}{\partial z}$; then, any vector $\vec{\zeta}$ will be given as

$$\vec{\zeta} = u^z \vec{v} \,.$$

By contrast, the homogeneous submanifold $\mathcal{M}$ is parametrised by means of the generators of the group, $\vec{\xi}_a$. As a result, any vector on $\mathcal{M}$ is expressed with respect to them as

$$\vec{u} = u^i \vec{\xi}_i \,,$$

where u^a take values in $\mathcal{G}$. We can similarly claim that every generator defines some "direction" in the group - or, which is the same, is tangent on some one-dimensional suborbit of the group action, that is parametrised by means of an affine parameter, w^i, which takes values in $\mathcal{G}$, such that $\vec{\xi}_a \sim \dfrac{\partial}{\partial w^a}$ - these are the canonical coordinates of group; whereas

$$\vec{u} = \tilde{u}^i \frac{\partial}{\partial w^i} \,.$$

We will not specify these affine parameters more, as they are always easy to disentangle (they largely depend on the Class of the group and whether the one-dimensional suborbits are directly distinguishable from the three-dimensional orbit); however, we claim that such w^i's always exist.

Let us proceed with the proof.

(1) Invariance of $\vec{\zeta}$ under the action of the group.
Choosing $\vec{\zeta}_1$ on some point P of a homogeneous hypersurface should be equivalent to choosing some $\vec{\zeta}_1' = \vec{\zeta}_1 + \lambda^i \vec{\xi}_i$, where $\lambda_i \in \mathbb{R}$, on point $P' = \mathfrak{g} \cdot P$, where $\mathfrak{g} \in \mathcal{G}$, so that P and P' belong to the same group orbit (see Fig. 1). Hence, the choice of $\vec{\zeta}$ along the the group should not be unique or special in any way. This means that $\vec{\zeta}$ should commute with the generators of the group.

The commutator of vectors $\vec{\zeta}$ and $\vec{u}$ is

$$\left[\vec{\zeta}, \vec{u}\right] = u^z u^i \left[\vec{v}, \vec{\xi}_i\right] + u^i \mathcal{L}_{\vec{\xi}_i} u^z \vec{v} + u^z \frac{\partial u^i}{\partial z} \vec{\xi}_i \,.$$

Given the vector $\vec{v}$ is on the quotient of the group, $\left[\vec{v}, \vec{\xi}_i\right] = 0$; and, given $u^i \in \mathcal{G}$, $\dfrac{\partial u^i}{\partial z} = 0$. Then, the commutator is

$$\left[\vec{\zeta}, \vec{u}\right] = u^i \mathcal{L}_{\vec{\xi}_i} u^z \vec{v} \,.$$

From here, we have the following result: the vector field $\vec{\zeta}$ is invariant under the action of the group if-f

$$\mathcal{L}_{\vec{\xi}_a} u^z = 0 \,,$$

that is, if-f the projection of $\vec{\zeta}$ on $\vec{v}$ (the 'length' of $\vec{\zeta}$) is independent of the group. Therefore, the vector field $\vec{\zeta}$ can be chosen to be invariant under the action of the group by choosing its "length" to be independent from the elements of the group. Then,

$$\nabla_{\vec{\xi}_a} \vec{\zeta} = \nabla_{\vec{\zeta}} \vec{\xi}_a \,,$$

or

$$[\vec{\zeta}, \vec{\xi}_a] = 0\,. \tag{44}$$

(2) Geodesic nature of $\vec{\zeta}$.

Choosing $\vec{\zeta}_1$ on some point P of a homogeneous hypersurface should be equivalent to choosing some $\vec{\zeta}_2$ on point Q on another homogeneous hypersurface, so that P and Q belong to different group orbits and the one cannot be "moved" to the other by means of the group elements (see Fig. 2.1). Hence, the $\vec{\zeta}$ should not behave differently at different points of the space-time. Given that $\vec{\zeta}$ "flows" from the one homogeneous hypersurface to another along a family of curves, C_z, on which it is tangent, its transport along these curves should not change its nature and behaviour; this may only happen if its transport along these curves is zero, hence if these curves are geodesics.

Assume Eq. (27) with $\vec{x}_\delta = \vec{x}_\gamma$ being transversal, thus equal to $\vec{\zeta}$, and $\vec{x}_\alpha$ being any of the generators. $\vec{x}_\beta$ is for the moment unspecified, and allowed to be either transveral or tangential to the group action and, thus, it always commutes with $\vec{\zeta}$; we denote it as $\vec{v}$. We also remember that $\mathcal{G}$ acts by homotheties; then

$$2\boldsymbol{g}\big(\mathcal{L}_{\vec{\xi}_a}\nabla_{\vec{\zeta}}\vec{\zeta}, \vec{v}\big) = -2\phi_a \boldsymbol{g}\big(\nabla_{\vec{\zeta}}\vec{\zeta}, \vec{v}\big) + 2\phi_a\big(\mathcal{L}_{\vec{\zeta}}\boldsymbol{g}\big)\big(\vec{v}, \vec{\zeta}\big) - \phi_a\big(\mathcal{L}_{\vec{v}}\boldsymbol{g}\big)\big(\vec{\zeta}, \vec{\zeta}\big)\,. \tag{45}$$

Then, from Eq. (20), the covariant derivative of $\vec{\zeta}$ with respect to itself is

$$2\boldsymbol{g}\big(\nabla_{\vec{\zeta}}\vec{\zeta}, \vec{v}\big) = 2\big(\mathcal{L}_{\vec{\zeta}}\boldsymbol{g}\big)\big(\vec{v}, \vec{\zeta}\big) - \big(\mathcal{L}_{\vec{v}}\boldsymbol{g}\big)\big(\vec{\zeta}, \vec{\zeta}\big)\,. \tag{46}$$

Combining Eqs. (45) and (46), we easily see that

$$\boldsymbol{g}\big(\mathcal{L}_{\vec{\xi}_a}\nabla_{\vec{\zeta}}\vec{\zeta}, \vec{v}\big) = 0\,,$$

independent of whether $\vec{v}$ is transversal or tangential. This is true if-f the Lie derivative of $\nabla_{\vec{\zeta}}\vec{\zeta}$ along the generators is orthogonal to $\vec{\zeta}$ or $\vec{\xi}_a$ - in fact, to both. So, it must be zero at any point of the space-time,

$$\mathcal{L}_{\vec{\xi}_a}\nabla_{\vec{\zeta}}\vec{\zeta} = 0\,.$$

Since the transversal and the generators commute if-f the length of $\vec{\zeta}$ is independent of the group ($\mathcal{L}_{\vec{\xi}_a} u^z = 0$), then the covariant derivative of $\vec{\zeta}$ along itself can only be tangent on C_ζ as well, since

$$\nabla_{\vec{\zeta}}\vec{\zeta} = \nabla_{u^z\vec{v}}(u^z\vec{v}) = \frac{\partial u^z}{\partial z}\vec{v}\,.$$

Thus, the Lie derivative of $\nabla_{\vec{\zeta}}\vec{\zeta}$ along $\vec{\xi}_a$ can be zero only as long as the covariant derivative itself is zero,

$$\nabla_{\vec{\zeta}}\vec{\zeta} = 0\,. \tag{47}$$

Therefore, $\dfrac{\partial u^z}{\partial z} = 0$; and the vector field $\vec{\zeta}$ is geodesic (the curves C_ζ being geodesic curves).

(3) The null character of $\vec{\zeta}$.

Choosing $\vec{\zeta}_1$ on some point P of a homogeneous hypersurface should be equivalent to choosing some $\vec{\zeta}'_2$ on point Q' on another homogeneous hypersurface, such that

- $\vec{\zeta}_2$ on point Q is related to $\vec{\zeta}_1$ on point P by means of "movement" along a curve C_ζ, and to $\vec{\zeta}'_2$ on point $Q' = \mathfrak{g} \cdot Q$ by means of "movement" along the group; and
- $\vec{\zeta}'_1$ on point $P' = \mathfrak{g} \cdot P$ is related to $vec\zeta_1$ on point P by means of "movement" along the group, and to $\vec{\zeta}'_2$ on point $Q' = \mathfrak{g} \cdot Q$ by means of "movement" along a curve C_ζ.

That is, the choice of the "starting point" is irrelevant, as any two points in the space-time can be connected by means of a "two-fold movements", along the group and along the geodesic curves C_ζ. Moreover, the order of the two "movements" does not matter and can easily be transposed. For this to be true, the vector field $\vec{\zeta}$ must retain its character along all such "movements"; or, which is the same, to obey some normalisation condition that is unchanged along these "movements".

There are several options we can choose from, of which the most logical are three: $\vec{\zeta}$ can be normal, orthogonal to the group, or null. Let us examine these cases.

- If the vector field is initially normal, then "moving" along the group will stretch or shrink it, due to the fact that the group acts by homotheties and, thus, it changes the length of vectors. Therefore, it will not remain normal as it "moves" along the group; so, this normalisation condition cannot hold.
- If the vector field is initially orthogonal to the group, then the orbits of the group cannot be null (either on the point or any other). Consequently, this normalisation condition cannot hold, if we hope to treat all possible actions of the Bianchi groups on space-times.
- If the vector field is null, none of the above problems exists; so, this normalisation condition can hold.

It must be noted that this normalisation condition is a simplifying option and can be omitted. It is one that seems to violate the usual 3+1 formalism, as time-like vectors are usually preferred; however, it is not so unusual a choice [**38, ?**] and, as we shall see, it simplifies calculations significantly.

Assume now that a point $P \in \mathcal{M}$ is chosen and $\vec{\zeta}$ is defined on it. The point can be "moved" along the curve C_{ζ_1} so that z increases to some interval $[0, z] \subset \mathbb{R}$; the new point Q is then mapped back to $\mathcal{M}$ by means of $\vec{\xi}_a$ and

z. Then, the point Q can be "moved" along the submanifold $\mathcal{M}$ in such a manner that w^i will increase (or decrease) within a subset of the group; the new point $Q' = \mathfrak{g} \cdot Q$ is then mapped back to C_{ζ_1} by means of w^i and $\vec{\zeta}$.[6] The mapping of this "motion" from $P \in \left[C_{\zeta_1} \times \mathcal{M}_1\right]$ to $Q' \in \left[C_{\zeta_2} \times \mathcal{M}_2\right]$ (where the indices correspond to different curves and hypersurfaces of the same families respectively) and back can be given by means of z and w^i. Therefore, this "motion" of P is described as a vector of the space-time that is a linear combination of $\vec{v}$ and $\vec{u}$. Consequently, there exists some function

$$J : [\mathbb{R} \times \mathcal{G}] \to [\mathbb{R} \times \mathcal{G}] \tag{48}$$

that tracks this "motion". This function constitutes the coordinate chart, while the affine parameters z ans w^i constitute the coordinates. □

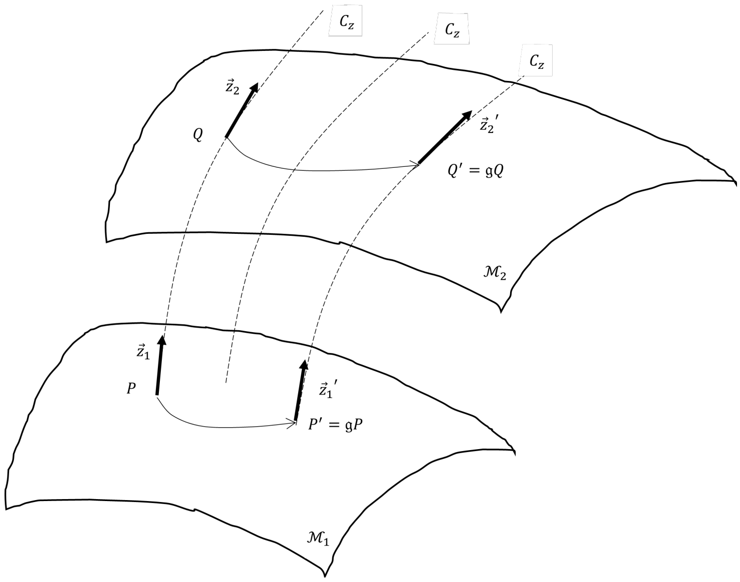

FIGURE 1. The schematic representation of Theorem 2.1.

What the theorem suggests is that a coordinate chart can always be constructed in a space-time where a Bianchi group acts freely and regularly by homotheties, by means of the generators of the group and a null, geodesic vector field that is invariant under the action of the group - what we may refer from now on as a transveral collineation. Such a coordinate chart always exist, at least locally, and the process of constructing it is exactly the process used to prove the theorem. It must be noted that similar coordinate charts can be found for space-times of this type without the strict prerequisites on the transversal collineation $\vec{\zeta}$ (namely, the geodesicity and the null character) and well-known examples in the literature can testify to this; however, what

[6]Of course, we could opt for the commutation of these two "motions"; point P could move first along the group to $P' = \mathfrak{g} \cdot P$, causing an increase (or decrease) of w^i - and then along C_{ζ_2} to $Q' = \mathfrak{g} \cdot Q$, causing an increase of z. The two cases would be identical.

we claim is that these restrictions on the transversal collinieation $\vec{\zeta}$ allow for an exhaustive and generic treatment of all space-times where the action of a Bianchi group admits homotheties and leads exactly to three-dimensional homogeneous submanifolds.[7]

3. Elements of the Space-Time

3.1. The metric. As usual, the metric $\boldsymbol{g}$ is covariantly constant, so the space-time connection is torsion-free Levi-Civita,

$$\nabla_{\vec{\zeta}}\boldsymbol{g} = 0\,. \tag{49}$$

At the same moment, the Lie derivative of the metric $\boldsymbol{g}$ along the invariant vector field $\vec{\zeta}$ is defined as

$$\mathcal{L}_{\vec{\zeta}}\boldsymbol{g} = \boldsymbol{k}\,. \tag{50}$$

In the usual treatment, $\boldsymbol{k}$ would be equivalent to the 'second fundamental form', as it denotes the extrinsic curvature of the homogeneous submanifold $\mathcal{M}$; in the same manner, the line element

$$\mathrm{d}s^2 = \boldsymbol{g}\big(\mathrm{d}x_i, \mathrm{d}x_i\big)$$

is the 'first fundamental form'. In our case, the latter is true, as it is derived from the metric; but the former is not, because $\vec{\zeta}$ is by definition null.[8] We can similarly define the Lie derivative of $\boldsymbol{k}$ along the invariant vector field $\vec{\zeta}$ as

$$\mathcal{L}_{\vec{\zeta}}\boldsymbol{k} = \boldsymbol{\ell}\,. \tag{51}$$

3.2. Einstein's field equations. Einstein proved that the afore-mentioned geometric objects are related with the matter distributed on the space-time. His theory states that the space-time is structured (curved) according to the density, energy and stress of the matter fields. Or, in the well-known phrasing, "Space-time tells matter how to move; matter tells space-time how to curve". Consequently, the Riemann and the Ricci curvature reflects the energy and momentum of matter, while the distribution and the evolution of matter reflects the curvature of the space-time.

The postulates on the foundations of Einstein's thought are:

(1) The laws of physics have the same form in all inertial reference frames.
As a result, there is no preferred frame (hence no preferred coordinate system) in the space-time and all observers are equivalent. All observers in all frames are to observe the same laws of physics and

[7] There appears to be an exception in this treatment, that could potentially make it non-exhaustive, but we will explain why it is not so; this case indeed restricts the generality of the theorem's application, but it has to be examined separately, as it does not fall under the conditions of the theorem altogether.

[8] Gourgoulhon uses a timelike or spacelike vector normal to a hypersurface to define the second fundamenta form, claiming that a similar construction is impossible with a null vector [**33**].

make equivalent measurements - while all deviations in the measurements are due to the difference between coordinate systems.

(2) Light propagates through empty space with a definite speed, independent of the speed of the observer (or the source).
Thus, there is only one point of common reference for all distinct observers; all of them yield the same measurement of the speed of light.

(3) In the limit of low speeds the gravity formalism should agree with Newtonian gravity.
As a result, the proposed theory of gravity should retain the results of the classical theory in the case of low speeds as, in this limit, the curvature of the space-time should decrease towards flatness.

From these hypotheses, Einstein related the curvature in a local scale with the stress-energy-momentum tensor, that gives the flux of the momentum vector across a hypersurface. Hilbert proposed an action in the form

$$S = \int \mathrm{d}^4 x \sqrt{-\det|\boldsymbol{g}|}\Big(\frac{R}{8\pi} - \Lambda\Big) - S_{matter} \ , \tag{52}$$

where S_{matter} is the action describing the properties of matter, Λ the cosmological constant and $\det|\boldsymbol{g}|$ is the determinant of the metric g - as usual, we assumed a geometric system of units, hence the speed of light and Newton's constant are equal to one. Varying the action with respect to the metric, one arrives to Einstein's field equations

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} - \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta} \ , \tag{53}$$

where

$$\boldsymbol{T} = \frac{\delta S_{matter}}{\delta \boldsymbol{g}} \tag{54}$$

the stress-energy-momentum tensor, a second-rank tensor derived as the variation of the S_{matter} with respect to the metric.

Usually, the first two terms of the left-hand side of Eq. (53) are considered a specific tensor, known as the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} \,. \tag{55}$$

The Einstein tensor contains the properties of the Ricci tensor (symmetry on the two indices). But, its most important property is that it divergence-free,

$$g^{-1}\left(\nabla_{\vec{x}}, \boldsymbol{G}\big(\vec{y}, \vec{z}\big)\right) = 0 \ , \tag{56}$$

or $g^{\mu\nu}\nabla_\mu G_{\nu\alpha} = 0$ in some local coordinate system. Interestingly, the third term of the left-hand side is also divergence-free, as the metric is covariantly constant. As a result, the entire left-hand side of Einstein's field equations is covariantly constant; hence, the right-hand side must be divergence-free,

$$g^{-1}\left(\nabla_{\vec{x}}, \boldsymbol{T}\big(\vec{y}, \vec{z}\big)\right) = 0 \ , \tag{57}$$

or $g^{\mu\nu}\nabla_\mu T_{\nu\alpha} = 0$ in some local coordinate system. This equation states that the stress-energy-momentum tensor is conserved along the geodesics. This reflects the well-known properties of physical systems: the conservation of energy and of momentum. Essentially, the components of Eq. (57) denote the conservation laws for energy and for momentum.

It is worth pointing out that there is a different version of the field equations. The trace of the Einstein tensor is easily proved to be equal to the negative of the Riccic scalar,

$$G = -R\,;$$

and given eq. (53), we can easily see that

$$G = 8\pi T\,,$$

where $T = g^{-1}\big(\boldsymbol{x}^\mu, \boldsymbol{x}^\nu\big)\boldsymbol{T}\big(\vec{x}_\mu, \vec{x}_\nu\big)$ the trace of the stress-energy-momentum tensor. Therefore,

$$R = -8\pi T\,. \tag{58}$$

Substituting to eq. (53), we easily obtain

$$R_{\alpha\beta} = 8\pi\Big(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\Big)\,. \tag{59}$$

This equation links directly the Ricci tensor with the stress-energy-momentum tensor; this form may prove useful in transforming the Einstein equations to a system of evolution equations and constraints.

3.3. The matter models. The specific form of S_{matter} and, by extent, of the stress-energy-momentum tensor depends on the particular matter model used to describe the sources of the Einstein equations. Here we will briefly present four fundamental matter models: the first one refers to the description of scalar fields - supposedly describing low-order approximations of quantum phenomena, or the aggregate effect of 'exotic' particles; the second concerns the description of (free or not) electromagnetic fields; and the third and fourth are two different approximations of usual (baryonic) matter - the former treats it as a classical macroscopic fluid, while the latter as an ensemble of particles described through kinetic (Vlasov) theory.

3.3.1. *The scalar field.* Let us assume a scalar field, Φ, and the corresponding potential, $V(\Phi)$.

The action related to the scalar field is composed of two terms: kinetic one (the derivatives of the scalar field) and the potential:

$$S_{matter} = -\frac{1}{2}\boldsymbol{g}\left(\nabla_{\vec{u}}\Phi, \nabla_{\vec{u}}\Phi\right) - V(\Phi)\,. \tag{60}$$

The corresponding stress-energy-momentum tensor is derived by varying this action with respect to the metric. When doing so, we obtain

$$\boldsymbol{T}(\vec{x}_\alpha, \vec{x}_\beta) = \nabla_{\vec{x}_\alpha}\Phi\nabla_{\vec{x}_\beta}\Phi - \frac{1}{2}\boldsymbol{g}(\vec{x}_\alpha, \vec{x}_\beta)\left(\boldsymbol{g}\left(\nabla_{\vec{x}_\mu}\Phi, \nabla_{\vec{x}_\mu}\Phi\right) - V(\Phi)\right)\,. \tag{61}$$

In some local coordinates, this expression becomes

$$T_{\alpha\beta} = \frac{\partial \Phi}{\partial x^\alpha}\frac{\partial \Phi}{\partial x^\beta} - \frac{1}{2} g_{\alpha\beta}\left(g^{\mu\nu}\frac{\partial \Phi}{\partial x^\mu}\frac{\partial \Phi}{\partial x^\nu} - V(\Phi)\right) .$$

The equations of motion of the scalar field are similarly derived by varying the action with respect to the scalar field (or, more appropriately, its covariant derivatives). When doing so, we obtain the following equation

$$\frac{1}{\sqrt{-\det|\boldsymbol{g}|}}\nabla_\mu\left(\sqrt{-\det|\boldsymbol{g}|}g^{\mu\nu}\nabla_\nu\Phi\right) - \frac{\partial V}{\partial \Phi} = 0 \tag{62}$$

3.3.2. *The electromagnetic field.* Let us consider the electromagnetic field described by two one-forms: the electric field intensity $\boldsymbol{E}$, and the magnetic flux density $\boldsymbol{B}$. The two vectors are determined through the Maxwell equations

$$\begin{aligned} \mathrm{div}_x(\boldsymbol{E}) &= \frac{\rho}{\epsilon_0} \\ \mathrm{div}_x(\boldsymbol{B}) &= 0 \\ \mathrm{curl}_x(\boldsymbol{E}) &= -\frac{\partial}{\partial t}\boldsymbol{B} \\ \mathrm{curl}_x(\boldsymbol{B}) &= \mu_0\boldsymbol{j} + \epsilon_0\mu_0\frac{\partial}{\partial t}\boldsymbol{E} , \end{aligned} \tag{63}$$

where ϵ_0 and μ_0 the electric permittivity and magnetic parmeability of vacuum respectivel; and ρ and $\boldsymbol{j}$ the electric charge and electric current densities respectively. Because of the over-determinacy of the Maxwell equations, both fields can be derived by means of a single four-potential, $\boldsymbol{A}$; and the entire description of the magnetic field can be given in terms of a tensor - the Faraday tensor,

$$\boldsymbol{F}(\vec{x},\vec{y}) = \nabla_{\vec{x}}\boldsymbol{A}(\vec{y}) - \nabla_{\vec{y}}\boldsymbol{A}(\vec{x}) , \tag{64}$$

which is skew symmetric (and thus trace-free). Using the Faraday tensor, the Maxwell equations are reduced to

$$\begin{aligned} \mathrm{d}\boldsymbol{F} &= 0 \\ \star\mathrm{d}(\star\boldsymbol{F}) &= \boldsymbol{J} , \end{aligned} \tag{65}$$

where $\boldsymbol{J}$ the four-current density and $\star$ the Hodge star operator [**33**].

With respect to the Faraday tensor, the stress-energy-momentum tensor for (free or not) electromagnetic fields is written as

$$\begin{aligned} \boldsymbol{T}(\vec{x}_\alpha,\vec{x}_\beta) =& g^{-1}\big(\mathbf{x}^\mu,\mathbf{x}^\nu\big)\boldsymbol{F}(\vec{x}_\alpha,\vec{x}_\mu)\boldsymbol{F}(\vec{x}_\beta,\vec{x}_\nu) \\ & -\frac{1}{4}\boldsymbol{g}(\vec{x}_\alpha,\vec{x}_\beta)g^{-1}\big(\mathbf{x}^\kappa,\mathbf{x}^\lambda\big)g^{-1}\big(\mathbf{x}^\mu,\mathbf{x}^\nu\big)\boldsymbol{F}(\vec{x}_\kappa,\vec{x}_\mu)\boldsymbol{F}(\vec{x}_\lambda,\vec{x}_\nu) . \end{aligned} \tag{66}$$

In some local coordinates, this expression becomes

$$T_{\alpha\beta} = g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu} - \frac{1}{4}g_{\alpha\beta}g^{\kappa\lambda}g^{\mu\nu}F_{\kappa\mu}F_{\lambda\nu} .$$

3.3.3. *The perfect fluid.* The first attempt to model realistic baryonic matter is that of a classical macroscopic fluid, described by two real-valued functions, the matter-energy density ρ the isotropic pressure P, which are scalars (zero-forms); the energy flux $\boldsymbol{q}$, which is a vector; and the anisotropic pressure (or viscosity) $\boldsymbol{\pi}$, which is a rank two tensor. These are measured by an observer, whose velocity $\vec{v}$, which is a one-form as well, is also required for the description. The stress-energy-momentum tensor is then simply given as [**33**]

(67)

$$\boldsymbol{T}(\vec{x},\vec{y}) = \big(\rho+P\big)\boldsymbol{v}(\vec{x})\otimes\boldsymbol{v}(\vec{y})+P\boldsymbol{g}\,(\vec{x},\vec{y})+\frac{1}{2}\big(\boldsymbol{q}(\vec{x})\otimes\boldsymbol{v}(\vec{y})-\boldsymbol{q}(\vec{y})\otimes\boldsymbol{v}(\vec{x})\big)+\boldsymbol{\pi}\,(\vec{x},\vec{y})\ ,$$

where $\otimes$ denotes tensor differentiation (the Kronecker product). In some local coordinate system, this becomes

$$T_{ab} = \big(\rho + P\big)v_a v_b + P g_{ab} + \frac{1}{2}\big(q_a v_b + q_b v_a\big) + \pi_{ab}\,.$$

We should note that the energy flux and anisotropic pressure are defined purely on the space-like submanifold of the space-time; thus, given the observer's velocity is a time-like vector, we have

$$g^{-1}\big(\boldsymbol{v},\boldsymbol{q}\big) = 0 \quad \text{and} \quad g^{-1}\big(\boldsymbol{v},\boldsymbol{\pi}\big) = 0\,; \tag{68}$$

moreover, the tensor of anisotropic pressure is trace-free and symmetric.

3.3.4. *The Liouville operator.* The second attempt to a realistic description of baryonic matter is that of collisionless matter distributed according to a non-negative real-valued function, f, defined on the mass shell that represents the density of particles with given space-time position and momentum; the mass shell is a hypersurface $\mathcal{P}$ in the cotangent bundle $\mathrm{T}\mathcal{M}$ [**26**]. It is important to note that each point in the cotangent bundle refers to the energy and momentum of a specific particle. The momenta of the particles, p_μ, are 1-forms defined on the cotangent bundle, whose dispersion relation

$$p_\mu p_\nu g^{\mu\nu} = -m^2\,, \tag{69}$$

where m is the mass of the particles, identifies the mass shell. Obviously, in the case of massive particles, this relation defines the mass shell as a hyperboloid within the past half of the light-cone - hence within all possible "futures" of the observer. In the case of massless particles, though, this relation is simplified to

$$p_\mu p_\nu g^{\mu\nu} = 0$$

and the mass shell coincides with the past half of the light-cone.

The Liouville operator describes the transport of the particles with positions (in local coordinates) x^α and momenta p_α, under the assumption that these particles move along geodesics; this operator can be defined as in [**28**],

$$\pounds = \frac{\partial\mathcal{H}}{\partial p_\mu}\frac{\partial}{\partial x^\mu} - \frac{\partial\mathcal{H}}{\partial x^\mu}\frac{\partial}{\partial p_\mu}\,,$$

where

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

is the Hamiltonian corresponding to the geodesic motion of (massive) particles. Hence, the Liouville operator is

$$\pounds = g^{\mu\nu} p_\mu \frac{\partial}{\partial x^\nu} - \frac{1}{2} p_\mu p_\nu \frac{\partial g^{\mu\nu}}{\partial x^\lambda} \frac{\partial}{\partial p_\lambda} \,. \tag{70}$$

Given that the particles are non-interacting and non-affected by any external force (*eg.* the electromagnetic force), then the action of the Liouville operator on the distribution function yields zero,

$$\pounds f = 0 \,. \tag{71}$$

What is important in our case is that the momenta are not necessarily spanned on the local coordinates; they can be spanned on the basis of the Bianchi group (the generators $\vec{\omega}_a$) and a transversal ($\vec{\zeta}$), so long as they are geodesic. As for the former, this is self-evident from the fact that the Bianchi group acts transitively on the space-time, hence the adjoint representations of its orbits are the geodesics on the homogeneous hypersurface; as for the latter, the theorem proposed in the Introduction and proved in Section 3 suggests that $\vec{\zeta}$ is also geodesic. Consequently, such a basis exists, hence the momenta can be spanned accordingly, and subsequently the partial derivatives with respect to the local coordinates in Eq. (70) can be substituted with Lie derivative along the basis of the group and the transversal.

The distribution of particles on the space-time functions as a source for the Einstein equations, hence it yields the stress-energy-momentum tensor. The usual variables of matter (mass-energy density, energy flux, isotropic pressure and viscosity) are respectively the first and second moments of f. Specifically, the stress-energy-momentum tensor is defined as

$$\boldsymbol{T}(\vec{x}, \vec{y}) = \int \mathrm{d}\Omega_p f \boldsymbol{p}(\vec{x}) \boldsymbol{p}(\vec{y}) \ , \tag{72}$$

where $\boldsymbol{p}(\vec{x}) = x^\mu p_\mu$ the span of the momenta along the vector $\vec{x}$ and $\mathrm{d}\Omega_p$ the volume in the momenta space. In some local coordinates, this would be

$$T_{\alpha\beta} = \int \mathrm{d}\Omega_p f(x,p) p_\alpha p_\beta \ .$$

We should note that $\mathrm{d}\Omega_p$ does not assume its usual form, as it significantly depends on the foliation of space-time according to $\vec{\zeta}$ and $\vec{\xi}_a$.

CHAPTER 3

The Field Equations on the Group Frame

1. Introduction

This chapter deals with the calculation of the connection and curvature of the space-time, derived on the frame of the group, so that the Einstein field equations can be written with explicit mention to the particular action, but with an explicit mention of a coordinate system. The first part deals with the form of the metric, while the second and third parts refer to the computation of the Riemann and the Ricci tensors in the case of a non-null and a null collineation respectively.

2. Construction of the Space-Time

The Theorem 2.1 as stated requires only that the Bianchi group acts freely and regularly, by homotheties on a pseudo-Riemannian manifold $(\mathcal{V}_4, \boldsymbol{g})$; hence, there are homogeneous hypersurfaces of the manifold that constitute a submanifold $(\mathcal{M}, \boldsymbol{\gamma})$ that is identified with the group. A result of the theorem 2.1 is that any transversal vector field $\vec{\zeta}$ in the quotient of the group action commutes with the generators of the group and it is tangent to some geodesic. If these hold, then the afore-mentioned space-time can be constructed as follows.

First of all, the space-time $\mathcal{V}$ is now decomposed to $[\mathbb{R} \times \mathcal{G}]$; *i.e.*, there can always be found such a coordinate system $\{z, w_i\}$ that is attached to the group and its quotient in such a manner that it respects the properties of both. In this system, every vector $\vec{v}$ is spanned as

$$\vec{v} = v_z \frac{\partial}{\partial z} + v_i \frac{\partial}{\partial w_i} , \tag{73}$$

where v_z is a $1-$vector tangent to the family of curves C_ζ and v_a is a $3-$vector spanned along the group. As expressed by the theorem,

- $\vec{\zeta} = \dfrac{\partial}{\partial z}$ and $\vec{\xi}_a$ are commuting, which make them $\vec{\zeta}$ invariant under the action of the group and, as such, independent from the $\vec{\xi}_a$ - thus, the entire space-time can be covered;
- $\vec{\zeta}$ is geodesic, while $\vec{\xi}_a$ follow the adjoint representations of the group, being geodesic by definition (through the action of the group being

free and regular)[1] - thus, the coordinate system is non-orthogonal but curvilinear; and

- $\vec{\zeta}$ is null.[2]

As a result, the coordinate system can be adjusted to each group and each action without any problem, maintaining at all times the principle that it must locally equivalent to an orthogonal (Cartesian) coordinate system.

As stated in previous chapter, the transversal needs not be null; this is an option that allows for a convenient, yet generic normalisation condition of $\vec{\zeta}$ that will be preserved throughout the different "movements" along the geodesics C_ζ and the orbits of the group. However, this choice has an interesting repercussion that somewhat limits the generality of the theorem. As we will see, this limitation is not so strong, as it implies a violation of the conditions of the theorem (namely, the free and regular action of the group); however, this repercussion needs to be discussed.[3]

Given the null causal character of $\vec{\zeta}$, we express the components of the metric as

$$\boldsymbol{g}(\vec{\zeta},\vec{\zeta}) = 0 \ , \ \ \boldsymbol{g}(\vec{\zeta},\vec{\xi}_a) = \boldsymbol{g}(\vec{\xi}_a,\vec{\zeta}) = \beta_\alpha \ \text{ and } \ \boldsymbol{g}(\vec{\xi}_a,\vec{\xi}_b) = \gamma_{ab}\,, \tag{74}$$

where γ_{ab} the $3-$metric on the submanifold $\mathcal{M}$ and β_a the 'shift' vectors. We can assume that the inverse metric has a similar form

$$g^{-1}(\boldsymbol{\zeta},\boldsymbol{\zeta}) = a \ , \ \ g^{-1}(\boldsymbol{\zeta},\boldsymbol{\xi}^a) = g^{-1}(\boldsymbol{\xi}^a,\boldsymbol{\zeta}) = b^a \ \text{ and } \ g^{-1}(\boldsymbol{\xi}^a,\boldsymbol{\xi}^b) = c^{ab}\,, \tag{75}$$

where $\boldsymbol{\zeta}$ the 1-form corresponding to $\vec{\zeta}$ and $\boldsymbol{\xi}^a$ the 1-forms corresponding to the generators $\vec{\xi}_a$;[4] and where a the inverse 'lapse' function, b^a the inverse 'shift' vectors and c^{ab} the inverse $3-$metric on the submanifold $\mathcal{M}$. Concerning the inversion of the metric, the following relations hold

$$\begin{aligned}
(i) &\quad \beta_i b^i = 1\\
(ii) &\quad \beta_i c^{ia} = 0\\
(iii) &\quad \gamma_{ai} b^i = -a\beta_a\\
(iv) &\quad \gamma_{ai} c^{bi} = \delta_a{}^b - \beta_a b^b\\
(v) &\quad \gamma_{ij} c^{ij} = 2\\
(vi) &\quad \gamma_{ij} b^i b^j = -a
\end{aligned} \tag{76}$$

[1]This does not mean that $\vec{\xi}_\alpha$ are geodesic; they can be geodesic if-f the orbits of the group $\mathcal{G}$ are geodesics on $\mathcal{M}$, which can be true if-f the group can be expressed as a product of one-parameter subgroups.

[2]We need to remember that this is optional; $\vec{\zeta}$ can be chosen not to be null, but other normalisation conditions for it are hard to be found.

[3]Some of this discussion will happen within the context of this thesis; some of it must be reserved for later.

[4]Given $\vec{\zeta}$ and $\vec{\xi}_a$ form a tetrad basis, then $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}^a$ form the dual basis, such that

$$\boldsymbol{\zeta}(\vec{\zeta}) = 1\,, \quad \boldsymbol{\zeta}(\vec{\xi}_a) = 0\,, \quad \boldsymbol{\xi}^b(\vec{\zeta}) = 0 \quad \text{and} \quad \boldsymbol{\xi}^b(\vec{\xi}_a) = \delta_a{}^b\,.$$

This implies that the inverse $3-$metric c^{ab} is degenerate. This does not affect the proper $3-$metric γ_{ab}, which is nondegenerate; however, there is an interesting repercussion. In the usual treatments, where the 3+1 formalism can be applied, the $3-$metric and its inverse have the same nondegeneracy and can function equally well as metrics, in *e.g.* 'lowering' indices of 1-forms and 'raising' indices of vectors, respectively, on the submanifold they are defined. This is not the case here for two reasons: First γ_{ab} and c_{ab} are not tensor, but matrices; so, neither can play the role of a metric for the submanifold $\mathcal{M}$.[5] Second, the degeneracy of c^{ab}, would make it impossible to use the respective inverse $3-$metric to 'raising' indices of vectors - as all vectors multiplied by it would yield 0.

We should note that the components of the metric and its inverse are functions of all four coordinates: both the transversal one (ζ), referring to the quotient, and the three unspecified tangential ones, referring to the group.

We should also note that, in the absence of specific coordinates on $\mathcal{M}$, Latin letters are used to describe the indices on the Bianchi group. Similarly, the components of the metric, the connection and the curvature tensors will be given with respect to the null vector and the generators of the group. If one wishes to express them with respect to coordinates on the space-time, z can be chosen as a coordinate along a null direction in the quotient of the group, with the remaining three being specified by the choice of a frame. Choosing the frame means specifying the form of the generators; if this is done, then the relations given henceforth are easily transformed. An special case would be that where the frame will be chosen so as the coordinate axis would coincide with the generators $\vec{\xi_a}$; in this case, the form of the metric, the connection and the curvature tensors would not change at all.

2.1. The case of null orbits. We should make a comment about the possibility of $a = 0$.

If we assume $a = 0$, right away, then the inversion conditions of the metric become

$$\begin{aligned} (i) &\quad \beta_i b^i = 1 \\ (ii) &\quad \beta_i c^{ia} = 0 \\ (iii) &\quad \gamma_{ai} b^i = 0 \\ (iv) &\quad \gamma_{ai} c^{bi} = \delta_a{}^b - \beta_a b^b \\ (v) &\quad \gamma_{ij} c^{ij} = 2 \end{aligned} \tag{77}$$

The fact that $\gamma_{ij} b^i b^j = 0$ may mean two things:

- If γ_{ab} is invertible, so that there are no degeneracy issues with the $3-$metric on manifold $\mathcal{M}$, then

$$b^a = 0\,,$$

[5] This is also true for β_a and b^a; they are not vectors and co-vectors, resepctively, in the usual sense.

and, therefore,

$$\beta_a = 0\,.$$

Thus, the orbits of the group are two-dimensional. This clearly contradicts the conditions of the theorem 2.1, namely of a free and regular action of the group. Therefore, this case is outside the limits of this thesis.

- If γ_{ab} is non-invertible, then the 3−metric on the homogeneous submanifold $\mathcal{M}$ has to be degenerate; that is, the orbits of the group are null. This is an interesting case that is within the scope of the thesis (as it does not violate the conditions of the theorem 2.1), but it is a 'degenerate' case. As we will see in the following chapters, this case reduces the Einstein equations to an underdetermined system, which is impossible to solve; the way to proceed is to assume additional constraints on the derivatives of the metric.

Interestingly, there are two courses of action that would allow us to overcome this difficulty.

The first is to consider the case of $a = 0$ separately, by admitting this as the normalisation condition. More precisely, we can work with a different version of the theorem that would requires that the null transversal $\vec{\zeta}$ commutes with the generators of the group, is geodesic, and its corresponding 1-form $\boldsymbol{\zeta}$ is null; this would imply that

$$g^{-1}(\boldsymbol{\zeta},\boldsymbol{\zeta})\,,$$

and that the 3−metric of the homogeneous submanifold $\mathcal{M}$ is degenerate, thus restricting the applicability of the second theorem to the one case the original theorem treats as degenerate - the orbits of the group being null. In this case, the metric will have the form

$$\boldsymbol{g}(\vec{\zeta},\vec{\zeta}) = \alpha \;\; , \;\; \boldsymbol{g}(\vec{\zeta},\vec{\xi}_a) = \boldsymbol{g}(\vec{\xi}_a,\vec{\zeta}) = \beta_\alpha \;\text{ and }\; \boldsymbol{g}(\vec{\xi}_a,\vec{\xi}_b) = \gamma_{ab}\,, \tag{78}$$

where γ_{ab} the 3−metric on the submanifold $\mathcal{M}$, β_a the 'shift' vector and α the lapse function. And the inverse metric will be

$$g^{-1}(\boldsymbol{\zeta},\boldsymbol{\zeta}) = 0 \;\; , \;\; g^{-1}(\boldsymbol{\zeta},\boldsymbol{\xi}^a) = g^{-1}(\boldsymbol{\xi}^a,\boldsymbol{\zeta}) = b^a \;\text{ and }\; g^{-1}(\boldsymbol{\xi}^a,\boldsymbol{\xi}^b) = c^{ab}\,, \tag{79}$$

with the following inversion conditions holding

$$\begin{aligned} (i) \quad & \beta_i b^i = 1 \\ (ii) \quad & \beta_i c^{ia} + \alpha b^a = 0 \\ (iii) \quad & \gamma_{ai} b^i = 0 \\ (iv) \quad & \gamma_{ai} c^{bi} + \beta_a b^b = \delta_a{}^b \end{aligned} \tag{80}$$

This would lead to a similar discussion like the one we will present in this thesis; however, it would consider a very limiting case, as it is equivalent to a theorem that treats only null homogeneous submanifolds. In contrast, what we will concern ourselves in the thesis is the current version of the theorem, keeping in mind that it can treat null orbits of the group only as a degenerate case.[6]

The second and more generic one would be to restate the theorem by removing the requirement of the null causal character of the transversal collineation. In this case, the metric will have the form

$$\boldsymbol{g}(\vec{\zeta},\vec{\zeta}) = \alpha \ , \ \ \boldsymbol{g}(\vec{\zeta},\vec{\xi}_a) = \boldsymbol{g}(\vec{\xi}_a,\vec{\zeta}) = \beta_\alpha \ \text{ and } \ \boldsymbol{g}(\vec{\xi}_a,\vec{\xi}_b) = \gamma_{ab} \, , \tag{81}$$

where γ_{ab} the 3−metric on the submanifold $\mathcal{M}$, β_a the 'shift' vector and α the lapse function. And assuming that the inverse metric has a similar form

$$g^{-1}(\boldsymbol{\zeta},\boldsymbol{\zeta}) = a \ , \ \ g^{-1}(\boldsymbol{\zeta},\boldsymbol{\xi}^a) = g^{-1}(\boldsymbol{\xi}^a,\boldsymbol{\zeta}) = b^a \ \text{ and } \ g^{-1}(\boldsymbol{\xi}^a,\boldsymbol{\xi}^b) = c^{ab} \, , \tag{82}$$

where the inversion conditions that must be met are

$$\begin{aligned} (i) \quad & \alpha a + \beta_i\beta^i = 1 \\ (ii) \quad & \beta_i c^{ia} + \alpha b^a = 0 \\ (iii) \quad & \gamma_{ai} b^i + a\beta_a = 0 \\ (iv) \quad & \gamma_{ai} c^{bi} + \beta_a b^b = \delta_a{}^b \end{aligned} \tag{83}$$

These conditions can be solved in general, given the 4−metric $\boldsymbol{g}$ is not degenerate, in the following form

$$a = \frac{\det(\gamma)}{\det(\boldsymbol{g})} \, , \quad b^a = \frac{\det(\gamma^a)}{\det(\boldsymbol{g})} \quad \text{and} \quad c^{ab} = \frac{\det(g^{a,b})}{\det(\boldsymbol{g})}$$

where γ^a denotes the matrix γ with the a−th column replaced by the 3−vector β, and $g^{a,b}$ denotes the matrix g with the a−th column replaced by the 4−vector (α,β) and the b-th row omitted.[7] From here, one can easily proceed, firstly, to restate the theorem removing the requirement $\vec{\zeta}$ to be null and, secondly, to compute the relevant curvature measures for the space-time (as we do for the metric of eq. (74) in the remainder of the chapter).

This track is probably preferable to the one we relied on, but it has a serious predicament: Either, there is no normalisation condition for $\vec{\zeta}$ (whereas theorem 2.1 would prove more general), but then the "movement" of $\vec{\zeta}$ along the group and the geodesics may alter its length to the point where the space-time will appear non-metric.[8] Or, some weaker and more complicated normalisation condition must be sought, to ensure that the space-time will remain metric.[9]

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[6]There is a different approach concerning specifically the analysis of null hypersurfaces by Gourgoulhon and Jaramillo [**38**].

[7]Despite appearances, this more generic version does not restore the usual 3+1 formalism. In the usual treatment, the fact that $\vec{\zeta}$ is strictly timelike, allows for a strictly non-degenerate γ and c matrices. Therefore, the inversion of the metric $\boldsymbol{g}$ leads to more familiar and easy to treat forms.

[8]This is easy to imagine. If $\vec{\zeta}$ does not have some normalisation condition, then $\boldsymbol{g}(\vec{\zeta},\vec{\zeta})$ will change when "moved" along some direction in the space-time; this may result to the covariant derivative of the metric being non-zero.

[9]Here, one should remember that normality and orthogonality cannot be used for the transversal, as they would violate the conditions of the theorem; and some other causal

3. The Structure of the Space-Time

If the Bianchi group acts freely and regularly so that a null transversal is invariant, then the orbits of the group are not restricted to spacelike or timelike character only; they can be null as well. However, we have argued that null orbits demonstrate a degeneracy, which requires separate treatment. In this section,m we will deal with the cases where the group acts by means of spacelike and timelike homogeneous submanifolds - as in the cases of Figs. 1a and 1b, respectively.

It is important to note that, according to a theorem laid out by McIntosh, such non-null homothetic vector fields -as the generators in this case- are always shear-free and their expansion is given as $\frac{3}{2}\phi_a$ [**39**].

This section is dedicated to the computation of the Christoffel symbols and of the Riemann and Ricci curvature tensors, so as to reach to Einstein's field equations.

3.1. Covariant and Lie derivatives of the metric. Before moving to the curvature tensors, the matter fields and the field equations, we shall calculate the necessary relations for these - namely, the covariant and the Lie derivatives of the metric. The components of the metric tensor and its inverse are given by eqs. (74) and (75) respectively; in this form, they are scalars, so the two types of derivatives coincide.

Using eqs. (18) in eqs. (74), we can obtain

$$\mathcal{L}_{\vec{\zeta}}\beta_a = k_a \ , \tag{84}$$

$$\mathcal{L}_{\vec{\xi}_b}\beta_a = \phi_b\beta_a - \beta_m C^m{}_{ab} \ , \tag{85}$$

$$\mathcal{L}_{\vec{\zeta}}\gamma_{ab} = k_{ab} \ \text{ and} \tag{86}$$

$$\mathcal{L}_{\vec{\xi}_c}\gamma_{ab} = \phi_c\gamma_{ab} - \gamma_{am}C^m{}_{bc} - \gamma_{bm}C^m{}_{ac} \, . \tag{87}$$

To obtain similar relations for the components of the inverse, we must keep in mind that the derivative of an inverse matrix follows the rule

$$\mathrm{d}(A^{ab}) = -A^{ai}A^{bj}\mathrm{d}(A_{ij}) \, .$$

Through these, we may reach

$$\mathcal{L}_{\vec{\zeta}}a = -2ab^i k_i - b^i b^j k_{ij} \ , \tag{88}$$

$$\mathcal{L}_{\vec{\xi}_a}a = -\phi_a a \ , \tag{89}$$

$$\mathcal{L}_{\vec{\zeta}}b^a = -\big(ac^{ai} - b^a b^i\big)k_i - c^{ai}b^j k_{ij} \ , \tag{90}$$

$$\mathcal{L}_{\vec{\xi}_b}b^a = -\phi_b b^a + b^n C^a{}_{nb} \ , \tag{91}$$

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character cannot be used as well, as it would limit its scope, perhaps even further than assuming $\vec{\zeta}$ to be null.

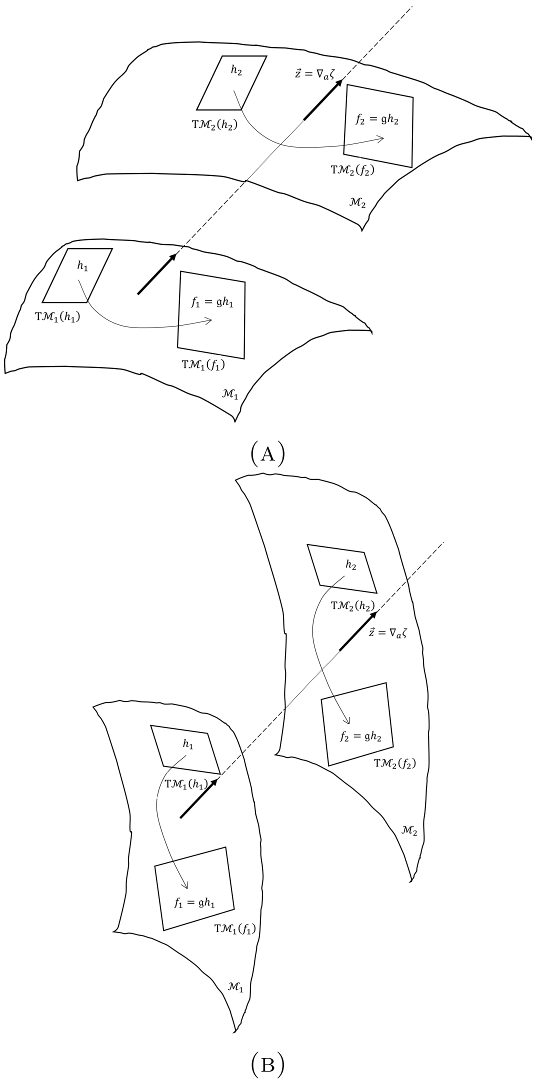

FIGURE 1. Two possible foliations of the space-time along a lightlike transversal.

(92) $$\mathcal{L}_{\vec{\zeta}} c^{ab} = -\big(c^{ai} b^b + c^{bi} b^a\big) k_i - c^{ai} c^{bj} k_{ij} \quad \text{and}$$

(93) $$\mathcal{L}_{\vec{\xi}_c} c^{ab} = -\phi_c c^{ab} + c^{an} C^b{}_{nc} + c^{bn} C^a{}_{nc} \,.$$

Furthermore, we should mention that the contractions of the inverse 3-metric, c^{ab}, with the shift vector, β_a, is zero; from this we may prove that the contraction of the Lie derivative of the inverse 3-metric with the shift vector is also zero - namely

(94) $$\beta_i \mathcal{L}_{\vec{\zeta}} c^{ia} = 0 \,.$$

Finally, the Lie derivatives of the k are also easy to define,

$$\mathcal{L}_{\vec{z}} k_{ab} = \ell_{ab} \,, \tag{95}$$

and derive

$$\mathcal{L}_{\vec{\xi}_c} k_{ab} = \phi_c k_{ab} - k_{am} C^m{}_{bc} - k_{bm} C^m{}_{bc} \,. \tag{96}$$

3.2. The Christoffel symbols. Specifying the arbitrary vectors as the invariant null vector field, $\vec{\zeta}$, and the generators of the Bianchi groups, $\vec{\xi}_a$, we may obtain the Christoffel symbols of the first kind directly from eq. (20).

(97)
$$\begin{aligned}
&\Gamma_{zzz} = 0 \ , \\
&\Gamma_{zza} = \Gamma_{zaz} = 0 \ , \\
&\Gamma_{zab} = \frac{1}{2}\big(- k_{ab} + \phi_a \beta_b + \phi_b \beta_a + \beta_m C^m{}_{ab}\big) \ , \\
&\Gamma_{azz} = k_a \ , \\
&\Gamma_{azb} = \Gamma_{abz} = \frac{1}{2}\big(k_{ab} - \phi_a \beta_b + \phi_b \beta_a - 3\beta_m C^m{}_{ab}\big) \text{ and} \\
&\Gamma_{abc} = \frac{1}{2}\big(- \phi_a \gamma_{bc} + \phi_b \gamma_{ac} + \phi_c \gamma_{ab} - \gamma_{am} C^m{}_{bc} + 3\gamma_{bm} C^m{}_{ca} - 3\gamma_{cm} C^m{}_{ab}\big) \,.
\end{aligned}$$

The Christoffel symbols of the second kind, as they are usually defined,

$$\Gamma^\alpha{}_{\beta\gamma} = g^{\alpha\mu}\Gamma_{\mu\beta\gamma}$$

Hence, the space-time connection has the following components

$$\Gamma^z{}_{zz} = b^i k_i \ , \tag{98}$$

$$\Gamma^z{}_{zb} = \Gamma^z{}_{bz} = \frac{1}{2}\big(b^i k_{ib} - \phi_i b^i \beta_b + \phi_b - 3b^n \beta_m C^m{}_{nb}\big) \ , \tag{99}$$

(100)
$$\Gamma^z{}_{bc} = \frac{1}{2}\left(-a k_{bc} - \phi_i b^i \gamma_{bc} + 2a\beta_m C^m{}_{bc} + 3\gamma_{bm} b^n C^m{}_{nc} + 3\gamma_{cm} b^n C^m{}_{nb}\right) \ ,$$

$$\Gamma^a{}_{zz} = c^{ai} k_i \ , \tag{101}$$

$$\Gamma^a{}_{zb} = \Gamma^a{}_{bz} = \frac{1}{2}\big(c^{ai} k_{ib} - \phi_i c^{ia} \beta_b - 3\beta_m c^{an} C^m{}_{nb}\big) \text{ and} \tag{102}$$

$$\begin{aligned}
\Gamma^a{}_{bc} =& \frac{1}{2}\Big(- b^a k_{bc} - \phi_i c^{ia} \gamma_{bc} + \phi_b \delta_c{}^a + \phi_c \delta_b{}^a \\
& - C^a{}_{bc} + b^a \beta_m C^m{}_{bc} + 3c^{an}\big(\gamma_{bm} C^m{}_{nc} + \gamma_{cm} C^m{}_{nb}\big)\Big) \,.
\end{aligned} \tag{103}$$

These Christoffel symbols refer to the basis of the group $\mathcal{G}$ rather than to the basis of some coordinates on the space-time. Hence, the latin indices do not refer to coordinates on $\mathcal{M}$, but to the basis of the generators $\vec{\xi}_a$.

Consequently, the covariant derivatives of $\vec{\zeta}$ and $\vec{\xi}_a$ are

$$\begin{aligned}
\nabla_{\vec{\zeta}}\vec{\zeta} =&\Gamma^z{}_{zz}\vec{\zeta} + \Gamma^i{}_{zz}\vec{\xi}_i = \\
=&b^i k_i\vec{\zeta} + c^{ai}k_i\vec{\xi}_i \ , \\
\nabla_{\vec{\xi}_a}\vec{\zeta} =&\nabla_{\vec{\zeta}}\vec{\xi}_a = \Gamma^z{}_{za}\vec{\zeta} + \Gamma^i{}_{za}\vec{\xi}_i = \\
=&\frac{1}{2}\big(b^i k_{ia} - \phi_i b^i\beta_a + \phi_a - 3b^n\beta_m C^m{}_{na}\big)\vec{\zeta} \\
&\qquad + \frac{1}{2}\left(c^{ij}k_{ja} - \phi_j c^{ij}\beta_a - 3c^{in}\beta_m C^m{}_{na}\right)\vec{\xi}_i \ \text{ and} \\
\nabla_{\vec{\xi}_b}\vec{\xi}_a =&\Gamma^z{}_{ab}\vec{\zeta} + \Gamma^i{}_{ab}\vec{\xi}_i = \\
=&\frac{1}{2}\Big(- ak_{ab} - \phi_i b^i\gamma_{ab} + 2a\beta_m C^m{}_{ab} + 3b^n\big(\gamma_{am}C^m{}_{nb} + \gamma_{bm}C^m{}_{na}\big)\Big)\vec{\zeta} \\
&+ \frac{1}{2}\Big(- b^i k_{ab} - \phi_j c^{ij}\gamma_{ab} + \phi_a\delta_b{}^i + \phi_b\delta_a{}^i \\
&\qquad - C^i{}_{ab} + b^i\beta_m C^m{}_{ab} + 3c^{in}\big(\gamma_{am}C^m{}_{nb} + \gamma_{bm}C^m{}_{na}\big)\Big)\vec{\xi}_i \,.
\end{aligned}$$

We notice that $\nabla_{\vec{\zeta}}\vec{\zeta} \neq 0$ as is enforced by the geometry of the space-time; hence, additional restrictions must be imposed to the metric in the form of

$$k_a = 0\,. \tag{104}$$

As for the independence of the transversal and the tangential directions, we notice that it is preserved.

3.3. The Riemann-Christoffel tensor. We define the following expressions

$$\mu_{ab} = k_{ab} + \phi_a\beta_b - \phi_b\beta_a + \beta_m C^m{}_{ab} \tag{105}$$

and

$$\nu_{abc} = \phi_c\gamma_{ab} - \gamma_{am}C^m{}_{bc} - \gamma_{bm}C^m{}_{bc}\ , \tag{106}$$

which appear in the covariant derivatives. Hence, the covariant derivatives of the transversal and tangential vectors with respect to one another become

(107)

$$\nabla_{\vec{\zeta}}\vec{\zeta} = 0$$

(108)

$$\nabla_{\vec{\zeta}}\vec{\xi}_a = \nabla_{\vec{\xi}_a}\vec{\zeta} = \frac{1}{2}\big(b^i\mu_{ai}\big)\vec{\zeta} + \frac{1}{2}\big(c^{ij}\mu_{aj}\big)\vec{\xi}_i$$

(109)

$$\nabla_{\vec{\xi}_a}\vec{\xi}_b = \frac{1}{2}\big(- \mathrm{a}k_{ab} - b^i\nu_{abi}\big)\vec{\zeta} + \frac{1}{2}\big(- b^i k_{ab} + \phi_a\delta_b{}^i + \phi_b\delta_a{}^i + C^i{}_{ab} - c^{ij}\nu_{abj}\big)\vec{\xi}_i \,.$$

The two expressions are differentiated as follows

$$\mathcal{L}_{\vec{\zeta}}\mu_{ab} = \mathcal{L}_{\vec{\zeta}}k_{ab} = \ell_{ab}\,, \tag{110}$$

$$\mathcal{L}_{\vec{\xi}_c}\mu_{ab} = \phi_c\mu_{ab} - \mu_{am}C^m{}_{bc} - \mu_{bm}C^m{}_{ac}\,, \tag{111}$$

and

$$\mathcal{L}_{\vec{\zeta}}\nu_{abc} = \phi_c k_{ab} - k_{am}C^m{}_{bc} - k_{bm}C^m{}_{bc}\,, \tag{112}$$

$$\mathcal{L}_{\vec{\xi}_d}\nu_{abc} = \phi_d\nu_{abc} - \nu_{abm}C^m{}_{cd} - \nu_{amc}C^m{}_{bd} - \nu_{mbc}C^m{}_{ad}\,. \tag{113}$$

However, important are also the differentiations of the contractions of these expressions with the inverse metric, specifically of the quantities $b^i\mu_{ai}$, $c^{ai}\mu_{bi}$ and $b^i\nu_{abi}$, $c^{di}\nu_{abi}$. The first pair is differentiated as

$$\mathcal{L}_{\vec{\zeta}}\big(b^i\mu_{ai}\big) = b^i\ell_{ai} - b^i c^{jk}k_{ij}\mu_{ak}\,, \tag{114}$$

$$\mathcal{L}_{\vec{\xi}_c}\big(b^i\mu_{ai}\big) = -b^i\mu_{im}C^m{}_{ac}\,, \tag{115}$$

and

$$\mathcal{L}_{\vec{\zeta}}\big(c^{bi}\mu_{ai}\big) = c^{bi}\ell_{ai} - c^{bi}c^{jk}k_{ij}\mu_{ak}\,, \tag{116}$$

$$\mathcal{L}_{\vec{\xi}_c}\big(c^{bi}\mu_{ai}\big) = c^{in}\mu_{ai}C^b{}_{nc} - c^{bi}\mu_{mi}C^m{}_{ac}\,; \tag{117}$$

the second pair is differentiated as

$$\mathcal{L}_{\vec{\zeta}}\big(b^i\nu_{abi}\big) = -b^i c^{jk}k_{ij}\nu_{abk} + \phi_i b^i k_{ab} - b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an}\big)\,, \tag{118}$$

$$\mathcal{L}_{\vec{\xi}_c}\big(b^i\nu_{abi}\big) = -b^i\big(\nu_{ami}C^m{}_{bc} + \nu_{bmi}C^m{}_{ac}\big)\,, \tag{119}$$

and

$$\mathcal{L}_{\vec{\zeta}}\big(c^{di}\nu_{abi}\big) = -c^{di}c^{jk}k_{ij}\nu_{abk} + \phi_i c^{di}k_{ab} + c^{dn}\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an}\big)\,, \tag{120}$$

$$\mathcal{L}_{\vec{\xi}_c}\big(c^{di}\nu_{abi}\big) = -c^{di}\big(\nu_{ami}C^m{}_{bc} + \nu_{bmi}C^m{}_{ac}\big) - c^{dn}\nu_{abm}C^m{}_{cn}\,. \tag{121}$$

Thus, taking the second-order covariant derivatives, we have

$$\nabla_{\vec{\zeta}}\nabla_{\vec{\zeta}}\vec{\zeta} = 0\,, \tag{122}$$

$$\begin{aligned}\nabla_{\vec{\zeta}}\nabla_{\vec{\zeta}}\vec{\xi}_a =& \frac{1}{2}\big(\mathcal{L}_{\vec{\zeta}}(b^i\mu_{ai})\big)\vec{\zeta} + \frac{1}{4}\big(b^i\mu_{ji}\big)\big(c^{jk}\mu_{ak}\big)\vec{\zeta}\\ &+ \frac{1}{2}\big(\mathcal{L}_{\vec{\zeta}}(c^{ij}\mu_{aj})\big)\vec{\xi}_i + \frac{1}{4}\big(c^{ik}\mu_{jk}\big)\big(c^{jl}\mu_{al}\big)\vec{\xi}_i =\\ =&\frac{1}{4}\Big(2b^i\ell_{ia} - b^i c^{jk}\big(2k_{ij} - \mu_{ji}\big)\mu_{ak}\Big)\vec{\zeta}\\ &+ \frac{1}{4}\Big(2c^{ij}\ell_{ja} - c^{ik}c^{jl}\big(2k_{jk} - \mu_{jk}\big)\mu_{al}\Big)\vec{\xi}_i\,,\end{aligned} \tag{123}$$

(124)

$$\begin{aligned}
\nabla_{\vec\zeta}\nabla_{\vec\xi_a}\vec\xi_b =&\frac{1}{2}\big(-\mathrm{a}\ell_{ab}+b^ib^jk_{ij}k_{ab}-\mathcal{L}_{\vec\zeta}(b^i\nu_{abi})\big)\vec\zeta\\
&\quad+\frac{1}{4}\big(-b^jk_{ab}+\phi_a\delta_b{}^j+\phi_b\delta_a{}^j+C^j{}_{ab}-c^{jl}\nu_{abl}\big)\big(b^i\mu_{ji}\big)\vec\zeta\\
&+\frac{1}{2}\big(-b^i\ell_{ab}+c^{ij}b^kk_{jk}k_{ab}-\mathcal{L}_{\vec\zeta}(c^{ij}\nu_{abj})\big)\vec\xi_i\\
&\quad+\frac{1}{4}\big(-b^jk_{ab}+\phi_a\delta_b{}^j+\phi_b\delta_a{}^j+C^j{}_{ab}-c^{jl}\nu_{abl}\big)\big(c^{ik}\mu_{jk}\big)\vec\xi_i=\\
=&\frac{1}{4}\Big(-2\mathrm{a}\ell_{ab}+b^ib^j\big(2k_{ij}-\mu_{ij}\big)k_{ab}+b^ic^{jk}\big(2k_{ij}-\mu_{ji}\big)\nu_{abk}\\
&\quad-2\phi_ib^ik_{ab}+b^i\big(\phi_a\mu_{bi}+\phi_b\mu_{ai}\big)+b^i\mu_{mi}C^m{}_{ab}\\
&\quad+2b^n\big(k_{am}C^m{}_{bn}+k_{bm}C^m{}_{an}\big)\Big)\vec\zeta\\
&+\frac{1}{4}\Big(-2b^i\ell_{ab}+c^{ij}b^k\big(2k_{jk}-\mu_{kj}\big)k_{ab}+c^{ik}c^{jl}\big(2k_{jk}-\mu_{jk}\big)\nu_{abl}\\
&\quad-2\phi_jc^{ij}k_{ab}+c^{ij}\big(\phi_a\mu_{bj}+\phi_b\mu_{aj}\big)+c^{ij}\mu_{mj}C^m{}_{ab}\\
&\quad+2c^{in}\big(k_{am}C^m{}_{bn}+k_{bm}C^m{}_{an}\big)\Big)\vec\xi_i\,,
\end{aligned}$$

(125)

$$\nabla_{\vec\xi_a}\nabla_{\vec\zeta}\vec\zeta=0\,,$$

(126)

$$\begin{aligned}
\nabla_{\vec\xi_a}\nabla_{\vec\xi_b}\vec\zeta =&\frac{1}{2}\big(\mathcal{L}_{\vec\xi_a}(b^i\mu_{bi})\big)\vec\zeta+\frac{1}{4}\big(b^i\mu_{bi}\big)\big(b^j\mu_{aj}\big)\vec\zeta\\
&\quad+\frac{1}{4}\big(c^{jl}\mu_{bl}\big)\big(-\mathrm{a}k_{ja}-b^i\nu_{aji}\big)\vec\zeta\\
&+\frac{1}{2}\big(\mathcal{L}_{\vec\xi_a}(c^{ij}\mu_{bj})\big)\vec\xi_i+\frac{1}{4}\big(b^k\mu_{bk}\big)\big(c^{ij}\mu_{aj}\big)\vec\xi_i\\
&\quad+\frac{1}{4}\big(c^{jl}\mu_{bl}\big)\big(-b^ik_{ja}+\phi_j\delta_a{}^i+\phi_a\delta_j{}^i+C^i{}_{ja}-c^{ik}\nu_{ajk}\big)\vec\xi_i=\\
=&\frac{1}{4}\Big(b^ib^j\mu_{ai}\mu_{bj}-\mathrm{a}c^{ij}k_{ai}\mu_{bj}-b^ic^{jk}\nu_{aji}\mu_{bk}+2b^i\mu_{mi}C^m{}_{ab}\Big)\vec\zeta\\
&+\frac{1}{4}\Big(c^{ij}b^k\mu_{aj}\mu_{bk}-b^ic^{jk}k_{ja}\mu_{bk}-c^{ik}c^{jl}\nu_{ajk}\mu_{bl}\\
&\quad+\big(\phi_ac^{ij}+\phi_kc^{jk}\delta^i{}_a\big)\mu_{bj}-c^{jn}\mu_{bj}C^i{}_{an}+2c^{ij}\mu_{jm}C^m{}_{ab}\Big)\vec\xi_i\quad\text{and}
\end{aligned}$$

(127)

$$
\begin{aligned}
\nabla_{\vec{\xi}_a}\nabla_{\vec{\xi}_b}\vec{\xi}_c =&\frac{1}{2}\big(\mathrm{a}(k_{bm}C^m{}_{ca}+k_{cm}C^m{}_{ba})-\mathcal{L}_{\vec{\xi}_a}(b^i\nu_{bci})\big)\vec{\zeta}\\
&+\frac{1}{4}\big(-\mathrm{a}k_{bc}-b^j\nu_{bcj}\big)\big(b^i\mu_{ai}\big)\vec{\zeta}\\
&+\frac{1}{4}\big(-b^jk_{bc}+\phi_b\delta_c{}^j+\phi_c\delta_b{}^j+C^j{}_{bc}-c^{jl}\nu_{bcl}\big)\big(-\mathrm{a}k_{aj}-b^i\nu_{aji}\big)\vec{\zeta}\\
&+\frac{1}{2}\big(-b^nk_{bc}C^i{}_{na}+b^i(k_{bm}C^m{}_{ca}+k_{cm}C^m{}_{ba})-\mathcal{L}_{\vec{\xi}_a}(c^{ij}\nu_{bcj})\big)\vec{\xi}_i\\
&+\frac{1}{4}\big(-\mathrm{a}k_{bc}-b^j\nu_{bcj}\big)\big(c^{ik}\mu_{ak}\big)\vec{\xi}_i\\
&+\frac{1}{4}\big(-b^jk_{bc}+\phi_b\delta_c{}^j+\phi_c\delta_b{}^j+C^j{}_{bc}-c^{jl}\nu_{bcl}\big)\times\\
&\big(-b^ik_{ja}+\phi_a\delta_j{}^i+\phi_j\delta_a{}^i+C^i{}_{ja}-c^{ik}\nu_{ajk}\big)\vec{\xi}_i=\\
=&\frac{1}{4}\Big(\mathrm{a}b^i\big(k_{ai}-\mu_{ai}\big)k_{bc}+\big(\mathrm{a}c^{ij}k_{ia}-b^ib^j\mu_{ai}\big)\nu_{bcj}-\mathrm{a}\big(\phi_bk_{ac}+\phi_ck_{ab}\big)\\
&+b^ib^j\nu_{aij}k_{bc}+c^{ij}b^k\nu_{ajk}\nu_{bci}-b^i\big(\phi_b\nu_{aci}+\phi_c\nu_{abi}\big)\\
&-\mathrm{a}k_{am}C^m{}_{bc}+2\mathrm{a}\big(k_{bm}C^m{}_{ca}+k_{cm}C^m{}_{ba}\big)\\
&-b^i\nu_{ami}C^m{}_{bc}+2b^i\big(\nu_{bmi}C^m{}_{ca}+\nu_{cmi}C^m{}_{ba}\big)\Big)\vec{\zeta}\\
&+\frac{1}{4}\Big(-\big(\mathrm{a}c^{ij}\mu_{aj}-b^ib^jk_{ja}\big)k_{bc}-\big(c^{ij}b^k\mu_{aj}-b^ic^{jk}k_{aj}\big)\nu_{bck}\\
&-b^i\big(\phi_ak_{bc}+\phi_bk_{ac}+\phi_ck_{ab}\big)-\phi_jb^j\delta^i{}_ak_{bc}+c^{ij}b^k\nu_{akj}k_{bc}\\
&-c^{ij}\big(\phi_a\nu_{bcj}+\phi_b\nu_{acj}+\phi_c\nu_{abj}\big)-\phi_jc^{jk}\delta^i{}_a\nu_{bck}+c^{ik}c^{jl}\nu_{ajk}\nu_{bcl}\\
&+\phi_a\phi_b\delta_c{}^i+\phi_a\phi_c\delta_b{}^i+2\phi_b\phi_c\delta_a{}^i-\phi_aC^i{}_{bc}-\phi_bC^i{}_{ca}+\phi_cC^i{}_{ab}\\
&+b^nk_{bc}C^i{}_{an}-b^ik_{am}C^m{}_{bc}+2b^i\big(k_{bm}C^m{}_{ca}+k_{cm}C^m{}_{ba}\big)\\
&+c^{jn}\nu_{bcj}C^i{}_{an}-2c^{in}\nu_{bcm}C^m{}_{an}\\
&-c^{ij}\nu_{amj}C^m{}_{bc}+2c^{ij}\big(\nu_{bmj}C^m{}_{ca}+\nu_{cmj}C^m{}_{ba}\big)+C^i{}_{an}C^n{}_{bc}\Big)\vec{\xi}_i\,.
\end{aligned}
$$

Hence, applying the definition of the Riemann-Christofell tensor,

(128) $$R\big(\vec{u},\vec{v}\big)\vec{w}=\nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w}-\nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w}-\nabla_{\ \vec{u},\vec{v}]}\vec{w}$$

we have

(129) $$R^z{}_{zza}=\frac{1}{4}\Big(2b^i\ell_{ia}-b^ic^{jk}\big(2k_{ij}-\mu_{ji}\big)\mu_{ak}\Big)\,,$$

(130) $$R^d{}_{zza}=\frac{1}{4}\Big(2c^{di}\ell_{ia}-c^{di}c^{jk}\big(2k_{ij}-\mu_{ij}\big)\mu_{ak}\Big)\,,$$

(131)

$$R^z{}_{cza} = \frac{1}{4}\Big(- 2\mathrm{a}\ell_{ac} + \big(\mathrm{a}c^{ij}k_{ai} - b^i b^j \mu_{ai}\big)\mu_{cj} + b^i b^j \big(2k_{ij} - \mu_{ij}\big)k_{ac} + b^i\big(\phi_a\mu_{ci} + \phi_c\mu_{ai}\big) - 2\phi_i b^i k_{ac} + b^i c^{jk}\big(2k_{ij} - \mu_{ij}\big)\nu_{ack} + b^i c^{jk}\nu_{aij}\mu_{ck} - b^i\mu_{mi}C^m{}_{ac} + 2b^n\big(k_{am}C^m{}_{cn} + k_{cm}C^m{}_{an}\big),$$

(132)

$$R^d{}_{cza} = \frac{1}{2}\Big(- 2b^d\ell_{ac} + c^{di}b^j\big(2k_{ij} - \mu_{ji}\big)k_{ac} - c^{di}b^j\mu_{ai}\mu_{cj} + b^d c^{ij}k_{ia}\mu_{cj} + c^{di}c^{jk}\big(2k_{ij} - \mu_{ij}\big)\nu_{abk} + c^{di}c^{jk}\nu_{aji}\mu_{ck} + \big(\phi_a c^{di} - \phi_j c^{ij}\delta^d{}_a\big)\mu_{ci} - 2\phi_i c^{di}k_{ac} - c^{di}\big(2\mu_{im} - \mu_{mi}\big)C^m{}_{ac} + c^{in}\mu_{ci}C^d{}_{an} + 2c^{dn}\big(k_{am}C^m{}_{cn} + k_{cm}C^m{}_{an}\big),$$

$$R^z{}_{zab} = \frac{1}{4}\Big(- \mathrm{a}c^{ij}\big(k_{ai}\mu_{bj} - k_{bi}\mu_{aj}\big) - b^i c^{jk}\big(\nu_{aji}\mu_{bk} - \nu_{bji}\mu_{ak}\big) + 2b^i\mu_{mi}C^m{}_{ab}\Big), \tag{133}$$

(134)

$$R^d{}_{zab} = \frac{1}{4}\Big(c^{di}b^j\big(\mu_{ai}\mu_{bj} - \mu_{bi}\mu_{aj}\big) - b^d c^{ij}\big(k_{ia}\mu_{bj} - k_{ib}\mu_{aj}\big) - c^{di}c^{jk}\big(\nu_{aij}\mu_{bk} - \nu_{bij}\mu_{ak}\big) + c^{di}\big(\phi_a\mu_{bi} - \phi_b\mu_{ai}\big) + \phi_j c^{ij}\big(\delta^d{}_a\mu_{bi} - \delta^d{}_b\mu_{ai}\big) + c^{in}\big(\mu_{ai}C^d{}_{bn} - \mu_{bi}C^d{}_{ak}\big) + c^{di}\big(4\mu_{im} - 2\mu_{mi}\big)C^i{}_{ab}\Big),$$

$$R^z{}_{cab} = \frac{1}{4}\Big(\mathrm{a}b^i\big(k_{ai} - \mu_{ai}\big)k_{bc} - \mathrm{a}b^i\big(k_{bi} - \mu_{bi}\big)k_{ac} + b^i b^j\big(\nu_{aij}k_{bc} - \nu_{bij}k_{ac}\big) - b^i b^j\big(\mu_{ai}\nu_{bcj} - \mu_{bi}\nu_{acj}\big) + \mathrm{a}c^{ij}\big(k_{ai}\nu_{bcj} - k_{bi}\nu_{acj}\big) + \mathrm{a}\big(\phi_a k_{bc} - \phi_b k_{ac}\big) + c^{ij}b^k\big(\nu_{ajk}\nu_{bci} - \nu_{bjk}\nu_{aci}\big) + b^i\big(\phi_a\nu_{bci} - \phi_b\nu_{aci}\big) + \mathrm{a}\big(k_{am}C^m{}_{bc} - k_{bm}C^m{}_{ac}\big) + 2\mathrm{a}k_{cm}C^m{}_{ab} + b^i\big(\nu_{ami}C^m{}_{bc} - \nu_{mbi}C^m{}_{ac}\big) + 2b^i\nu_{cmi}C^m{}_{ab}\Big) \quad \text{and} \tag{135}$$

(136)

$$
\begin{aligned}
R^d{}_{cab} = \frac{1}{4}\Big(& b^d b^i\big(k_{ia}k_{bc} - k_{ib}k_{ac}\big) - \mathrm{a}c^{di}\big(\mu_{ai}k_{bc} - \mu_{bi}k_{ac}\big) \\
&+ c^{di}b^j\big(\nu_{aij}k_{bc} - \nu_{bij}k_{ac}\big) - c^{di}b^j\big(\mu_{ai}\nu_{bcj} - \mu_{bi}\nu_{acj}\big) + b^d c^{ij}\big(k_{ia}\nu_{bcj} - k_{ib}\nu_{acj}\big) \\
&- \phi_i b^i\big(\delta^d{}_a k_{bc} - \delta^d{}_b k_{ac}\big) - \phi_j c^{ij}\big(\delta^d{}_a \nu_{bci} - \delta^d{}_b \nu_{aci}\big) \\
&+ c^{di}c^{jk}\big(\nu_{aji}\nu_{bck} - \nu_{bji}\nu_{ack}\big) - \phi_c\big(\phi_a \delta_b{}^d - \phi_b \delta_a{}^d\big) \\
&- b^n\big(k_{ac}C^d{}_{bn} - k_{bc}C^d{}_{an}\big) + b^d\big(k_{am}C^m{}_{bc} - k_{bm}C^m{}_{ac}\big) + 2b^d k_{cm}C^m{}_{ab} \\
&- c^{in}\big(\nu_{aci}C^d{}_{bn} - \nu_{bci}C^d{}_{an}\big) + 2c^{dn}\big(\nu_{acm}C^m{}_{bn} - \nu_{bcm}C^m{}_{an}\big) \\
&+ c^{di}\big(\nu_{ami}C^m{}_{bc} - \nu_{bmi}C^m{}_{ac}\big) + 2c^{di}\nu_{cmi}C^m{}_{ab} \\
&+ C^d{}_{an}C^n{}_{bc} - C^d{}_{bn}C^n{}_{ac} - 2C^d{}_{cn}C^n{}_{ab}\Big).
\end{aligned}
$$

Lowering the first index of the Riemann-Christoffel tensor, we have

$$
\begin{aligned}
R_{zcza} =& \beta_i R^i{}_{cza} = \\
=& \frac{1}{4}\Big(- 2\ell_{ac} + c^{ij}k_{ai}\mu_{cj} - \phi_j c^{ij}\beta_a\mu_{ci} + c^{in}\beta_m\mu_{ci}C^m{}_{na}\Big),
\end{aligned}
\tag{137}
$$

$$
\begin{aligned}
R_{dzza} =& \beta_d R^z{}_{zza} + \gamma_{di}R^i{}_{zza} = \\
=& \frac{1}{4}\Big(2\ell_{ad} - c^{ij}\big(2k_{id} - \mu_{id}\big)\mu_{aj}\Big),
\end{aligned}
\tag{138}
$$

(139)

$$
\begin{aligned}
R_{dcza} =& \beta_d R^z{}_{cza} + \gamma_{di}R^i{}_{cza} = \\
=& \frac{1}{4}\Big(b^i\big(2k_{di} - \mu_{di}\big)k_{ac} - b^i\mu_{ad}\mu_{ci} \\
&+ \phi_a b^i\beta_d\mu_{ci} - 2\phi_d k_{ac} + \phi_c\mu_{ad} - \phi_j c^{ij}\gamma_{ad}\mu_{ci} \\
&+ c^{ij}\big(2k_{di} - \mu_{di}\big)\nu_{acj} + c^{ij}\nu_{iad}\mu_{cj} - 2\phi_d k_{ac} \\
&+ \big(\mu_{md} - 2\mu_{dm}\big)C^m{}_{ac} + c^{in}\gamma_{dm}\mu_{ci}C^m{}_{an} - 2b^i\beta_d\big(\mu_{mi} - \mu_{im}\big)C^m{}_{ac} \\
&+ 2\big(k_{am}C^m{}_{cd} + k_{cm}C^m{}_{ad}\big)\Big),
\end{aligned}
$$

(140)

$$
\begin{aligned}
R_{zcab} =&\beta_i R^i{}_{cab} = \\
=&\frac{1}{4}\Big(b^i\big(k_{ia}k_{bc} - k_{ib}k_{ac}\big) - c^{ij}\big(k_{ia}\nu_{bcj} - k_{ib}\nu_{acj}\big) - \phi_i b^i\big(\beta_a k_{bc} - \beta_b k_{ac}\big) \\
&- 3b^n\beta_m\big(k_{ac}C^m{}_{bn} - k_{bc}C^m{}_{an}\big) \\
&- k_{am}C^m{}_{bc} + k_{bm}C^m{}_{ac} - 2k_{cm}C^m{}_{ab} \\
&- \phi_j c^{ij}\big(\beta_a\nu_{bci} - \beta_b\nu_{aci}\big) - \phi_c\big(\phi_a\beta_b - \phi_b\beta_a\big) \\
&- c^{in}\beta_m\big(\nu_{aci}C^m{}_{bn} - \nu_{bci}C^m{}_{an}\big) \\
&+ \beta_m\big(C^m{}_{na}C^n{}_{bc} - C^m{}_{nb}C^n{}_{ac} + 2C^m{}_{nc}C^n{}_{ab}\big)\Big) \quad \text{and}
\end{aligned}
$$

(141)

$$
\begin{aligned}
R_{dcab} =& \beta_d R^z{}_{cab} + \gamma_{di}R^i{}_{cab} = \\
=& \frac{1}{4}\Big(- \mathrm{a}\big(\mu_{ad}k_{bc} - \mu_{bd}k_{ac}\big) + b^i\big(\nu_{aid}k_{bc} - \nu_{bid}k_{ac}\big) - b^i\big(\mu_{ad}\nu_{bci} - \mu_{bd}\nu_{aci}\big) \\
&+ \mathrm{a}\beta_d\big(\phi_a k_{bc} - \phi_b k_{ac}\big) - \phi_i b^i\big(\gamma_{ad}k_{bc} - \gamma_{bd}k_{ac}\big) \\
&- \phi_i c^{ij}\big(\gamma_{ad}\nu_{bci} - \gamma_{bd}\nu_{aci}\big) + b^i\beta_d\big(\phi_a\nu_{bci} - \phi_b\nu_{aci}\big) - \phi_c\big(\phi_a\gamma_{bd} - \phi_b\gamma_{ad}\big) \\
&- b^n\gamma_{dm}\big(k_{ac}C^m{}_{bn} - k_{bc}C^m{}_{an}\big) - c^{in}\gamma_{dm}\big(\nu_{aci}C^m{}_{bn} - \nu_{bci}C^m{}_{an}\big) \\
&+ \big(\nu_{amd}C^m{}_{bc} - \nu_{bmd}C^m{}_{ac} + 2\nu_{cmd}C^m{}_{ab}\big) \\
&- 2b^n\beta_d\big(\nu_{acm}C^m{}_{bn} - \nu_{bci}C^m{}_{an}\big) + 2\big(\nu_{acm}C^m{}_{bd} - \nu_{bcm}C^m{}_{ad}\big) + \\
&+ \gamma_{dm}\big(C^m{}_{na}C^n{}_{bc} - C^m{}_{nb}C^n{}_{ac} + 2C^m{}_{nc}C^n{}_{ab}\big)\Big)\,.
\end{aligned}
$$

3.4. The Ricci tensor and the Ricci scalar; the Einstein tensor. The Ricci tensor is given from Eq. (39), we easily find that the components of the Ricci tensor are given as

$$
\begin{aligned}
R_{zz} &= R^z{}_{zzz} + R^i{}_{ziz}\ , \\
R_{za} &= R^z{}_{zza} + R^i{}_{zia} \quad \text{and} \\
R_{ab} &= R^z{}_{azb} + R^i{}_{aib}\,.
\end{aligned}
$$

Using the afore-mentioned components of the Riemann tensor; hence

$$
\begin{aligned}
R_{zz} =&\frac{1}{4}\Big(- 2c^{ij}\ell_{ij} + c^{ik}c^{jl}\big(2k_{ij} - \mu_{ij}\big)\mu_{lk}\Big) \\
=&\frac{1}{4}\Big(- 2c^{ij}\ell_{ij} + c^{ik}c^{jl}k_{ij}k_{kl} + c^{ik}c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl}\Big) \quad ,
\end{aligned}
\tag{142}
$$

(143)

$$
\begin{aligned}
R_{za} = R_{az} =& \frac{1}{4}\Big(2b^i\ell_{ia} - b^i c^{jk}\big(2k_{ij} - \mu_{ij}\big)\mu_{ak} + b^i c^{jk}\big(k_{ak} - \mu_{ak}\big)k_{ij} \\
&\quad - c^{ik}c^{jl}\nu_{ijk}\mu_{al} + c^{ik}c^{jl}\nu_{aik}k_{jl} + 3\phi_j c^{ij}\mu_{ai} - \phi_a c^{ij}k_{ij} \\
&\quad + c^{in}\big(3\mu_{mi} - 4\mu_{im}\big)C^m{}_{an} - c^{in}\mu_{ai}C^m{}_{mn} = \\
=& \frac{1}{4}\Big(2b^i\ell_{ia} - c^{ij}b^l k_{jl}k_{ia} - \phi_k b^i c^{jk}\beta_a k_{ij} - 3\phi_j c^{ij}k_{ai} \\
&\quad - 2c^{ik}b^l\beta_m k_{ai}C^m{}_{kl} - c^{in}k_{im}C^m{}_{an} + c^{ij}k_{ij}C^m{}_{ma} \\
&\quad - 3\phi_i\phi_j c^{ij}\beta_a \\
&\quad + 2\phi_i c^{ik}b^l\beta_a\beta_m C^m{}_{kl} + 3\phi_i c^{in}\beta_m C^m{}_{an} + 2\phi_i c^{in}\beta_a C^m{}_{mn} \\
&\quad - 2c^{jk}b^l\beta_m\beta_n C^m{}_{aj}C^n{}_{kl} + 7c^{kl}\beta_m C^m{}_{nk}C^n{}_{al}\Big) \quad \text{and}
\end{aligned}
$$

(144)

$$
\begin{aligned}
R_{ab} =& \frac{1}{4}\Big(-2\mathrm{a}\ell_{ab} - \big(\mathrm{a}c^{ij} - 2b^i b^j\big)k_{ij}k_{ab} + 2\phi_k\big(b^i c^{jk} - c^{ij}b^k\big)\gamma_{ab}k_{ij} \\
&\quad - 3\phi_i b^i k_{ab} + b^i\big(\phi_a k_{bi} - \phi_b k_{ai}\big) \\
&\quad - 2\big(b^i c^{jn} - c^{ij}b^n\big)k_{ij}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an}\big) \\
&\quad - 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an} + k_{ab}C^m{}_{mn}\big) \\
&\quad + \phi_i\phi_j\big(c^{ij}\gamma_{ab} - b^i b^j\beta_a\beta_b\big) + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi_b \\
&\quad + 2\phi_i c^{ik}b^l\gamma_{ab}\beta_m C^m{}_{kl} + 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{bm} - \beta_b\gamma_{am}\big)C^m{}_{kl} \\
&\quad - 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} - \gamma_{ab}C^m{}_{mn}\big) \\
&\quad + c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} - 2c^{jk}b^l\beta_n\big(\gamma_{am}C^n{}_{bj} + \gamma_{bm}C^n{}_{aj}\big)C^m{}_{kl} \\
&\quad - 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{ak}\big)C^m{}_{nl} - 2c^{kl}\gamma_{mn}C^m{}_{ak}C^n{}_{bl} \\
&\quad - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{ak}\big)C^n{}_{nl} \\
&\quad - b^l\beta_m C^m{}_{an}C^n{}_{bl} + b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{an}\big)C^n{}_{ml} \\
&\quad - 3C^m{}_{an}C^n{}_{bm} - C^m{}_{ma}C^n{}_{nb}\,.
\end{aligned}
$$

In the same manner, the Ricci scalar is defined in Eq. (40) and given Eqs. (142), (143) and (144), we have,

$$
R = aR_{zz} + 2b^i R_{zi} + c^{ij}R_{ij}\,,
$$

so

(145)

$$
\begin{aligned}
R =& \frac{1}{4}\Big(-4\big(\mathrm{a}c^{ij}-b^i b^j\big)\ell_{ij} + c^{ik}\big(\mathrm{a}c^{jl}-2b^j b^l\big)k_{ij}k_{kl} + \phi_k\big(2b^i c^{jk}+7c^{ij}b^k\big)k_{ij} \\
& - 3c^{ik}b^l k_{im}C^m{}_{kl} - \big(4b^i c^{jn}-5c^{ij}b^n\big)k_{ij}C^m{}_{mn} \\
& + 2\phi_i\phi_j c^{ij} - \phi_i c^{ik}b^l\beta_m C^m{}_{kl} \\
& - c^{ik}c^{jl}\big(\gamma_{mn}+\mathrm{a}\beta_m\beta_n\big)C^m{}_{ij}C^n{}_{kl} \\
& - 3c^{jk}b^l\beta_m C^m{}_{nj}C^n{}_{kl} - 7c^{kl}C^m{}_{nk}C^n{}_{ml} - 5c^{kl}C^m{}_{mk}C^n{}_{nl}\,.
\end{aligned}
$$

We should again mention the first two terms of the Ricci tensor (eqs. 142-144) and the first term of the Ricci scalar (eq. 184) are the ones specified by Lifshitz and Khalatnikov in the case of a Bianchi group acting by isometries on synchronous coordinates [**40**]; the remaining terms are either due to the action by homotheties (whereas the homothety constants ϕ_a are present), or due to the non-synchronous frame (whereas the structure constants $C^c{}_{ab}$ are present).

Having defined the Einstein tensor in Eq. (53), using Eqs. (142), (143), (144) and (184), we can calculate its components to be

$$
G_{zz} = \frac{1}{4}\Big(-2c^{ij}\ell_{ij} + c^{ik}c^{jl}k_{ij}k_{kl} + c^{ik}c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl}\Big)\ , \tag{146}
$$

(147)

$$
\begin{aligned}
G_{za} = G_{az} =& \frac{1}{4}\Big(2b^i\ell_{ia} + 2\big(\mathrm{a}c^{ij}-b^i b^j\big)\beta_a\ell_{ij} \\
& - b^i c^{jk}k_{ij}k_{ak} - \frac{1}{2}c^{ik}\big(\mathrm{a}c^{jl}-2b^j b^l\big)\beta_a k_{ij}k_{kl} \\
& - 3\phi_j c^{ij}k_{ai} - \frac{1}{2}\phi_k\big(4b^i c^{jk}+7c^{ij}b^k\big)\beta_a k_{ij} - 4\phi_i\phi_j c^{ij}\beta_a \\
& - 2c^{ik}b^l\beta_m k_{ia}C^m{}_{kl} - \frac{3}{2}c^{ik}b^l\beta_a k_{im}C^m{}_{kl} \\
& + \frac{1}{2}\big(5c^{ij}b^n - 4b^i c^{jn}\big)\beta_a k_{ij}C^m{}_{mn} - c^{in}k_{im}C^m{}_{an} + c^{ij}k_{ij}C^m{}_{ma} \\
& - \frac{5}{2}\phi_i c^{ik}b^l\beta_a\beta_m C^m{}_{kl} + 3\phi_i c^{in}\beta_m C^m{}_{an} + 2\phi_i c^{in}\beta_a C^m{}_{mn} \\
& - 2c^{jk}b^l\beta_m\beta_n C^m{}_{aj}C^n{}_{kl} + \frac{3}{2}c^{jk}b^l\beta_a\beta_m C^m{}_{nj}C^n{}_{kl} \\
& + \frac{1}{2}c^{ik}c^{jl}\beta_a\big(\gamma_{mn}+\mathrm{a}\beta_m\beta_n\big)C^m{}_{ij}C^n{}_{kl} \\
& + 7c^{kl}\beta_m C^m{}_{nk}C^n{}_{al} \\
& + \frac{7}{2}c^{kl}\beta_a C^m{}_{nk}C^n{}_{ml} + \frac{5}{2}c^{kl}\beta_a C^m{}_{mk}C^n{}_{nl}\Big) \quad \text{and}
\end{aligned}
$$

(148)

$$
\begin{aligned}
G_{ab} =& \frac{1}{4}\Big(- 2\mathrm{a}\ell_{ab} + 2\big(\mathrm{a}c^{ij} - b^i b^j\big)\gamma_{ab}\ell_{ij} \\
& - \big(\mathrm{a}c^{ij} - 2b^i b^j\big)k_{ij}k_{ab} + \frac{1}{2}c^{ik}\big(\mathrm{a}c^{jl} - b^j b^l\big)\gamma_{ab}k_{ij}k_{kl} \\
& + \frac{1}{2}\phi_k\big(2b^i c^{jk} - 11b^k c^{ij}\big)\gamma_{ab}k_{ij} - 3\phi_i b^i k_{ab} + b^i\big(\phi_a k_{bi} + \phi_b k_{ai}\big) \\
& - 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an} + k_{ab}C^m{}_{mn}\big) \\
& - 2\big(b^i c^{jn} - c^{ij}b^n\big)k_{ij}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} + \gamma_{ab}C^m{}_{mn}\big) \\
& + \frac{1}{2}\gamma_{ab}\big(3c^{ik}b^l k_{im}C^m{}_{kl} + c^{ij}b^n k_{ij}C^m{}_{mn}\big) \\
& + \phi_i\phi_j b^i b^j\beta_a\beta_b + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi_b \\
& + 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{am} + \beta_b\gamma_{am}\big)C^m{}_{kl} + \frac{5}{2}\phi_i c^{ik}b^l\gamma_{ab}\beta_m C^m{}_{kl} \\
& - 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} + \gamma_{ab}C^m{}_{mn}\big) \\
& + c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} + \frac{1}{2}c^{ik}c^{jl}\big(\gamma_{mn} + \mathrm{a}\beta_m\beta_n\big)\gamma_{ab}C^m{}_{ij}C^n{}_{kl} \\
& - 2c^{jk}b^l\beta_n\big(\gamma_{am}C^n{}_{bj} + \gamma_{bm}C^n{}_{aj}\big)C^m{}_{kl} + \frac{3}{2}c^{jk}b^l\gamma_{ab}\beta_m C^m{}_{nj}C^n{}_{kl} \\
& - 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{ak}\big)C^m{}_{nl} - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{ak}\big)C^n{}_{nl} \\
& - 2c^{kl}\gamma_{mn}C^m{}_{ak}C^n{}_{bl} + \frac{7}{2}c^{kl}\gamma_{ab}C^m{}_{nk}C^n{}_{ml} + \frac{5}{2}c^{kl}\gamma_{ab}C^m{}_{mk}C^n{}_{nl} \\
& - b^l\beta_m C^m{}_{an}C^n{}_{bl} + b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{na}\big) \\
& - 3C^m{}_{an}C^n{}_{bm} - C^m{}_{ma}C^n{}_{nb}\Big) .
\end{aligned}
$$

It is worth pointing out that the trace of the Einstein tensor, say the "Einstein scalar", is simply the negative of the Ricci scalar, since

$$G = \mathrm{a}G_{zz} + 2b^i G_{zi} + c^{ij}G_{ij} = \mathrm{a}R_{zz} + 2b^i R_{zi} - R + c^{ij}R_{ij} - R = -R\,;$$

and, therefore,

(149)

$$
\begin{aligned}
G = -R = -\frac{1}{4}\Big(& - 4\big(\mathrm{a}c^{ij} - b^i b^j\big)\ell_{ij} - c^{ik}\big(\mathrm{a}c^{jl} - 2b^j b^l\big)k_{ij}k_{kl} \\
& - \phi_k\big(2b^i c^{jk} + 7c^{ij}b^k\big)k_{ij} - 2\phi_i\phi_j c^{ij} \\
& + 3c^{ik}b^l k_{im}C^m{}_{kl} + \big(4b^i c^{jn} - 5c^{ij}b^n\big)k_{ij}C^m{}_{mn} \\
& + \phi_i c^{ik}b^l\beta_m C^m{}_{kl} + c^{ik}c^{jl}\big(\gamma_{mn} + \mathrm{a}\beta_m\beta_n\big)C^m{}_{ij}C^n{}_{kl} \\
& + 3c^{jk}b^l\beta_m C^m{}_{nj}C^n{}_{kl} + 7c^{kl}C^m{}_{nk}C^n{}_{ml} + 5c^{kl}C^m{}_{mk}C^n{}_{nl} \,.
\end{aligned}
$$

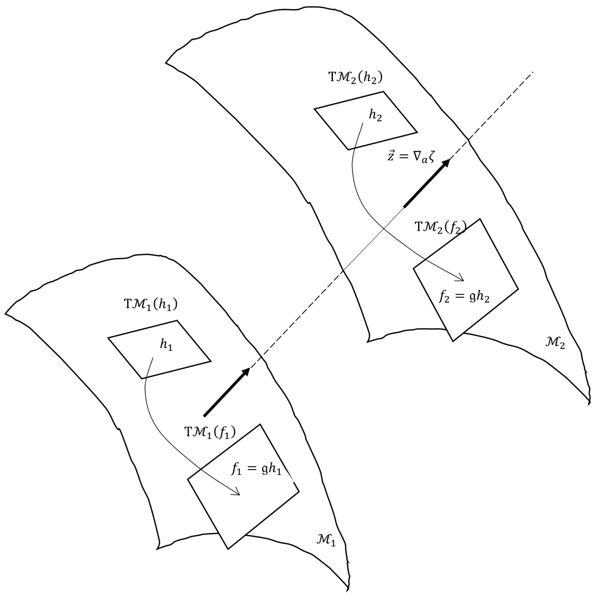

FIGURE 2. The remaining possible foliation of the space-time along a lightlike transversal.

4. The Structure of the Space-Time for Null Orbits

In this section, we will show the specific form the curvature elements of the space-time take in the case the group acts by means of lightlike homogeneous submanifolds. This case is merely an extension (or a specification) of the previous section, not a separate one; however, we decided to present it separately, as, under the theorem 2.1, it becomes a degenerate case.

In this case (shown in Fig. 2), the result by McIntosh does not hold; however, a similar theorem holds, deeming the null homothetic vectors shear-free and their expansion equal to $\dfrac{\phi_a}{\psi_a}$, where ψ_a the proportionality constant of the generetor $\vec{\xi}_a$ over a principal null transversal [**39**].

4.1. The Christoffel symbols. Admitting that a = 0, we can obtain the following relations for the Lie derivatives of the components of the metric, given in eq. (81), as

$$\mathcal{L}_{\vec{\zeta}}\beta_a = 0 \ , \tag{150}$$

$$\mathcal{L}_{\vec{\xi}_b}\beta_a = \phi_b\beta_a - \beta_m C^m{}_{ab} \ , \tag{151}$$

$$\mathcal{L}_{\vec{\zeta}}\gamma_{ab} = k_{ab} \ \text{ and} \tag{152}$$

$$\mathcal{L}_{\vec{\xi}_c}\gamma_{ab} = \phi_c\gamma_{ab} - \gamma_{am}C^m{}_{bc} - \gamma_{bm}C^m{}_{ac} \ . \tag{153}$$

As for the Lie derivatives of the inverse metric, given in eq. (82), we have

(154) $$\mathcal{L}_{\vec{\zeta}} b^a = -c^{ai} b^j k_{ij} \ ,$$

(155) $$\mathcal{L}_{\vec{\xi}_b} b^a = -\phi_b b^a + b^n C^a{}_{nb} \ ,$$

(156) $$\mathcal{L}_{\vec{\zeta}} c^{ab} = -c^{ai} c^{bj} k_{ij} \ \text{ and}$$

(157) $$\mathcal{L}_{\vec{\xi}_c} c^{ab} = -\phi_c c^{ab} + c^{an} C^b{}_{nc} + c^{bn} C^a{}_{nc} \, .$$

Here, we should remind that, when the orbits of the group action are null, both matrices γ_{ab} and c^{ab} are singular, since

$$\gamma_{ai} b^i = 0 \quad \text{and} \quad c^{ai} \beta_i = 0 \, ,$$

for non-zero β_a and b^a. Differentiating the former with respect to $\vec{\zeta}$, we have

$$\mathcal{L}_{\vec{\zeta}} \big(\gamma_{ai} b^i \big) = k_{ai} b^i - \gamma_{ai} c^{ij} b^k k_{jk} = 0 \, ,$$

and, remembering that $\gamma_{ai} c^{ij} = \delta_a{}^j - \beta_a b^j$, we can easily derive[10]

(158) $$k_{ij} b^i b^j = 0 \, .$$

For a non-zero b^a, this implies that the matrix k_{ab} is indefinite, since it is symmetric; given it has at least one zero eigenvalue, it also becomes singular. Thus, in general,

(159) $$k_{ai} b^i = 0 \, .$$

This immediately implies that

(160) $$\mathcal{L}_{\vec{\zeta}} b^a = 0 \, .$$

So, not only β_a, but also b^a is invariant under the "movement" along the geodesics in the transversal. Interestingly, repeating the process, we also obtain

(161) $$b^i \ell_{ia} = 0 \, .$$

Therefore, the Christoffel symbols of the first kind are unchanged:

(162)
$$\begin{aligned}
&\Gamma_{zzz} = 0 \ , \\
&\Gamma_{zza} = \Gamma_{zaz} = 0 \ , \\
&\Gamma_{zab} = \frac{1}{2} \big(- k_{ab} + \phi_a \beta_b + \phi_b \beta_a + \beta_m C^m{}_{ab} \big) \ , \\
&\Gamma_{azz} = k_a \ , \\
&\Gamma_{azb} = \Gamma_{abz} = \frac{1}{2} \big(k_{ab} - \phi_a \beta_b + \phi_b \beta_a - 3 \beta_m C^m{}_{ab} \big) \ \text{ and} \\
&\Gamma_{abc} = \frac{1}{2} \big(- \phi_a \gamma_{bc} + \phi_b \gamma_{ac} + \phi_c \gamma_{ab} - \gamma_{am} C^m{}_{bc} + 3 \gamma_{bm} C^m{}_{ca} - 3 \gamma_{cm} C^m{}_{ab} \big) \, .
\end{aligned}$$

---

[10] Another way to reach the same conclusion is to use eq. (88) and substitute a = 0; then the right-hand side must be equal to zero as well.

While those of the second kind become

$$
\begin{aligned}
&\Gamma^{z}{}_{zz} = 0 \ , \\
&\Gamma^{z}{}_{zb} = \Gamma^{z}{}_{bz} = \frac{1}{2}\big(- \phi_i b^i \beta_b + \phi_b - 3b^n \beta_m C^m{}_{nb}\big) \ , \\
&\Gamma^{z}{}_{bc} = \frac{1}{2}\left(-\phi_i b^i \gamma_{bc} + 3\gamma_{bm} b^n C^m{}_{nc} + 3\gamma_{cm} b^n C^m{}_{nb}\right) \ , \\
&\Gamma^{a}{}_{zz} = 0 \ , \\
&\Gamma^{a}{}_{zb} = \Gamma^{a}{}_{bz} = \frac{1}{2}\big(c^{ai} k_{ib} - \phi_i c^{ia} \beta_b - 3\beta_m c^{an} C^m{}_{nb}\big) \text{ and} \\
&\Gamma^{a}{}_{bc} = \frac{1}{2}\Big(- b^a k_{bc} - \phi_i c^{ia} \gamma_{bc} + \phi_b \delta_c{}^a + \phi_c \delta_b{}^a \\
&\qquad - C^a{}_{bc} + b^a \beta_m C^m{}_{bc} + 3c^{an}\big(\gamma_{bm} C^m{}_{nc} + \gamma_{cm} C^m{}_{nb}\big)\Big) \, .
\end{aligned}
\tag{163}
$$

4.2. The Riemann-Christoffel tensor. Following the same process as in the previous section, we obtain the second order covariant derivatives of $\vec{\zeta}$ and $\vec{\xi}_a$ with respect to themselves. It is interesting to note that, given the degeneracy in both $\boldsymbol{\gamma}$ and $\boldsymbol{k}$, the similarly defined tensors μ_{ab} and ν_{abc} have the following additional properties: Firstly,

$$
b^i b^j \mu_{ij} = 0 \quad \text{and} \quad b^i b^j \nu_{aij} = 0 \, ; \tag{164}
$$

note that this does not imply they are singular, or even indefinite, because (unlike k_{ab}) they are not symmetric. Secondly,

$$
\mathcal{L}_{\vec{\zeta}}\big(b^i \mu_{ai} \, , \tag{165}
$$

hence the contraction $b^i \mu_{ai}$ is not altered by a "movement" along the family of geodesics C_ζ.

Knowing those, it is easy to compute the second-order covariant derivatives:

$$
\nabla_{\vec{\zeta}} \nabla_{\vec{\zeta}} \vec{\zeta} = 0 \, , \tag{166}
$$

$$
\begin{aligned}
\nabla_{\vec{\zeta}} \nabla_{\vec{\zeta}} \vec{\xi}_a = &\frac{1}{4}\Big(b^i c^{jk} \mu_{ji} \mu_{ak}\Big)\vec{\zeta} \\
&+ \frac{1}{4}\Big(2c^{ij} \ell_{ja} - c^{ik} c^{jl}\big(2k_{jk} - \mu_{jk}\big)\mu_{al}\Big)\vec{\xi}_i \, ,
\end{aligned}
\tag{167}
$$

(168)
$$
\begin{aligned}
\nabla_{\vec{\zeta}}\nabla_{\vec{\xi}_a}\vec{\xi}_b =&\frac{1}{4}\Big(- b^i c^{jk}\mu_{ji}\nu_{abk} \\
&- 2\phi_i b^i k_{ab} + b^i\big(\phi_a\mu_{bi} + \phi_b\mu_{ai}\big) + b^i\mu_{mi}C^m{}_{ab} \\
&+ 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an}\big)\Big)\vec{\zeta} \\
&+ \frac{1}{4}\Big(- 2b^i\ell_{ab} + c^{ij}b^k\big(2k_{jk} - \mu_{kj}\big)k_{ab} + c^{ik}c^{jl}\big(2k_{jk} - \mu_{jk}\big)\nu_{abl} \\
&- 2\phi_j c^{ij}k_{ab} + c^{ij}\big(\phi_a\mu_{bj} + \phi_b\mu_{aj}\big) + c^{ij}\mu_{mj}C^m{}_{ab} \\
&+ 2c^{in}\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an}\big)\Big)\vec{\xi}_i\,,
\end{aligned}
$$

(169)
$$\nabla_{\vec{\xi}_a}\nabla_{\vec{\zeta}}\vec{\zeta} = 0\,,$$

(170)
$$
\begin{aligned}
\nabla_{\vec{\xi}_a}\nabla_{\vec{\xi}_b}\vec{\zeta} =&\frac{1}{4}\Big(b^i b^j\mu_{ai}\mu_{bj} - b^i c^{jk}\nu_{aji}\mu_{bk} + 2b^i\mu_{mi}C^m{}_{ab}\Big)\vec{\zeta} \\
&+ \frac{1}{4}\Big(c^{ij}b^k\mu_{aj}\mu_{bk} - b^i c^{jk}k_{ja}\mu_{bk} - c^{ik}c^{jl}\nu_{ajk}\mu_{bl} \\
&+ \big(\phi_a c^{ij} + \phi_k c^{jk}\delta^i{}_a\big)\mu_{bj} - c^{jn}\mu_{bj}C^i{}_{an} + 2c^{ij}\mu_{jm}C^m{}_{ab}\Big)\vec{\xi}_i \quad \text{and}
\end{aligned}
$$

(171)
$$
\begin{aligned}
\nabla_{\vec{\xi}_a}\nabla_{\vec{\xi}_b}\vec{\xi}_c =&\frac{1}{4}\Big(- b^i b^j\mu_{ai}\nu_{bcj} + c^{ij}b^k\nu_{ajk}\nu_{bci} - b^i\big(\phi_b\nu_{aci} + \phi_c\nu_{abi}\big) \\
&- b^i\nu_{ami}C^m{}_{bc} + 2b^i\big(\nu_{bmi}C^m{}_{ca} + \nu_{cmi}C^m{}_{ba}\big)\Big)\vec{\zeta} \\
&+ \frac{1}{4}\Big(- \big(c^{ij}b^k\mu_{aj} - b^i c^{jk}k_{aj}\big)\nu_{bck} - b^i\big(\phi_a k_{bc} + \phi_b k_{ac} + \phi_c k_{ab}\big) \\
&- \phi_j b^j\delta^i{}_a k_{bc} + c^{ij}b^k\nu_{akj}k_{bc} \\
&- c^{ij}\big(\phi_a\nu_{bcj} + \phi_b\nu_{acj} + \phi_c\nu_{abj}\big) - \phi_j c^{jk}\delta^i{}_a\nu_{bck} + c^{ik}c^{jl}\nu_{ajk}\nu_{bcl} \\
&+ \phi_a\phi_b\delta_c{}^i + \phi_a\phi_c\delta_b{}^i + 2\phi_b\phi_c\delta_a{}^i - \phi_a C^i{}_{bc} - \phi_b C^i{}_{ca} + \phi_c C^i{}_{ab} \\
&+ b^n k_{bc}C^i{}_{an} - b^i k_{am}C^m{}_{bc} + 2b^i\big(k_{bm}C^m{}_{ca} + k_{cm}C^m{}_{ba}\big) \\
&+ c^{jn}\nu_{bcj}C^i{}_{an} - 2c^{in}\nu_{bcm}C^m{}_{an} \\
&- c^{ij}\nu_{amj}C^m{}_{bc} + 2c^{ij}\big(\nu_{bmj}C^m{}_{ca} + \nu_{cmj}C^m{}_{ba}\big) + C^i{}_{an}C^n{}_{bc}\Big)\vec{\xi}_i\,.
\end{aligned}
$$

From these, applying the definition of the Riemann-Christofell tensor,

(172)
$$R\big(\vec{u},\vec{v}\big)\vec{w} = \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} - \nabla_{\ \vec{u},\vec{v}]}\vec{w}$$

we have

$$
\begin{aligned}
R^z{}_{zza} =& \frac{1}{4}\Big(b^i c^{jk}\mu_{ji}\mu_{ak}\Big) = \\
=& \frac{1}{4}\Big(\phi_j c^{ij} k_{ia} - \phi_i\phi_j c^{ij}\beta_a + \phi_i c^{in}\beta_m C^m{}_{an}\Big)\,,
\end{aligned}
\tag{173}
$$

$$
\begin{aligned}
R^d{}_{zza} &= \frac{1}{4}\Big(2c^{di}\ell_{ia} - c^{di}c^{jk}\big(2k_{ij} - \mu_{ij}\big)\mu_{ak}\Big) = \\
&= \frac{1}{4}\Big(2c^{di}\ell_{ia} - c^{di}c^{jk}k_{ij}k_{ak} - \phi_k c^{di}c^{jk}\beta_a k_{ij} \\
&\quad - c^{dk}c^{il}\beta_m k_{ia}C^m{}_{kl} + c^{di}c^{jn}\beta_a k_{ij}C^m{}_{an} \\
&\quad + \phi_i c^{dk}c^{il}\beta_a\beta_m C^m{}_{kl} - c^{dk}c^{jl}\beta_m\beta_n C^m{}_{aj}C^n{}_{kl}\Big)\,,
\end{aligned}
\tag{174}
$$

(175)

$$
\begin{aligned}
R^z{}_{cza} = \frac{1}{4}\Big(&- b^i b^j\mu_{ai}\mu_{cj} + b^i\big(\phi_a\mu_{ci} + \phi_c\mu_{ai}\big) - 2\phi_i b^i k_{ac} \\
&- b^i c^{jk}\mu_{ij}\nu_{ack} + b^i c^{jk}\nu_{aij}\mu_{ck} \\
&- b^i\mu_{mi}C^m{}_{ac} + 2b^n\big(k_{am}C^m{}_{cn} + k_{cm}C^m{}_{an}\big) = \\
= \frac{1}{4}\Big(&- 2\phi_i b^i k_{ac} + 2b^n\big(k_{am}C^m{}_{cn} + k_{cm}C^m{}_{an}\big) - c^{ik}b^l\gamma_{am}k_{ic}C^m{}_{kl} \\
&+ \phi_i\phi_j c^{ij}\gamma_{ac} + \phi_i\phi_j b^i b^j\beta_a\beta_c + \phi_i b^i\big(\phi_a\beta_c + \phi_c\beta_a\big) - 2\phi_a\phi_c \\
&+ \phi_i c^{ik}b^l\gamma_{ac}\beta_m C^m{}_{kl} - \phi_i c^{in}\big(\gamma_{am}C^m{}_{cn} + \gamma_{cm}C^m{}_{an}\big) \\
&- \phi_i b^i b^n\beta_m\big(\beta_a C^m{}_{cn} + \beta_c C^m{}_{an}\big) + b^n\beta_m\big(\phi_a C^m{}_{cn} + \phi_c C^m{}_{an}\big) \\
&+ \phi_i b^i\beta_m C^m{}_{ac} \\
&- c^{jk}b^l\beta_n\big(\gamma_{am}C^m{}_{cj} + \gamma_{cm}C^m{}_{aj}\big)C^n{}_{kl} - b^k b^l\beta_m\beta_n C^m{}_{ak}C^n{}_{cl} \\
&- b^l\beta_m C^m{}_{nl}C^n{}_{ac}\Big)\,,
\end{aligned}
$$

(176)

$$
\begin{aligned}
R^d{}_{cza} = \frac{1}{4}\Big(& - 2b^d \ell_{ac} - c^{di} b^j \mu_{ji} k_{ac} - c^{di} b^j \mu_{ai}\mu_{cj} + b^d c^{ij} k_{ia}\mu_{cj} \\
& + c^{di} c^{jk} \big(2k_{ij} - \mu_{ij}\big)\nu_{abk} + c^{di} c^{jk} \nu_{aji}\mu_{ck} \\
& + \big(\phi_a c^{di} - \phi_j c^{ij} \delta^d{}_a\big)\mu_{ci} - 2\phi_i c^{di} k_{ac} \\
& - c^{di}\big(2\mu_{im} - \mu_{mi}\big)C^m{}_{ac} + c^{in}\mu_{ci} C^d{}_{an} \\
& + 2c^{dn}\big(k_{am} C^m{}_{cn} + k_{cm} C^m{}_{an}\big) = \\
= \frac{1}{4}\Big(& - 2b^d \ell_{ac} + b^d c^{ij} k_{ia} k_{jc} + phi_k c^{di} c^{jk} \gamma_{ac} k_{ij} - 2\phi_i c^{di} k_{ac} \\
& - \phi_i c^{di} b^j \big(\beta_a k_{ci} - \beta_c k_{ai}\big) + c^{di}\big(\phi_a k_{ci} - \phi_c k_{ai}\big) - \phi_i c^{ij} \delta^d{}_a k_{ci} \\
& - c^{di} c^{jn} k_{ij} \big(\gamma_{am} C^m{}_{cn} + \gamma_{cm} C^m{}_{an}\big) - 2c^{dn}\big(k_{am} C^m{}_{cn} + k_{cm} C^m{}_{an}\big) \\
& + c^{dk} b^l \beta_m k_{ac} C^m{}_{kl} + \big(b^d c^{in} - c^{di} b^n\big)\beta_m k_{ai} C^m{}_{cn} \\
& + c^{in} k_{ci} C^d{}_{an} - c^{di} k_{im} C^m{}_{ac} \\
& - \phi_a \phi_j c^{ij} \delta^d{}_a \beta_c + \phi_c \phi_j i c^{di} \beta_a \\
& + \phi_k c^{dk} c^{il} \gamma_{ac} \beta_m C^m{}_{kl} + \phi_i c^{dk} c^{il} \gamma_{am} \beta_c C^m{}_{kl} \\
& + \phi_i c^{dn} b^i \beta_m \big(\beta_a C^m{}_{cn} - \beta_c C^m{}_{an}\big) + c^{dn}\beta_m\big(\phi_a C^m{}_{cn} + \phi_c C^m{}_{an}\big) \\
& - \phi_i c^{in} \delta^d{}_a \beta_m C^m{}_{cn} - \phi_i c^{in} \beta_c C^d{}_{an} - 4\phi_i c^{di} \beta_m C^m{}_{ac} \\
& - c^{dk} c^{jl} \gamma_{cm} \beta_n C^m{}_{aj} C^n{}_{kl} - c^{kl} \beta_m C^d{}_{ak} C^m{}_{cl} + 3c^{kl} \beta_m C^m{}_{ln} C^n{}_{ac}\,,
\end{aligned}
$$

(177)

$$
\begin{aligned}
R^z{}_{zab} = \frac{1}{4}\Big(& - b^i c^{jk} \big(\nu_{aji}\mu_{bk} - \nu_{bji}\mu_{ak}\big) + 2b^i \mu_{mi} C^m{}_{ab}\Big) = \\
= \frac{1}{4}\Big(& c^{jk} b^l \big(\gamma_{am} k_{bi} - \gamma_{bm} k_{ai}\big) C^m{}_{kl} + b^n \big(k_{am} C^m{}_{bn} - k_{bm} C^m{}_{an}\big) \\
& - \phi_i c^{ik} b^l \big(\gamma_{am}\beta_b - \gamma_{bm}\beta_a\big) C^m{}_{kl} + 2\phi_i b^i b^n \beta_m \big(\beta_a C^m{}_{bn} - \beta_b C^m{}_{an}\big) \\
& - 4\phi_i b^i \beta_m C^m{}_{ab} \\
& + c^{jk} b^l \beta_m \big(\gamma_{am} C^n{}_{bj} - \gamma_{bm} C^n{}_{aj}\big) + b^n \beta_m \big(C^m{}_{an} C^n{}_{bm} - C^m{}_{bn} C^n{}_{am}\big) \\
& + 2b^l \beta_m C^m{}_{nl} C^n{}_{ab}\Big)\,,
\end{aligned}
$$

(178)

$$
\begin{aligned}
R^{d}{}_{zab} = \frac{1}{4}\Big(& c^{di}b^{j}\big(\mu_{ai}\mu_{bj}-\mu_{bi}\mu_{aj}\big)-b^{d}c^{ij}\big(k_{ia}\mu_{bj}-k_{ib}\mu_{aj}\big)\\
& -c^{di}c^{jk}\big(\nu_{aij}\mu_{bk}-\nu_{bij}\mu_{ak}\big)+c^{di}\big(\phi_{a}\mu_{bi}-\phi_{b}\mu_{ai}\big)+\phi_{j}c^{ij}\big(\delta^{d}{}_{a}\mu_{bi}-\delta^{d}{}_{b}\mu_{ai}\big)\\
& +c^{in}\big(\mu_{ai}C^{d}{}_{bn}-\mu_{bi}C^{d}{}_{ak}\big)+c^{di}\big(4\mu_{im}-2\mu_{mi}\big)C^{i}{}_{ab}\Big)=\\
=\frac{1}{4}\Big(& \phi_{j}b^{d}c^{ij}\big(\beta_{a}k_{bi}-\beta_{b}k_{ai}\big)+2c^{di}k_{im}C^{m}{}_{ab}\\
& +c^{dk}c^{il}\big(\gamma_{am}k_{bi}-\gamma_{bm}k_{ai}\big)C^{m}{}_{kl}+c^{di}b^{n}\beta_{m}\big(k_{ai}C^{m}{}_{bn}-k_{bi}C^{m}{}_{an}\big)\\
& +\phi_{i}c^{dk}c^{il}\big(\beta_{a}\gamma_{bm}-\beta_{b}\gamma_{am}\big)C^{m}{}_{kl}\\
& -\phi_{i}\big(c^{di}b^{n}+b^{d}c^{in}\big)\beta_{m}\big(\beta_{a}C^{m}{}_{bn}-\beta_{b}C^{m}{}_{an}\big)\\
& +c^{dk}c^{jl}\beta_{n}\big(\gamma_{am}C^{n}{}_{bj}-\gamma_{bm}C^{n}{}_{aj}\big)\\
& +c^{dk}b^{l}\beta_{m}\beta_{n}\big(C^{m}{}_{ak}C^{n}{}_{bl}-C^{m}{}_{bk}C^{n}{}_{al}\big)\\
& +6c^{dl}\beta_{m}C^{m}{}_{nl}C^{n}{}_{ac}\Big),
\end{aligned}
$$

(179)

$$
\begin{aligned}
R^{z}{}_{cab} = \frac{1}{4}\Big(& -b^{i}b^{j}\big(\mu_{ai}\nu_{bcj}-\mu_{bi}\nu_{acj}\big)+c^{ij}b^{k}\big(\nu_{ajk}\nu_{bci}-\nu_{bjk}\nu_{aci}\big)+b^{i}\big(\phi_{a}\nu_{bci}-\phi_{b}\nu_{aci}\big)\\
& +b^{i}\big(\nu_{ami}C^{m}{}_{bc}-\nu_{mbi}C^{m}{}_{ac}\big)+2b^{i}\nu_{cmi}C^{m}{}_{ab}\Big)=\\
=\frac{1}{4}\Big(& -2\phi_{i}\phi_{i}jb^{i}b^{j}\big(\beta_{a}\gamma_{bc}-\beta_{b}\gamma_{ac}\big)+\phi_{i}b^{i}\big(\phi_{a}\gamma_{bc}-\phi_{b}\gamma_{ac}\big)\\
& -\phi_{i}c^{ik}b^{l}\big(\gamma_{am}\gamma_{bc}-\gamma_{bm}\gamma_{ac}\big)C^{m}{}_{kl}+2\phi_{i}b^{i}\big(\gamma_{am}C^{m}{}_{bc}-\gamma_{bm}C^{m}{}_{ac}\big)\\
& +2\phi_{i}b^{i}\big(\gamma_{am}C^{m}{}_{bc}-\gamma_{bm}C^{m}{}_{ac}+\gamma_{cm}C^{m}{}_{ab}\big)\\
& -c^{jk}b^{l}\big(\gamma_{am}\gamma_{bc}-\gamma_{bm}\gamma_{ac}\big)C^{m}{}_{cj}C^{n}{}_{kl}+c^{jk}b^{l}\gamma_{cm}\big(\gamma_{an}C^{m}{}_{bj}-\gamma_{bn}C^{m}{}_{aj}\big)C^{n}{}_{kl}\\
& -b^{l}\big(\gamma_{am}C^{n}{}_{bc}-\gamma_{bm}C^{n}{}_{ac}\big)C^{m}{}_{nl}-b^{l}\big(\gamma_{am}C^{n}{}_{bl}-\gamma_{bm}C^{n}{}_{al}\big)C^{m}{}_{cn}\\
& -b^{l}\gamma_{cm}\big(C^{m}{}_{an}C^{n}{}_{bl}-C^{m}{}_{bn}C^{n}{}_{al}+2C^{m}{}_{nl}C^{n}{}_{ab}\big)\\
& -b^{l}\gamma_{mn}\big(C^{m}{}_{al}C^{n}{}_{bc}-C^{m}{}_{bl}C^{n}{}_{ac}+2C^{m}{}_{cl}C^{n}{}_{ab}\big)\quad\text{and}
\end{aligned}
$$

(180)

$$
\begin{aligned}
R^{d}{}_{cab} = \frac{1}{4}\Big(& c^{di}b^{j}\big(\nu_{aij}k_{bc} - \nu_{bij}k_{ac}\big) - c^{di}b^{j}\big(\mu_{ai}\nu_{bcj} - \mu_{bi}\nu_{acj}\big) \\
&+ b^{d}c^{ij}\big(k_{ia}\nu_{bcj} - k_{ib}\nu_{acj}\big) \\
&- \phi_{i}b^{i}\big(\delta^{d}{}_{a}k_{bc} - \delta^{d}{}_{b}k_{ac}\big) - \phi_{j}c^{ij}\big(\delta^{d}{}_{a}\nu_{bci} - \delta^{d}{}_{b}\nu_{aci}\big) \\
&+ c^{di}c^{jk}\big(\nu_{aji}\nu_{bck} - \nu_{bji}\nu_{ack}\big) - \phi_{c}\big(\phi_{a}\delta_{b}{}^{d} - \phi_{b}\delta_{a}{}^{d}\big) \\
&- b^{n}\big(k_{ac}C^{d}{}_{bn} - k_{bc}C^{d}{}_{an}\big) + b^{d}\big(k_{am}C^{m}{}_{bc} - k_{bm}C^{m}{}_{ac}\big) + 2b^{d}k_{cm}C^{m}{}_{ab} \\
&- c^{in}\big(\nu_{aci}C^{d}{}_{bn} - \nu_{bci}C^{d}{}_{an}\big) + 2c^{dn}\big(\nu_{acm}C^{m}{}_{bn} - \nu_{bcm}C^{m}{}_{an}\big) \\
&+ c^{di}\big(\nu_{ami}C^{m}{}_{bc} - \nu_{bmi}C^{m}{}_{ac}\big) + 2c^{di}\nu_{cmi}C^{m}{}_{ab} \\
&+ C^{d}{}_{an}C^{n}{}_{bc} - C^{d}{}_{bn}C^{n}{}_{ac} - 2C^{d}{}_{cn}C^{n}{}_{ab}\Big) = \\
= \frac{1}{4}\Big(& \phi_{j}\big(c^{di}b^{j} - b^{d}c^{ij}\big)\big(\gamma_{ac}k_{bi} - \gamma_{bc}k_{ai}\big) - \phi_{i}b^{d}b^{i}\big(\beta_{a}k_{bc} - \beta_{b}k_{ac}\big) \\
&- c^{dk}b^{l}\big(\gamma_{am}k_{bc} - \gamma_{bm}k_{ac}\big)C^{m}{}_{kl} - c^{di}b^{n}\big(\gamma_{am}k_{bi} - \gamma_{bm}k_{ai}\big)C^{m}{}_{cn} \\
&+ c^{di}b^{n}\gamma_{cm}\big(k_{ai}C^{m}{}_{bn} - k_{bi}C^{m}{}_{an}\big) - b^{d}b^{n}\beta_{m}\big(k_{ac}C^{m}{}_{bn} - k_{bc}C^{m}{}_{an}\big) \\
&+ b^{d}\big(k_{am}C^{m}{}_{bc} - k_{bm}C^{m}{}_{ac} + 2k_{cm}C^{m}{}_{ab} \\
&+ \phi_{i}c^{di}\big(\phi_{a}\gamma_{bc} - \phi_{b}\gamma_{ac}\big) - \phi_{i}\phi_{j}c^{ij}\big(\delta^{d}{}_{a}\gamma_{bc} - \delta^{d}{}_{b}\gamma_{ac}\big) \\
&+ \phi_{i}c^{dk}c^{il}\big(\gamma_{am}\gamma_{bc} - \gamma_{bm}\gamma_{ac}\big)C^{m}{}_{kl}\big) \\
&+ 2\phi_{i}c^{di}b^{n}\big(\beta_{a}\gamma_{bm} - \beta_{b}\gamma_{am}\big)C^{m}{}_{cn} - 2\phi_{i}c^{di}b^{n}\gamma_{cm}\big(\beta_{a}C^{m}{}_{bn} - \beta_{b}C^{m}{}_{an}\big) \\
&- \phi_{i}c^{in}\big(\delta^{d}{}_{a}\gamma_{bm} - \delta^{d}{}_{b}\gamma_{am}\big)C^{m}{}_{cn} - \phi_{i}c^{in}\gamma_{cm}\big(\delta^{d}{}_{a}C^{m}{}_{bn} - \delta^{d}{}_{b}C^{m}{}_{an}\big) \\
&+ c^{dk}c^{jl}\big(\gamma_{am}\gamma_{bn} - \gamma_{bm}\gamma_{an}\big)C^{m}{}_{cj}C^{n}{}_{kl} \\
&- c^{dk}c^{jl}\gamma_{cm}\big(\gamma_{an}C^{m}{}_{bj} - \gamma_{bn}C^{m}{}_{aj}\big)C^{n}{}_{kl} \\
&+ \big(c^{dk}b^{l} + c^{dl}b^{k}\big)\beta_{n}\big(\gamma_{am}C^{n}{}_{bl} - \gamma_{bm}C^{n}{}_{al}\big)C^{m}{}_{ck} \\
&- c^{dk}b^{l}\gamma_{cm}\beta_{n}\big(C^{m}{}_{ak}C^{n}{}_{bl} - C^{m}{}_{bk}C^{n}{}_{al}\big) \\
&+ c^{kl}\big(\gamma_{am}C^{d}{}_{bk} - \gamma_{bm}C^{d}{}_{ak}\big)C^{m}{}_{cl} - c^{kl}\gamma_{cm}\big(C^{d}{}_{ak}C^{m}{}_{bl} - C^{d}{}_{bk}C^{m}{}_{al}\big) \\
&- 3c^{dl}\big(\gamma_{am}C^{n}{}_{bl} - \gamma_{bm}C^{n}{}_{al}\big)C^{m}{}_{cn} - c^{dl}\big(\gamma_{am}C^{n}{}_{bc} - \gamma_{bm}C^{n}{}_{ac}\big)C^{m}{}_{nl} \\
&+ 2c^{dl}\gamma_{cm}\big(C^{m}{}_{an}C^{n}{}_{bl} - C^{m}{}_{bn}C^{n}{}_{al} - C^{m}{}_{nl}C^{n}{}_{ab}\big) \\
&+ c^{dl}\gamma_{mn}\big(C^{m}{}_{al}C^{n}{}_{bc} - C^{m}{}_{bl}C^{n}{}_{ac} - 2C^{m}{}_{cl}C^{n}{}_{ab}\big) \\
&+ C^{d}{}_{an}C^{n}{}_{bc} - C^{d}{}_{bn}C^{n}{}_{ca} + 2C^{d}{}_{cn}C^{n}{}_{ab}\Big)\,.
\end{aligned}
$$

4.3. The Ricci tensor and the Ricci scalar; the Einstein tensor. The Ricci tensor is derived in the same way - by subtracting the upper with the second lower index of the Riemann tensor.

(181)
$$\begin{aligned} R_{zz} =& \frac{1}{4}\Big(-2c^{ij}\ell_{ij} + c^{ik}c^{jl}\big(2k_{ij} - \mu_{ij}\big)\mu_{lk}\Big) \\ =& \frac{1}{4}\Big(-2c^{ij}\ell_{ij} + c^{ik}c^{jl}k_{ij}k_{kl} + c^{ik}c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl}\Big) \quad , \end{aligned}$$

(182)
$$\begin{aligned} R_{za} = R_{az} =& \frac{1}{4}\Big(b^i c^{jk}\mu_{ij}\mu_{ak} - c^{ik}c^{jl}\nu_{ijk}\mu_{al} + c^{ik}c^{jl}\nu_{aik}k_{jl} + 3\phi_j c^{ij}\mu_{ai} - \phi_a c^{ij}k_{ij} \\ & + c^{in}\big(3\mu_{mi} - 4\mu_{im}\big)C^m{}_{an} - c^{in}\mu_{ai}C^m{}_{mn} = \\ =& \frac{1}{4}\Big(-3\phi_j c^{ij}k_{ai} - 3\phi_i\phi_j c^{ij}\beta_a \\ & - 2c^{ik}b^l\beta_m k_{ai}C^m{}_{kl} - c^{in}k_{im}C^m{}_{an} + c^{ij}k_{ij}C^m{}_{ma} \\ & + 2\phi_i c^{ik}b^l\beta_a\beta_m C^m{}_{kl} + 3\phi_i c^{in}\beta_m C^m{}_{an} + 2\phi_i c^{in}\beta_a C^m{}_{mn} \\ & - 2c^{jk}b^l\beta_m\beta_n C^m{}_{aj}C^n{}_{kl} + 7c^{kl}\beta_m C^m{}_{nk}C^n{}_{al}\Big) \quad \text{and} \end{aligned}$$

(183)
$$\begin{aligned} R_{ab} =& \frac{1}{4}\Big(-2\phi_k c^{ij}b^k\gamma_{ab}k_{ij} - 3\phi_i b^i k_{ab} \\ & + 2c^{ij}b^n k_{ij}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an}\big) \\ & - 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an} + k_{ab}C^m{}_{mn}\big) \\ & + \phi_i\phi_j\big(c^{ij}\gamma_{ab} - b^i b^j\beta_a\beta_b\big) + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi_b \\ & + 2\phi_i c^{ik}b^l\gamma_{ab}\beta_m C^m{}_{kl} + 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{bm} + \beta_b\gamma_{am}\big)C^m{}_{kl} \\ & - 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} - \gamma_{ab}C^m{}_{mn}\big) \\ & + c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} - 2c^{jk}b^l\beta_n\big(\gamma_{am}C^n{}_{bj} + \gamma_{bm}C^n{}_{aj}\big)C^m{}_{kl} \\ & - 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{ak}\big)C^m{}_{nl} - 2c^{kl}\gamma_{mn}C^m{}_{ak}C^n{}_{bl} \\ & - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{ak}\big)C^n{}_{nl} \\ & - b^l\beta_m C^m{}_{an}C^n{}_{bl} + b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{an}\big)C^n{}_{ml} \\ & - 3C^m{}_{an}C^n{}_{bm} - C^m{}_{ma}C^n{}_{nb} \,. \end{aligned}$$

In the same manner, the Ricci scalar is defined in Eq. (40); so, we have,

$$R = aR_{zz} + 2b^i R_{zi} + c^{ij}R_{ij} \,,$$

so

(184)
$$\begin{aligned} R =& \frac{1}{4}\Big(7\phi_k c^{ij}b^k k_{ij} - 3c^{ik}b^l k_{im}C^m{}_{kl} + 5c^{ij}b^n k_{ij}C^m{}_{mn} \\ & + 2\phi_i\phi_j c^{ij} - \phi_i c^{ik}b^l\beta_m C^m{}_{kl} \\ & - c^{ik}c^{jl}\gamma_{mn}C^m{}_{ij}C^n{}_{kl} - 3c^{jk}b^l\beta_m C^m{}_{nj}C^n{}_{kl} \\ & - 7c^{kl}C^m{}_{nk}C^n{}_{ml} - 5c^{kl}C^m{}_{mk}C^n{}_{nl} \,. \end{aligned}$$

Having defined the Einstein tensor in Eq. (53), we can calculate its components to be

$$G_{zz} = \frac{1}{4}\Big(- 2c^{ij}\ell_{ij} + c^{ik}c^{jl}k_{ij}k_{kl} + c^{ik}c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl}\Big) \ , \tag{185}$$

(186)

$$\begin{aligned}
G_{za} = G_{az} =& \frac{1}{4}\Big(- 3\phi_j c^{ij}k_{ai} - \frac{7}{2}\phi_k c^{ij}b^k\beta_a k_{ij} - 4\phi_i\phi_j c^{ij}\beta_a \\
& - 2c^{ik}b^l\beta_m k_{ia}C^m{}_{kl} - \frac{3}{2}c^{ik}b^l\beta_a k_{im}C^m{}_{kl} \\
& + \frac{5}{2}c^{ij}b^n\beta_a k_{ij}C^m{}_{mn} - c^{in}k_{im}C^m{}_{an} + c^{ij}k_{ij}C^m{}_{ma} \\
& - \frac{5}{2}\phi_i c^{ik}b^l\beta_a\beta_m C^m{}_{kl} + 3\phi_i c^{in}\beta_m C^m{}_{an} + 2\phi_i c^{in}\beta_a C^m{}_{mn} \\
& - 2c^{jk}b^l\beta_m\beta_n C^m{}_{aj}C^n{}_{kl} + \frac{3}{2}c^{jk}b^l\beta_a\beta_m C^m{}_{nj}C^n{}_{kl} \\
& + \frac{1}{2}c^{ik}c^{jl}\beta_a\gamma_{mn}C^m{}_{ij}C^n{}_{kl} \\
& + 7c^{kl}\beta_m C^m{}_{nk}C^n{}_{al} \\
& + \frac{7}{2}c^{kl}\beta_a C^m{}_{nk}C^n{}_{ml} + \frac{5}{2}c^{kl}\beta_a C^m{}_{mk}C^n{}_{nl}\Big) \quad \text{and}
\end{aligned}$$

(187)

$$
\begin{aligned}
G_{ab} =& \frac{1}{4}\Big(-\frac{11}{2}\phi_k b^k c^{ij}\gamma_{ab}k_{ij} - 3\phi_i b^i k_{ab} \\
& - 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an} + k_{ab}C^m{}_{mn}\big) \\
& + 2c^{ij}b^n k_{ij}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} + \gamma_{ab}C^m{}_{mn}\big) \\
& + \frac{1}{2}\gamma_{ab}\big(3c^{ik}b^l k_{im}C^m{}_{kl} + c^{ij}b^n k_{ij}C^m{}_{mn}\big) \\
& + \phi_i\phi_j b^i b^j \beta_a\beta_b + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi_b \\
& + 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{am} + \beta_b\gamma_{am}\big)C^m{}_{kl} + \frac{5}{2}\phi_i c^{ik}b^l\gamma_{ab}\beta_m C^m{}_{kl} \\
& - 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} + \gamma_{ab}C^m{}_{mn}\big) \\
& + c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} + \frac{1}{2}c^{ik}c^{jl}\gamma_{mn}\gamma_{ab}C^m{}_{ij}C^n{}_{kl} \\
& - 2c^{jk}b^l\beta_n\big(\gamma_{am}C^n{}_{bj} + \gamma_{bm}C^n{}_{aj}\big)C^m{}_{kl} \\
& + \frac{3}{2}c^{jk}b^l\gamma_{ab}\beta_m C^m{}_{nj}C^n{}_{kl} \\
& - 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{ak}\big)C^m{}_{nl} \\
& - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{ak}\big)C^n{}_{nl} \\
& - 2c^{kl}\gamma_{mn}C^m{}_{ak}C^n{}_{bl} + \frac{7}{2}c^{kl}\gamma_{ab}C^m{}_{nk}C^n{}_{ml} + \frac{5}{2}c^{kl}\gamma_{ab}C^m{}_{mk}C^n{}_{nl} \\
& - b^l\beta_m C^m{}_{an}C^n{}_{bl} + b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{na}\big) \\
& - 3C^m{}_{an}C^n{}_{bm} - C^m{}_{ma}C^n{}_{nb}\Big).
\end{aligned}
$$

CHAPTER 4

Vacuum Solutions: The Einstein System

1. Introduction

This chapter deals with the Einstein system when the source (the stress-energy-momentum tensor) is zero; *i.e.*, when the space-time is deprived of all matter and energy. In this case, the Einstein equations take their simpler form; and, when the coordinate system is chosen appropriately so as to coincide with the group and its quotient, they are reduced to a set of ordinary differential equations. Of course, not all ordinary differential solutions are integrable, but the existence and uniqueness of a solution is easy to prove locally (*i.e.*, within a certain range of value of the independent variable) using the Picard–Lindelöf theorem and the Banach fixed point theorem.

The first part of the chapter concerns the form the Einstein system takes in the vacuum case and the existence and uniqueness of its solutions. It is followed by two major examples: the application of Bianchi I and Bianchi II groups in the Minkowski space-time. We show how the appropriate coordinate system can be found for each case; moreover, we examine a peculiarity appearing in the case of Bianchi I, which distinguishes this particular class of space-times from other similar treatments. Finally, we discuss the solution of the Einstein system in a general case, so as to consider solutions near the ones examined as examples; as part of this discussion, we show that the peculiarity emerging in one of the examples is not necessarily a isolated event, but may appear in several space-times.

2. The Einstein System

The Einstein system is composed of the Einstein tensor is equal to zero:

$$\boldsymbol{G} = 0\,.^{1} \tag{188}$$

This system refers to space-times 'deprived' of any form of matter or energy; space-times that are referred to as 'vacua'. Some of these space-times are also known as 'flat', but this is not a safe designation in General Relativity. In general, it is assumed that space-time is curved because of the presence of matter (and/or energy); however, this is true only to some extent. There are

[1] This is equivalent to setting the Ricci tensor equal to zero:

$$\boldsymbol{R} = 0\,.$$

However, the Einstein tensor is preferred, because it is the 'natural' choice in the context of General Relativity.

space-times that are 'empty' (in that they contain no matter and/or energy), but are not 'flat'. The best examples of such space-times are the Schwarzschild and Kerr black hole solutions: while, they contain no matter, the intrinsic curvature of the foliated (spatial) submanifolds is non-zero - in fact, it grows as the singularity is approached and it blows up there. As a result, these space-times, although 'empty', are not 'flat' in the manner usually perceived by three-dimensional observers (such as us). On the contrary, the Friedmann-Lemaître-Robertson-Walker cosmologies are an example of space-times that are non-empty (they contain a homogeneous perfect fluid), but possess spatial manifolds with zero intrinsic curvature. As a result, a clarification needs to be made:

- There are space-times that are flat, because all (extrinsic and intrinsic) measures of curvature are zero everywhere; that is, the Riemann-Chistoffel tensor is zero. An obvious example of this is the Minkowski space-time.
- There are space-times that are flat because they contain no matter and/or energy; thus, their Ricci tensor is zero, but not necessarily the Riemann-Christoffel tensor. The Schwarzschild and the Kerr black holes are such solutions. These spaces are denoted *Ricci-flat*.
- There are space-times that are flat only with respect to some particular measure of curvature (*e.g.*, the Weyl curvature), but they contain matter and/or energy, thus their Ricci tensor is non-zero. An example of this is the Friedmann-Lemaître-Robertson-Walker cosmologies. These spaces are denoted *conformally flat*.
- Finally, there are space-times that are not 'empty' or 'flat' in any of the afore-mentioned senses, however, they approach (Ricci-)flatness either for large distances (away from any centre of symmetry), or for large times. These are known as *asymptotically flat*. The Schwarzschild and the Kerr space-times are examples of this.

It should be obvious that our case is that of Ricci-flatness.

2.1. The evolution equations and the constraints. The Einstein system in vacuum consist of a system of differential equations for the metric, $\boldsymbol{g}$, and its first derivative along z, $\boldsymbol{k}$, that are written in the compact form

$$\begin{aligned} \mathcal{L}_{\vec{\zeta}}\boldsymbol{g} &= \boldsymbol{k} \quad \text{and} \\ \boldsymbol{T}(g,k) &= 0\,. \end{aligned} \tag{189}$$

Given the derivatives of both $\boldsymbol{g}$ and $\boldsymbol{k}$ along the generators of the group are given by

$$\begin{aligned} \mathcal{L}_{\vec{\xi}_a}\boldsymbol{g} &= \phi_a\boldsymbol{g} \quad \text{and} \\ \mathcal{L}_{\vec{\xi}_a}\boldsymbol{k} &= \phi_a\boldsymbol{k}\,, \end{aligned} \tag{190}$$

the Einstein system is reduced to a system of ordinary differential equations, whose independent parameter is z.

Using eqs. (84), (85) and (88), (90), (92), we can express the derivatives of the metric and its inverse, as

$$\frac{\partial \beta_a}{\partial z} = 0\,, \tag{191}$$

$$\frac{\partial \gamma_{ab}}{\partial z} = k_{ab}\,, \tag{192}$$

$$\frac{\partial \mathrm{a}}{\partial z} = -b^i b^j k_{ij}\,, \tag{193}$$

$$\frac{\partial b^a}{\partial z} = -c^{ai} b^j k_{ij} \quad \text{and} \tag{194}$$

$$\frac{\partial c^{ab}}{\partial z} = -c^{ai} c^{bj} k_{ij}\,. \tag{195}$$

Using eqs. (185), (186) and (187), and setting them equal to zero, we can express the derivatives of $\boldsymbol{k}$ as

$$2c^{ij}\frac{\partial k_{ij}}{\partial z} = c^{ik}c^{jl}k_{ij}k_{kl} + c^{ik}c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl}\,, \tag{196}$$

$$\begin{aligned}
2b^i\frac{\partial k_{ia}}{\partial z} &+ 2\big(\mathrm{a}c^{ij} - b^i b^j\big)\beta_a\frac{\partial k_{ij}}{\partial z} = \\
&= b^i c^{jk}k_{ij}k_{ak} + \frac{1}{2}c^{ik}\big(\mathrm{a}c^{jl} - 2b^j b^l\big)\beta_a k_{ij}k_{kl} \\
&+ 3\phi_j c^{ij}k_{ai} + \frac{1}{2}\phi_k\big(4b^i c^{jk} + 7c^{ij}b^k\big)\beta_a k_{ij} \\
&+ 2c^{ik}b^l\beta_m k_{ia}C^m{}_{kl} + \frac{3}{2}c^{ik}b^l\beta_a k_{im}C^m{}_{kl} \\
&- \frac{1}{2}\big(5c^{ij}b^n - 4b^i c^{jn}\big)\beta_a k_{ij}C^m{}_{mn} + c^{in}k_{im}C^m{}_{an} - c^{ij}k_{ij}C^m{}_{ma} \\
&+ 4\phi_i\phi_j c^{ij}\beta_a \\
&+ \frac{5}{2}\phi_i c^{ik}b^l\beta_a\beta_m C^m{}_{kl} - 3\phi_i c^{in}\beta_m C^m{}_{an} - 2\phi_i c^{in}\beta_a C^m{}_{mn} \\
&+ 2c^{jk}b^l\beta_m\beta_n C^m{}_{aj}C^n{}_{kl} - \frac{3}{2}c^{jk}b^l\beta_a\beta_m C^m{}_{nj}C^n{}_{kl} \\
&- \frac{1}{2}c^{ik}c^{jl}\beta_a\big(\gamma_{mn} + \mathrm{a}\beta_m\beta_n\big)C^m{}_{ij}C^n{}_{kl} \\
&- 7c^{kl}\beta_m C^m{}_{nk}C^n{}_{al} \\
&- \frac{7}{2}c^{kl}\beta_a C^m{}_{nk}C^n{}_{ml} - \frac{5}{2}c^{kl}\beta_a C^m{}_{mk}C^n{}_{nl} \quad \text{and}
\end{aligned} \tag{197}$$

(198)

$$
\begin{aligned}
&2\mathrm{a}\frac{\partial k_{ab}}{\partial z} - 2\big(\mathrm{a}c^{ij} - b^i b^j\big)\gamma_{ab}\frac{\partial k_{ij}}{\partial z} = \\
&\quad = -\big(\mathrm{a}c^{ij} - 2b^i b^j\big)k_{ij}k_{ab} + \frac{1}{2}c^{ik}\big(\mathrm{a}c^{jl} - b^j b^l\big)\gamma_{ab}k_{ij}k_{kl} \\
&\quad + \frac{1}{2}\phi_k\big(2b^i c^{jk} - 11b^k c^{ij}\big)\gamma_{ab}k_{ij} - 3\phi_i b^i k_{ab} + b^i\big(\phi_a k_{bi} + \phi_b k_{ai}\big) \\
&\quad - 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an} + k_{ab}C^m{}_{mn}\big) \\
&\quad - 2\big(b^i c^{jn} - c^{ij}b^n\big)k_{ij}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} + \gamma_{ab}C^m{}_{mn}\big) \\
&\quad + \frac{1}{2}\gamma_{ab}\big(3c^{ik}b^l k_{im}C^m{}_{kl} + c^{ij}b^n k_{ij}C^m{}_{mn}\big) \\
&\quad + \phi_i\phi_j b^i b^j \beta_a\beta_b + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi_b \\
&\quad + 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{am} + \beta_b\gamma_{am}\big)C^m{}_{kl} + \frac{5}{2}\phi_i c^{ik}b^l\gamma_{ab}\beta_m C^m{}_{kl} \\
&\quad - 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} + \gamma_{ab}C^m{}_{mn}\big) \\
&\quad + c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} + \frac{1}{2}c^{ik}c^{jl}\big(\gamma_{mn} + \mathrm{a}\beta_m\beta_n\big)\gamma_{ab}C^m{}_{ij}C^n{}_{kl} \\
&\quad - 2c^{jk}b^l\beta_n\big(\gamma_{am}C^n{}_{bj} + \gamma_{bm}C^n{}_{aj}\big)C^m{}_{kl} + \frac{3}{2}c^{jk}b^l\gamma_{ab}\beta_m C^m{}_{nj}C^n{}_{kl} \\
&\quad - 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{ak}\big)C^m{}_{nl} - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{ak}\big)C^n{}_{nl} \\
&\quad - 2c^{kl}\gamma_{mn}C^m{}_{ak}C^n{}_{bl} + \frac{7}{2}c^{kl}\gamma_{ab}C^m{}_{nk}C^n{}_{ml} + \frac{5}{2}c^{kl}\gamma_{ab}C^m{}_{mk}C^n{}_{nl} \\
&\quad - b^l\beta_m C^m{}_{an}C^n{}_{bl} + b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{na}\big) - 3C^m{}_{an}C^n{}_{bm} - C^m{}_{ma}C^n{}_{nb}\,.
\end{aligned}
$$

However, we need to note that, unlike eqs. (191)-(195), the derivatives of the dependent variable are not isolated in eqs. (196)-(198). To isolate them, we proceed to the following: First of all, we contract eq. (197) with b^a and then we subtract it from eq. (196) - the result is the following

(199)

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{a}c^{ik}c^{jl}k_{ij}k_{kl} + \frac{1}{2}\phi_k\big(10b^i c^{jk} - 7c^{ij}b^k\big)k_{ij} \\
&\quad + \frac{1}{2}c^{ik}b^l k_{im}C^m{}_{kl} + \frac{1}{2}\big(4b^i c^{jn} - 7c^{ij}b^n\big)k_{ij}C^m{}_{mn} \\
&\quad + 4\phi_i\phi_j c^{ij} + \frac{11}{2}\phi_i c^{ik}b^l\beta_m C^m{}_{kl} - 2\phi_i c^{in}C^m{}_{mn} \\
&\quad - c^{ik}\big(\mathrm{a}c^{jl} - 2b^j b^l\big)\beta_m\beta_n C^m{}_{ij}C^n{}_{kl} - \frac{1}{2}c^{ik}c^{jl}\big(\gamma_{mn} + \mathrm{a}\beta_m\beta_n\big)C^m{}_{ij}C^n{}_{kl} \\
&\quad + 7c^{jk}b^l\beta_m C^m{}_{nj}C^n{}_{kl} - \frac{7}{2}c^{kl}C^m{}_{nk}C^n{}_{ml} - \frac{5}{2}c^{kl}C^m{}_{mk}C^n{}_{nl} = 0\,,
\end{aligned}
$$

which serves as a constraint equation for the components of $\boldsymbol{g}$ and $\boldsymbol{k}$. This constraint

Second, we contract eq. (197) with b^a and eq. (198) with c^{ab}, adding them afterwards. The result is

(200)

$$\begin{aligned}2\big(\mathrm{a}c^{ij} - b^i b^j\big)\frac{\partial k_{ij}}{\partial z} =& \frac{1}{2}\big(\mathrm{a}c^{ij} - b^i b^j\big)c^{kl}k_{ij}k_{kl} - \frac{1}{4}c^{ik}\big(\mathrm{a}c^{jl} - 3b^j b^l\big)k_{ij}k_{kl}\\ &+ \frac{1}{2}\phi_k\big(6b^i c^{jk} + 7c^{ij}b^k\big)k_{ij}\\ &+ \frac{1}{4}\big(10b^i c^{jn} - 9c^{ij}b^n\big)k_{ij}C^m{}_{mn} + \frac{3}{4}c^{ik}b^l k_{im}C^m{}_{kl}\\ &+ \frac{1}{2}\phi_i\phi_j c^{ij} + \frac{5}{2}\phi_i c^{ik}b^l\beta_m C^m{}_{kl} + 2\phi_i c^{in}C^m{}_{mn}\\ &- \frac{1}{2}c^{ik}c^{jl}\gamma_{mn}C^m{}_{ij}C^n{}_{kl} - \frac{1}{4}c^{ik}\big(3\mathrm{a}c^{jl} - 4b^j b^l\big)\beta_m\beta_n C^m{}_{ij}C^n{}_{kl}\\ &+ \frac{3}{4}c^{jk}b^l\beta_m C^m{}_{nj}C^n{}_{kl} + 2c^{jk}b^l\beta_m C^m{}_{kl}C^n{}_{nj}\\ &- \frac{7}{4}c^{kl}C^m{}_{nk}C^n{}_{ml} - \frac{5}{4}c^{kl}C^m{}_{mk}C^n{}_{nl}\,,\end{aligned}$$

which contains the derivative of $\boldsymbol{k}$ subtracted with the rank-2 tensor $\mathrm{a}c^{ij} - b^i b^j$ (a 'block determinant' of the inverse matrix). Subtracting from eq. (197), we obtain

(201)

$$\begin{aligned}2b^i\frac{\partial k_{ia}}{\partial z} =& b^i c^{jk}k_{ij}k_{ak} + \frac{1}{4}c^{ik}\big(\mathrm{a}c^{jl} - 3b^j b^l\big)k_{ij}k_{kl}\\ &+ 3\phi_j c^{ij}k_{ai} - \phi_k b^i c^{jk}\beta_a k_{ij}\\ &+ \frac{3}{4}c^{ik}b^l\beta_a k_{im}C^m{}_{kl} + 2c^{ik}b^l\beta_m k_{ia}C^m{}_{kl}\\ &- \frac{1}{4}\big(2b^i c^{jn} + c^{ij}b^n\big)\beta_a k_{ik}C^m{}_{mn} + c^{in}k_{im}C^m{}_{na} - c^{ij}k_{ij}C^m{}_{ma}\\ &+ \frac{7}{2}\phi_i\phi_j c^{ij}\beta_a - 3\phi_i c^{in}\beta_m C^m{}_{na} - 4\phi_i c^{in}\beta_a C^m{}_{mn}\\ &+ \frac{1}{4}c^{ik}\big(\mathrm{a}c^{jl} - 4b^j b^l\big)\beta_a\beta_m\beta_n C^m{}_{ij}C^n{}_{kl}\\ &+ 2c^{jk}b^l\beta_m\beta_n C^m{}_{aj}C^n{}_{kl} - \frac{9}{4}c^{jk}b^l\beta_a\beta_m C^m{}_{nj}C^n{}_{kl}\\ &- 2c^{jk}b^l\beta_a\beta_m C^m{}_{kl}C^n{}_{nj} - 7c^{kl}\beta_m C^m{}_{nk}C^n{}_{al}\\ &- \frac{7}{4}c^{kl}\beta_a C^m{}_{nk}C^n{}_{ml} - \frac{5}{4}c^{kl}\beta_a C^m{}_{mk}C^n{}_{nl}\,;\end{aligned}$$

and adding to eq. (198), we obtain

(202)

$$
\begin{aligned}
2\mathrm{a}\frac{\partial k_{ab}}{\partial z} =& -\big(\mathrm{a}c^{ij} - 2b^i b^j\big)k_{ij}k_{ab} + \frac{1}{4}c^{ik}\big(\mathrm{a}c^{jl} + b^j b^l\big)\gamma_{ab}\big(2k_{ik}k_{jl} + k_{ij}k_{kl}\big) \\
&+ 4\phi_k\big(b^i c^{jk} - c^{ij}b^k\big)\gamma_{ab}k_{ij} - 2\phi_i b^i k_{ab} + b^i\big(\phi_a k_{bi} + \phi_b k_{ai}\big) \\
&- 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an} + k_{ab}C^m{}_{mn}\big) \\
&+ \frac{1}{4}\big(2b^i c^{jn} + c^{ij}b^n\big)\gamma_{ab}k_{ij}C^m{}_{mn} + \frac{9}{4}c^{ik}b^l\gamma_{ab}k_{im}C^m{}_{kl} \\
&- 2\big(b^i c^{jn} - c^{ij}b^n\big)k_{ij}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an}\big) \\
&- \frac{1}{2}\phi_i c^{ij}\gamma_{ab} + \phi_i\phi_j b^i b^j\beta_a\beta_b + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi_b \\
&+ 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{bm} + \beta_b\gamma_{am} + 2\beta_m\gamma_{ab}\big)C^m{}_{kl} \\
&- 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an}\big) \\
&+ c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} - \frac{1}{4}c^{ik}\big(\mathrm{a}c^{jl} - 4b^j b^l\big)\gamma_{ab}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl} \\
&+ \frac{9}{4}c^{jk}b^l\gamma_{ab}\beta_m C^m{}_{nj}C^n{}_{kl} + 2c^{jk}b^l\gamma_{ab}\beta_m C^m{}_{kl}C^n{}_{nj} \\
&- 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{ak}\big) - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{ak}\big)C^n{}_{nl} \\
&+ \frac{7}{4}c^{kl}\gamma_{ab}C^m{}_{nk}C^n{}_{ml} + \frac{5}{4}c^{kl}\gamma_{ab}C^m{}_{mk}C^n{}_{nl} \\
&- b^l\beta_m C^m{}_{na}C^n{}_{bl} + b^l\beta_m\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{an}\big)C^n{}_{ml} \\
&- 3C^m{}_{na}C^n{}_{mb} - C^m{}_{ma}C^n{}_{nb}\,.
\end{aligned}
$$

The system of eqs. (196), (201) and (202) are a series of evolution equations for k_{ab}, each of whom corresponds to a different subtraction of the derivative $\dfrac{\partial k_{ab}}{\partial z}$ with the metric. Counting equations and unknowns, we have 10 equations (as expected from the Einstein system) for 6 unknowns (the independent components of k_{ab}). It is clear that the system is overdetermined and eqs. (196) and (201) are redundant, while eq. (202) should be sufficient to account for the evolution of all components of $\boldsymbol{k}$; given $\mathrm{a} \neq 0$, the easily obtain the usual structure of an evolution equation.[2] This is not an uncommon result in the Einstein system, especially if we account for the particular structure of the metric (namely, no zz-component and a constant with respect to z $za-$component). Moreover, the two additional equations are not exactly redundant; in the same manner that eq. (199) provides a constraint for the components of $\boldsymbol{k}$ (particularly for the initial conditions), they can provide constraints for the derivative of it.

[2]We remind here that the case $\mathrm{a} = 0$ should not be considered within this formalism, as it leads to pathologies.

It is noteworthy that eqs. (201) and (202) lead back to eqs. (197) and (198) respectively if we use eq. (200), in the same manner they were derived; thus, they are equivalent to the Einstein equations. However, there is a stronger claim that these equations are equivalent to the Einstein equations: The same equations can be derived using the fact that

$$R_{zz} = 0\,, \quad R_{za} = 0 \quad \text{and} \quad R_{ab} = 0\,,$$

which is true for vacuum space-times. Notably, $R_{zz} = 0$ yields eq. (196) immediately, while $R_{za} = 0$ and $R_{ab} = 0$ may yield eqs. (201) and (202) if the constraint eq. (199) is used appropriately.

One final word with regards to the constraint equations. Apart from their use to recover the Ricci tensor from the evolution equations, the constraint equations serve another purpose in the Initial Value Problem of eqs. (196), (201) and (202): any initial condition $\{\boldsymbol{g}_0, \boldsymbol{k}_0\}$ must fulfill the constraint eq. (199). Moreover, if the constraint propagates, then the solution of the system for any 'moment' along z will also fulfill the constraint. To show that this is possible, we can differentiate the left-hand side of eq. (199) and show, by means of the evolution equations, that the derivative is equal to zero.

2.2. The case of null orbits. Let us go consider now the special case where $a = 0$ and the 3−metric on the homogeneous submanifold $\mathcal{M}$ is degenerate. Firstly, we must consider what happens to the components of the metric and their derivatives, given both γ_{ab} and c^{ab} are singular.

The first evolution equations come from eqs. (84), (85) and (88), (90), (92) and refer to the derivatives of the components of the metric and its inverse:

$$\frac{\partial \beta_a}{\partial z} = 0\,, \tag{203}$$

$$\frac{\partial \gamma_{ab}}{\partial z} = k_{ab}\,, \tag{204}$$

$$\frac{\partial b^a}{\partial z} = 0 \quad \text{and} \tag{205}$$

$$\frac{\partial c^{ab}}{\partial z} = -c^{ai}c^{bj}k_{ij}\,. \tag{206}$$

Next, we need evolution equations (and constraints) for the components of $\boldsymbol{k}$. Using the Einstein or tensor does not help much, because there appears to be no way to express the derivative of k_{ab} separately. Namely, the only equation that involves derivatives of k_{ab} comes from $G_{zz} = 0$ and is given as

$$2c^{ij}\frac{\partial k_{ij}}{\partial z} = c^{ik}c^{jl}k_{ij}k_{kl} + c^{ik}c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl}\,; \tag{207}$$

Taking $G_{za} = 0$, we obtain

$$
\begin{aligned}
3\phi_j c^{ij} k_{ai} &+ \frac{7}{2}\phi_k c^{ij} b^k \beta_a k_{ij} + 4\phi_i\phi_j c^{ij}\beta_a \\
&+ 2c^{ik} b^l \beta_m k_{ia} C^m{}_{kl} + \frac{3}{2} c^{ik} b^l \beta_a k_{im} C^m{}_{kl} - \frac{5}{2} c^{ij} b^n \beta_a k_{ij} C^m{}_{mn} \\
&+ c^{in} k_{im} C^m{}_{an} - c^{ij} k_{ij} C^m{}_{ma} \\
&+ \frac{5}{2}\phi_i c^{ik} b^l \beta_a \beta_m C^m{}_{kl} - 3\phi_i c^{in}\beta_m C^m{}_{an} - 2\phi_i c^{in}\beta_a C^m{}_{mn} \\
&+ 2c^{jk} b^l \beta_m \beta_n C^m{}_{aj} C^n{}_{kl} - \frac{3}{2} c^{jk} b^l \beta_a \beta_m C^m{}_{nj} C^n{}_{kl} \\
&- \frac{1}{2} c^{ik} c^{jl} \beta_a \gamma_{mn} C^m{}_{ij} C^n{}_{kl} \\
&- 7c^{kl}\beta_m C^m{}_{nk} C^n{}_{al} - \frac{7}{2} c^{kl}\beta_a C^m{}_{nk} C^n{}_{ml} - \frac{5}{2} c^{kl}\beta_a C^m{}_{mk} C^n{}_{nl} = 0\,,
\end{aligned}
\tag{208}
$$

and taking $G_{ab} = 0$, we obtain

$$
\begin{aligned}
&- \frac{11}{2}\phi_k b^k c^{ij}\gamma_{ab} k_{ij} - 3\phi_i b^i k_{ab} \\
&\quad - 2b^n \big(k_{am} C^m{}_{bn} + k_{bm} C^m{}_{an} + k_{ab} C^m{}_{mn}\big) \\
&\quad + 2c^{ij} b^n k_{ij} \big(\gamma_{am} C^m{}_{bn} + \gamma_{bm} C^m{}_{an} + \gamma_{ab} C^m{}_{mn}\big) \\
&\quad + \frac{1}{2}\gamma_{ab}\big(3c^{ik} b^l k_{im} C^m{}_{kl} + c^{ij} b^n k_{ij} C^m{}_{mn}\big) \\
&\quad + \phi_i\phi_j b^i b^j \beta_a\beta_b + 2\phi_i b^i \big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi_b \\
&\quad + 2\phi_i c^{ik} b^l \big(\beta_a\gamma_{am} + \beta_b\gamma_{am}\big) C^m{}_{kl} + \frac{5}{2}\phi_i c^{ik} b^l \gamma_{ab}\beta_m C^m{}_{kl} \\
&\quad - 2\phi_i c^{in}\big(\gamma_{am} C^m{}_{bn} + \gamma_{bm} C^m{}_{an} + \gamma_{ab} C^m{}_{mn}\big) \\
&\quad + c^{ik} c^{jl}\gamma_{am}\gamma_{bn} C^m{}_{ij} C^n{}_{kl} + \frac{1}{2} c^{ik} c^{jl}\gamma_{ab}\gamma_{mn} C^m{}_{ij} C^n{}_{kl} \\
&\quad - 2c^{jk} b^l \beta_n \big(\gamma_{am} C^n{}_{bj} + \gamma_{bm} C^n{}_{aj}\big) C^m{}_{kl} + \frac{3}{2} c^{jk} b^l \gamma_{ab}\beta_m C^m{}_{nj} C^n{}_{kl} \\
&\quad - 2c^{kl}\big(\gamma_{am} C^n{}_{bk} + \gamma_{bm} C^n{}_{ak}\big) C^m{}_{nl} - 2c^{kl}\big(\gamma_{am} C^m{}_{bk} + \gamma_{bm} C^m{}_{ak}\big) C^n{}_{nl} \\
&\quad - 2c^{kl}\gamma_{mn} C^m{}_{ak} C^n{}_{bl} + \frac{7}{2} c^{kl}\gamma_{ab} C^m{}_{nk} C^n{}_{ml} + \frac{5}{2} c^{kl}\gamma_{ab} C^m{}_{mk} C^n{}_{nl} \\
&\quad - b^l \beta_m C^m{}_{an} C^n{}_{bl} + b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{na}\big) - 3C^m{}_{an} C^n{}_{bm} - C^m{}_{ma} C^n{}_{nb}\,.
\end{aligned}
\tag{209}
$$

It is noteworthy that the latter two are constraints, while the former is a contracted evolution equation for k_{ab}, which cannot be separated from c^{ab}. It is also noteworthy that using the Ricci tensor, we will not yield very different results.[3] Therefore, finding an evolution equation for $\boldsymbol{k}$ is not possible within this context.

[3] In fact, $R_{zz} = 0$ will result to eq. (207), while $R_{za} = 0$ and $R_{ab} = 0$ will result to eqs. (208) and (209) if the constraint eq. (199) is used.

What we can observe though is that, defining $\kappa = c^{ij} k_{ij}$, then

$$\frac{\partial \kappa}{\partial z} = c^{ij} \frac{\partial k_{ij}}{\partial z} - c^{ik} c^{jl} k_{ij} k_{kl} \,,$$

and, using eq. (207),

$$\frac{\partial \kappa}{\partial z} = c^{ik} c^{jl} k_{ij} k_{kl} + 2 c^{ik} c^{jl} \beta_m \beta_n C^m{}_{ij} C^n{}_{kl} \,.$$

Now, defining $K^a{}_b = c^{ai} k_{ib}$, the latter become

$$\frac{\partial \kappa}{\partial z} = K^i{}_j K^j{}_i + 2 c^{ik} c^{jl} \beta_m \beta_n C^m{}_{ij} C^n{}_{kl} \,, \tag{210}$$

which does not contain k_{ab} whatsoever. Moreover, if the evolution of the new variable $K^a{}_b$ can be expressed explicitly, then we have an alternative set of evolution equations that can be used instead of those for k_{ab}. Notably, knowing $K^a{}_b$, it is easy to determine k_{ab} as

$$\gamma_{ai} K^i{}_b = k_{ab} \,. \tag{211}$$

Differentiating $K^a{}_b = c^{ai} k_{ib}$ with respect to z, we have

$$\frac{\partial K^a{}_b}{\partial z} = c^{ai} \frac{\partial k_{ib}}{\partial z} - K^a{}_i K^i{}_b \,.$$

The difficult part here is identifying the term $c^{ai} \frac{\partial k_{ib}}{\partial z}$ and replacing it by something "meaningful". At first, this term seems odd as neither the components of the Ricci tensor nor those of the Einstein tensor contain it; however, looking at the components of the Riemann-Christoffel tensor, we can easily see that $R^a{}_{zzb}$ contains this term. The problem now is to express the Riemann-Cristoffel tensor as something meaningful - something that is given in relation to the variables we have (β_a, γ_{ab}, b^a, c^{ab} and k_{ab} - or κ and $K^a{}_b$ instead of k_{ab}) and the stress-energy-momentum tensor. This is possible if we recall eq. (43), whereas the Riemann-Christoffel tensor can by definition be written as

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} = & g^{\alpha\mu} W_{\mu\beta\gamma\delta} + \frac{1}{2} \big(\delta^\alpha{}_\gamma R_{\beta\delta} + g^{\alpha\mu} g_{\beta\delta} R_{\mu\gamma} - g^{\alpha\mu} g_{\beta\gamma} R_{\mu\delta} - \delta^\alpha{}_\delta R_{\beta\gamma} \big) \\ & - \frac{1}{6} \big(\delta^\alpha{}_\gamma g_{\beta\delta} - \delta^\alpha{}_\delta g_{\beta\gamma} \big) R \,; \end{aligned} \tag{212}$$

and if we recall from eq. (59), that the Ricci tensor and the Ricci scalar can be expressed directly with respect to the stress-energy-momentum tensor as

$$\begin{aligned} R_{\alpha\beta} &= 8\pi \Big(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \Big) \\ R &= -8\pi T \end{aligned}$$

Therefore, the Riemann-Christoffel tensor is

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} = & g^{\alpha\mu} W_{\mu\beta\gamma\delta} + 4\pi \big(\delta^\alpha{}_\gamma T_{\beta\delta} - \delta^\alpha{}_\delta T_{\beta\gamma} + g^{\alpha\mu} g_{\beta\delta} T_{\mu\gamma} - g^{\alpha\mu} g_{\beta\gamma} T_{\mu\delta} \big) \\ & - \frac{20\pi}{3} \big(\delta^\alpha{}_\gamma g_{\beta\delta} - \delta^\alpha{}_\beta g_{\beta\gamma} \big) T \,. \end{aligned} \tag{213}$$

Specifying this for $\alpha = a$, $\beta = \gamma = z$ and $\delta = b$, we have

$$R^a{}_{zzb} = c^{ai} W_{izzb} - 4\pi\big((\delta^a{}_b - b^a \beta_b)\, T_{zz} + c^{ai}\beta_b T_{zi}\big)\,. \tag{214}$$

Of course, in the case of vacuum $T_{zz} = T_{zi} = 0$; and in the case of a flat space-time (where both the Ricci and the Weyl curvature vanish), the whole expression reduces to $R^a{}_{zzb} = 0$. But, in the more generic case, eq. (214) along with the general form of the specific component in Chapter 3, give us the way to express the evolution of $K^a{}_b$ with respect to our state variables.

The general form of this evolution equation, then, become

$$\begin{aligned}\frac{\partial K^a{}_b}{\partial z} =& K^a{}_i K^i{}_b + 2\phi_j c^{ai}\beta_b K_i{}^j + 4c^{ak}\beta_m K^l{}_a C^m{}_{kl} \\ & - 2\phi_i c^{ak} c^{il}\beta_b\beta_m C^m{}_{kl} + 2c^{ak}c^{il}\beta_m\beta_n C^m{}_{bi} C^n{}_{kl} \\ & + c^{ai}W_{izzb} - 4\pi\big((\delta^a{}_b - b^a\beta_b)\, T_{zz} + c^{ai}\beta_b T_{zi}\big)\,.\end{aligned} \tag{215}$$

This equation serves as the evolution equation for our new variable $K^a{}_b$; similarly, eq. (471) serves as the evolution equation for its trace, κ, in the general case. Specifying this for vacuum($T_{zz} = T_{za} = 0$), gives us the following equation

$$\begin{aligned}\frac{\partial K^a{}_b}{\partial z} =& K^a{}_i K^i{}_b + 2\phi_j c^{ai}\beta_b K_i{}^j + 4c^{ak}\beta_m K^l{}_a C^m{}_{kl} \\ & - 2\phi_i c^{ak} c^{il}\beta_b\beta_m C^m{}_{kl} + 2c^{ak}c^{il}\beta_m\beta_n C^m{}_{bi} C^n{}_{kl} \\ & + c^{ai}W_{izzb}\,.\end{aligned} \tag{216}$$

Using κ and $K^a{}_b$ to replace k_{ab} in eqs. (208) and (209), we can also determine the two constraints that complete the system. The first one is

(217)

$$\begin{aligned}& 3\phi_i K^i{}_a + \frac{7}{2}\phi_i b^i \beta_a \kappa \\ & \quad - 2b^k\beta_m K^l{}_a C^m{}_{kl} - \frac{3}{2} b^k \beta_a K^l{}_m C^m{}_{kl} - \frac{5}{2} b^n \beta_a \kappa C^m{}_{mn} + K^n{}_m C^m{}_{an} - \kappa C^m{}_{ma} \\ & \quad + 4\phi_i\phi_j c^{ij}\beta_a + \frac{5}{2}\phi_i c^{ik} b^l \beta_a\beta_m C^m{}_{kl} - 3\phi_i c^{in}\beta_m C^m{}_{an} - 2\phi_i c^{in}\beta_a C^m{}_{mn} \\ & \quad + 2c^{jk}b^l\beta_m\beta_n C^m{}_{aj} C^n{}_{kl} - \frac{3}{2} c^{jk} b^l \beta_a\beta_m C^m{}_{nj} C^n{}_{kl} - \frac{1}{2} c^{ik} c^{jl}\beta_a \gamma_{mn} C^m{}_{ij} C^n{}_{kl} \\ & \quad - 7c^{kl}\beta_m C^m{}_{nk} C^n{}_{al} - \frac{7}{2} c^{kl}\beta_a C^m{}_{nk} C^n{}_{ml} - \frac{5}{2} c^{kl}\beta_a C^m{}_{mk} C^n{}_{nl} = 0\,,\end{aligned}$$

and the second one

(218)

$$
\begin{aligned}
&-\frac{11}{2}\phi_i b^i \gamma_{ab}\kappa - 3\phi_i b^i \gamma_{ai} K^i{}_b \\
&\quad - 2b^n K^i{}_m \big(\gamma_{ai} C^m{}_{bn} + \gamma_{bi} C^m{}_{an}\big) - 2b^n \gamma_{ai} K^i{}_b C^m{}_{mn}\big) \\
&\quad + 2b^n \kappa \big(\gamma_{am} C^m{}_{bn} + \gamma_{bm} C^m{}_{an} + \gamma_{ab} C^m{}_{mn}\big) \\
&\quad + \frac{1}{2}\gamma_{ab}\big(- 3b^k K^l{}_m C^m{}_{kl} + b^n \kappa C^m{}_{mn}\big) \\
&\quad + \phi_i \phi_j b^i b^j \beta_a \beta_b + 2\phi_i b^i \big(\phi_a \beta_b + \phi_b \beta_a\big) + 3\phi_a \phi_b \\
&\quad + 2\phi_i c^{ik} b^l \big(\beta_a \gamma_{am} + \beta_b \gamma_{am}\big) C^m{}_{kl} + \frac{5}{2}\phi_i c^{ik} b^l \gamma_{ab} \beta_m C^m{}_{kl} \\
&\quad - 2\phi_i c^{in} \big(\gamma_{am} C^m{}_{bn} + \gamma_{bm} C^m{}_{an} + \gamma_{ab} C^m{}_{mn}\big) \\
&\quad + c^{ik} c^{jl} \gamma_{am} \gamma_{bn} C^m{}_{ij} C^n{}_{kl} + \frac{1}{2} c^{ik} c^{jl} \gamma_{ab} \gamma_{mn} C^m{}_{ij} C^n{}_{kl} \\
&\quad - 2c^{jk} b^l \beta_n \big(\gamma_{am} C^n{}_{bj} + \gamma_{bm} C^n{}_{aj}\big) C^m{}_{kl} + \frac{3}{2} c^{jk} b^l \gamma_{ab} \beta_m C^m{}_{nj} C^n{}_{kl} \\
&\quad - 2c^{kl} \big(\gamma_{am} C^n{}_{bk} + \gamma_{bm} C^n{}_{ak}\big) C^m{}_{nl} - 2c^{kl} \big(\gamma_{am} C^m{}_{bk} + \gamma_{bm} C^m{}_{ak}\big) C^n{}_{nl} \\
&\quad - 2c^{kl} \gamma_{mn} C^m{}_{ak} C^n{}_{bl} + \frac{7}{2} c^{kl} \gamma_{ab} C^m{}_{nk} C^n{}_{ml} + \frac{5}{2} c^{kl} \gamma_{ab} C^m{}_{mk} C^n{}_{nl} \\
&\quad - b^l \beta_m C^m{}_{an} C^n{}_{bl} + b^l \big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{na}\big) - 3C^m{}_{an} C^n{}_{bm} - C^m{}_{ma} C^n{}_{nb} \,.
\end{aligned}
$$

The system of eqs. (467), (468), (469), (470), (471) and (472) is the system of evolution equations for the metric components and their derivatives, with eqs. (473) and (474) serving as constraints, in the case the group acts by null orbits, resulting to both the induced 3−metric γ_{ab} and its inverse c^{ab} being degenerate. It must be noted, however, that this is an incomplete system, in the sence that the term $c^{ai}W_{izzb}$ and its dynamics have not been specified; this appears as a 'source term' in the evolution equation for $K^a{}_b$, which conveys the effects of 'tidal forces' or 'long-range curvature' to the evolution of the 'second fundamental form'.[4] There are two ways of dealing with this term:

(1) If we restrict our focus to space-times that are conformally symmetric, then this term will vanish identically. Interestingly, this course immediately prevents us from studying any case where the lightlike homogeneous manifolds are somehow justified by the physical manifestations (*e.g.*, the case of solutions for homogeneous gravitational waves).
(2) If we do not, then the evolution equations and constraints for the Weyl tensor must also be included in the system. To specify these

[4]The presence of this term in eq. (215) is actually very interesting. One of the fundamental examples for the case where the group acts by means of null orbits is that of homogeneous gravitational waves; in this example, even if the waves propagate in vacuum, tidal forces are present. Therefore, the fact that the Weyl tensor affects the evolution of the derivatives of the metric appears to be entirely logical.

equations, we can use the second (differential) Bianchi identities

$$\nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0\,,$$

which can be rewritten as

$$\begin{aligned}\nabla_\alpha W_{\beta\gamma\delta\epsilon} &+ \nabla_\beta W_{\gamma\alpha\delta\epsilon} + \nabla_\gamma W_{\alpha\beta\delta\epsilon} = \\ &+ \frac{1}{2}\big(g_{\alpha\delta}\left(\nabla_\beta R_{\gamma\epsilon} - \nabla_\gamma R_{\beta\epsilon}\right) - g_{\alpha\epsilon}\left(\nabla_\beta R_{\gamma\delta} - \nabla_\gamma R_{\beta\delta}\right) \\ &\quad - g_{\beta\delta}\left(\nabla_\alpha R_{\gamma\epsilon} - \nabla_\gamma R_{\alpha\epsilon}\right) + g_{\beta\epsilon}\left(\nabla_\alpha R_{\gamma\delta} - \nabla_\gamma R_{\alpha\delta}\right) \\ &\quad + g_{\gamma\delta}\left(\nabla_\alpha R_{\beta\epsilon} - \nabla_\beta R_{\alpha\epsilon}\right) - g_{\gamma\epsilon}\left(\nabla_\alpha R_{\beta\delta} - \nabla_\beta R_{\alpha\epsilon}\right)\big) \\ &+ \frac{1}{6}\big((g_{\alpha\delta}g_{\beta\epsilon} - g_{\alpha\epsilon}g_{\beta\delta})\nabla_\gamma R - (g_{\alpha\delta}g_{\gamma\epsilon} - g_{\alpha\epsilon}g_{\gamma\delta})\nabla_\beta R \\ &\quad + (g_{\beta\delta}g_{\gamma\epsilon} - g_{\beta\epsilon}g_{\gamma\delta})\nabla_\alpha R\big)\,.\end{aligned} \tag{219}$$

This relates the covariant derivative of the Weyl tensor to the covariant derivatives of the Ricci tensor and the Ricci scalar; given the latter are already expressed with respect to the components of the metric, its inverse and its derivative, we can easily express the change of the Weyl tensor with respect to z (that would yield the evolution equation for it) and with respect to the group (that would yield the respective constraints) in relation to them, thus closing the system.[5]

Let us consider the second, more general case. First, it is important to note that any homothetic vector field in a space-time is also a curvature collineation, meaning that

$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{R}\big(\vec{u},\vec{w}\big) = 0\,; \tag{220}$$

and a Ricci collineation, meaning that

$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{R} = 0\,. \tag{221}$$

Therefore, it is also a Weyl collineation,

$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{W}\big(\vec{u},\vec{w}\big) = 0\,. \tag{222}$$

This means that the constraint equations that are derived from the Bianchi identities are simply

$$\mathcal{L}_{\vec{\xi}_f} W_{azbz} = 0\,, \tag{223}$$

$$\mathcal{L}_{\vec{\xi}_f} W_{azbc} = 0\,, \tag{224}$$

$$\mathcal{L}_{\vec{\xi}_f} W_{acbz} = 0 \quad \text{and} \tag{225}$$

[5] The alternative would be to mimic the usual 3+1 formalism by admitting some observer's velocity vector (that spans over $\vec{\zeta}$ and $\vec{\xi}_a$) and decompose the Weyl tensor to its 'electric' and 'magnetic' part, writing an evolution equation and a constraint for each of them [**43**]. However, this implies additional complexity that is not required at this point.

$$\mathcal{L}_{\vec{\xi}_f} W_{acbd} = 0\,. \tag{226}$$

As for the evolution equations, they are given as

$$\mathcal{L}_{\vec{\zeta}} W_{azzb} = \frac{1}{2}\big(\gamma_{ab}\mathcal{L}_{\vec{\zeta}} R_{zz} - \beta_a \mathcal{L}_{\vec{\zeta}} R_{zb}\big)\,, \tag{227}$$

$$\mathcal{L}_{\vec{\zeta}} W_{azbc} + \mathcal{L}_{\vec{\zeta}} W_{bzac} = \frac{1}{2}\beta_c \mathcal{L}_{\vec{\zeta}} R_{ab}\,, \tag{228}$$

$$\mathcal{L}_{\vec{\zeta}} W_{zabc} + \mathcal{L}_{\vec{\zeta}} W_{azbc} = 0 \quad \text{and} \tag{229}$$

$$\begin{aligned}\mathcal{L}_{\vec{\zeta}} W_{abcd} =& \frac{1}{2}\big(\gamma_{ac}\mathcal{L}_{\vec{\zeta}} R_{bd} - \gamma_{ad}\mathcal{L}_{\vec{\zeta}} R_{bc} + \gamma_{bc}\mathcal{L}_{\vec{\zeta}} R_{ad} - \gamma_{bd}\mathcal{L}_{\vec{\zeta}} R_{ad} \\ &+ \frac{1}{6}\big(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}\big)\mathcal{L}_{\vec{\zeta}} R\,.\end{aligned} \tag{230}$$

The Lie derivatives of the Ricci tensor components and of the Ricci scalar with respect to the transversal $\vec{\zeta}$ can be given directly from the matter terms, remembering that

$$R_{\alpha\beta} = 8\pi\Big(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\Big)\,,$$

and

$$R = -8\pi T\,.$$

Of course, of all these evolution equations and constraints, only one is of importance to us: the evolution equation for the component W_{azzb}. In the case of vacuum, the evolution equation is simply

$$\frac{\partial W_{azzb}}{\partial z} = 0\,. \tag{231}$$

This system of equations (471), (472), (473), (474) and (407) is still a system that can be treated in the same manner as the more generic case of non-null orbits that was introduced in the previous subsection.[6] So, the argument of a general local solution that is put forth in the following subsection holds equally well for both.

There is, nevertheless, a minor problem that one needs to consider

[6]The main difference of the two system is the fact that, in the case of non-null orbits, one can work directly with the components of the metric and the ‘second fundamental form’ - while, in the case of null orbits, we have to rely on the auxiliary variables κ and $K^a{}_b$. Yet, it is easy to compute the solution for the actual variable (k_{ab}) once the solution for these two is known.

2.3. A fixed point argument for a general local solution. The eqs. (191), (192), (193), (194), (195) and (202) are a set of evolution equations of the form

$$\frac{\partial}{\partial z}\vec{X}(z) = \boldsymbol{F}\left(\vec{X}(z)\right) \; ; \tag{232}$$

i.e., they are an autonomous system of ordinary differential equations.[7] This system can be integrated in a certain range of values of z, say $\left[z_0, z_1\right]$, when a set of initial conditions

$$\vec{X}(z_0) = \vec{X}_0 \, , \tag{233}$$

is given, and when the right-hand side, $\boldsymbol{F}(\vec{X})$ fulfills some criteria (*e.g.*, it is Lipschitz).

As for eq. (199), it serves as a constraint of $\boldsymbol{g}$ and $\boldsymbol{k}$, in the form of

$$\boldsymbol{G}_2\left(\vec{X}\right) = 0 \, . \tag{234}$$

This constraint must be fulfilled for the initial conditions, before the integration is attempted.

The existence and uniqueness of solutions of eq. (232) under generic initial conditions of eq. 233) that satisfy the constraint of eq. (234) can be proved by means of the Picard-Lindelöf theorem, so long as the solution is local, *i.e.* confined within an interval $z \in \left[z_1, z_2\right]$. The theorem is states as

Theorem 2.1 (Picard-Lindelöf (existence and uniqueness) theorem). *Let Ω be a n open subset of $\mathbb{R}^n$ and $\boldsymbol{F}$ be a continuous function from Ω to $\mathbb{R}^n$; and let $\vec{X}$ be a function defined on $\mathbb{R}^n$ that satisfied the Initial Value Problem (232-234) for $z \in \left[z_1, z_2\right]$, such that $z_0 \in \left[z_1, z_2\right]$. Then,there exists at least one solution to the Initial Value Problem in some closed interval $\left[z_0 - h, z_0 + h\right]$ where $h > 0$ and $z_0 - h \geq z_1$ and $z_0 + h \leq z_2$.*
Moreover, if $\boldsymbol{F}$ is continuously differentiable, the exists a unique solution, $\bar{X}$, to the Initial Value Problem in the closed interval $\left[x_0 - h, x_0 + h\right]$.
Finally, the Picard iteration, defined as

$$\vec{X}_{(n+1)}(z) = \vec{X}_0 + \int_{z_0}^{z} \boldsymbol{F}\left(\vec{X}_{(n)}(s)\right) \mathrm{d}s \, , \tag{235}$$

produces a sequence of functions $\left\{\vec{X}_{(n)}\right\}$ that converges to $\bar{X}$ uniformly within the interval $\left[z_1, z_2\right]$.

The conditions required to prove the theorem are

(1) Some metric space $\left(\mathcal{D}, d\right)$, where all functions $\vec{X}\,, \vec{Y}$ belong; the metric of the space can be defined by the uniform norm

$$d\left(\vec{X}, \vec{Y}\right) = \left\|\vec{X}(z) - \vec{Y}(z)\right\|_\infty = \sup_{z \in \left[z_1, z_2\right]} |\vec{X}(z) - \vec{Y}(z)| \, . \tag{236}$$

[7] Here, $\vec{X} = \left\{\mathrm{a}, b^a, c^{ab}, \gamma_{ab}, k_{ab}\right\}$ and $\boldsymbol{F}$ is the right-hand side of the afore-mentioned equations.

(2) The Lipschitz continuity of $\boldsymbol{F}$.

(3) The Lipschitz condition for the operator $\mathbb{H} = \vec{X}_0 + \int_{z_0}^{z} \boldsymbol{F}\left(\vec{X}(s)\right) \mathrm{d}s$, so that

$$\left\|\mathbb{H}\vec{X}(z) - \mathbb{H}\vec{Y}(z)\right\|_\infty \leq q\left\|\vec{X}(z) - \vec{Y}(z)\right\|_\infty , \tag{237}$$

for a real parameter $q \in [0,1]$ and all functions $\vec{X}, \vec{Y}$ in the metric space $\left(\mathcal{D}, d\right)$.

(4) The Banach fixed-point theorem, applied to prove the existence of a solution, by proving that the application of the operator $\mathbb{H}$ is a contraction

$$\left\|\mathbb{H}\vec{X}_{(n+1)}(z) - \mathbb{H}\vec{X}_{(n)}(z)\right\|_\infty \leq qp_1 \left\|\vec{X}_{(n+1)}(z) - \vec{X}_{(n)}(z)\right\|_\infty \tag{238}$$

for $p_1 \in \mathbb{R}$ an integration constant; so, as $n \to \infty$, the solutions converge to $\bar{X}$.

(5) The Grönwall lemman, applied to prove the uniqueness of the solution, by proving that any two function $\vec{X}(z)$ and $\vec{Y}(z)$ that have been derived from the Picard iteration have to be equal.

$$\left\|\vec{X}(z) - \vec{Y}(z)\right\|_\infty = \left\|\int_{z_0}^{z} \left(\boldsymbol{F}\left(\vec{X}_{(n)}(s) - \vec{Y}_{(n)}(s)\right)\right)\mathrm{d}s\right\|_\infty \leq qp_2 \left\|\vec{X} - \vec{Y}\right\|_\infty \tag{239}$$

for $p_2 \in \mathbb{R}$ an integration constant; so, as $n \to \infty$, $\vec{X} = \vec{Y}$.

We shall omit a more detailed proof, as the theorem is relatively well-known.

Given that the conditions of the Picard-Lindelöf theorem are generally met int he case of eqs. (232), we can immediately claim that the Einstein system in vacuum is integrable within an interval of z, such that constraint is always satisfied. The question of extended the interval $\left[z_1, z_2\right]$ so that the entire manifold $\mathcal{V}_4$ is covered by the resulting metric $\boldsymbol{g}$, may remain unanswered for the moment; the reasons are two:

- The Picard-Lindelöf theorem proves local and not global existence; to prove global existence, it is likely that additional assumptions on the Initial Value Problem must be imposed [**41, 42**]. However, this global existence refers to the particular choice of z, so the space-time may not be covered; therefore, in this thesis, we are concerned with local existence.
- As we will attempt to show by means of examples, it is not always possible to find a quotient $\vec{\zeta}$ such that the entire space-time $\left(\mathcal{V}_4, \boldsymbol{g}\right)$ can be foliated accordingly; so, global existence may not exist for all possible space-times.

3. Examples

This section provides some examples of vacuum space-times that are solutions to the Einstein system.

3.1. Bianchi I acting on the Minkowski space-time. Let the Minkoski space-time, describing a vacuum solution of General Relativity with maximal symmetries. Its metric is given as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \,. \tag{240}$$

Let a group $\mathcal{G}$ acting on this space-time, whose algebra yields the following generators:

(1) Boost along the x direction:

$$\vec{\xi}_1 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial x_3} \,. \tag{241}$$

(2) Rotation on the $x_1 - x_2$ plane:

$$\vec{\xi}_2 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} \,. \tag{242}$$

(3) Dilation:

$$\vec{\xi}_3 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} \,. \tag{243}$$

The commutators of the group are

$$[\vec{\xi}_1, \vec{\xi}_2] = 0 \;\;, \quad [\vec{\xi}_2, \vec{\xi}_3] = 0 \quad \text{and} \quad [\vec{\xi}_3, \vec{\xi}_1] = 0 \,, \tag{244}$$

which deems $\mathcal{G}$ a Bianchi I group. Furthermore, it is relatively easy to see that the action of the group on the Minkowski space-time is as follows

$$\begin{aligned} &\mathcal{L}_{\vec{\xi}_1}\boldsymbol{g} = 0 \;\;, \\ &\mathcal{L}_{\vec{\xi}_2}\boldsymbol{g} = 0 \quad \text{and} \\ &\mathcal{L}_{\vec{\xi}_3}\boldsymbol{g} = 2\boldsymbol{g} \,, \end{aligned} \tag{245}$$

hence the boost and the rotation can be classified as isometries of the Minkowski space-time, while dilation is a homothety with $\phi_3 = 2$.

In order to rewrite the metric according to the group acting on it, we wish to obtain a set of coordinates, three of which are the canonical coordinates of the group, while the fourth refers to the quotient of its action. Given the Ricci-flatness of the Minkowski space-time, the group action can be approximated by the exponential map of the corresponding algebra defined as

$$e^A = e^{q_1 \mathrm{D}\vec{\xi}_1 + q_2 \mathrm{D}\vec{\xi}_2 + q_3 \mathrm{D}\vec{\xi}_3} \,, \tag{246}$$

where $\mathrm{D}\vec{\xi}_i$ the Jacobian matrix of each generator and q_i a respective parameter; these parameters will help us define the canonical coordinates. Then, to obtain a transversal to the group, we choose an arbitrary point on the space-time, *e.g.*

$$P = (1, 1, 0, 0) \,, \tag{247}$$

and a null direction for the Minkowski space-time that will serve as the initial transversal geodesic; then, applying the exponential map on this geodesic, we shall obtain the general form of the transversal geodesic.

Concerning the exponential map, it is easily constructed as

$$A = q_1 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + q_2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + q_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \tag{248}$$
$$= \begin{pmatrix} q_3 & 0 & 0 & q_1 \\ 0 & q_3 & q_2 & 0 \\ 0 & -q_2 & q_3 & 0 \\ q_1 & 0 & 0 & q_3 \end{pmatrix},$$

hence

$$e^A = \begin{pmatrix} e^{q_3}\cosh q_1 & 0 & 0 & e^{q_3}\sinh q_1 \\ 0 & e^{q_3}\cos q_2 & e^{q_3}\sin q_2 & 0 \\ 0 & -e^{q_3}\sin q_2 & e^{q_3}\cos q_2 & 0 \\ e^{q_3}\sinh q_1 & 0 & 0 & e^{q_3}\cosh q_1 \end{pmatrix}. \tag{249}$$

Now, having chosen the initial point of the initial transversal geodesic as $P_0(1,1,0,0)$ and its initial "velocity" as $\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2}$, the initial transversal geodesic is simply the line described parametrically as

$$\epsilon : \{ \quad t = 1 + z \ , \quad x_1 = 1 \ , \quad x_2 = z \quad \text{and} \quad x_3 = 0 \, , \tag{250}$$

where z a parameter increasing along the $\vec{e}_2$ direction. This parameter will function as the transversal coordinate. For the canonical coordinates of the group, let them be w_1, w_2 and w_3, we may choose the algebraic expressions related to the parameters q_1, q_2 and q_3,

$$\begin{gathered} \frac{1+w_1^2}{1-w_1^2} = \cosh q_1 \quad \text{and} \quad \frac{2w_1}{1-w_1^2} = \sinh q_1 \\ \frac{1-w_2^2}{1+w_2^2} = \cos q_2 \quad \text{and} \quad \frac{2w_2}{1+w_2^2} = \sin q_2 \\ w_3 = e^{q_3} \, . \end{gathered} \tag{251}$$

Now, the exponential map is written as

$$e^A = \begin{pmatrix} w_3\frac{1+w_1^2}{1-w_1^2} & 0 & 0 & w_3\frac{2w_1}{1-w_1^2} \\ 0 & w_3\frac{1-w_2^2}{1+w_2^2} & -w_3\frac{2w_2}{1+w_2^2} & 0 \\ 0 & w_3\frac{2w_2}{1+w_2^2} & w_3\frac{1-w_2^2}{1+w_2^2} & 0 \\ w_3\frac{2w_1}{1-u^2} & 0 & 0 & w_3\frac{1+w_1^2}{1-w_1^2} \end{pmatrix}. \tag{252}$$

The action of the exponential map on the straight line ϵ yields the general form of the transversal line with respect to the new set of coordinates

(253)
$$\epsilon : \begin{cases} t = \dfrac{w_3(1+w_1^2)(1+z)}{1-w_1^2}\,, & x_1 = \dfrac{w_3\big((1-w_2^2)-2zw_2\big)}{1+w_2^2}\,, \\ x_2 = \dfrac{w_3\big(2w_2+(1-w_2^2)z\big)}{1+w_2^2} \quad \text{and} & x_3 = \dfrac{2w_3w_1(1+z)}{1-w_1^2}\,. \end{cases}$$

These function as the transformation rule from the Cartesian coordinates $\{t, x_1, x_2, x_3\}$ to the group-specific coordinates $\{z, w_1, w_2, w_3\}$. What we need to do now is to specify the inverse transformation, from the group-specific coordinates to the Cartesian coordinates. To do so, we assume two points on the space-time, $P_1(\tau, \chi_1, \chi_2, \chi_3)$ and $P_2(t, x_1, x_2, x_3)$, that are related by the group action. In this, we distinguish between the quotient of the group action (that corresponds to the choice of z) and the group action itself; furthermore, we distinguish the group action to its two isometries (boost and rotation, related to w_1 and w_2 respectively) and the homothety (dilation, related to w_3).

(1) The point P_1 is the initial point of the action, so its coordinates can be described with respect to z as

(254)
$$\tau = 1+z\,, \quad \chi_1 = 1\,, \quad \chi_2 = z \quad \text{and} \quad \chi_3 = 0\,.$$

(2) Distinguishing between the group action and its quotient, we can identify the coordinate on the latter, z, through the invariant of the group,
$$\frac{-t^2+x_3^2}{x_1^2+x_2^2}\,.$$
Moving along z, the above quantity remains constant. Hence,

(255)
$$\frac{-t^2+x_3^2}{x_1^2+x_2^2} = \frac{-\tau^2+\chi_3^2}{\chi_1^2+\chi_2^2}\,.$$

Thus, we have moved from point P_1 to point P_2; now we take the way back through the group action.

(3) Distinguishing the group action to the isometries and the homothety, we can isolate the effects of each symmetry and the respective invariants. This is expressed as if the move "back" from P_2 to P_1 is decomposed by a motion from P_2 to $P_2'(t', x_1', x_2', x_3')$ by means of the homothety (dilation), and then a motion from P_2' to P_2 by means of the isometries (boost and rotation). This means that the coordinates of the two points are related as

(256)
$$t' = \frac{t}{w_3}\,, \quad x_1' = \frac{x_1}{w_3}\,, \quad x_2' = \frac{x_2}{w_3} \quad \text{and} \quad x_3' = \frac{x_3}{w_3}\,.$$

(4) Identifying w_3 with the dilation $(\vec{\xi}_3)$, this remains constant under the isometries, hence

$$\begin{aligned} &-\left(t'\right)^2+\left(x_3'\right)^2=-\left(\frac{t}{w_3}\right)^2+\left(\frac{x_3}{w_3}\right)^2=-\tau^2+\chi_3^2 \quad \text{and} \\ &\left(x_1'\right)^2+\left(x_2'\right)^2=\left(\frac{x_1}{w_3}\right)^2+\left(\frac{x_2}{w_3}\right)^2=\chi_1^2+\chi_2^2 \,. \end{aligned} \tag{257}$$

(5) Following, the rotation symmetry, identified with w_2, results by moving along the an angle, θ in a circle on the $x_1 - x_2$ plane, in such a manner that

$$\begin{aligned} \cos\theta &= \frac{(x_1',x_2')\cdot(\chi_1,\chi_2)}{|(x_1',x_2')||(\chi_1,\chi_2)|}=\frac{x_1'\chi_1+x_2'\chi_2}{(x_1')^2+(x_2')^2}=w_3\frac{x_1\chi_1+x_2\chi_2}{x_1^2+x_2^2} \quad \text{and} \\ \sin\theta &= \frac{\big((x_1',x_2')\times(\chi_1,\chi_2)\big)\dot{\vec{n}}}{|(x_1',x_2')||(\chi_1,\chi_2)|}=\frac{-x_1'\chi_2+x_2'\chi_1}{(x_1')^2+(x_2')^2}=w_3\frac{-x_1\chi_2+x_2\chi_1}{x_1^2+x_2^2}\,, \end{aligned} \tag{258}$$

where $\cdot$, $\times$ and $|\;|$ are the Euclidean inner product, cross product and norm respectively; and $\vec{n}$ any unit vector perpendicular to the $x_1 - x_2$ plane. Following, we may take

$$w_2=\tan\left(\frac{\theta}{2}\right)=\frac{\sin\theta}{1+\cos\theta}\,. \tag{259}$$

(6) Following, the boost symmetry, identified with w_1, results by moving along the an angle, ϕ, along a hyperbola on the $t - x_3$ plane, in such a manner that

$$\begin{aligned} \cosh\phi &= \frac{(t',x_3')\odot(\tau,\chi_3)}{\wr(t',x_3')\wr\wr(\tau,\chi_3)\wr}=\frac{-t'\tau+x_3'\chi_3}{-\left(t'\right)^2+\left(x_3'\right)^2}=w_3\frac{-t\tau+x_3\chi_3}{-t^2+\chi_3^2} \quad \text{and} \\ \sinh\phi &= \frac{\big((t',x_3')\otimes(\tau,\chi_3)\big)\odot\vec{m}}{\wr(t',x_3')\wr\wr(\tau,\chi_3)\wr}=\frac{t'\chi_3-x_3'\tau}{-\left(t'\right)^2+\left(x_3'\right)^2}=w_3\frac{t\chi_3-x_3\tau}{-t^2+x_3^2} \end{aligned} \tag{260}$$

where $\odot$, $\otimes$ and $\wr\;\wr$ are the Minkowskian inner product, cross product and norm respectively (that is, the inner product, cross product and norm in hyperbolic geometry);[8] and $\vec{m}$ any unit vector perpendicular to the $t - x_3$ plane. Following, we may take

$$w_1=\tanh\left(\frac{\phi}{2}\right)=\frac{\sinh\phi}{1+\cosh\phi}\,. \tag{261}$$

[8]The Minkowskian (or hyperbolic) operations are defined in accordance to their Euclidean ones, as

$$\begin{aligned} \vec{a}\odot\vec{b} &= -a_0b_0+a_3b_3 \\ \vec{a}\otimes\vec{b} &= (a_0b_3-a_3b0)\,\vec{m} \\ \wr\,\vec{a}\wr &= \vec{a}\odot\vec{a}=-(a_0)^2+(a_3)^2 \end{aligned}$$

for any two vectors of the form $\vec{a}=a_0\dfrac{\partial}{\partial t}+a_3\dfrac{\partial}{\partial x_3}$ and $\vec{b}=b_0\dfrac{\partial}{\partial t}+b_3\dfrac{\partial}{\partial x_3}$ with $a_\kappa\,,b_\kappa\in\mathbb{R}$; and for $\vec{m}$ a unit vector perpendicular to the $t-x_3$ plane.

Substituting τ, χ_1, χ_2 and χ_3 to the invariant of the group, we arrive at a quadratic equation with respect to s,

$$\frac{-t^2+x_1^2+x_2^2+x_3^2}{x_1^2+x_2^2}z^2+2z+\frac{-t^2+x_1^2+x_2^2+x_3^2}{x_1^2+x_2^2}=0\,, \tag{262}$$

whose solutions are

$$z=\frac{x_1^2+x_2^2\pm\sqrt{(t^2-x_3^2)(-t^2+2x_1^2+2x_2^2+x_3^2)}}{-t^2+x_1^2+x_2^2+x_3^2}\,. \tag{263}$$

From the other two invariants (for the boosts and the rotations respectively), we obtain the following solutions for w_3,

$$w_3=\pm\frac{\sqrt{t^2-x_3^2}}{1+z}=\pm\sqrt{\frac{x_1^2+x_2^2}{1+z^2}}\,. \tag{264}$$

Finally, from the rotation, we obtain the following transformation rule

$$w_2=\tan\left(\frac{\theta}{2}\right)=\frac{-x_1z+x_2}{\sqrt{(1+z^2)(x_1^2+x_2^2)}+x_1+x_2z}\,, \tag{265}$$

and from the boost, we obtain the following transformation rule

$$w_1=\tanh\left(\frac{\phi}{2}\right)=\frac{-t(1+z^2)}{\sqrt{(1+z^2)(t^2-x_3^2)}+x_3(1+z^2)} \tag{266}$$

The generators may now take the form

$$\begin{aligned}
\vec{\xi}_1&=\frac{2(z+1)w_1w_3}{1-w_1^2}\frac{\partial}{\partial t}+\frac{(z+1)(1+w_1^2)w_3}{1-w_1^2}\frac{\partial}{\partial x_3}\,,\\
\vec{\xi}_2&=\frac{\big(2w_2+z(1-w_2^2)\big)\,w_3}{1+w_2^2}\frac{\partial}{\partial x_1}-\frac{\big((1-w_2^2)-2zw_2\big)\,w_3}{1+w_2^2}\frac{\partial}{\partial x_2}\quad\text{and}\\
\vec{\xi}_3&=\frac{(z+1)(1+w_1^2)w_3}{1-w_1^2}\frac{\partial}{\partial t}+\frac{\big((1-w_2^2)-2zw_2\big)}{1+w_2^2}\frac{\partial}{\partial x_1}\\
&\quad+\frac{\big(2w_2+z(1-w_2^2)\big)\,w_3}{1+w_2^2}\frac{\partial}{\partial x_2}+\frac{2(z+1)w_1w_3}{1-w_3^2}\frac{\partial}{\partial x_3}\,,
\end{aligned}$$

and in the new coordinate system they are simplified to

$$\begin{aligned}
\vec{\xi}_1&=\frac{2}{1-w_1^2}\frac{\partial}{\partial w_1}\,,\\
\vec{\xi}_2&=-\frac{2}{1+w_2^2}\frac{\partial}{\partial w_2}\quad\text{and}\\
\vec{\xi}_3&=w_3\frac{\partial}{\partial w_3}\,.
\end{aligned} \tag{267}$$

At the same time, the ray on the quotient is tangent to

$$\vec{\zeta}=e^A\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x_2}\Big)=\frac{\big(1+w_1^2\big)\,w_3}{1-w_1^2}\frac{\partial}{\partial t}-\frac{2w_2w_3}{1+w_2^2}\frac{\partial}{\partial x_1}+\frac{\big(1-w_2^2\big)\,w_3}{1+w_2^2}\frac{\partial}{\partial x_2}+\frac{2w_1w_3}{1-w_1^2}\frac{\partial}{\partial x_3}\,,$$

while in the new coordinate system it becomes

$$\vec{\zeta} = \frac{\partial}{\partial z}\,. \tag{268}$$

The four vectors of the basis are linearly independent given $w_3 \neq 0$, since the determinant with $\vec{\zeta}$, $\vec{\xi}_1$, $\vec{\xi}_2$ and $\vec{\xi}_3$ as column-entries is $-\dfrac{4w_3}{\left(1-w_1^2\right)\left(1+w_2^2\right)}$.

Now the metric can be expressed as the Minkowskian inner products of the four vectors,

$$\begin{aligned}
g_{zz} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\zeta}\big) = \vec{\zeta}\odot\vec{\zeta} = 0\\
g_{z1} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_1\big) = \vec{\zeta}\odot\vec{\xi}_1 = 0\\
g_{z2} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_2\big) = \vec{\zeta}\odot\vec{\xi}_2 = w_3^2\\
g_{z3} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_3\big) = \vec{\zeta}\odot\vec{\xi}_3 = -w_3^2\\
g_{11} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_1\big) = \vec{\xi}_1\odot\vec{\xi}_1 = (1+z)^2 w_3^2\\
g_{12} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_2\big) = \vec{\xi}_1\odot\vec{\xi}_2 = 0\\
g_{13} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_3\big) = \vec{\xi}_1\odot\vec{\xi}_3 = 0\\
g_{22} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_2\big) = \vec{\xi}_2\odot\vec{\xi}_2 = (1+z^2) w_3^2\\
g_{23} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_3\big) = \vec{\xi}_2\odot\vec{\xi}_3 = 0\\
g_{33} &= \boldsymbol{g}\big(\vec{\xi}_3,\vec{\xi}_3\big) = \vec{\xi}_3\odot\vec{\xi}_3 = -2zw_3^2
\end{aligned}$$

or

$$\boldsymbol{g} = \begin{pmatrix} 0 & 0 & w_3^2 & -w_3^2 \\ 0 & (z+1)^2 w_3^2 & 0 & 0 \\ w_3^2 & 0 & \left(z^2+1\right) w_3^2 & 0 \\ -w_3^2 & 0 & 0 & -2zw_3^2 \end{pmatrix}, \tag{269}$$

while the inverse is

$$g^{-1} = \begin{pmatrix} \frac{2z\left(z^2+1\right)}{(z-1)^2 w_3^2} & 0 & -\frac{2z}{(z-1)^2 w_3^2} & -\frac{z^2+1}{(z-1)^2 w_3^2} \\ 0 & \frac{1}{(z+1)^2 w_3^2} & 0 & 0 \\ -\frac{2z}{(z-1)^2 w_3^2} & 0 & \frac{1}{(z-1)^2 w_3^2} & \frac{1}{(z-1)^2 w_3^2} \\ -\frac{z^2+1}{(z-1)^2 w_3^2} & 0 & \frac{1}{(z-1)^2 w_3^2} & \frac{1}{(z-1)^2 w_3^2} \end{pmatrix}. \tag{270}$$

The Lie derivative of the metric along $\vec{\zeta}$ is

$$\mathcal{L}_{\vec{\zeta}}\boldsymbol{g} = \frac{\partial}{\partial z}\boldsymbol{g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2(z+1)w_3^2 & 0 & 0 \\ 0 & 0 & 2zw_3^2 & 0 \\ 0 & 0 & 0 & -2w_3^2 \end{pmatrix}, \tag{271}$$

and the second Lie derivative of the metric is

$$\mathcal{L}_{\vec{\zeta}}(\mathcal{L}_{\vec{\zeta}}\boldsymbol{g}) = \frac{\partial^2}{\partial z^2}\boldsymbol{g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2w_3^2 & 0 & 0 \\ 0 & 0 & 2w_3^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{272}$$

Now, the Lie derivative along $\vec{\xi}_1$ is

$$\mathcal{L}_{\vec{\xi}_1}\boldsymbol{g} = \frac{2}{1-w_1^2}\frac{\partial}{\partial w_1}\boldsymbol{g} = 0\,, \tag{273}$$

the Lie derivative along $\vec{\xi}_2$ is

$$\mathcal{L}_{\vec{\xi}_2}\boldsymbol{g} = -\frac{2}{1+w_2^2}\frac{\partial}{\partial w_2}\boldsymbol{g} = 0\,, \tag{274}$$

and the Lie derivative along $\vec{\xi}_3$ is

$$\mathcal{L}_{\vec{\xi}_3}\boldsymbol{g} = w_3\frac{\partial}{\partial w_3}\boldsymbol{g} = 2\boldsymbol{g}\,. \tag{275}$$

The present example is particularly interesting, as it reveals a problem peculiar to this class of space-times. The quotient of the group is determined by means of a quadratic equation (262); so, there is no unique quotient that respects all criteria of the Theorem **??**, but there can be multiple (up to two) quotients that fulfill these criteria and yield the same foliation. In this sense, no clear distinction between 'past' and 'future' along the range of values of z exists - the parameter can 'bifurcate' between the collineations while the foliation is unaltered. However, the more interesting issue arises in the points where this 'bifurcation' can occur. Let us consider that the evolution of the coordinates is such that $-t^2+x_1^2+x_2^2+x_3^2 \to 0$; then, $z \to 0$ regardless of the 'nearby' values. Similarly, when $-t^2+x_3^2 \to 0$, $z \to 1$. Essentially, z 'jumps' non-smoothly from whatever value it may have to 0 when $-t^2+x_1^2+x_2^2+x_3^2 \to 0$ and to 1 when $-t^2+x_3^2 \to 0$, although $t\,, x_1\,, x_2\,, x_3$ are evolving smoothly. This makes z a pathological function, and leads to the result that the transversal collineation is *non-Hausdorff*.

Fig. 1 depicts this non-Hausdorff nature of the transversal in the following way: Different values of z are depicted as different hypersurfaces in the space-time. Alternatively, each hypersurface corresponds to all the points of space-time that yield the same value for z; and, of course, it represents a 'slice' of the embedded homogeneous submanifold for the said value of z.[9] It is easy to see that the $t - x_3$ hypersurface in the first case, and the $x_1 = x_2 = 0$ light-cone in the second, do not are pathologies of z, as the coordinates approach to 0 along them; these are exactly the cases where $z \to 0$ and $z \to 1$, respectively, regardless of the 'nearby' values.

[9]The two subfigures correspond to the two different solutions of eq. (262).

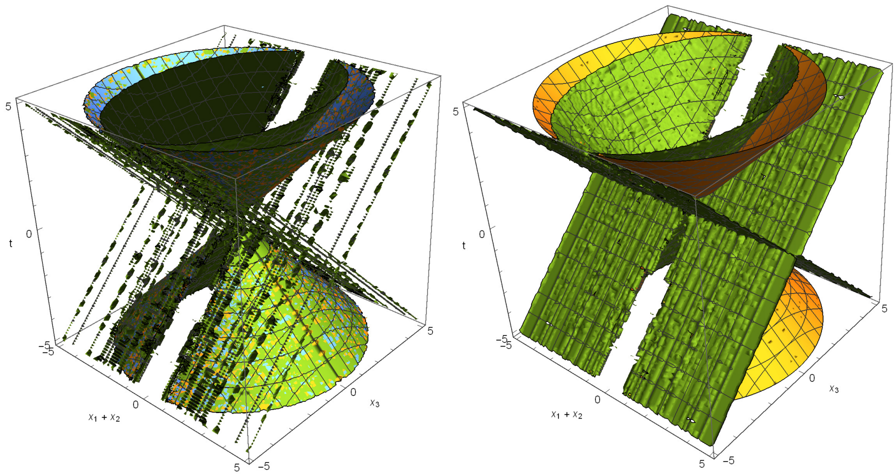

FIGURE 1. Different values of z as a function of the coordinates of space-time.

This pathology is intriguing (and paradoxical) as it is not expected to occur in a subspace of a well-known, well-studied, and relatively simple case as the Minkowski space-time is. It is not clear, at first, if this depends on the particular example (the group chosen to act on the Minkowski space-time) and, thus, it is an 'uncovered' feature of the space-time - or, if it depends on our particular formalism and, thus, it is a "coordinate pathology" rather than a "topological" one.[10] However, it is noteworthy that both pathologies occur on a light-cone, the one of which is associated with $\vec{\xi}_2$ and the other with $\vec{\xi}_3$. It is a logical conclusion, then, that these pathologies are built-in the example, playing the role of 'cosmic censorship'; z cannot behave smoothly near these regions, because it runs the risk violating some physical property (*e.g.*, allowing the system to evolve in such a way as to meet its 'past' self in the 'future').

With regards to pathologies of this sort, we will discuss more in the final section of this chapter.

3.2. Bianchi III acting on the Minkowski space-time. Let the Minkoski space-time, describing a vacuum solution of General Relativity with maximal symmetries. Its metric is given as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \,. \tag{276}$$

[10]This remark can be clearly compared to the question of a singularity. A point or moment in space-time is not characterides as a singularity when the particular coordinates lead to a 'blow-up' of the metric (a "coordinate singularity", like the event horizon in the Schwarzschild coordinates in the Schwarzschild solution) - but when invariant geometric or topological variables 'blow up' (a "topological singularity", like the 'centre' of rotation in the same example).

Let a group $\mathcal{G}$ acting on this space-time, whose algebra yields the following generators:

(1) Boost along the x_3 direction:

$$\vec{\xi}_1 = x_3\frac{\partial}{\partial t} + t\frac{\partial}{\partial x_3} \,. \tag{277}$$

(2) Loxodromy on the $t - x_3$ plane:

$$\vec{\xi}_2 = x_1\frac{\partial}{\partial t} + (t - x_3)\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3} \,. \tag{278}$$

(3) Dilation:

$$\vec{\xi}_3 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} \,. \tag{279}$$

The commutators of the group are

$$[\vec{\xi}_1, \vec{\xi}_2] = -\vec{\xi}_2 \ , \quad [\vec{\xi}_2, \vec{\xi}_3] = 0 \quad \text{and} \quad [\vec{\xi}_3, \vec{\xi}_1] = 0 \,, \tag{280}$$

which deems $\mathcal{G}$ a Bianchi III group with the non-vanishing structure constants ${C^2}_{21} = -{C^2}_{12} = 1$. Furthermore, it is relatively easy to see that the action of the group on the Minkowski space-time is as follows

$$\begin{aligned} &\mathcal{L}_{\vec{\xi}_1}\boldsymbol{g} = 0 \ , \\ &\mathcal{L}_{\vec{\xi}_2}\boldsymbol{g} = 0 \quad \text{and} \\ &\mathcal{L}_{\vec{\xi}_3}\boldsymbol{g} = 2\boldsymbol{g} \,, \end{aligned} \tag{281}$$

hence the boost and the loxodromy can be classified as isometries of the Minkowski space-time, while dilation is a homothety with $\phi_3 = 2$.

In order to rewrite the metric according to the group acting on it, we wish to obtain a set of coordinates, three of which are the canonical coordinates of the group, while the fourth is transversal to it. To trace the canonical coordinates of the group, we rely on the exponential map of the algebra defined as

$$e^A = e^{q_1 \mathrm{D}\vec{\xi}_1} e^{q_2 \mathrm{D}\vec{\xi}_2} e^{q_3 \mathrm{D}\vec{\xi}_3} \,, \tag{282}$$

where $\mathrm{D}\vec{\xi}_i$ the Jacobian matrix of each generator and q_i a respective parameter. This leads to the following exponential map

$$e^A = \begin{pmatrix} \frac{1}{2}e^{q_3}\left(e^{q_1}\left(q_2^2+1\right)+e^{-q_1}\right) & e^{q_1}e^{q_3}q_2 & 0 & -\frac{1}{2}e^{q_3}\left(e^{q_1}\left(q_2^2-1\right)+e^{-q_1}\right) \\ e^{q_3}q_2 & e^{q_3} & 0 & -e^{q_3}q_2 \\ 0 & 0 & e^{q_3} & 0 \\ \frac{1}{2}e^{q_3}\left(e^{q_1}\left(q_2^2+1\right)-e^{-q_1}\right) & e^{q_1}e^{q_3}q_2 & 0 & -\frac{1}{2}e^{q3}\left(e^{q_1}\left(q_2^2-1\right)-e^{-q_1}\right) \end{pmatrix} . \tag{283}$$

Consider now the following transformations

$$\frac{1+w_1}{1-w_1} = e^{q_1} \ , \quad w_2 = q_2 \quad \text{and} \quad w_3 = e^{q_3} \,, \tag{284}$$

the exponential map is rewritten as

$$e^A = \begin{pmatrix} \frac{w_3}{2}W_- & \frac{1+w_1}{1-w_1}w_2w_3 & 0 & -\frac{w_3}{2}W_- \\ w_2w_3 & w_3 & 0 & -w_2w_3 \\ 0 & 0 & w_3 & 0 \\ \frac{w}{2}W_+ & \frac{1+w_1}{1-w_1}w_2w_3 & 0 & -\frac{w_3}{2}W_+ \end{pmatrix}, \tag{285}$$

where

$$W_- = \left(\frac{1+w_1}{1-w_1}\left(w_2^2-1\right) + \frac{1-w_1}{1+w_1}\right).$$

and

$$W_+ = \left(\frac{1+w_1}{1-w_1}\left(w_2^2+1\right) - \frac{1-w_1}{1+w_1}\right).$$

Then, to obtain a transversal to the group, we choose an arbitrary point on the space-time, that is invariant under the group, *e.g.* $P_0(1,0,1,0)$, and a null direction for the Minkowski space-time that will serve as the initial transversal geodesic, *e.g.* $\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}$; then, applying the exponential map on this geodesic, we shall obtain the general form of the transversal geodesic. This transversal geodesic begins a simple line, described parametrically as

$$\epsilon : \{ \quad t = 1+z \;, \quad x_1 = z \;, \quad x_2 = 1 \quad \text{and} \quad x_3 = 0\,, \tag{286}$$

where z a parameter increasing along the $\vec{e}_2$ direction. This parameter will function as the transversal coordinate; the canonical cooridnates of the group are chosen as in the previous example to be u, r and w - algebraic transformations of q_1, q_2 and q_3. The action of the exponential map on the straight line ϵ yields the general form of the transversal line with respect to the new set of coordinates

(287)

$$\epsilon : \begin{cases} t = \frac{w_3}{2}\left(\frac{(1+w_1)\left(w_2^2+1+(w_2+1)^2 z\right)}{1-w_1} + \frac{(z+1)(1-w_1)}{1+w_1}\right) \;, \\ x_1 = w_1\left((w_2+1)z + w_2\right) \;, \\ x_2 = w_3 \quad \text{and} \\ x_3 = \frac{w_3}{2}\left(\frac{(1+w_1)\left(w_2^2+1+(w_2+1)^2 z\right)}{1-w_1} - \frac{(z+1)(1-w_1)}{1+w_1}\right). \end{cases}$$

To obtain the reverse transformation from the group-specific coordinates to the Cartesian coordinates, we will -as in the previous example- consider two points on the space-time, $P_1(\tau, \chi_1, \chi_2, \chi_3)$ and $P_2(t, x_1, x_2, x_3)$, that are related by the group action. In this, we distinguish between the quotient of the group action (that corresponds to the choice of z) and the group action itself; furthermore, we distinguish the group action to its two isometries (boost and loxodromy, related to w_1 and w_2 respectively) and the homothety (dilation, related to w_3).

(1) The point P_1 is the initial point of the action, so its coordinates can be described with respect to z as

$$\tau = 1 + z\,, \quad \chi_1 = z\,, \quad \chi_2 = 1 \quad \text{and} \quad \chi_3 = 0\,. \tag{288}$$

(2) Distinguishing between the group action and its quotient, we can identify the coordinate on the latter, s, through the invariant of the group,

$$\frac{x_2^2}{-t^2 + x_1^2 + x_2^2 + x_3^2}\,.$$

Moving along s, the above quantity remains constant. Hence,

$$\frac{x_2^2}{-t^2 + x_1^2 + x_2^2 + x_3^2} = \frac{\chi_2^2}{-\tau^2 + \chi_1^2 + \chi_2^2 + \chi_3^2}\,. \tag{289}$$

Thus, we have moved from point P_1 to point P_2; now we take the way back through the group action.

(3) Distinguishing the group action to the isometries and the homothety, we can isolate the effects of each symmetry and the respective invariants. This is expressed as if the move "back" from P_2 to P_1 is decomposed by a motion from P_2 to $P_2'(t', x_1', x_2', x_3')$ by means of the homothety (dilation), and then a motion from P_2' to P_2 by means of the isometries (boost and rotation). This means that the coordinates of the two points are related as

$$t' = \frac{t}{w_3}\,, \quad x_1' = \frac{x_1}{w_3}\,, \quad x_2' = \frac{x_2}{w_3} \quad \text{and} \quad x_3' = \frac{x_3}{w_3}\,. \tag{290}$$

(4) Identifying w_3 with the dilation $(\vec{\xi}_3)$, this remains constant under the isometries. The coordinate not affected at all by the isometries, hence which remains constant under their action, is x_2; we know already that $x_2 = w_3$. Hence, this is given immediately as

$$w_3 = x_2\,. \tag{291}$$

(5) Following, the boost symmetry, identified with w_1, results by moving along the an angle, ϕ, along a hyperbola on the $t - x_3$ plane, in such a manner that

$$\begin{aligned} \cosh\phi &= \frac{(t', x_3') \odot (\tau, \chi_3)}{\wr(t', x_3')\wr\,\wr(\tau, \chi_3)\wr} = \frac{-t'\tau - x_3'\chi_3}{-(t')^2 + (x_3')^2} = w_3\frac{-t\tau - x_3\chi_3}{-t^2 + x_3^2} \quad \text{and} \\ \sinh\phi &= \frac{(t', x_3') \otimes (\tau, \chi_3)}{\wr(t', x_3')\wr\,\wr(\tau, \chi_3)\wr} = \frac{t'\chi_3 + x_3'\tau}{-(t')^2 + (x_3')^2} = w_3\frac{t\chi_3 + x_3\tau}{-t^2 + x_3^2} \end{aligned} \tag{292}$$

where $\odot$, $\otimes$ and $\wr\,\wr$ are the Minkowskian inner product, cross product and norm respectively. Following, we may take

$$w_1 = \tanh\left(\frac{\phi}{2}\right) = \frac{\sinh\phi}{1 + \cosh\phi}\,. \tag{293}$$

(6) Finally, knowing w_3 and x_3, we may use the exponential map - especially the action on x_1- to determine w_2, since

(294) $$x_1 = w_3\left(w_2\tau + \chi_1 - w_2\chi_3\right) .$$

Substituting τ, χ_1, χ_2 and χ_3 to the invariant of the group, we easily get

$$\frac{x_2^2}{-t^2 + x_1^2 + x_2^2 + x_3^2} = \frac{1}{2z} ,$$

which is immediately solved to

(295) $$z = \frac{t^2 - x_1^2 - x_2^2 - x_3^2}{2x_2^2} .$$

The dilation is directly given as

(296) $$w_3 = x_2 .$$

From the boost, we obtain

$$w_1 = \frac{w_3(t\chi_3 + x_3\tau)}{-t^2 + x_3^2 + w_3(-t\tau - x_3\chi_3)} = \frac{x_2x_3(1+z)}{-t^2 + x_3^2 - tx_2(1+z)} ,$$

hence

(297) $$w_1 = -\frac{x_3\left(t^2 - x_1^2 + x_2^2 - x_3^2\right)}{2x_2\left(t^2 - x_3^2\right) + t\left(t^2 - x_1^2 + x_2^2 - x_3^2\right)} .$$

Finally, from the exponential map, we obtain

$$x_1 = x_2\left(w_2(1+z) + z\right) ,$$

which is easily solved to

(298) $$w_2 = -\frac{t^2 - x_1^2 - 2x_1x_2 - x_2^2 - x_3^2}{t^2 - x_1^2 + x_2^2 - x_3^2} .$$

The generators may now take the form

$$\begin{aligned}
\vec{\xi}_1 =& \frac{w_3}{2}\left(\frac{(1+w_1)\left((1+z)w_2^2 + 2zw_2 + z + 1\right)}{1-w_1} + \frac{(1+z)(1-w_1)}{1+w_1}\right)\frac{\partial}{\partial t} \\
&+ \frac{w_3}{2}\left(\frac{(1+w_1)\left((1+z)w_2^2 + 2zw_2 + z + 1\right)}{1-w_1} - \frac{(1+z)(1-w_1)}{1+w_1}\right)\frac{\partial}{\partial x_3} , \\
\vec{\xi}_2 =& w_3\left((1+z)w_2 + z\right)\frac{\partial}{\partial t} + \frac{(1+z)(1-w_1)w_3}{1+w_1}\frac{\partial}{\partial x_1} + w_3\left((1+z)w_2 + z\right)\frac{\partial}{\partial x_3} \quad \text{and} \\
\vec{\xi}_3 =& \frac{w_3}{2}\left(\frac{(1+u_1)\left((1+z)w_2^2 + 2zw_2 + z + 1\right)}{1-w_1} - \frac{(1+z)(1-w_1)}{1+w_1}\right)\frac{\partial}{\partial t} \\
&+ w_3\left((1+z)w_2 + z\right)\frac{\partial}{\partial x_1} + w_3\frac{\partial}{\partial x_2} \\
&+ \frac{w_3}{2}\left(\frac{(1+w_1)\left((1+z)w_2^2 + 2zw_2 + z + 1\right)}{1-w_1} + \frac{(1+z)(1-w_1)}{1+w_1}\right)\frac{\partial}{\partial x_3} ,
\end{aligned}$$

and in the new coordinate system they are simplified to

$$\begin{aligned} \vec{\xi}_1 &= \frac{1-w_1^2}{2}\frac{\partial}{\partial w_1}\ , \\ \vec{\xi}_2 &= \frac{1-w_1}{1+w_1}\frac{\partial}{\partial w_2} \quad \text{and} \\ \vec{\xi}_3 &= w_3\frac{\partial}{\partial w_3}\,. \end{aligned} \tag{299}$$

At the same time, the ray on the quotient is tangent to

$$\begin{aligned} \vec{\zeta} &= e^A\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x_1}\Big) = \\ &= \frac{w_3}{2}\left(\frac{(1+w_2)^2(1+w_1)}{1-w_1}+\frac{1-w_1}{1+w_1}\right)\frac{\partial}{\partial t} + (1+w_2)w_3\frac{\partial}{\partial x_1} \\ &\quad + \frac{w_3}{2}\left(\frac{(1+w_2)^2(1+w_1)}{1-w_1}-\frac{1-w_1}{1+w_1}\right)\frac{\partial}{\partial x_3}\ , \end{aligned}$$

while in the new coordinate system it becomes

$$\vec{\zeta} = \frac{\partial}{\partial z}\,. \tag{300}$$

The four vectors of the basis are linearly independent, since the determinant with $\vec{\zeta}$, $\vec{\xi}_1$, $\vec{\xi}_2$ and $\vec{\xi}_3$ as column-entries is $(1+z)\dfrac{(1-w_1)w_3^4}{1+z_1}$.

Now the metric can be expressed as the Minkowskian inner products of the four vectors,

$$\begin{aligned} g_{ss} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\zeta}\big) = \vec{\zeta}\odot\vec{\zeta} = 0 \\ g_{s1} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_1\big) = \vec{\zeta}\odot\vec{\xi}_1 = w_2w_3^2 \\ g_{s2} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_2\big) = \vec{\zeta}\odot\vec{\xi}_2 = \frac{1-w_1}{1+w_1}w_3^2 \\ g_{s3} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_3\big) = \vec{\zeta}\odot\vec{\xi}_3 = -w_3^2 \\ g_{11} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_1\big) = \vec{\xi}_1\odot\vec{\xi}_1 = (1+z)\left(1+w_2^2+(1+w_2)^2z\right)w_3^2 \\ g_{12} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_2\big) = \vec{\xi}_1\odot\vec{\xi}_2 = (1+z)\left(z+(1+z)w_2\right)\frac{1-w_1}{1+w_1}w_3^2 \\ g_{13} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_3\big) = \vec{\xi}_1\odot\vec{\xi}_3 = 0 \\ g_{22} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_2\big) = \vec{\xi}_2\odot\vec{\xi}_2 = (1+z)^2\frac{(w_1-1)^2}{(w_1+1)^2}w_3^2 \\ g_{23} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_3\big) = \vec{\xi}_2\odot\vec{\xi}_3 = 0 \\ g_{33} &= \boldsymbol{g}\big(\vec{\xi}_3,\vec{\xi}_3\big) = \vec{\xi}_3\odot\vec{\xi}_3 = -2zw_3^2 \end{aligned}$$

or

(301)
$$\boldsymbol{g} = \begin{pmatrix} 0 & w_2 w_3^2 & \dfrac{(1-w_1)w_3^2}{1+w_1} & -w_3^2 \\ w_2 w_3^2 & (z+1)\left(w_2^2+(w_2+1)^2 z+1\right) w_3^2 & \frac{(z+1)(zw_2+w_2+z)(1-w_1)w_3^2}{1+w_1} & 0 \\ \frac{(1-w_1)w_3^2}{w_1+1} & \frac{(z+1)(zw_2+w_2+z)(1-w_1)w_3^2}{1+w_1} & \frac{(z+1)^2(1-w_1)^2 w_3^2}{(1+w_1)^2} & 0 \\ -w_3^2 & 0 & 0 & -2zw_3^2 \end{pmatrix}.$$

The Lie derivative of the metric along $\vec{\zeta}$ is

(302)
$$\mathcal{L}_{\vec{\zeta}}\boldsymbol{g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\left(w_2^2+w_2+(w_2+1)^2 z+1\right) w_3^2 & \frac{(2z+2(1+z)w_2+1)(1-w_1)w_3^2}{1+w_1} & 0 \\ 0 & \frac{(2z+2(1+z)w_2+1)(1-w_1)w_3^2}{1+w_1} & \frac{2(1+z)(1-w_1)^2 w_3^2}{(1+w_1)^2} & 0 \\ 0 & 0 & 0 & -2w_3^2 \end{pmatrix},$$

and the second Lie derivative of the metric is

$$\mathcal{L}_{\vec{\zeta}}(\mathcal{L}_{\vec{\zeta}}\boldsymbol{g}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2(w_2+1)^2 w_3^2 & \frac{2(w_2+1)(1-w_1)w_3^2}{1+w_1} & 0 \\ 0 & \frac{2(w_2+1)(1-w_1)w_3^2}{1+w_1} & \frac{2(1-w_1)^2 w_3^2}{(1+w_1)^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{303}$$

Now, the Lie derivative along $\vec{\xi}_1$ is

(304)
$$\mathcal{L}_{\vec{\xi}_1}\boldsymbol{g} = \frac{1-w_1^2}{2}\frac{\partial}{\partial w_1}\boldsymbol{g} = \begin{pmatrix} 0 & 0 & \frac{(1-3_1)w_2^2}{1+w_1} & 0 \\ 0 & 0 & -\frac{(1+z)(zw_2+w_2+z)(1-w_1)w_3^2}{1+w_1} & 0 \\ -\frac{(1-w_1)w_3^2}{1+w_1} & -\frac{(1+z)(zw_2+w_2+z)(1-w_1)w_3^2}{1+w_1} & -\frac{2(1+z)^2(1-w_1)^2 w_3^2}{(1+w_1)^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the Lie derivative along $\vec{\xi}_2$ is

(305)
$$\mathcal{L}_{\vec{\xi}_2}\boldsymbol{g} = \frac{1-w_1}{1+w_1}\frac{\partial}{\partial w_2}\boldsymbol{g} =$$
$$= \begin{pmatrix} 0 & \frac{(1-w_1)w_3^2}{1+w_1} & 0 & 0 \\ \frac{(1-w_1)w_3^2}{1+w_1} & \frac{2(1+z)(zw_2+w_2+z)(1-w_1)w_3^2}{1+w_1} & \frac{(1+z)^2(1-w_1)^2 w_3^2}{(1+w_1)^2} & 0 \\ 0 & \frac{(1+z)^2(1-w_1)^2 w_3^2}{(1+w_1)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the Lie derivative along $\vec{\xi}_3$ is
(306)

$$\mathcal{L}_{\vec{\xi}_3}\boldsymbol{g} = w_3\frac{\partial}{\partial w_3}\boldsymbol{g} = 2\boldsymbol{g} = \begin{pmatrix} 0 & 2w_2w_3^2 & \frac{2(1-w_1)w_3^2}{1+w_1} & -2w \\ 2w_2w_3^2 & 2(1+z)\left(w_2^2+(1+w_2)^2z+1\right)w_3^2 & \frac{2(1+z)(zw_2+w_2+z)(1-w_1)w_2^2}{1+w_1} & 0 \\ \frac{2(1-w_1)w_3^2}{1+w_1} & \frac{2(1+z)(zw_2+w_2+z)(1-w_1)w^2}{1+w_1} & \frac{2(1+z)^2(1-w_1)^2w_3^2}{(1+w_2)^2} & 0 \\ -2w_3^2 & 0 & 0 & -4z \end{pmatrix}$$

4. Solving the Einstein System

As discussed, a general solution to the Einstein system is possible only locally, within an interval of z. A global solution would be possible only if additional conditions are imposed; the simplest such case is the derivatives of the right-hand side, $\boldsymbol{F}(\vec{X})$ are bounded in the entire domain [**51**].

However, what the examples revealed is that the foliation considered in this work may not always fulfill any additional conditions (*e.g.*, the geodesics may be incomplete, or the quotient may be non-Hausdorf). As a result, global existence may not exist for several examples. The question that is immediately brought up, then, is whether these examples are 'isolated events', or whether they are a generic feature of this class of space-times. In an attempt to answer this question, we proceed in proving a theorem about 'neighbouring solutions'.

4.1. The existence of local solutions near known space-times. The theorem can be stated as follows:

THEOREM 4.1 (Existence of neighbouring solutions). *Let Ω be an open subset of $\mathbb{R}^n$ and $\boldsymbol{F}$ be a continuously differentiable function from Ω to $\mathbb{R}^n$. Suppose that $\vec{X}$ satisfied the ordinary differential equation*

$$\frac{\partial}{\partial z}\vec{X}(z) = \boldsymbol{F}\left(\vec{X}(z)\right) ,$$

for $z \in \left[z_1, z_2\right]$. Suppose finally that $\mathcal{U}_1 \subset \Omega$ and $\mathcal{U}_2 \subset \Omega$ are two open subsets of Ω, such that

$$\vec{X}(z_1) \in \mathcal{U}_1 \quad \textit{and} \quad \vec{X}(z_2) \in \mathcal{U}_2 \,.$$

Then, there exists some $\delta > 0$ such that all $\vec{W} \in \mathfrak{B}\big(\vec{X}(z_1), \delta\big)$ there is a continuously differentiable solutions $\vec{Y}$ to the Initial Value Problem

$$\frac{\partial}{\partial z}\vec{Y}(z) = \boldsymbol{F}\left(\vec{Y}(z)\right)$$
$$\vec{Y}(z_1) = \vec{W} \,,$$

for $z \in \left[z_1, z_2\right]$. This solution $\vec{Y}$ also satisfies

$$\vec{Y}(z_1) \in \mathcal{U}_1 \quad \textit{and} \quad \vec{Y}(z_2) \in \mathcal{U}_2 \,.$$

This theorem follows from the usual well-posedeness theorem of the Initial Value Problem for ordinary differential equations. However, not the entire theorem follows trivially; the main conclusion that does not is that the solution is defined on the netire interval $\big[z_1, z_2\big]$, since the existence part of the Picard-Lindelöf theorem is purely local. To overcome this difficulty, we consider a well-known fact from the theory of metric spaces:

LEMMA 4.2. *Let $\mathcal{K}_1$ be a compact subset of Ω. Then there is some $\mu > 0$ and a compact subset $\mathcal{K}_2 \subset \Omega$, such that $\mathrm{B}\big(\vec{V}, \mu\big) \subset \mathcal{K}_2$ for every $\vec{V} \in \mathcal{K}_1$.*

This lemma can be applied to $\mathcal{K}_1 = \vec{X}\big[z_1, z_2\big] = \Big\{\vec{X}(z) : z \in \big[z_1, z_2\big]\Big\}$, which is a compact subset of $\mathrm{I\!R}^n$, since $\vec{X}$ is continuous and the interval $\big[z_1, z_2\big]$ is compact.

PROOF. Let $\mathcal{K}_1 = \vec{X}\big[z_1, z_2\big] = \Big\{\vec{X}(z) : z \in \big[z_1, z_2\big]\Big\}$ a compact subset of $\mathrm{I\!R}^n$, ans q be the corresponding positive number from the application of Lemma 4.2. Remembering that $\mathcal{U}_1$ and $\mathcal{U}_2$ are open, then there are $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$\mathrm{B}\big(\vec{X}(z_1), \epsilon_1\big) \subset \mathcal{U}_1 \quad \text{and} \quad \mathrm{B}\big(\vec{X}(z_2), \epsilon_2\big) \subset \mathcal{U}_2\,. \tag{307}$$

Now, let us define the maximum norm of the Jacobian of $\boldsymbol{F}$, which we have assumed to be continuous,

$$\lambda = \max_{\vec{V} \in \mathcal{K}_2} \left\| \frac{\partial \boldsymbol{F}}{\partial \vec{X}}(\vec{V}) \right\| \,, \tag{308}$$

where $\dfrac{\partial \boldsymbol{F}}{\partial \vec{X}}$ is an abuse of notation to refer to all derivatives of $\boldsymbol{F}$ with respect to all arguments $\vec{X}$ (a usual symbol for the Jacobian). Let

$$\eta = \min\big\{\epsilon_2, q\big\}\,,$$

whereas we can show that theorem holds for

$$\delta = \min\big\{\epsilon_1, \eta \exp\left(\lambda(z_1 - z_2)\right)\,. \tag{309}$$

We define

$$\vec{Y}_{(0)}(z) = \vec{X}(z) + \vec{W} - \vec{X}(z_1) \tag{310}$$

and

$$\vec{T}_{(0)}(z) = \vec{W} - \vec{X}(z_1)\,, \tag{311}$$

so that $\vec{Y}_{(0)} - \vec{T}_{(0)} = \vec{X}(z)$ is a solution to the Initial Value Problem; we also define

$$\vec{Y}_{(n+1)}(z) = \vec{W} + \int_{z_1}^{z} \boldsymbol{F}\left(\vec{Y}_{(n)}(s)\right) \mathrm{d}s \tag{312}$$

and

$$\vec{T}_{(n+1)}(z) = \vec{W} - \vec{X}(z_1) + \int_{z_1}^{z} \boldsymbol{F}\left(\vec{T}_{(n)}(s)\right) \mathrm{d}s \tag{313}$$

the Picard iterates of $\vec{Y}_{(0)}$ and $\vec{T}_{(0)}$ for all $n \geq 0$.

It follows trivially that

- For all $z \in \big[z_1, z_2\big]$,
$$\Big\|\vec{Y}_{(0)}(z) - \vec{X}(z)\Big\| \leq \Big\|\vec{W} - \vec{X}(z_1)\Big\|$$
and
$$\vec{Y}_{(0)}(z) \in \mathrm{B}\big(\vec{X}(z), \mu\big)\,.$$
- For all $z \in \big[z_1, z_2\big]$,
$$\Big\|\vec{T}_{(0)}\Big\| \leq \Big\|\vec{W} - \vec{X}(z_1)\Big\|$$
and
$$\vec{T}_{(0)}(z) \in \mathrm{B}\big(\vec{X}(z_1), \mu\big)\,.$$

However, we are interested in proving similar relations for the Picard iterates, *i.e.* for any j. This proof is cyclical as it requires to prove these relation for the previous step so as to extend it to the next. Let us collect the process in a list of statements.

(a) For all $n \geq 0$ and $z \in \big[z_1, z_2\big]$,
$$\Big\|\vec{Y}_{(n)}(z) - \vec{X}(z)\Big\| \leq \Big\|\vec{W} - \vec{X}(z_1)\Big\| \sum_{i=0}^{n} \frac{\lambda^i (z - z_1)^i}{i!}\,. \tag{314}$$

(b) For all $n \geq 0$ and $z \in \big[z_1, z_2\big]$,
$$\Big\|\vec{T}_{(n)}(z)\Big\| \leq \Big\|\vec{W} - \vec{X}(z_1)\Big\| \frac{\lambda^n (z - z_1)^n}{n!}\,. \tag{315}$$

(c) For all $n \geq 0$ and $z \in \big[z_1, z_2\big]$,
$$\Big\|\vec{T}_{(n)}(z)\Big\| \leq \Big\|\vec{W} - \vec{X}(z_1)\Big\| \frac{\lambda^n (z_2 - z_1)^n}{n!}\,. \tag{316}$$

(d) For all $n \geq 0$ and $z \in \big[z_1, z_2\big]$,
$$\vec{Y}_{(n)}(z) \in \mathrm{B}\big(\vec{X}(z), \mu\big)\,. \tag{317}$$

(e) For all $r \in \big[0, 1\big]$ and all $z \in \big[z_1, z_2\big]$,
$$\vec{P}_{(n)}(r, z) \in \mathrm{B}\big(\vec{X}(z), \mu\big)\,, \tag{318}$$
where $\vec{P}_{(n)} = r\mathrm{Y}_{(n)}(z) + (1 - r)\mathrm{X}(z)$.

(f) If $n > 0$, then for all $z \in \big[z_1, z_2\big]$,
$$\vec{Y}_{(n-1)}(z) \in \mathrm{B}\big(\vec{X}(z), \mu\big)\,. \tag{319}$$

(g) If $n > 0$, then for all $r \in \big[0, 1\big]$ and for all $z \in \big[z_1, z_2\big]$,
$$\vec{Q}_{(n)}(r, z) \in \mathrm{B}\big(\vec{X}(z), \mu\big)\,, \tag{320}$$
where $\vec{Q}_{(n)}(r, z) = r\vec{Y}_{(n)}(z) + (1 - r)\vec{Y}_{(n-1)}$.

(h) For all $r \in \big[0, 1\big]$ and all $z \in \big[z_1, z_2\big]$,
$$\vec{P}_{(n)}(r, z) \in \mathcal{K}_2\,. \tag{321}$$

(i) If $j > 0$, then for all $r \in \big[0, 1\big]$ and all $z \in \big[z_1, z_2\big]$,

$$\vec{Q}_{(n)}(r, z) \in \mathcal{K}_2 \,. \tag{322}$$

(j) For all $z \in \big[z_1, z_2\big]$,

$$\vec{Y}_{(n)} \in \Omega \,. \tag{323}$$

(k) The function $\vec{Y}_{(n+1)}(z) = \vec{X}(z) + \vec{W} - \vec{X}(z_1) + \int_{z_1}^{z} \Big(\boldsymbol{F}\big(\vec{Y}_{(n)}(s)\big) - \boldsymbol{F}\big(\vec{X}(s)\big)\Big)\,\mathrm{d}s$ is well-defined for all $z \in \big[z_1, z_2\big]$.

(l) The function $\vec{T}_{(n+1)}(z) = \vec{Y}_{(n+1)}(z) - \vec{Y}_{(n)}(z)$ is well-defined for all $z \in \big[z_1, z_2\big]$.

(m) The function M_n given by

$$M_n(r, s) = \begin{cases} \dfrac{\partial \boldsymbol{F}}{\partial \vec{X}}\big(\vec{P}_{(j)}(r, s)\big) & \text{if } n = 0 \\ \dfrac{\partial \boldsymbol{F}}{\partial \vec{X}}\big(\vec{Q}_{(j)}(r, s)\big) & \text{if } n > 0 \end{cases}, \tag{324}$$

is well-defined for all $r \in \big[0, 1\big]$ and all $z \in \big[z_1, z_2\big]$ and satisfies

$$\|M_n(r, s)\| \leq \lambda \tag{325}$$

(n) For all $z \in \big[z_1, z_2\big]$,

$$\vec{T}_{(n+1)}(z) = \int_{z_1}^{z} \int_0^1 M_j(r, s) \vec{T}_{(n)}(s) \mathrm{d}r \mathrm{d}s \,. \tag{326}$$

(o) For all $z \in \big[z_1, z_2\big]$,

$$\Big\|\vec{T}_{(n+1)}(s)\Big\| \leq \Big\|\vec{W} - \vec{X}(z_1)\Big\| \, \frac{\lambda^{n+1}(z - z_1)^{n+1}}{(n+1)!} \,. \tag{327}$$

(p) For all $z \in \big[z_1, z_2\big]$,

$$\Big\|\vec{Y}_{(n+1)}(z) - \vec{X}(z)\Big\| \leq \Big\|\vec{W} - \vec{X}(z_1)\Big\| \sum_{i=0}^{n+1} \frac{\lambda^i (z - z_1)^i}{i!} \,. \tag{328}$$

These statements can be proved one at a time by inducing the index n.

Statement (a) is proved trivially for $n = 0$; in the inductive case, where $n > 0$, it follows from statement (p).

Similarly, statement (b) is proved trivially for $n = 0$; in the inductive case, where $n > 0$, it follows from statement (o).

Statement (c) follows from statement (b), given $z - z_1 \leq z_2 - z_1$ for all $z \in \big[z_1, z_2\big]$.

Statement (d) follows from statement (a), given

$$\begin{aligned}
&\left\|\vec{W}-\vec{X}(z_1)\right\|\sum_{i=0}^{n}\frac{\lambda^i(z-z_1)^i}{i!}\leq\left\|\vec{W}-\vec{X}(z_1)\right\|\sum_{i=0}^{\infty}\frac{\lambda^i(z-z_1)^i}{i!}\\
&\Rightarrow\left\|\vec{W}-\vec{X}(z_1)\right\|\sum_{i=0}^{n}\frac{\lambda^i(z-z_1)^i}{i!}\leq\left\|\vec{W}-\vec{X}(z_1)\right\|\exp\left(\lambda(z-z_1)\right)\\
&\Rightarrow\left\|\vec{W}-\vec{X}(z_1)\right\|\sum_{i=0}^{n}\frac{\lambda^i(z-z_1)^i}{i!}<\delta\exp\left(\lambda(z-z_1)\right)\leq\mu\,.
\end{aligned}$$

Statement (e) follows from statement (d) and the fact that balls are convex, so $\vec{X}(z)\in\mathrm{B}\big(\vec{X}(z),\mu\big)$.

Statement (f) is vacuously true for $n=0$; in the inductive case, where $n>0$, it follows from statement (d).

Statement (g) follows directly from statements (d) and (f), since balls are convex.

Statement (h) follows from statement (e), given $\mathrm{B}\big(\vec{X}(z),\mu\big)\subset\mathcal{K}_2$.

Similarly, statement (i) follows from statement g, given $\mathrm{B}\big(\vec{X}(z),\mu\big)\subset\mathcal{K}_2$.

Statement (j) follows from statement (i), since $\mathcal{K}_2\subset\Omega$.

Statement (k) follows from statement (j) and the Picard iteration, given the integrand is well-defined and continuous (so, it satisfies the assumptions of the Picard-Lindelöf theorem).

Statement (l) follows directly from statement (k) and the Picard iteration.

Statement (m) is proved using the fact that M_j is well defined; the latter follows from statement (h) if $n=0$ (the base case), or from statement (i) if $n>0$ (the inductive case).

Statement (n) is proved through Picard iteration. Since $\vec{T}_{(n+1)}(z)=\vec{Y}_{(n+1)}(z)-\vec{Y}_{(n)}$, we have

$$\vec{T}_{(n+1)}(z)=\vec{X}(z)+\vec{W}-\vec{X}(z_1)+\int_{z_1}^{z}\Big(\boldsymbol{F}\big(\vec{Y}_{(n)}(s)\big)-\boldsymbol{F}\big(\vec{X}(z)\big)\Big)\,\mathrm{d}s-\vec{Y}_{(n)}\,.$$

For $n=0$ (the base case), this reduces to

$$\vec{T}_{(1)}(z)=\int_{z_1}^{z}\Big(\boldsymbol{F}\big(\vec{Y}_{(0)}(s)\big)-\boldsymbol{F}\big(\vec{X}(z)\big)\Big)\,\mathrm{d}s\,;$$

while for $n>0$ (the inductive case), this becomes

$$\vec{T}_{(n+1)}(z)=\int_{z_1}^{z}\Big(\boldsymbol{F}\big(\vec{Y}_{(n)}(s)\big)-\boldsymbol{F}\big(\vec{Y}_{(n-1)}(z)\big)\Big)\,\mathrm{d}s\,.$$

If $n=0$, from the Fundamental Theorem of Calculus, we can see that

$$\begin{aligned}\boldsymbol{F}\big(\vec{Y}_{(0)}(s)\big)-\boldsymbol{F}\big(\vec{X}(z)\big)&=\boldsymbol{F}\big(\vec{P}_{(0)}(1,s)\big)-\boldsymbol{F}\big(\vec{P}_{(0)}(0,z)\big)=\\&=\int_0^1\left(\frac{\partial\boldsymbol{F}}{\partial\vec{X}}\big(\vec{P}_{(0)}(r,s)\big)\frac{\partial\vec{P}_{(0)}}{\partial r}(r,s)\right)\mathrm{d}r=\\&=\int_0^1 M_0(r,s)\big(\vec{Y}_{(0)}(s)-\vec{X}(s)\big)\mathrm{d}r=\\&=\int_0^1 M_0(r,s)\vec{T}_{(0)}\mathrm{d}r\,;\end{aligned}$$

while, for $n>0$, we can use a similar reasoning

$$\begin{aligned}\boldsymbol{F}\big(\vec{Y}_{(n)}(s)\big)-\boldsymbol{F}\big(\vec{Y}_{(n-1)}(z)\big)&=\boldsymbol{F}\big(\vec{Q}_{(n)}(1,s)\big)-\boldsymbol{F}\big(\vec{Q}_{(n)}(0,z)\big)=\\&=\int_0^1\left(\frac{\partial\boldsymbol{F}}{\partial\vec{X}}\big(\vec{Q}_{(n)}(r,s)\big)\frac{\partial\vec{Q}_{(n)}}{\partial r}(r,s)\right)\mathrm{d}r=\\&=\int_0^1 M_n(r,s)\big(\vec{Y}_{(n)}(s)-\vec{Y}_{(n-1)}(s)\big)\mathrm{d}r=\\&=\int_0^1 M_n(r,s)\vec{T}_{(n)}\mathrm{d}r\,.\end{aligned}$$

In either case, integrating over the interval $\big[z_1,z_2\big]$ results to (n).

Statement (o) can be proved using the submultiplicativity of the norm,

$$\left\|M_n(r,s)\vec{T}_{(n)}(s)\right\|\le\|M_n(r,s)\|\left\|\vec{T}_{(n)}(s)\right\|\,,$$

and statements (b) and (m) to obtain

$$\left\|M_n(r,s)\vec{T}_{(n)}(s)\right\|\le\left\|\vec{W}-\vec{X}(z_1)\right\|\frac{\lambda^{n+1}(z-z_1)^n}{n!}\,.$$

Using statement (n) and since the norm of an integral is less than the integral of the norm, we can prove that

$$\begin{aligned}\left\|\int_{z_1}^z\int_0^1 M_n(r,s)\vec{T}_{(n)}(s)\mathrm{d}r\mathrm{d}s\right\|&\le\int_{z_1}^z\int_0^1\left\|M_n(r,s)\vec{T}_{(n)}(s)\right\|\mathrm{d}r\mathrm{d}s\\&\le\int_{z_1}^z\int_0^1\left\|\vec{W}-\vec{X}(z_1)\right\|\frac{\lambda^{n+1}(z-z_1)^n}{n!}\mathrm{d}r\mathrm{d}s\\\Rightarrow\vec{T}_{(n+1)}(z)\le\left\|\vec{W}-\vec{X}(z_1)\right\|\frac{\lambda^{n+1}(z-z_1)^{n+1}}{(n+1)!}\end{aligned}$$

whence statement (o) is proved.

Statement (p) follows simply from statements (a) and (o) and the Triangle Inequality.

Given the infinite sum $\sum_{i=0}^{\infty} \dfrac{\lambda^i (z_2 - z_1)^i}{i!}$ always converges to $\exp\left(\lambda(z_2 - z_1)\right)$; and given statement (c), we get a uniform convergence for

$$\sum_{i=0}^{\infty} \vec{T}_{(i)}(z)$$

on the interval $\left[z_1, z_2\right]$ by the comparison test; that is, $\lim_{k\to\infty} \sum_{i=0}^{k} \vec{Z}_i(s)$ is convergent. However, this sum is equal to

$$\sum_{i=0}^{k} \vec{T}_{(i)}(z) = \vec{Y}_{(k)}(z) + \vec{W} - \vec{X}(z_1)\,,$$

where only the first summand depends on k; so $\lim_{k\to\infty} \vec{Y}_{(k)}(z)$ is also uniformly convergent. Let $\vec{Y}(z)$ be its limit. Then, assuming $r = 1$ in statement (h), we obtain $\vec{Y}_{(k)}(z) \subset \mathcal{K}_2$; and, since $\mathcal{K}_2$ is compact and, by extent, closed, we have

$$\vec{Y}(z) \in \mathcal{K}_2\,,$$

and thus

$$\vec{Y}(z) \in \Omega\,,$$

for all $z \in \left[z_1, z_2\right]$.

Due to uniform convergence, we can show that

$$\vec{Y}(z) = \vec{X}(z) + \vec{W} - \vec{X}(z_1) + \int_{z_1}^{z} \Big(\boldsymbol{F}\big(\vec{Y}(s)\big) - \boldsymbol{F}\big(\vec{X}(s)\big)\Big)\, \mathrm{d}s\,.$$

Using the Fundamental Theorem of Calculus, it is easy to conclude that

$$\frac{\partial}{\partial z}\vec{Y}(z) - \frac{\partial}{\partial z}\vec{X}(z) = \boldsymbol{F}\big(\vec{Y}(z)\big) - \boldsymbol{F}\big(\vec{X}(z)\big)\,,$$

for all $z \in \left[z_1, z_2\right]$. Given statement (k), whereas $\vec{Y}_{(k+1)}(z_1) = \vec{Y}(z) = \vec{W}$; and given $\vec{X}$ satisfies the Initial Value Problem, it is easy to conclude that

$$\frac{\partial}{\partial z}\vec{Y}(z) = \boldsymbol{F}\big(\vec{Y}(z)\big)\,;$$

that is, $\vec{Y}(z)$ also satisfies the Initial Value Problem for all $z \in \left[z_1, z_2\right]$.

Remembering that

$$\left\|\vec{Y}(z_1) - \vec{X}(z_1)\right\| = \left\|\vec{W} - \vec{X}(z_1)\right\| < \delta < \epsilon_1$$

and $\mathrm{B}\big(\vec{X}(z_1), \epsilon_1\big) \subset \mathcal{U}_1$, so that $\vec{Y}(z_1) \in \mathcal{U}_1$. Taking limits in statement (a), we easily obtain

$$\left\|\vec{Y}(z) - \vec{X}(z)\right\| \leq \left\|\vec{W} - \vec{X}(z_1)\right\| \exp\left(\lambda(z - z_1)\right)\,,$$

and hence

$$\left\|\vec{Y}(z_2) - \vec{X}(z_2)\right\| \leq \left\|\vec{W} - \vec{X}(z_1)\right\| \exp\left(\lambda(z_2 - z)1)\right) < \delta \exp\left(\lambda(z_2 - z_1)\right) \leq \eta \leq \epsilon_1\,.$$

Remembering that $\mathrm{B}\big(\vec{X}(z_2), \epsilon_2\big) \subset \mathcal{U}_2$, we can see that $\vec{Y}(z_2) \in \mathcal{U}_2$. □

Let us now consider the following special case: If φ is a continuous function on Ω, then $\mathcal{U}_1 = \varphi^{-1}((-\infty, 0))$ and $\mathcal{U}_2 = \varphi^{-1}((0, \infty))$, we see that, if there is a solution $\vec{X}$ of the Initial Value Problem on the interval $\big[z_1, z_2\big]$, such that

$$\varphi\left(\vec{X}(z_1)\right) < 0 \quad \text{and} \quad \varphi\left(\vec{X}(z_2)\right) > 0\,,$$

then, there always exists some $\delta > 0$ such that, for all $\vec{W} \in \mathrm{B}\left(\vec{X}(z_1), \delta\right)$, there is a solution to the Initial Value Problem

$$\begin{aligned} &\frac{\partial}{\partial z}\vec{Y}(z) = \boldsymbol{F}\left(\vec{Y}(z)\right) \\ &\vec{Y}(z_1) = \vec{W} \end{aligned}$$

on the interval $\big[z_1, z_2\big]$, such that

$$\varphi\left(\vec{Y}(z_1)\right) < 0 \quad \text{and} \quad \varphi\left(\vec{Y}(z_2)\right) > 0\,,$$

Of course, the Intermediate Value Theorem shows that

$$\varphi\left(\vec{X}(z)\right) = 0 \quad \text{and} \quad \varphi\left(\vec{Y}(z')\right) = 0$$

for some $z\,, z' \in \big[z_1, z_2\big]$.

CHAPTER 5

Pseudo-Vacuum Solution: The Einstein-Klein-Gordon System

1. Introduction

This chapter is concerned with the first extension of the previous result. We consider the case of free scalar fields as source for the Einstein equations; that is, we consider the Einstein-Klein-Gordon system. The latter is composed by the Einstein equations, where the stress-energy-momentum tensor is derived from the kinetic term and the potential of a scalar field, and the Klein Gordon equation for an arbitrary potential. The fundamental assumption here is that the scalar fields (being free) propagate along the geodesics of the space-time, and thus they inherit the symmetries (*i.e.*, the homotheties) of the space-time; as a result, the Klein-Gordon equation is also reduced to an ordinary differential equation in the same manner as the Einstein equations. This means that the Picard-Lindelöf theorem still holds and existence and uniqueness of solutions can be proved locally simply by extending the argument from the previous chapter.

In the first part, we deal with

- providing an exact form for the stress-energy-momentum tensor in the case of scalar fields; and
- expressing the Klein-Gordon equation in the foliation that the Einstein equations are expressed.

Then, we discuss the existence and uniqueness of solutions of the system.

2. The Einstein-Klein-Gordon System

The Einstein-Klein-Gordon system is composed of the Einstein tensor equal to the stress-energy-momentum tensor for a scalar field, Φ:

$$\boldsymbol{G} = 8\pi \boldsymbol{T}(\Phi)\,, \tag{329}$$

where the stress-energy-momentum tensor is written as

$$\boldsymbol{T}(\vec{x}_\alpha, \vec{x}_\beta) = \nabla_{\vec{x}_\alpha}\Phi \nabla_{\vec{x}_\beta}\Phi - \frac{1}{2}\boldsymbol{g}(\vec{x}_\alpha, \vec{x}_\beta)\left(\boldsymbol{g}\left(\nabla_{\vec{x}_\mu}\Phi, \nabla_{\vec{x}_\mu}\Phi\right) - V(\Phi)\right)\,, \tag{330}$$

where $\otimes$ denotes the tensor multiplication (Kronecker product). This system refers to space-times whose sole energy and/or momentum is produced by the presence of some scalar field. This field is of unspecified physical meaning, as it does not seek to describe a well-defined classical force (*e.g.*, electromagnetism) or well-defined macroscopic matter (*e.g.*, a fluid); however, it is often used to mimic the low-order effects of a modified, even quantum theory of gravity,[1] or the low-order effects of an 'exotic' particle that may be in abundance in the early or the late Universe (*e.g.*, the particles or fields that caused inflation, or the particles or fields that cause the accelerated expansion).[2] It often leads to unexpected results, such as the violation of the energy conditions. Despite this limited and rather controversial use, we shall explore the case for the sake of completeness.

2.1. The scalar fields. First of all, let us assume that the stress-energy-momentum tensor in the form

$$\boldsymbol{T}\big(\vec{x},\vec{y}\big) = \nabla_{\vec{x}}\Phi, \nabla_{\vec{y}}\Phi + \boldsymbol{g}\big(\vec{x},\vec{y}\big)\Big(g^{-1}\big(\nabla_{\vec{u}}\Phi, \nabla_{\vec{u}}\Phi\big) - V(\Phi)\Big)\,, \tag{331}$$

is projected along the group and the quotient in the same manner that the metric and the curvature tensors are. Thus, we define

(332)

$$\begin{aligned}
T_{zz} &= \boldsymbol{T}\big(\vec{\zeta},\vec{\zeta}\big) = \big(\nabla_{\vec{\zeta}}\Phi\big)^2 \\
T_{za} &= \boldsymbol{T}\big(\vec{\zeta},\vec{\xi}_a\big) \\
&= \nabla_{\vec{\zeta}}\Phi\nabla_{\vec{\xi}_a}\Phi - \frac{1}{2}\beta_a\left(\mathrm{a}\big(\nabla_{\vec{\zeta}}\Phi\big)^2 + 2b^i\nabla_{\vec{\zeta}}\Phi\nabla_{\vec{\xi}_i}\Phi + c^{ij}\nabla_{\vec{\xi}_i}\Phi\nabla_{\vec{\xi}_j}\Phi - V(\Phi)\right) \\
T_{ab} &= \boldsymbol{T}\big(\vec{\xi}_a,\vec{\xi}_b\big) \\
&= \nabla_{\vec{\xi}_a}\Phi\nabla_{\vec{\xi}_b}\Phi + \frac{1}{2}\gamma_{ab}\left(\mathrm{a}\big(\nabla_{\vec{\zeta}}\Phi\big)^2 + 2b^i\nabla_{\vec{\zeta}}\Phi\nabla_{\vec{\xi}_i}\Phi + c^{ij}\nabla_{\vec{\xi}_i}\Phi\nabla_{\vec{\xi}_j}\Phi - V(\Phi)\right)
\end{aligned}$$

We also know that the stress-energy-momentum tensor inherits the symmetries of the space-time. In our case, its Lie derivative along a homothety vanishes,

$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{T} = 0\,.$$

Applying this to the definition of eq. (331), we can prove that scalar fields also inherit the symmetries of the space-time.

THEOREM 2.1 (Symmetry Inheritance of Scalar Fields). *Let the stress-energy-momentum tensor of a scalar field, as in eq. (331); and let a space-time $\big(\mathcal{V}_4,\boldsymbol{g}\big)$ where a Bianchi group $\mathcal{G}$ acts freely and regularly by means of homotheties. Then, the kinetic term and the potential of the scalar field are propagated along the scalar fields as*

$$\mathcal{L}_{\vec{\xi}_a}\big(\nabla_{\vec{x}}\Phi\big) = 0 \tag{333}$$

and

$$\mathcal{L}_{\vec{\xi}_a}V(\Phi) = -\phi_a V(\Phi)\,. \tag{334}$$

[1] The scalar field can also be used to describe the low-order effects of string or brane theories.

[2] The term pseudo-vacuum solutions used here, usually refers to the Einstein-cosmological constant system, where the Einstein tensor is proportional to the metric by some constant Λ. This can be seen as a limit case of this particular type, where the scalar field is constant, or reaches a constant value in the late Universe.

PROOF. Taking the Lie derivative of eq. (331), we have

$$
\begin{aligned}
&\mathcal{L}_{\vec{\xi}_a}\boldsymbol{T} = 0\\
&\quad\Rightarrow 2\mathrm{grad}(\Phi)\mathcal{L}_{\vec{\xi}_a}\mathrm{grad}(\Phi) + \phi_a\boldsymbol{g}\left(g^{-1}\big(\mathrm{grad}(\Phi),\mathrm{grad}(\Phi)\big) - V(\Phi)\right)\\
&\qquad\qquad + \boldsymbol{g}\left(-\phi_a g^{-1}\big(\mathrm{grad}(\Phi),\mathrm{grad}(\Phi)\big) - \mathcal{L}_{\vec{\xi}_a}V(\Phi)\right) = 0\\
&\quad\Rightarrow 2\mathrm{grad}(\Phi)\mathcal{L}_{\vec{\xi}_a}\mathrm{grad}(\Phi) + \boldsymbol{g}\big(-\mathcal{L}_{\vec{\xi}_a}V(\Phi) - \phi_a V(\Phi)\big) = 0\,.
\end{aligned}
$$

Since the total expression is equal to 0, each part should also be:

$$2\mathrm{grad}(\Phi)\mathcal{L}_{\vec{\xi}_a}\mathrm{grad}(\Phi) = 0 \quad\Rightarrow\quad \mathcal{L}_{\vec{\xi}_a}\mathrm{grad}(\Phi) = 0$$

and

$$\mathcal{L}_{\vec{\xi}_a}V(\Phi) = -\phi_a V(\Phi)\,.$$

□

The theorem leads to a further restriction in the propagation of the scalar field along the orbits of the group.

COROLLARY 2.1.1. *Given the scalar field inherits the homotheties of the space-time* $(\mathcal{V}_4, \boldsymbol{g})$ *as described by theorem 2.1, then*

$$\mathcal{L}_{\vec{\xi}_a}\Phi = \Phi_a\,, \tag{335}$$

where $\Phi_a \in \mathbb{R}$.

The results of these is that the scalar field and its kinetic term are invariant under the action of the group; the depend merely on z (the parameter along the quotient). Moreover, a further restriction exists for the potential.

COROLLARY 2.1.2. *Given the scalar field inherits the homotheties of the space-time* $(\mathcal{V}_4, \boldsymbol{g})$ *as described by theorem 2.1, then*

$$V(\Phi) = V_0 \exp\left(-\phi_a\frac{\Phi}{\Phi_a}\right)\,. \tag{336}$$

Interestingly, if we choose $\Phi_a = 0$, without any loss of generality, then $V(\Phi) = 0$. This result is not paradoxical; given a free scalar field (one that inherits the symmetries of the space-time), the evolution along the orbits of the group should be minimal. If the action of the group was by isometries, whereas $\mathcal{L}_{\vec{\xi}_a}\Phi = 0$, the potential would vanish identically and the scalar field would be constant. The reason some evolution of the scalar field, even minimal, is possible is due to the fact that the group acts by homotheties.

Of course, these results imply that the stress-energy-momentum tensor changes to

$$
\begin{aligned}
T_{zz} &= \Psi^2\\
T_{za} &= -\frac{1}{2}\mathrm{a}\beta_a\Psi^2 + \left(\Phi_a - \Phi_i b^i\beta_a\right)\Psi - \frac{1}{2}\Phi_i\Phi_j c^{ij}\beta_a + \frac{1}{2}\beta_a V(\Phi)\\
T_{ab} &= -\frac{1}{2}\mathrm{a}\gamma_{ab}\Psi^2 - \Phi_i b^i\gamma_{ab}\Psi + \left(\Phi_a\Phi_b - \frac{1}{2}\Phi_i\Phi_j c^{ij}\gamma_{ab}\right) + \frac{1}{2}\gamma_{ab}V(\Phi)\,,
\end{aligned}
\tag{337}
$$

where $\Psi = \nabla_{\bar{\zeta}}\Phi$. It is also worth noting that such a potential (the exponential one) has been used to describe the early Universe and the incidence of inflation - or the late Universe and the accelerated expansion; however, in these uses, the energy conditions are often violated.

2.2. The evolution and constraint equations. The Einstein-Klein-Gordon system is an extension of the Einstein system we considered in the previous chapter; it consists of the Einstein equations and the Klein-Gordon equations.

The former are derived in the same manner that they were derived in Chapter 4, with the exception that the right-hand side of eq. (185), (186) and (187) is not zero, but equal to $8\pi T_{zz}$, $8\pi T_{za}$ and $8\pi T_{ab}$ respectively. Therefore, the evolution equations for the metric and its inverse are

(338) $$\frac{\partial \beta_a}{\partial z} = 0\,,$$

(339) $$\frac{\partial \gamma_{ab}}{\partial z} = k_{ab}\,,$$

(340) $$\frac{\partial \mathrm{a}}{\partial z} = -b^i b^j k_{ij}\,,$$

(341) $$\frac{\partial b^a}{\partial z} = -c^{ai} b^j k_{ij} \quad \text{and}$$

(342) $$\frac{\partial c^{ab}}{\partial z} = -c^{ai} c^{bj} k_{ij}\,.$$

And the evolution equation for the derivatives, $\boldsymbol{k}$, is

(343)

$$
\begin{aligned}
2\mathrm{a}\frac{\partial k_{ab}}{\partial z} =& -\big(\mathrm{a}c^{ij} - b^i b^j\big)k_{ij}k_{ab} + \frac{1}{2}\big(\mathrm{a}c^{ik} - b^i b^k\big)c^{jl}\gamma_{ab}k_{ij}k_{kl} + \frac{3}{4}\mathrm{a}c^{ij}c^{kl}\gamma_{ab}k_{ij}k_{kl} \\
& - 3\phi_i b^i k_{ab} + b^i\big(\phi_a k_{bi} + \phi_b k_{ai}\big) + \frac{1}{4}\phi_k\big(4b^i c^{jk} + 3c^{ij}b^k\big)\gamma_{ab}k_{ij} \\
& - 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an} + k_{ab}C^m{}_{mn}\big) \\
& - 2\big(b^i c^{jn} - c^{ij}b^n\big)k_{ij}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an}\big) \\
& - \frac{1}{2}\big(3b^i c^{jn} - 5c^{ij}b^n\big)\gamma_{ab}k_{ij}C^m{}_{mn} + \frac{1}{2}c^{ik}b^l\gamma_{ab}k_{im}C^m{}_{kl} \\
& + \frac{1}{4}\phi_i\phi_j\big(11c^{ij}\gamma_{ab} - 4b^i b^j\beta_a\beta_b\big) + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi_b \\
& + \frac{5}{2}\phi_i c^{ik}b^l\gamma_{ab}\beta_m C^m{}_{kl} + 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{bm} + \beta_b\gamma_{bm}\big)C^m{}_{kl} \\
& - 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} - 2\gamma_{ab}C^m{}_{mn}\big) \\
& + c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} + \frac{3}{2}c^{ik}c^{jl}\big(\gamma_{mn} + \mathrm{a}\beta_m\beta_n\big)\gamma_{ab}C^m{}_{ij}C^n{}_{kl} \\
& - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{bk}\big)C^n{}_{nl} - 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{bk}\big)C^m{}_{nl} \\
& + \frac{7}{4}c^{kl}\gamma_{ab}C^m{}_{nk}C^n{}_{mk} - \frac{5}{4}c^{kl}\gamma_{ab}C^m{}_{mk}C^n{}_{nl} - 4c^{kl}\gamma_{mn}C^m{}_{ak}C^n{}_{bl} \\
& - b^l\beta_m C^m{}_{an}C^n{}_{bl} - b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{an}\big)C^n{}_{ml} \\
& - 3C^m{}_{an}C^n{}_{bm} - C^m{}_{ma}C^n{}_{nb} \\
& + 4\pi\mathrm{a}\gamma_{ab}\Psi^2 + 8\pi\Phi_a\Phi_b\,,
\end{aligned}
$$

And the constraint becomes

(344)

$$
\begin{aligned}
&\big(\mathrm{a}c^{ik} - b^i b^k\big)c^{jl}k_{ij}k_{kl} + \frac{1}{2}\mathrm{a}c^{ij}c^{kl}k_{ij}k_{kl} + \frac{7}{2}\phi_k c^{ij}b^k k_{ij} \\
&\quad - 2c^{ik}b^j b^l\beta_m k_{ij}C^m{}_{kl} + c^{ik}b^l k_{im}C^m{}_{kl} - \big(c^{ij}b^n - 2b^i c^{jn}\big)k_{ij}C^m{}_{mn} \\
&\quad - \frac{5}{2}\phi_i\phi_j c^{ij} - 7\phi_i c^{ik}b^l\beta_m C^m{}_{kl} \\
&\quad + \big(\mathrm{a}c^{ik} - 2b^i b^k\big)c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl} + \frac{7}{2}c^{kl}C^m{}_{nk}C^n{}_{ml} + \frac{5}{2}c^{kl}C^m{}_{mk}C^n{}_{nl} \\
&\quad + 6\pi\mathrm{a}\Psi^2 + 4\pi\Psi\Phi_i b^i\,,
\end{aligned}
$$

Interestingly, choosing $\Phi_a = 0$ and $\Psi = 0$, the terms related to the scalar field vanish. This is not surprising though. A constant scalar field does not interact with the space-time.

The Klein-Gordon equation is derived by varying the action with respect to the scalar field (or, more appropriately, its derivative). In the absence of a potential and given the scalar field depends only on z, the Klein-Gordon takes the form

$$\mathrm{a}\frac{\partial}{\partial z}\Psi - \frac{1}{2}\big(\mathrm{a}c^{ij} + 2b^i b^j\big)k_{ij}\Psi - \frac{\partial V}{\partial \Phi} = 0\,, \tag{345}$$

and

$$\frac{\partial}{\partial z}\Phi = \Psi\,. \tag{346}$$

2.3. The case of null orbits. Let us go consider now the special case where $a = 0$ and the 3−metric on the homogeneous submanifold $\mathcal{M}$ is degenerate. Firstly, we must consider what happens to the components of the metric and their derivatives, given both γ_{ab} and c^{ab} are singular.

The first evolution equations come from eqs. (84), (85) and (88), (90), (92) and refer to the derivatives of the components of the metric and its inverse:

$$\frac{\partial \beta_a}{\partial z} = 0\,, \tag{347}$$

$$\frac{\partial \gamma_{ab}}{\partial z} = k_{ab}\,, \tag{348}$$

$$\frac{\partial b^a}{\partial z} = 0 \quad \text{and} \tag{349}$$

$$\frac{\partial c^{ab}}{\partial z} = -c^{ai}c^{bj}k_{ij}\,. \tag{350}$$

Next, we need evolution equations (and constraints) for the components of $\boldsymbol{k}$. Following the same idea, we define

$$\kappa = c^{ij}k_{ij} \quad \text{and} \quad K^a{}_b = c^{ai}k_{ib}\,.$$

we can then produce evolution equations and constraints for the derivatives of the metric.

First of all, given $R_{zz} = 8\pi T_{zz}$, we have the first evolution equation

$$\frac{\partial \kappa}{\partial z} = K^i{}_j K^j{}_i + 2c^{ik}c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl} + 8\pi\Psi^2\,, \tag{351}$$

Then, rom eq. (215), we obtain the second evolution equation

$$\begin{aligned}\frac{\partial K^a{}_b}{\partial z} =& K^a{}_i K^i{}_b + 2\phi_j c^{ai}\beta_b K_i{}^j + 4c^{ak}\beta_m K^l{}_a C^m{}_{kl} \\ &- 2\phi_i c^{ak}c^{il}\beta_b\beta_m C^m{}_{kl} + 2c^{ak}c^{il}\beta_m\beta_n C^m{}_{bi}C^n{}_{kl} \\ &+ c^{ai}W_{izzb} - 4\pi\big(\delta^a{}_b - b^a\beta_b\big)\Psi^2 - 4\pi c^{ai}\beta_b\Phi_i\Psi\,.\end{aligned} \tag{352}$$

From $G_{za} = 8\pi T_{za}$, we obtain the first constraint,

(353)

$$\begin{aligned}
&3\phi_i K^i{}_a + \frac{7}{2}\phi_i b^i \beta_a \kappa \\
&\quad - 2b^k \beta_m K^l{}_a C^m{}_{kl} - \frac{3}{2} b^k \beta_a K^l{}_m C^m{}_{kl} - \frac{5}{2} b^n \beta_a \kappa C^m{}_{mn} + K^n{}_m C^m{}_{an} - \kappa C^m{}_{ma} \\
&\quad + 4\phi_i \phi_j c^{ij} \beta_a + \frac{5}{2}\phi_i c^{ik} b^l \beta_a \beta_m C^m{}_{kl} - 3\phi_i c^{in} \beta_m C^m{}_{an} - 2\phi_i c^{in} \beta_a C^m{}_{mn} \\
&\quad + 2c^{jk} b^l \beta_m \beta_n C^m{}_{aj} C^n{}_{kl} - \frac{3}{2} c^{jk} b^l \beta_a \beta_m C^m{}_{nj} C^n{}_{kl} \\
&\quad - \frac{1}{2} c^{ik} c^{jl} \beta_a \gamma_{mn} C^m{}_{ij} C^n{}_{kl} \\
&\quad - 7c^{kl} \beta_m C^m{}_{nk} C^n{}_{al} - \frac{7}{2} c^{kl} \beta_a C^m{}_{nk} C^n{}_{ml} - \frac{5}{2} c^{kl} \beta_a C^m{}_{mk} C^n{}_{nl} = \\
&\qquad = 8\pi \Psi \Phi_a - 4\pi \beta_a \big(\mathrm{a}\Psi^2 + 2b^i \Phi_i \Psi + c^{ij} \Phi_i \Phi_j - V(\Phi)\big)\,,
\end{aligned}$$

and, from $G_{ab} = 8\pi T_{ab}$, the second one

(354)

$$\begin{aligned}
&-\frac{11}{2}\phi_i b^i \gamma_{ab} \kappa - 3\phi_i b^i \gamma_{ai} K^i{}_b \\
&\quad - 2b^n K^i{}_m \big(\gamma_{ai} C^m{}_{bn} + \gamma_{bi} C^m{}_{an}\big) - 2b^n \gamma_{ai} K^i{}_b C^m{}_{mn}\big) \\
&\quad + 2b^n \kappa \big(\gamma_{am} C^m{}_{bn} + \gamma_{bm} C^m{}_{an} + \gamma_{ab} C^m{}_{mn}\big) \\
&\quad + \frac{1}{2}\gamma_{ab}\big(-3b^k K^l{}_m C^m{}_{kl} + b^n \kappa C^m{}_{mn}\big) \\
&\quad + \phi_i \phi_j b^i b^j \beta_a \beta_b + 2\phi_i b^i \big(\phi_a \beta_b + \phi_b \beta_a\big) + 3\phi_a \phi_b \\
&\quad + 2\phi_i c^{ik} b^l \big(\beta_a \gamma_{am} + \beta_b \gamma_{am}\big) C^m{}_{kl} + \frac{5}{2}\phi_i c^{ik} b^l \gamma_{ab} \beta_m C^m{}_{kl} \\
&\quad - 2\phi_i c^{in} \big(\gamma_{am} C^m{}_{bn} + \gamma_{bm} C^m{}_{an} + \gamma_{ab} C^m{}_{mn}\big) \\
&\quad + c^{ik} c^{jl} \gamma_{am} \gamma_{bn} C^m{}_{ij} C^n{}_{kl} + \frac{1}{2} c^{ik} c^{jl} \gamma_{ab} \gamma_{mn} C^m{}_{ij} C^n{}_{kl} \\
&\quad - 2c^{jk} b^l \beta_n \big(\gamma_{am} C^n{}_{bj} + \gamma_{bm} C^n{}_{aj}\big) C^m{}_{kl} + \frac{3}{2} c^{jk} b^l \gamma_{ab} \beta_m C^m{}_{nj} C^n{}_{kl} \\
&\quad - 2c^{kl} \big(\gamma_{am} C^n{}_{bk} + \gamma_{bm} C^n{}_{ak}\big) C^m{}_{nl} - 2c^{kl} \big(\gamma_{am} C^m{}_{bk} + \gamma_{bm} C^m{}_{ak}\big) C^n{}_{nl} \\
&\quad - 2c^{kl} \gamma_{mn} C^m{}_{ak} C^n{}_{bl} + \frac{7}{2} c^{kl} \gamma_{ab} C^m{}_{nk} C^n{}_{ml} + \frac{5}{2} c^{kl} \gamma_{ab} C^m{}_{mk} C^n{}_{nl} \\
&\quad - b^l \beta_m C^m{}_{an} C^n{}_{bl} + b^l \big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{na}\big) - 3C^m{}_{an} C^n{}_{bm} - C^m{}_{ma} C^n{}_{nb} = \\
&\qquad = 8\pi \Phi_a \Phi_b - 4\pi \gamma_{ab} \big(\mathrm{a}\Psi^2 + 2b^i \Phi_i \Psi + c^{ij} \Phi_i \Phi_j - V(\Phi)\big)\,.
\end{aligned}$$

As for the evolution of the Weyl tensor, using the components of the stress-energy-momentum tensor, we have

$$\frac{\partial W_{azzb}}{\partial z} = 8\pi \gamma_{ab} \Psi \frac{\partial \Psi}{\partial z} - 4\pi \Phi_a \frac{\partial \Psi}{\partial z} + \pi \beta_a \beta_b \Psi \frac{\partial V}{\partial \Phi}\,. \tag{355}$$

Finally, for the evolution of the scalar field, the equations become

$$\frac{\partial V}{\partial \Phi} = 0\,, \tag{356}$$

and

$$\frac{\partial}{\partial z}\Phi = \Psi\,. \tag{357}$$

These equations mean that the potential is constant, even zero without loss of generality; hence, the scalar field may evolve according to

$$\frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{2}\kappa\frac{\partial \Phi}{\partial z} = 0\,, \tag{358}$$

which solves to

$$\Phi = \Phi_0(w_i) + \Phi_1(w_i)\int_1^z \exp\left(\frac{1}{2}\int_1^\chi \kappa \mathrm{d}\chi\right)\mathrm{d}\chi\,, \tag{359}$$

where $\Phi_0\,, \Phi_1$ depend on the initial conditions; and χ an auxiliary variable for the integration. If the proper initial conditions are chosen (such that $\Phi_1 = 0$), or if the intrinsic curvature is trace-free ($\kappa = 0$), then the scalar field will be constant with respect to z.

2.4. A fixed point argument for a general local solution. The extension of the previous chapter's results on local existence and uniqueness is immediate. The Einstein-Klein-Gordon system of eqs. (338), (339), (340), (341), (342), (343), (345) and (346) can be written in the form of the Initial Value Problem

$$\begin{aligned} &\frac{\partial}{\partial z}\vec{X}(z) = \boldsymbol{F}\big(\vec{X}(z)\big) \\ &\vec{X}(z_0) = \vec{X}_0 \end{aligned} \tag{360}$$

for z in some interval $\big[z_1, z_2\big]$; where $\vec{X} = \big\{\mathrm{a}, b^a, c^{ab}, \gamma_{ab}, k_{ab}, \Psi, \Phi\big\}$ and $\boldsymbol{F}$ the right-hand side of the corresponding equations (which is continuously differentiable in the interval); and assuming that the initial conditions satisfy the constraint of eq. (344).

Then, the Picard-Lindelöf theorem holds and the Einstein-Klein-Gordon system has a unique solution $\bar{X}$ in the interval $\big[z_0 - h, z_0 + h\big]$, which can be extended to the entire $\big[z_1, z_2\big]$ (but not beyond than that).[3] Given the choice $\Phi_a = 0$ and $V(\Phi) = 0$, this solution will be the same for $\boldsymbol{g}$ and $\boldsymbol{k}$ as in Chapter 4, as they will not be affected by the evolution of the scalar field; as for the evolution of the scalar field it will be determined immediately by integrating

$$\mathrm{a}\frac{\partial^2}{\partial z^2}\Phi = \frac{1}{2}\big(\mathrm{a}c^{ij} + 2b^i b^j\big)k_{ij}\frac{\partial}{\partial z}\Phi\,.$$

[3]Here $\big[z_1, z_2\big]$, as in theorem 2.1, refers to the interval in which we know that the Initial Value Problem is well-defined, *i.e.*, $\boldsymbol{F}$ meets the criteria of the theorem (namely, Lipschitz continuity).

3. Examples

An example of a solution of the Einstein-Klein-Gordon can be found in Carot and Coligne [**52**], that we will briefly discuss here. Both examples belong in the class of self-similar Wainwright solutions of General Relativity.

3.1. Bianchi V and VI_0 acting on self-similar Wainwright $B-ii$ space-times. Let the metric of a scalar field space-time

$$\mathrm{d}s^2 = f^2 t^{-\frac{16}{7}} \left[-2x_1^{\frac{1}{2}} \mathrm{d}t^2 + 2x_1^{\frac{1}{2}} \mathrm{d}x_1^2 + x_1 \big(\mathrm{d}x_2^2 + \mathrm{d}x_3^2\big) \right] , \tag{361}$$

where f a real constant; notably, this metric retains the correct signature only for $x_1 , t > 0$. The space-time in question admits three Killing fields: two translations along the directions of x_1 and x_2,

$$\vec{\xi}_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad \vec{\xi}_2 = \frac{\partial}{\partial x_2} \tag{362}$$

and a rotation on the $x_1 - x_2$ plane,

$$\vec{\xi}_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \, . \tag{363}$$

These vectors are the generators of a Bianchi VII_0 group:

$$\begin{aligned} \big[\vec{\xi}_1, \vec{\xi}_2\big] &= 0 \, , \\ \big[\vec{\xi}_2, \vec{\xi}_3\big] &= \vec{\xi}_1 \quad \text{and} \\ \big[\vec{\xi}_3, \vec{\xi}_1\big] &= \vec{\xi}_2 \, . \end{aligned} \tag{364}$$

However, this group acts on the space-time in an inappropriate manner; the orbits of the group are two-dimensional submanifolds. So, instead of the rotation, we will choose a Lorentzian boost along x_1 as the third vector:

$$\vec{\xi}_3 = 2t \frac{\partial}{\partial t} + 2x_1 \frac{\partial}{\partial x_1} + \frac{3}{2} x_2 \frac{\partial}{\partial x_2} + \frac{3}{2} x_3 \frac{\partial}{\partial x_3} \, , \tag{365}$$

which is a homothetic vector of the space-time, such that

$$\mathcal{L}_{\vec{\xi}_3} \boldsymbol{g} = -\frac{16}{7} \boldsymbol{g} \, . \tag{366}$$

Moreover, the group now is a Bianchi V:

$$\begin{aligned} \big[\vec{\xi}_1, \vec{\xi}_2\big] &= 0 \, , \\ \big[\vec{\xi}_2, \vec{\xi}_3\big] &= \frac{3}{2} \vec{\xi}_2 \quad \text{and} \\ \big[\vec{\xi}_3, \vec{\xi}_1\big] &= -\frac{3}{2} \vec{\xi}_1 \, . \end{aligned} \tag{367}$$

This group acts freely and regularly on the space-time, allowing only for three-dimensional orbits, and admitting at least one homothetic vector. However, it is difficult to find the appropriate collineation on the quotient that would complete the foliation. For example, let us start by attempting a general form

$$\vec{\zeta} = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x_1} + C\frac{\partial}{\partial x_2} + D\frac{\partial}{\partial x_3} \,,$$

and imposing the conditions required: by demanding that $\vec{\zeta}$ commutes with $\vec{\xi}_1$ and $\vec{\xi}_2$, the coefficients A, B, C and D are independent of x_1 and x_2 - and demanding that it commutes with $\vec{\xi}_3$, we have

$$\begin{aligned} t\frac{\partial A}{\partial t} - A = 0 \quad &\Rightarrow \quad A = A_0 t \\ t\frac{\partial B}{\partial t} - B = 0 \quad &\Rightarrow \quad B = B_0 t \\ t\frac{\partial C}{\partial t} - C = 0 \quad &\Rightarrow \quad C = C_0 t \\ t\frac{\partial D}{\partial t} - D = 0 \quad &\Rightarrow \quad D = D_0 t \end{aligned}$$

Now, imposing geodesicity, we have

$$\begin{aligned} &\nabla_{\vec{\zeta}}\vec{\zeta} = A_0^2 t\frac{\partial}{\partial t} + A_0 B_0 t\frac{\partial}{\partial x_1} + A_0 C_0 t\frac{\partial}{\partial x_2} + A_0 D_0 t\frac{\partial}{\partial x_3} = 0 \\ &\Rightarrow \quad A_0 = 0\,. \end{aligned}$$

However, imposing nullity, we get

$$\boldsymbol{g}\big(\vec{\zeta},\vec{\zeta}\big) = 2x_1^{\frac{1}{2}} B_0^2 t^2 + x_1^2\big(C_0^2 + D_0^2\big)t^2 \,,$$

which is impossible, since B_0, C_0 and D_0 are independent of x_1. As a result, no null and geodesic collineation invariant to the action of the group exists in this case.

Let us now consider the scalar field space-time described by the metric

(368) $$\mathrm{d}s^2 = e^{2\alpha t}H^{2c_1}\big(-\mathrm{d}t^2 + \mathrm{d}x_1^2\big) + e^{2\beta t}H\big(e^{2\gamma t}H^{2c_2}\mathrm{d}x_2^2 + e^{-2\gamma t}H^{-2c_2}\mathrm{d}x_3^2\big)\,,$$

where α, β, γ, c_1 and c_2 are real constants; and $H(x_1)$ a function equal to either $\cosh a(x_1 - x_{1(0)})$, or $\cos a(x_1 - x_{1(0)})$, or $\ln a(x_1 - x_{1(0)})$. This space-time admits a homothetic vector,

(369) $$\vec{\xi}_1 = \frac{k}{2\alpha}\frac{\partial}{\partial t} + k\Big(1 - \frac{\beta + 2\gamma}{\alpha}\Big)x_2\frac{\partial}{\partial x_2} + k\Big(1 - \frac{\beta - 2\gamma}{\alpha}\Big)x_3\frac{\partial}{\partial x_3}\,,$$

such that

(370) $$\mathcal{L}_{\vec{\xi}_1}\boldsymbol{g} = \frac{k}{2}\boldsymbol{g}\,;$$

and two isometries,

(371) $$\vec{\xi}_2 = \frac{\partial}{\partial t} \quad \text{and} \quad \vec{\xi}_3 = \frac{\partial}{\partial x_3}\,.$$

The algebra of these vectors is that of the Bianchi VI_0 group.[4]

$$[\vec{\xi}_1,\vec{\xi}_2] = 0\,,\qquad [\vec{\xi}_2,\vec{\xi}_3] = 0 \quad \text{and} \qquad [\vec{\xi}_3,\vec{\xi}_1] = k\Big(1-\frac{\beta-2\gamma}{\alpha}\Big)\vec{\xi}_3\,. \tag{372}$$

Interestingly, this action is free and regular (the orbits of the group are three-dimensional submanfiolds) and a very simple vector can be defined on the quotient, that is null, geodesic and invariant under the action of the group - a peculiar 'translation' along the $t - x_1$ plane of the form

$$\vec{\zeta} = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x_1}\,. \tag{373}$$

Demanding that this vector commutes with $\vec{\xi}_2$, we obtain

$$\frac{\partial A}{\partial t} = \frac{\partial B}{\partial t} = 0\,;$$

and demanding that it also commutes with $\vec{\xi}_3$,

$$\frac{\partial A}{\partial x_3} = \frac{\partial B}{\partial x_3} = 0\,.$$

So, A and B are both independent of t and x_3. Finally, demanding that it commutes with $\vec{\xi}_1$, we obtain

$$\frac{\partial A}{\partial x_2} = \frac{\partial B}{\partial x_2} = 0\,.$$

Thus, A and B depend only on x_1.

Imposing the null condition on the proposed $\vec{\zeta}$, we get

$$A^2 = B^2\,,$$

which can be solved to either $A = B$ or to $A = -B$.

Finally, invoking geodesicity, we have,[5]

$$\begin{aligned}\nabla_{A\frac{\partial}{\partial t}+B\frac{\partial}{\partial x_1}}&\Big(A\frac{\partial}{\partial t}+B\frac{\partial}{\partial x_1}\Big) = \\ &= \Big(B\frac{\partial A}{\partial x_1} + \alpha(A^2+B^2) + 2c_1\frac{\partial_{x_1}H}{H}AB\Big)\frac{\partial}{\partial t} \\ &\quad + \Big(B\frac{\partial B}{\partial x_1} + c_1\frac{\partial_{x_1}H}{H}(A^2+B^2) + 2\alpha AB\Big)\frac{\partial}{\partial x_1} = 0\,,\end{aligned}$$

---

[4]It is easier to see if $k = 1 - \dfrac{\beta-2\gamma}{\alpha}$.

[5]It is useful to remind that the Christoffel symbols of interest for this metric as

$$\Gamma^t{}_{tt} = \Gamma^t{}_{x_1x_1} = \Gamma^{x_1}{}_{tx_1} = a \quad \text{and} \quad \Gamma^t{}_{tx_1} = \Gamma^{x_1}{}_{tt} = \Gamma^{x_1}{}_{x_1x_1} = c_1\frac{\partial_{x_1}H}{H}\,.$$

which means that

$$B\frac{\partial A}{\partial x_1} = -\alpha(A^2+B^2) - 2c_1\frac{\partial_{x_1}H}{H}AB$$
$$B\frac{\partial B}{\partial x_1} = -c_1\frac{\partial_{x_1}H}{H}(A^2+B^2) + 2\alpha AB$$

For $A = -B$, both equations can be true either if $A = B = 0$, or if $\frac{\partial_{x_1}H}{H} = \frac{c_1}{\alpha}$; however, the former implies there is no transversal (of this form, at least), while the latter cannot hold for any of the possible forms for $H(x_1)$. For $A = B$, both equations can be true and they solve to

$$A = B = c_3 e^{\alpha x_1 + c_1\int_1^{x_1}\frac{\partial_{\chi_1}H}{H}\mathrm{d}\chi_1},$$

for any integration constant c_3 (we can assume $c_3 = 1$, for simplicity) and χ_1 being an auxiliary variable for the integration.

Then, the transversal collineation can take the form

$$\vec{\zeta} = e^{\alpha x_1 + c_1\int_1^{x_1}\frac{\partial_{\chi_1}H}{H}\mathrm{d}\chi_1}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right). \tag{374}$$

Then, the metric will have the form

(375)

$$\begin{aligned}
g_{zz} &= \boldsymbol{g}(\vec{\zeta},\vec{\zeta}) = 0\\
g_{z1} &= \boldsymbol{g}(\vec{\zeta},\vec{\xi}_1) = \frac{kH^{2c_1}}{2\alpha}e^{2\alpha^2 t x_1 + 2\alpha c_1 t\int_1^{x_1}\frac{\partial_{\chi_1}H}{H}\mathrm{d}\chi_1}\\
g_{z2} &= \boldsymbol{g}(\vec{\zeta},\vec{\xi}_2) = H^{2c_1}e^{2\alpha^2 t x_1 + 2\alpha c_1 t\int_1^{x_1}\frac{\partial_{\chi_1}H}{H}\mathrm{d}\chi_1}\\
g_{z3} &= \boldsymbol{g}(\vec{\zeta},\vec{\xi}_3) = 0\\
g_{11} &= \boldsymbol{g}(\vec{\xi}_1,\vec{\xi}_1) = -\frac{k^2}{4\alpha^2}e^{2\alpha t}H^{2c_1} + x_2^2\Big(-1+\frac{\beta}{\alpha}+2\frac{\gamma}{\alpha}\Big)^2 H^{1+2c_2}e^{2(\beta+\gamma)t}\\
&\qquad + x_2^2\Big(1-\frac{\beta}{\alpha}+2\frac{\gamma}{\alpha}\Big)^2 H^{1-2c_2}e^{2(\beta-\gamma)t}\\
g_{12} &= \boldsymbol{g}(\vec{\xi}_1,\vec{\xi}_2) = -\frac{ke^{2\alpha t}H^{2c_1}}{2\alpha}\\
g_{13} &= \boldsymbol{g}(\vec{\xi}_1,\vec{\xi}_3) = \frac{k(\alpha-\beta+2\gamma)x_3H^{1-2c_2}e^{2(\beta-\gamma)t}}{\alpha}\\
g_{22} &= \boldsymbol{g}(\vec{\xi}_2,\vec{\xi}_2) = -e^{2\alpha t}H^{2c_1}\\
g_{23} &= \boldsymbol{g}(\vec{\xi}_2,\vec{\xi}_3) = \frac{k(\alpha-\beta+2\gamma)x_3H^{1-2c_2}e^{2(\beta-\gamma)t}}{\alpha}\\
g_{33} &= \boldsymbol{g}(\vec{\xi}_3,\vec{\xi}_3) = H^{1-2c_2}e^{2(\beta-\gamma)t}
\end{aligned}$$

4. The Existence of Neighbouring Solutions in the Einstein-Klein-Gordon System

The example examined resulted to the impossibility to construct the foliation on the space-time according to the Theorem 2.1, even if the Bianchi group was acting freely and regularly. This brings up the possibility that space-times that are solution to the Einstein-Klein-Gordon system and allow for a Bianchi group to act freely and regularly do not belong to the class we study. The question as of whether these space-times are unique or not is easy to answer by invoking the Theorem 4.1 proved in the previous chapter. No further amendment is needed, as the Eintein-Klein-Gordon system has been stated as a system of evolution equations in the same manner the Einsteins system was.

CHAPTER 6

Electro-Vacuum Solution: The Einstein-Maxwell System

1. Introduction

This chapter is concerned with another extension of the previous result. We consider the case of free electromagnetic fields as source for the Einstein equations; that is, we consider the Einstein-Maxwell system. The latter is composed by the Einstein equations, where the stress-energy-momentum tensor is derived from the Faraday electromagnetic stress tensor, and the Maxwell equations, whose sources (the electric charge and currency densities) are zero. The fundamental assumption here is that the electromagnetic fields (being free) propagate along the geodesics of the space-time, and thus they inherit the symmetries (*i.e.*, the homotheties) of the space-time; as a result, the Maxwell equations are also reduced to ordinary differential equations in the same manner as the Einstein equations. This means that the Picard-Lindelöf theorem still holds and existence and uniqueness of solutions can be proved locally simply by extending the argument from the previous chapter.

In the first part, we deal with

- providing an exact form for the stress-energy-momentum tensor in the case of electromagnetic fields; and
- expressing the Maxwell equations in the foliation that the Einstein equations are expressed.

Then, we discuss the existence and uniqueness of solutions of the system. In the second part, we present an example: the Bianchi III group acting on the Ehlers-Kundt *pp*-wave space-time, which has been used to describe plane electromagnetic or gravitational waves propagating freely in an (otherwise) vacuum space-time. This example presents a similar peculiarity to the case of the Bianchi I action in a Minkowski space-time presented in the previous chapter. Subsequently, we attempt to extend the result of the previous chapter (for vacuum space-times), that 'unknown' solutions exist near 'known' ones, that inherit the same paculiarities.

2. The Einstein-Maxwell System

The Einstein-Maxwell system is composed of the Einstein tensor equal to the stress-energy-momentum tensor for a electromagnetic field, $\boldsymbol{F}$:

$$\boldsymbol{G} = 8\pi \boldsymbol{T}(F)\,, \tag{376}$$

where the stress-energy-momentum tensor is written as

(377)
$$\boldsymbol{T} = g^{-1}\big(\mathbf{u},\mathbf{w}\big)\boldsymbol{F}\big(\vec{x},\vec{u}\big)\boldsymbol{F}\big(\vec{y},\vec{w}\big) - \frac{1}{4}\boldsymbol{g}\big(\vec{x},\vec{y}\big)g^{-1}\big(\mathbf{u},\mathbf{s}\big)g^{-1}\big(\mathbf{v},\mathbf{t}\big)\boldsymbol{F}\big(\vec{u},\vec{v}\big)\boldsymbol{F}\big(\vec{s},\vec{t}\big) \ .$$

This system refers to space-times whose sole energy and/or momentum is produced by the presence of free electromagnetic fields. This could be the propagation of electromagnetic waves (*i.e.* light) in an otherwise empty space-time (these space-times are usually described by the Ehlers-Kundt class of plane-wave solutions); but, it could also be the concentration of an electric field in a very small (almost infinitesimal) region, giving birth to a spherically symmetric asymptotically flat space-time, like the Reissner-Nordström solution. These cases can be more physical than the case of a scalar field; though they may lead to very similar results (from a mathematical perspective).

2.1. The electromagnetic fields. First of all, let us assume that the stress-energy-momentum tensor in the form of eq. (377) is projected along the group and the quotient in the same manner that the metric and the curvature tensors are. Thus, we define

(378)
$$\begin{aligned}
T_{zz} &= \boldsymbol{T}\big(\vec{\zeta},\vec{\zeta}\big) = c^{ij}F_{zi}F_{zj}\\
T_{za} &= \boldsymbol{T}\big(\vec{\zeta},\vec{\xi}_a\big) = b^iF_{za}F_{zi} + c^{ij}F_{zi}F_{aj}\\
&\qquad - \frac{1}{4}\beta_a\left(\mathrm{a}c^{ij}F_{zi}F_{zj} + b^ib^jF_{zi}F_{zj} + 2b^ic^{jk}F_{zk}F_{ij} + c^{ik}c^{jl}F_{ij}F_{kl}\right)\\
T_{ab} &= \boldsymbol{T}\big(\vec{\xi}_a,\vec{\xi}_b\big) = \mathrm{a}F_{za}F_{zb} - 2b^iF_{za}F_{ib} + c^{ij}F_{ai}F_{bj}\\
&\qquad - \frac{1}{4}\gamma_{ab}\left(\mathrm{a}c^{ij}F_{zi}F_{zj} + b^ib^jF_{zi}F_{zj} + 2b^ic^{jk}F_{zk}F_{ij} + c^{ik}c^{jl}F_{ij}F_{kl}\right)
\end{aligned}$$

We also know that the stress-energy-momentum tensor inherits the symmetries of the space-time. In our case, its Lie derivative along a homothety vanishes,

$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{T} = 0\,.$$

Applying this to the definition of eq. (377), we can prove that electromagnetic fields also inherit the symmetries of the space-time.

THEOREM 2.1 (Symmetry Inheritance of Free Electromagnetic Fields). *Let the stress-energy-momentum tensor of a free electromagnetic field, as in eq. (377); and let a space-time* $\big(\mathcal{V}_4,\boldsymbol{g}\big)$ *where a Bianchi group* $\mathcal{G}$ *acts freely and regularly by means of homotheties. Then, the Faraday tensor inherits the symmetry as*

(379)
$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{F} = \frac{\phi_a}{2}\boldsymbol{F}\,.$$

PROOF. Taking the Lie derivative of eq. (377), we have

$$
\begin{aligned}
&\mathcal{L}_{\vec{\xi}_a}\boldsymbol{T} = 0\\
\Rightarrow\quad & -\phi_a g^{-1}(\mathbf{u},\mathbf{w})\boldsymbol{F}(\vec{x},\vec{u})\boldsymbol{F}(\vec{y},\vec{u}) + 2g^{-1}(\mathbf{u},\mathbf{v})\boldsymbol{F}(\vec{x},\vec{u})\mathcal{L}_{\vec{\xi}_a}\boldsymbol{F}(\vec{y},\vec{w})\\
&\quad - \frac{1}{4}\phi_a\boldsymbol{g}(\vec{x},\vec{y})g^{-1}(\mathbf{u},\mathbf{s})g^{-1}(\mathbf{v},\mathbf{t})\boldsymbol{F}(\vec{u},\vec{v})\boldsymbol{F}(\vec{s},\vec{t})\\
&\quad + \frac{1}{2}\phi_a\boldsymbol{g}(\vec{x},\vec{y})g^{-1}(\mathbf{u},\mathbf{s})g^{-1}(\mathbf{v},\mathbf{t})\boldsymbol{F}(\vec{u},\vec{v})\boldsymbol{F}(\vec{s},\vec{t})\\
&\quad - \frac{1}{2}\boldsymbol{g}(\vec{x},\vec{y})g^{-1}(\mathbf{u},\mathbf{s})g^{-1}(\mathbf{v},\mathbf{t})\boldsymbol{F}(\vec{u},\vec{v})\mathcal{L}_{\vec{\xi}_a}\boldsymbol{F}(\vec{s},\vec{t}) = 0\\
\Rightarrow\quad & g^{-1}(\mathbf{u},\mathbf{w})\left(2\mathcal{L}_{\vec{\xi}_a}\boldsymbol{F}(\vec{y},\vec{w}) - \phi_a\boldsymbol{F}(\vec{y},\vec{w})\right)\boldsymbol{F}(\vec{x},\vec{u})\\
&\quad + \frac{1}{4}\boldsymbol{g}(\vec{x},\vec{y})g^{-1}(\mathbf{u},\mathbf{s})g^{-1}(\mathbf{v},\mathbf{t})\left(-2\mathcal{L}_{\vec{\xi}_a}\boldsymbol{F}(\vec{v},\vec{t}) - \phi_a\boldsymbol{F}(\vec{v},\vec{t})\right)\boldsymbol{F}(\vec{u},\vec{s}) = 0\,.
\end{aligned}
$$

Since the total expression is equal to 0, each part should also be; thus,

$$2\mathcal{L}_{\vec{\xi}_a}\boldsymbol{F} - \phi_a\boldsymbol{F} = 0 \quad\Rightarrow\quad \mathcal{L}_{\vec{\xi}_a}\boldsymbol{F} = \frac{\phi_a}{2}\boldsymbol{F}\,.$$

□

Thus, the electromagnetic fields are differentiating along the group in the same manner as the metric.

(380)
$$
\begin{aligned}
\mathcal{L}_{\vec{\xi}_b}F_{za} &= \frac{\phi_b}{2}F_{za} - F_{zm}C^m{}_{ab}\\
\mathcal{L}_{\vec{\xi}_c}F_{ab} &= \frac{\phi_c}{2}F_{ab} - F_{am}C^m{}_{bc} - F_{mb}C^m{}_{ac}\,.
\end{aligned}
$$

Also, raising the indices leads to similar differentiation relations,

(381)
$$
\begin{aligned}
\mathcal{L}_{\vec{\xi}_b}F^{za} &= -\frac{\phi_b}{2}F^{za} + F^{zn}C^a{}_{nb}\\
\mathcal{L}_{\vec{\xi}_c}F^{zb} &= -\frac{\phi_c}{2}F^{ab} + F^{an}C^b{}_{nc} + F^{nb}C^a{}_{nc}\,.
\end{aligned}
$$

2.2. The evolution and constraint equations. The Einstein-Maxwell system is an extension of the Einstein system we considered in the previous chapter; it consists of the Einstein equations and the source-free Maxwell equations.

The former are derived in the same manner that they were derived in Chapter 4, with the exception that the right-hand side of eq. (185), (186) and (187) is not zero, but equal to $8\pi T_{zz}$, $8\pi T_{za}$ and $8\pi T_{ab}$ respectively. Therefore, the evolution equations for the metric and its inverse are

(382)
$$\frac{\partial\beta_a}{\partial z} = 0\,,$$

(383)
$$\frac{\partial\gamma_{ab}}{\partial z} = k_{ab}\,,$$

$$\frac{\partial \mathrm{a}}{\partial z} = -b^i b^j k_{ij}\,, \tag{384}$$

$$\frac{\partial b^a}{\partial z} = -c^{ai} b^j k_{ij} \quad \text{and} \tag{385}$$

$$\frac{\partial c^{ab}}{\partial z} = -c^{ai} c^{bj} k_{ij}\,. \tag{386}$$

And the evolution equation for the derivatives, $\boldsymbol{k}$, is

(387)

$$\begin{aligned}
2\mathrm{a}\frac{\partial k_{ab}}{\partial z} =& -\big(\mathrm{a}c^{ij} - b^i b^j\big)k_{ij}k_{ab} + \frac{1}{2}\big(\mathrm{a}c^{ik} - b^i b^k\big)c^{jl}\gamma_{ab}k_{ij}k_{kl} + \frac{3}{4}\mathrm{a}c^{ij}c^{kl}\gamma_{ab}k_{ij}k_{kl} \\
& - 3\phi_i b^i k_{ab} + b^i\big(\phi_a k_{bi} + \phi_b k_{ai}\big) + \frac{1}{4}\phi_k\big(4b^i c^{jk} + 3c^{ij}b^k\big)\gamma_{ab}k_{ij} \\
& - 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an} + k_{ab}C^m{}_{mn}\big) \\
& - 2\big(b^i c^{jn} - c^{ij}b^n\big)k_{ij}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an}\big) \\
& - \frac{1}{2}\big(3b^i c^{jn} - 5c^{ij}b^n\big)\gamma_{ab}k_{ij}C^m{}_{mn} + \frac{1}{2}c^{ik}b^l\gamma_{ab}k_{im}C^m{}_{kl} \\
& + \frac{1}{4}\phi_i\phi_j\big(11c^{ij}\gamma_{ab} - 4b^i b^j\beta_a\beta_b\big) + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi)b \\
& + \frac{5}{2}\phi_i c^{ik}b^l\gamma_{ab}\beta_m C^m{}_{kl} + 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{bm} + \beta_b\gamma_{bm}\big)C^m{}_{kl} \\
& - 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} - 2\gamma_{ab}C^m{}_{mn}\big) \\
& + c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} + \frac{3}{2}c^{ik}c^{jl}\big(\gamma_{mn} + \mathrm{a}\beta_m\beta_n\big)\gamma_{ab}C^m{}_{ij}C^n{}_{kl} \\
& - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{bk}\big)C^n{}_{nl} - 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{bk}\big)C^m{}_{nl} \\
& + \frac{7}{4}c^{kl}\gamma_{ab}C^m{}_{nk}C^n{}_{mk} - \frac{5}{4}c^{kl}\gamma_{ab}C^m{}_{mk}C^n{}_{nl} - 4c^{kl}\gamma_{mn}C^m{}_{ak}C^n{}_{bl} \\
& - b^l\beta_m C^m{}_{an}C^n{}_{bl} - b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{an}\big)C^n{}_{ml} \\
& - 3C^m{}_{an}C^n{}_{bm} - C^m{}_{ma}C^n{}_{nb} \\
& + 24\pi\left(\beta_a\beta_b\gamma_{ij} - \gamma_{ab}\gamma_{ij} - \frac{1}{2}\gamma_{ab}\beta_i\beta_j\right)F^{zi}F^{zj} \\
& + 24\pi\big(\mathrm{a}F_{za}F_{zb} - 2b^i F_{za}F_{ib} + c^{ij}F_{ai}F_{bj}\big) \\
& - 12\pi\big(ac^{ij}F_{zi}F_{zj} + b^i b^j F_{zi}F_{zj} - b^i c^{jk}F_{zk}F_{ij}\big)\gamma_{ab}\,,
\end{aligned}$$

And the constraint becomes

(388)
$$\begin{aligned}&\big(\mathrm{a}c^{ik}-b^ib^k\big)c^{jl}k_{ij}k_{kl}+\frac{1}{2}\mathrm{a}c^{ij}c^{kl}k_{ij}k_{kl}+\frac{7}{2}\phi_kc^{ij}b^kk_{ij}\\&\quad-2c^{ik}b^jb^l\beta_mk_{ij}C^m{}_{kl}+c^{ik}b^lk_{im}C^m{}_{kl}-\big(c^{ij}b^n-2b^ic^{jn}\big)k_{ij}C^m{}_{mn}\\&\quad-\frac{5}{2}\phi_i\phi_jc^{ij}-7\phi_ic^{ik}b^l\beta_mC^m{}_{kl}\\&\quad+\big(\mathrm{a}c^{ik}-2b^ib^k\big)c^{jl}\beta_m\beta_nC^m{}_{ij}C^n{}_{kl}+\frac{7}{2}c^{kl}C^m{}_{nk}C^n{}_{ml}+\frac{5}{2}c^{kl}C^m{}_{mk}C^n{}_{nl}\\&\quad+4\pi\big(2\mathrm{a}c^{ij}F_{zi}F_{zj}-4b^ib^jF_{zi}F_{zj}+b^ic^{jk}F_{zk}F_{ij}-c^{ik}c^{jl}F_{ij}F_{kl}\big)\,.\end{aligned}$$

The Maxwell equations are derived through

$$\mathrm{d}\boldsymbol{F}=0$$
$$\star\mathrm{d}\big(\star\boldsymbol{F}\big)=0$$

that can be written in the more familiar form as

(389) $$g^{-1}\left(\nabla_{\vec{x}},\boldsymbol{F}\big(\vec{x},\vec{y}\big)\right)=0\quad\text{and}\quad\nabla_{\vec{x}}\boldsymbol{F}\big(\vec{y},\vec{u}\big)+\nabla_{\vec{y}}\boldsymbol{F}\big(\vec{u},\vec{x}\big)+\nabla_{\vec{u}}\boldsymbol{F}\big(\vec{x},\vec{y}\big)=0\,,$$

for arbitrary vector fields $\vec{x}$, $\vec{y}$ and $\vec{u}$. Specifying the latter to be either the generators ($\vec{\xi}_a$) or the transversal ($\vec{\zeta}$), we have the following relations:

$$\begin{aligned}&\mathcal{L}_{\vec{\zeta}}F^{za}=0\ ,\\&\mathcal{L}_{\vec{\xi}_i}F^{iz}=0\ ,\\&\mathcal{L}_{\vec{\xi}_i}F^{ia}=0\ ,\\&\mathcal{L}_{\vec{z}}F_{ab}+\mathcal{L}_{\vec{\xi}_a}F_{bz}+\mathcal{L}_{\vec{\xi}_b}F_{za}=0\quad\text{and}\\&\mathcal{L}_{\vec{\xi}_a}F_{bc}+\mathcal{L}_{\vec{\xi}_b}F_{ca}+\mathcal{L}_{\vec{\xi}_c}F_{ab}=0\,.\end{aligned}$$

The first three can be rewritten with respect to F_{za} and F_{ab} as

(390)
$$\begin{aligned}&\big(\mathrm{a}c^{ai}-b^ab^i\big)\frac{\partial}{\partial z}F_{zi}-c^{ai}b^j\frac{\partial}{\partial z}F_{ij}=\\&\quad=\big(\mathrm{a}c^{ai}-b^ab^i\big)c^{jk}k_{ij}F_{zk}+\big(c^{ak}b^j-b^ac^{jk}\big)b^ik_{ij}F_{zk}+\big(c^{ai}c^{jl}b^k+b^ic^{jk}c^{al}\big)k_{ij}F_{kl}\,,\end{aligned}$$

which contains the derivatives with respect to z - so, it can help in deriving an evolutionary equation;

(391) $$\frac{3}{2}\phi_i\big(\mathrm{a}c^{ij}F_{zi}+c^{ij}b^kF_{jk}\big)=\big(\mathrm{a}c^{jn}F_{zj}+b^ic^{jn}F_{ij}\big)C^m{}_{mn}\,,$$

which can function as a constraint for F_{za} and F_{ab}; and

(392) $$\frac{3}{2}\phi_i\big(b^ic^{ja}F_{zj}-c^{ai}b^jF_{zj}+c^{ij}c^{ka}F_{jk}\big)=c^{ik}b^lF_{zi}C^a{}_{kl}+c^{ik}c^{jl}F_{ij}C^a{}_{kl}\,,$$

which can also serve as a constraint.

Using the Lie derivatives of the Faraday tensor from eqs. (380), the first one reduces to

(393) $$\frac{\partial}{\partial z}F_{ab}=\frac{1}{2}\big(\phi_aF_{zb}-\phi_bF_{za}\big)-2F_{zm}C^m{}_{ab}\,,$$

which is the evolution equation for F_{ab}. What we need now is an evolution equation for F_{za}, which can be derived from eq. (390); substituting the derivative of F_{ab}, we get

$$\begin{aligned}\left(\mathrm{a}c^{ai} - b^a b^i\right)\frac{\partial}{\partial z}F_{zi} =& \frac{1}{2}c^{ai}b^j\left(\phi_i F_{zj} - \phi_j F_{zi}\right) + 2c^{ak}b^l F_{zm}C^m{}_{kl} \\ &+ \left(\mathrm{a}c^{ai} - b^a b^i\right)c^{jk}k_{ij}F_{zk} + \left(c^{ak}b^j - b^a c^{jk}\right)b^i k_{ij}F_{zk} \\ &+ \left(c^{ai}c^{jl}b^k + b^i c^{jk}c^{al}\right)k_{ij}F_{kl}\,,\end{aligned}$$

and substracting with γ_{ab}, we obtain

$$\begin{aligned}\mathrm{a}\frac{\partial}{\partial z}F_{za} =& \frac{1}{2}b^i\left(\phi_a F_{zi} - \phi_i F_{za}\right) - 2b^n F_{zm}C^m{}_{an} \\ &+ \left(\mathrm{a}c^{ij} - b^i b^j\right)k_{ja}F_{zi} + b^i b^j k_{ij}F_{za} \\ &- b^i c^{jk}\left(k_{ij}F_{ak} + k_{ak}F_{ij}\right) - \left(c^{ik}b^j b^l - b^i b^k c^{jl}\right)\beta_a k_{kl}F_{ij}\,,\end{aligned} \tag{394}$$

which serves as the second evolution equation.

2.3. The case of null orbits. Let us go consider now the special case where $a = 0$ and the 3−metric on the homogeneous submanifold $\mathcal{M}$ is degenerate. Firstly, we must consider what happens to the components of the metric and their derivatives, given both γ_{ab} and c^{ab} are singular.

The first evolution equations come from eqs. (84), (85) and (88), (90), (92) and refer to the derivatives of the components of the metric and its inverse:

$$\frac{\partial \beta_a}{\partial z} = 0\,, \tag{395}$$

$$\frac{\partial \gamma_{ab}}{\partial z} = k_{ab}\,, \tag{396}$$

$$\frac{\partial b^a}{\partial z} = 0 \quad \text{and} \tag{397}$$

$$\frac{\partial c^{ab}}{\partial z} = -c^{ai}c^{bj}k_{ij}\,. \tag{398}$$

Next, we need evolution equations (and constraints) for the components of $\boldsymbol{k}$. Following the same idea, we define

$$\kappa = c^{ij}k_{ij} \quad \text{and} \quad K^a{}_b = c^{ai}k_{ib}\,.$$

we can then produce evolution equations and constraints for the derivatives of the metric.

First of all, given $R_{zz} = 8\pi T_{zz}$, we have the first evolution equation

$$\frac{\partial \kappa}{\partial z} = K^i{}_j K^j{}_i + 2c^{ik}c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl} + 8\pi c^{ij}F_{zi}F_{zj}\,, \tag{399}$$

Then, rom eq. (215), we obtain the second evolution equation

$$
\begin{aligned}
\frac{\partial K^a{}_b}{\partial z} =& K^a{}_i K^i{}_b + 2\phi_j c^{ai}\beta_b K_i{}^j + 4c^{ak}\beta_m K^l{}_a C^m{}_{kl} \\
& - 2\phi_i c^{ak} c^{il}\beta_b\beta_m C^m{}_{kl} + 2c^{ak}c^{il}\beta_m\beta_n C^m{}_{bi} C^n{}_{kl} \\
& + c^{ai} W_{izzb} - 4\pi\big(\delta^a{}_b - b^a\beta_b\big)c^{ij}F_{zi}F_{zj} \\
& - 4\pi c^{ai} b^j \beta_b F_{zi}F_{zj} + 4\pi c^{aj}c^{ik}\beta_a F_{zk}F_{ij}\,.
\end{aligned} \tag{400}
$$

From $G_{za} = 8\pi T_{za}$, we obtain the first constraint,

(401)

$$
\begin{aligned}
& 3\phi_i K^i{}_a + \frac{7}{2}\phi_i b^i \beta_a \kappa \\
& \quad - 2b^k\beta_m K^l{}_a C^m{}_{kl} - \frac{3}{2}b^k\beta_a K^l{}_m C^m{}_{kl} - \frac{5}{2}b^n\beta_a\kappa C^m{}_{mn} + K^n{}_m C^m{}_{an} - \kappa C^m{}_{ma} \\
& \quad + 4\phi_i\phi_j c^{ij}\beta_a + \frac{5}{2}\phi_i c^{ik} b^l \beta_a\beta_m C^m{}_{kl} - 3\phi_i c^{in}\beta_m C^m{}_{an} - 2\phi_i c^{in}\beta_a C^m{}_{mn} \\
& \quad + 2c^{jk}b^l\beta_m\beta_n C^m{}_{aj} C^n{}_{kl} - \frac{3}{2}c^{jk}b^l\beta_a\beta_m C^m{}_{nj} C^n{}_{kl} \\
& \quad - \frac{1}{2}c^{ik}c^{jl}\beta_a\gamma_{mn} C^m{}_{ij} C^n{}_{kl} \\
& \quad - 7c^{kl}\beta_m C^m{}_{nk} C^n{}_{al} - \frac{7}{2}c^{kl}\beta_a C^m{}_{nk} C^n{}_{ml} - \frac{5}{2}c^{kl}\beta_a C^m{}_{mk} C^n{}_{nl} = \\
& \qquad = 8\pi\big(b^i F_{za}F_{zj} + c^{ij}F_{zi}F_{aj}\big) \\
& \qquad\quad - 2\pi\beta_a\big(\mathrm{a}c^{ij}F_{zi}F_{zj} + b^i b^j F_{zi}F_{zj} + 2b^i c^{jk}F_{zk}F_{ij} + c^{ik}c^{jl}F_{ij}F_{kl}\big)\,,
\end{aligned}
$$

and, from $G_{ab} = 8\pi T_{ab}$, the second one

(402)
$$
\begin{aligned}
&-\frac{11}{2}\phi_i b^i\gamma_{ab}\kappa - 3\phi_i b^i\gamma_{ai}K^i{}_b\\
&\quad - 2b^n K^i{}_m\big(\gamma_{ai}C^m{}_{bn} + \gamma_{bi}C^m{}_{an}\big) - 2b^n\gamma_{ai}K^i{}_b C^m{}_{mn}\big)\\
&\quad + 2b^n\kappa\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} + \gamma_{ab}C^m{}_{mn}\big)\\
&\quad + \frac{1}{2}\gamma_{ab}\big(- 3b^k K^l{}_m C^m{}_{kl} + b^n\kappa C^m{}_{mn}\big)\\
&\quad + \phi_i\phi_j b^i b^j\beta_a\beta_b + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi_b\\
&\quad + 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{am} + \beta_b\gamma_{am}\big)C^m{}_{kl} + \frac{5}{2}\phi_i c^{ik}b^l\gamma_{ab}\beta_m C^m{}_{kl}\\
&\quad - 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} + \gamma_{ab}C^m{}_{mn}\big)\\
&\quad + c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} + \frac{1}{2}c^{ik}c^{jl}\gamma_{ab}\gamma_{mn}C^m{}_{ij}C^n{}_{kl}\\
&\quad - 2c^{jk}b^l\beta_n\big(\gamma_{am}C^n{}_{bj} + \gamma_{bm}C^n{}_{aj}\big)C^m{}_{kl} + \frac{3}{2}c^{jk}b^l\gamma_{ab}\beta_m C^m{}_{nj}C^n{}_{kl}\\
&\quad - 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{ak}\big)C^m{}_{nl} - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{ak}\big)C^n{}_{nl}\\
&\quad - 2c^{kl}\gamma_{mn}C^m{}_{ak}C^n{}_{bl} + \frac{7}{2}c^{kl}\gamma_{ab}C^m{}_{nk}C^n{}_{ml} + \frac{5}{2}c^{kl}\gamma_{ab}C^m{}_{mk}C^n{}_{nl}\\
&\quad - b^l\beta_m C^m{}_{an}C^n{}_{bl} + b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{na}\big) - 3C^m{}_{an}C^n{}_{bm} - C^m{}_{ma}C^n{}_{nb} =\\
&\qquad = 8\pi\big(\mathrm{a}F_{za}F_{zb} - 2b^i b^j F_{za}F_{ib} + c^{ij}F_{ai}F_{bj}\big)\\
&\qquad\quad - 2\pi\gamma_{ab}\big(\mathrm{a}c^{ij}F_{zi}F_{zj} + b^i b^j F_{zi}F_{zj} + 2b^i c^{jk}F_{zk}F_{ij} + c^{ik}c^{jl}F_{ij}F_{kl}\big)\,.
\end{aligned}
$$

As for the evolution of the scalar field, the first equation becomes

$$\frac{\partial}{\partial z}F_{ab} = \frac{1}{2}\big(\phi_a F_{zb} - \phi_b F_{za}\big) - 2F_{zm}C^m{}_{ab}\,, \tag{403}$$

but the second one turns a constraint:

$$\frac{1}{2}b^i\big(\phi_a F_{zi} - \phi_i F_{za}\big) = 2b^n F_{zm}C^m{}_{an} + b^i c^{jk}k_{ak}F_{ij}\,. \tag{404}$$

And for the constraints, we have

$$\frac{3}{2}\phi_i c^{ij}b^k F_{jk} = b^i c^{jn}F_{ij}C^m{}_{mn}\,, \tag{405}$$

and

$$\frac{3}{2}\phi_i\big(b^i c^{ja}F_{zj} - c^{ai}b^j F_{zj} + c^{ij}c^{ka}F_{jk}\big) = c^{ik}b^l F_{zi}C^a{}_{kl} + c^{ik}c^{jl}F_{ij}C^a{}_{kl}\,. \tag{406}$$

Effectively, the Einstein-Maxwell system, when the group acts by means of null orbits, has three constraints. These three constraints are not equivalent; as a result, the system is overdetermined.

Finally, the component of the Weyl tensor evolves as

$$
\begin{aligned}
\frac{\partial W_{azzb}}{\partial z} =& 4\pi\big(2c^{ij}\gamma_{ab}+b^ib^j\beta_a\beta_b\big)F_{zi}\frac{\partial F_{zj}}{\partial z}+4\pi b^ic^{jk}\beta_a\beta_b\Big(F_{zk}\frac{\partial F_{ij}}{\partial z}+F_{ij}\frac{\partial F_{zk}}{\partial z}\Big)\\
&+4\pi c^{ik}c^{jl}\beta_a\beta_bF_{ij}\frac{\partial F_{kl}}{\partial z}\\
&-8\pi\gamma_{ab}c^{ik}K^j{}_kF_{zi}F_{zj}-2\pi\beta_a\beta_b\big(c^{ik}c^{jr}K^l{}_r+c^{jl}c^{ir}K^k{}_r\big)F_{ij}F_{kl}\\
&-4\pi\beta_a\beta_bb^ic^{jl}K^k{}_lF_{zk}F_{ij}+4\pi c^{ik}\beta_bK^j{}_kF_{zi}F_{aj}\,.
\end{aligned}
\tag{407}
$$

2.4. A fixed point argument for a general local solution. The extension of the previous chapter's results on local existence and uniqueness is immediate. The Einstein-Klein-Gordon system of eqs. (382), (383), (384), (385), (386), (387), (393) and (394) can be written in the form of the Initial Value Problem

$$
\begin{aligned}
&\frac{\partial}{\partial z}\vec{X}(z)=\boldsymbol{F}\big(\vec{X}(z)\big)\\
&\vec{X}(z_0)=\vec{X}_0
\end{aligned}
\tag{408}
$$

for z in some interval $\big[z_1,z_2\big]$; where $\vec{X}=\big\{\mathrm{a},b^a,c^{ab},\gamma_{ab},k_{ab},F_{za},F_{ab}\big\}$ and $\boldsymbol{F}$ the right-hand side of the corresponding equations (which is continuously differentiable in the interval); and assuming that the initial conditions satisfy the constraints of eqs. (388), (391) and (392).

Then, the Picard-Lindelöf theorem holds and the Einstein-Klein-Gordon system has a unique solution $\bar{X}$ in the interval $\big[z_0-h,z_0+h\big]$, which can be extended to the entire $\big[z_1,z_2\big]$ (but not further than that).

3. Examples

3.1. Bianchi *III* acting on Ehlers-Kundt *pp*-wave space-times. The solutions for plane-fronted parallelly-propagated waves were initially discovered by Ehlers and Kundt in 1962 as a vacuum solution. In the general case, the metric can be given in Brinkmann coordinates as follows

$$
\mathrm{d}s^2=H(u,x_1,x_2)\mathrm{d}u^2-2\mathrm{d}u\mathrm{d}v-\mathrm{d}x_1^2-\mathrm{d}x_2^2\,,
\tag{409}
$$

where $H(u,x_1,x_2)$ is an arbitrary smooth function, v a null coordinate (as ∂_v is a light-like vector), x_1 and x_2 two spatial coordinates (as ∂_{x_1} and ∂_{x_2} are two space-like vectors) and u an either temporal, null or spatial coordinate (as ∂_u can be either a time-like, light-like or space-like vector respectively). In the case of vacuum, as discussed, the smooth function is specified as it must be harmonic over the spatial coordinates, *i.e.*

$$
\frac{\partial^2H}{\partial x_1^2}+\frac{\partial^2H}{\partial x_2^2}=0\,.
\tag{410}
$$

One of the particular cases falling into this is when H is quadratic over the two spatial variables, or

$$
H(u,x_1,x_2)=h_{ij}(u)x^ix^j\,,
\tag{411}
$$

for $i + j = 2$.

In this case, two Killing vectors are easy to be found as

$$\vec{\xi}_1 = \frac{\partial}{\partial v} \qquad \text{and} \qquad \vec{\xi}_2 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}\,, \tag{412}$$

that is, the translation along the null coordinate v and the rotation on the space-like plane $x_1 - x_2$, are isometries of the space-time, *i.e.*

$$\mathcal{L}_{\vec{\xi}_1} \boldsymbol{g} = 0 \qquad \text{and} \qquad \mathcal{L}_{\vec{\xi}_2} \boldsymbol{g} = 0 \tag{413}$$

Moreover, the metric scales in a specific way - hence, a scaling symmetry exists along the null and spatial coordinates. Consequently, the dilation along

$$\vec{\xi}_3 = 2v \frac{\partial}{\partial v} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \tag{414}$$

rescales the metric by a factor of 2, *i.e.*

$$\mathcal{L}_{\vec{\xi}_3} \boldsymbol{g} = 2\boldsymbol{g}\,; \tag{415}$$

or equivalently, the dilation $\vec{\xi}_3$ is a homothety of the space-time.

The two isometries and the one homothety form a Lie algebra whose commutators are

$$\left[\vec{\xi}_1, \vec{\xi}_2\right] = 0\,, \qquad \left[\vec{\xi}_2, \vec{\xi}_3\right] = 0 \qquad \text{and} \qquad \left[\vec{\xi}_3, \vec{\xi}_1\right] = -2\vec{\xi}_1\,, \tag{416}$$

hence the structure constants of this algebra are zero apart from

$$C^1{}_{13} = 2\,. \tag{417}$$

This Lie algebra corresponds to the Bianchi *III* real group.

Let us consider a vector

$$\vec{\zeta} = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial x_1} + d\frac{\partial}{\partial x_2}\,, \tag{418}$$

where a, b, c and d four arbitrary functions of the four Brinkmann coordinates to be specified later.

First, we need to show the conditions for these four functions for which the vector z commuted with all three generators of Bianchi *III* group. Taking the translation along v , we have

$$\left[\vec{\xi}_1, \vec{\zeta}\right] = 0\,,$$

which leads to

$$\frac{\partial a}{\partial v} = \frac{\partial b}{\partial v} = \frac{\partial c}{\partial v} = \frac{\partial d}{\partial v} = 0\,, \tag{419}$$

which means that none of the four functions is dependent on the null coordinate. Moving to the rotation on the $x_1 - x_2$ plane, we have

$$\left[\vec{\xi}_2, \vec{\zeta}\right] = 0\,,$$

which leads to

$$
x_2\frac{\partial a}{\partial x_1} - x_1\frac{\partial a}{\partial x_2} = 0
$$
$$
x_2\frac{\partial b}{\partial x_1} - x_1\frac{\partial b}{\partial x_2} = 0
\tag{420}
$$
$$
x_2\frac{\partial c}{\partial x_1} - x_1\frac{\partial c}{\partial x_2} = -d
$$
$$
x_2\frac{\partial d}{\partial x_1} - x_1\frac{\partial d}{\partial x_2} = c\,.
$$

Of these four partial differential equations, the first two are independent of the rest and integrable to $a = a_1\big(\frac{1}{2}(x_1^2 + x_2^2)\big)$ and $b = b_1\big(\frac{1}{2}(x_1^2 + x_2^2)\big)$, which means they are constant along concentric circles of the $x_1 - x_2$ plane, whose centre is the origin.

Finally, taking the dilation along v, x_1 and x_2, we have

$$
\big[\vec{\xi}_3, \vec{\zeta}\big] = 0\,,
$$

which leads to

$$
x_1\frac{\partial a}{\partial x_1} + x_2\frac{\partial a}{\partial x_2} = 0
$$
$$
x_1\frac{\partial b}{\partial x_1} + x_2\frac{\partial b}{\partial x_2} = 2b
\tag{421}
$$
$$
x_1\frac{\partial c}{\partial x_1} + x_2\frac{\partial c}{\partial x_2} = c
$$
$$
x_1\frac{\partial d}{\partial x_1} + x_2\frac{\partial d}{\partial x_2} = d\,.
$$

These partial differential equations are independent and integrable as $a = a_2\big(\frac{x_2}{x_1}\big)$, $b = x_1^2 b_2\big(\frac{x_2}{x_1}\big)$, $c = x_1 c_1\big(\frac{x_2}{x_1}\big)$ and $d = x_1 d_1\big(\frac{x_2}{x_1}\big)$, which means that a is constant along the lines $x_1 = x_2$, while b increases as a quadratic on x_1 and c and d increase linearly on x_1 along the same lines.

Comparing these results with the results found previously, we arrive at the following problem

$$
a_1\Big(\frac{1}{2}(x_1^2 + x_2^2)\Big) = a_2\Big(\frac{x_2}{x_1}\Big)
\tag{422}
$$
$$
b_1\Big(\frac{1}{2}(x_1^2 + x_2^2)\Big) = x_1^2 b_2\Big(\frac{x_2}{x_1}\Big)\,,
$$

where the functions on the left-hand side must be equal to the function of the right-hand side - otherwise, the solutions of the first system are not compatible with the solution of the second. If one tries the simpler form for a_1, a_2 and b_1 , b_2 (the linear), it becomes obvious that the first equation leads to

$$
x_1 = x_2\,,
$$

while the second to

$$x_2 = \frac{1}{x_1} \pm \frac{\sqrt{1-x_1^4}}{x_1}.$$

The two solutions coincide only when $x_1 = x_2 = \pm 1$; which means that, in the linear case of a_1, a_2 and b_1, b_2, ζ will commute with $\vec{\xi_2}$ and $\vec{\xi_3}$ only at these supersurfaces. A possible solution would be to assume that the quotient belongs to this submanifold.

Another approach would be to combine eqs. (420) and (421), getting

$$\begin{aligned}
&\frac{\partial a}{\partial x_1} = \frac{\partial a}{\partial x_2} = 0 \\
&\frac{\partial b}{\partial x_1} = \frac{2x_1}{x_1^2+x_2^2} b \quad \text{and} \quad \frac{\partial b}{\partial x_2} = \frac{2x_2}{x_1^2+x_2^2} b \\
&\frac{\partial c}{\partial x_1} = \frac{x_1 c - x_2 d}{x_1^2+x_2^2} \quad \text{and} \quad \frac{\partial c}{\partial x_2} = \frac{x_2 c + x_1 d}{x_1^2+x_2^2} \\
&\frac{\partial d}{\partial x_1} = \frac{x_2 c + x_1 d}{x_1^2+x_2^2} \quad \text{and} \quad \frac{\partial d}{\partial x_2} = \frac{x_1 c - x_2 d}{x_1^2+x_2^2}.
\end{aligned} \tag{423}$$

The solution to these is that a is independent of x_1 and x_2; and b is proportional to x^2+y^2.

Now, using the fact that $\vec{\zeta}$ must be geodesic, we have

$$\begin{aligned}
\nabla_{\vec{\zeta}}\vec{\zeta} =& a\nabla_{\partial_u}\left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial x_1} + d\frac{\partial}{\partial x_2}\right) \\
&+ b\nabla_{\partial_v}\left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial x_1} + d\frac{\partial}{\partial x_2}\right) \\
&+ c\nabla_{\partial_{x_1}}\left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial x_1} + d\frac{\partial}{\partial x_2}\right) \\
&+ d\nabla_{\partial_{x_2}}\left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial x_1} + d\frac{\partial}{\partial x_2}\right) = 0
\end{aligned}$$

which leads to

$$\begin{aligned}
a\frac{\partial a}{\partial u} + c\frac{\partial a}{\partial x_1} + d\frac{\partial a}{\partial x_2} &= 0 \\
a\frac{\partial b}{\partial u} + c\frac{\partial b}{\partial x_1} + d\frac{\partial b}{\partial x_2} &= 0 \\
a\frac{\partial c}{\partial u} + c\frac{\partial c}{\partial x_1} + d\frac{\partial c}{\partial x_2} &= 0 \\
a\frac{\partial d}{\partial u} + c\frac{\partial d}{\partial x_1} + d\frac{\partial d}{\partial x_2} &= 0
\end{aligned} \tag{424}$$

Substituting previous results, we have

$$
\begin{aligned}
& a\frac{\partial a}{\partial u} = 0 \\
& a\frac{\partial b}{\partial u} + \frac{4x_1}{x_1^2+x_2^2}bc + \frac{4x_2}{x_1^2+x_2^2}bd = 0 \\
& a\frac{\partial c}{\partial u} + \frac{x_1}{x_1^2+x_2^2}c^2 + \frac{x_2}{x_1^2+x_2^2}d^2 = 0 \\
& a\frac{\partial d}{\partial u} + \frac{x_2}{x_1^2+x_2^2}\left(c^2-d^2\right) + \frac{2x_1}{x_1^2+x_2^2}dc = 0
\end{aligned} \tag{425}
$$

The first signifies that a is a constant. If we choose $a = 0$ for simplicity, we reach a very interesting result, where

$$
\begin{aligned}
& \frac{4x_1}{x_1^2+x_2^2}bc + \frac{4x_2}{x_1^2+x_2^2}bd = 0 \quad \Rightarrow \quad b = 0 \quad \text{or} \quad x_1c + x_2d = 0 \\
& \frac{x_1}{x_1^2+x_2^2}c^2 + \frac{x_2}{x_1^2+x_2^2}d^2 = 0 \quad \Rightarrow \quad x_1c^2 + x_2d^2 = 0 \\
& \frac{x_2}{x_1^2+x_2^2}\left(c^2-d^2\right) + \frac{2x_1}{x_1^2+x_2^2}dc = 0 \quad \Rightarrow \quad x_2\left(c^2-d^2\right) + 2x_1cd = 0
\end{aligned}
$$

The easiest way for all these relations to hold is $a = b = c = d = 0$, whereas there is no geodesic vector field in the quotient that commutes with the Bianchi group. However, avoiding $a = 0$, we can still take $b = 0$ (for simplicity); in which case, the second equation is always satisfied, but the third and fourth must be solved.

Finally, for the transversal to be null, we have

$$
\boldsymbol{g}\big(\vec{\zeta},\vec{\zeta}\big) = 0 \quad \Rightarrow \quad a^2H(u,x_1,x_2) - c^2 - d^2 = 0\,. \tag{426}
$$

This leads to

$$
c^2 + d^2 = a^2H(u,x_1,x_2)\,.
$$

It becomes obvious, now, that for the conditions for $H(u,x_1,x_2)$ to be met, the remaining two functions should

$$
c = c_0(u)c_1(x_1^2+x_2^2) \quad \text{and} \quad d = d_0(u)d_1(x_1^2+x_2^2)\,. \tag{427}
$$

Differentiating $H(u,x_1,x_2)$ with respect to x_1 and x_2 twice,

$$
\begin{aligned}
\frac{\partial^2 H}{\partial x_1^2} &= \frac{2}{a^2}\,\frac{c^2}{x_1^2+x_2^2} \\
\frac{\partial^2 H}{\partial x_2^2} &= \frac{2}{a^2}\,\frac{c^2-d^2}{x_1^2+x_2^2}
\end{aligned}
$$

and using eq. (410), we see that

$$
2c^2 = d^2\,. \tag{428}
$$

This means that the null vector on the quotient has the form

$$
\vec{\zeta} = a\frac{\partial}{\partial u} + c_0(u)c_1(x_1^2+x_2^2)\frac{\partial}{\partial x_1} + \frac{\sqrt{2}}{2}c_0(u)c_1(x_1^2+x_2^2)\frac{\partial}{\partial x_2}\,, \tag{429}
$$

where $c_0(u)$ an unspecified function. Specifying the particular solution further is possible; the metric can be computed as

$$
\begin{aligned}
g_{zz} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\zeta}\big) = 0\\
g_{z1} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_1\big) = a\\
g_{z2} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_2\big) = \frac{\sqrt{2}}{2}\big(2x_1 - \sqrt{2}x_2\big)c_0(u)c_1(x_1^2+x_2^2)\\
g_{z3} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_3\big) = -2av + \frac{\sqrt{2}}{2}\big(2x_1 + \sqrt{2}x_2\big)c_0(u)c_1(x_1^2+x_2^2)\\
g_{11} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_1\big) = 0\\
g_{12} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_2\big) = 0\\
g_{13} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_3\big) = 0\\
g_{22} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_2\big) = x_1^2 + x_2^2\\
g_{23} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_3\big) = x_1^2 - x_2^2\\
g_{33} &= \boldsymbol{g}\big(\vec{\xi}_3,\vec{\xi}_3\big) = x_1^2 + x_2^2
\end{aligned} \tag{430}
$$

However, computing the precise coordinates for the metric is also important.

Particular examples of Bianchi groups acting on the Ehlers-Kundt space-time that also admit a homothetic vector field have been found by Tupper *et al.* [**47**].

4. The Existence of Neighbouring Solutions in the Einstein-Maxwell System

Interestingly, a similar result with Chapter 4 is present here. Determining the precise coordinates for the space-time, where the metric is adjusted to the frame of the group, implies specifying $c_0(u)$, $d_0(u)$ and $H(u, x_1, x_2)$; also, it implies picking a particular solution for c and d with respect to x_1 and x_2 among the many that may satisfy the conditions required. The latter means that there is not a unique vector in the quotient and that may interfere in determining the corresponding parameter z. It should be made clear that the result stated in Theorem 4.1 still holds here. No further amendment is needed, as the Eintein-Maxwell system has been stated as a system of evolution equations in the same manner the Einsteins system was.

CHAPTER 7

Perfect Fluids Solution: The Einstein-Euler System

1. Introduction

This chapter is concerned with the final extension of the previous result, where we finally depart from an 'empty' space-time, by allowing the presence of fluids. In particular, we consider the simple case of perfect fluids with with barotropic equation of state; that is, we consider the simples and most usual form of the Einstein-Euler system. The latter is composed by the Einstein equations, where the stress-energy-momentum tensor is derived from the energy density and pressure of the fluid, and the Euler equations that describe the evolution of the velocity and the density of the fluid. The fundamental result here is that the particles of the fluid travel along the geodesics of the space-time, and thus the energy density, the pressure and the velocity of the fluid inherit the symmetries (*i.e.*, the homotheties) of the space-time; as a result, the Euler equations are also reduced to ordinary differential equations in the same manner as the Einstein equations. This means that the Picard-Lindelöf theorem still holds and existence and uniqueness of solutions can be proved locally simply by extending the argument from the previous chapter.

In the first part, we deal with

- providing an exact form for the stress-energy-momentum tensor in the case of a perfect fluid; and
- expressing the Euler equations in the foliation that the Einstein equations are expressed.

Then, we discuss the existence and uniqueness of solutions of the system. In the second part, we present an example: the Bianchi VI_h group acting on the Kasner space-time - one of the well-known solutions of anisotropic homogeneous cosmology (usually classified as a Bianchi I cosmology, since its Killing vectors form the Bianchi I algebra). Another example that is considered is the Bianchi VI_h acting on the Gödel space-time - a very early solution of the Einstein-Euler system that describes an inhomogeneous rotating universe filled with dust.

2. The Einstein-Euler System

The Einstein-Euler system is composed of the Einstein tensor equal to the stress-energy-momentum tensor for a fluid, that depends on the energy density (ρ), the pressure (P) and the velocity ($\boldsymbol{v}$) of the fluid:

$$\boldsymbol{G} = 8\pi \boldsymbol{T}(\rho, P, \upsilon)\,, \tag{431}$$

assuming the fluid is perfect, whereas the non-isotropic pressure (viscosity) and the energy flux vanish. Then, the stress-energy-momentum tensor is written as

$$\boldsymbol{T} = \big(\rho + P\big)\boldsymbol{v}(\vec{x}) \otimes \boldsymbol{v}(\vec{y}) + P\boldsymbol{g}\,(\vec{x},\vec{y}) \ . \tag{432}$$

For the system to be complete, the Euler equations are required, which can be derived from the conservation of energy and momentum, given as

$$\begin{aligned} & g^{-1}\left(\nabla_{\vec{x}}, \boldsymbol{T}\big(\vec{y},\vec{z}\big)\right) = 0 \\ \Rightarrow \quad & \nabla_{\vec{x}}\rho\boldsymbol{v}(\vec{x}) \otimes \boldsymbol{v}(\vec{y}) + \nabla_{\vec{x}}P\left(\boldsymbol{v}(\vec{x}) \otimes \boldsymbol{v}(\vec{y}) + \boldsymbol{g}\big(\vec{x},\vec{y}\big)\right) \\ & + \boldsymbol{v}(\vec{x}) \otimes \nabla_{\vec{x}}\boldsymbol{v}(\vec{y}) + \boldsymbol{v}(\vec{y}) \otimes \nabla_{\vec{x}},\boldsymbol{v}(\vec{x}) + \nabla_{\vec{x}}\left(\boldsymbol{g}\big(\vec{x},\vec{y}\big)\right) \ . \end{aligned} \tag{433}$$

One should note that the system is incomplete even then, as the conservation provides four additional differential equations (usually, one for the energy density and three for the spatial components of the velocity)[1], while there are five unknown variables (the pressure is not specified internally). The system can then be complete only by an additional assumption: that the perfect fluid has a barotropic equation of state, whereas the pressure is not independent, but depends on the energy density of the fluid:

$$P = P(\rho)\,. \tag{434}$$

Space-times described by this system are often found in relation to either cosmological or astrophysical problems. With regards to the former, the Friedmann-Lemaˆitre-Robinson-Walker solution describes a fluid space-time which is homogeneous and isotropic (thus, having the maximum number of possible isometries); famous solutions that belong to the Bianchi cosmologies (where the Bianchi groups act by isometries restricted to space-like obrits), like the Collins, the Collins-Stewart and the Jacobs solutions, are also examples of perfect fluids; finally, space-times without that describe inhomogeneous cosmologies, like the Lemaître-Tolman-Bondi or the Gödel solutions, use perfect fluids as the source for the Einstein equations. With regards to the latter, the most solutions that describe a star in equilibrium (*e.g.*, the Hartle-Thorne metric) or a collapsing start (*e.g.*, the Oppenheimer-Snyder metric) consider perfect fluids as the source of the gravity.

2.1. The perfect fluids. The first thing to do is to project the stress-energy-momentum tensor of a perfect fluid, as in eq. (432), along the group and the quotient in the same manner that the metric and the curvature tensors are. Thus, we define

$$\begin{aligned} T_{zz} &= \boldsymbol{T}\big(\vec{\zeta},\vec{\zeta}\big) = \big(\rho + P\big)(\upsilon_z)^2 \\ T_{za} &= \boldsymbol{T}\big(\vec{\zeta},\vec{\xi}_a\big) = \rho\upsilon_z\upsilon_a + P\big(\upsilon_z\upsilon_a + \beta_a\big) \\ T_{ab} &= \boldsymbol{T}\big(\vec{\xi}_a,\vec{\xi}_b\big) = \rho\upsilon_a\upsilon_b + P\big(\upsilon_a\upsilon_b + \gamma_{ab}\big) \end{aligned} \tag{435}$$

[1]For a slowly moving (non-relativistic) fluid, these would be sufficient as the time-like component can be assumed constant; for a relativistic fluid, the spatial components of velocity become irrelevant.

We also know that the stress-energy-momentum tensor inherits the symmetries of the space-time. In our case, its Lie derivative along a homothety vanishes,

$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{T} = 0\,.$$

Applying this to the definition of eq. (432), we can prove that scalar fields also inherit the symmetries of the space-time.

THEOREM 2.1 (Symmetry Inheritance of Perfect Fluids). *Let the stress-energy-momentum tensor of a perfect fluid, as in eq. (432); and let a space-time* $(\mathcal{V}_4, \boldsymbol{g})$ *where a Bianchi group* $\mathcal{G}$ *acts freely and regularly by means of homotheties. Then, the symmetries are inherited by the velocity*

$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{v} = \frac{\phi_a}{2}\boldsymbol{v}\,, \tag{436}$$

the matter-energy density

$$\mathcal{L}_{\vec{\xi}_a}\rho = -\phi_a\rho\,, \tag{437}$$

and the pressure

$$\mathcal{L}_{\vec{\xi}_a}P = -\phi_a P\,. \tag{438}$$

PROOF. Taking the Lie derivative of eq. (432), we have

$$\begin{aligned}
&\mathcal{L}_{\vec{\xi}_a}\boldsymbol{T} = 0\\
\Rightarrow\quad &\mathcal{L}_{\vec{\xi}_a}\rho\boldsymbol{v}\otimes\boldsymbol{v} + \mathcal{L}_{\vec{\xi}_a}P\big(\boldsymbol{v}\otimes\boldsymbol{v}+\boldsymbol{g}\big)\\
&\quad + 2\big(\rho+P\big)\boldsymbol{v}\otimes\mathcal{L}_{\vec{\xi}_a}\boldsymbol{v} + P\mathcal{L}_{\vec{\xi}_a}\boldsymbol{g} = 0\\
\Rightarrow\quad &\boldsymbol{v}\otimes\big(\mathcal{L}_{\vec{\xi}_a}\rho\boldsymbol{v} + 2\rho\mathcal{L}_{\vec{\xi}_a}\big)\\
&\quad + \boldsymbol{v}\otimes\big(\boldsymbol{v}\mathcal{L}_{\vec{\xi}_a}P\boldsymbol{v} + 2P\mathcal{L}_{\vec{\xi}_a}\boldsymbol{v}\big)\\
&\quad + \big(\mathcal{L}_{\vec{\xi}_a}P + \phi_a P\big)\boldsymbol{g} = 0\,.
\end{aligned}$$

Since the sum of this equation is zero, so should each summand be;[2] starting from the last, we can immediately prove that

$$\mathcal{L}_{\vec{\xi}_a}P = -\phi_a P\,. \tag{439}$$

and substituting this to the second summand, we have

$$\mathcal{L}_{\vec{\xi}_a}\boldsymbol{v} = \frac{\phi_a}{2}\boldsymbol{v}\,, \tag{440}$$

and, finally, substituting this to the first summand,

$$\mathcal{L}_{\vec{\xi}_a}\rho = -\phi_a\rho\,. \tag{441}$$

Therefore, the proof is complete. □

This symmetry inheritance has two interesting results. The first concerns a constraint on the equation of state.

[2]We should note that this is possible only due to the fact that the equation is tensorial. Therefore, the term along $\boldsymbol{g}$ is independent of the terms along $\boldsymbol{v}$.

Corollary 2.1.1 (Linear Equation of State). *Let there be a barotropic equation of state, $P = P(\rho)$.*
Given both the matter-energy density and the pressure inherit the symmetries of the space-time as in the Theorem 2.1, then the equation of state has to be linear.

Proof. Given $P = P(\rho)$ and $\mathcal{L}_{\vec{\xi}_a} P = -\phi_a P$, we have

$$\mathcal{L}_{\vec{\xi}_a} P = -\phi_a P \quad \Rightarrow \quad \frac{\partial P}{\partial \rho} \mathcal{L}_{\vec{\xi}_a} \rho = -\phi_a P \,,$$

but, we know that $\mathcal{L}_{\vec{\xi}_a} \rho = -\phi_a \rho$; so

$$-\phi_a \frac{\partial P}{\partial \rho} \rho = -\phi_a P \quad \Rightarrow \quad \frac{\partial P}{\partial \rho} = \frac{P}{\rho} \quad \Rightarrow \quad P = (w-1)\rho \,,$$

up to the addition of a constant. □

Here $w-1$ is the barotropic index. Notably, the same result can also come from the Strong and the Dominant Energy Conditions, so it is considered a universal result to all fluid space-times that admit (at least) one homothetic vector [**49, 50**]. For example, McIntosh proved that a barotropic fluid must be strictly stiff for a homothetic vector field to exist if the flow is orthogonal to the group [**53**]; similarly, Wainwright proved that the equation of state is always linear whenever a perfect fluid space-time admits a non-trivial homothetic vector - and, if the homothetic vector is parallel to the fluid's velocity, then the fluid has to be stiff [**48**].

The second result concerns the velocity and its differentiation along the group and its quotient.

Corollary 2.1.2. *Given the velocity spans as $\boldsymbol{\upsilon} = \upsilon_z \boldsymbol{\zeta} + \upsilon_i \boldsymbol{\xi}^i$.*
Since the velocity one-form is differentiated as stated by Theorem 2.1, then its components are differentiated in the same manner:

$$\begin{aligned} &\mathcal{L}_{\vec{\xi}_a} \upsilon_z \frac{\phi_a}{2} \upsilon_z \quad \textit{and} \\ &\mathcal{L}_{\vec{\xi}_a} \upsilon_b = \frac{\phi_a}{2} \upsilon_b - \upsilon_m C^m{}_{ab} \,. \end{aligned} \tag{442}$$

Proof. Let $\mathcal{L}_{\vec{\xi}_a} \boldsymbol{\upsilon} = \frac{\phi_a}{2} \boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon} = \upsilon_z \boldsymbol{\zeta} + \upsilon_i \boldsymbol{\xi}^i$. Then,

$$\mathcal{L}_{\vec{\xi}_a} \left(\upsilon_z \boldsymbol{\zeta} + \upsilon_i \boldsymbol{\xi}^i \right) = \frac{\phi_a}{2} \left(\upsilon_z \boldsymbol{\zeta} + \upsilon_i \boldsymbol{\xi}^i \right)$$

or

$$\left(\mathcal{L}_{\vec{\xi}_a} \upsilon_z - \frac{\phi}{2} \upsilon_z \right) \boldsymbol{\zeta} + \left(\mathcal{L}_{\vec{\xi}_a} \upsilon_i - \frac{\phi}{2} \upsilon_i \right) \boldsymbol{\xi}^i = -\upsilon_z \mathcal{L}_{\vec{\xi}_a} \boldsymbol{\zeta} - \upsilon_i \mathcal{L}_{\vec{\xi}_a} \boldsymbol{\xi}^i \,, \tag{443}$$

Projecting this equation along $\vec{\zeta}$, we get

$$\left(\mathcal{L}_{\vec{\xi}_a} \upsilon_z - \frac{\phi}{2}\upsilon_z\right)\boldsymbol{\zeta}(\vec{\zeta}) + \left(\mathcal{L}_{\vec{\xi}_a} \upsilon_i - \frac{\phi}{2}\upsilon_i\right)\boldsymbol{\xi}^i = -\upsilon_z\big(\mathcal{L}_{\vec{\xi}_a}\boldsymbol{\zeta}\big)(\vec{\zeta}) - \upsilon_i\big(\mathcal{L}_{\vec{\xi}_a}\boldsymbol{\xi}^i\big)(\vec{\zeta})$$

$$\Rightarrow \quad \mathcal{L}_{\vec{\xi}_a} \upsilon_z - \frac{\phi}{2}\upsilon_z = -\upsilon_z\big(\mathcal{L}_{\vec{\xi}_a}\zeta\big)(\vec{\zeta})\,.$$

And, given that the transversal collineation on the quotient commutes with the group, $\mathcal{L}_{\vec{\xi}_a}\boldsymbol{\zeta} = 0$, it is obvious that the left-hand side must be zero. Or

$$\mathcal{L}_{\vec{\xi}_a} \upsilon_z = \frac{\phi}{2}\upsilon_z\,. \tag{444}$$

Similarly, projecting eq. (443) along $vec\xi_b$, we get

$$\left(\mathcal{L}_{\vec{\xi}_a} \upsilon_z - \frac{\phi}{2}\upsilon_z\right)\boldsymbol{\zeta}(\vec{\xi}_b) + \left(\mathcal{L}_{\vec{\xi}_a} \upsilon_i - \frac{\phi}{2}\upsilon_i\right)\boldsymbol{\xi}^i(\vec{\xi}_b) = -\upsilon_z\big(\mathcal{L}_{\vec{\xi}_a}\boldsymbol{\zeta}\big)(\vec{\xi}_b) - \upsilon_i\big(\mathcal{L}_{\vec{\xi}_a}\boldsymbol{\xi}^i\big)(\vec{\xi}_b)$$

$$\Rightarrow \quad mathcalL_{\vec{\xi}_a} \upsilon_b - \frac{\phi}{2}\upsilon_b = -\upsilon_i\big(\mathcal{L}_{\vec{\xi}_a}\boldsymbol{\xi}^i\big)(\vec{\xi}_b)\,.$$

Now, given that a one-form commutes with its flow along a vector field of the same algebra, $\big(\mathcal{L}_{\vec{\xi}_a}\boldsymbol{\xi}^b\big)(\vec{\xi}_c) = -\boldsymbol{\xi}^b\left(\big[\xi_a,\xi_c\big]\right) = -C^b{}_{ac}$, then

$$\mathcal{L}_{\vec{\xi}_a} \upsilon_b = \frac{\phi}{2}\upsilon_b - \upsilon_m C^m{}_{ab}\,. \tag{445}$$

□

Raising the indices of the components of the velocity by use of the metric, we can easily prove that a similar differentiation holds:

$$\begin{aligned} &\mathcal{L}_{\vec{\xi}_a} \upsilon^z = -\phi_a \upsilon^z \quad \text{and} \\ &\mathcal{L}_{\vec{\xi}_a} \upsilon^b = -\phi_a \upsilon^b + \upsilon^n C^b{}_{na}\,. \end{aligned} \tag{446}$$

2.2. The evolution and constraint equations. The Einstein-Euler system is an extension of the Einstein system we considered in the previous chapter; it consists of the Einstein equations and the Euler equations.

The Einstein equations are derived in the usual manner, with the exception that the right-hand side of eq. (185), (186) and (187) is not zero, but equal to $8\pi T_{zz}$, $8\pi T_{za}$ and $8\pi T_{ab}$ respectively. Therefore, the evolution equations for the metric and its inverse are

$$\frac{\partial \beta_a}{\partial z} = 0\,, \tag{447}$$

$$\frac{\partial \gamma_{ab}}{\partial z} = k_{ab}\,, \tag{448}$$

$$\frac{\partial \mathrm{a}}{\partial z} = -b^i b^j k_{ij}\,, \tag{449}$$

$$\frac{\partial b^a}{\partial z} = -c^{ai} b^j k_{ij} \quad \text{and} \tag{450}$$

$$\frac{\partial c^{ab}}{\partial z} = -c^{ai} c^{bj} k_{ij} \,. \tag{451}$$

And the evolution equation for the derivatives, $\boldsymbol{k}$, is

(452)

$$\begin{aligned}
2\mathrm{a}\frac{\partial k_{ab}}{\partial z} =& -\big(\mathrm{a}c^{ij} - b^i b^j\big)k_{ij}k_{ab} + \frac{1}{2}\big(\mathrm{a}c^{ik} - b^i b^k\big)c^{jl}\gamma_{ab}k_{ij}k_{kl} + \frac{3}{4}\mathrm{a}c^{ij}c^{kl}\gamma_{ab}k_{ij}k_{kl} \\
& - 3\phi_i b^i k_{ab} + b^i\big(\phi_a k_{bi} + \phi_b k_{ai}\big) + \frac{1}{4}\phi_k\big(4b^i c^{jk} + 3c^{ij}b^k\big)\gamma_{ab}k_{ij} \\
& - 2b^n\big(k_{am}C^m{}_{bn} + k_{bm}C^m{}_{an} + k_{ab}C^m{}_{mn}\big) \\
& - 2\big(b^i c^{jn} - c^{ij}b^n\big)k_{ij}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an}\big) \\
& - \frac{1}{2}\big(3b^i c^{jn} - 5c^{ij}b^n\big)\gamma_{ab}k_{ij}C^m{}_{mn} + \frac{1}{2}c^{ik}b^l\gamma_{ab}k_{im}C^m{}_{kl} \\
& + \frac{1}{4}\phi_i\phi_j\big(11c^{ij}\gamma_{ab} - 4b^i b^j\beta_a\beta_b\big) + 2\phi_i b^i\big(\phi_a\beta_b + \phi_b\beta_a\big) + 3\phi_a\phi)b \\
& + \frac{5}{2}\phi_i c^{ik}b^l\gamma_{ab}\beta_m C^m{}_{kl} + 2\phi_i c^{ik}b^l\big(\beta_a\gamma_{bm} + \beta_b\gamma_{bm}\big)C^m{}_{kl} \\
& - 2\phi_i c^{in}\big(\gamma_{am}C^m{}_{bn} + \gamma_{bm}C^m{}_{an} - 2\gamma_{ab}C^m{}_{mn}\big) \\
& + c^{ik}c^{jl}\gamma_{am}\gamma_{bn}C^m{}_{ij}C^n{}_{kl} + \frac{3}{2}c^{ik}c^{jl}\big(\gamma_{mn} + \mathrm{a}\beta_m\beta_n\big)\gamma_{ab}C^m{}_{ij}C^n{}_{kl} \\
& - 2c^{kl}\big(\gamma_{am}C^m{}_{bk} + \gamma_{bm}C^m{}_{bk}\big)C^n{}_{nl} - 2c^{kl}\big(\gamma_{am}C^n{}_{bk} + \gamma_{bm}C^n{}_{bk}\big)C^m{}_{nl} \\
& + \frac{7}{4}c^{kl}\gamma_{ab}C^m{}_{nk}C^n{}_{mk} - \frac{5}{4}c^{kl}\gamma_{ab}C^m{}_{mk}C^n{}_{nl} - 4c^{kl}\gamma_{mn}C^m{}_{ak}C^n{}_{bl} \\
& - b^l\beta_m C^m{}_{an}C^n{}_{bl} - b^l\big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{an}\big)C^n{}_{ml} \\
& - 3C^m{}_{an}C^n{}_{bm} - C^m{}_{ma}C^n{}_{nb} \\
& - 12\pi(\rho + P)\big(2\upsilon_a\upsilon_b - (\mathrm{a}\upsilon_z + b^i\upsilon_i)\upsilon_z\gamma_{ab}\big) - 12\pi P\gamma_{ab} \,,
\end{aligned}$$

And the constraint becomes

(453)

$$\begin{aligned}
&\big(\mathrm{a}c^{ik} - b^i b^k\big)c^{jl}k_{ij}k_{kl} + \frac{1}{2}\mathrm{a}c^{ij}c^{kl}k_{ij}k_{kl} + \frac{7}{2}\phi_k c^{ij}b^k k_{ij} \\
&\quad - 2c^{ik}b^j b^l\beta_m k_{ij}C^m{}_{kl} + c^{ik}b^l k_{im}C^m{}_{kl} - \big(c^{ij}b^n - 2b^i c^{jn}\big)k_{ij}C^m{}_{mn} \\
&\quad - \frac{5}{2}\phi_i\phi_j c^{ij} - 7\phi_i c^{ik}b^l\beta_m C^m{}_{kl} \\
&\quad + \big(\mathrm{a}c^{ik} - 2b^i b^k\big)c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl} + \frac{7}{2}c^{kl}C^m{}_{nk}C^n{}_{ml} + \frac{5}{2}c^{kl}C^m{}_{mk}C^n{}_{nl} \\
&\quad - 12\pi(\rho + P)\left(\mathrm{a}\upsilon_z - b^i\upsilon_i\right)\upsilon_z - 12\phi P \,.
\end{aligned}$$

The Euler equations are derived from the derivatives of the stress-energy-momentum tensor. We know that the gradient of the tensor is constant - that is,

$$\begin{aligned} \nabla_{\vec{\zeta}} T_z{}^z = 0 \ , \quad \nabla_{\vec{\zeta}} T_a{}^z \ , \\ \nabla_{\vec{\xi}_i} T_z{}^i = 0 \quad \text{and} \quad \nabla_{\vec{\xi}_i} T_a{}^i \ . \end{aligned} \tag{454}$$

Taking the first two of these relations, and remembering that $T_z{}^z = \mathrm{a}T_{zz} + 2b^i T_{zi}$ and $T_a{}^z = 2\mathrm{a}T_{zz} + b^i T_{ai}$, we can reach to two equations for the evolution of the matter-energy density, the pressure and the observers' velocity components;[3] the first equations is

$$\begin{aligned} &\big(\mathrm{a}(\upsilon_z)^2 + 2b^i\upsilon_i\upsilon_z\big)\mathcal{L}_{\vec{\zeta}}\rho + \big(\mathrm{a}(\upsilon_z)^2 + 2b^i\upsilon_i\upsilon_z + 2\big)\mathcal{L}_{\vec{\zeta}}P \\ &\quad + 2\big(\rho+P\big)\big(\mathrm{a}\upsilon_z + b^i\upsilon_i\big)\mathcal{L}_{\vec{\zeta}}\upsilon_z + 2\big(\rho+P\big)b^i\upsilon_z\mathcal{L}_{\vec{\zeta}}\upsilon_i \\ &\quad - \big(\rho+P\big)\big(b^ib^jk_{ij}\upsilon_z + 2c^{ij}b^kk_{jk}\upsilon_i\big)\upsilon_z = 0\,, \end{aligned}$$

and the second

$$\begin{aligned} &2\big(\mathrm{a}\upsilon_z + b^i\upsilon_i\big)\upsilon_a\mathcal{L}_{\vec{\zeta}}\rho + \big(2\mathrm{a}\upsilon_z\upsilon_a + 2b^i\upsilon_i\upsilon_a + \mathrm{a}\beta_a\big)\mathcal{L}_{\vec{\zeta}}P \\ &\quad \big(\rho+P\big)\upsilon_a\big(2\mathrm{a}\mathcal{L}_{\vec{\zeta}}\upsilon_z + b^i\mathcal{L}_{\vec{\zeta}}\upsilon_i\big) + \big(\rho+P\big)\big(2\mathrm{a}\upsilon_z + b^i\upsilon_i\big)\mathcal{L}_{\vec{\zeta}}\upsilon_a \\ &\quad + \big(\rho+P\big)\big(2b^ib^jk_{ij}\upsilon_z\upsilon_a + c^{ij}b^kk_{ij}\upsilon_i\upsilon_a\big) + P\big(2b^ik_{ai} + b^ib^j\beta_ak_{ij}\big) = 0\,. \end{aligned}$$

Remembering that $P = (w-1)\rho$ and that the Lie derivatives along $\vec{\zeta}$ reduce to simple partial derivatives with respect to the parameter z - and defining

$$U_z = \mathcal{L}_{\vec{\zeta}}\upsilon_z = \frac{\partial}{\partial z}\upsilon_z \quad \text{and} \quad U_a = \mathcal{L}_{\vec{\zeta}}\upsilon_a = \frac{\partial}{\partial z}\upsilon_a\,, \tag{455}$$

we can rewrite the two equations as

$$\begin{aligned} &\Big(w\big(\mathrm{a}(\upsilon_z)^2 + 2b^i\upsilon_i\upsilon_z + 2\big) - 2\Big)\,\frac{\partial\rho}{\partial z} \\ &\quad + 2w\big(\mathrm{a}\upsilon_z + b^i\upsilon_i\big)U_z\rho + 2wb^i\upsilon_zU_i\rho \\ &\quad - w\big(b^ib^jk_{ij}\upsilon_z + 2c^{ij}b^kk_{jk}\upsilon_i\big)\upsilon_z\rho = 0\,, \end{aligned} \tag{456}$$

and

(457)

$$\begin{aligned} &\Big(w\big(2\mathrm{a}\upsilon_z + 2b^i\upsilon_i\big)\upsilon_a + (w-1)\mathrm{a}\beta_a\Big)\,\frac{\partial\rho}{\partial z} \\ &\quad + w\big(2\mathrm{a}U_z + b^iU_i\big)\upsilon_a\rho + w\big(2\mathrm{a}\upsilon_z + b^i\upsilon_i\big)U_a\rho + (w-1)\big(2b^ik_{ai} + b^ib^j\beta_ak_{ij}\big)\rho \\ &\quad + 2w\big(b^ib^jk_{ij}\upsilon_z + c^{ij}b^kk_{jk}\upsilon_i\big)\upsilon_a\rho = 0\,. \end{aligned}$$

[3]It should be obvious that $\vec{\upsilon} = \upsilon^z\vec{\zeta} + \upsilon^i\vec{\xi}_i$, and that $\boldsymbol{\upsilon} = \upsilon_z\boldsymbol{\zeta} + \upsilon_i\boldsymbol{\xi}^i$; where $\upsilon_z = \beta_i\upsilon^i$ and $\upsilon_i = \beta_i\upsilon^z + \gamma_{ij}\upsilon^j$.

Collecting the terms from eq. (457) and assuming $\upsilon_a \neq 0$ and $U_a \neq= 0$, we have

$$w\big(2\mathrm{a}\upsilon_z + b^i\upsilon_i\big)\rho$$
$$\Rightarrow \quad w = 0 \quad \text{or} \quad \rho = 0 \quad \text{or} \quad 2\mathrm{a}\upsilon_z = -b^i\upsilon_i$$
$$w\left(2\big(\mathrm{a}\upsilon_z + b^i\upsilon_i\big)\frac{\partial\rho}{\partial z} + \big(2\mathrm{a}U_z + b^iU_i\big)\rho - 2c^{ij}b^kk_{jk}\upsilon_i\rho\right)$$
$$\Rightarrow \quad w = 0 \quad \text{or} \quad 2\big(\mathrm{a}\upsilon_z + b^i\upsilon_i\big)\frac{\partial\rho}{\partial z} = -\big(2\mathrm{a}U_z + b^iU_i - 2c^{ij}b^kk_{jk}\upsilon_i\big)\rho$$
$$(w-1)\Big(\mathrm{a}\beta_a\frac{\partial}{\partial z}\rho - b^ib^j\beta_ak_{ij}\rho\Big) + 2(w-1)b^ik_{ia}\rho = 0$$
$$\Rightarrow \quad w = 1 \quad \text{or} \quad \mathrm{a}\frac{\partial\rho}{\partial z} = b^ib^jk_{ij}\rho$$

The following trivial cases exist:

1. Let $w = 0$ (the case of 'dark energy'), and $\rho = \dfrac{\partial\rho}{\partial z} = 0$.
2. Let $w = 1$ (the case of presureless dust), and $\rho = \dfrac{\partial\rho}{\partial z} = 0$.

Both of these cases are degenerate in the sense that the fluid disappears ($\rho = 0$ and $P = 0$) and the Einstein-Euler system reverts to vacuum. The following non-trivial cases also exist:

3. Let $w = 1$ (the case of presurelss dust), and $2\mathrm{a}\upsilon_z = -b^i\upsilon_i$ (a constraint on the fluid velocity); and

$$\mathrm{a}\upsilon_z\frac{\partial\rho}{\partial z} = -\Big(\mathrm{a}U_z + \frac{1}{2}b^iU_i\Big)\rho - \big(b^ib^jk_{ij}\upsilon_z + c^{ij}b^kk_{jk}\upsilon_i\big)\rho\,. \tag{458}$$

Then, eq. (456) becomes

$$\mathrm{a}\upsilon_z\frac{\partial\rho}{\partial z} = \Big(\mathrm{a}U_z - \frac{2}{3}b^iU_i\Big)\rho + \frac{1}{3}\big(b^ib^jk_{ij}\upsilon_z + 2c^{ij}b^kk_{jk}\upsilon_i\big)\rho\,. \tag{459}$$

Differentiating the relation $2\mathrm{a}\upsilon_z = -b^i\upsilon_i$ with respect to $\vec{\zeta}$, we obtain

$$2\mathrm{a}U_z - b^iU_i = 2b^ib^jk_{ij}\upsilon_z + c^{ij} + c^{ij}b^kk_{jk}\upsilon_i\,,$$

and substituting to eq. (459), we obtain

$$\mathrm{a}\upsilon_z\frac{\partial\rho}{\partial z} = \Big(\frac{5}{3}\mathrm{a}U_z - b^iU_i\Big)\rho\,, \tag{460}$$

which serves as the equation of continuity, *i.e.*, the equation that refers to the conservation of energy for the fluid. Returning to eq. (458) and combining with eq. (460), we can eliminate $\dfrac{\partial\rho}{\partial z}$; and, given $\rho \neq 0$, we obtain the following equation for the motion of the fluid

$$2\mathrm{a}U_z + \frac{9}{2}b^iU_i = -b^ib^jk_{ij}\upsilon_z - c^{ij}b^kk_{jk}\upsilon_i\,.$$

Comparing this with the derivative of $2\mathrm{a}\upsilon_z = -b^i\upsilon_i$, we obtain the following equations

$$\mathrm{a}U_z = \mathrm{a}\frac{\partial \upsilon_z}{\partial z} = \frac{15}{22}b^ib^jk_{ij}\upsilon_z + \frac{7}{22}c^{ij}b^kk_{jk}\upsilon_i \quad \text{and}$$
$$b^iU_i = b^i\frac{\partial \upsilon_i}{\partial z} - \frac{6}{11}b^ib^jk_{ij}\upsilon_z - \frac{4}{11}c^{ij}b^kk_{ij}\upsilon_i\,. \tag{461}$$

These equations denote the evolution of the velocity of the fluid.

4. If w is not specified, then

$$2\mathrm{a}\upsilon_z = -b^i\upsilon_i\,,$$

and

$$\mathrm{a}\frac{\partial \rho}{\partial z} = b^ib^jk_{ij}\rho\,. \tag{462}$$

This contribute to eq. (457) yielding

$$3\mathrm{a}U_z - 2b^iU_i = -4b^ib^jk_{ij}\upsilon_z - c^{ij}b^kk_{jk}\upsilon_i\,; \tag{463}$$

and, differentiating the relation $2\mathrm{a}\upsilon_z = -b^i\upsilon_i$, we have

$$2\mathrm{a}U_z - b^iU_i = 2b^ib^jk_{ij}\upsilon_z + c^{ij}b^kk_{ij}\upsilon_i\,. \tag{464}$$

Combining these two, we obtain the following two equations for the evolution of the velocity field,

$$\mathrm{a}U_z = \mathrm{a}\frac{\partial u_z}{\partial z} = 6b^ib^jk_{ij}\upsilon_z - 2c^{ij}b^kk_{jk}\upsilon_i \quad \text{and}$$
$$b^iU_i = b^i\frac{\partial u_i}{\partial z} = -10b^ib^jk_{ij} + 3c^{ij}b^kk_{jk}u_i\,. \tag{465}$$

Also, adding eqs. (463) and (464), we obtain

$$\frac{5}{2}\mathrm{a}U_z - \frac{3}{2}b^iU_i = b^ib^jk_{ij}\upsilon_z$$

and substituting this to eq. (462), we obtain the continuity equation for the fluid

$$\mathrm{a}\upsilon_z\frac{\partial \rho}{\partial z} = \frac{5}{2}\mathrm{a}U_z - \frac{3}{2}b^iU_i\,. \tag{466}$$

2.3. The case of null orbits. Let us go consider now the special case where $a = 0$ and the $3-$metric on the homogeneous submanifold $\mathcal{M}$ is degenerate. Firstly, we must consider what happens to the components of the metric and their derivatives, given both γ_{ab} and c^{ab} are singular.

The first evolution equations come from eqs. (84), (85) and (88), (90), (92) and refer to the derivatives of the components of the metric and its inverse:

$$\frac{\partial \beta_a}{\partial z} = 0\,, \tag{467}$$

$$\frac{\partial \gamma_{ab}}{\partial z} = k_{ab}\,, \tag{468}$$

$$\frac{\partial b^a}{\partial z} = 0 \quad \text{and} \tag{469}$$

$$\frac{\partial c^{ab}}{\partial z} = -c^{ai}c^{bj}k_{ij}\,. \tag{470}$$

Next, we need evolution equations (and constraints) for the components of $\boldsymbol{k}$. Following the same idea, we define

$$\kappa = c^{ij}k_{ij} \quad \text{and} \quad K^a{}_b = c^{ai}k_{ib}\,.$$

we can then produce evolution equations and constraints for the derivatives of the metric.

First of all, given $R_{zz} = 8\pi T_{zz}$, we have the first evolution equation

$$\frac{\partial \kappa}{\partial z} = K^i{}_j K^j{}_i + 2c^{ik}c^{jl}\beta_m\beta_n C^m{}_{ij}C^n{}_{kl} + 8\pi(\rho+P)(\upsilon_z)^2\,, \tag{471}$$

Then, from eq. (215), we obtain the second evolution equation

(472)
$$\begin{aligned}\frac{\partial K^a{}_b}{\partial z} =& K^a{}_i K^i{}_b + 2\phi_j c^{ai}\beta_b K_i^{\,j} + 4c^{ak}\beta_m K^l{}_a C^m{}_{kl} \\ &- 2\phi_i c^{ak}c^{il}\beta_b\beta_m C^m{}_{kl} + 2c^{ak}c^{il}\beta_m\beta_n C^m{}_{bi}C^n{}_{kl} \\ &+ c^{ai}W_{izzb} - 4\pi\big(\delta^a{}_b - b^a\beta_b\big)(\rho+P)(\upsilon_z)^2 - 4\pi c^{ai}\beta_b(\rho+P)\upsilon_z\upsilon_i\,.\end{aligned}$$

From $G_{za} = 8\pi T_{za}$, we obtain the first constraint,

(473)
$$\begin{aligned}&3\phi_i K^i{}_a + \frac{7}{2}\phi_i b^i\beta_a\kappa \\ &\quad - 2b^k\beta_m K^l{}_a C^m{}_{kl} - \frac{3}{2}b^k\beta_a K^l{}_m C^m{}_{kl} - \frac{5}{2}b^n\beta_a\kappa C^m{}_{mn} + K^n{}_m C^m{}_{an} - \kappa C^m{}_{ma} \\ &\quad + 4\phi_i\phi_j c^{ij}\beta_a + \frac{5}{2}\phi_i c^{ik}b^l\beta_a\beta_m C^m{}_{kl} - 3\phi_i c^{in}\beta_m C^m{}_{an} - 2\phi_i c^{in}\beta_a C^m{}_{mn} \\ &\quad + 2c^{jk}b^l\beta_m\beta_n C^m{}_{aj}C^n{}_{kl} - \frac{3}{2}c^{jk}b^l\beta_a\beta_m C^m{}_{nj}C^n{}_{kl} \\ &\quad - \frac{1}{2}c^{ik}c^{jl}\beta_a\gamma_{mn}C^m{}_{ij}C^n{}_{kl} \\ &\quad - 7c^{kl}\beta_m C^m{}_{nk}C^n{}_{al} - \frac{7}{2}c^{kl}\beta_a C^m{}_{nk}C^n{}_{ml} - \frac{5}{2}c^{kl}\beta_a C^m{}_{mk}C^n{}_{nl} = \\ &\quad = 8\pi\big((\rho+P)\upsilon_z\upsilon_a + P\beta_a\big)\,,\end{aligned}$$

and, from $G_{ab} = 8\pi T_{ab}$, the second one

(474)
$$
\begin{aligned}
&-\frac{11}{2}\phi_i b^i \gamma_{ab}\kappa - 3\phi_i b^i \gamma_{ai} K^i{}_b \\
&\quad - 2b^n K^i{}_m \big(\gamma_{ai} C^m{}_{bn} + \gamma_{bi} C^m{}_{an}\big) - 2b^n \gamma_{ai} K^i{}_b C^m{}_{mn}\big) \\
&\quad + 2b^n \kappa\big(\gamma_{am} C^m{}_{bn} + \gamma_{bm} C^m{}_{an} + \gamma_{ab} C^m{}_{mn}\big) \\
&\quad + \frac{1}{2}\gamma_{ab}\big(-3b^k K^l{}_m C^m{}_{kl} + b^n \kappa C^m{}_{mn}\big) \\
&\quad + \phi_i \phi_j b^i b^j \beta_a \beta_b + 2\phi_i b^i \big(\phi_a \beta_b + \phi_b \beta_a\big) + 3\phi_a \phi_b \\
&\quad + 2\phi_i c^{ik} b^l \big(\beta_a \gamma_{am} + \beta_b \gamma_{am}\big) C^m{}_{kl} + \frac{5}{2}\phi_i c^{ik} b^l \gamma_{ab} \beta_m C^m{}_{kl} \\
&\quad - 2\phi_i c^{in}\big(\gamma_{am} C^m{}_{bn} + \gamma_{bm} C^m{}_{an} + \gamma_{ab} C^m{}_{mn}\big) \\
&\quad + c^{ik} c^{jl} \gamma_{am} \gamma_{bn} C^m{}_{ij} C^n{}_{kl} + \frac{1}{2} c^{ik} c^{jl} \gamma_{ab} \gamma_{mn} C^m{}_{ij} C^n{}_{kl} \\
&\quad - 2c^{jk} b^l \beta_n \big(\gamma_{am} C^n{}_{bj} + \gamma_{bm} C^n{}_{aj}\big) C^m{}_{kl} + \frac{3}{2} c^{jk} b^l \gamma_{ab} \beta_m C^m{}_{nj} C^n{}_{kl} \\
&\quad - 2c^{kl}\big(\gamma_{am} C^n{}_{bk} + \gamma_{bm} C^n{}_{ak}\big) C^m{}_{nl} - 2c^{kl}\big(\gamma_{am} C^m{}_{bk} + \gamma_{bm} C^m{}_{ak}\big) C^n{}_{nl} \\
&\quad - 2c^{kl} \gamma_{mn} C^m{}_{ak} C^n{}_{bl} + \frac{7}{2} c^{kl} \gamma_{ab} C^m{}_{nk} C^n{}_{ml} + \frac{5}{2} c^{kl} \gamma_{ab} C^m{}_{mk} C^n{}_{nl} \\
&\quad - b^l \beta_m C^m{}_{an} C^n{}_{bl} + b^l \big(\beta_a C^m{}_{bn} + \beta_b C^m{}_{na}\big) - 3C^m{}_{an} C^n{}_{bm} - C^m{}_{ma} C^n{}_{nb} = \\
&\qquad = 8\pi\big((\rho + P)\upsilon_a \upsilon_b + P\gamma_{ab}\big)\,.
\end{aligned}
$$

With regards to the evolution of the fluid, we can go back to eqs. (456) and (457), which are now rewritten as

$$
\big(2w(b^i\upsilon_i + 1) - 2\big)\frac{\partial \rho}{\partial z} + 2wb^i\big(\upsilon_i U_z + \upsilon_z U_i\big)\rho = 0\,, \tag{475}
$$

and as

$$
\Big(wb^i\upsilon_i \frac{\partial \rho}{\partial z} + wb^i U_i \rho\Big)\upsilon_a + \big(wb^i \upsilon_i\big) U_a = 0\,. \tag{476}
$$

Collecting the terms from eq. (476) and assuming $\upsilon_a \neq 0$ and $U_a \neq= 0$, we have

$$
\begin{aligned}
&2wb^i\Big(\upsilon_i \frac{\partial \rho}{\partial z} + U_i \rho\Big) = 0 \\
&\qquad \Rightarrow \quad w = 0 \quad \text{or} \quad b^a = 0 \quad \text{or} \quad \upsilon_a \frac{\partial \rho}{\partial z} = -U_a \rho \\
&wb^i \upsilon_i \rho = 0 \\
&\qquad \Rightarrow \quad w = 0 \quad \text{or} \quad b^i \upsilon_i = 0 \quad \text{or} \quad \rho = 0
\end{aligned}
$$

The following cases exist:

1. Let $w = 0$ (the case of 'dark energy'), and $\rho = \dfrac{\partial \rho}{\partial z} = 0$. This case is trivial and we can discard it.

2. Let $w = 0$ (the case of 'dark energy'). and $\upsilon_a \dfrac{\partial \rho}{\partial z} = -U_z \rho$. From eq. (475), we obtain

(477)
$$\frac{\partial \rho}{\partial z} = 0\,,$$

which can serve as the continuity equation; immediately,

$$U_a = \frac{\partial u_a}{\partial z} = 0\,.$$

This case is interesting because it implies a fluid with negative pressure (hence, negative self-gravity) that has a steady and incompressible flow.

3. Let $b^i \upsilon_i = 0$, which results to $b^i U_i = 0$; thus, both the group-component of the velocity vector and its derivative with respect to z (the 'acceleration') are either zero or null. Then, the continuity equation for the fluid is[4]

(478)
$$\upsilon_a \frac{\partial \rho}{\partial z} = -U_a \rho\,.$$

From eq. (475), we obtain

$$2(w-1)\frac{\partial \rho}{\partial z} = 0\,.$$

This equation can result to: either $w = 1$ (the case of dust), or $\dfrac{\partial \rho}{\partial z} = 0$ and $U_a = 0$ (the case of steady incompressible flow). Both cases are non-trivial in the sense that the fluid does not 'disappear'; yet, both of them are remarkably simple, in the sense that the dynamics of the fluid are well-defined without solving the full Einstein-Euler system.

4. The case of $b^a = 0$ is not of particular interest. This case would imply that the null orbits are two-dimensional, whereas the properties of the theorem 2.1 no longer hold.

Finally, the component of the Weyl tensor evolves as

(479)
$$\begin{aligned}\frac{\partial W_{abcd}}{\partial z} =& 4\pi(1+w)\big(\gamma_{ab}(\upsilon_z)^2 + \beta_a\beta_b(b^i\upsilon_i\upsilon_z + c^{ij}\upsilon_i\upsilon_j) + \beta_a\upsilon_b\upsilon_z\big)\frac{\partial \rho}{\partial z} + 12\pi\beta_a\beta_b w\frac{\partial \rho}{\partial z}\\ &+ 4\pi(1+w)\big(2\gamma_{ab}\upsilon_z + \beta_a\beta_b b^i\upsilon_i + \beta_a\upsilon_b\big)\rho U_z\\ &+ 4\pi(1+w)\beta_a\beta_b c^{ij}\rho\upsilon_i U_j + 2\pi(1+w)\beta_a\rho\upsilon_z U_b\\ &- 2\pi(1+w)\beta_a\beta_b c^{ik}K^j{}_k\rho\upsilon_i\upsilon_j\,.\end{aligned}$$

[4]The similarity of this equation to the continuity equation in the Friedman-Robertson-Walker case is remarkable.

2.4. A fixed point argument for a general local solution. Once again, the extension of the results from Chapter 4 on local existence and uniqueness is trivial. The Einstein-Euler system of eqs. (447), (448), (449), (450), (451), (452), (460) and (461) - or (466) and (465) can be written in the form of the Initial Value Problem

$$\begin{aligned} &\frac{\partial}{\partial z}\vec{X}(z) = \boldsymbol{F}\big(\vec{X}(z)\big) \\ &\vec{X}(z_0) = \vec{X}_0 \end{aligned} \tag{480}$$

for z in some interval $\big[z_1, z_2\big]$; where $\vec{X} = \big\{\mathrm{a}, b^a, c^{ab}, \gamma_{ab}, k_{ab}, \rho, \upsilon_z, \upsilon_a\big\}$ and $\boldsymbol{F}$ the right-hand side of the corresponding equations (which is continuously differentiable in the interval); and assuming that the initial conditions satisfy the constraints of eq. (453).

3. Examples

3.1. Bianchi VI_h and III acting on Kasner space-times. The Kasner solutions originally described a spatially homogeneous vacuum space-time expanding under shear, that can be expressed in the following metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \sum_{i=1}^{3} t^{2p_i} (\mathrm{d}x_i)^2 \,, \tag{481}$$

where p_i three real constants, that fulfill $\sum_{i=1}^{3} p_i = 1$ and $\sum_{i=1}^{3}(p_i)^2 = 1$; however, these last conditions need not hold apart from the strict case of a vacuum.

Of course, such a space-time has three Killing vectors at all times; three translations along the space-like directions,

$$\vec{\xi}_1 = \frac{\partial}{\partial x_1}\,, \qquad \vec{\xi}_2 = \frac{\partial}{\partial x_2} \qquad \text{and} \qquad \vec{\xi}_3 = \frac{\partial}{\partial x_3}\,, \tag{482}$$

for which

$$\mathcal{L}_{\vec{\xi}_1} g = \mathcal{L}_{\vec{\xi}_2} g = \mathcal{L}_{\vec{\xi}_3} g = 0\,. \tag{483}$$

These isometries constitute a Lie algebra with zero structure constants, *i.e.* a Lie algebra corresponding to the Bianchi I real group. However, if we replace the last one with the following dilation:

$$\vec{\xi}_3 = \frac{1}{1-p_2}\Big(t\frac{\partial}{\partial t} + (1-p_1)x_1\frac{\partial}{\partial x_1} + (1-p_2)x_2\frac{\partial}{\partial x_2} + (1-p_3)x_3\frac{\partial}{\partial x_3}\Big)\,, \tag{484}$$

for which

$$\mathcal{L}_{\vec{\xi}_3}\boldsymbol{g} = \frac{2}{1-p_2}\boldsymbol{g}\,; \tag{485}$$

consequently, $\vec{\xi}_3$ is a homothety of the spacetime that rescales the metric accordingly.

The three vectors, the two translations $\vec{\omega}_1$ and $\vec{\xi}_2$ and the dilation $\vec{\xi}_3$, form a Lie algebra with commutators

$$[\vec{\xi}_1, \vec{\xi}_2] = 0\,, \qquad [\vec{\xi}_2, \vec{\xi}_3] = \vec{\xi}_2 \qquad \text{and} \qquad [\vec{\xi}_3, \vec{\xi}_1] = -\frac{1-p_1}{1-p_2}\vec{\xi}_1\,, \tag{486}$$

or equivalently, they form a Lie algebra with non-zero structure constants

$$C^1{}_{13} = \frac{1-p_1}{1-p_2} \qquad \text{and} \qquad C^2{}_{23} = 1\,. \tag{487}$$

This Lie algebra corresponds to the Bianchi VI_h real group, where $h = \dfrac{1-p_1}{1-p_2}$; this group reverts to Bianchi III in the extreme case of $p_1 = 1$ and $p_2 = p_3 = 0$ (expansion of the spacetime only along the x_1 direction) and to Bianchi V in the case of $p_1 = p_2$ (expansion or contraction along the x_1 and x_2 directions is uniform).

Let us consider a vector

$$\vec{\zeta} = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2} + d\frac{\partial}{\partial x_3}\,, \tag{488}$$

where a, b, c and d four arbitrary functions of the four coordinates.to be specified later.

First, we need to show the conditions for these four functions for which the vector $\vec{\zeta}$ commuted with all three generators of Bianchi III group. Taking the translation along x_1, we have

$$[\vec{\xi}_1, \vec{\zeta}] = 0\,,$$

which leads to

$$\frac{\partial a}{\partial x_1} = \frac{\partial b}{\partial x_1} = \frac{\partial c}{\partial x_1} = \frac{\partial d}{\partial x_1} = 0\,; \tag{489}$$

and taking the translation along x_2, we have

$$[\vec{\xi}_2, \vec{\zeta}] = 0\,,$$

which leads to

$$\frac{\partial a}{\partial x_2} = \frac{\partial b}{\partial x_2} = \frac{\partial c}{\partial x_2} = \frac{\partial d}{\partial x_2} = 0\,. \tag{490}$$

Both of these mean that the components of the vector $\vec{z}$ are independent of x_1 and x_2. Then, taking then translation along x_3, that is,

$$[\vec{\xi}_3, \vec{\zeta}] = 0\,,$$

which leads to

$$
\begin{aligned}
t\frac{\partial a}{\partial t} + (1-p_3)x_3\frac{\partial a}{\partial x_3} &= a \\
t\frac{\partial b}{\partial t} + (1-p_3)x_3\frac{\partial b}{\partial x_3} &= (1-p_1)b \\
t\frac{\partial c}{\partial t} + (1-p_3)x_3\frac{\partial c}{\partial x_3} &= (1-p_2)c \\
t\frac{\partial d}{\partial t} + (1-p_3)x_3\frac{\partial d}{\partial x_3} &= (1-p_3)d\,.
\end{aligned} \tag{491}
$$

These partial differential equations are independent and integrable as

$$
\begin{aligned}
a(t,x_3) &= tA\Big(\frac{x_3}{t^{1-p_3}}\Big) \\
b(t,x_3) &= t^{1-p_1}B\Big(\frac{x_3}{t^{1-p_3}}\Big) \\
c(t,x_3) &= t^{1-p_2}C\Big(\frac{x_3}{t^{1-p_3}}\Big) \\
d(t,x_3) &= t^{1-p_3}D\Big(\frac{x_3}{t^{1-p_3}}\Big)\,,
\end{aligned} \tag{492}
$$

which means that they grow over time along the world-lines $x_3 = t^{1-p_3}$.

Furthermore, requiring that the vectors $\vec{\zeta}$ is geodesic, we have

$$
\begin{aligned}
\nabla_{\vec{\zeta}}\vec{\zeta} =& a\nabla_{\partial_t}\Big(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2} + d\frac{\partial}{\partial x_3}\Big) \\
&+ b\nabla_{\partial_{x_1}}\Big(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2} + d\frac{\partial}{\partial x_3}\Big) \\
&+ c\nabla_{\partial_{x_2}}\Big(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2} + d\frac{\partial}{\partial x_3}\Big) \\
&+ d\nabla_{\partial_{x_3}}\Big(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2} + d\frac{\partial}{\partial x_3}\Big) = 0
\end{aligned} \tag{493}
$$

which leads to

$$
\begin{aligned}
& a\frac{\partial a}{\partial t} + d\frac{\partial a}{\partial x_3} + \frac{p_1}{t^{1-2p_1}}b^2 + \frac{p_2}{t^{1-2p_2}}c^2 + \frac{p_3}{t^{1-2p_3}}d^2 = 0 \\
& a\frac{\partial b}{\partial t} + d\frac{\partial b}{\partial x_3} + 2\frac{p_1}{t}ab = 0 \\
& a\frac{\partial c}{\partial t} + d\frac{\partial c}{\partial x_3} + 2\frac{p_2}{t}ac = 0 \\
& a\frac{\partial d}{\partial t} + d\frac{\partial d}{\partial x_3} + 2\frac{p_3}{t}ad = 0\,.
\end{aligned} \tag{494}
$$

Using the solutions of equations (492), we obtain

(495)
$$\begin{aligned}
\frac{\partial a}{\partial t} &= A(y) - (1-p_3)y\frac{\partial A}{\partial y} &&\text{and} && \frac{\partial a}{\partial x_3} = t^{p_3}\frac{\partial A}{\partial y}\\
\frac{\partial b}{\partial t} &= t^{-p_1}\Big((1-p_1)B(y) - (1-p_3)y\frac{\partial B}{\partial y}\Big) &&\text{and} && \frac{\partial b}{\partial x_3} = t^{p_3-p_1}\frac{\partial B}{\partial y}\\
\frac{\partial c}{\partial t} &= t^{-p_2}\Big((1-p_2)C(y) - (1-p_3)y\frac{\partial C}{\partial y}\Big) &&\text{and} && \frac{\partial c}{\partial x_3} = t^{p_3-p_2}\frac{\partial C}{\partial y}\\
\frac{\partial d}{\partial t} &= t^{-p_3}\Big((1-p_3)D(y) - (1-p_3)y\frac{\partial D}{\partial y}\Big) &&\text{and} && \frac{\partial d}{\partial x_3} = \frac{\partial D}{\partial y}\,,
\end{aligned}$$

where $y = \frac{x_3}{t^{1-p_3}}$. And, then, substituting to eq. (494), we arrive to

(496)
$$\begin{aligned}
\big((1-p_3)yA - D\big)\frac{\partial A}{\partial y} &= A^2 + p_1B^2 + p_2C^2 + p_3D^2\\
\big((1-p_3)yA - D\big)\frac{\partial B}{\partial y} &= (1+p_1)AB\\
\big((1-p_3)yA - D\big)\frac{\partial C}{\partial y} &= (1+p_2)AC\\
\big((1-p_3)yA - D\big)\frac{\partial D}{\partial y} &= (1+p_3)AD\,.
\end{aligned}$$

Interestingly, "dividing" the second and third equation by the fourth (and slightly abusing notation), yields

$$\begin{aligned}
\frac{\partial B}{\partial D} &= \frac{1+p_1}{1+p_3}\frac{B}{D}\\
\frac{\partial C}{\partial D} &= \frac{1+p_2}{1+p_3}\frac{C}{D}
\end{aligned}$$

which can be solved to

(497)
$$B = B_oD^{\frac{1+p_1}{1+p_3}} \quad \text{and} \quad C = C_oD^{\frac{1+p_1}{1+p_3}}\,,$$

for any two real parameters $B_o\,, C_o$; and substituting these to the first equation, after dividing by the fourth, one obtains

$$\frac{\partial A}{\partial D} = \frac{1}{1+p_3}\frac{a}{D} + \frac{p_3}{1+p_3}\frac{D}{a}\Big(1 + \frac{p_1}{p_3}B_oD^{-\frac{2p_1}{1+p_3}} + \frac{p_2}{p_3}C_oD^{-\frac{2p_2}{1+p_3}}\Big)\,,$$

This equation, although integrable, leads to a very long and complicated relation for A and D. However, assuming (without loss of generality) that $B_o = C_o = 0$, we can easily solve the latter to

(498)
$$A = A_o(1+p_3)D^{\frac{1}{1+p_3}}\,.$$

Then, using the null condition, we can easily obtain

$$a^2 = r^{2p_3}d^2\,,$$

which leads to

$$A^2 = D^2\,. \tag{499}$$

From where

$$D = A_o^{\frac{1+p_3}{p_3}}(1+p_3)^{\frac{1+p_3}{p_3}} = \text{const.}\,; \tag{500}$$

And

$$A = A_o^{1+p_3}(1+p_3)^{1+p_3} = \text{const.}\,. \tag{501}$$

So, eventually, the transversal collineation can be given as

$$\vec{\zeta} = a_o(1+p_3)^{1+p_3}t\frac{\partial}{\partial t} + a_o^{\frac{1}{p_3}}(1+p_3)^{\frac{1+p_3}{p_3}}t^{1-p_3}\frac{\partial}{\partial x_3}\,. \tag{502}$$

Given the result cannot be specified further, we will simply denote the metric

(503)

$$\begin{aligned}
g_{zz} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\zeta}\big) = 0\\
g_{z1} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_1\big) = 0\\
g_{z2} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_2\big) = 0\\
g_{z3} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_3\big) = a_0\frac{(1+p_3)^{1+p_3}}{1-p_2}t^2 + a_0^{\frac{1}{p_3}}\frac{(1+p_3)^{\frac{1+2p_3}{p_3}}}{1-p_2}t^{1+p_3}x_3\\
g_{11} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_1\big) = t^{2p_1}\\
g_{12} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_2\big) = 0\\
g_{13} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_3\big) = (1-p_1)x_2t^{2p_1}\\
g_{22} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_2\big) = \frac{t^{2p_2}}{(1-p_2)^2}\\
g_{23} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_3\big) = x_2t^{2p_2}\\
g_{33} &= \boldsymbol{g}\big(\vec{\xi}_3,\vec{\xi}_3\big) = -t^2 + (1-p_1)^2x_1^2t^{2p_1} + (1-p_2)^2x_2^2t^{2p_2} + (1-p_3)^2x_3^2t^{2p_3}
\end{aligned}$$

Interestingly, if $p_2 = p_3 = 0$ (the Bianchi III case), then $D = 1$ and $A = 1$; whereas, the transversal collineation becomes

$$\vec{\zeta} = t\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_3}\Big)\,. \tag{504}$$

Then, the metric becomes

$$
\begin{aligned}
g_{zz} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\zeta}\big) = 0\\
g_{z1} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_1\big) = 0\\
g_{z2} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_2\big) = 0\\
g_{z3} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_3\big) = t^2 + tx_3\\
g_{11} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_1\big) = t^2\\
g_{12} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_2\big) = 0\\
g_{13} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_3\big) = 0\\
g_{22} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_2\big) = 1\\
g_{23} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_3\big) = x_2\\
g_{33} &= \boldsymbol{g}\big(\vec{\xi}_3,\vec{\xi}_3\big) = -t^2 + x_2^2 + x_3^2
\end{aligned}
\tag{505}
$$

Interestingly, this transition as $p_2\,, p_3 \to 0$, which, on the one hand, changes the Bianchi group (from VI_h to III) and, on the other hand, changes non-smoothly the transversal collineation, is another interesting example of a peculiarity that may emerge under this particular formulation. However, this case is not directly a problem of the space-time, but merely the manifestation that a different transveral will be needed if Bianchi III acts instead of Bianchi VI_h, despite the fact that the action of the former group seems to be derived smoothly from the action of the latter.

The same problem does not apply if $p_1 = p_2 = 0$ and $p_3 = 1$, which is a simple version of Bianchi V. Then, the transversal simply becomes

$$
\vec{\zeta} = 4a_0\Big(t\frac{\partial}{\partial t} + \frac{\partial}{\partial x_3}\Big)\,. \tag{506}
$$

Then, the metric becomes

$$
\begin{aligned}
g_{zz} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\zeta}\big) = 0\\
g_{z1} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_1\big) = 0\\
g_{z2} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_2\big) = 0\\
g_{z3} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi}_3\big) = 4a_0t^2\big(1 + 2x_3\big)\\
g_{11} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_1\big) = 1\\
g_{12} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_2\big) = 0\\
g_{13} &= \boldsymbol{g}\big(\vec{\xi}_1,\vec{\xi}_3\big) = x_2\\
g_{22} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_2\big) = 1\\
g_{23} &= \boldsymbol{g}\big(\vec{\xi}_2,\vec{\xi}_3\big) = x_2\\
g_{33} &= \boldsymbol{g}\big(\vec{\xi}_3,\vec{\xi}_3\big) = t^2\big(-1 + x_1^2 + x_2^2\big)
\end{aligned}
\tag{507}
$$

3.2. Bianchi VI_h acting on Gödel space-times. The Gödel space-time is characterised by pressure-less dust rotating around an axis (here assumed to be x_2); the metric is given in a comoving observers coordinate system as

$$\mathrm{d}s^2 = \frac{1}{2\omega^2}\left[-\left(\mathrm{d}t + e^{mx_1}\mathrm{d}x_2\right)^2 + \mathrm{d}x_1^2 + \frac{1}{2}e^{2mx_1}\mathrm{d}x_2^2 + \mathrm{d}x_3^2\right] , \tag{508}$$

where ω is the circular velocity of the particles of the fluid, and m is a real constant. This space-time admits a total of five Killing vectors: three translations,

$$\vec{\xi}_1 = \frac{\partial}{\partial t} \ , \quad \vec{\xi}_2 = \frac{\partial}{\partial x_2} \quad \text{and} \quad \vec{\xi}_4 = \frac{\partial}{\partial x_3} \, , \tag{509}$$

and two more

$$\begin{aligned} \vec{\xi}_3 &= \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} \quad \text{and} \\ \vec{\xi}_5 &= -\frac{2}{m}e^{-mx_1}\frac{\partial}{\partial t} + y\frac{\partial}{\partial x_2} + \left(\frac{1}{m}e^{-2mx_1} - \frac{1}{2m}x_2^2\right)\frac{\partial}{\partial x_2} \, . \end{aligned} \tag{510}$$

Of these, we will pick $\vec{\xi}_1$, $\vec{\xi}_2$ and $\vec{\xi}_3$ as a Group that acts freely and regularly (by isometries, rather than homotheties) on the space-time; these three form a Bianchi VI_h algebra, since

$$\begin{aligned} \left[\vec{\xi}_1, \vec{\xi}_2\right] &= 0 \ , \\ \left[\vec{\xi}_2, \vec{\xi}_3\right] &= -m\vec{\xi}_2 \quad \text{and} \\ \left[\vec{\xi}_3, \vec{\xi}_1\right] &= 0 \, , \end{aligned} \tag{511}$$

with $h = m$.

It is not difficult to find a vector that commutes with all three; we consider the vector,

$$\vec{\zeta} = c\frac{\partial}{\partial x_2} + d\frac{\partial}{\partial x_3} \, , \tag{512}$$

and we impose that it should commuted with all three generators of Bianchi VI_h; that is,

$$\begin{aligned} \left[\vec{\zeta}, \vec{\xi}_1\right] = 0 \ &\Rightarrow \ \frac{\partial d}{\partial t} = \frac{\partial d}{\partial t} = 0 \\ \left[\vec{\zeta}, \vec{\xi}_2\right] = 0 \ &\Rightarrow \ \frac{\partial d}{\partial x_2} = \frac{\partial d}{\partial x_2} = 0 \\ \left[\vec{\zeta}, \vec{\xi}_3\right] = 0 \ &\Rightarrow \ \frac{\partial c}{\partial x_1} = -mc \quad \text{and} \quad \frac{\partial d}{\partial x_1} = 0 \end{aligned} \tag{513}$$

Therefore, d is constant (assumed to be $d = 1$ for convenience) and $c = c_0 e^{-mx_1}$, where c_0 some constant (again, it can be assumed 1 for simplicity). Demanding that the vector is null, we conclude to

$$\boldsymbol{g}\left(\vec{\zeta}, \vec{\zeta}\right) = 0 \ \Rightarrow \ \frac{3}{2}c_0^2 - 1 = 0 \, , \tag{514}$$

whereas

$$c_0 = \sqrt{\frac{2}{3}}\,.$$

Finally, it is easy to see that it is geodesic. Thus, the null geodesic collineation in the quotient is

$$\vec{\zeta} = \sqrt{\frac{2}{3}} e^{-mx_1} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\,. \tag{515}$$

The metric is easily constructed as follows:

$$\begin{aligned}
g_{zz} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\zeta}\big) = 0\\
g_{z1} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi_1}\big) = \sqrt{\frac{2}{3}}\\
g_{z2} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi_2}\big) = \sqrt{\frac{3}{2}} e^{mx_1}\\
g_{z3} &= \boldsymbol{g}\big(\vec{\zeta},\vec{\xi_3}\big) = -\sqrt{\frac{3}{2}} m x_2 e^{mx_1}\\
g_{11} &= \boldsymbol{g}\big(\vec{\xi_1},\vec{\xi_1}\big) = 1\\
g_{12} &= \boldsymbol{g}\big(\vec{\xi_1},\vec{\xi_2}\big) = e^{mx_1}\\
g_{13} &= \boldsymbol{g}\big(\vec{\xi_1},\vec{\xi_3}\big) = -m x_2 e^{mx_1}\\
g_{22} &= \boldsymbol{g}\big(\vec{\xi_2},\vec{\xi_2}\big) = \frac{3}{2} e^{2mx_1}\\
g_{23} &= \boldsymbol{g}\big(\vec{\xi_2},\vec{\xi_3}\big) = -\frac{3}{2} m x_2 e^{2mx_1}\\
g_{33} &= \boldsymbol{g}\big(\vec{\xi_3},\vec{\xi_3}\big) = \frac{3}{2} m^2 x_2^2 e^{2mx_1} - 1
\end{aligned} \tag{516}$$

4. The Existence of Neighbouring Solutions in the Einstein-Euler System

The first example examined resulted to the possibility of transitioning from one Bianchi group to another by means of a smooth change of the parameters of the metric, Moreover, this transition leads to a different transversal collineation, in such a way that the two vector fields are not smoothly related. This brings up the possibility that space-times that are solution to the Einstein-Euler system and allow for a Bianchi group to act freely and regularly can interchange between one another in a non-smooth and pathological manner.[5] The question as of whether these space-times are unique or not is easy to answer by invoking the Theorem 4.1 proved in the previous chapter. No further amendment is needed, as the Eintein-Euler system has been stated as a system of evolution equations in the same manner the Einsteins system was.

[5]This transition reminds that of a 'blue sky catastrophe' in the context of dynamical systems.

CHAPTER 8

Conclusions

The work presented in this thesis refers to the application of Bianchi groups on space-times that are solutions to Einstein's General Theory of Relativity. Unlike other similar studies, this one drops the usual assumptions that:

(1) the Bianchi group acts transitively (thus, admitting at least one fixed point, or allowing for orbits of dimension other than its own);
(2) the Bianchi group acts by isometries (thus, its generators are Killing vectors of the space-time); and
(3) the quotient contains at least one vector field orthogonal to the group, which is used to obtain an orthogonal slicing.

When these assumptions are used, the Bianchi group acts on either space-like submanifolds (resulting to a time-like quotient) or time-like surfaces (resulting to a space-like quotient); such cases are well-studied in the literature (for example, the spatially homogeneous cosmological models result from the first case, while several inhomogeneous models from the second). Dropping these assumptions, we face the possibility of a homogeneous submanifold that is strictly of dimension 3 and whose signature may vary. The assumptions we impose are that:

(1) the Bianchi group acts freely and regularly (whereas the dimension of the homogeneous submanifold is 3); and
(2) the Bianchi group acts by homotheties (whereas the metric is not conserved when "moved" along them).

Under these assumptions, we hope to provide a general framework for the treatment of space-times that contain one homogeneous submanifold. There is a certain disadvantage of this course: usual solutions of the Einstein equations, whose symmetries (either isometries or homotheties) include the generators of a Bianchi group, are *á priori* excluded, because the homogeneous submanifold they yield is of dimension less than 3 - the best example is that of the Schwarzschild space-time, that contains a two-dimensional homogeneous submanifold as result of the action of the Bianchi IX group. Nevertheless, we hope that two advantages of equal importance may exist: On the one hand, a number of space-times (either ready known or not) is bound to contain homogeneous submanifolds whose causal structure is not fixed, neither does it fall in one of the usual categories (always space-like or always time-like) - an easy example is the case of homogeneous gravitational waves; these space-times, despite their apparent similarity with the usual Bianchi space-times cannot be studied in that framework. On the other hand, there is a number of solutions (some of them known in the literature) where homothetic vectors play an important role - the cases of perfect fluids have been studied for many decades, while the cases of imperfect fluids and of Vlasov matter (particularly of massless particles) are still largely unexplored); these solutions would also greatly benefit from the treatment offered in this thesis.

The core of the thesis lies in the proof that such a space-time can always admit (at least, locally) an appropriate coordinate chart, adapted to the group and a null geodesic transversal. We prove as a theorem that whenever a null geodesic vector field exists in the quotient of the group action (one that is invariant to the group), such a coordinate chart is always possible. Interestingly, this vector field must be

- invariant to the group, so that it may define a direction in the space-time independent to the generators of the group;
- geodesic, so that this direction can be defined at any point in the space-time; and
- null, so that its length does not change when “moved” either along the group orbits or the transversal.

It is important to stretch out that such a vector field may not exist (or may not be easy to find) for every space-time on which a Bianchi group acts as described - in fact, the thesis offers some counter-examples. In this case, a coordinate chart may still be found, but not by following the steps of this theorem; moreover, this coordinate chart is likely one that is not adapted to the group, *i.e.*, it does not contain the canonical coordinates of the group.

Following that, we proceeded by attempting to actually construct such a space-time, by computing the connection and the curvature tensor of it. In Chapter 3, we presented these calculations concluding that such a space-time is possible. The Einstein equations can be decomposed to ‘transversal’, ‘group’ and ‘mixed’ components, as is the case with the (usual and somewhat simpler) 3+1 formalism; moreoever, we can easily prove (as is done in the following chapters) that the Einstein equations can always take the form of an Initial Value Problem with evolution and constraint equations, whose integrability can be discussed in the usual fashion. There is though one problem. The removal of the orthogonality condition allows us to explore situations where the homogeneous submanifold is light-like; however, the substitution of this condition with that of the null character of thansveral collineation makes this interesting case a degenerate one. More specifically, attempting to construct a coordinate chart for a null transversal and a null homogeneous submanifold, when the coordinates are adapted to them, makes both the induced metric, its inverse and its derivatives degenerate Thus, the Einstein equations take a peculiar form, one in which the identification of the appropriate Initial Value Problem is not immediate. This case we treated separately, by employing auxiliary variables and by relying to the Bianchi identities to recover any "missing information" about the curvature that is not revealed in the degenerate form the Einstein equations take.

What remains is to prove that these equations indeed admit solutions of some interest.

We start by exploring the simplest case: vacuum (Ricci-flat) space-times. In Chapter 4, we consider the case of a vanishing Ricci tensor, where the Einstein equations take the simplest form. We show that the Einstein equations are reduced to system of ordinary differential equations, given the fact that the derivatives of the metric were unspecified in only one direction of the frame - that of the quotient; the derivatives along the orbits of the group are constrained by the fact that the group acts by homotheties. In this case, the Einstein system can take the form of evolution equations and constraints, that falls under the conditions of the Picard-Lindelöf theorem; as a result, the existence and uniqueness of solutions is easy to prove, at least locally. This result was followed by some examples; in particular, two re-parametrisations of the Minkowski space-time in such a manner that a Bianchi group (Bianchi I in the first case, Bianchi III in the second) is applied such a way that it admits at least one homothety (a Lorentzian dilation). For these examples, we are able to locate the appropriate transversal collineation and specify the coordinates that adapt to it and the group, thus expressing the metric in its canonical form.

However, the interesting result that we reached brings forth an additional peculiarit of our method: the direction found in the quotient that satisfies the conditions of (i) invariance under the group action, (ii) geodesicity, and (iii) nullity, is not necessary Hausdorf. More specifically, we find that in one of the examples (ironically, the simpler one), the quotient contains 'holes' as it approaches certain light-cones. This result seems unnatural and it poses the following question: are these pathologies a result of our treatment and, thus, they would vanish when a different coordinate system would be chosen - or are they an inherent feature of the action of Bianchi groups on pseudo-Riemannian manifolds? If the case is the former, then it is a regrettable problem of our methodology (as the canonical coordinates cannot be always used), but it is a removable one. However, if the case is the latter, then the framework proposed in this thesis carries the advantage of being able to identify these pathologies whenever they appear.

Given an exhaustive analysis of these pathologies was not possible within the limitations of the thesis, we considered the following way to answer this question. We considered the case of 'neighbouring' solutions; that is, of solutions of a similar Initial Value Problem from different, yet not entirely dissimilar initial conditions. Thus, we proved a theorem stating that such 'neighbouring' solutions result to solutions that share the results of the original one (solutions with similar, even if not identical features).[1] As a result, such a 'pathological' situation (for example, a non-Hausdorf quotient, or a non-unique quotient) is not an isolated event, but may exist in many possible space-times that follow our construction. This is not a conclusive answer as to whether these 'pathologies' are actual topological features of the space-times; but, it definitely points to the existence of many such cases for different solutions.

The following chapters are, to some extent, repetitive of the analysis of Chapter 4, but with different (and progressively more complicated) matter models being considered. The common features in all situations is that all the matter models chosen can inherit the symmetries of the space-time; that is, their evolution along the group orbits can be constrained by the homotheties of the space-time. This simplifies the situation analytically to (more or less) an extended version of the vacuum solutions we have examined. Essentially, all these matter models lead to a similar Initial Value Problem, whose evolution equations are strictly ordinary differential equations. Hence, the two theorems proved in Chapter 4 (the existence and uniqueness one by Picard and Lindelöf, and the 'extended stability' one) can be used almost unaltered.

Chapter 5 is concerned with the case of the Einstein-Klein-Gordon system, where the source of the Einstein equations is a scalar field. In this case, it is proved that the kinetic term and the potential of the scalar field inherit the symmetries (*i.e.*, the homotheties) of the space-time and, by extent, the evolution equation of the scalar field (the Klein-Gordon equation) reduces to an ordinary differential equation as well. This makes the system integrable locally in the same manner that the Einstein system is. Interstingly, such a scalar field may take very specific forms (*e.g.*, its potential is either zero or en exponential), yet it is not always free in the usual sense; although it is 'free' in the sense that it is restricted along the group orbits, the very fact that the group acts by homotheties allows the kinetic term and the potential to grow or shrink sufficiently to make this 'free' scalar field dynamic rather than constant. The examples considered are derived from the work of Carot and Coligne in the Wainwright $B-ii$ space-times and allow for one counter-example, where a transversal collineation that meets the requirements of theorem 2.1 cannot be found.

Following, the case of the Einstein-Maxwell system was also examined in Chapter 6, where the course of the Einstein equations is a free electromagnetic

[1] This is a result that mimics similar stability results , proving that solutions of the same Initial Value Problem, albeit with perturbed initial conditions will converge to the same solution. However, in a certain extent, it generalises this notion of stability, as its scope is not to show that solutions converge, but that they share similar characteristics.

field. In a mirroring way, it is proved that the Faraday tensor inherits the symmetries (*i.e.*, the isometries) of the space-time and that restricts the behaviour of its derivatives - and, consequently, the form of the Maxwell equations. Once again, the 'freedom' of the electromagnetic fields should be perceived as relative, as the symmetries inherited are homotheties, therefore the strength of the electromagnetic field may grow or shrink as it propagates along the orbits. The stress-energy-momentum tensor for an electromagnetic field is 'attached' to the Einstein equations and the specific form of the Maxwell equations is considered; it is not surprising that these were proved to be ordinary differential equations as well and, thus, that the Picard-Lindelöf theorem also applies here, guaranteeing local existence and uniqueness of solutions. We also examine a particular example, that of the Ehlers-Kundt plane-wave space-time, usually invoked to describe electromagnetic or gravitational waves propagating in an 'empty' Universe. The study of the example and the attempt to construct a metric also reveals a number of 'pathogenic' issues in the spirit of those revealed in Chapter 4; as a result, we can simply extend the theorem proved there to show that such 'pathogenies' may exist in many space-times that are solutions to the Einstein-Maxwell system under our foliation, given their initial conditions are close to the example mentioned.

Finally, in Chapter 7, we considered the case of the Einstein-Euler system, where classical baryonic fluids may be used as a source for the gravitational field. It is also proved that the matter-energy density and the isotropic pressure of the fluid, as well as the observer's velocity inherit the symmetries (*i.e.*, the homotheties) of the space-time. This result leads to the consideration of the barotropic equation of state (required to close the system of differential equations); it is proved (in accordance to the literature and the energy conditions) that only the linear barotropic equation is possible, whereas the pressure is always proportional to the matter-energy density. Following this, we can show that the equations for the fluid are also written as a set of ordinary differential equations and, by extent, the Einstein-Euler system falls also in the premises of the Picard-Lindelöf theorem. As a result, proving the existence and uniqueness of solutions is trivial. The example of the Kasner and the Gödel space-times in the presence of some perfect barotropic fluid are examined; finding a precise vector field in the quotient may prove difficult, but it exists in all cases examined. Interestingly, the case of the Kasner universes provides several examples that seem to emerge naturally as the values of the exponents p_1, p_2 and p_3 change smoothly; nevertheless, what is observed is that the transversal vector field does not change smoothly with them, but undergoes 'jumps' as we move from one Bianchi group to another. This can be considered a case of 'pathology', therefore the theorem of similar neighbouring solutions is still valid here; yet, unlike the case of Bianchi I acting on the Minksowski space-time in Chapter 4, this 'pathology' is likely the result of our choice of trasversal, rather than an inherent feature of these solutions.

This work seems to open several possibilities, as the advantage of the methodology proposed is that it can deal with a number of solutions of General Relativity that cannot be treated under the usual 3+1 formalism. Indeed, there is a serious disadvantage, that cases with group orbits of dimension less than 3 cannot be treated; but, we should not see this as restricting. We can see that the advantages are such that specific cases can be treated in a much simpler way if this methodology is adopted.

One possible continuation that could show the benefits of this work is the exploration of gravitational wave space-times. In the present thesis, we revisited the Ehlers-Kundt solution, which refers to plane waves in an 'empty' universe. However, more examples of homogeneous gravitational waves (that travel along light-like homogeneous submanifolds) may be possible - for example, waves that propagate with spherical, cylindrical, *etc.* fronts. Moreover, we could focus on cases of gravitational waves that propagate through matter; or cases of interaction between gravitational waves, or between gravitational and electromagnetic waves. Although some of these cases have been examined in the literature, revisiting them may be important, given the recent advances of Gravitational Wave Astronomy. This exploration may be also be important from a purely theoretical perspective as well, as dealing with light-like homogeneous submanifolds appears as one of the main advantages of our method. Of course, this would imply a further discussion of the degenerate case; but, such a discussion is possible as the main points have been already stressed out and the system of equations can still be written as an Initial Value Problem.

Another possible continuation of this work, that can also exploit the benefits stemming from its methodology, lies in the field not examined here, yet which served as the original inspiration of the thesis. When the work started, we hoped to examine the Einstein-Vlasov system under this foliation - and, maybe, to extend the study to the Einstein-Boltzmann. The case of massless particles, in particular, seems very promising, as they would allow for homothetic generators of the group dictating the homogeneity of the space-time; this could even be related to the question of Penrose's Conformal Cyclic Cosmology, where one 'universe' is conformally rescaled to the next, when it has met its 'thermodynamic death'. Given the many shortcomings of the work (and the limitations of time, space and effort), these systems were not considered. Nevertheless, these systems are of greater interest, since they are not (always) reducible to systems of ordinary differential equations (the presence of integrals in the Vlasov or Boltzmann equations being part of the problem; also, the difficulty in constraining all particles in motion along the orbits of the group in the Einstein-Boltzman case); hence, the existence and uniqueness of solutions cannot be proved trivially by means of the Picard-Lindelöf theorem, but must be sought in each case separately. The question of whether similar 'pathologies' exist in these cases is also of interest; but, this too can only be answered after the stability of solutions has been proved.

The same can be true for the case of imperfect fluids - the full Einstein-Euler system. In the present work, we treated the case of perfect fluids, but other works (using the 3+1 formalism) have already examined the case of imperfect fluids, that include non-isotropic pressure and energy flows. However, the question of the existence and the role of homothetic vectors in solutions with imperfect fluids has remained largely unexplored; as a result, tilted and self-similar solutions for spatially homogeneous or inhomogeneous cosmologies with an imperfect fluid have been rarely (if ever at all) considered. Given our methodology emphasises on the existence of at least one homothetic vector in the symmetry group, we could attempt a generalisation of the works pursued by Eardley, Wainwright, Carot, and others in the previous decades, by extending the scope to imperfect fluids. In a similar manner, the case of the Einstein-Maxwell-Euler system can also be examined.

The last possible continuations come to highlight an interesting possibility for the proposed framework. Both the Einstein-Vlasov (or Einstein-Boltzmann) system, particularly with massless particles, and the Einstein-Euler system with imperfect fluids are attempts to a more detailed and realistic description of the universe - especially of earlier stages, where

- the behaviour of the matter may have not been as simple as perceived in more recent stages; and
- the currently observed, very small inhomogeneities and anisotropies may have been large enough to violate the Cosmological Axis.

Such situations have been examined so far by means of the simple action of the Bianchi groups on the space-time (where the generators are isometries and the homogeneity is restrained to space-like or time-like submanifolds with an orthogonal transversal) and have yielded interesting theoretical and empirical results. It is a good question to what extent the proposed methodology can offer an extensive counter-proposal to this literature. The prospect of examining in detail whether solutions that model the earlier stages of the universe are plausible and whether they can lead us to the currently observed stage is a fruitful one and should definitely be examined.

Another interesting continuation is the pursue of global existence. We know this cannot be true always, as some of the examples considered present problems, or do not posses a maximally extended geodesic spray (thus, containing potential singularities). However, it would still be interesting to examine some of these cases - particularly in the Einstein-Maxwell and the Einstein-Euler cases; the consideration of the joint case of Einstein-Maxwell-Euler may also consist a continuation of our research.

Finally, the extension of this work to the case of conformal motions can also be considered. But, one should be too careful in undertaking such a task, as (we fear) the inherent difficulties might prove greater than the anticipated benefits.

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