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Curvature Inheritance Symmetry in Ricci Flat Spacetimes

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Abstract: In this article, we study curvature inheritance symmetry in Ricci flat spacetimes. We show that, if Ricci flat spacetimes are not of Petrov type N, and admit curvature inheritance symmetries, then the only existing symmetries are conformal motions. We also prove that the only Ricci flat spacetime that admits a proper curvature inheritance symmetry and is of Petrov type other than N is the flat spacetime. Next, we find that the vacuum *pp*-waves of Petrov type N if admit curvature inheritance symmetry, then conformal motion implies homothetic motion.

Keywords: curvature symmetry inheritance; Ricci flat spacetimes; gravitational waves; null tetrad

MSC: 53B20; 83C20; 83C35; 83C60; 53A45

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1. Introduction and Preliminaries

The general theory of relativity, an important field in theory of gravitation, can be described by Einstein field equations with a cosmological constant. These equations, whose fundamental constituent is the spacetime metric g, are highly nonlinear partial differential equations of the second order; therefore, it is not so straightforward to obtain their exact solutions. They become still more difficult to solve if the spacetime metric depends on all coordinates [1–3]. This problem, however, can be simplified to some extent if some geometric symmetry properties are assumed to be possessed by the metric tensor. These geometric symmetry properties are described by Killing vector fields and lead to conservation laws in the form of first integrals of a dynamical system [4]. There exist a reasonably large number of solutions to Einstein field equations possessing different symmetry structures [5]. These solutions were further classified according to their properties and groups of motion [1].

The Bianchi identities describe the interaction between matter and free gravitational parts of the gravitational field, which is characterized by the curvature tensor in general theory of relativity. In gravitational physics, the main objective of all investigations is the construction of the gravitational potential, which satisfies Einstein field equations (EFEs). The interaction of matter and gravitation through Einstein field equations with a cosmological term is given as

$$R_{ij} - \frac{1}{2}Rg_{ij} + \wedge g_{ij} = k\, T_{ij}, \tag{1}$$

where R_{ij} denotes the Ricci tensor, g_{ij} denotes the metric tensor, scalar curvature denoted by R, k is a constant, T_{ij} represents the energy-momentum tensor, and $\wedge$ is a cosmological constant [1–3].

For a chosen distribution of matter, we impose symmetry assumptions that are compatible with the dynamics of matter. These geometrical symmetries [2,3,6] of spacetime are represented by the following equation:

$$\pounds_{\xi} A = 2\Omega A, \tag{2}$$

where $\pounds_\xi$ stands for the Lie derivative along vector field ξ^i, 'A' denotes a geometrical or physical quantity, and Ω is a scalar function. Vector field ξ^i is either spacelike ($\xi^i\xi_i > 0$), timelike ($\xi^i\xi_i < 0$), or null ($\xi^i\xi_i = 0$).

A simple example can be provided here as the metric inheritance symmetry or conformal motion (Conf M) along a conformal Killing vector (CKV) ξ^i for $A = g_{ij}$ in Equation (2). Most primary symmetry on (V_4, g) is motion (M) or isometry that is obtained by setting $A = g_{ij}$, $\Omega = 0$ in Equation (2); then ξ^i is called a Killing vector [2,6], and the corresponding equation is called a Killing equation. To date, more than 30 geometric symmetries have been found in the literature. For a detailed study of symmetry inheritance, see [7–10]. Many other studies were also conducted on curvature inheritance symmetry in various research subfields of mathematics and physics (for more details, see [11–18]).

In 1992, the notion of curvature inheritance (CI) was introduced by K.L. Duggal in [8]. Curvature inheritance is the generalization of curvature collineation (CC), which was defined by Katzin in 1969 [19].

Definition 1. *A smooth vector field ξ^i on spacetime $(V_4,\ g)$ (where g is smooth (C^∞) metric on V_4) generates inheritance symmetry if it satisfies the following equation*

$$\pounds_\xi D = 2\Omega D, \tag{3}$$

where $\pounds_\xi D$ works as the Lie derivative of the geometrical/physical quantity D of spacetime V_4 with respect to ξ^i and $\Omega = \Omega(x^i)$ is an inheriting factor (scalar function of spacetime coordinates).

If 'D' is by Riemann curvature tensor R^h_{ijk}, then Equation (3) defines [8] curvature inheritance. If $\Omega = 0$, then $\pounds_\xi R^h_{ijk} = 0$ and ξ^i follows a curvature symmetry on V_4 or it simply generates a curvature collineation (CC). CI Equation (3) can be written in local coordinates as

$$R^h_{ijk;l}\xi^l - R^l_{ijk}\xi^h_{;l} + R^h_{ljk}\xi^l_{;i} + R^h_{ilk}\xi^l_{;j} + R^h_{ijl}\xi^l_{;k} = 2\Omega R^h_{ijk}. \tag{4}$$

A $(1,3)$-type conformal curvature tensor (or Weyl tensor) C^h_{ijk} on spacetime (V_4, g) is defined by [20]

$$C^h_{ijk} = R^h_{ijk} + \frac{1}{2}(\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_j g_{ik} - R^h_k g_{ij}) + \frac{R}{6}(g_{ij}\delta^h_k - g_{ik}\delta^h_j). \tag{5}$$

where δ^i_j is Kronecker delta.

Ref. [21] A spacetime is conformally flat if

$$C^h_{ijk} = 0, \tag{6}$$

The most natural way to define conformally flatness of the space can be through $g_{ij} = \phi^2\eta_{ij}$, where η_{ij} is the metric of flat spacetime and ϕ is known as the conformal function.

Example 1 ([22])**.** *The line element for a generalized pp-wave spacetime can be written as*

$$ds^2 = -2dudv - 2H(u,y,z)du^2 + dy^2 + dz.^2 \tag{7}$$

In the special case of a null Einstein–Maxwell generalized plane wave spacetime, function H can be

$$2H = A(u)y^2 + 2B(u)yz + C(u)z.^2 \tag{8}$$

If $A(u) = -C(u)$, the spacetime is a vacuum, and if $A(u) = C(u)$ and $B(u) = 0$, the spacetime is conformally flat.

Example 2 ([22])**.** *The conformally flat generalized plane wave spacetimes have metrics of the form*

$$ds^2 = -A(u)(y^2 + z^2)du^2 - 2dudv + dy^2 + dz.^2 \quad (9)$$

However, there are three specializations of function A(u) for which one of the proper CKV becomes an SCKV. Thus, the maximal dimension of the inheriting algebra for conformally flat generalized plane wave spacetimes is eight.

Now, we introduce the notion of conformal curvature inheritance (Conf CI) symmetry. If 'D' is replaced by Weyl tensor C^h_{ijk}, then Equation (3) defines "conformal curvature Inheritance" and represents the following equation

$$\pounds_\xi C^h_{ijk} = 2\Omega C^h_{ijk}, \quad (10)$$

where $\Omega = \Omega(x^i)$ can be called conformal inheritance function.

Projective curvature tensor W^h_{ijk} on spacetime (V_4, g) is given by the following Equation [20]

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{3}[\delta^h_j R_{ik} - \delta^h_k R_{ij}]. \quad (11)$$

Now, we introduce the notion of projective curvature inheritance (PCI) symmetry. If 'D' is replaced by projective curvature tensor C^h_{ijk}, then Equation (3) defines "projective curvature Inheritance" and represents the following equation

$$\pounds_\xi W^h_{ijk} = 2\Omega W^h_{ijk}, \quad (12)$$

where $\Omega = \Omega(x^i)$ can be called projective inheritance function.

Recent studies [8–19] exhibited a deep interest in the study of the different symmetries (in particular, curvature, Ricci, projective, matter, semiconformal symmetry [23,24], and conharmonic curvature inheritance [9]). These geometrical symmetries appear strongly beneficial towards the exact solutions of Einstein field Equation (1).

The literature [25] on the classification of spacetimes is very wide and still expanding with a result of elegance according to its isometries, groups of motion, null tetrad methods, spinor or generating techniques. These studies [22,26–30] on symmetries played a significant role in the classification of spacetimes, generating many interesting results with fruitful applications. The geometric structure of metrics (well-explained by curvature and Ricci tensors) and the physical structure of spacetime is enabled through the energy-momentum tensor. Ricci flat spaces [7] or conformally flat spaces have attracted the attention of many researchers. The readers may also see the work on the symmetries of curvature and Weyl tensors by G. S. Hall et al. [31].

In this paper, we consider the curvature inheritance symmetry in [7] Ricci flat/empty spacetime ($R_{ij} = 0$) or vacuum spacetime ($T_{ij} = 0$) in terms of the basis of null tetrad classification [32].

The plan of this paper is as follows: In Section 2 the main results are given with proof and its consequence result. The curvature inheritance symmetries admitted by vacuum pp-waves of Petrov type N are given in Section 3. We further explain well the steps or terms used in the proofs of the results or examples, and all related calculations are arranged as apendices in the paper.

2. Main Results

In recent years, various researchers have studied different spacetime symmetries and obtained many geometrically and physically important results. However, the application of such inheritance symmetries to the Ricci flat spacetime of general relativity is restricted by the following theorem:

Theorem 1. *If Ricci flat spacetimes (V_4, g), which are not of Petrov type N, admit curvature inheritance symmetry, then the only existing symmetries are conformal motions.*

Proof. Duggal [8] obtained the necessary condition for a curvature inheritance symmetry in Riemannian spaces, given by

$$R^a_{jkl}\hbar_{ia} + R^a_{ikl}\hbar_{ja} = 0, \tag{13}$$

where

$$\hbar_{ij} = £_\xi g_{ij} = \xi_{i;j} + \xi_{j;i}. \tag{14}$$

Since Equation (13) is also applicable in spacetimes of general relativity, we can explicitly write all the components of a null tetrad [33] for Equation (13) (see Appendix B). Then, the obtained equations can be easily solved for Ricci flat spacetimes. On first choosing the appropriate canonical forms $(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4)$ for the tetrad components of the Weyl conformal tensor, the resulting equations would be simplified for each Petrov type (except type N, Appendix B). After simple calculations, we can have

$$\hbar_{ij} = £_\xi g_{ij} = 2\phi g_{ij}, \tag{15}$$

where ϕ is a scalar function. Since Equation (15) represents a conformal motion, this leads to the proof of the theorem. □

Corollary 1. *If Ricci flat spacetimes (V_4, g) that are not of Petrov type N admit a proper curvature inheritance symmetry along a vector field ξ, then (V_4, g) is a flat spacetime.*

Proof. From [6], a spacetime (V_4, g) admits a conformal motion generated by conformal Killing vector ξ satisfying Equation (15), and Weyl tensor (5) is invariant under conformal transformations. It follows that every conformal motion must satisfy

$$£_\xi C^h_{ijk} = 0. \tag{16}$$

Curvature inheritance symmetry in Ricci flat spacetime $(C^h_{ijk} = R^h_{ijk})$ is defined as follows:

$$£_\xi C^h_{ijk} = 2\Omega C^h_{ijk}. \tag{17}$$

Comparing Equations (16) and (17), we obtain

$$C^h_{ijk} = 0. \qquad (\Omega \neq 0) \tag{18}$$

Thus, from Equation (6), (V_4, g) is conformally flat. Since the Ricci tensor also is zero, this means that the only possibility is flat spacetimes. This completes the proof. □

From the definition of conformal curvature tensor C^h_{ijk} (as in Equation (5)), for a Ricci flat spacetime, we obtain

$$C^h_{ijk} = R^h_{ijk}. \tag{19}$$

As mentioned in Section 1, if a (V_4, g) admits a Conf CI, then vector ξ defining the Conf CI must satisfy $£_\xi C^h_{ijk} = 2\Omega C^h_{ijk}$. Thus, in a Ricci flat spacetime, every conformal curvature inheritance is a curvature inheritance.

Next, we consider the Weyl projective curvature tensor W^h_{ijk} (defined in (11)) to obtain the following (in a Ricci flat spacetime):

$$W^h_{ijk} = R^h_{ijk}. \tag{20}$$

As mentioned in Section 1, if a (V_4, g) admits a projective curvature inheritance, then vector ξ defining the projective curvature inheritance must satisfy $£_\xi W^h_{ijk} = 2\Omega W^h_{ijk}$. Thus, in a Ricci flat spacetime, every projective curvature inheritance is a curvature inheritance.

Remark 1. *It is obvious that, in empty spacetimes, different types of curvature tensors (such as conformal and projective curvature tensors) are equivalent to Riemann curvature tensors, and their respective inheritance symmetry implies curvature inheritance symmetry.*

For Petrov type N gravitational fields, we have

$$\hbar_{ij} = 2\phi g_{ij} + \alpha l_i l_j, \tag{21}$$

where ϕ and α are real-valued smooth functions on V_4, and l_i represents a principal vector associated to the Weyl tensor satisfying

$$C^i_{jkl} l_i = 0. \tag{22}$$

There arises a natural question as to whether or not the Petrov type N gravitational fields in empty spacetime can admit CI other than Conf M? The answer is affirmative for pp-waves.

3. Vacuum pp-Waves

The line element or metric of the [1,34] plane fronted gravitational waves can be written in terms of the usual coordinates u, v, x, y in the form

$$ds^2 = 2dudv - 2H(v,x,y)dv^2 - 2dxdy, \tag{23}$$

where $H : V_4 \longrightarrow \Re$ is a smooth function on V_4, such that (V_4, g) is a vacuum and Petrov type N. Vector field $\frac{\partial}{\partial u} = l$ spans the principal null vectors of flat spacetime and is covariantly constant. Vector $\frac{\partial}{\partial v}$ is non-null. Null vector $\frac{\partial}{\partial u}$ points in the direction of the pp-wave. Unit vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are spacelike and span a 2-dimensional wave surface along $\frac{\partial}{\partial u}$, where u, v are real, and x, y are the complex conjugates, such that $y = \bar{x}$. The conditions for a purely gravitational waves are

$$\frac{\partial H}{\partial u} = \frac{\partial^2 H}{\partial x \partial y} = 0. \tag{24}$$

These plane-fronted gravitational waves are characterized by any of the following properties: (i) covariantly constant (ii) type N with shear-free, non diverging rays (iii) type N with one-parameter group of affine collineations trajectories (iv) Einstein field mapped conformally onto another Einstein field equation.

We have vectors [29] $l^i = \delta^i_1$, $n^i = \delta^i_2 + H\delta^i_1$, $m^i = \delta^i_0$ and $\bar{m}^i$ where $(x^1, x^2, x^0, x^{\bar{0}}) = (u, v, x, y)$, satisfy the orthonormality conditions $l^i n_i = -m^i \bar{m}_i = 1$, all other contractions being zero. Such a tetrad forms a basis of null vectors in the sense of Newman and Penrose [33], and the intrinsic derivatives for the tetrad vectors are

$$D\phi = l^i \phi_{,i} = l^i \frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial u}, \tag{25}$$

$$\triangle\phi = n^i \phi_{,i} = n^i \frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial v} + H\frac{\partial \phi}{\partial u}, \tag{26}$$

and

$$\delta\phi = m^i \phi_{,i} = m^i \frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial y}. \tag{27}$$

Substituting the coordinates into the commutation relations held by these intrinsic derivatives and using (24), it makes all spin coefficients zero except only

$$\nu = -\frac{\partial H}{\partial x}, \tag{28}$$

where the Newman–Penrose field equations then yield

$$\psi_0 = \psi_1 = \psi_2 = \psi_3 = 0, \;\; \psi_4 = -\frac{\partial^2 H}{\partial x^2}. \tag{29}$$

where ψ_i $(i = 0,1,2,3,4)$ are five independent complex tetrad components of the Weyl tensor C^h_{ijk}. Equation (21) is expressed as

$$£_\xi g_{ij} = 2\phi g_{ij} + \alpha l_i l_j. \tag{30}$$

where ϕ and α are real-valued smooth functions on V_4. The curvature inheritance symmetry in empty-space can be written as $£_\xi C^i_{jkl} = 2\Omega C^i_{jkl}$. Using Equations (22) and (30), this leads to

$$£_\xi C_{ijkl} = 2(\Omega + \phi) C_{ijkl} = \frac{1}{2}\mho C_{ijkl}. \tag{31}$$

Let us assume that $2(\Omega + \phi) = \frac{1}{2}\mho$ and then null tetrad components for Equations (30) and (31) (when $\alpha = 0$) are obtained in [30]. For $\alpha \neq 0$, we write null tetrad components for both equations in Appendix A as Equations (A1) and (A2) respectively. Now, substituting Equations (28) and (29) and zero instead of all spin coefficients except ν and $\bar{\nu}$ into the null tetrad equations of Appendix A from (A3) and (A11), we obtain

$$D\xi_1 = 0, \tag{32}$$

$$\triangle\xi_2 = -\frac{\partial H}{\partial x}\xi_3 - \frac{\partial H}{\partial y}\bar{\xi}_3 + \frac{1}{2}\alpha, \tag{33}$$

$$\delta\xi_3 = 0, \tag{34}$$

$$\triangle\xi_1 + D\xi_2 - \frac{1}{2}\mho = 0, \tag{35}$$

$$\delta\xi_1 + D\xi_3 = 0, \tag{36}$$

$$\delta\xi_2 + \triangle\xi_3 = -\frac{\partial H}{\partial y}\xi_1, \tag{37}$$

$$\bar{\delta}\xi_3 + \delta\bar{\xi}_3 + \frac{1}{2}\mho = 0, \tag{38}$$

$$D\xi_3 = 0, \tag{39}$$

$$-\frac{\partial^3 H}{\partial v \partial x^2}\xi_1 + \frac{\partial^3 H}{\partial x^3}\xi_3 - 2\frac{\partial^2 H}{\partial x^2}(\triangle\xi_1 - \bar{\delta}\xi_3 - \frac{1}{4}\mho) = 0. \tag{40}$$

Equations (32) and (34)–(39) can be integrated, and we obtain

$$\xi_1 = \xi_1(v), \tag{41}$$

$$\xi_2 = -[a + \bar{a} + \dot{\xi}_1]u - xy\dot{a} - \dot{b}y - \dot{\bar{b}}x - H\xi_1 + c(v), \tag{42}$$

$$\xi_3 = a(v)x + b(v), \tag{43}$$

and

$$\frac{1}{2}\mho = -(a + \bar{a}), \tag{44}$$

where $a + \bar{a} = 2k_1$ (constant). Equation (40) is reconsidered by separating variables and rearranging the terms. There are several cases, but in this section, we only discuss the case corresponding to $\frac{\partial^3 H}{\partial x^3} = 0$ (this corresponds to a plane gravitational wave). We also chose a coordinate system, so that

$$H = c_1 x^2 + c_2 y^2. \tag{45}$$

The form of H in Equation (45) is the common solution for the partial differential equations:

(I) $\frac{(\partial^2 H)}{(\partial x \partial y)} = 0$

(II) $\frac{(\partial^3 H)}{(\partial x^3)} = 0$

(III) $\partial H/\partial u = 0$.

where $c_1 = f(v)$ and $c_2 = \bar{f}(v)$ are scalar functions of 'v ' and also $f(v)$ is same as in Equation (30) of α. Further, imposing the condition $(c_1 c_2)^{-\frac{1}{4}}$ $[\log \frac{c_1}{c_2}] \neq$ constant, Equation (40) yields $\xi_1 = 0$, $a = \bar{a}$, and for the general curvature inheritance, the null tetrad components are given by

$$\xi_1 = 0, \tag{46}$$

$$\xi_2 = -2au - xy\dot{a} - \dot{b}y - \dot{\bar{b}}x + c(v\ ), \tag{47}$$

and

$$\xi_3 = a(v)x + b(v). \tag{48}$$

Putting these components in Equation (33), we obtain

$$\frac{1}{2}\alpha = -2\dot{a}u - \ddot{a}xy - y(\ddot{b} - 2c_2\bar{b}) - x(\ddot{\bar{b}} - 2c_1 b) + \dot{c}. \tag{49}$$

If $\alpha = 0$, Equation (21) represents conformal motion. Not all CI are Conf M due to the fact that α does not vanish identically. For $\alpha = 0$, we can easily conclude that the plane gravitational waves (23) admit Conf M along a conformal Killing vector ξ. It is evident that

$$\ddot{b} - 2c_2\bar{b} = 0, \tag{50}$$

$$a = k_1 \quad \text{and} \quad c = k_2,$$

where k_1 and k_2 are arbitrary constants. Equation (50) exhibits four independent solutions, and there exists a six-parameter group of Conf M, while Conf M with $k_1 = 0$ forms a group of motion with five parameters. For $k_2 = b = 0$ and from the results of [30] (Theorems 1 and 2), we conclude that Conf M implies homothetic motion (HM).

In either case, if Ricci flat spacetimes are other than Petrov type N (such as I, II, D, III), then equations of Appendix B from (A13) to (A20) yielding $D\phi$, $\triangle\phi$, and $\delta\phi$ all vanish; thus, ϕ is a constant, and the conformal motion reduces to homothetic motion.

Remark 2. *If the vacuum pp-wave spacetime of Petrov type N admits curvature inheritance symmetry along vector field ξ, then every conformal motion is homothetic.*

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Appendix A

The null tetrad components [30] of Equation (30) are as follows:

$$\xi_{i;j} + \xi_{j;i} = \frac{1}{2}\mho\eta_{ij} + \xi^k(\gamma_{ikj} + \gamma_{jki}) + \alpha l_i l_j, \tag{A1}$$

and Equation (31) as follows

$$\begin{aligned} C_{ijkl;m}\xi^m + C_{mjkl}\xi^m_{;i} + C_{imkl}\xi^m_{;j} + C_{ijml}\xi^m_{;k} + C_{ijkm}\xi^m_{;l} = \frac{1}{2}C_{ijkl}\mho + \xi^m[C_{njkl}(\gamma^n_{im} + \gamma^n_{mi}) \\ + C_{inkl}(\gamma^n_{jm} + \gamma^n_{mj}) + C_{ijnl}(\gamma^n_{km} + \gamma^n_{mk}) + C_{ijkn}(\gamma^n_{lm} + \gamma^n_{ml})]. \end{aligned} \tag{A2}$$

where γ^i_{jk} denotes the Ricci rotation coefficients [20]. Equations (A1) and (A2) can be written explicitly in empty spacetime as follows:

$$D\xi_1 = (\epsilon + \bar{\epsilon})\xi_1 - \bar{\kappa}\xi_3 - \kappa\bar{\xi}_3, \tag{A3}$$

$$\triangle\xi_2 = -(\gamma + \bar{\gamma})\xi_2 + \nu\xi_3 + \bar{\nu}\bar{\xi}_3 + \frac{1}{2}\alpha, \tag{A4}$$

$$\delta\xi_3 = \bar{\lambda}\xi_1 - \sigma\xi_2 - (\bar{\theta} - \beta)\xi_3, \tag{A5}$$

$$\triangle\xi_1 + D\xi_2 - \frac{1}{2}\mho = (\gamma + \bar{\gamma})\xi_1 - (\epsilon + \bar{\epsilon})\xi_2 + (\pi - \bar{\tau})\xi_3 + (\bar{\pi} - \tau)\bar{\xi}_3, \tag{A6}$$

$$\delta\xi_1 + D\xi_3 = (\bar{\theta} + \beta + \bar{\pi})\xi_1 - \kappa\xi_2 + (\epsilon - \bar{\epsilon} - \bar{\rho})\xi_3 - \sigma\bar{\xi}_3, \tag{A7}$$

$$\delta\xi_2 + \triangle\xi_3 = \bar{\nu}\xi_1 - (\bar{\theta} + \beta + \tau)\xi_2 + (\mu + \gamma - \bar{\gamma})\xi_3 + \bar{\lambda}\bar{\xi}_3, \tag{A8}$$

$$\delta\xi_3 + \bar{\delta}\bar{\xi}_3 + \frac{1}{2}\mho = (\mu + \bar{\mu})\xi_1 - (\rho + \bar{\rho})\xi_2 + (\theta - \bar{\beta})\xi_3 + (\bar{\theta} - \beta)\bar{\xi}_3, \tag{A9}$$

$$-\psi_4 D\xi_3 = 0, \tag{A10}$$

$$\xi_1\triangle\psi_4 + \xi_2 D\psi_4 - \xi_3\bar{\delta}\psi_4 - \bar{\xi}_3\delta\psi_4 - \frac{1}{2}\mho\psi_4 + 2\psi_4(\triangle\xi_1 - \delta\xi_3) = 0. \tag{A11}$$

where ξ_i vector fields and $\epsilon, \kappa, \gamma, \nu, \beta, \theta, \tau, \rho, \mu, \pi, \sigma$, and its conjugates represent the spin coefficients. For Petrov type N empty spacetime [30] (pp-waves), the only nonzero spin coefficient is ν.

Appendix B

The null tetrad components [30] of Equation (13) are as follows:

$$\begin{aligned} R_{ijkl;m}\xi^m + R_{mjkl}\xi^m_{;i} + R_{imkl}\xi^m_{;j} + R_{ijml}\xi^m_{;k} + R_{ijkm}\xi^m_{;l} = \frac{1}{2}R_{ijkl}\phi + \xi^m[R_{njkl}(\gamma^n_{im} + \gamma^n_{mi}) \\ + R_{inkl}(\gamma^n_{jm} + \gamma^n_{mj}) + R_{ijnl}(\gamma^n_{km} + \gamma^n_{mk}) + R_{ijkn}(\gamma^n_{lm} + \gamma^n_{ml})]. \end{aligned} \tag{A12}$$

Equation (A12) in empty spacetime can be written explicitly as

$$\psi_2 D\phi - \psi_1\bar{\delta}\phi = 0, \tag{A13}$$

$$\psi_2\triangle\phi - \psi_3\delta\phi = 0, \tag{A14}$$

$$\psi_2\delta\phi - \psi_1\triangle\phi = 0, \tag{A15}$$

$$\psi_2\bar{\delta}\phi - \psi_3 D\phi = 0, \tag{A16}$$

$$\psi_1 D\phi - \psi_0\bar{\delta}\phi = 0, \tag{A17}$$

$$\psi_3\triangle\phi - \psi_4\delta\phi = 0, \tag{A18}$$

$$\psi_0\triangle\phi - \psi_1\delta\phi = 0, \tag{A19}$$

$$\psi_4 D\phi - \psi_3\bar{\delta}\phi = 0. \tag{A20}$$

From the above equations, D, $\triangle$, δ and $\bar{\delta}$ are the intrinsic derivatives defined by Newman and Penrose that satisfy the commutation relations

$$\triangle D - D\triangle = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\triangle - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta, \tag{A21}$$

$$\delta D - D\delta = (\bar{\theta} + \beta - \bar{\pi})D + \kappa\triangle - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta, \tag{A22}$$

$$\delta\triangle - \triangle\delta = -\bar{v}D + (\tau - \bar{\theta} - \beta)\triangle + \bar{\lambda}\bar{\delta} + (\mu + \gamma - \bar{\gamma})\delta, \tag{A23}$$

$$\bar{\delta}\delta - \delta\bar{\delta} = (\mu - \bar{\mu})D + (\bar{\rho} - \rho)\triangle - (\bar{\theta} - \beta)\bar{\delta} - (\bar{\beta} - \theta)\delta. \tag{A24}$$

If the spacetime is of type N ($\psi_0 = 0, \psi_1 = 0, \psi_2 = 0, \psi_3 = 0, \psi_4 \neq 0$), then Equations (A13)–(A20) yield $D\phi = \delta\phi = 0$ (the canonical form for the ψ's is used in each Petrov type). Putting ϕ in (A24) gives

$$(\rho - \bar{\rho})\triangle\phi = 0. \tag{A25}$$

Hence, if $\rho \neq \bar{\rho}$, $\triangle\phi$ must be zero. Now $D\phi$, $\triangle\phi$, and $\delta\phi$ are all zero only if ϕ is a constant and the conformal motion is homothetic, which is a contradiction. Thus, the conclusion is straightforward. If the spacetime is not of Petrov type N, then the only possible spacetime symmetries are conformal motions.

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