Hidden symmetries on Kerr-NUT-(A)dS metrics of Einstein-Sasaki type

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Abstract. The hidden symmetries of higher dimensional Euclideanised Kerr-NUT-(A)dS metrics are investigated. In certain scaling limits these metrics are related to the Einstein-Sasaki ones. The complete set of Killing-Yano tensors of the Einstein-Sasaki spaces are presented. For this purpose the Killing forms of the Calabi-Yau cone over the Einstein-Sasaki manifold are constructed. Two new Killing forms on Einstein-Sasaki manifolds are identified associated with the complex volume form of the cone manifolds. As a concrete example we present the complete set of Killing-Yano tensors on the five-dimensional Einstein-Sasaki $Y(p,q)$ spaces. The corresponding hidden symmetries are not anomalous and the geodesic equations are superintegrable.

1. Introduction

The usual spacetimes symmetries are represented by isometries connected with the Killing vector fields. Slightly more generally, the conformal Killing vector field preserve a given conformal class of metrics. For each of the (conformal) Killing vector fields there exists a conserved quantity for the (null) geodesic motions.

Besides them a spacetime may also possess hidden symmetries generated by higher order symmetric or antisymmetric tensor fields. The symmetric Stäckel-Killing tensors give rise to conserved quantities of higher order in particle momenta. A natural generalization of (conformal) Killing vector fields is given by the antisymmetric (conformal) Killing-Yano tensors. Killing-Yano tensors are also called Yano tensors or Killing forms, and conformal Killing-Yano tensors are sometimes referred as conformal Yano tensors, conformal Killing forms or twistor forms.

In physics, Yano tensors play a fundamental role being related to the separability of field equations with spin, pseudoclassical spinning models, the existence of quantum symmetry operators, supersymmetries, etc.

In this paper we want to take a closer look at the Killing forms of Euclideanised Kerr-NUT-(A)dS metrics which are related to Einstein-Sasaki metrics. Recently Einstein-Sasaki geometries have been the object of much attention in connection with the supersymmetric backgrounds relevant to the AdS/CFT correspondence. On the other hand a lot of interest focuses on higher dimensional black hole spacetimes. The search of hidden symmetries generated by the Killing forms in rotating black hole geometries has an important role for describing the properties of black holes in various dimensions.

The Kerr-NUT-AdS metrics in all dimensions have been constructed in [1]. The general Kerr-NUT-AdS metrics have $(2n-1)$ non-trivial parameters where the spacetime dimension is $(2n+1)$

in the odd-dimensional case and $(2n)$ in the even dimensional case. It was also considered the BPS, or supersymmetric, limits of these metrics. After Euclideanisation, these limits yield in odd dimensions new families of Einstein-Sasaki metrics, whereas the even-dimensional metrics result in the Ricci-flat Kähler manifolds. An alternative procedure was proposed in [2] generalizing the scaling limit of Martelli and Sparks [3]. More precisely, in a certain limit one gets an Einstein-Kähler metric from an even-dimensional Kerr-NUT-(A)dS spacetime and the Einstein-Sasaki space is constructed as a $U(1)$ bundle over this metric. On the other hand, performing the scaling limit of the odd-dimensional Kerr-NUT-(A)dS spacetimes one gets directly the same Einstein-Sasaki space obtained as a $U(1)$ bundle over the Einstein-Kähler metric [2].

The Kerr-NUT-(A)dS metrics possess explicit and hidden symmetries encoded in a series of Stäckel-Killing tensors and Killing vectors [1]. These symmetries allow one constructs a set of quantities conserved along geodesics. Moreover they are functionally independent, in involution and guarantee complete integrability of the geodesic motions [4, 5, 6].

The structure of the hidden symmetries for a Sasaki space is derived from the characteristic Sasakian 1-form. Thus a tower of Killing-Yano and conformal Killing-Yano tensors can be constructed [2]. The corresponding hidden symmetries are purely geometrical, irrespective of the fact whether the Einstein equations are satisfied or not.

The main purposes of this paper is to point out the special case of the higher dimensional Kerr-NUT-(A)dS metrics which are related to the Einstein-Sasaki ones. In this case there are two additional Killing-Yano tensors taking into account that the metric cone is Calabi-Yau [7, 8]. These two exceptional Killing forms can be also described using the Killing spinors of an Einstein-Sasaki manifold [9].

In Section 2 we review some basic facts about the Einstein-Sasaki spaces and their cone manifolds. In the next Section we discuss the Killing forms on Einstein-Sasaki spaces which proceed from Euclideanised Kerr-NUT-(A)dS metrics in certain scaling limits. We identity two new Killing forms associated with the complex volume form of the cone manifold. In Section 4 we restrict to the five-dimensional $Y(p, q)$ manifolds and present the complete set of Killing forms. Finally we give our conclusions in Section 5.

2. Mathematical preliminaries
For convenience we shall only give the briefest account of the mathematical concepts and results needed to study the hidden symmetries on Einstein-Sasaki spaces.

2.1. Killing vector fields and their generalizations
A vector field X on a Riemannian manifold (M, g) is said to be a *Killing vector field* if the Lie derivative of the metric g with respect to X vanishes or, equivalently, if the Levi-Civita connection ∇ of g satisfies

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0,$$

for all vector fields Y, Z on M. A natural generalization of Killing vector fields is given by the *conformal Killing vector fields*, i.e. vector fields with a flow preserving a given conformal class of metrics [10]. On the other hand, a *conformal Killing-Yano tensor* of rank p on a Riemannian manifold (M, g) is a p-form ω which satisfies:

$$\nabla_X \omega = \frac{1}{p+1} X \lrcorner\, d\omega - \frac{1}{n-p+1} X^* \wedge d^* \omega, \tag{1}$$

for any vector field X on M, where ∇ is the Levi-Civita connection of g, n is the dimension of M, X^* is the 1-form dual to the vector field X with respect to the metric g, $\lrcorner$ is the operator dual to the wedge product and d^* is the adjoint of the exterior derivative d. If ω is co-closed in (1), then we obtain the definition of a *Killing-Yano tensor* [10]. It is easy to see that for

$p = 1$, they are dual to Killing vector fields. Moreover, a Killing form ω is said to be a *special Killing form* if it satisfies for some constant c the additional equation

$$\nabla_X(d\omega) = cX^* \wedge \omega \,, \tag{2}$$

for any vector field X on M.

Besides the antisymmetric generalization of the Killing vectors one might also consider higher order symmetric tensors. A symmetric tensor $K_{(i_1...i_k)}$ obeying the equation

$$K_{(i_1...i_k;j)} = 0 \,,$$

is called a *Stäckel-Killing tensor*. For any geodesic with a tangent vector u^i the following object

$$P_K = K_{i_1...i_k} u^{i_1} \cdots u^{i_k} \,,$$

is conserved.

These two generalizations of the Killing vectors could be related. Given two Killing-Yano tensors $\omega^{i_1,...,i_k}$ and $\sigma^{i_1,...,i_k}$ it is possible to associate with them a Stäckel-Killing tensor of rank 2:

$$K_{ij}^{(\omega,\sigma)} = \omega_{ii_2...i_k}\sigma_j^{\ i_2...i_k} + \sigma_{ii_2...i_k}\omega_j^{\ i_2...i_k} \,. \tag{3}$$

Therefore a method to generate higher order integrals of motion is to identify the complete set of Killing-Yano tensors. The existence of enough integrals of motion leads to complete integrability or even superintegrability of the mechanical system when the number of functionally independent constants of motion is larger than its number of degrees of freedom. Let us mention that when a Stäckel-Killing tensor is of the form (3), there are no quantum anomalies thanks to an integrability condition satisfied by the Killing-Yano tensors [11, 12, 13].

2.2. Kähler and Sasakian manifolds

An almost Hermitian structure on a smooth manifold M is a pair (g, J), where g is a Riemannian metric on M and J is an almost complex structure on M, which is compatible with g, i.e.

$$g(JX, JY) = g(X, Y) \,,$$

for all vector fields X, Y on M. In this case, the triple (M, J, g) is called an *almost Hermitian manifold*. Moreover, if J is parallel with respect to the Levi-Civita connection ∇ of g, then (M, J, g) is said to be a *Kähler manifold*. We remark that on a Kähler manifold, the associated Kähler form, i.e the alternating 2-form Ω defined by

$$\Omega(X, Y) = g(JX, Y) \,,$$

is closed. In local holomorphic coordinates $(z^1, ..., z^m)$, the associated Kähler form Ω can be written as

$$\Omega = ig_{j\bar{k}}dz^j \wedge d\bar{z}^k = \sum X_j^* \wedge Y_j^* = \frac{i}{2}\sum Z_j^* \wedge \bar{Z}_j^* \,,$$

where $(X_1, Y_1, ..., X_m, Y_m)$ is an adapted local orthonormal field (i.e. such that $Y_j = JX_j$), and $(Z_j, \bar{Z}_j)$ is the associated complex frame given by

$$Z_j = \frac{1}{2}(X_j - iY_j), \ \bar{Z}_j = \frac{1}{2}(X_j + iY_j) \,.$$

There is an intimate connection between its Kähler form and the volume form (which is just the Riemannian volume form determined by the metric) as follows

$$d\mathcal{V} = \frac{1}{m!}\Omega^m \,,$$

where $d\mathcal{V}$ denotes the volume form of M, Ω^m is the wedge product of Ω with itself m times, m being the complex dimension of M [14]. Hence the volume form is a real (m, m)-form on M.

On the other hand, if the volume of a Kähler manifold is written as

$$d\mathcal{V} = dV \wedge d\bar{V},$$

then dV is the complex volume holomorphic $(m, 0)$ form of M. It is now clear that the complex volume form of a Kähler manifold can be written in a simple way with respect to any orthonormal basis, using complex vierbeins $e_i + Je_i$. In fact, the complex volume form of a Kähler manifold M is, up to a power factor of the imaginary unit i, the exterior product of these complex vierbeins.

2.3. The Kähler cone of an Einstein-Sasaki manifold

Let M be a smooth manifold equipped with a triple (φ, ξ, η), where φ is a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on M. If we have:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

then we say that (φ, ξ, η) is an *almost contact structure* on M [15].

A Riemannian metric g on M is said to be *compatible* with the almost contact structure (φ, ξ, η) if and only if the relation

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

holds for all pair of vector fields X, Y on M.

An almost contact metric structure (φ, ξ, η, g) is a *Sasakian structure* if and only if the Levi-Civita connection ∇ of the metric g satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{4}$$

for all vector fields X, Y on M [16].

A Sasakian structure may also be reinterpreted and characterized in terms of the metric cone as follows. The metric cone of a Riemannian manifold (M, g) is the Riemannian manifold $C(M) = (0, \infty) \times M$ with the metric given by

$$\bar{g} = dr^2 + r^2 g,$$

where r is a coordinate on $(0, \infty)$. Then M is a Sasaki manifold if and only if its metric cone $C(M)$ is Kähler [17]. In particular, the cone $C(M)$ is equipped with an integrable complex structure J and a Kähler 2-form Ω, both of which are parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$. Moreover, M has odd dimension $2n+1$, where $n+1$ is the complex dimension of the Kähler cone. We note that the Sasakian manifold (M, g) is naturally isometrically embedded into the cone via the inclusion

$$M = \{r = 1\} = \{1\} \times M \subset C(M),$$

and the Kähler structure of the cone $(C(M), \bar{g})$ induces an almost contact metric structure (ϕ, ξ, η, g) on M satisfying (4).

A *Einstein-Sasaki manifold* is a Riemannian manifold (M, g) that is both Sasaki and Einstein, i.e. a Sasakian manifold satisfying the Einstein condition

$$Ric_g = \lambda g,$$

for some real constant λ, where Ric_g denotes the Ricci tensor of g. Einstein manifolds with $\lambda = 0$ are called *Ricci-flat manifolds*. Similarly, an *Einstein-Kähler manifold* is a Riemannian manifold (M, g) that is both Kähler and Einstein. The most important subclass of Einstein-Kähler manifolds are the *Calabi-Yau manifolds*, which are Kähler and Ricci-flat.

It is also very important to note that a Sasaki manifold M is Einstein if and only if the cone metric $C(M)$ is Kähler Ricci-flat. In particular the Kähler cone of an Einstein-Sasaki manifold has trivial canonical bundle and the restricted holonomy group of the cone is contained in $SU(m)$, where m denotes the complex dimension of the Kähler cone [7, 18].

2.4. Progression from Einstein-Kähler to Einstein-Sasaki to Calabi-Yau manifolds
Suppose we have an Einstein-Sasaki metric g_{ES} on a manifold M_{2n+1} of odd dimension $2n+1$. An Einstein-Sasaki manifold can always be written as a fibration over an Einstein-Kähler manifold M_{2n} with the metric g_{EK} twisted by the overall $U(1)$ part of the connection [19]

$$ds^2_{ES} = (d\psi_n + 2A)^2 + ds^2_{EK} \,, \tag{5}$$

where dA is given as the Kähler form of the Einstein-Kähler base. This can be easily seen when we write the metric of the cone manifold $M_{2n+2} = C(M_{2n+1})$ as

$$ds^2_{cone} = dr^2 + r^2 ds^2_{ES} = dr^2 + r^2 \left((d\psi_n + 2A)^2 + ds^2_{EK} \right) \,.$$

The cone manifold is Calabi-Yau and its Kähler form can be written as

$$\Omega_{cone} = rdr \wedge (d\psi_n + 2A) + r^2 \Omega_{EK} \,, \tag{6}$$

and the Kähler condition $d\Omega_{cone} = 0$ implies

$$dA = \Omega_{EK} \,, \tag{7}$$

where Ω_{EK} is Kähler form of the Einstein-Kähler base manifold M_{2n}.

The Sasakian 1-form of the Einstein-Sasaki metric is

$$\eta = 2A + d\psi_n \,, \tag{8}$$

which is a special unit-norm Killing 1-form obeying for all vector fields X [8]

$$\nabla_X \eta = \frac{1}{2} X \lrcorner \, d\eta \,, \quad \nabla_X (d\eta) = -2X^* \wedge \eta \,.$$

3. Killing forms on Kerr-NUT-(A)dS space in a certain scaling limit
In recent time new Einstein-Sasaki spaces have been constructed by taking certain BPS [20] or scaling limits [2, 3] of the Euclideanised Kerr-de Sitter metrics.

In even dimensions, performing the scaling limit on the Euclideanised Kerr-NUT-(A)dS spaces, the Einstein-Kähler metric g_{EK} and the Kähler potential A are [2]:

$$g_{EK} = \frac{\Delta_\mu dx_\mu^2}{X_\mu} + \frac{X_\mu}{\Delta_\mu} \left(\sum_{j=0}^{n-1} \sigma_\mu^{(j)} d\psi_j \right)^2 \,,$$

with

$$X_\mu = -4 \prod_{i=1}^{n+1} (\alpha_i - x_\mu) - 2b_\mu \,, \quad A = \sum_{k=0}^{n-1} \sigma^{(k+1)} d\psi_k \,,$$

and

$$\Delta_\mu = \prod_{\nu \neq \mu}(x_\nu - x_\mu)\,, \quad \sigma_\mu^{(k)} = \sum_{\substack{\nu_1 < \cdots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}\ldots x_{\nu_k}\,, \quad \sigma^{(k)} = \sum_{\nu_1 < \cdots < \nu_k} x_{\nu_1}\ldots x_{\nu_k}\,.$$

Here, coordinates $x_\mu\,(\mu = 1,\ldots,n)$ stands for the Wick rotated radial coordinate and longitudinal angles and the Killing coordinates $\psi_k\,(k = 0,\ldots,n-1)$ denote time and azimuthal angles with Killing vectors $\xi^{(k)} = \partial_{\psi_k}$. Also $\alpha_i\,(i = 1,\ldots,n+1)$ and b_μ are constants related to the cosmological constant, angular momenta, mass and NUT parameters [1].

We mention that in the case of odd-dimensional Kerr-NUT-(A)dS spaces the appropriate scaling limit leads to the same Einstein-Sasaki metric (5).

The hidden symmetries of the Sasaki manifold M_{2n+1} are described by the special Killing $(2k+1)$−forms [8]:

$$\Psi_k = \eta \wedge (d\eta)^k \quad , \quad k = 0, 1, \cdots, n-1\,. \tag{9}$$

Semmelmann obtained in [8] that special Killing forms on a Riemannian manifold M are exactly those forms which translate into parallel forms on the metric cone $C(M)$. Therefore, the metric cone being either flat or irreducible, the problem of finding all special Killing forms is reduced to a holonomy problem [21]. In the case of holonomy $U(n+1)$, i.e. the cone $M_{2n+2} = C(M_{2n+1})$ is Kähler, or equivalently M_{2n+1} is Sasaki, it follows that all special Killing forms are spanned by the forms Ψ_k (9). Besides these Killing forms, there are n closed conformal Killing forms (also called $*$-Killing forms)

$$\Phi_k = (d\eta)^k \quad , \quad k = 1, \cdots, n\,. \tag{10}$$

In the case of holonomy $SU(n+1)$, i.e. the cone $M_{2n+2} = C(M_{2n+1})$ is Kähler and Ricci-flat, or equivalently M_{2n+1} is Einstein-Sasaki, there are *two additional* Killing forms of degree n on the manifold M_{2n+1}. These additional Killing forms are connected with the additional parallel forms of the Calabi-Yau cone manifold M_{2n+2} given by the complex volume form and its conjugate [8].

In order to write explicitly these additional Killing forms, we introduce the complex vierbeins on the Einstein-Kähler manifold M_{2n}. First of all we shall write the metric g_{EK} in the form

$$g_{EK} = o^{\hat\mu}o^{\hat\mu} + \tilde{o}^{\hat\mu}\tilde{o}^{\hat\mu}\,,$$

and the Kähler 2-form

$$\Omega = dA = o^{\hat\mu} \wedge \tilde{o}^{\hat\mu}\,,$$

where

$$o^{\hat\mu} = \sqrt{\frac{\Delta_\mu}{X_\mu(x_\mu)}}\,dx_\mu\,, \quad \tilde{o}^{\hat\mu} = \sqrt{\frac{X_\mu(x_\mu)}{\Delta_\mu}}\sum_{j=0}^{n-1}\sigma_\mu^{(j)}d\psi_j\,.$$

We introduce the following complex vierbeins on Einstein-Kähler manifold M_{2n} [22]:

$$\zeta_\mu = o^{\hat\mu} + i\tilde{o}^{\hat\mu} \quad , \quad \mu = 1, \cdots, n\,.$$

On the Calabi-Yau cone manifold M_{2n+2} we take $\Lambda_\mu = r\zeta_\mu$ for $\mu = 1, \cdots, n$ and

$$\Lambda_{n+1} = dr + i\,r\,\eta\,.$$

The standard complex volume form of the Calabi-Yau cone manifold M_{2n+2} is [22]

$$dV = \Lambda_1 \wedge \Lambda_2 \wedge \cdots \wedge \Lambda_{n+1}\,.$$

As real forms we obtain the real respectively the imaginary part of the complex volume form. For example, writing

$$\Lambda_j = \lambda_{2j-1} + i\lambda_{2j}, \ j = 1, ..., n+1 \, ,$$

we obtain that the real part of the complex volume is given by

$$\mathcal{R}e \ dV = \sum_{p=0}^{[\frac{n+1}{2}]} \sum_{\substack{1 \le i_1 < i_2 < ... < i_{n+1} \le 2n+2 \\ (C)}} (-1)^p \lambda_{i_1} \wedge \lambda_{i_2} \wedge ... \wedge \lambda_{i_{n+1}} \, , \tag{11}$$

where the condition (C) in (11) means that in the second sum are taken only the indices $i_1, ..., i_{n+1}$ such that $i_1 + ... + i_{n+1} = (n+1)^2 + 2p$ and $(i_k, i_{k+1}) \ne (2j-1, 2j)$, for all $k \in \{1, ..., n\}$ and $j \in \{1, ..., n+1\}$.

On the other hand, we obtain that the imaginary part of the complex volume is given by

$$\mathcal{I}m \ dV = \sum_{p=0}^{[\frac{n}{2}]} \sum_{\substack{1 \le i_1 < i_2 < ... < i_{n+1} \le 2n+2 \\ (C')}} (-1)^p \lambda_{i_1} \wedge \lambda_{i_2} \wedge ... \wedge \lambda_{i_{n+1}} \, , \tag{12}$$

where the condition (C') in (12) means that in the second sum are considered only the indices $i_1, i_2, ..., i_{n+1}$ such that $i_1 + ... + i_{n+1} = (n+1)^2 + 2p + 1$ and $(i_k, i_{k+1}) \ne (2j-1, 2j)$, for all $k \in \{1, ..., n\}$ and $j \in \{1, ..., n+1\}$.

To find the additional Killing forms on the Einstein-Sasaki spaces we connect them with the additional parallel forms on the metric cone. As real forms they are given by the real and respectively imaginary part of the complex volume form. For this purpose we make use of the fact that for any p-form ω^M on the space M_{2n+1} we can define an associated $(p+1)$-form ω^C on the cone $C(M_{2n+1})$

$$\omega^C := r^p dr \wedge \omega^M + \frac{r^{p+1}}{p+1} d\omega^M \, . \tag{13}$$

Moreover ω^C is parallel if and only if ω^M is a special Killing form (2) with constant $c = -(p+1)$ [8]. Therefore the 1-1-correspondence between special Killing p-forms on M_{2n+1} and parallel $(p+1)$-forms on the metric cone $C(M_{2n+1})$ allows us to describe the additional Killing forms on Einstein-Sasaki spaces.

In order to find the additional Killing forms on the Einstein-Káhler manifold M_{2n+1} we must identify the ω^M form in the complex volume form of the Calabi-Yau cone. An explicit example is presented in the next Section.

4. $Y(p,q)$ **manifolds**

The infinite family $Y(p,q)$ of Einstein-Sasaki metrics on $S^2 \times S^3$ [3, 20, 23, 24]. Such manifolds provide supersymmetric backgrounds relevant to the AdS/CFT correspondence. The total space $Y(p,q)$ of an S^1-fibration over $S^2 \times S^2$ with relative prime winding numbers p and q is topologically $S^2 \times S^3$.

The starting point is the explicit local metric of the 5-dimensional $Y(p,q)$ manifold given by the line element [23, 24, 25]

$$ds_{ES}^2 = \frac{1-c\,y}{6}(d\theta^2 + \sin^2\theta\,d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}(d\psi - \cos\theta\,d\phi)^2$$
$$+ w(y)\left[d\alpha + \frac{ac - 2y + c\,y^2}{6(a-y^2)}[d\psi - \cos\theta\,d\phi]\right]^2 \, , \tag{14}$$

where

$$w(y) = \frac{2(a - y^2)}{1 - cy}, \quad q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2},$$

and a, c are constants. A detailed discussion of the range of these parameters is given in [24] in connection with the regularity properties of the $Y(p, q)$ metrics. For $c = 0$ the metric takes the local form of the standard homogeneous metric on $T^{1,1}$ [26]. Otherwise the constant c can be rescaled by a diffeomorphism and in what follows we assume $c = 1$.

The coordinate change $\alpha = -\frac{1}{6}\beta - \frac{1}{6}c\psi'$, $\psi = \psi'$ takes the line element (14) to the following form

$$ds^2_{ES} = \frac{1 - y}{6}(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{p(y)}dy^2 + \frac{1}{36}p(y)(d\beta + \cos\theta\, d\phi)^2$$
$$+ \frac{1}{9}[d\psi' - \cos\theta\, d\phi + y(d\beta + \cos\theta\, d\phi)]^2,$$

with $p(y) = w(y)\, q(y)$.

From (8) in the case of the $Y(p, q)$ space the Sasakian 1-form is

$$\eta = \frac{1}{3}d\psi' + 2A,\tag{15}$$

with

$$A = \frac{1}{6}[-\cos\theta\, d\phi + y(d\beta + \cos\theta\, d\phi)],$$

connected with local Kähler form Ω_{EK} as in (7).

The form of the metric (14) with the 1-form (15) is the standard one for a locally Einstein-Sasaki metric with $\frac{\partial}{\partial\psi'}$ the Reeb vector field. Note also that the holomorphic (2,0)-form of the Einstein-Kähler base manifold is

$$dV_{EK} = \sqrt{\frac{1 - y}{6p(y)}}(d\theta + i\sin\theta\, d\phi) \wedge \left[dy + i\frac{p(y)}{6}(d\beta + \cos\theta\, d\phi)\right].$$

From the isometries $SU(2) \times U(1) \times U(1)$ the momenta P_ϕ, P_ψ, P_α and the Hamiltonian describing the geodesic motions are conserved [25, 27]. P_ϕ is the third component of the $SU(2)$ angular momentum, while P_ψ and P_α are associated with the $U(1)$ factors. Additionally, the total $SU(2)$ angular momentum given by

$$J^2 = P_\theta^2 + \frac{1}{\sin^2\theta}(P_\phi + \cos\theta P_\psi)^2 + P_\Psi^2,\tag{16}$$

is also conserved.

In what follows we are looking for further conserved quantities specific for motions in Einstein-Sasaki spaces. First of all, according to (9), the Killing 1-form η (15) together with the third rank form

$$\Psi = \eta \wedge d\eta$$
$$= \frac{1}{9}[(1 - y)\sin\theta\, d\theta \wedge d\phi \wedge d\psi' + dy \wedge d\beta \wedge d\psi'$$
$$+ \cos\theta\, dy \wedge d\phi \wedge d\psi' - \cos\theta\, dy \wedge d\beta \wedge d\phi + (1 - y)y\sin\theta\, d\beta \wedge d\theta \wedge d\phi],$$

are special Killing forms (2) with constants $c = -2$ and $c = -4$ respectively. Let us note also that Φ_k (10) with $k = 1, 2$ are closed conformal Killing forms.

On the Calabi-Yau manifold $C(M_{2n+2})$ the Kähler form (6) with the Sasakian 1-form (15) is

$$\Omega_{cone} = r^2 \frac{1-cy}{6} \sin\theta \, d\theta \wedge d\phi + \frac{r^2}{6} dy \wedge (d\beta + \cos\theta \, d\phi)$$
$$+ \frac{1}{3} r dr \wedge [y \, d\beta + d\psi' - (1-cy)\cos\theta \, d\phi], .$$

The complex volume holomorphic $(3,0)$ form on the metric cone is [26, 28]

$$dV_{cone} = e^{i\psi'} r^2 dV_{EK} \wedge (dr + ir \wedge \eta)$$
$$= e^{i\psi'} r^2 \sqrt{\frac{1-cy}{6p(y)}} (d\theta + i\sin\theta \, d\phi)$$
$$\wedge \left[dy + i\frac{p(y)}{6}(d\beta + \cos\theta \, d\phi) \right]$$
$$\wedge \left[dr + i\frac{r}{3}[y \, d\beta + d\psi' - (1-cy)\cos\theta \, d\phi] \right]. \tag{17}$$

Extracting from the complex volume form (17) the form ω^M on the Einstein-Sasaki space according to (13) for $p = 2$ we get the following additional Killing 2-forms of the $Y(p,q)$ spaces written as real forms:

$$\Xi = \operatorname{Re}\omega^M = \sqrt{\frac{1-y}{6\, p(y)}}$$
$$\times \left(\cos\psi' \left[-dy \wedge d\theta + \frac{p(y)}{6}\sin\theta \, d\beta \wedge d\phi \right] - \right.$$
$$\left. - \sin\psi' \left[-\sin\theta \, dy \wedge d\phi - \frac{p(y)}{6} d\beta \wedge d\theta + \frac{p(y)}{6}\cos\theta \, d\theta \wedge d\phi \right] \right),$$

$$\Upsilon = \operatorname{Im}\omega^M = \sqrt{\frac{1-y}{6\, p(y)}}$$
$$\times \left(\cos\psi' \left[-\sin\theta \, dy \wedge d\phi - \frac{p(y)}{6} d\beta \wedge d\theta + \frac{p(y)}{6}\cos\theta \, d\theta \wedge d\phi \right] - \right.$$
$$\left. + \sin\psi' \left[-dy \wedge d\theta + \frac{p(y)}{6}\sin\theta \, d\beta \wedge d\phi \right] \right),$$

The Stäckel-Killing tensors associated with the Killing forms Ψ, Ξ, Υ are constructed as in (3). The list of the non vanishing components of these Stäckel-Killing tensors is quite long and will be given elsewhere. Together with the Killing vectors P_ϕ, P_ψ, P_α and the total angular momentum J^2 (16) these Stäckel-Killing tensors provide the superintegrability of the $Y(p,q)$ geometries.

5. Concluding remarks
In this paper we presented the complete set of Killing forms on Einstein-Sasaki spaces associated with Euclideanised Kerr-NUT-(A)dS spaces in a certain scaling limit. The multitude of Killing-Yano and Stäckel-Killing tensors makes possible a complete integrability of geodesic equations. In the case of geodesic and Klein-Gordon equations, the existence of separable coordinates is connected with Stäckel-Killing tensors. On the other hand from (conformal) Killing-Yano tensors one can construct first order differential operators which commute with Dirac operators

[29]. In [30, 31] it was shown that the solutions of Dirac equation in general higher dimensional Kerr-NUT-(A)dS spacetimes can be found by separating variables and the resulting ordinary differential equations can be completely decoupled. It is interesting to study separability and eigenvalues of Dirac operators on Einstein-Sasaki manifolds. Let us note also that in the higher dimensional Kerr-NUT-(A)dS spacetimes the stationary string equations are completely integrable [32]. An important open question is a separability problem for the gravitational perturbations in higher dimensional rotating black holes spacetimes, some preliminary results being achieved recently [33].

Another important direction of research is whether the Killing forms are also intrinsically linked to other higher spin perturbations. It is still an open question whether massless field equations, e. g. the Maxwell field, allow separation of variables in Kerr-NUT-(A)dS spaces.

These remarkable properties of higher dimensional black hole solutions offer new perspectives in investigation of hidden symmetries of other spacetimes structures. A possible extension of these techniques can be performed on the spaces with mixed 3-structures which appear in many modern studies [17, 22]. Finally we mention some recent extensions of the Killing-Yano symmetry in the presence of skew-symmetric torsion. Preliminary results [34, 35] indicate that Killing forms in the presence of torsion preserve most of the properties of the standard Killing forms.

As an exemplification of the general framework we have presented the complete set of Killing forms on five-dimensional Einstein-Sasaki $Y(p,q)$ spaces. The multitude of Stäckel-Killing tensors associated with these Killing forms implies the superintegrability of the geodesic motions.

In connection with the third rank Killing-Yano tensors on the $Y(p,q)$ spaces let us note an interesting geometrical interpretation of the Lax representation [36, 37, 38].

These remarkable properties of the Killing forms offer new perspectives in the investigation of the supersymmetries, separability of Hamilton-Jacobi, Klein-Gordon, Dirac equations on Einstein-Sasaki spaces.

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