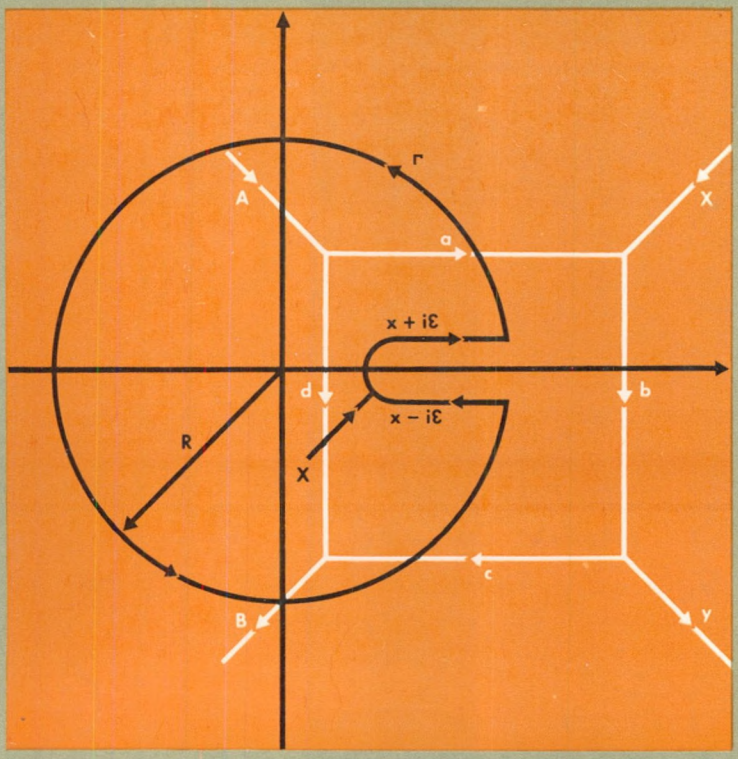

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SELECTED TOPICS IN NUCLEAR THEORY

LECTURES PRESENTED
AT AN
INTERNATIONAL SUMMER SCHOOL,
LOW TATRA MOUNTAINS,
20 AUGUST - 9 SEPTEMBER 1962,
ORGANIZED BY THE
NUCLEAR RESEARCH INSTITUTE
OF THE CZECHOSLOVAK ACADEMY
OF SCIENCES,
WITH THE CO-OPERATION
OF THE IAEA

INTERNATIONAL ATOMIC ENERGY AGENCY, VIENNA 1963

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FOREWORD

An International Summer School on Selected Topics in Nuclear Theory was held during the period 20 August to 9 September 1962 in the Low Tatra Mountains, Czechoslovakia, under the auspices of the Nuclear Research Institute of the Czechoslovak Academy of Sciences, with financial support from the International Atomic Energy Agency.

In view of the wide interest of the seven topics considered there and of the speed with which the field of theoretical physics is developing, the Agency decided to make available its facilities for rapid publication and to publish the lectures under its own imprint; however, all editorial and composition work has been performed under the supervision of the general editor, Dr. F. Janouch of the Nuclear Research Institute of the Czechoslovak Academy of Sciences.

The problem of keeping in touch with the rapidly changing but fundamental field of theoretical physics is a difficult one, particularly for scientists in the developing countries. It is hoped that such publications as the present one and the companion volume containing the lectures presented at the Agency's Seminar on Theoretical Physics at Trieste will help, at least in a modest fashion, to overcome these difficulties.

SIGVARD EKLUND
Director General

December 1962

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INTRODUCTION

The lectures published in the present volume under the general title "Selected Topics in Nuclear Theory" were given from 20 August to 8 September 1962 at the International Summer School organized in the Low Tatra Mountains, Czechoslovakia, by the Nuclear Research Institute of the Czechoslovak Academy of Sciences with the co-operation of the International Atomic Energy Agency.

The lectures are devoted exclusively to theoretical low-energy nuclear physics. Low-energy nuclear physics is, of course, of great interest to countries that are unable to construct the expensive equipment necessary for experimental research in high-energy physics. Moreover, theoretical physicists specializing in low-energy physics are everywhere lacking. One aim of the Summer School was to give young scientists an opportunity of taking a short course in low-energy physics and learning something about achievements in regard to it.

The title "Selected Topics in Nuclear Theory" correctly reflects the principles upon which the programme was drawn up: it is quite clearly impossible, in three weeks, to cover the whole of theoretical nuclear physics in any great detail. The organizers tried to select topics that would reveal where great progress has recently been made or where the prospects are particularly good.

A few words should be said about the lectures themselves, which divide roughly into three groups.

Firstly, the theory of direct nuclear reactions, which in recent years has begun to play an ever greater role in the study of nuclear structures.

One of the characteristic features of direct nuclear reactions (e.g. stripping and pick-up) is that only a small number of degrees of freedom, from the many which characterize the atomic nucleus, takes part in the reaction. This fact makes the theoretical treatment of direct reactions more hopeful.

The lectures of Professor Austern show why direct nuclear reactions are so convenient for the study of nuclear structures and present a systematic explanation of distorted wave methods, their application and achievements. The concluding section discusses correlations between direct nuclear reactions and reactions occurring via the compound nucleus, together with their distinguishing characteristics.

The successes obtained by means of the so-called dispersion relations in work on the theory of elementary particles led to the idea of applying this method also to the theory of nuclear reactions, i.e. in work on low-energy physics.

Professor Shapiro describes the dispersion method and shows how to apply it to direct nuclear reactions. The dispersion method does not make use of wave functions, but only of certain general features of reaction amplitudes, e.g. analyticity and unitarity. In addition to providing interesting results (the expression of the amplitudes of the most diverse reactions by means of the amplitudes of the simplest), it also permits a better understanding of some previous results as, for instance, Butler's theory of stripping.

The second group of lectures is concerned with models of the atomic nucleus. As the nucleus is a many-particle system and the forces acting between the particles are not exactly known, its theoretical description is necessarily based on model representations.

One of the most widely used atomic nucleus models is the shell model, constructed on the analogy of the electron shell of the atom. The nucleons are regarded as free and moving in a certain average nucleon field. Despite the large numbers of other models that have appeared recently, the shell model remains one of the most developed and the one most often used by physicists for specific computations.

In Professor Elliott's lectures the present state of shell model theory is formulated and the connection between the shell model and other models of the atomic nucleus is discussed. Professor Elliott does not restrict himself to the usual elements of shell model theory, which can be found in any text book on nuclear theory, but discusses lesser-known facts, e.g. the theory of multiplets and supermultiplets, a method of calculating the properties of light nuclei without using genealogical coefficients, and so on. Group theory methods are adopted for the classification of the wave functions of a many-nucleon system. Professor Elliott's lectures also show how the fundamental principles of the shell model (taking configuration mixing into account) make it possible to obtain wave functions that have some of the characteristic properties of the wave functions of deformed nuclei.

Professor de-Shalit deals with the closely allied theory of electromagnetic transitions in nuclei, the study of which provided the main basis for the shell model, yielding important data on excited states of nuclei (their energies, moments, parities, etc.). Such data are generally very reliable, since electromagnetic interactions in the nucleus are well known. Apart from electromagnetic transitions, he discusses other (static) properties of nuclei, i.e. their magnetic and electric moments, and shows which measurements of these values make it possible to reach model-independent conclusions about the structure of the nucleus and the forces acting between nucleons.

The theory of pairing correlations in nuclei is dealt with by Professors Soloviev and Belyaev. In the first models in which axial and non-axial nuclei were examined, deformations were considered phenomenologically as parameters of the theory. In these lectures it is assumed that nucleons move in a self-consistent field with some residual interaction. This residual (pairing) interaction is taken into account by a method similar to that used in the modern theory of superconductivity - a method first proposed by Professor Bogolyubov (who was unfortunately prevented by illness from participating in the School), which is now known as the superconductive model of atomic nuclei.

It is very interesting to find that certain very specific properties of nuclei can be explained in terms of extremely general concepts. Professor Belyaev shows, for instance, that the superconductivity of metals at very low temperatures and the non-sphericity of nuclei are a consequence of one and the same physical law.

The model examined in Professor Soloviev's lectures is used to explain the characteristics of fundamental and low-lying excited levels of medium

and heavy particles (energy, spin, parity); considerable space is devoted to methods of calculating α, β and γ-transition probabilities in this model.

Professor Belyaev uses the superconductivity model to calculate collective effects in nuclei, discussing the dependence of equilibrium conditions in atomic nuclei on the number of nucleons in the nucleus, inertia moments in deformed nuclei, and the genesis of vibrational levels in spherical nuclei.

In the third group, Professor Tolhoek discusses in his lectures the theory of weak interactions between nucleons and leptons. Systematically and in detail, he explains the modern theory of weak interactions as newly formulated after the discovery of parity non-conservation in weak interactions. He gives a detailed review of nuclear β-decay and deals with some general aspects of weak interactions (V-A-interaction, intermediate bosons, the electron and muon neutrino, etc.).

Great attention is devoted to the fundamentals of muon nuclear physics. Although muon nuclear physics stands on the boundary between the physics of elementary particles and nuclear physics, experimental and theoretical study of μ-capture in atomic nuclei can not only facilitate the elucidation of various weak interaction problems but also facilitates study of the structure of the atomic nucleus itself; hence its inclusion among the selected topics in nuclear theory is justified at present.

The lectures are being printed from manuscripts received beforehand or actually during the Summer School, with minor and mainly unimportant alterations and amendments. No uniform system of symbols has been adopted in the book (for example for Clebsch-Gordan coefficients, spherical functions, etc.), various systems of units are used, and so on. Uniformity would have demanded substantial alterations in the manuscripts and a corresponding delay in publication; the editor hopes that any deficiences there may be in this respect - unfortunately by no means rare in the physics literature - will not unduly worry the reader.

It was not considered necessary to include papers read at the seminars by individual participants, as in most cases they have been or will be published in scientific journals.

The editor thanks the International Atomic Energy Agency for the understanding spirit with which it has undertaken the complicated business of publishing the lectures, and also for its assistance in preparing the book for publication. He must also express his gratitude to Mrs. H. Watney-Kaczér for her help in preparing the manuscript for the press and to P. Winternitz and P. Vogel for many suggestions which were of assistance in getting the manuscript ready for printing.

F. Janouch

INTRODUCTION

Le présent ouvrage réunit, sous le titre «Certains aspects de la physique nucléaire théorique», les conférences données du 20 août au 8 septembre 1962 au Cours international d'été, qui a été organisé dans la Basse Tatra (Tchécoslovaquie) par l'Institut de recherches nucléaires de l'Académie tchécoslovaque des sciences, en collaboration avec l'Agence internationale de l'énergie atomique.

Les conférences portent exclusivement sur la physique nucléaire théorique des basses énergies. En effet, cette partie de la physique nucléaire présente un grand intérêt pour les petits pays qui n'ont pas la possibilité de construire les installations coûteuses indispensables pour les recherches expérimentales sur la physique des hautes énergies. On sait qu'il y a un peu partout pénurie de théoriciens de la physique des basses énergies. C'est pourquoi un des objectifs du Cours d'été était de permettre à de jeunes théoriciens de suivre un enseignement général sur la physique des basses énergies et de se faire une idée des progrès accomplis dans ce domaine.

Le titre de l'ouvrage, qui était aussi celui du Cours, fait parfaitement ressortir le principe qui avait présidé à l'établissement du programme du Cours d'été; il était manifestement impossible d'aborder en trois semaines, d'une façon quelque peu approfondie, tous les aspects de la physique nucléaire théorique. Les organisateurs du Cours s'étaient donc efforcés de choisir certains domaines où de grands progrès ont pu être accomplis ces temps derniers ou qui offrent de grandes possibilités pour l'avenir.

Les conférences peuvent être classées en trois groupes.

Le premier groupe de conférences porte sur la théorie des réactions nucléaires directes qui, depuis quelques années, a joué un rôle de plus en plus important dans l'étude de la structure du noyau atomique.

Rappelons d'abord que les réactions nucléaires directes, soient les réactions dites d'épuisement («stripping») ou de ramassage («pick-up»), sont caractérisées par le fait qu'un petit nombre des degrés de liberté propres aux noyaux atomiques est utilisé dans les réactions. Ce fait permet d'envisager une étude théorique des réactions nucléaires.

Les conférences du Professeur Austern expliquent pourquoi ce sont précisément les réactions nucléaires directes qui facilitent l'étude de la structure du noyau. Elles décrivent d'une manière détaillée la méthode des ondes distordues ainsi que ses applications et ses succès. Dans ses conclusions, le conférencier a examiné les relations entre les réactions nucléaires directes et celles qui donnent lieu à la formation transitoire d'un noyau composé, ainsi que leurs caractéristiques respectives.

Les progrès accomplis dans la théorie des particules élémentaires au moyen des relations de dispersion ont fait penser à la possibilité d'appliquer cette méthode à la théorie des réactions nucléaires, c'est-à-dire au domaine des basses énergies.

Dans ses conférences, le Professeur Shapiro a exposé la méthode des relations de dispersion et indiqué comment elle peut être appliquée aux réactions nucléaires directes. Cette méthode qui, au lieu d'utiliser des fonctions d'ondes, ne fait appel qu'à certaines propriétés de toutes les amplitudes de réactions, savoir: l'analyticité et l'unitarité, donne non seule-

ment un certain nombre de résultats intéressants (expression des amplitudes de réactions très diverses par celles de réactions très simples), mais encore la possibilité de mieux comprendre certains résultats obtenus antérieurement, par exemple la théorie de l'épuisement («stripping») de Butler.

Le deuxième groupe de conférences porte sur les modèles du noyau atomique. Etant donné que le noyau comprend de nombreuses particules dont les forces d'interaction sont encore mal connues, sa description théorique se fonde nécessairement sur des modèles.

On sait qu'un des modèles du noyau atomique les plus utilisés est un modèle en couches construit en s'inspirant des couches électroniques de l'atome; dans ce modèle, les nucléons sont libres et se déplacent dans un champ nucléonique moyen. Un grand nombre d'autres modèles ont été imaginés récemment, mais le modèle en couches reste un des plus perfectionnés et c'est celui que les physiciens emploient habituellement dans les calculs.

Au cours de ses conférences, le Professeur Elliott a décrit l'état actuel de la théorie du modèle en couches ainsi que ses relations avec d'autres modèles. Il ne s'est pas borné à exposer des faits bien connus dont on peut trouver une description dans n'importe quel livre de théorie nucléaire, mais il a expliqué des éléments moins connus tels que la théorie des multiplets et des supermultiplets ainsi qu'une méthode de calcul des propriétés des atomes légers permettant de ne pas utiliser les coefficients de parenté relative, etc. Pour la classification des fonctions d'ondes d'un système à plusieurs nucléons, il utilise la théorie des groupes. En plus de ces questions, le Professeur Elliott a expliqué comment les principes fondamentaux du modèle en couches (compte tenu du mélange des configurations) permettent d'obtenir des fonctions d'ondes présentant certaines propriétés caractéristiques qui correspondent aux fonctions d'ondes des noyaux déformés.

Un enseignement étroitement lié aux conférences du Professeur Elliott est celui que le Professeur de-Shalit a dispensé et qui porte sur la théorie des transitions électromagnétiques dans les noyaux. On sait que la définition du modèle en couches est fondée avant tout sur l'étude des transitions électromagnétiques dans les noyaux, qui avait permis d'obtenir des renseignements importants sur les états d'excitation des noyaux, leurs énergies, moments nucléaires, parités, etc. Ce sont, d'une manière générale, des données très sûres, du fait que les interactions électromagnétique dans le noyau sont parfaitement connues. Le Professeur de-Shalit étudie non seulement les transitions électromagnétiques, mais encore certaines autres propriétés (statiques) des noyaux (c'est-à-dire leurs moments magnétiques et électriques) et indique les mesures qui permettent de dégager des conclusions - autant que possible sans tenir compte des modèles concrets - sur la structure du noyau et les forces qui s'exercent entre les nucléons.

La théorie des corrélations de paires dans les noyaux a été exposée par les Professeurs Soloviev et Belyaev. Dans les premiers modèles, où l'on distinguait les noyaux axiaux et non axiaux, les déformations avaient été introduites d'une manière phénoménologique, en tant que paramètres de la théorie. MM. Soloviev et Belyaev admettent que les nucléons se déplacent dans un champ autoconsistant avec une certaine interaction résiduelle. Pour tenir compte de cette interaction (par paires) résiduelle, ils ont recours

à une méthode analogue à celle qui est utilisée dans la théorie moderne de la supraconductivité. Cette méthode, qui avait été proposée tout d'abord par le Professeur Bogolyubov (que la maladie a malheureusement empêché de prendre part aux travaux du Cours), a reçu le nom de modèle supraconducteur du noyau atomique.

Il est intéressant de constater que plusieurs propriétés très particulières des noyaux peuvent être expliquées à l'aide de considérations très générales; ainsi, le Professeur Belyaev démontre que la supraconductivité des métaux à des températures très basses et l'asphéricité des noyaux résultent d'une même loi physique.

Dans ses conférences, le Professeur Soloviev s'est servi du modèle étudié pour expliquer les propriétés des niveaux fondamentaux et des niveaux excités inférieurs des noyaux moyens et lourds (énergies, spins et parités). Il a consacré une grande partie de son temps aux méthodes permettant de calculer les probabilités des transitions α, β et γ pour ce modèle.

De son côté, le Professeur Belyaev a utilisé le modéle supraconducteur du noyau pour calculer les effets collectifs dans les noyaux. Il a examiné dans quelle mesure l'état d'équilibre des noyaux atomiques dépend du nombre des nucléons du noyau; il a étudié en outre les moments d'inertie des noyaux déformés et l'origine des niveaux de vibration dans des noyaux sphériques.

Le troisième groupe de conférences comprend notamment celles que le Professeur Tolhoek a consacrées à la théorie des interactions faibles entre nucléons et leptons. Le conférencier a présenté d'une façon systématique et minutieuse la théorie moderne des interactions faibles dans son aspect nouveau, telle qu'elle a été formulée après la découverte du principe de la non-conservation de la parité dans les interactions faibles. Après avoir fait un exposé détaillé de la désintégration nucléaire β, il a traité certaines questions générales relatives aux interactions faibles (interaction V-A, bosons intermédiaires, neutrino-électron et neutrino-muon, etc.).

Le Professeur Tolhoek a accordé une grande place à un exposé des principes de la physique nucléaire du muon. Bien que cet aspect de la physique nucléaire se trouve à la limite du sujet et appartienne déjà à la physique des particules élémentaires, l'étude expérimentale et théorique de la capture du muon dans les noyaux atomiques pourra non seulement permettre d'élucider certaines questions relatives aux interactions faibles, mais se révéler utile pour l'analyse de la structure du noyau atomique même. C'est pourquoi on a jugé opportun de retenir cet aspect de la physique nucléaire théorique.

. Pour la publication des conférences, on a utilisé les manuscrits reçus avant ou pendant le Cours d'été, en y apportant quelques modifications ou rectifications peu importantes. On n'a pas unifié les diverses notations (par exemple celles des coefficients de Clebsch-Gordan ou des fonctions sphériques), on a conservé les différents systèmes d'unités, etc. Il n'aurait pas été possible d'opérer une unification sans modifier sensiblement le texte des manuscrits, ce qui eût nécessairement retardé la publication. Le rédacteur espère que ce défaut (hélas assez fréquent dans les publications consacrées à la physique) n'embarrassera pas trop le lecteur.

Le rédacteur n'a pas jugé nécessaire de faire figurer dans le volume les mémoires que certains participants ont présentés lors des réunions

scientifiques; en effet, dans la plupart des cas, il s'agit de travaux qui ont été publiés depuis ou qui vont paraître prochainement dans des revues scientifiques.

Il félicite l'Agence internationale de l'énergie atomique d'avoir entrepris la tâche ardue que constitue la publication des conférences, et il la remercie du concours qu'elle a bien voulu lui apporter pour la mise au point du volume. Il exprime également sa reconnaissance à Mme H. Watney-Kaczér pour l'assistance fournie dans la préparation des manuscrits, ainsi qu'à MM. P. Winternitz et P. Vogel pour leurs nombreuses suggestions, qui ont permis d'améliorer les textes avant de les envoyer à l'impression.

F. Janouch

ВВЕДЕНИЕ

В настоящей книге опубликованы под общим заглавием "Избранные главы теории ядра" лекции, прочитанные в период с 20 августа по 8 сентября 1962 года в Международной Летней школе, организованной Институтом ядерных исследований АН ЧСР в сотрудничестве с МАГАТЭ в Низких Татрах (ЧССР).

Лекции посвящены исключительно теоретической ядерной физике низких энергий. Ядерная физика низких энергий, как известно, представляет большой интерес для малых стран, у которых нет возможности строить дорогостоящие установки, необходимые для экспериментальных исследований в области физики высоких энергий. Вместе с тем известно, что повсеместно чувствуется недостаток теоретиков, специализирующихся в области физики низких энергий. Ввиду этого одной из целей Летней школы было предоставить молодым теоретикам возможность прослушать несколько обзорных курсов по физике низких энергий и получить представление о достижениях в этой области.

Название "Избранные главы теории ядра" лучше всего характеризует принцип, по которому составлялась программа Летней школы, ибо вполне ясно, что невозможно в течение трех недель сколь-нибудь глубоко затронуть все области теоретической ядерной физики. Организаторы школы старались выбрать несколько областей, в которых за последнее время либо достигнуть большой прогресс, либо наметились хорошие перспективы на будущее.

Несколько слов о самих лекциях, которые можно разделить примерно на три группы.

Первая группа касается теории прямых ядерных реакций, которые в последние годы начинают играть все большую роль при изучении структуры атомного ядра.

Как известно, отличительной чертой прямых ядерных реакций (например, реакции срыва и подхвата) является участие в них лишь небольшого числа степеней свободы из тех многих, которые характеризуют атомное ядро. Это обстоятельство делает теоретическое рассмотрение прямых ядерных реакций весьма привлекательным.

В лекциях проф. Остерна было показано, почему именно прямые ядерные реакции удобны для изучения структуры ядра. В этих лекциях излагается метод искаженных волн. В заключительной части обсуждается соотношение между прямыми ядерными реакциями и реакциями, проходящими через составное ядро, и их отличительные особенности.

Успехи, которые были достигнуты в теории элементарных частиц с помощью так называемых дисперсионных ссотношений, привели к идее использовать этот метод и в теории ядерных реакций, т.е. в области низких энергий.

Профессор Шапиро в своих лекциях излагает метод дисперсионных соотношений и показывает возможности его применения к прямым ядерным реакциям. Дисперсионный метод, который использует не волновые функции, а лишь некоторые общие свойства амплитуд реакций,

как, например, аналитичность и унитарность, дает не только ряд интересных результатов (выражение амплитуд самых разнообразных реакций через амплитуды реакций простейших), но и возможность лучше понять некоторые прежние результаты (например, теорию срыва Батлера).

Вторая группа лекций посвящена моделям атомного ядра. Ввиду того, что ядро - система многих частиц, силы взаимодействия между которыми точно не известны, теоретическое его описание неизбежно основано на модельных представлениях.

Как известно, одной из самых распространенных моделей атомного ядра является оболочечная модель, построенная по аналогии с электронной оболочкой атома; нуклоны рассматриваются как свободные и движущиеся в некотором среднем нуклонном поле. Несмотря на большое количество других моделей, появившихся в последнее время, оболочечная модель остается одной из самых разработанных и наиболее часто применяемых среди физиков для конкретных расчетов.

В лекциях профессора Эллиотта дается современная формулировка оболочечной модели и обсуждается ее взаимоотношение с другими моделями атомного ядра. Профессор Эллиотт не ограничивается обычными сведениями по этому вопросу, которые можно в настоящее время найти в любом учебнике по теории ядра, но приводит менее известные данные, как, например, теорию мультиплетов, методы расчета свойств легких ядер без применения генеалогических коэффициентов и т.д. Для классификации волновых функций системы многих нуклонов используются методы теории групп. Показывается, как основные идеи оболочечной модели (с учетом конфигурационных примесей) позволяют получать волновые функции, имеющие некоторые характерные свойства, соответствующие волновым функциям деформированных ядер.

С лекциями профессора Эллиотта очень тесно связан курс профессора де-Шалита, который занимается теорией электромагнитных переходов в ядрах. Как известно, одним из главных оснований для создания оболочечной модели послужило изучение электромагнитных переходов в ядрах, которые давали важные сведения о возбужденных состояниях ядер, их энергиях, моментах, четностях и.т.д. Эти сведения обычно весьма достоверны, так как свойства электро-магнитного взаимодействия в ядре хорошо известны. Профессор де-Шалит обсуждает не только электромагнитные переходы, но и другие свойства ядер (статические), - т.е. их магнитные и электрические моменты, - и показывает, какие измерения этих величин позволяют делать заключения, по возможности независимые от конкретных моделей, о структуре ядра и о силах, действующих между нуклонами.

Теории парных корреляций в ядрах посвящены лекции профессоров Соловьева и Беляева. В первых моделях, в которых рассматривались аксиальные и неаксиальные ядра, деформации вводились феноменологически, как параметры теории. В лекциях профессоров Соловьева и Беляева предполагается, что нуклоны движутся в самосогласованном поле с некоторым остаточным взаимодействием. Учет этого ос-

таточного (парного) взаимодействия производится методом, аналогичным методу, использованному в современной теории сверхпроводимости. Этот метод был предложен впервые профессором Боголюбовым (которому болезнь, к сожалению, помешала принять участие в работе Школы) и получил название сверхпроводимой модели атомного ядра.

Очень интересно отметить, что некоторые весьма специфические свойства ядер можно объяснить с помощью весьма общих соображений: например, в лекциях проф. Беляева показано, что сверхпроводимость металлов при очень низких температурах и несферичность ядер являются следствием одного и того же физического закона.

В лекциях профессора Соловьева рассматриваемая модель применяется для объяснения свойств основных и низко лежащих возбужденных уровней средних и тяжелых ядер (их энергий, спинов и четностей). Значительная часть лекций посвящена методам расчета вероятностей α-, β-, и γ-переходов в этой модели.

Профессор Беляев использует в своих лекциях сверхпроводимую модель ядра для расчета коллективных эффектов в ядрах, обсуждает зависимость равновесного состояния атомных ядер от числа нуклонов в ядре, моменты инерции деформированных ядер и возникновение вибрационных уровней в сферических ядрах.

К третьей части относятся лекции профессора Тольхука, которые посвящены теории слабых взаимодействий между нуклонами и лептонами. В этих лекциях систематически и очень подробно изложена современная теория слабых взаимодействий в том новом виде, как она была сформулирована после открытия несохранения четности в слабых взаимодействиях. Кроме подробного обзора ядерного β-распада в этих лекциях содержатся главы, посвященные некоторым общим вопросам слабых взаимодействий (V-A-взаимодействие, промежуточные бозоны, электронное и мюонное нейтрино и т.д.).

Большое внимание уделено изложению основ мюонной ядерной физики. Хотя мюонная ядерная физика и находится на границе между физикой элементарных частиц и физикой ядра, экспериментальное и теоретическое изучение μ-захвата в атомных ядрах может способствовать выяснению не только ряда вопросов, касающихся слабых взаимодействий, но и оказаться полезным для изучения структуры самого атомного ядра. По этой причине представляется в настоящее время целесообразным включение этих вопросов в избранные главы по теории ядра.

Лекции печатаются по рукописям, полученным либо до начала работы Летней школы, либо в процессе ее работы, с небольшими, в основном несущественными, изменениями и исправлениями. В книге не приведены к единому виду различные обозначения, - например, коэффициентов Клебша-Гордана, сферических функций, применяются разные системы единиц и т.д. Приведение к единому виду нельзя было бы проделать без существенного вмешательства в рукопись, а это привело бы, конечно, к значительному увеличению срока издания. Издатель надеется, что этот недостаток (в физической литературе, к сожалению, не редкий) не вызовет особых затруднений у читателя.

Издатель не считал необходимым включать в книгу доклады, прочитанные на семинарах отдельными участниками Школы, ввиду того, что в большинстве случаев эти работы опубликованы или готовятся к опубликованию в научных журналах.

Издатель благодарит МАГАТЭ за понимание, с которым оно взялось за сложное дело печатания лекций и за помощь в подготовке выпуска книги в свет. Издатель также приносит благодарность г-же Г. Уотней-Кацер за помощь при подготовке рукописи в печать и П. Винтерницу и П. Вогелю за многие замечания, которые помогли улучшить рукопись, подготовленную к печати.

Ф. Яноух

INTRODUCCION

La presente publicación contiene bajo el título general "Temas escogidos de teoría nuclear" las conferencias pronunciadas del 20 de agosto al 8 de septiembre de 1962 en el curso internacional de verano organizado en el Bajo Tatra (Checoeslovaquia) por la Academia de Ciencias de la República Socialista Checoeslovaca en colaboración con el Organismo Internacional de Energía Atómica.

Estas conferencias se dedicaron exclusivamente a la física nuclear teórica de las bajas energías. Como es sabido, esta rama de la física nuclear presenta gran interés para los países pequeños que no están en condiciones de construir las costosas instalaciones indispensables para efectuar investigaciones experimentales sobre física de las energías elevadas. Asimismo, nadie ignora que escasean los teóricos especializados en física de las bajas energías. Por ello, uno de los objetivos del curso de verano fue permitir que los jóvenes teóricos siguiesen una enseñanza general de la física de las bajas energías, haciéndose una idea de los progresos realizados en esta materia.

El título "Temas escogidos de teoría nuclear" caracteriza perfectamente el principio que presidió la elaboración del programa del curso de verano, ya que es evidente que en un curso de tres semanas de duración no es posible tratar a fondo todo el campo de la física nuclear teórica. Por lo tanto, los organizadores del curso procuraron elegir algunas materias en las cuales últimamente se registraron grandes progresos, o que presentan perspectivas favorables para el futuro.

Conviene decir algunas palabras sobre las conferencias, que se pueden clasificar en tres grupos.

El primer grupo se consagró a la teoría de las reacciones nucleares que en estos últimos años están desempeñando un papel cada vez más importante en el estudio de la estructura del núcleo atómico.

Como es bien sabido, las reacciones nucleares directas, es decir, las reacciones de agotamiento ("stripping") o de captación ("pick-up"), se caracterizan por el hecho de que en ellas se utiliza un pequeño número de los grados de libertad propios del núcleo atómico. Este hecho permite abordar teóricamente el estudio de las reacciones nucleares.

En sus conferencias el Profesor Austern explicó por qué dichas reacciones precisamente facilitan la investigación de la estructura del núcleo. Expuso detalladamente el método de las ondas deformadas, sus aplicaciones y resultados. Por último, estudió la relación entre las reacciones nucleares directas y aquellas reacciones en las que se forman núcleos compuestos, así como sus características respectivas.

Los progresos logrados en la teoría de las partículas elementales mediante las relaciones de dispersión, indujeron a aplicar el mismo método a las teorías de las reacciones nucleares, es decir, en la esfera de las bajas energías.

El Profesor Shapiro expuso el método de las relaciones de dispersión y la posibilidad de aplicarlo a las reacciones nucleares directas. Con este método, que en lugar de utilizar funciones ondulatorias recurre a ciertas propiedades generales de las amplitudes de las reacciones, tales como

su analiticidad y unitariedad, se obtienen no sólo una serie de resultados interesantes (como la expresión de las amplitudes de una grande variedad de reacciones mediante las amplitudes de las reacciones muy sencillas), sino que se puede comprender mejor algunos resultados logrados previamente, por ejemplo, la teoría de Butler, o teoría del agotamiento ("stripping").

El segundo grupo de conferencias versó sobre los modelos del núcleo atómico. Como el núcleo es un sistema integrado por muchas partículas, cuyas fuerzas de interacción no se conocen con exactitud, su descripción teórica se basa inevitablemente en modelos.

Uno de los modelos de núcleo atómico más difundido es el modelo de capas, concebido inspirándose en las capas electrónicas del átomo; según dicho modelo, los nucleones son libres y se mueven en un campo nucleónico medio. A pesar de que últimamente se han propuesto muchos otros modelos, dicho modelo sigue siendo uno de los más perfeccionados y el que los físicos utilizan con más frecuencia para sus cálculos.

En sus conferencias, el Profesor Elliott describió el estado actual de la teoría del modelo de las capas, así como sus relaciones con otros modelos. No se limitó a exponer hechos ya conocidos, cuya descripción puede hallarse en cualquier libro de teoría nuclear, sino que explicó temas menos difundidos, tales como la teoría de los multipletes y de los supermultipletes, así como un método de cálculo de las propiedades de los átomos ligeros que permite evitar el uso de los coeficientes de ascendencia relativa, etc. Para clasificar las funciones ondulatorias de un sistema de varios nucleones se utiliza la teoría de los grupos. Igualmente, el Profesor Elliott demostró cómo los principios básicos del modelo de capas (habida cuenta de la mezcla de configuraciones) permiten deducir funciones ondulatorias que poseen algunas de las propiedades características correspondientes a las funciones ondulatorias de los núcleos deformados.

Las conferencias del Profesor de-Shalit, que versaron sobre la teoría de las transiciones electromagnéticas en los núcleos, tuvieron muchos puntos de contacto con las del Profesor Elliott. Como es sabido, la determinación del modelo de capas está basada sobre todo en el estudio de las transiciones electromagnéticas en los núcleos, que proporcionó datos importantes acerca de los estados excitados del núcleo, de sus energías, momentos, paridades, etc. Dichos datos son generalmente muy fidedignos, ya que se conocen bien las interacciones electromagnéticas en el núcleo. El Profesor de-Shalit examinó no solamente las transiciones electromagnéticas, sino también otras propiedades (estáticas) del núcleo, tales como sus momentos magnéticos y eléctricos, y demostró que mediciones de esas magnitudes permiten deducir conclusiones - en lo posible independientes de los modelos concretos - acerca de la estructura del núcleo y de las fuerzas que se ejercen entre los nucleones.

Las conferencias de los Profesores Soloviev y Beliaev trataron de la teoría de las correlaciones de los pares en los núcleos. En los primeros modelos, en los que se distinguían los núcleos axiales y los no axiales, las deformaciones se introdujeron, por razones fenomenológicas, como parámetros de la teoría. Dichos profesores suponen que los nucleones se mueven en un campo autoconsistente con cierta interacción residual. El cálculo de esta interacción residual (por pares) se efectúa por un método

análogo al que se aplica en la teoría moderna de la superconductividad. Este método fue propuesto por primera vez por el Profesor Bogolyubov (que, desgraciadamente, no pudo asistir al curso por razones de salud) y se ha denominado modelo superconductor del núcleo atómico.

Es interesante observar que algunas propiedades muy especiales del núcleo se pueden explicar con ayuda de consideraciones generales: por ejemplo, el Profesor Beliaev demostró que la superconductividad de los metales a temperaturas muy bajas y la no esfericidad del núcleo son el resultado de una misma ley física.

El Profesor Soloviev recurrió en sus conferencias a dicho modelo para explicar las propiedades de los niveles excitados básicos e inferiores de los núcleos medios y pesados (energías, espines y paridades). Se extendió en particular sobre los métodos que permiten calcular la probabilidad de las transiciones alfa, beta y gamma en este modelo.

El Profesor Beliaev empleó el modelo superconductor del núcleo para calcular los efectos colectivos en los núcleos. Estudió las variaciones del estado de equilibrio de los núcleos en función del número de nucleones del núcleo, los momentos de inercia de los núcleos deformados y el origen de los niveles vibratorios en los núcleos esféricos.

En el último grupo de conferencias figuran las del Profesor Tolhoek, consagradas a la teoría de las interacciones débiles entre nucleones y leptones. El citado profesor expuso en forma sistemática y muy detallada la teoría moderna de las interacciones débiles, en una nueva forma, como se volvió a formular después del descubrimiento de la no conservación de la paridad en dichas interacciones débiles. Además de examinar detenidamente la desintegración nuclear beta, explicó algunos problemas generales de las interacciones débiles (interacciones V-A, bosones intermedios, neutrino-electrón y neutrino-muón, etc.).

Explicó detalladamente los fundamentos de la física nuclear del muón. Si bien esta parte de la física nuclear se encuentra en el límite entre la física de las partículas elementales y la física nuclear, el estudio teórico y experimental de la captura del muón en los núcleos atómicos puede servir no sólo para sclarar una serie de cuestiones relacionadas con las interacciones débiles, sino también para estudiar la estructura del propio núcleo atómico. Por este motivo, se ha juzgado conveniente incluir estas cuestiones en una selección de temas de teoría nuclear.

Para publicar las lecciones se han utilizado los originales recibidos antes de empezar el curso de verano o durante el mismo, con pequeñas modificaciones o correcciones secundarias. En el libro no se han unificado las diversas notaciones -por ejemplo, las de los coeficientes de Clebsch-Gordan o de las funciones esféricas- y se emplean distintos sistemas de unidades, etc. Dicha unificación hubiese exigido modificar considerablemente los originales con el consiguiente retardo en su publicación. El editor espera que este defecto (que lamentablemente es frecuente en las obras de física) no supondrá una molestia para el lector.

El editor no juzgó necesario incluir en el volumen los informes leídos en los seminarios por ciertos participantes, puesto que en su mayoría esos trabajos han aparecido ya o aparecerán en breve en revistas científicas.

El editor agradece al OIEA la buena disposición con que se dedicó a la complicada tarea de publicar estas conferencias y la asistencia prestada en la preparación del volumen. Asimismo expresa su reconocimiento a la Sra. H. Watney-Kaczér por su ayuda en la preparación de los manuscritos, y a los señores P. Winternitz y P. Vogel por las numerosas observaciones que permitieron mejorar los textos antes de su impresión.

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DIRECT REACTIONS

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INTRODUCTION

In order to give a unified presentation of one point of view, these lectures are devoted only to a detailed development of the standard theories of direct reactions, starting from basic principles. Discussion is given of the present status of the theories, of the techniques used for practical calculation, and of possible future developments.

The direct interaction (DI) aspects of a reaction are those which involve only a few of the many degrees of freedom of a nucleus. In fact the minimum number of degrees of freedom which must be involved in a reaction are those required to describe the initial and final channels, and DI studies typically consider these degrees of freedom and no others. Because of this simplicity DI theories may be worked out in painstaking detail.

DI processes concern only part of the wave function for a problem. The other part involves complicated excitations of many degrees of freedom, and gives the compound nucleus (CN) effects. While it is extremely interesting to learn how to separate DI and CN effects in an orderly manner, if they are both present in a reaction, no suitable method has yet been found. Instead, current work stresses the kinds of reactions and the kinds of final states in which DI effects dominate and in which CN effects may almost be forgotten. The DI cross-sections which are studied are often extremely large, comparable to elastic scattering cross-sections.

Why DI effects are often strong is a question deeply bound up with our understanding of nuclear structure. For reactions in which the incident and outgoing projectiles are both nucleons the answer is found in the nuclear properties which cause the independent-particle model to be a good approximation. The attractive part of the two-nucleon force is moderately weak and long range, and goes mostly to setting up an average one-body potential well. In first approximation this well gives, at negative energies, the levels of the independent-particle shell model and it gives, at positive energies, the optical model elastic scattering. In second approximation, nucleons interact with each other and we get level splittings or, in the continuum, transitions to other states. Now most CN excited states overlap badly with the ground state, and therefore are excited weakly. Transition to the CN is gradual. On the other hand, low-lying excited states which are in the same shell model configuration as the ground state can be excited strongly in just the first step of interaction. Because they are low-lying they have large decay widths for particle emission. Thus strong DI transitions appear. The study of these transitions is the natural extension of the shell model into the continuum. The DI transitions to low excited states tend to be pure, because the flux which goes into CN excitations is mostly used up in populating the much more numerous final states at high energy. The flux which goes into CN excitations is accounted for in DI calculations by the use of an imaginary term in the optical potential. (Unfortunately, this ima-

ginary term is normally used in a manner which is not very self-consistent.)

Independent-particle motion in the nucleus is not an important aspect of reactions initiated by composite projectiles, such as deuterons, tritons, alpha particles, etc. In these cases excitation of the CN is strong. However, again, low-lying states of the product nucleus are not populated strongly by the CN mode, and do overlap well with the incident channel. Such states can be excited with appreciable probability by incident partial waves of high angular momentum, which are shielded by their angular momentum (through the associated centrifugal barrier) from too rapid transition into the CN. In this way we get surface reactions. Rearrangement collisions, involving transfer of one or two nucleons, take place easily by such surface reactions.

At high bombarding energy DI transitions are enhanced with respect to CN excitation merely because the process of particle emission is fast compared with the process of forming excited states. This is true whatever the nature of the projectile.

CN theories assume a complete equilibrium of the reacting system, with the excitation distributed statistically among all energetically-possible levels. Obviously DI theories are the opposite extreme assumption. One can also imagine in-between kinds of circumstances, which appear as the nucleus gradually relaxes toward complete equilibrium. IZUMO [1] recently presented such a theory, based on the shell model, in which it is assumed that, prior to the development of the full CN equilibrium of all the particles of the nucleus, there always appears a partial equilibrium among only the nucleons in the outermost major shell. It may be that matrix elements connecting the outermost shell with inner shells really are small enough for this effect to appear, at least on an energy-averaged basis.

On the whole I will follow the customary practice of concentrating attention on the DI mode of reaction, and ignoring the CN mode. This means that the analysis only concerns certain kinds of reactions in certain regions of energy. Rather than attempt to define the region of application of the theory, I will just say that it is normally obvious from experiment when we have a case for which a DI theory should work well. Cases of competition between DI and CN have never yet been treated successfully. There will be more about this subject later.

It is possible to say a little bit here about reactions which populate the higher states of the residual nucleus, so that the continuous spectrum of particles emerging from the reaction is studied. Often (n, n') reactions, using 14 MeV neutrons, are of this type. It is obvious that the continuum produced in these reactions comes from a mixture of the DI and CN modes. It is also obviously very difficult to imagine how to determine from experiment the relative amounts of these two modes. Detecting devices measure the flux of particles into a given final state, not the number of degrees of freedom in the wave function. Experimenters often attempt to solve this problem by adopting the very crude idea that because DI cross-sections tend to peak in the forward hemisphere, a division of the cross-section into parts which are fore-aft symmetric or antisymmetric is therefore tantamount to a division into the CN or DI parts. Indeed, in a DI, forward peaking usually does predominate, because the incident projectile interacts with only a small part of the entire mass of the target nucleus. Therefore the momentum transfer in the reaction tends to be of the order of, or less than, the momentum of the incident projectile. But DI cross-sections are not zero in the backward hemisphere, and occasionally are quite large there. Furthermore,

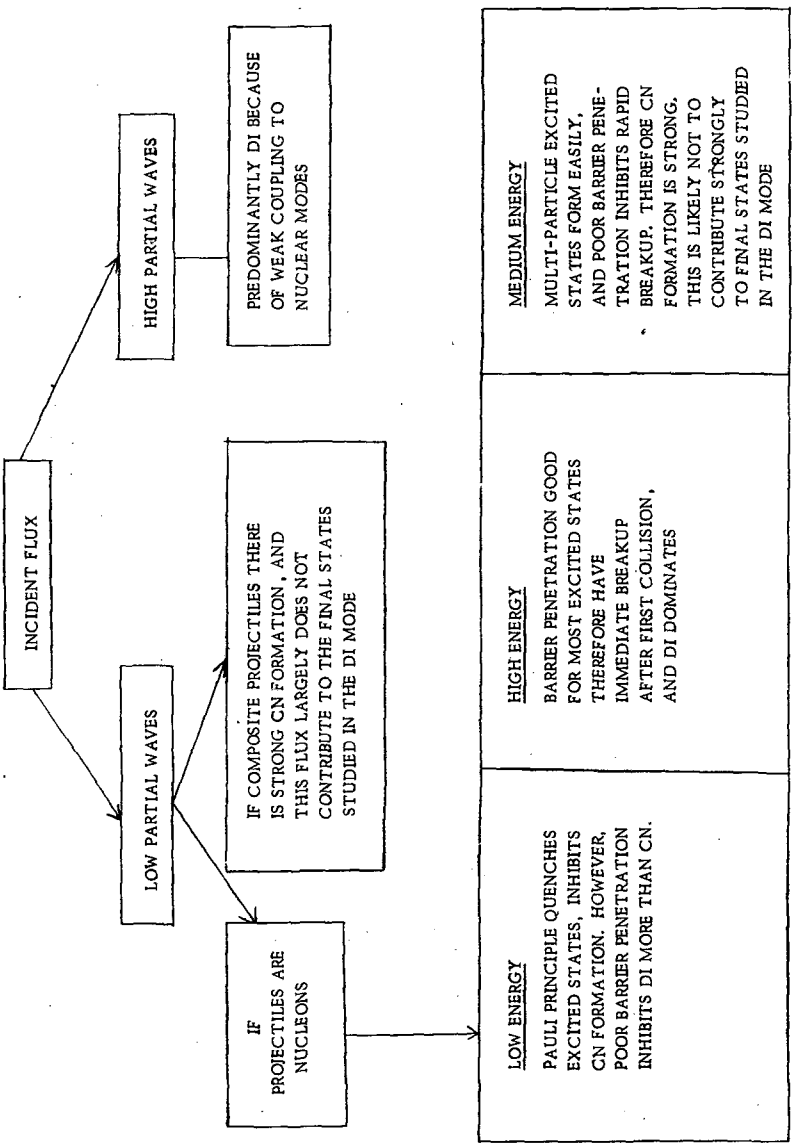

Fig. 1

The DI and CN processes compete for the incident flux

even the well-known strong forward peaks in DI angular distributions occupy small solid angle and do not carry all of the DI flux, much of which is found near 90° scattering angle. For these reasons, the customary procedure of the experimenters necessarily underestimates the DI part of the continuum. They estimate admixtures of 5% to 10%; the correct result may be 10% to 20%. The problem is difficult. What I wish to point out is that an improved analysis may be possible. It may be possible to use some of the more reliable modern techniques for computing DI angular distributions, to generate a set of typical DI curves for the cases in question and to predict from these the average fore-aft asymmetry of the DI part alone. Then, inasmuch as the CN part in the continuous spectrum certainly is isotropic, a separation on the basis of angular distribution could be attempted.

Probably the most reliable way to determine from experiment whether a reaction is of DI type, and to separate DI and CN parts, is to study the reaction as a function of bombarding energy. Those motions which involve just a few degrees of freedom are not usually expected to give rise to narrow resonances. Therefore the DI parts of a reaction should change their properties slowly and smoothly with energy. Even if there are fluctuations, the DI part is related to the average of the cross-section over a range of energy. This line of thought leads us to consider wave packet ideas, and to a further understanding of the meaning of the DI part of a many-body wave function.

1. WAVE PACKETS

The correspondence between scattering experiments and the solutions of the Schrödinger equation is obtained by the use of wave packets. Ordinarily one gives only enough consideration to this method to observe that under limiting conditions an incident wave packet, if expanded in stationary solutions of the problem, leads to an outgoing wave packet whose detailed properties may be factored out from the calculation, and ignored. The outgoing wave packet multiplies the outgoing stationary-state amplitude; this latter factor is non-trivial, and we study it in great detail. I wish to review these ideas, and to observe how they are changed if we do not have the usual limiting conditions.

In the asymptotic region, far from the scattering centre, the stationary state wave function is

$$\psi^{(\pm)}(\vec{r},\ \xi_i,\ \vec{k}) = v_i(\xi_i)\, e^{i(\vec{k}\cdot\vec{r})} + \sum_j f_j^{(\pm)}(\hat{r}_j,\ \vec{k})\ [e^{\pm ik_j r_j}/r_j\ ]v_j(\xi_j). \quad (1.1)$$

The index j labels the channel in which a scattered wave appears. The variable $\vec{r}_j$ is the vector distance between separating fragments in channel j, and ξ_j is the internal variables of the fragments. These variables may have different meanings in different channels, according as the reaction may lead to rearrangements. The functions $v_j(\xi_j)$ are normalized internal wave functions for the fragments. Finally, $f_j^{(\pm)}$ is the scattering amplitude in channel j; it is a function of the unit vector $\hat{r}_j = \vec{r}_j/r_j$, which indicates the direction of observation. Although I have indicated only two-body breakup, the considerations which follow really are more general.

Only two of the possible types of boundary conditions are indicated in Eq. (1.1): either outgoing scattered waves in all channels, or ingoing scat-

tered waves in all channels. In fact we may have outgoing waves in some channels, and ingoing waves in others, or any linear combination of such boundary conditions, perhaps giving standing waves in some channels. With any one choice of such boundary conditions the set of all solutions of the Schrödinger equation, for all values of the incident momentum $\vec{k}$, is a complete set for the scattering problem. We may use the most convenient choice of boundary conditions, in any one case.

Let us consider an incident wave packet, which at time t = 0 is located at the point $\vec{z}_0$, on the negative z axis, and asymptotically far from the scattering centre. It has the form

$$\Psi(\vec{r}, \xi_i; t = 0) = v_i(\xi_i)\, e^{i \vec{k}_0 \cdot (\vec{r} - \vec{z}_0)} \quad Z(\vec{r} - \vec{z}_0). \tag{1.2}$$

The function $Z(\vec{r} - \vec{z}_0)$ expresses the localization near $\vec{z}_0$. It is the <u>envelope</u> of the wave packet at t = 0.

To understand the propagation of the wave packet for t > 0, we expand (1.2) in terms of the energy eigenfunctions (1.1), and make use of the simple time-dependence

$$e^{i [E(k)/\hbar]\, t}$$

which these functions have, where $E(k) = \hbar^2 k^2 / 2M_i^*$, and M_i^* is the reduced mass in channel i. Provided the (+) boundary conditions are used in all channels, corresponding to outgoing waves, none of the scattered wave parts in (1.1) plays any role in this expansion. This is the important result of using outgoing wave solutions *. In the asymptotic region it is obviously true that channels $j \neq i$ play no role in the expansion of the incident wave packet, merely because the internal arrangements of the fragments in j are orthogonal to those in i. In channel i the scattered wave near $\vec{z}_0$, on the negative z axis, is locally of the form $\exp[-i\vec{k}_0 \cdot \vec{r}]$ if outgoing waves are used, but is of the form $\exp[i\vec{k}_0 \cdot \vec{r}]$ if ingoing waves are used. Now if we consider wave packets whose envelope function $Z(\vec{r} - \vec{z}_0)$ changes slowly over distances of the order of the wavelength k_0^{-1}, then the outgoing scattered wave expression obviously has negligible overlap with the incident packet. The ingoing scattered wave expression has excellent overlap with the incident packet, and we avoid much confusion by not using it.

Our first physical result has been obtained, namely, it is only meaningful to work with stationary state eigenfunctions $\psi^{(+)}$, provided

$$|\nabla Z| \ll k_0 Z. \tag{1.3}$$

In fact this condition is difficult to violate, under any circumstances, because the natural durations of the wave packets produced by accelerators are very great.

The expansion in terms of the functions $\psi^{(+)}$ is asymptotically the Fourier expansion

$$\phi(\vec{k} - \vec{k}_0) = (2\pi)^{-3} \int d^3 r' Z(\vec{r}') \, e^{-i\vec{r}' \cdot (\vec{k} - \vec{k}_0)} \tag{1.4}$$

* Wave functions having ingoing waves in channels $j \neq i$ are linearly independent of those having ingoing waves in channel i and in no other channel. It is therefore possible to choose such linear combinations that we deal with functions either all of whose ingoing waves are in channel i, or none are. Functions of the latter type have amplitude zero in the expansion of the incident wave packet.

giving

$$\Psi(\vec{r}, \xi_i; t = 0) = \int d^3k\, e^{-i(\vec{k}\cdot\vec{z}_0)}\, \phi(\vec{k} - \vec{k}_0)\, \psi^{(+)}(\vec{r}, \xi_i; \vec{k}). \qquad (1.5)$$

The propagation of this wave function for future times is given by

$$\Psi(\vec{r}, \xi_i; t) = \int d^3k\, e^{-i(\vec{k}\cdot\vec{z}_0)}\, \phi(\vec{k} - \vec{k}_0)\, e^{i\hbar^{-1}E(k)t}\, \psi^{(+)}(\vec{r}, \xi_i; \vec{k}). \qquad (1.6)$$

Useful results for the scattering problem are computed in terms of the exact solution, (1.6).

We may consider the asymptotic outgoing amplitude in channel j,

$$A_j(\vec{r}_j, t) = \int d^3k\, e^{-i(\vec{k}\cdot\vec{z}_0)}\, \phi(\vec{k} - \vec{k}_0)\, e^{i\hbar^{-1}E(k)t}\, f_j^{(+)}(\hat{r}_j, \vec{k})\, e^{ik_j r_j}. \qquad (1.7)$$

It is understood that k_j is a function of k, in terms of the excitation energy of channel j. Now the customary analysis makes the assumption that $f_j^{(+)}$ depends on $\vec{k}$ much more slowly than do the other factors in the integrand. In particular, the wave packet is assumed to have the property that $\phi(\vec{k} - \vec{k}_0)$ is localized within a range of values $\vec{k}$ so near $\vec{k}_0$, that $f_j^{(+)}$ may be evaluated at $\vec{k} = \vec{k}_0$, and factored out from the integral. We also expand

$$e^{ik_j r_j} \approx e^{ik_j(0)r_j + i\mu k r_j} \qquad (1.8)$$

where $k_j(o)$ is the value of k_j when $k = k_0$, and $\mu = dk_j/dk$. Under these approximations

$$A_j(\vec{r}_j, t) \approx f_j^{(+)}(\hat{r}_j, \vec{k}_0)\, e^{ik_j(0)r_j} \int d^3k\, e^{-i(\vec{k}\cdot\vec{z}_0)}\, \phi(\vec{k} - \vec{k}_0)\, e^{i\hbar^{-1}E(k)t}\, e^{i\mu k r_j}. \qquad (1.9)$$

It is recognized that the integral remaining in (1.9) restores the original wave packet, at a displaced position, corresponding to the velocities in channels i and j, and with the usual quantum-mechanical spreading. The time taken for a packet to travel from the asymptotic location where it is formed to the asymptotic location where it is observed is normally so small that the spreading, proportional to $(t)^{\frac{1}{2}}$, is negligible. Under the circumstance that (1.9) is a valid approximation to (1.7) the amplitude in channel j is proportional to the stationary state amplitude $f_j^{(+)}(\hat{r}_j; \vec{k}_0)$, and the detailed shape of the wave packet does not matter. Therefore a stationary state analysis of the scattering is equivalent to a wave packet analysis.

Now it is not at all obvious that the scattered amplitude $f_j^{(+)}$ varies so slowly with $\vec{k}$ that it may be factored out of the integral in (1.7). There may be resonances. Suppose the length in time of the original wave packet is Δt. Then the energy spread in $\phi(\vec{k} - \vec{k}_0)$ is of the order $\hbar(\Delta t)^{-1}$, and we may factor (1.7) to give the form (1.9), in which the structure of the original wave packet does not matter, only provided the resonance widths are not narrower than $\hbar(\Delta t)^{-1}$. In fact the natural lengths of wave packets from accelerators are of the order 10^{-9} sec or greater and $\hbar \times 10^{9}$ sec$^{-1} \approx 10^{-6}$ eV, so that the approximations required to achieve Eq. (1.9), and thus to justify a stationary state analysis of scattering, are very accurately ful-

filled for practical cases. Lifetimes of resonant states are reliably much smaller than the durations of the wave packets which excite them.

As a matter of interest one may not wish to ignore the details of the outgoing wave packet in (1.7). It is perfectly feasible to introduce physically-motivated explicit forms for $\phi(\vec{k} - \vec{k}_0)$ and $f_j^{(+)}$, and study just how resonances, or a mixture of resonance and potential scattering (DI), say, do influence the shape of the outgoing wave packet. One should use wave packets $Z(\vec{r} - \vec{z}_0)$ of smoothed rectangular shape, for good physical correspondence. Not only are Gaussian shapes non-physical, for packets of the very great duration which we are considering here, but they also tend to mask all the interesting details of onset and straggling effects in the scattered wave packet, which are the useful results of a careful investigation of (1.7). Investigations of this kind have been performed, for example, by SASAKAWA [2] and by NUSSENZWEIG [3]. One wonders whether such details of the structure of wave packets can be studied experimentally.

Superficially, the discussion just given makes the usual time-dependent interpretation of direct reactions, as given by FRIEDMAN and WEISSKOPF [4], appear erroneous. In that interpretation we imagine an incident wave packet whose duration in time is very small, so small that its spread in energy extends over many resonances. Then we separate the outgoing amplitude in the fashion

$$f_j^{(+)}(\hat{r}_j, \vec{k}) = f_j^{(+)}(\hat{r}_j, \vec{k})_{DI} + f_j^{(+)}(\hat{r}_j, \vec{k})_{FLUC}. \qquad (1.10)$$

In this discussion, now, the DI term is defined to vary slowly with $\vec{k}$. The FLUC term fluctuates as a function of $\vec{k}$, such that its average value is zero. When Eq. (1.10) is introduced into Eq. (1.7), the resulting DI term of that equation may be factored into the form (1.9), corresponding to a time of interaction of the packet with the nucleus which is of the same order as its very small duration in time. The FLUC term cannot be factored. Instead, the shape of the outgoing wave packet is altogether changed, and one finds a "very long" exponential tail, corresponding to decay of the compound nucleus. Because these two parts of the outgoing amplitude are emitted at very different times they do not interfere. We have thus defined direct interaction and compound nucleus amplitudes, which contribute independently to the cross-section. One amplitude is emitted rapidly ("directly", in fact) while the other is emitted slowly. The fallacy in this entire picture is, of course, that no such wave packets as we have been imagining actually do exist in the laboratory. Actual wave packets are so long that the DI and FLUC amplitudes are emitted almost exactly simultaneously. By deliberately spoiling the energy resolution in an experiment we do not produce a wave packet of small duration, but only an incoherent superposition of packets of long duration. The entire idea of relying upon wave packets to separate direct and compound effects appears irrelevant to actual experiments.

Nevertheless some very clear wave packet effects can be identified, and may become the basis of actual experiments. Although an outgoing packet has constant intensity over most of its duration, the detailed reaction mechanisms do influence how it starts and stops. The DI amplitude especially influences the shape of the wave packet at its start. Such details of shape have a chance of being detected experimentally, because they influence the spectrum of low-energy bremsstrahlung produced when charged particles are scattered [5].

Although the DI and FLUC amplitudes cannot be separated in time, using physically available wave packets, there is a separation of their contributions to the cross-section, a separation which is a result of the incoherent energy average obtained in an actual low resolution experiment. An incoherent average is an average of the square

$$\left| f_j^{(+)}(\hat{r}_j, \vec{k})_{DI} + f_j^{(+)}(\hat{r}_j, \vec{k})_{FLUC} \right|^2 \tag{1.11}$$

In the cross term the FLUC amplitude appears linearly, and this cross term therefore averages to zero, even though the average is achieved by a superposition of independent wave packets. The average cross-section is proportional to the sum

$$\left| f_j^{(+)}(\hat{r}_j, \vec{k})_{DI} \right|^2 + \left| f_j(\hat{r}_j, \vec{k})_{FLUC} \right|^2 , \tag{1.12}$$

in which the DI and FLUC terms now appear separately. This separation is the same one which would appear if very short time-packets could be used, and we may therefore retain and use the interpretation of the DI and FLUC terms, which the consideration of such wave packets suggested.

It would appear that DI and CN effects are clearly separated if we deal with energy-averaged cross-sections. The two terms in the amplitude (1.10) may be dealt with by different theoretical methods. This analysis is used by many authors of fundamental papers about the low-energy optical model, and about its extension to treat direct reactions. These authors take as a fundamental definition of their optical potentials that these should reproduce the average amplitude, and then compute the potentials accordingly. It is therefore interesting to emphasize that a definition of the DI as that process which reproduces the average amplitude, and gives rapid re-emission of short time-packets, is not the same as the definition that it is that part of the reaction process which involves very few degrees of freedom. Auxiliary potentials which correctly reproduce the average amplitude are likely to be in part simulating average effects due to the excitation of many degrees of freedom. We are thrown back again on the physical fact that reaction processes often contain important simplicities of DI type. But these are consequences of the special dynamical properties of the system, and cannot merely be extracted from general reaction theory. Energy averaging does help, by separating off a simplified part of the reaction amplitude, in which we find the DI effects if they are present.

2. DISTORTED WAVES

Consider the Schrödinger equation

$$\left\{(-\hbar^2/2M_f^*)\,\nabla_f^2 + U_f(r_f) + \mathcal{H}_f(\xi_f) - E\right\}\,\Psi = -V_f(\vec{r_f}, \xi_f)\,\Psi \tag{2.1}$$

where we specify on the left-hand side the kinetic energy and optical interaction of the separating fragments, and the Hamiltonian $\mathcal{H}_f(\xi_f)$ which describes their internal motions. On the right-hand side we have the residual interaction with respect to the final state channel f. The optical interaction $U_f(r_f)$ has a negative imaginary part, to simulate the transitions to compound nucleus states of motion, which we shall not include in our Ψ. The variables

ξ_f(or ξ_i in the initial channel) are those few internal degrees of freedom which are considered. Typically, they describe the motions of one or two nucleons, or one or two collective variables. More details about ξ_f and ξ_i are given in section 7.

To solve Eq. (2.1) we first multiply on the left by the complex conjugate of the wave function for the internal motion of the final state, $v_f(\xi_f)$, and integrate. We may define

$$E_f v_f = (E - \mathcal{U}_f)\, v_f, \tag{2.2}$$

and

$$k_f^2 = (2M_f^* E_f/\hbar^2), \tag{2.3}$$

and

$$\mathcal{U}_f = (2M_f^* U_f/\hbar^2).$$

Then

$$\left\{-\nabla_f^2 + \mathcal{U}_f - k_f^2\right\} \Psi_f = \frac{-2M_f^*}{\hbar^2} \left\langle v_f \middle| V_f \middle| \Psi \right\rangle \tag{2.4}$$

where $\psi_f(\vec{r}_f)$ is the coefficient of v_f in Ψ,

$$\psi_f(\vec{r}_f) = \left\langle v_f \middle| \Psi \right\rangle . \tag{2.5}$$

We want to understand in considerable detail the eigensolutions of the left-hand side of Eq. (2.4). These may then be used to build the Green's function for this equation, to obtain its solution.

One interesting eigensolution of the left-hand side of Eq. (2.4) is the elastic scattering wave function $\chi_f^{(+)}$. The notation which will be used for functions of this sort is now presented:

$$\left[-\nabla^2 + \mathcal{U} - k^2\right] \chi^{(+)} = 0 \tag{2.6}$$

$$\chi^{(+)} = \frac{4\pi}{kr} \sum_{\ell m} i^\ell e^{i\sigma_\ell} f_\ell(k, r)\, Y_\ell^m(\hat{r})\, Y_\ell^{m*}(\Theta, \Phi). \tag{2.7}$$

The phases and normalizations in (2.7) are arranged so that asymptotically

$$\chi^{(+)} \rightarrow e^{i(\vec{k}\cdot\vec{r})} + \text{outgoing scattered waves.}$$

The angles (Θ, Φ) give the direction of $\vec{k}$ with respect to the coordinate axes. The asymptotic form of the radial function f_ℓ is

$$f_\ell \rightarrow \frac{i}{2}\left[H_\ell^* - \eta_\ell H_\ell\right], \tag{2.8}$$

where η_ℓ is the reflection coefficient for the ℓ^{th} partial wave. The function $H_\ell(kr)$ is that defined by HULL and BREIT [6] to be the Coulomb analogue of $ikrh_\ell^{(1)}$, where $h_\ell^{(1)}$ is the outgoing spherical Hankel function. In terms of the regular and irregular radial Coulomb functions

$$H_\ell(kr) = G_\ell(kr) + iF_\ell(kr),$$

where asymptotically

$$F_\ell \to \sin \theta_\ell,$$
$$G_\ell \to \cos \theta_\ell,$$

and

$$\theta_\ell = kr - n\,[\ln (2kr)] - (\ell\pi/2) + \sigma_\ell$$
$$\sigma_\ell = \arg \Gamma(\ell + 1 + in),$$
$$n = ZZ'M^* e^2/\hbar^2 k.$$

The Coulomb potential, in our equations, is part of the optical potential U_f.

It is not inevitable to represent $\chi^{(+)}$ in spherical harmonic expansion, and at energies of 100 MeV or greater one tends to avoid doing so. However, the expansion is orderly, and easy to work with, and is well adapted for use on an automatic computing machine.

So far as computation goes, it is interesting to note that by numerical integration of the radial Schrödinger equation it is easy to get the shape of f_ℓ at small r, but difficult to get the normalization. Elastic scattering calculations only need the shape.

To build a Green's function for Eq. (2.4) it is necessary to supplement the regular radial wave functions f_ℓ of Eq. (2.6), which vanish at the origin, with other solutions of Eq. (2.6) which do not vanish at r = 0. It is convenient to consider solution functions $h_\ell(k, r)$, which go over asymptotically to pure outgoing functions

$$h_\ell \to H_\ell, \text{ at large } r. \tag{2.9}$$

Then a suitable Green's function for the radial differential equation is

$$k^{-1} f_\ell(k, r_<) h_\ell(k, r_>), \tag{2.10}$$

where $r_<$, $r_>$ are as usual the lesser or greater of r, r'. The function (2.10) satisfies the equation

$$\left\{ -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + U - k^2 \right\} k^{-1} f_\ell(k, r_<) h_\ell(k, r_>) = \delta(\vec{r} - \vec{r}\,'). \tag{2.11}$$

Finally, the Green's function for the entire differential equation, Eq. (2.4) is

$$K_f(\vec{r}_f, \vec{r}_f') = \sum_{\ell m} \frac{f_\ell(k_f, r_<)\, h_\ell(k_f, r_>)\, Y_\ell^m(\hat{r}_f)\, Y_\ell^{m*}(\hat{r}_f')}{k_f r_f r_f'}. \tag{2.12}$$

We mainly wish to use Eq. (2.12) to find the asymptotic outgoing solution of Eq. (2.4), for $r_f \to \infty$. Therefore $r_f \gg r_f'$ and Eq. (2.12) may be simplified. The asymptotic form of h_ℓ may be used, and

$$K_f \to \frac{e^{i(k_f r_f) - in[\ln(2k_f r_f)]}}{k_f r_f r_f'} \sum_{\ell, m} i^{-\ell} e^{i\sigma_\ell} f_\ell(k_f, r') Y_\ell^m(\hat{r}_f) Y_\ell^{m*}(\hat{r}_f'). \tag{2.13}$$

It is customary to recognize that Eq. (2.13) very closely resembles the original $\chi_f^{(+)}$, and to rewrite Eq. (2.13) in the form

$$K_f \rightarrow \frac{e^{i(k_f r_f - n[\ln(2k_f r_f)])}}{4\pi r_f} \chi_f^{(-)*}(\vec{k}_f, \vec{r}_f) , \qquad (2.14)$$

where

$$\chi_f^{(-)*}(\vec{k}_f, \vec{r}_f) = \frac{4\pi}{k_f r_f} \sum_{\ell,m} i^{-\ell} e^{i\sigma_\ell} f_\ell(k_f, r_f)\, Y_\ell^m(\hat{k}_f) Y_\ell^{m*}(\hat{r}_f), \qquad (2.15)$$

and the vector $\vec{k}_f$, now introduced, has the direction of $\hat{r}_f$ in Eq. (2.13). The function $\chi^{(-)}$ is known as the "time-reversed scattering wave function". It evidently fulfils the Wigner relation

$$\chi^{(-)*}(\vec{k}, \vec{r}) = \chi^{(+)}(-\vec{k}, \vec{r}). \qquad (2.16)$$

The time-reversed function $\chi^{(-)}$ is a solution of the scattering problem which uses ingoing scattered wave boundary conditions, and the complex conjugate of the potential U_f. However, this function is best generated by using standard calculations for $\chi^{(+)}$, followed by the transformation (2.16). The intention of the long discussion up to this point has been to make clear the properties and the purpose of the function $\chi_f^{(-)}$. It is usually introduced much more briefly. However, there is some confusion about this function, because in the presence of the complex potential U_f there is loss of flux from the elastic scattering channel, and because f_ℓ is a complex function, rather than merely a real function with a complex coefficient. We see that the complex conjugate of f_ℓ is never used, that the Wigner relation (2.16) is exactly fulfilled, and that this relation only concerns relative phases of different partial waves.

Now Eq. (2.14) is used to get the solution ψ_f of Eq. (2.4) in the asymptotic region. It is

$$\psi_f = \chi_f^{(+)}\delta_{f,i} - \frac{2M_f^*}{\hbar^2} \int K_f(\vec{r}_f, \vec{r}_f')\langle v_f | V_f | \Psi\rangle d^3 r_f', \qquad (2.17)$$

where the first term carries the ingoing-wave part which is necessary in case channel f should be the incident channel i, this would be the case of elastic scattering. In the present work the first term of (2.17) normally is not needed. The coefficient of the outgoing wave in (2.17) is the scattering amplitude

$$f_f(\hat{k}_f, \vec{k}_i) = f_0 \delta_{f,i} - \frac{M_f^*}{2\pi\hbar^2} \left\langle v_f \chi_f^{(-)} \middle| V_f \middle| \Psi \right\rangle . \qquad (2.18)$$

It is customary not to use the scattering amplitude but rather the T-matrix element

$$T_{fi} = (-2\pi\hbar^2/M_f^*)\, f_f ,$$

so that

$$T_{fi} = \left\langle v_f e^{i\vec{k}_f \cdot \vec{r}_f} \middle| U_f \middle| v_i \chi_i^{(+)} \right\rangle + \left\langle v_f \chi_f^{(-)} \middle| V_f \middle| \Psi \right\rangle . \qquad (2.19)$$

Equation (2.19) is known as the "Gell-Mann, Goldberger relation", for a problem in which we consider two potentials. It may be considered that we have derived this relation by transformation of the expression

$$T_{fi} = \left\langle v_f e^{i(\vec{k}_f \cdot \vec{r}_f)} \mid U_f + V_f \mid \Psi \right\rangle, \tag{2.20}$$

which is more commonly employed. Introduction of the distorted final wave $\chi_f^{(-)}$ has removed the potential U_f from explicit appearance in the off-diagonal term of (2.19). (It has been considered, in the present work, that Ψ only involves direct reaction degrees of freedom, and that U_f is an optical potential, in order to compensate for the omitted CN degrees of freedom. This attitude will be maintained. However, the transformation from (2.20) to (2.19) is completely general, and U_f may be any part of the Hamiltonian, or any auxiliary potential, which we may wish to isolate.)

The differential cross-section is expressed in terms of T_{fi} by computing the outgoing current and dividing it by the incoming current per unit area, giving

$$d\sigma/d\Omega = [M_i^* M_f^* / (2\pi\hbar^2)^2] \, (k_f/k_i) \sum_{Av} \mid T_{fi} \mid^2 \tag{2.21}$$

where $\sum_{Av}$ is a sum over final spin states and an average over initial spin states.

Physically useful results are obtained by approximating Ψ in the expressions for the transition amplitude. Distorted waves Born approximation (DWB) is obtained by introducing the approximation

$$\Psi \approx \chi_i^{(+)} (\vec{k}_i, \vec{r}_i) \, v_i (\xi_i) \tag{2.22}$$

in Eq. (2.19). One then finds

$$T_{fi}(\mathrm{DWB}) = \left\langle v_f (\xi_f) \chi_f^{(-)} (\vec{k}_f, \vec{r}_f) \mid V_f (\xi_f, \vec{r}_f) \mid \chi_i^{(+)} (\vec{k}_i, \vec{r}_i) \, v_i (\xi_i) \right\rangle. \tag{2.23}$$

The distorted wave $\chi_i^{(+)}$ is computed in terms of the optical interaction $U_i(r_i)$ between the colliding particles in channel i. That (2.22) should be a good approximation is based on the idea that the average interaction U_i of the colliding particles is primarily responsible for determining the wave function, that the interaction which causes transitions may be treated as a perturbation. Eq. (2.23) is at present the standard method for DI calculations, and considerable effort is being devoted to calculating cross-sections by this method. Often excellent agreement with experiment is obtained.

In case the incident and emerging projectiles are nucleons, Eq. (2.23) takes on an obvious interpretation as an extension of ordinary shell model calculations. The wave functions $v_f \chi_f^{(-)}$ and $v_i \chi_i^{(+)}$ are zero-order independent particle model eigenfunctions in the shell model potential well, and T_{fi} is a matrix element of the residual interaction. Presumably T_{fi} should give the magnitude of DI cross-sections about as well as corresponding bound state calculations usually give level splittings.

It is not correct to use the approximation (2.22) in Eq. (2.20). One sees this by inserting in Eq. (2.20) an improved version of this approximation, in which the term of Ψ which is first-order in V_f, is also carried. Formally,

$$T_{fi} \approx \left\langle v_f e^{i(\vec{k}_f \cdot \vec{r}_f)} \mid (U_f + V_f) \left\{1 + (E - H_f + i\eta)^{-1} V_f\right\} \mid \chi_i^{(+)} v_i \right\rangle, \quad (2.24)$$

where $H_f = H - V_f$, as usual. The product of U_f times the second term of the wave function is of first-order in V_f, comparable with the terms which are obtained by using Eq. (2.22) without improvement. It is only in Eq. (2.19) that our simple approximation of Ψ may be used correctly. We shall see later that the "adiabatic" method uses Eq. (2.20). It must therefore use a better approximation of Ψ than does the DWB method.

The present discussion has largely been ignoring spins. However, these are easy to put in when needed. Also it has been ignoring (anti-) symmetrization of the projectile with the target nucleus. The usual justification for that is that projectile energies are rather far above the Fermi energy of the nucleus.

3. ZERO-RANGE APPROXIMATION — QUALITATIVE RESULTS — SURFACE REACTIONS

I want to start a new section in order to emphasize the importance of introducing "zero-range approximation". Apart from this, this new section merely continues the discussion of DWB calculations.

Four integration variables are displayed in Eq. (2.23). These are $\vec{r}_f$, $\vec{r}_i$, ξ_f, ξ_i. Each of these is a vector, or contains a vector in some fashion. Now, although only two of these four vector variables really are independent, the properties of the wave functions are usually not such as to permit making any exact transformation to eliminate the redundant variables. It would be particularly helpful to have only one $\vec{r}$ variable in the two distorted waves, because of the difficulty of treating these wave functions. With plane waves this would be easy to achieve, but then, plane waves are known to be poor representations of the physics of a DI.

For want of an exact transformation we simply introduce the approximation $\vec{r}_f \approx \vec{r}_i \approx \vec{r}$. This approximation is based upon the principle that the wave functions do not change appreciably over distances of the order of the ranges of the nuclear potentials which make up the interaction V_f. By implication the zero-range approximation somewhat alters the relation between ξ_f and ξ_i, but I prefer to carry these variables intact, in order to be reminded of their physical meaning. When we consider stripping, the treatment of the $\xi_{f,i}$ will become clear. With use of the zero-range approximation, Eq. (2.23) becomes

$$T_{fi}\,(\mathrm{DWB}) = \int d^3r\, \chi_f^{(-)*}(\vec{k}_f, \vec{r}) \left\langle v_f(\xi_f) \mid V_f(\xi_f, \vec{r}) \mid v_i(\xi_i) \right\rangle \chi_i^{(+)}(\vec{k}_i, \vec{r}). \quad (3.1)$$

It should be noted that the zero-range approximation does not make the approximation, condemned by WILKINSON [7], of treating the ranges of the internal wave functions v_f, v_i as being small.

The sensitivity of DWB calculations to the range of V_f has been tested to some extent in inelastic scattering calculations, in which it is quite true that $\vec{r}_f = \vec{r}_i$, and $\xi_f = \xi_i$, without use of the zero-range approximation. Errors of the order of factors of two are found, especially at larger scattering angles. The problem has also been studied carefully in deuteron stripping, and the errors there are much smaller, and tend to vary slowly with scattering angle. Improved calculations are being prepared.

Certain general physical properties of DWB calculations may be seen immediately in Eq. (3.1), merely as a result of introducing zero-range approximation. The factor $\langle v_f | V_f | v_i \rangle$ is in some sense, to be discussed, the "reduced width" for the reaction. It influences the magnitude of the cross-section. It depends upon the detailed model of the reaction process, and incorporates all the knowledge about the bound state wave functions which one can obtain from spectroscopic studies. The distorted waves $\chi_f^{(-)*}$ and $\chi_i^{(+)}$ are primarily responsible for determining the angular distribution for the reaction. If strongly-absorbed projectiles are used, the distorted waves only overlap with the factor $\langle v_f | V_f | v_i \rangle$ at large r, at a radius where this factor has a shape which is almost model-independent. Evidently the reaction analysis has been separated into distinct steps. The step involving detailed spectroscopic calculations mainly determines the magnitude of the cross-section. The step involving distorted waves mainly determines the angular distribution.

A further property of DWB calculations is related to the fact that Eq. (3.1) is exact for the treatment of inelastic scattering, as already mentioned. Therefore one can think of the zero-range approximation as having the great importance of reducing all DWB calculations to the same mathematical structure which appears in the calculation of inelastic scattering. Even more strongly: All direct interaction calculations which use the zero-range approximation give equivalent formulas for the angular distribution. Different physical models or different DI reactions only give different results for the magnitude of the cross-section, or give different values for the parameters in the same old angular distribution formulas.

The "reduced width" factor $\langle v_f | V_f | v_i \rangle$ is put into standard form by expanding it in multipoles of the vector $\vec{r}$. The expansion of V_f is

$$V_f(\vec{r}, \xi) = \sum_{L,M} V_{LM}(r, \xi)[i^L Y_L^M(\hat{r})]^* . \tag{3.2}$$

Since V_f is an overall scalar function of the space coordinates (spins are ignored), the V_{LM} must behave under rotations of coordinates like the spherical harmonics Y_L^M, and have parity $(-)^L$. The factor i^L is included to ensure the reality of reduced matrix elements. Applying the Wigner-Eckart theorem to the matrix elements of the interaction (3.2), we get for the "reduced width" factor

$$\langle v_f | V_f | v_i \rangle = \sum_{L,M} \langle J_f M_f | J_i L, M_i M \rangle \langle J_f || V_L || J_i \rangle [i^L Y_L^M(\hat{r})]^* , \tag{3.3}$$

where J_i, J_f are the initial and final nuclear spins, and M_i, M_f are their z-components. We then see that the Lth multipole in the expansion (3.2) corresponds to transfer of angular momentum L to the nucleus. The Clebsch-Gordan coefficient limits L, and ensures conservation of angular momentum

$$|J_i - J_f| \leq L \leq J_i + J_f , \tag{3.4}$$

while the spherical harmonic determines the change in parity to be $(-)^L$. The reduced matrix element is now a function of radius only, and it is convenient to write it as the product of a "strength" times a "form factor",

$$\langle J_f || V_L || J_i \rangle = A_L F_L(r). \tag{3.5}$$

By convention F_L is presumed to be dimensionless, and therefore A_L has the dimensions of energy. The separation in (3.5) is one of convenience, so that for example universal form factors with simple normalization may be used in computation. To understand possible shapes of DI angular distributions it is not necessary to explore many different sorts of shapes for F_L. The wave functions of strongly-absorbed projectiles only overlap effectively with F_L in the surface region, as already mentioned, where the principal property of F_L is that it drops off exponentially with radius, in a manner determined by the binding energies of the internal, bound-state wave functions v_f, v_i. Even the general case is not too complicated, and F_L is not ever a rapidly-varying function of radius, because v_f and v_i are not rapidly-varying. In the case of deuteron stripping F_L is especially easy to interpret. It is the radial wave function of the captured particle. More details about these questions will become apparent later. What is chiefly worth noting is that it is not difficult to understand F_L well enough to predict DI angular distributions. Magnitudes of cross-sections, as expressed by the factor A_L, are considerably more sensitive to the spectroscopic model of the nucleus, and require much future theoretical work (see section 7).

Upon substitution of Eqs. (3.3) and (3.5) into the amplitude, Eq. (3.1), the result may be written

$$T_{fi}\,(\mathrm{DWB}) = \sum_{LM} A_L \langle J_f M_f \mid J_i L,\ M_i M \rangle\, \mathcal{J}^{LM} \tag{3.6}$$

$$\mathcal{J}^{LM} = \int d^3 r\ \chi_f^{(-)*}\,(\vec{k}_f, \vec{r})\,[i^{-L} Y_L^{M*}(\hat{r})\ F_L(r)]\ \chi_i^{(+)}\,(\vec{k}_i, \vec{r}). \tag{3.7}$$

The differential cross-section is obtained as indicated in Eq. (2.21). The sum over M_f, and the average over M_i, only concern the Clebsch-Gordan coefficients, and we get the general result

$$d\sigma/d\Omega = (M_i^* M_f^*/(2\pi\hbar^2)^2)\ (k_f/k_i)\ ((2J_f + 1)/(2J_i + 1))\ \sum_L \mid A_L \mid^2 (2L + 1)^{-1} \times \sum_{M=-L}^{L} \mid \mathcal{J}^{L,M} \mid^2 . \tag{3.8}$$

We see in Eq. (3.8) the important fact that different values of the angular momentum transfer L do not interfere in determining the angular distribution. This is another consequence of the zero-range approximation, and of the almost-factored form thereby obtained for the transition amplitude, Eq. (3.1). For any given scattering angle for the DI differential cross-section, it is possible to think of the nuclear transition from state v_i to state v_f as being a total cross-section!

Reactions encountered in the laboratory normally require consideration of only one value of L. Not only are the allowed values of L in a reaction limited by angular momentum conservation, Eq. (3.4), but also by the special properties of whatever nuclear model describes the states v_f and v_i. Small admixture terms which may be allowed are not important in the differential cross-section, Eq. (3.8), because there is no interference. Most suggestions of L-mixing have been based upon angular distributions predicted with inadequate wave functions $\chi_i^{(+)}$ and $\chi_f^{(-)}$.

Our last remaining problem, in evaluating the cross-section, lies in computing the integral $\mathcal{J}^{L,M}$ of Eq. (3.7). To do this it is necessary to have explicit expressions for the waves $\chi_i^{(+)}$, $\chi_f^{(-)}$. The normal, reliable method

for treating these wave functions is to use the spherical harmonic expansions, already given. Plane waves, although unsuitable, have been very popular. At high energies WKB wave functions may be used. Methods of analytic function theory also give some insight into the structure of the integral.

It is interesting to present the plane wave result, because it is easy to derive, and in order to have it available for comparison with other work. In this case

$$\mathcal{J}^{L,M} \approx \int d^3r \; e^{i(\vec{q}\cdot\vec{r})} \; [i^{-L} Y_L^{M*} F_L] \; , \tag{3.9}$$

where $\vec{q} = \vec{k}_i - \vec{k}_f$ is the "momentum transfer" to the nucleus, a characteristic parameter of plane wave theories. We may expand in spherical waves, using $\vec{q}$ as the z-axis, for now, and carry through the angle integration. Then

$$\mathcal{J}^{L,M} = \sqrt{4\pi(2L+1)} \;\; \delta_{M,0} \int_0^\infty j_L(qr) \; F_L(r) \; r^2 dr. \tag{3.10}$$

If F_L were proportional to $\delta(r - R_0)$ one would get a cross-section proportional to $[j_L(qR_0)]^2$. If F_L were of the form of a bound state radial wave function, but cut off inside the nucleus, thus if it were of the form

$$F_L = 0 \qquad r < R_0$$

$$F_L \propto h_L^{(1)}(iKr), \quad r > R_0 \tag{3.11}$$

where $\hbar^2K^2/2M_f^*$ is the nuclear binding energy, then one would obtain an explicit analytic expression for the integral, just the familiar Butler Wronskian.

It is now well known that the results obtained in this way are extremely inaccurate. This inaccuracy is irremediable if the magnitude of the cross-section is considered. However, if only the angular distribution is considered the inaccuracy may often be largely compensated for by alteration of the numerical parameters in the theory. From this point of view the plane wave result becomes a sort of convenient interpolation formula, and in the case of deuteron stripping it has been widely used in this manner for finding relations among different experiments, and thereby for getting useful spectroscopic information. The success of that work is testimony to the correctness of the basic stripping assumption, and to the general similarity of the many low-energy stripping experiments, rather than to any sort of reasonableness of the plane-wave approximation. Nevertheless, it is pleasant to have such a simple formula as Eq. (3.10), and its consequences.

At the opposite extreme of accuracy one uses the spherical harmonic expansions, Eqs. (2.7), (2.15), of the distorted waves, and computes the overlap integrals numerically. With the z-axis along the incident direction $\vec{k}_i$ and the y-axis along $\vec{k}_i \times \vec{k}_f$ the result may be written

$$\mathcal{J}^{L,M} = (-)^M \mathcal{J}^{L,-M} = [4\pi(2L+1)^{\frac{1}{2}}/k_f k_i] \sum_{\ell'\ell} i^{\ell-\ell'-L} \; e^{i(\sigma_\ell(i) + \sigma_{\ell'}(f))} \sqrt{2\ell'+1}$$

$$\times Y_{\ell'}^M(\Theta, 0) \langle \ell' L, -MM \,|\, \ell 0 \rangle \langle \ell' L, 00 \,|\, \ell 0 \rangle \beta_{\ell'\ell}^L \; . \tag{3.12}$$

The radial integrals are given by

$$\beta^{L}_{\ell',\ell} = \int_0^\infty f^{(f)}_{\ell'}(k_f, r)\, F_L(r)\, f^{(i)}_{\ell}(k_i, r)\, dr. \qquad (3.13)$$

The procedure for using these equations is the following: First one finds the optical potentials which fit the elastic scattering in channels i and f. Then one computes numerically the radial functions $f_{\ell'}^{(f)}$ and $f_{\ell}^{(i)}$, and then the overlap integrals $\beta_{\ell'\ell}^{L}$. These integrals are used in Eq. (3.12) to find the DWB amplitude, and this is then squared and summed over M to obtain the cross-section. (Sometimes authors attempt to use Racah coefficients to obtain in closed form the result of the summation on M. This is a mistake. Not only is $\mathcal{T}^{L,M}$ inherently of physical interest, and worth computing explicitly, but also the ℓ', ℓ sum is more difficult than the M sum, and therefore the use of Racah methods actually increases the amount of work.)

The first application of the above method in nuclear reaction work was made in 1953 by Horowitz and Messiah. Extensive calculations were later performed by Tobocman and collaborators, and by Levinson and Banerjee. At present Satchler and collaborators are engaged in a very large programme in the methodical application of this method, and in the exploration of its consequences in nuclear physics.

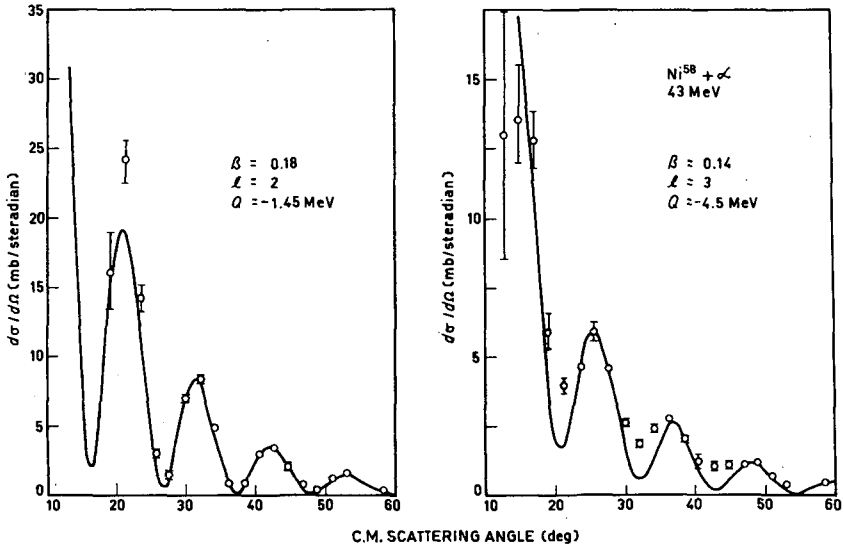

Fig. 2

A recent successful application of the DWB method in inelastic scattering. Optical parameters obtained from elastic scattering

Figures 2 and 3 show some results from a recent paper by Rost, for the inelastic scattering of alpha particles from Ni^{58} and Mg^{24}. The only adjustable parameter in these calculations, after fitting the elastic scattering, is the deformation parameter β which multiplies the cross-section. This parameter has the same meaning as in other studies of deformed nuclei, as

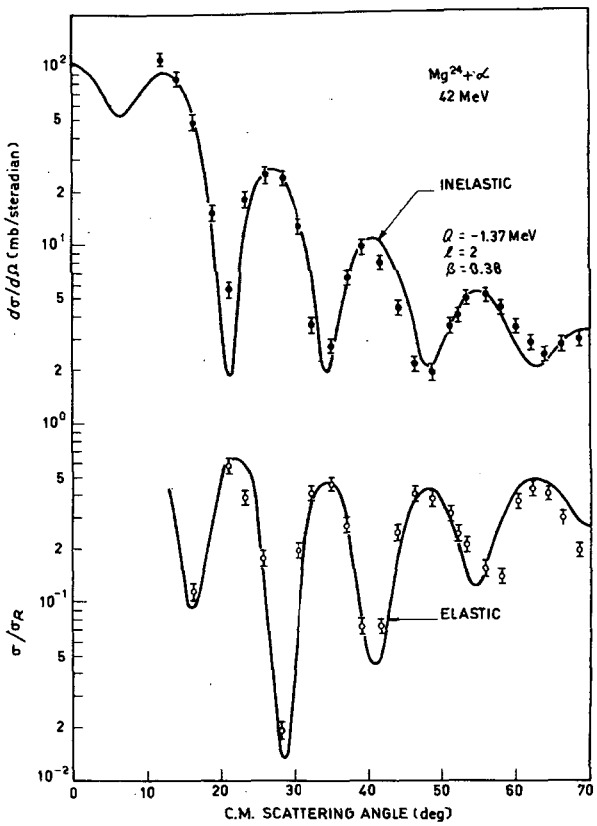

Fig. 3

Same as figure 2

will be discussed later. The values obtained for β in the (α, α') work agree with those normally considered. These results make the theory look very good.

I may remark that such investigations have a fundamental interest only if the optical potentials used to compute the radial wave functions have been obtained from elastic scattering studies. Since these potentials sometimes are under-determined, it may be possible to use some of the freedom of their parameters to adjust for a best fit to reaction cross-sections. However, the DWB theory of reactions is inherently less accurate than the optical model theory of elastic scattering, and therefore parameter adjustment, which improves DWB fits at the expense of elastic scattering fits, must be regarded as non-physical.

I would also like to present, at this point, some pictures of actual optical model wave functions, to show what kinds of complications are contained in DWB calculations. Figs. 4 and 5 show a three-dimensional model of the modulus $| \chi^{(+)} (\vec{k}, \vec{r}) |$ of the elastic scattering wave function for alpha particles incident on Ca^{40} at 18 MeV. The dark zone is the 10% - 90% region of the optical potential. Strong diffraction oscillations are seen. Fig. 6 shows lines

of constant phase for the function $\chi^{(+)}$ for 40 MeV protons on Ca^{40}. Again, complicated interference effects are seen. These complications are probably not very dependent on precise parameter values, but seem instead to be necessary effects of the scattering of waves by semi-transparent obstacles. What the pictures show is that short-cut approximation of these wave functions is dangerous.

Another dangerous approximation is the use of square-well distorting potentials. We will see below that some of the most characteristic DWB results may be understood as consequences of the reflection coefficients η_ℓ. It is just in reflection effects that square wells are at their worst.

It is often possible to obtain simple understanding of the series (3.12). In the case of inelastic scattering of strongly absorbed projectiles, as in

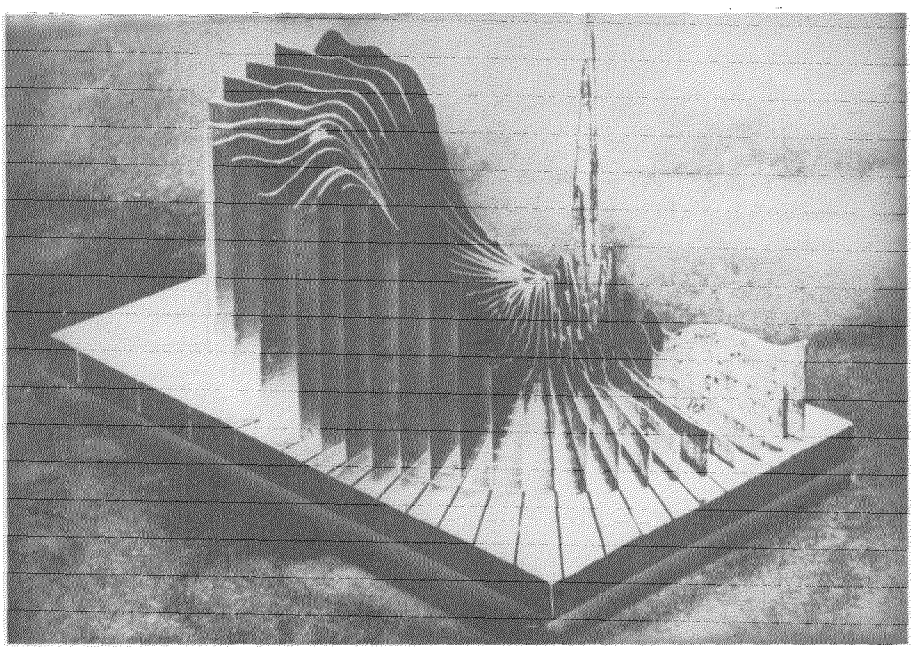

Fig. 4

Three-dimensional model of $|\chi^{(+)}|$, the optical model wave function, for 18 MeV alpha particles bombarding Ca^{40}. The beam is incident from the left. The dark zone is the 10% - 90% region of the optical potential

(α, α') reactions at medium energy, it is especially easy to see the physical content of Eq. (3.12). One achieves considerable understanding of the entire method by studying this case.

Inelastic scattering to low-lying final states of the target nucleus has the convenient property $k_f \approx k_i$. Let us consider this case, and also L = 0. Then Eqs. (3.12), (3.13) simplify to

$$\mathcal{T}^{0,0} = \frac{(4\pi)^{\frac{1}{2}}}{k^2} \sum_\ell e^{i 2\sigma_\ell} (2\ell + 1) P_\ell (\Theta) \beta^0_{\ell\ell} \qquad (3.12')$$

$$\beta^0_{\ell\ell} = \int_0^\infty f_\ell^2(k, r)\, F_0(r)\, dr\ . \qquad (3.13')$$

The classical cut-off angular momentum $\ell_0 = kR_0$ plays a large role in understanding these equations. For $\ell > \ell_0$ centrifugal repulsion causes f_ℓ to be very small in the nuclear interior, and therefore $\beta^0_{\ell\ell}$ is very small if $\ell > \ell_0$.

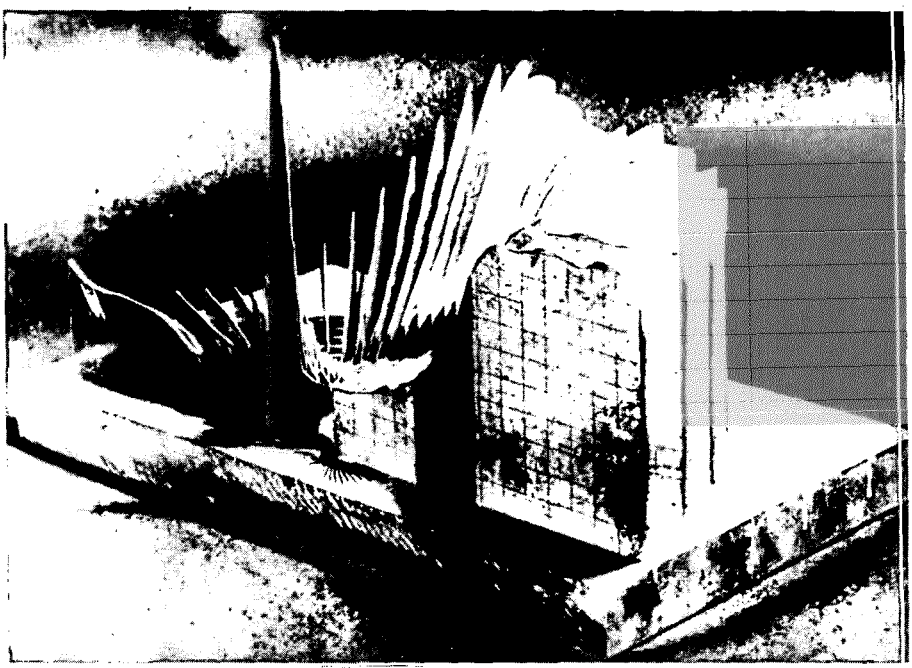

Fig. 5

Same as figure 4, another view. Beam now incident from the right

For $\ell < \ell_0$, in the case of strong absorption, the reflection coefficient η_ℓ is very small. A "sharp cut-off model"

$$\eta_\ell = 1\ , \quad \ell > \ell_0$$
$$\eta_\ell = 0\ , \quad \ell < \ell_0 \qquad (3.14)$$

is frequently employed. In this case it is apparent that outside the nucleus the radial wave function for $\ell < \ell_0$ reduces to

$$f_\ell = (i/2)\, H_\ell^*,\ r > R_0, \qquad (3.15)$$

a purely ingoing travelling wave. Because $\eta_\ell \approx 0$ we know that this wave experiences negligible reflection at the nuclear surface, and therefore that it may be continued into the nuclear interior, according to WKB ideas, as an ingoing travelling wave of modified wavelength. Eventually the travelling

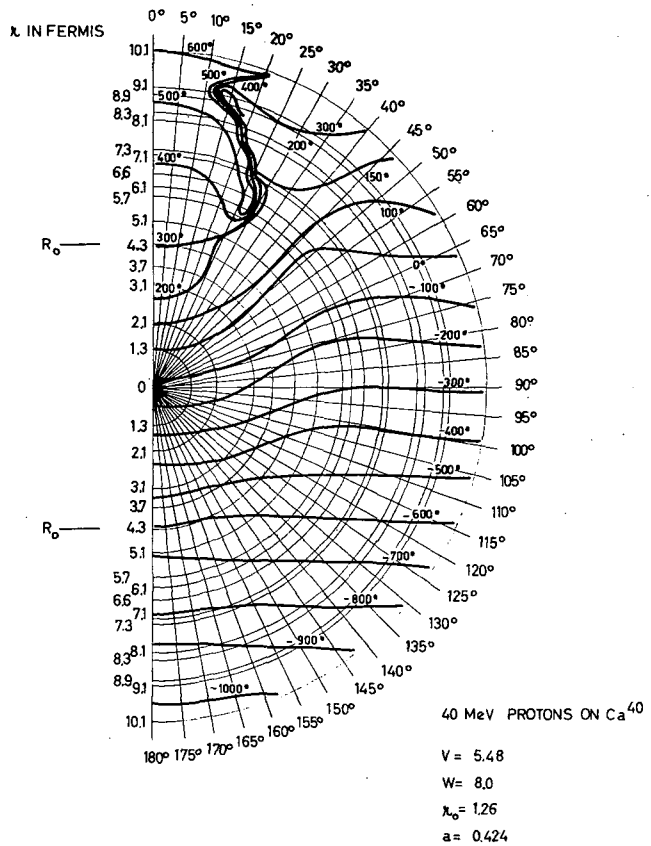

Fig. 6

Contours of constant phase, in the optical wave function $\chi^{(+)}$ for 40 MeV protons bombarding Ca^{40}, plotted in intervals of 100 degrees. The beam is incident from the bottom. Note the region of very rapid phase change

wave is absorbed, so that at $r \ll R_0$ its amplitude becomes negligible. Generally this absorption is not too rapid, and the important fact about the overlap integral (3.13') for $\ell < \ell_0$ is that it is the integral of the product of a rather smooth function F_0 multiplied by the square of a rapidly-oscillating travelling wave expression. Evidently $\beta^0_{\ell\ell}$ integrates to a rather small value if $\ell < \ell_0$. Therefore the overlap integral is appreciable only if $\ell \approx \ell_0$. In Fig. 7 this radial integral is shown for the case of 43 MeV (α,α') on Ni^{58}, as computed by Satchler's group at Oak Ridge. It is seen that this integral is very well localized about ℓ_0.

Because of the above properties of $\beta^0_{\ell\ell}$ the amplitude $\mathcal{J}^{0,0}$ is of the form of an average of $P_\ell(\Theta)$ for a few values of ℓ near ℓ_0,

$$\mathcal{J}^{0,0} \propto \langle P_\ell(\Theta) \rangle_{\ell \approx \ell_0}. \qquad (3.16)$$

The cross-section is

$$\sigma \propto \left| \left\langle P_\ell(\Theta) \right\rangle_{\ell \approx \ell_0} \right|^2 . \qquad (3.17)$$

Successive P_ℓ functions very closely resemble each other at small Θ, but gradually drift out of phase as Θ becomes large. Therefore the average in

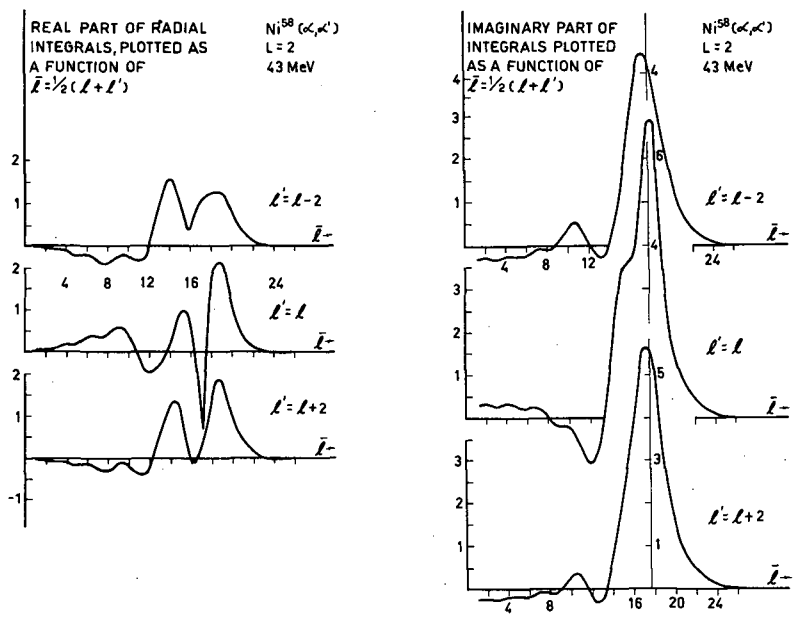

Fig. 7

Real and imaginary parts of radial integrals for the reaction $Ni^{58}(\alpha, \alpha')$, L = 2, at 43 MeV incident energy

(3.16) gives constructive interference at small Θ, and the angular distribution is a function closely resembling $P_{\ell_0}(\Theta)$. Destructive interference gradually sets in as Θ becomes large, and the entire expression (3.16) may be thought of as $P_{\ell_0}(\Theta)$ multiplied by an envelope factor which goes to zero at large Θ. The cross-section has the familiar form with peaks equally spaced in Θ.

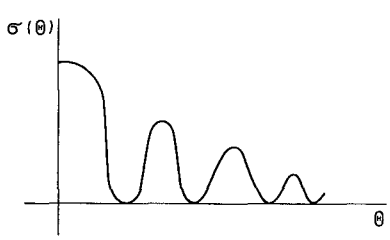

Fig. 8

Form of the angular distribution in the case of strongly-absorbing projectiles and small momentum change ("surface reactions")

Of course $P_{\ell_0}(\Theta)$ may be approximated in terms of a Bessel function,

$$P_{\ell_0}(\Theta) \approx J_0([2\ell_0 + 1]\ \sin\frac{\Theta}{2}) \tag{3.18}$$

at very small values of Θ.

If $L \neq 0$ a corresponding analysis of Eqs. (3.12) and (3.13) may be carried through. The overlap integral $\beta^{L}_{\ell'\ell}$ is non-vanishing in the rather broader class of cases $\ell \approx \ell_0$, $\ell' < \ell_0$ or $\ell' \approx \ell_0$, $\ell < \ell_0$. A good approximation if $|\ell - \ell'|$ is not too large is

$$\beta^{L}_{\ell'\ell} \approx \beta^{0}_{\bar{\ell}\bar{\ell}}, \quad \bar{\ell} = \tfrac{1}{2}(\ell + \ell').$$

However, the decisive observation required for simple understanding of Eq. (3.12) is that if $\{|\ell - \ell'|/\ell_0\} \ll 1$, so that only the Clebsch-Gordan coefficients in Eq. (3.12) are rapidly-varying functions of ℓ, then

$$\begin{aligned}\sum_{\ell} i^{\ell - \ell' - L} &\langle \ell' L, -MM \,|\, \ell 0 \rangle \langle \ell' L, 00 \,|\, \ell 0 \rangle \\ &\approx \frac{[(L-M)!\,(L+M)!\,]^{\frac{1}{2}}}{(L-M)!!\,(L+M)!!}, \text{if } (L+M) \text{ even}, \\ &\approx 0 \qquad\qquad\qquad\qquad , \text{if } (L+M) \text{ odd}.\end{aligned} \tag{3.19}$$

Here $N!! \equiv (N)(N-2)(N-4)\ldots(1 \text{ or } 2)$, as usual. The simplified form of Eq. (3.12) therefore becomes

$$\begin{aligned}\mathcal{J}^{L,M} &\approx \frac{4\pi(2L+1)^{\frac{1}{2}}}{k^2} \frac{[(L-M)!\,(L+M)!\,]^{\frac{1}{2}}}{(L-M)!!\,(L+M)!!} \\ &\times \sum_{\ell} e^{i2\sigma_\ell}\ (2\ell+1)^{\frac{1}{2}}\ \beta^{0}_{\ell\ell}\ Y^{M}_{\ell}(\Theta, 0), \text{ if } (L+M) \text{ even} \\ &\approx 0 \qquad\qquad\qquad\qquad , \text{ if } (L+M) \text{ odd}.\end{aligned} \tag{3.20}$$

Only the terms having $|M| = L, L-2, \ldots$ contribute to the inelastic scattering, and these combine in just the same way as in the case $L = 0$, as is seen in Eq. (3.20). Therefore

$$\sigma \propto \sum_{M,\,(L+M)\text{ even}} \left| \langle Y^{M}_{\ell}(\Theta, 0) \rangle_{\ell \approx \ell_0} \right|^2. \tag{3.21}$$

Successive Y^{M}_{ℓ} are obtained from each other by differentiation. For $M \geq 0$,

$$Y^{M}_{\ell}(\Theta, 0) = \left[\left(\frac{2\ell+1}{4\pi}\right)\frac{(\ell-M)!}{(\ell+M)!}\right]^{\frac{1}{2}} (\sin\Theta)^{M} \frac{d^{M}}{d(\cos\Theta)^{M}} P_{\ell}(\cos\Theta). \tag{3.22}$$

It is seen that in the region away from $\Theta \approx 0$, where P_ℓ has become simply oscillatory, each differentiation reverses the phase of oscillation. However, in any one application of Eq. (3.21) only all odd or all even values of M appear; therefore all terms of (3.21) oscillate in phase. We therefore verify one part of the Blair phase rule: all cross-sections with even L have oscillations that are in phase with each other, and that are out of phase with the oscillations of cross-sections having odd L. We also see in (3.22) that near small Θ the amplitudes are proportional to $(\sin\Theta)^{M}$. Because of this the

first peak of the angular distribution tends to be displaced to larger Θ, as L increases, an effect well known in deuteron stripping.

In summary, what we learn from these qualitative studies of the partial wave series is that near small Θ, in the case of inelastic scattering of strongly-absorbed projectiles, the partial wave expansion, Eq. (3.12), is not very sensitive to interferences among its various terms. The terms combine constructively, and each term has the same general form as the final answer. It is to be expected that numerical calculations using this expansion will be very reliable at small Θ.

These results are related to the question of surface reactions. It has been seen that the overlap integrals $\beta_{\ell'\ell}^{L}$ vanish altogether if $\ell' < \ell_0$ and $\ell < \ell_0$. There is a localization of the reaction in angular momentum space. The distortion does not merely set equal to zero the integrands of the low partial waves in the region $r < R_0$. In that circumstance there would still be non-vanishing contributions from $r > R_0$. Instead, the entire integrals for the low partial waves vanish. In this sense the usual picture of surface reactions is wrong. Fig. 9 illustrates this point. It is taken from the work of Rost, pre-

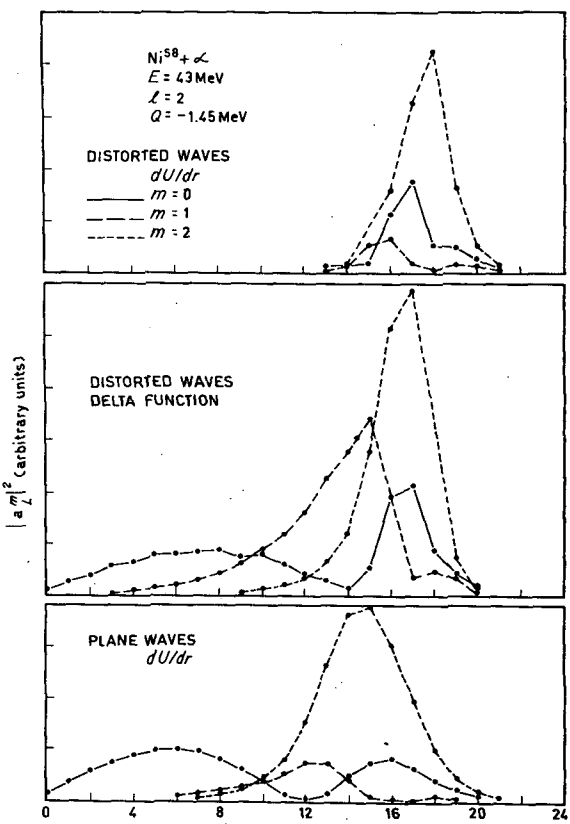

Fig. 9

Absolute square of radial integrals for the reaction $Ni^{58}(\alpha, \alpha')$, L = 2, at 43 MeV. Correct DWB calculation (top) is compared with delta function radial integration (centre) and with plane wave calculation (bottom). Note the extreme localization in ℓ of the exact calculation

viously mentioned. The moduli of the coefficients in the Legendre expansion of the scattering amplitude are plotted, for the L = 2 excitation of Ni^{58} by bombardment with 43 MeV alpha particles. Coefficients obtained in the full DWB calculation are seen to group closely. If the same distorted waves are used, but the form factor F_L is replaced by a δ-function at the surface, then the contributions for the low partial waves are seen to become much larger. Plane wave results are also shown, for comparison.

Although localization in angular momentum is the distinctive property, there is also localization in radius. The few radial integrals which are large correspond to $\ell \approx \ell_0$. Wave functions for these ℓ values vanish inside the nucleus because of centrifugal repulsion, and have their first large peak in the surface region, about at the sharp cut-off radius R_0. This is of the order of 2f further out than the half-point of the Saxon potential. The region of the peak of j_{ℓ_0} is the region of localization in radius. (In rough approximation these important radial waves may be regarded as so well shielded by the centrifugal potential as to be undistorted.)

The manner in which the absorption mechanism operates to give surface reactions is also seen by considering the flux patterns associated with the three-dimensional wave functions $\chi_i^{(+)}$ and $\chi_f^{(-)}$, in the amplitude $\mathcal{J}^{L,M}$ of Eq. (3.7). For short wavelengths, $kR_0 \gg 1$, these flux patterns are essentially classical (Fig. 10). Because of absorption the flux inside the nucleus is either exclusively radially ingoing or exclusively radially outgoing. Accordingly, an overlap integral in the nuclear interior implies a large momentum transfer. Because the form factor F_L is smooth, it cannot supply large momenta, and the over-

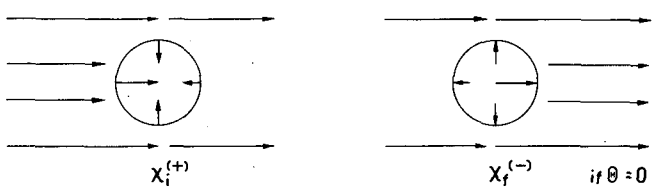

Fig. 10

Semi-classical flux patterns for the distorted waves $\chi_i^{(+)}$ and $\chi_f^{(-)}$, showing the effects of absorption

lap integral therefore receives only small contributions from the nuclear interior. This result is general, What makes it important is that in addition at small Θ the average momenta of $\chi_i^{(+)}$ and $\chi_f^{(-)}$ in the surface region, near the nuclear equator, are very nearly equal. The overlap integral in this region therefore requires only small momentum transfer, and is large. We get a surface reaction.

This section may be concluded with some further remarks on the very important subject of momentum transfer. With strong absorption we get surface reactions if $k_f \approx k_i$ and $\Theta \approx 0$, because under these circumstances the momentum transfer near the nuclear surface is small. If $k_f \approx k_i$ but Θ becomes large, then the momentum transfer in the surface region also becomes large; then the overlap contributions from the surface region become small, and are no more important than the overlap contributions from the nuclear interior. For this reason, at large Θ the reaction is no longer a surface reaction. Likewise, if $|k_f - k_i|$ is large the reaction is not a sur-

face reaction. Generally, reactions in which there is a mass change have large $|k_f - k_i|$. An interesting exception is deuteron stripping to bound final states, at incident energies of the order 8 - 20 MeV. In this case $k_f \approx k_i$ is often fulfilled very closely, and therefore these reactions are surface reactions. In contrast, at high bombarding energy deuteron stripping is dominated by the mass change, and $k_i \approx \sqrt{2} k_f$ if low-lying final states are formed. In this case entirely different methods of approximation must be used, as developed in recent work by Kerman.

4. POLES - HIGH ENERGY

The two topics of this section are related only in that both are natural outgrowths of the DWB discussion. Both concern the use of interesting approximations for the distorted waves $\chi_i^{(+)}$, $\chi_f^{(-)}$. Let us only consider the amplitude for angular momentum transfer L,

$$\mathcal{J}^{L,M} = \int d^3r\, \chi_f^{(-)*}(\vec{k_f}, \vec{r}) \left[i^{-L} Y_L^{M*} F_L \right] \chi_i^{(+)}(\vec{k_i}, \vec{r}). \tag{3.'1}$$

Several authors have stressed the possibility of gaining valuable information about the reaction by studying a pole which the above amplitude has, when considered as a function of the momentum transfer, $q = |\vec{k_i} - \vec{k_f}|$, if this q is treated as a complex variable. The pole corresponds to contributions to the integral from large values of r. At large r the form factor always falls exponentially, at some rate determined by the binding energies of the initial and final nuclear states,

$$F_L \to (\text{const})\, e^{-Kr}/r. \tag{4.1}$$

For stripping, for example, the parameter K is related to the binding energy of the captured particle, $\hbar^2 K^2 / 2M^*$. Now if K should be very small, then large values of r would dominate in $\mathcal{J}^{L,M}$; these could be such large radii that the distorted waves would simplify. If small r are disregarded entirely, then the amplitude may be written

$$\mathcal{J}^{L,M} \approx \text{const} \int d^3r \left\{ e^{-i(\vec{k}_f \cdot \vec{r})} + \frac{(\text{scattered waves})_f}{r} \right\} \left[i^{-L} Y_L^{M*} \frac{e^{-Kr}}{r} \right]$$

$$\times \left\{ e^{i(\vec{k}_i \cdot \vec{r})} + \frac{(\text{scattered waves})_i}{r} \right\}. \tag{4.2}$$

The scattered parts of $\chi_f^{(+)}$ and $\chi_f^{(-)}$ always decrease as r^{-1}, to conserve flux, and this property has been exhibited explicitly in Eq. (4.2). If very large r do dominate in Eq. (4.2) then the r^{-1} dependence of the scattered wave parts is sufficient to make the contributions from these parts negligible. What is left is the elementary plane wave Born approximation, with no distortion effects at all! Therefore

$$\mathcal{J}^{L,M} \approx \delta_{M,0}(\text{const}) \int_{}^{\infty} j_L(qr)\, e^{-Kr}\, r dr, \qquad (4.3)$$

where the lower limit is not specified but must be large enough to justify the approximations already made. It is seen that if q = 0 and K = 0 then the result of the integration in Eq. (4.3) is infinite, much larger than the contribution from any finite volume of space immediately around the nucleus, and therefore justifying in this case the approximations that have already been made. If these approximations are correct, it is also useful to observe from Eq. (4.1) that the integral (4.3) is proportional to the asymptotic amplitude of the form factor ("reduced width"), and therefore measures this quantity.

One need not have started with the DWB scattering amplitude. It is clear that the exact amplitude possesses all of the properties just described.

Of course, q = 0 and K = 0 are parameter values which are not likely to be realized in an actual experiment. The situation q = 0 is not impossible, and is often closely realized at $\Theta = 0°$ in deuteron stripping experiments, in which the energy change can balance the mass change. However, $K \approx 0$ is not likely. Values of K which permit radii of the order 5 f to be important are not unknown, but to justify our approximations, in deriving Eq. (4.3), we would want large contributions from radii of the order 50 f, say, and would require correspondingly small values of K.

Although Eq. (4.3) is not a good approximation under actual physical circumstances, it is always a good approximation at certain non-physical values of q. For example, at large r the asymptotic form of j_L is that of a sine function, so let us consider

$$\int_0^{\infty} \sin qr\, e^{-Kr}\, dr = q\,(K^2 + q^2)^{-1}.$$

The r = 0 lower limit simplifies the integral and adds only a finite contribution. It is seen that this integral has poles at $q = \pm iK$, corresponding to the contribution of infinite impact parameters in Eq. (4.3), when q has these imaginary values. One might hope to extrapolate the physically observed cross-sections to q = iK, and thereby obtain reduced widths that are not influenced by distortion.

The method can be improved somewhat if the leading, plane-wave terms in Eq. (4.2) are replaced by Coulomb waves.

Present indications are that the above method has no practical utility. Actual values of K are so large that for physical values of q the integral (4.2) is dominated by the scattered waves. The integral (3.7) is dominated by the behaviour at small r. As a result one has no idea how to perform the extrapolation to the pole.

Deuteron stripping experiments at very low bombarding energy have small q at all angles, if K is small. A series of such experiments have been performed [8], and it has been claimed that these give good fits to plane wave formulas, according to Eq. (4.3). But the plane wave formulas have adjustable parameters, so it is hard to know how one would get bad fits to the rather featureless cross-sections that are observed. "Experiments" have also been performed with computing machines [9], using DWB codes,

to see whether plane wave properties would emerge if K and q were taken small enough. The pole was found to be too remote to be reached by these DWB calculations.

* * * * * * * * *

A very useful and practical approximation of Eq. (3.7) is the "high-energy approximation", for example, as presented by McCAULEY and BROWN [10], or by GLAUBER [11]. In this method, WKB approximation is used for the distorted waves $\chi_i^{(+)}$ and $\chi_f^{(-)}$. Because the WKB wave functions are used only in the limited region of space in which the overlap integral is being computed, the errors caused by the neglected quantum mechanical diffraction effects are not large. It would appear that the method is reliable for nucleons at energies, $E \gtrsim 100$ MeV, or for heavier projectiles at lower energies. A straightforward way to test the high-energy approximation is to compare it with results of exact calculations which are based upon the partial wave expansion. Calculations performed by BROWN, and by JACKSON bear out the accuracy of the method [12]. Other calculations [13], of a preliminary sort, find it at fault at an energy of 150 MeV. The meaning of the latter calculations is not clear.

In the simplest WKB treatment we make the substitution

$$\chi^{(+)}(\vec{k}, \vec{r}) = e^{iS(\vec{r})}, \tag{4.4}$$

where the phase function $S(\vec{r})$ is approximated by an integral, taken along the classical trajectory passing through the point $\vec{r}$,

$$S(\vec{r}) \approx \int^{\vec{r}} [k^2 - 2M^* U h^{-2}]^{\frac{1}{2}} \, d\tau. \tag{4.5}$$

The integrand is the local momentum, at each point along the classical trajectory. At high energy the potential does not cause large changes of the local momentum, but may very well cause important changes in S, because in S the effects are cumulative.

If the energy is high then particle trajectories do not deviate very much from straight lines in the direction $\vec{k}$; the deviation is particularly slight for the short segments of the trajectories which lie within the optical potential, or in its immediate vicinity. The integration in Eq. (4.5) may be simplified to

$$S(\vec{r}) \approx \int_{-\infty}^{0} [k^2 - 2M^* \hbar^{-2} U(\vec{r} + \hat{k}\rho)]^{\frac{1}{2}} \, d\rho, \tag{4.6}$$

where $\hat{k}$ is the unit vector in the direction of $\vec{k}$. It is customary to simplify the integral further by expanding the square root, on the basis that at high energy not only does k^2 become large but the potential U becomes weakened. We then obtain

$$S(\vec{r}) \approx (\vec{k} \cdot \vec{r}) - (M^*/\hbar^2 k) \int_{-\infty}^{0} U(\vec{r} + \hat{k}\rho) \, d\rho. \tag{4.7}$$

Because this formula is intended for use at high energy, it is worthwhile to remark that the result (4.7) is also obtained if one starts from the Klein-Gordon equation and makes the same steps of approximation used in deriving (4.7) from the Schrödinger-equation starting point. However, in this case M^* is replaced by (E/c^2).

The distorted waves we need for DWB work may be written, using Eq. (4.7) and the time-reversal relation, Eq. (2.16), as

$$\chi_i^{(+)}(\vec{k}_i, \vec{r}) \approx \exp i \{ (\vec{k}_i \cdot \vec{r}) - \left(\frac{M_i^*}{\hbar^2 k_i}\right) \int_{-\infty}^{0} U_i(\vec{r} + \hat{k}_i \rho)\, d\rho \}, \qquad (4.8a)$$

$$\chi_f^{(-)*}(\vec{k}_f, \vec{r}) \approx \exp i \{ -(\vec{k}_f \cdot \vec{r}) - \left(\frac{M_f^*}{\hbar^2 k_f}\right) \int_{-\infty}^{0} U_f(\vec{r} - \hat{k}_f \rho)\, d\rho \}. \qquad (4.8b)$$

It is instructive to change the sign of the dummy variable, ρ, in Eq. (4.8b), so that

$$\chi_f^{(-)*}(\vec{k}_f, \vec{r}) \approx \exp i \{ -(\vec{k}_f \cdot \vec{r}) - \left(\frac{M_f^*}{\hbar^2 k_f}\right) \int_{0}^{\infty} U_f(\vec{r} + \hat{k}_f \rho)\, d\rho \}. \qquad (4.8b')$$

Now (4.8a) and (4.8b') are inserted in the distorted waves matrix element, Eq. (3.7). It is seen that the optical potential U_i modulates the wave function of the incident particle, from $-\infty$ to the point $\vec{r}$, where the reaction occurs. Beyond the point $\vec{r}$ the wave function of the emerging particle is modulated by the potential U_f. Yet further simplifications appear in the case of inelastic scattering, as illustrated in articles by SQUIRES and SANDERSON [14]. In this case $M_i^* = M_f^*$ and $U_i = U_f$. Furthermore, the cross-section at high energy is large only at small scattering angles, for which $\hat{k}_i \approx \hat{k}_f$. Under these circumstances, the two phase-shift integrals of Eqs. (4.8a), (4.8b') combine when the product $[\chi_f^{(-)*} \chi_i^{(+)}]$ is formed, and give a single integral over the range $-\infty < \rho < \infty$. This integral depends only on the impact parameter,

$$\vec{b} = \vec{r} - \hat{k}\,(\vec{r} \cdot \hat{k}), \qquad (4.9)$$

of the classical trajectory, and may conveniently be incorporated with the form factor F_L of the DWB calculation, to give an altered form factor. The remaining calculation, beyond this step, merely uses plane wave Born approximation with the altered form factor. The inclusion of distortion effects by this method is quite easy, and does not require the use of computing machines. Sanderson, for example, studied the inelastic scattering of 185 MeV protons by C^{12}, and found that distortion reduced various differential cross-sections by factors of 2 or 3, and altered the angular distribution in some cases.

The generalizations required to include spin-orbit distorting potentials are straightforward, and are described in the papers of McCauley and Brown, and of Glauber, already cited, and in an article by KÖHLER [15]. The latter author considers the polarization of inelastically scattered protons.

The papers of McCauley and Brown, and Glauber, do not merely use the high energy method to generate distorted waves corresponding to given

optical potentials. Instead the phase function $S(\vec{r})$ is summed over the encounters of the given projectile with each of the individual nucleons in the target nucleus. In this manner a microscopic model of the basic physics behind the optical potential is made. Optical potentials are thereby introduced which are appropriate for the individual inelastic process which may be considered. Optical potentials for excited states are slightly different from those for ground states, but this effect appears to be negligible in comparison with other errors of the DWB method.

If the internal degrees of freedom of the target are correctly taken into consideration, then the method of generating the phase function S by summing the encounters of the incident projectile with individual nucleons actually generates the entire wave function for the system. This wave function describes all inelastic processes, as well as the elastic scattering. It is not merely a distorted wave for the entrance channel. To get that distorted wave the entire wave function would have to be projected onto the entrance channel. However, projection onto other channels gives other processes. These ideas are used in the Drozdov-Blair theory of inelastic scattering.

Finally, it is interesting to contrast the high-energy method with the semi-classical model of direct reactions, of BUTLER, AUSTERN and PEARSON [16]. In that model, ray-tracing ideas are used, as with WKB wave functions. However, that model goes on to picture the angular momentum transfer to the nucleus as taking place precisely at those points $\vec{r}$ which satisfy the relation $L = \left| \vec{r} \times \{(\vec{K}_i(\vec{r}) - \vec{K}_f(\vec{r}))\} \right|$, where $\vec{K}_i$ and $\vec{K}_f$ are the local values of the momenta inside the nucleus. Such a localization of the angular momentum transfer not only implies using the semi-classical approximation for the projectile wave functions $\chi_i^{(+)}$ and $\chi_f^{(-)}$, but also for the nuclear wave functions v_i and v_f which determine the form factor F_L. Physically interesting values of L are not large enough to justify this further approximation unless distortion effects are very weak [17].

5. ADIABATIC THEORY - COMPROMISE METHOD

This method is chiefly of interest in problems of inelastic scattering, for which, at the hands of Drozdov and Blair, it hs been of great use.

We go back to the exact transition amplitude in its most primitive form, Eq. (2.20). For the problem of inelastic scattering we set $\xi_i = \xi_f = \xi$, $\vec{r}_i = \vec{r}_f = \vec{r}$ and $U_f = U_i = U$, $V_i = V_f = V$. Therefore

$$T_{fi} = \left\langle v_f(\xi)\, e^{i(\vec{k}_f \cdot \vec{r})} \,\middle|\, U(r) + V(\vec{r}, \xi) \,\middle|\, \Psi(\vec{r}, \xi) \right\rangle . \qquad (5.1)$$

The method of distorted waves Born approximation replaces Ψ by

$$\Psi \approx v_i(\xi)\, \chi_i^{(+)}(\vec{k}_i, \vec{r}),$$

after making one step of transformation to isolate the dependence on V. In the adiabatic method we directly replace Ψ in Eq. (5.1) with the expression

$$\Psi(\vec{r}, \xi) \approx v_i(\xi)\, \psi_i^{(+)}(\vec{k}_i, \vec{r}; \xi). \qquad (5.2)$$

The wave function $\psi_i^{(+)}$ is a solution of the equation

$$\{ - \frac{\hbar^2}{2 M^*} \nabla^2 + U(r) + V(\vec{r}, \vec{\xi}) - E \} \psi_i^{(+)} = 0 \qquad (5.3)$$

in which the Hamiltonian $\mathcal{H}(\xi)$ which governs the motion of the internal variables ξ does not appear. As a result the variables ξ may be treated as c-numbers in Eq. (5.3), as constants, upon which the solution function $\psi_i^{(+)}$ depends only parametrically. The calculation of T_{fi} has been separated into two steps. In the first step the differential equation (5.3) involving only the one vector-variable, $\vec{r}$, is solved. The solution is then multiplied by $v_i(\xi)$, and Eq. (5.1) is used to project from this product the amplitude for state f.

The method is obviously based on the principle that the initial state of the system, $v_i(\xi)$, changes so slowly with time that the scattering wave function $\psi_i^{(+)}$ for the projectile is determined by the instantaneous condition of $v_i(\xi)$ at the moment the projectile first appears. The Hamiltonian $\mathcal{H}(\xi)$ may be neglected in the Green's function for the scattering process. Evidently this approximation is good only under special circumstances. Of course, the projectile velocity must be high, but it always is. What is more important is that ξ must be a variable which does change slowly, for which $V(\vec{r}, \xi)$ does not easily give coupling to excited states of high energy. Collective variables are of this kind. They vary slowly because of the large associated mass. The adiabatic method is used almost entirely in studies of collective excitations.

The method is analogous to that used in computing the collision between two classical bodies, one light and one heavy. We compute the change of state of the light body as if the heavy body remained stationary (Fig. 11). However, the change of momentum of the light body is recognized as the

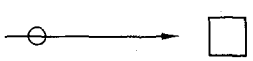

Fig. 11

Classical collision of two bodies, one heavy and one light

momentum given to the heavy body, and thereby we determine its new state.

The time-dependent interpretation of the adiabatic method is developed at length by Glauber, in his Boulder lectures. He carries the Hamiltonian $\mathcal{H}(\xi)$ as giving time-dependent operators in the variable $\vec{r}$, and shows what errors are made when this time-dependence is omitted.

Two properties of the adiabatic method are helpful in finding interpretations of the inelastic scattering theory. The first is seen by noting that if $f = i$ in Eq. (5.1), then the elastic scattering amplitude is found from the same approximate wave function $v_i \psi_i^{(+)}$ which is used to compute inelastic scattering. Therefore elastic and inelastic scattering are easily compared. The other helpful property is seen by noting that if $k_f = k_i$, so that the energy loss is reglected, then

$$\langle e^{i(\vec{k}_f \cdot \vec{r})} | U + V | \psi_i^{(+)} \rangle \qquad (5.4)$$

is the elastic scattering amplitude for the projectile for fixed values of ξ.

Such approximate methods as we may know for this amplitude may then be employed.

It is necessary to realize that an exact solution for the wave function $\psi_i^{(+)}$ is not normally at all easy to obtain. If the angular momentum transfer is non-vanishing, $L \neq 0$, then the "potential" $[U(\vec{r}) + V(\vec{r}, \xi)]$ in Eq. (5.3) is not a scalar function of $\vec{r}$ and most normal methods of solving this equation therefore fail. Approximations must be used.

The early versions of the adiabatic method use the high-energy approximation to compute $\psi_i^{(+)}$. They have found their greatest application in studies of the scattering of medium and high-energy alpha particles, in which case the momentum is large, and the high-energy approximation is justified. To use this approximation we write the scattered amplitude of Eq. (5.4) in the form

$$f(\vec{k}_f, \vec{k}_i; \xi) = -\left(\frac{M^*}{2\pi\hbar^2}\right) \int e^{-i(\vec{k}_f \cdot \vec{r})} \left[U(\vec{r}) + V(\vec{r}, \xi)\right] \psi_i^{(+)} d^3r. \qquad (5.5)$$

Since $\psi_i^{(+)}$ is presumed known it is convenient to use Green's theorem to transform away the potentials $[U(\vec{r}) + V(\vec{r}, \xi)]$. It shall be presumed that $[U(\vec{r}) + V(\vec{r}, \xi)]$ is non-vanishing only within some finite volume, bounded by a surface Σ. Then Eq. (5.3) is used to substitute out the potentials, and we obtain

$$f(\vec{k}_f, \vec{k}_i; \xi) = -\frac{1}{4\pi} \int_{\text{vol}} e^{-i(\vec{k}_f \cdot \vec{r})} \left[\nabla^2 + k_i^2\right] \psi_i^{(+)} d^3r. \qquad (5.6)$$

This is of the form

$$f = -\frac{1}{4\pi} \int_{\text{vol}} e^{-i(\vec{k}_f \cdot \vec{r})} \left[k_i^2 - k_f^2\right] \psi_i^{(+)} d^3r$$

$$-\frac{1}{4\pi} \int_{\text{vol}} \left[e^{-i(\vec{k}_f \cdot \vec{r})} (\nabla^2 \psi_i^{(+)}) - (\nabla^2 e^{-i(\vec{k}_f \cdot \vec{r})}) \psi_i^{(+)}\right] d^3r. \qquad (5.7)$$

The first term is a correction to the adiabatic amplitude, if the energy loss is not negligible. It shall be ignored henceforth, as is customary. The second term becomes, by Green's theorem,

$$f = -\frac{1}{4\pi} \int_{\Sigma} \left[e^{-i(\vec{k}_f \cdot \vec{r})} (\nabla \psi_i^{(+)}) - (\nabla e^{-i(\vec{k}_f \cdot \vec{r})}) \psi_i^{(+)}\right] \cdot d\vec{\Sigma}. \qquad (5.8)$$

This expression is the scattered amplitude. The incident plane wave part of $\psi_i^{(+)}$ contributes only to the ingoing amplitude, if $k_f = k_i$, and it may be verified that it makes a vanishing contribution in Eq. (5.8). Therefore an equivalent equation is

$$f = -\frac{1}{4\pi} \int_{\Sigma} \left[e^{-i(\vec{k}_f \cdot \vec{r})} (\nabla \psi_{\text{scatt}}) - (\nabla e^{-i(\vec{k}_f \cdot \vec{r})}) \psi_{\text{scatt}}\right] \cdot d\vec{\Sigma}. \qquad (5.9)$$

It is in this equation that we introduce the high-energy approximation.

In the high-energy approximation the incident particles are assumed to follow straight-line paths in the direction $\vec{k}_i$, and to suffer only a change of phase as they move along these paths. Therefore Σ is conveniently taken to be a cylinder, whose axis lies along the direction $\vec{k}_i$ (Fig. 12). Evidently

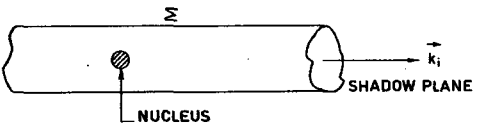

Fig. 12

The surface Σ on which we apply Green's theorem

the wave ψ_{scatt} then vanishes everywhere on the surface Σ except on the "shadow plane", which is the end of the cylinder on the downstream side. In fact $\psi_{scatt} \neq 0$ only within the area of the geometrical shadow of the nucleus. The element of area on the shadow plane is $\hat{k}_i\, dA$. Then

$$f \approx -\frac{1}{4\pi} \int_{\text{shadow}} e^{-i(\vec{k}_f \cdot \vec{r})} \left[(\hat{k}_i \cdot \nabla) + i\,(\hat{k}_i \cdot \vec{k}_f) \right] \psi_{scatt}\, dA\,, \tag{5.10}$$

or

$$f = -\frac{1}{4\pi} \int_{\text{shadow}} e^{-i(\vec{k}_f \cdot \vec{r})} \left[(\hat{k}_i \cdot \nabla) + i\,(\hat{k}_i \cdot \vec{k}_f) \right] \left[e^{iS(\infty)} - e^{i(\vec{k}_i \cdot \vec{r})} \right] dA, \tag{5.11}$$

where $S(\infty)$ is the WKB phase factor, integrated from one side of the nucleus to the other. The most interesting special case of Eq. (5.11) is that in which the nucleus is "black" to the incident particles, in which case $\exp[iS(\infty)] = 0$ over the region of the shadow. The integral very much simplifies. It becomes

$$f_{black} = \frac{ik}{4\pi} [1 + \cos\Theta] \int_{\text{shadow}} e^{i[(\vec{k}_i - \vec{k}_f) \cdot \vec{r}]}\, dA\,, \tag{5.12}$$

where Θ is the scattering angle. Equation (5.12) is the basic equation of the "Fraunhofer" approximation of the adiabatic method. Generally it is used with the additional approximation $[1 + \cos\Theta] \approx 2$, for the "obliquity factor". This additional approximation is consistent with those already made and shall be adopted.

In the black nucleus limit the collective coordinates ξ do nothing except change the size and shape of the nuclear shadow. Therefore this is a theory of collective deformations of the surface. Suppose the nuclear surface is described by the equation

$$R(\theta, \phi) = R_0 + \sum_{L,M} \xi_{L,M}\, Y_L^{M*}(\theta, \phi)\,. \tag{5.13}$$

The $\xi_{L,M}$ are operators for the collective motions, and are the deformation distances for the L, M multipoles. The corresponding geometrical shadow is described by

$$\rho(\phi) = R_0 + \delta(\phi) , \tag{5.14}$$

and to first order in the deformations $\xi_{L,M}$, if these are small, we have

$$\rho(\phi) \approx R_0 + \sum_{L,M} \xi_{L,M} Y_L^{M*}(\tfrac{\pi}{2}, \phi), \tag{5.15}$$

in terms of the deformation at the nuclear equator. These expressions are used to compute the Fraunhofer amplitude, which I take at small Θ as

$$f_{black} = \frac{ik}{2\pi} \int_0^{2\pi} d\phi \int_0^{R_0 + \delta(\phi)} \rho d\rho \; e^{-ik\rho\Theta\cos\phi} . \tag{5.16}$$

To first order in the deformation the results of integration are

$$f_{black} = ikR_0^2 \left[\frac{J_1(kR_0\Theta)}{kR_0\Theta}\right] + ikR_0 \sum_{\substack{L,M \\ (L+M)\ \text{even}}} \left(\frac{2L+1}{4\pi}\right)^{\frac{1}{2}} \left(\frac{iM}{|M|}\right)^L \xi_{L,M} \frac{[(L-M)!(L+M)!]^{\frac{1}{2}}}{(L-M)!!(L+M)!!} J_{|M|}(kR\Theta). \tag{5.17}$$

The first term is the familiar result for elastic scattering from a black disc. Its square is the differential elastic scattering cross-section. To get the inelastic cross-sections we multiply $f(\theta, \phi; \xi)$ by $v_i(\xi)$, according to the prescription given earlier, and compute the projection on to $v_f(\xi)$. Then

$$\frac{d\sigma(i\to f)}{d\Omega} = (kR_0)^2 \left(\frac{2L+1}{4\pi}\right) \left| \langle v_f^M | \xi_{L,M} | v_i \rangle \right|^2 \times \sum_{M=-L,-L+2,\ldots}^{L} \frac{[(L-M)!(L+M)!]}{[(L-M)!!(L+M)!!]^2} J^2_{|M|}(kR_0\Theta) , \tag{5.18}$$

for a given multipole L, exciting a spin-zero target nucleus. The matrix element $\langle v_f^M | \xi_{L,M} | v_i \rangle$ in fact is independent of M, because of our definition of the $\xi_{L,M}$, and may be regarded as the reduced matrix element $\langle v_f \| \xi_L \| v_i \rangle$. For a permanently deformed rotating nucleus this matrix element, for example, is

$$\langle v_f \| \xi_L \| v_i \rangle = (\beta R_0) / (2L+1)^{\frac{1}{2}} , \tag{5.19}$$

where β is the usual deformation parameter. (The product (βR_0) is more meaningful than β itself, because the parameter R_0 is somewhat vague.)

Equation (5.18) is the basic result of the Blair-Drozdov theory of inelastic scattering. The reduced matrix element is the only adjustable parameter in this theory. The radius R_0 is determined by fitting the black disc formula to the elastic scattering, within the context of the same basic theory.

For medium-energy alpha particle inelastic scattering, Eq. (5.18) often gives excellent fits to experiment, and reduced matrix elements determined with its use agree well with those determined by electromagnetic means. The most characteristic property of this formula is that, for even L values the cross-section is a linear combination of

$$J_0^2,\ J_2^2,\ \ldots,\ J_L^2,$$

whereas for odd L values it is a linear combination of

$$J_1^2,\ J_3^2,\ \ldots,\ J_L^2.$$

Bessel functions very rapidly approach their asymptotic forms, so that

$$J_L^2 \rightarrow \sin^2(k R_0 \Theta + \frac{\pi}{4} - \frac{L\pi}{2}).$$

Therefore, beyond the first oscillation, or so, all even parity transitions have angular distributions of the same shape, and all odd parity transitions have the same shape. However, the two oscillate precisely 180° out of phase with each other. It is further noted that the odd-parity angular distribution is in phase with the elastic angular distribution, giving a calibration of the parity determinations. These statements constitute the "Blair phase rule". We have previously encountered the first part of this phase rule as a consequence of the DWB calculation for a strongly-absorbing nucleus. The phase rule is usually extremely well obeyed in experiment, and is in agreement with exact DWB calculations. Figures 13 and 14 show some typical experimental results and their comparison with the Blair formula.

It is seen in Figs 13 and 14 that the Fraunhofer curves follow the period of the oscillations of the experimental curves quite well, but that the magnitudes of the Fraunhofer cross-sections drop off much too slowly with angle. This is a characteristic difficulty of the application of the high-energy method for a sharp-edged obstacle. Some kind of smoothing is called for. To determine reduced matrix elements the Fraunhofer curve usually is fitted to experiment at the smaller angles. There is some vagueness in fitting magnitudes because the experimental and theoretical curves have different shape, but this is not too bad.

At very small angles Eq. (5.18) may in principle be used to determine not just parities of transitions, but also precise L values. The formula is inaccurate at very small angles, because it omits Coulomb excitation. Also effects due to non-zero energy loss are noticeable at these angles. Nevertheless it may yet turn out that the difficult experiments at the very small angles will lead to useful results.

Also, we may note that the high-energy approximation of the adiabatic theory easily gives the cross-section for double excitation, if the expansion of the amplitude in Eq. (5.16) is carried to second order in the $\xi_{L,M}$. I will not pursue this.

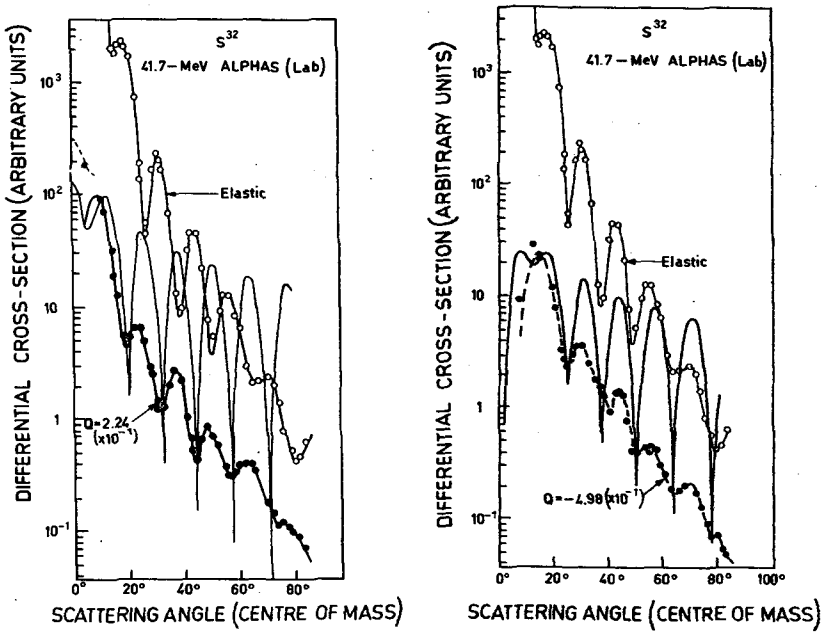

Fig. 13

Comparison of the Blair formula with experiment

The fact that the adiabatic theory normally is carried to only first order in the $\xi_{L,M}$ is a clue to further understanding and improvement of the method. This *first-order expansion is necessary because of the difficulties of solving* for the wave function $\psi_i^{(+)}(\vec{K}_i, \vec{r}, \xi)$, which I mentioned earlier. Because $\psi_i^{(+)}$ is computed in first order, the entire exact wave function $\Psi(\vec{r}, \xi)$ is being used to only first order in the interaction V. The resulting approximation, as you will recall, is then equivalent to the DWB approximation. DWB consistently carries all terms first order in V. It uses a different arrangement of the calculation, such that the first-order term of Ψ, which gives us so much trouble in the adiabatic method, is removed by the use of the Gell-Mann, Goldberger transformation. As a result the operator $U(r)$ is removed from T_{fi}, and we obtain a distorted final state wave function. The normal, first-order adiabatic result may be regarded as a simplified restatement of the DWB result. In the Fraunhofer approximation it brings a very great improvement in convenience.

It is difficult to base improved adiabatic results upon the surface integral of Eq. (5.9). It is only in the high-energy limit that the integrand is non-vanishing over only the shadow plane. I will describe briefly some recent work of Blair and myself which exploits the relation with the DWB method to obtain improved results. Much of the simplicity of the present adiabatic results is retained, as well as the close relation with elastic scattering.

To do the DWB calculation we must be more precise about the interaction $V(\vec{r}, \xi)$. If only the shape of the nucleus is to be important, then in an adiabatic theory there is very little alternative to supposing that for the deformed nucleus the projectile encounters just the normal optical potential,

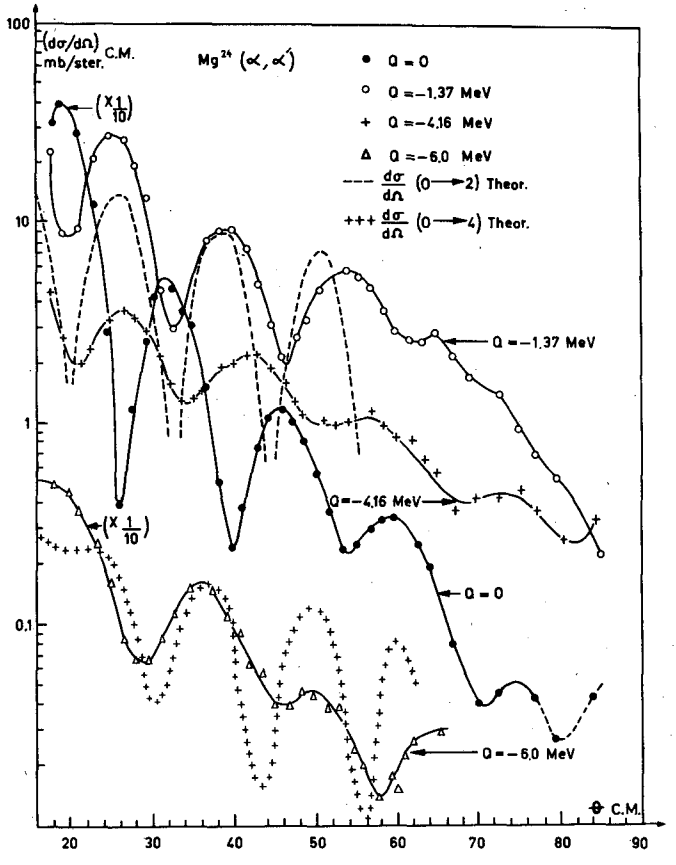

Fig. 14

More comparison of the Blair formula with experiment. Note the phase rule relations

but with a deformed shape. The depth and diffuseness must remain unchanged. Therefore the net interaction with the projectile is

$$U(r) + V(\vec{r}, \xi) = U(r,\ R_0 + \alpha), \tag{5.20}$$

where

$$\alpha = \sum_{L,M} \xi_{L,M}\, Y_L^{M*}(\theta,\ \phi) \tag{5.21}$$

as before. The nuclear radius parameter in the optical potential is increased by the angle-dependent increment α. To first order in α we get

$$V(\vec{r}, \xi) \approx \alpha \frac{\partial U}{\partial R_0}, \tag{5.22}$$

the derivative being evaluated at $\alpha = 0$. For definite multipole order L the DWB result is now found to be

$$T_{fi}^{M}(\mathrm{DWB}) = \left\langle v_f \middle|\middle| \xi_L \middle| v_i \right\rangle \int d^3r\, \chi_f^{(-)*} \left[Y_L^{M*} \frac{\partial U}{\partial R_0} \right] \chi_i^{(+)} \tag{5.23}$$

$$= \left\langle v_f \middle|\middle| \xi_L \middle|\middle| v_i \right\rangle \left[\frac{4\pi\,(2L+1)}{k^2} \right]^{\frac{1}{2}} \sum_{\ell,\ell'} i^{\ell - \ell'}\, e^{i(\sigma_\ell + \sigma_{\ell'})} \sqrt{2\ell'+1}$$

$$\times\, Y_{\ell'}^{M}(\Theta, 0) \left\langle \ell' L, -M M \middle| \ell\, 0 \right\rangle \left\langle \ell' L, 0\, 0 \middle| \ell\, 0 \right\rangle \int_0^\infty f_{\ell'} \frac{\partial U}{\partial R_0} f_\ell\, dr. \tag{5.24}$$

I have introduced $k_f = k_i = k$, because we are seeking an adiabatic result.

Much of the difficulty of Eq. (5.24) lies in computing the radial integrals. For these we introduce an approximation

$$\int_0^\infty f_{\ell'} \frac{\partial U}{\partial R_0} f_\ell\, dr \approx \int_0^\infty f_{\bar{\ell}} \frac{\partial U}{\partial R_0} f_{\bar{\ell}}\, dr, \tag{5.25}$$

where $\bar{\ell} = \frac{1}{2}(\ell + \ell')$. This approximation is based on the principle that for strongly absorbing projectiles, and for the partial waves having $\ell, \ell' \approx \ell_0$, the radial wave functions have not yet begun to oscillate, and are slowly varying functions of ℓ. Besides $|\ell - \ell'| \leq L$, and is not a large number.

An exact theorem relates the single-index radial integral of Eq. (5.25) to the η_ℓ amplitudes of elastic scattering. It is

$$\int_0^\infty f_\ell^2 \frac{\partial U}{\partial R_0}\, dr = \frac{iE}{2k} \frac{\partial \eta_\ell}{\partial R_0}. \tag{5.26}$$

For strongly-absorbing projectiles this theorem may be made more useful with the aid of two further good approximations:

$$\eta_\ell \approx \eta\,(\ell - \ell_0), \tag{5.27a}$$

$$\ell_0 \approx k R_0. \tag{5.27b}$$

From these we obtain

$$\frac{\partial \eta_\ell}{\partial R_0} \approx -k \frac{\partial \eta_\ell}{\partial \ell} \tag{5.28}$$

and therefore

$$\int_0^\infty f_\ell^2 \frac{\partial U}{\partial R_0}\, dr \approx -\frac{iE}{2} \frac{\partial \eta_\ell}{\partial \ell}. \tag{5.29}$$

Substitution of the approximations, Eqs (5.25) and (5.29), into Eq. (5.24) yields the amplitude $T_{fi}^{M}(\mathrm{DWB})$. I will take one more step, and multiply this by $(-M^*/2\pi\hbar^2)$ to get the inelastic "scattered amplitude" for the state f, because this is generally used in adiabatic discussions. Then, at last,

$$f^M(i \to f) \approx \frac{i}{2} \langle v_f || \xi_L || v_i \rangle (2L+1)^{\frac{1}{2}} \sum_{\ell,\ell'} i^{\ell-\ell'} e^{(\sigma_\ell + \sigma_{\ell'})} \sqrt{2\ell'+1}\; Y^M_{\ell'}(\Theta, 0)$$

$$\times \langle \ell' L, -MM | \ell 0 \rangle \langle \ell' L, 0\,0 | \ell 0 \rangle \left[\frac{\partial \eta_{\bar{\ell}}}{\partial \bar{\ell}} \right]. \qquad (5.30)$$

Equation (5.30) may be proved to reduce to the Fraunhofer formula as the parameters reduce to the set of limits: $(L/\ell_0) \to 0$, and $[\partial \eta_\ell / \partial \ell\,] \neq 0$ only over a range of ℓ small compared with ℓ_0. If these limiting conditions are not met, then a physically-based smoothed-cut-off approximation may instead be introduced for the function η_ℓ, and we see that the series in Eq. (5.30) is not difficult to compute numerically, and that results much better than those given by the Fraunhofer formula are to be expected.

For the case L = 2 a formula similar to Eq. (5.30) was previously derived by BLAIR, SHARP and WILETS [18]. They found that the replacement of η_ℓ by a simple two-parameter, real function (Fig. 15) was sufficiently accurate to

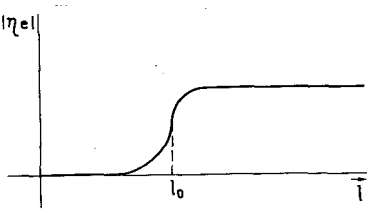

Fig. 15

The smoothed-cutoff form for η_ℓ, here taken to be real

match almost perfectly the exact DWB results of Satchler's group. Figure 16 shows a comparison between the formula of Blair, Sharp and Wilets, and an exact DWB result for the same case. This figure, taken from the recent paper of Rost, also shows a comparison with the older Fraunhofer prediction. Obviously there has been great improvement. The BSW results are especially useful, because simple "universal curves" could be based on their two parameters, and these curves enable very easy analysis of experiment. It may be that this method can be extended, with the application of the new result just derived.

Considerable generalization of this "compromise method" is possible.

6. COUPLED CHANNELS

This will be the last bit of discussion concerning techniques of solving the differential equations of the DI theory.

I have been emphasizing the DWB method. This method may be employed for all types of DI reactions. and therefore is valuable as being a unified approach for many different situations. For nucleons as projectiles it is related to the basic physics of the shell model. For projectiles such as medium-energy alpha particles it is equivalent to the very successful adiabatic method and is much better adapted for accurate evaluation with the aid of computing machines. At present the principal problem with the DWB

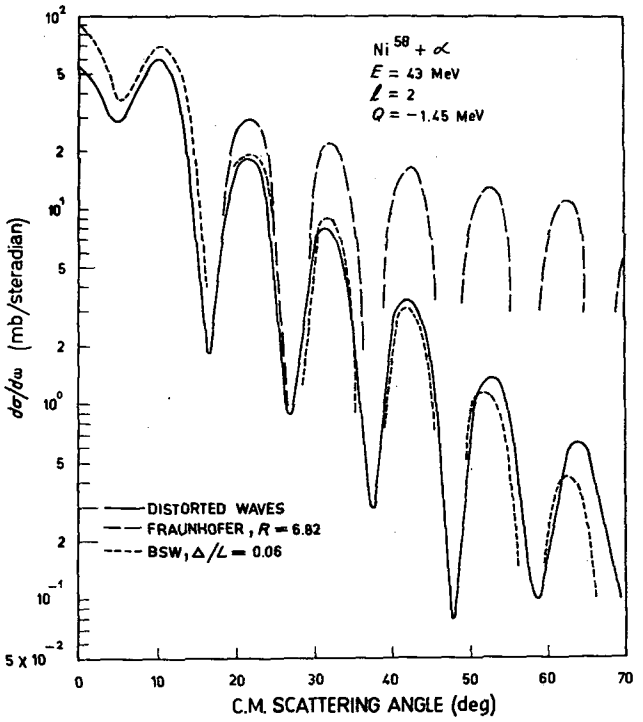

Fig. 16

Improvement of the adiabatic theory achieved by using the smoothed-cutoff η_ℓ function. Note the very good agreement with the DWB curve

method seems to be merely that it has not yet been applied sufficiently widely with sufficient accuracy. However, DWB is a Born approximation. The interaction which causes transitions between channels is carried to only first order. Inasmuch as DI cross-sections frequently are very large, one wonders whether a first-order theory is sufficiently accurate. It is desirable to test the basic accuracy of DWB.

Some rough estimates may be made. First, suppose a reaction cross-section equals the nuclear area, πR^2; this is a very large cross-section. We may estimate $(d\sigma/d\Omega) \approx R^2$ for this case. However in DWB

$$\frac{d\sigma}{d\Omega} = (M^*/2\pi\hbar^2)^2 (k_f/k_i) \Sigma_{Av} \left| \int d^3r \; \chi_f^{(-)*} V^{fi} \chi_i^{(+)} \right|^2 \qquad (6.1a)$$

where

$$V^{fi} \equiv \langle v_f | V_f(\xi, \vec{r}) | v_i \rangle . \qquad (6.1b)$$

Now large cross-sections only occur if the momentum transfer is small, so we may approximate $\chi_f^{(-)*}\chi_i^{(+)} \approx 1$, and have

$$R^2 \approx (M^*/2\pi\hbar^2)^2 [\, \bar{V}^{fi} \times \text{volume} \,]^2 \ . \qquad (6.2)$$

Here the matrix element of Eq. (6.1) has been replaced by the product of an average V^{fi}, multiplied by the volume over which it is effective. If the latter is the full nuclear volume, then

$$\bar{V}^{fi} \approx (2\pi\hbar^2 R/M^*)(\text{volume})^{-1} \ ,$$

$$\approx 3(\hbar^2/2\, M^* R^2) \ \approx 30\ \text{MeV}/A^{\frac{2}{3}} \ .$$

Except at very small A, this effective interaction is seen to be quite weak, and for medium and high energies may properly be treated by Born approximation. This is even more true if we consider the much smaller cross-sections which normally are in question.

Equation (6.2) predicts a large cross-section as a consequence of a weak potential, because of a surface-to-volume effect. The entire nuclear volume is made to cooperate coherently in removing flux from a given area of the incident beam. However, with strong distortions not all of the nuclear volume is able to participate on equal terms. An estimate suited for this case may be based on the formulas of the preceding section, for strongly-absorbed projectiles. The average interaction potential connecting two partial waves of the incident and emerging projectiles is

$$\bar{V}^{fi} = \left| \langle v_f || \xi_L || v_i \rangle (\Delta R)^{-1} \int_0^\infty f_\ell (\partial U/\partial R_0)\, f_\ell \, dr \right|$$

$$\approx \langle v_f || \xi_L || v_i \rangle (\Delta R)^{-1} \ (E/2) \left| \partial \eta_\ell / \partial \ell \right| \ . \qquad (6.3)$$

Here ΔR is the interval over which the integrand is large. For a highly-deformed nucleus the average amplitude of the (coherent) deformation is comparable with ΔR. By experience the maximum value of $|\partial \eta_\ell / \partial \ell|$ is known to be

$$\left| \partial \eta_\ell / \partial \ell \right|_{max} \approx \tfrac{1}{2} \ \text{or} \ \tfrac{1}{3} \ .$$

Therefore the maximum value of the average potential is found to be about $(E/5)$, say. Evidently Born approximation is likely to be reliable, although not always of ideal accuracy. The surface-to-volume effect *does not* appear in Eq. (6.3).

Rost has considered [19] a somewhat more formal question, related to the ideas in the preceding paragraph. He extracted numerical values of the radial matrix elements, in explicit calculations for strong transitions induced by medium-energy alpha particles, and found them to be small. He also computed the associated S-matrix elements, which are multiples of the radial integrals, and he tested the S-matrix for unitarity. The S-matrix elements for elastic scattering are the η_ℓ, and we have $|\eta_\ell| < 1$ because

the optical potential has a negative imaginary part. Rost finds that the squares of the off-diagonal S-matrix elements always are very much smaller than $\{1 - |\eta_\ell|^2\}$; this is true for each partial wave separately. The problem he treats is sufficiently general that it is convincing that at medium and high energies the DWB method has no difficulties with unitarity.

At low energy the DWB method fails, because the kinetic energy then is not large compared with the potential $V(\vec{r}, \xi)$. There will be more about this subject later.

The most satisfactory way to test DWB is to perform a more accurate calculation and to compare its results with those of DWB. The more accurate calculation, which has been considered by a number of authors, is the "Tamm-Dancoff" or "coupled-channels" method. †) I will describe it briefly. This method has no difficulties with unitarity, and works as well at low energy as at medium and high energies. However, it is very complicated to carry out in practice. To avoid excessive complications I will treat only the one special case of inelastic scattering, and for a nucleus whose ground state has zero spin.

The exact wave function $\Psi(\vec{r}, \xi)$ for the coupled system (projectile + target) may be expanded in terms of the functions $v_{In}^{M}(\xi)$; the eigenstates of the target nucleus. A member of this set is the n^{th} eigenstate of angular moment I and projection M. The coefficients of $v_{In}^{M}(\xi)$ in the expansion of Ψ are functions of the projectile coordinate $\vec{r}$. We take the z-axis in the direction of the incident beam, and therefore the z-projection of the angular momentum of Ψ is zero.

Now we treat Ψ by introducing it into the Schrödinger equation, to get a system of coupled differential equations from which to determine the channel wave functions, the coefficients of v_{In}^{M} in Ψ. Before displaying the equations, it is desirable to identify the good quantum numbers in Ψ. These quantum numbers indicate how to simplify the differential equations by separation of variables. Evidently neither I nor orbital angular momentum is a good quantum number, because the coupling potential $V(\vec{r}, \xi)$ normally is not a scalar in $\vec{r}$ or ξ, and therefore the ordinary partial wave expansion fails to give separation of variables. However $V(\vec{r}, \xi)$ is a scalar in $\vec{r}$ and ξ jointly, and therefore J and parity are good quantum numbers. Alternatively, because we have assumed I = 0 in the incident channel, the orbital angular momentum of that channel is equivalent to J and parity of the whole system, and it is a good quantum number. ‡) We therefore expand Ψ in terms of the orbital angular momentum of the incident channel:

$$\Psi(\vec{r}, \xi) = (\sqrt{4\pi}/kr) \sum_{\ell} i^{\ell} (2\ell + 1)^{\frac{1}{2}} \psi^{\ell}(\vec{r}, \xi), \tag{6.4a}$$

$$\psi^{\ell}(\vec{r}, \xi) = \sum_{\ell', I, n} u^{\ell}_{\ell' In}(r) \; \underline{y}^{\ell 0}_{\ell' In}(\hat{r}, \xi), \tag{6.4b}$$

†) Recently SCHWARTZ [20] criticized the use of the coupled-channels method to calculate the scattering of electrons by atoms, on the grounds of its having very slow convergence. However, the difficulties in his case appear to be caused by the dense nature of the spectrum of a single electron near the threshold for ionization. Nuclear spectra do not have this property.

‡) In the general case questions of "eigenstates of the S-matrix" arise, and the methods discussed by BLATT and BIEDENHARN [21], must be used.

$$y^{\ell 0}_{\ell' In}(\hat{r},\xi) = \sum_M \langle \ell' I, -MM | \ell 0 \rangle Y_{\ell'}^{-M}(\hat{r})\, v_{In}^{M}(\xi). \qquad (6.4c)$$

The Coulomb phase factors have been absorbed into the radial wave functions here, for compactness. Evidently the channel wave function in the entrance channel is simple. It is the coefficient of v_{00}^{0}, and is

$$(-\sqrt{4\pi}/kr)\sum_\ell i^\ell (2\ell+1)^{\frac{1}{2}}\, u^{\ell}_{\ell 00}(r)\, Y^0_\ell(\hat{r}).$$

Asymptotically

$$u^{\ell}_{\ell 00} \to e^{i\sigma_\ell}\left(\frac{i}{2}\right)[H^*_\ell - \eta_\ell H_\ell] \qquad (6.5)$$

as usual. All the other radial functions are purely outgoing functions, asymptotically, and for I, n $\neq$ 0, 0,

$$u^{\ell}_{\ell' In} \to [k/k(In)]^{\frac{1}{2}}\, \eta^{\ell}_{\ell' In}\, (e^{i\sigma_{\ell'}}/2i)\, H_\ell, \qquad (6.6)$$

where $\eta_{\ell' In}{}^{\ell}$ are the S-matrix elements between radial Coulomb wave functions normalized to unit radial flux. All cross-sections are expressed in terms of these S-matrix elements.

Upon substitution of Ψ into the Schrödinger equation we get the coupled system of equations from which the $u^{\ell}_{\ell' In}$ are to be determined. For a given ℓ this is:

$$\left\{\frac{\hbar^2}{2M^*}\left[-\frac{d^2}{dr^2} + \frac{\ell'(\ell'+1)}{r^2}\right] + U(r) + \epsilon(I,n) - E\right\} u^{\ell}_{\ell' In}(r)$$

$$= -\sum_{\ell'' I' n'} \langle y^{\ell}_{\ell' In} | V(\vec{r},\xi) | y^{\ell}_{\ell'' I' n'} \rangle\, u^{\ell}_{\ell'' I' n'}(r) \qquad (6.7)$$

where $\epsilon(I,n)$ is the excitation energy of state I, n.

To reduce Eq. (6.7) to DWB we would have to make two steps of simplification. In the first step, in the equations for the excited states, I, n $\neq$ 0, 0, we would ignore on the right-hand side all terms except I' = 0, n' = 0. The incident channel thereby would be presumed to be the largest part of the wave function. Then the set of coupled equations would become

$$\left\{\frac{\hbar^2}{2M^*}\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}\right] + U - E\right\} u^{\ell}_{\ell 00} = -\sum_{\ell'' I' n'} \langle \ell 00;\ \ell | V | \ell'' I' n';\ \ell \rangle\, u^{\ell}_{\ell'' I' n'} \qquad (6.8a)$$

$$\left\{\frac{h^2}{2M^*}\left[-\frac{d^2}{dr^2} + \frac{\ell'(\ell'+1)}{r^2}\right] + U + \epsilon(I,n) - E\right\} u^{\ell}_{\ell' In}$$

$$= -\langle \ell' In;\ \ell | V | \ell 00;\ell \rangle\, u^{\ell}_{\ell 00}, \qquad (6.8b)$$

where (6.8b) is used for I, n $\neq$ 0, 0. Evidently if $u^{\ell}_{\ell 00}$ were known, then Eq. (6.8b) could be solved by use of the Green's function for the left-hand side. However, if the coupling is strong then Eq. (6.8a) shows that $u^{\ell}_{\ell 00}$ depends on the detailed properties of the excited state wave functions, and is not all easy to obtain. Now in DWB we make the one further approximation

$$u^{\ell}_{\ell 00} \approx e^{i\sigma_\ell} f_\ell(k, r)$$

for use in Eq. (6.8b), where f_ℓ is the ground state radial wave function computed with the phenomenological optical potential which fits elastic scattering. This potential does not have the peculiar shape or parameters which might be needed to account correctly for the reaction of the excited states on $u^{\ell}_{\ell 0}$. The reaction is buried in the imaginary term of the optical potential, in an averaged way. This appears to be the principal error of DWB. But normally such an average treatment of the "radiation damping" should not be at all bad.

To obtain the method of coupled channels a Green's function solution is not attempted. None of the channel wave functions is assumed known in advance. Instead, a few values of I, n are regarded as especially important (the "chosen channels"), all channels other than these few are ignored, and the coupled differential equations for the few chosen channels are solved exactly by integrating them out to large r numerically, step-by-step, starting from r = 0. If the number of coupled equations is N, then this numerical integration must be performed N times, starting from N linearly-independent initial conditions. That linear combination of the N solutions is then selected which satisfies the boundary condition of having an ingoing wave in only the ground state channel. Evidently the method becomes prohibitive if many chosen channels are used. The number of coupled equations equals the number of terms of the sum in the right-hand side of Eq. (6.7). If L is the highest multipole which is carried in the expansion of $V(\vec{r}, \xi)$, as in Eq. (3.2), then the number of coupled equations is of the order of (2L + 1) times the number of chosen channels. In typical practical applications [22] of the method the chosen channels are only the three lowest collective states of the nucleus, and the calculation is nevertheless very lengthy.

Figure 17 shows a famous graph from the paper of Chase et al., showing the failure of DWB in the calculation of the excitation of the collective I = 2 level by 1 MeV neutrons. Here β is the strength of the interaction V. It is the usual deformation parameter. DWB corresponds to β = 0. Evidently for larger β the cross-section departs drastically from the DWB values.

In contrast, at higher energies, Buck finds fairly good agreement with DWB, in agreement with our qualitative estimates at the beginning of this section.

Two important effects contribute to the failure of DWB at low energies, both of which go away as we reach the region of medium energies, in which most experiments are done. One of these effects is merely that Born approximation always fails at low energy. This is seen in the factor k^{-1} in the Green's function, if we are careful to watch that meaningful normalizations are used. Mott and Massey stress this point in their discussion of coupled channels. If the potential V is to be regarded as small there must be some other quantity in the differential equations, compared to which it is small, and which tends to determine the wave function $u^{\ell}_{\ell 00}$ of the ground state channel. Despite the optical potential U, the only control over the shape of the

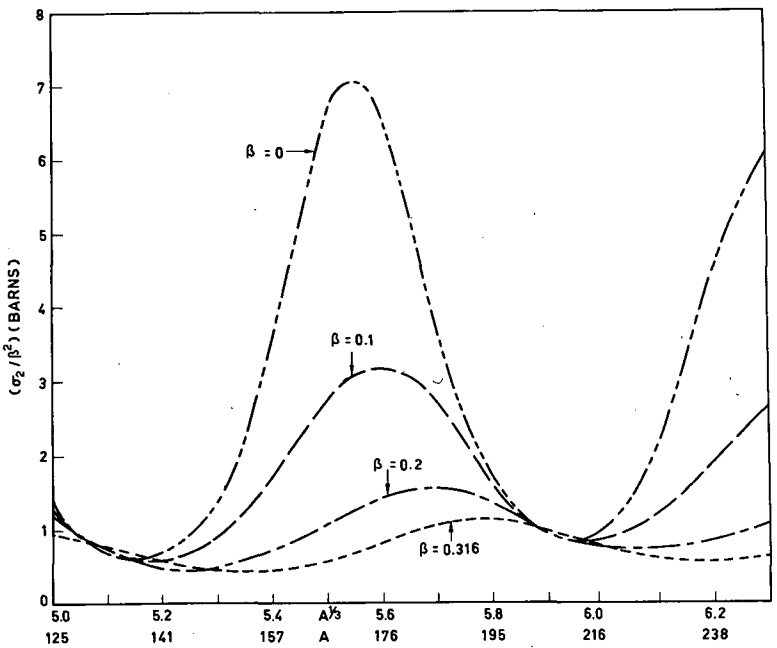

Fig. 17

Results of coupled-channel calculation for excitation of I = 2 rotational level by inelastic scattering of 1 MeV neutrons, showing failure of DWB at this low energy. The graph shows cross section divided by square of coupling constant β. If DWB were correct all curves would coincide with the curve labelled β = 0

wave function at large radii is the kinetic energy. At low energy the wave function is not "stiff" against perturbations, and Born approximations fail. However, the actual DI perturbing potentials are not very strong, and therefore it is only necessary to go to energies $E \gtrsim 10$ MeV before strong reactions on the ground state channel become unimportant. (We recall that DWB carries these reactions in an averaged fashion.) †

The other effect which makes DWB fail at low energy is more specific for nuclei. It is related to the occurrence of shape resonances in the optical potential. At a shape resonance the DWB radial wave function f_ℓ becomes very large inside and near the nucleus, and large transition amplitudes appear. It is not surprising that the result of Chase et al. show large errors of the DWB method in the regions of resonance. However, shape resonances are inherently a low energy phenomenon. They are caused by multiple reflections at the nuclear surface, which trap part of the wave function inside the nucleus. Actual nuclear surface thicknesses are such that the reflectivity (for nucleons) becomes negligible for energies $E \gtrsim 10$ MeV. Therefore

† The careful reader will have noticed that the radial integral in the strongly-absorbing case increases with bombarding energy. This effect is real, within the range of energies in which strong absorption occurs. It occurs because as the energy rises the wave functions become able to overlap effectively with the nuclear potential at greater depths, where it is stronger. However, our estimate of these integrals is accurate, and they are small compared with the kinetic energy. In addition, these radial integrals, as defined, do not yet carry the k^{-1} factor of the Green's function.

shape resonances are not a consideration at these energies. (Some effort is being made by Melkanoff to track down residual effects due to shape resonances at medium energies. These effects should be small, but not yet zero.)

In summary, except at very low energies, DWB is not drastically in error. Such corrections as may be needed probably can be introduced well enough by perturbative methods, rather than by use of the very difficult coupled-channels calculation. The coupled-channels calculations of Buck did yield the first accurate results for the double excitation process. However, it seems likely that adiabatic calculations, with or without Fraunhofer approximation, will be sufficiently accurate for the analysis of that process. We may conclude that there is not too much trouble in getting accurate solutions of the basic equations of DI theories. What is more interesting is to learn what physical information to build into those theories.

7. SPECTROSCOPY: COHERENCE, PARENTAGE

All of the discussion until now has concerned the effects involving the distorted waves $\chi_i^{(+)}$, $\chi_f^{(-)}$, for the relative motion of the colliding fragments in the incident and emerging channels. This discussion has been directed primarily toward the angular distribution in the reaction. In DWB, using zero-range approximation, the internal structures of the colliding fragments influence the angular distribution fairly weakly, through the "form factor" F_L. We were therefore able largely to put aside consideration of the matrix element for the internal wave functions

$$\langle v_f(\xi_f) \mid V_f(\xi_f, \vec{r}) \mid v_i(\xi_i) \rangle \tag{7.1}$$

of Eq. (3.1). Now it is necessary to discuss this matrix element, and thereby to consider the magnitude of the cross-section.

One interesting case was already considered in section 5, where for the interaction of a projectile with a deformed nucleus we considered the model that the projectile interacts with a deformed potential well. Eqs. (5.20) and (5.21) especially present this model. This model is of the same sort as other phenomenological models of deformed nuclei, and deformation parameters obtained by using it for inelastic scattering experiments also agree with those obtained by other means. This is not a trivial fact. The distorting potentials establish the region where the product $\chi_f^{(-)*}\ \chi_i^{(+)}$ is large. The magnitude of the cross-section would be altogether wrong unless the interaction which excites the nucleus were to overlap with this important region of the distorted waves in just the right way. Satchler and collaborators consider this question in a recent article.

However, greater interest attaches to the analysis of the nuclear states in terms of individual nucleons, and to the consideration of $V_f(\xi_f, \vec{r})$ as a sum of two-nucleon interactions. We wish to treat inelastic scattering, and also reactions involving the transfer of one or more nucleons. Therefore a more complete notation is now introduced:

$$v_i(\xi_i) = v_a(1, 2, \ldots, n)\ v_b(n+1, \ldots, A) \tag{7.2a}$$

$$v_f(\xi_f) = v_c(1, 2, \ldots, m)\, v_d(m+1, \ldots, A) \tag{7.2b}$$

$$V_f(\xi_f, \vec{r}) = \{ \sum_{\substack{i=1\to m \\ j=m+1\to A}} V(i,j) - U_f(r) \}. \tag{7.3}$$

The labels $1, \ldots, A$ are taken to refer to coordinates of individual nucleons. We are a little redundant here, because the distance $\vec{r}$ between the centres of mass of a, b or of c, d has been removed already into the distorted waves $\chi^{(\pm)}$. This is not usually an important difficulty. The interaction V_f has been written as the sum of all two-nucleon interactions between c and d, minus the optical interaction from which $\chi_f^{(-)}$ is generated. The optical potential U_f cancels most of the first term of Eq. (7.3). Generally one of the nuclei, c, d, is much lighter than the other, say d is lighter than c, so $(A-m) \ll m$; some simplification is then possible. In the reaction, only a few of the nucleons of c change their orbits. Nucleus c may be regarded as composed of a massive "core", plus a few "active" nucleons. If the core is sufficiently massive then the interaction of d with the core does not change the states of the active particles, but only contributes to the elastic scattering. Except for small recoil corrections which arise in actual cases, the interaction of d with the core is precisely cancelled by part of U_f, and need not be considered at all (see section 8). The remainder of U_f is the small part which is contributed by elastic matrix elements of the interactions of d with the active particles. But by treating the inelastic matrix elements in Born approximation we are by implication treating the elastic matrix elements in Hartree approximation, and make very little error if we give them no further thought. Therefore

$$V_f \approx \sum_{\substack{j>m \\ \text{active particles } i \leqslant m}} V(i,j), \tag{7.4}$$

which is what we would naturally have written anyhow.

The matrix element (7.1) becomes

$$\sum_{\substack{i\leqslant m \\ j>m}} \langle v_c(1, \ldots, m)\, v_d(m+1, \ldots, A) \,|\, V(i,j) \,|\, v_a(1, \ldots, n)\, v_b(n+1, \ldots, A) \rangle. \tag{7.5}$$

The coordinate $\vec{r}_f$, the displacement of the separating fragments, is buried in this matrix element, and must be disentangled and displayed. Also we wish to employ zero-range approximation, and this implies relations among the coordinates in Eq. (7.5). These questions are best left for special cases.

What is even more interesting is to ask how many terms appear in the sum in Eq. (7.5). This is the question having to do with coherence, or collective effects, and also concerns the treatment of special cases. We therefore consider in turn three important special cases.

First, suppose $m = n$, so the reaction is inelastic scattering. For simplicity we may take $A = m + 1$, and thereby consider inelastic scattering of a single nucleon. Then j has the one value $j = m + 1$, and the channel coordinate $\vec{r}$ is the distance from $m + 1$ to the centre of mass of particles $1, \ldots, m$. The matrix element of Eq. (7.5) becomes

$$\sum_i \langle v_c(1, \ldots, m)\, v_d(m+1) \,|\, V(i, \vec{r}) \,|\, v_a(1, \ldots, m) v_b(m+1) \rangle. \tag{7.6}$$

The sum now is the sum over the active particles of nucleus c, in the transition $v_a \rightarrow v_c$. The functions v_b, v_d may be regarded as spin, isotopic spin functions. For definiteness it is also interesting to introduce the multipole expansion of V, of Eq. (3.2), and to carry only one multipole. We then have

$$\left[i^L Y_L^M(\hat{r}) \right]^* \sum_i \langle v_c(1, \ldots, m)\, v_d \,|\, V_{LM}(\vec{r_i}, r) \,|\, v_a(1, \ldots, m) v_b \rangle . \quad (7.7)$$

In Eq. (7.7) the vector properties of $\vec{r}$ have been extracted from the matrix element. The resulting functions V_{LM} are vector functions of $\vec{r_i}$ and must behave under rotations and reflections of coordinates like the spherical harmonics $Y_L^M(\hat{r}_i)$. These V_{LM} are typical one-body operators of nuclear spectroscopy. Each one of these operators can change the orbit of only one nucleon, in coupling v_a to v_c. The number of terms in the sum in Eq. (7.7) is therefore equal to the number of nucleons whose orbits are changed.

Equation (7.7) clearly shows strong coherence properties. Many low-lying states of nuclei are linear superpositions of many alternative excitations of a single nucleon from the ground state. These are the particle-hole excitations. Eq. (7.7) shows that inelastic scattering excites all the terms in the linear combination, and is very sensitive to the appearance of coherence in this linear combination. PINKSTON and SATCHLER [23] stressed this point. They emphasized that the matrix element (7.7) has the same basic structure as electromagnetic matrix elements, and may be compared directly with electromagnetic matrix elements B(EL), B(ML). There are differences. The sum here extends over neutrons and protons, whereas the electromagnetic case emphasizes protons. Also B(ML) are strongly spin-dependent, whereas spin-dependent forces are weaker in inelastic scattering, and in any case are different. Another difference lies in the dependence of V_{LM} on radius. In the electromagnetic case this is proportional to $[Y_L^M(\hat{r}_i) r_i^L]$. In nucleon scattering, if V is taken to have zero range then r_i depends on radius in the same way as does r, and this is as do the important terms of $[\chi_f^{(-)} * \chi_i^{(+)}]$. Often this is as the spherical Bessel function $j_{\ell_0}(kr)$, where ℓ_0 is the cut-off in the strongly absorbing case. Although these two radial dependences differ, they both emphasize the nuclear surface, and the linear combination of terms in the excited state wave functions is not sensitive to details in that region. Inelastic scattering therefore compares closely with electromagnetic excitation. One should look for both similarities and differences.

Figure 18 shows a famous graph from a paper by COHEN and RUBIN [24], for the excitation of the lowest 2^+ states of nuclei in (p, p) reactions. The inelastic cross-section shows a strong correlation with B(E2).

Not only is there this correlation, but the (p, p′) cross-sections also are very large. The mechanism which makes the cross-sections large was studied by ROST [19] for the case Mg^{24} (p, p′), and also by Satchler for a number of cases. Briefly, in collective excitations not only does the projectile interact with the outermost few nucleons, but also it interacts to an important extent with all the nucleons in the next lower-lying closed shell, because that is slightly deformed by the valence particles. One easily gets a factor two enhancement of the transition amplitude in this way. Of course, shell model wave functions or lowest-order Nilsson wave functions miss this effect. In the case of C^{12} (p, p′) it was the omission of this effect which

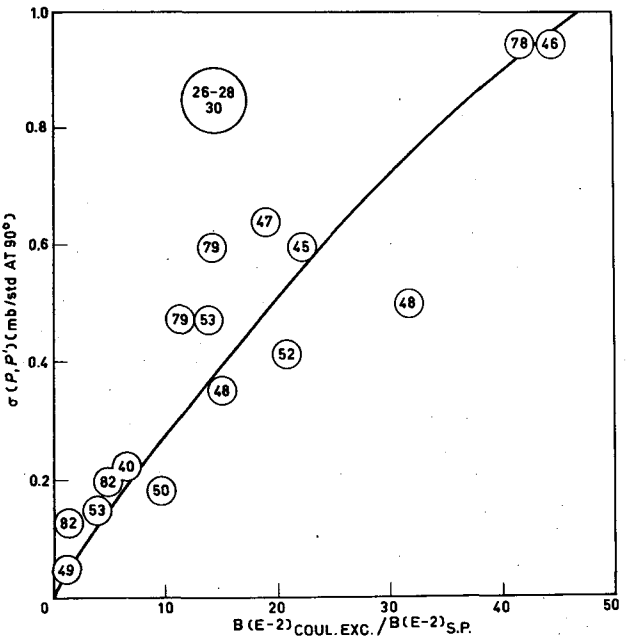

Fig. 18

Correlation between B(E 2) and cross-section for inelastic scattering of protons

caused the failure of the LEVINSON-BANERJEE calculation [25] to fit the absolute cross-section, using two-body forces of normal strength. They treated C^{12} as a core plus two active particles. Properly, all the nucleons of C^{12} participate coherently in the inelastic scattering, and rather good wave functions are needed to discuss this process.

Of course, this new emphasis of DI calculations on the bound state wave functions enhances their interest. Recently, SANDERSON and WALL [26] have used the wave functions computed by Gillet and Sanderson for the level structure of Ca^{40}, to estimate the ratios of cross-sections for (α, α') scattering to several excited states having widely differing properties and cross-sections. Their rough computations resemble experiment. More such work will be done. It must be considered now that one of the goals of nuclear spectroscopy is the calculation of form factors F_L for inelastic scattering.

Collective states of nuclei are not always found at low excitation energies. The 2^+ and 3^- states of T = 0 are at low energy and appear strongly in (p, p') and (α, α') reactions. Such states are not excited in (p, n) reactions, and most of the (p, n) cross-section lies at higher energies. A very interesting part of the (p, n) cross-section was recently found in experiments at Livermore [27], in which the analogues of the ground state and 2^+ excited state of the target nucleus were detected. The excitation energy for these analogue states is the Coulomb energy for adding a proton. Despite the high energy the states are narrow and the cross-sections are large.

The next case of Eq. (7.5) which we consider is that for m = n + 1, corresponding to the transfer of a single nucleon in a stripping process. The

transferred nucleon is the only active particle of nucleus c, and i has the one value i = m = n + 1. Therefore the matrix element reduces to

$$\sum_{j>m} \langle v_c(1, \ldots, n, m) v_d(m+1, \ldots, A) \mid V(\vec{r}, j) \mid v_a(1, \ldots, n) v_b(m, \ldots, A) \rangle. \tag{7.8}$$

Nucleus b is a deuteron, or triton, or alpha particle, say, hence nucleus d is a proton, or deuteron, or triton, respectively. The sum over j therefore runs over a trivial system, and is not interesting. There is no sum at all over particles in c and only that one part of nucleus c in which particle m is bound to nucleus a as a core even enters into the reaction. Coherence effects are completely absent. Single-nucleon stripping is of a type opposite to that of inelastic scattering.

(This discussion has been a little extreme. It is possible that in nucleus d the particle m is in part bound to an excited state of the core. Particles of the core then may be active particles, and the matrix element for stripping may contain interactions between nucleus d and the active core particles. However, this effect must be weak, because it requires both particle m and nucleus d to have simultaneous good overlap with the core. Some effects of this kind were considered by Yoshida in a recent preprint. He notes that states of c which are excited strongly in inelastic scattering may sometimes also be formed with large probability in stripping, by adding a single nucleon to nucleus a.)

As an example, let us evaluate Eq. (7.8) for the case of deuteron stripping, A = m + 1 = n + 2. The matrix element is

$$\left\langle v_c(1, \ldots, n, \vec{r}) \mid V\left(\left|\vec{r} - \vec{r}_{m+1}\right|\right) \mid v_a(1, \ldots, n)\, \phi_D\left(\left|\vec{r} - \vec{r}_{m+1}\right|\right) \right\rangle. \tag{7.9}$$

I have ignored spins, so v_d does not appear, and have replaced v_b by the deuteron internal wave function ϕ_D. In zero range

$$V\phi_D = \{(\hbar^2/M)\nabla^2 - \hbar^2\gamma^2/M\}\, \phi_D \approx -\sqrt{4\pi}(\hbar^2/M)\, N\, \delta(\vec{r} - \vec{r}_{m+1}), \tag{7.10}$$

where M is the mass of a nucleon, N is the normalization constant of the radial wave function of ϕ_D, and $(\hbar^2\gamma^2/M)$ is the binding energy of ϕ_D. (It is well known from the work of Bethe and Longmire that even if a zero-range wave function ϕ_D is used, the normalization N must be carried in effective range approximation, to avoid large errors.) The matrix element (7.9) now reduces to

$$-\sqrt{4\pi}\,(\hbar^2/M)\, N \langle v_c(1, \ldots, n, \vec{r}) \mid v_a(1, \ldots, n) \rangle, \tag{7.11}$$

a simple overlap integral.

From the point of view of the shell model, the above overlap integral is the product of the shell model orbital for the m^{th} nucleon, multiplied by a fractional parentage coefficient. Let us write

$$\langle v_c(1,\ldots,n,\vec{r}) \mid v_a(1,\ldots,n)\rangle = \sum_L \left[Y_L^M(\hat{r}) F_L(r)\right] \langle c: J_c T_c \mid a: J_a T_a \rangle_L . \tag{7.12}$$

Our old friend the "form factor" (see Eq. (3.5)) reappears here as the normalized radial wave function of the transferred particle. These wave functions are readily calculable as eigenfunctions of, say, a potential of Saxon shape. Therefore stripping experiments measure the last factor in Eq. (7.12), the c.f.p. Thereby valuable spectroscopic information is obtained. A discussion of the determination of this factor from experiment is given by FRENCH [28]. A summary of all work of this kind up to 1960 is given in a review by MacFARLANE and FRENCH [29], along with a discussion of procedures for the shell model interpretation of such data.

The above discussion requires one important correction, connected with the identity of particles. We are neglecting the Pauli principle, on the grounds that exchange integrals between bound and continuum particles are small. But because of the identity of particles the same (physical) reaction can take place in many mathematically different ways. Which particle is number m is not prescribed by the physics. This question is discussed in the Appendix of the article by French. He shows that we should make the replacement

$$\langle v_c(1,\ldots,n,\vec{r}) \mid v_a(1,\ldots,n)\rangle \rightarrow [(A-n)\,S(L)]^{\frac{1}{2}}\ [Y_L^M(\hat{r})F(r)], \tag{7.13a}$$

$$S(L) \equiv m \langle c: J_c T_c \mid a: J_a T_a \rangle_L^2 . \tag{7.13b}$$

The quantity S (L) is known as the "relative reduced width" or "spectroscopic factor" for the transition. It is the useful result in studies of the magnitudes of stripping reactions. Often authors multiply S by a so-called "single-particle reduced width". This latter step only refers to calculations with the crude, cut-off Butler theory. It is meaningless in a DWB calculation. However S itself is a purely spectroscopic quantity, and is equally meaningful in all theories.

S = 1 for a "single-particle state", i.e. for the case in which an inequivalent particle is added to a closed-shell core. Larger values of S may appear if equivalent particles are already present. Smaller values of S appear if the single particle state L should be distributed among a group of final states, by coupling of L with the angular moment J_a of the core, or by the introduction of excitations of the core. In this manner S is distributed among the states of a "giant resonance". Because of the absence of coherence effects, the single-particle strength is easily recovered if all the strength in the giant resonance is summed. Extensive work of this kind has been done recently by B. L. Cohen and collaborators.

DWB calculations agree very well with the magnitudes of cross-sections for the stripping of medium energy deuterons by light and medium mass nuclei. The agreement deteriorates if Coulomb effects become very strong, apparently because DWB does not fully take into account polarization (it does in part!). As in inelastic scattering there is some worry about consistently computing bound wave functions and distorted waves in the same potential wells, in order that overlap integrals be meaningful.

Finally, I remark again, as in section 3, that the analysis of experiment for spectroscopic results has until recently been done entirely empirically, using the "plane wave theory" as a rough guide. This has been possible because within a given shell model configuration, within a small range of bombarding energies, the reaction dynamics tends to change slowly with experimental conditions. It therefore factors out if experiments are compared, and the interesting quantity S may be determined.

The remaining special case of Eq. (7.5) which must be treated, is that corresponding to two-nucleon transfer. Coherence effects now appear again, as YOSHIDA recently remarked [30], although not of the same kind as in inelastic scattering. Different special cases of Eq. (7.5) explore many different kinds of nuclear structure effects.

For the present case m = n + 2; therefore the matrix element becomes

$$\sum_{j>n+2} \langle v_c(1,\ldots,n+2) v_d(n+3,\ldots,A) \,|\, V(n+1,j) + V(n+2,j) \,|\, v_a(1,\ldots,n) \times v_b(n+1,\ldots,A) \rangle . \tag{7.14}$$

It is not immediately obvious yet how to display the channel coordinate $\vec{r}$, the displacement of c from d, or of a from b. Zero range for the interaction V does not eliminate all integrations over the internal structure of nucleus b. Let us, therefore, introduce a systematic set of internal coordinates, and treat the special case A = n + 3, for the stripping of two nucleons from a triton, and use methods paralleling those of Eqs. (7.9), et seq., for deuteron stripping. The coordinates are shown in Fig. 19. The triton wave

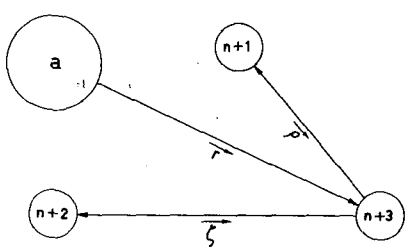

Fig. 19

Coordinate system for a (H^3, p) reaction

function $\phi_T(\vec{\rho},\vec{\zeta})$ is a function of the vectors $\vec{\rho}$ and $\vec{\zeta}$. The coordinate $\vec{r}$ has been introduced somewhat arbitrarily. Zero-range approximation (see Eq. (3.1)) leaves the meaning of $\vec{r}$ slightly vague. For the moment, the definition above is convenient, and no worse than any other. Eq. (7.14) now becomes

$$\langle v_c(1,\ldots n,\ \vec{r}+\vec{\rho},\ \vec{r}+\vec{\zeta}) \,|\, V(\rho) + V(\zeta) \,|\, v_a\, \phi_T(\vec{\rho},\ \vec{\zeta}) \rangle . \tag{7.15}$$

Now if the three potentials which govern the wave function ϕ_T have zero range, then the two which appear in Eq. (7.15) can be substituted out exactly in terms of pseudopotentials, just as was done in Eq. (7.10) for the deuteron

stripping problem. These pseudopotentials are of the form of δ-functions multiplied by coefficients which are obtained by operations on ϕ_T, and are as accurate as ϕ_T itself. Interactions chosen in this way are consistent with the forms used for the wave functions. I will not give here any expressions for the pseudopotentials, but will assume them to be known, and will make the replacement

$$V(r) \rightarrow V_0 \delta(\vec{r}). \tag{7.16}$$

The constant V_0 has the dimensions of (energy x volume). Upon making the replacement (7.16), Eq. (7.15) becomes

$$V_0 \langle v_c(1\ldots n, \vec{r}, \vec{r}+\vec{\zeta}) \mid v_a \phi_T(0, \vec{\zeta}) \rangle + V_0 \langle v_c(1\ldots n, \vec{r}+\vec{\rho}, \vec{r}) \mid v_a \phi_T(\vec{\rho}, 0) \rangle . \tag{7.17}$$

The consequences of the internal structure of ϕ_T are seen in Eq. (7.17). Because of our use of zero-range approximation these effects do not influence the angular distribution (see section 3). Authors often worry at great length about such influences. Unfortunately, in order to do so they use plane waves for the $\chi_i^{(+)}$ and $\chi_f^{(-)}$ and undoubtedly commit worse errors than they cure. I believe that the introduction of Eq. (7.17) into Eq. (3.1), for the nuclear matrix element, gives our best available theory of two-nucleon stripping.

If the factors $\phi_T(0,\vec{\zeta})$ or $\phi_T(\vec{\rho}, 0)$ in Eq. (7.17) are approximately of the form of the deuteron wave function, then we may think of Eq. (7.17) as picking out the deuteron-like part of the relative motion of the two nucleons, n + 1 and n + 2, of nucleus c. The cross-section for the reaction can be large if this type of correlation between these two nucleons is large. This correlation can be studied carefully if the dependence of v_c in Eq. (7.17) on $\vec{\zeta}$ or $\vec{\rho}$, respectively, should be separated by the use of the Talmi transformation, or some such manoeuvre. However, the correlation in question also may be approximated as the correlation for the two nucleons just to be near each other, and the variables $\vec{\rho}$ and $\vec{\zeta}$ may be set equal to zero. This approximation is based on the small range of ϕ_T. It goes beyond the usual zero-range approximation. We then have instead of Eq. (7.17)

$$2 \langle V_0 \phi_T \rangle \langle v_c(1\ldots n, \vec{r}, \vec{r} \mid v_a(1\ldots n) \rangle . \tag{7.18}$$

This is the matrix element studied by Yoshida.

To treat the overlap integral in Eq. (7.18) we expand v_c in eigenstates of the nucleus a, multiplied by shell model orbitals for the two active nucleons. For brevity denote these nucleons as 1 and 2. Then

$$v_c(J_c M_c) = \Sigma v_{a'}(J_{a'} M_{a'}) \{ R_{n_1 \ell_1}(r_1) R_{n_2 \ell_2}(r_2) Y_{\ell_1}^{m_1}(1) Y_{\ell_2}^{m_2}(2) \chi_S^{M_S}(1,2) \}$$

$$\times \langle \ell_1 \ell_2 m_1 m_2 \mid L M_L \rangle \langle L S M_L M_S \mid J M \rangle \langle J_{a'} J M_{a'} M \mid J_c M_c \rangle$$

$$\times A(a' J_{a'}; JLS; n_1 \ell_1, n_2 \ell_2). \tag{7.19}$$

The sum is over all repeated indices. The factor A is the expansion coefficient, essentially a fractional parentage coefficient for two nucleons. All other symbols have obvious meaning.

When (7.19) is used in (7.18) only the one term having $a' = a$, $J_{a'} = J_a$, $M_{a'} = M_a$ is selected, as is usual in stripping. Then Eq. (7.18) becomes

$$2\langle V_0\phi_T\rangle \Sigma\, A(aJ_a;\ JLS;\ 1,2)\langle J_aJM_aM \mid J_cM_c\rangle R_1(r)R_2(r)\mathcal{Y}^{JM}_{LS}(\hat{r}_1 = \hat{r}_2), \tag{7.20}$$

$$\mathcal{Y}^{JM}_{LS}(1,2) = \Sigma Y^{m_1}_{\ell_1}(1)\, Y^{m_2}_{\ell_2}(2)\chi^{M_s}_{S}(1,2)\langle \ell_1\ell_2 m_1 m_2 \mid LM_L\rangle\langle LSM_L M_S\ JM\rangle. \tag{7.21}$$

I will go one step further than usual in these notes, and substitute Eq. (7.20) into the formula for the differential cross-section, Eq. (3.8), in order to show which summation indices lead to coherent sums, and which to incoherent sums. Then

$$d\sigma/d\Omega = (M_i^* M_f^*/(2\pi\hbar^2)^2)(k_f/k_i)(2J_f + 1/2J_i + 1)[2V_0\langle\phi_T\rangle]^2$$

$$\times \Sigma_{JLSM}(2J+1)^{-1}\Big|\Sigma_{1,2} A(aJ_a;JLS;1,2)\int d^3r\, \chi_f^{(-)*} R_1R_2\mathcal{Y}^{JM}_{LS}\chi_i^{(+)}\Big|^2. \tag{7.22}$$

For completeness, it should be understood that the distorted waves are multiplied by their respective spin wave functions, for the incident and emerging projectiles. Yoshida reduces this formula much further. He shows that it may be interpreted in terms of a matrix element for the transition of a single particle from orbital 1 to orbital 2, and gives explicit evaluations for several interesting cases.

Equation (7.22) shows the usual incoherent sum over different multipoles of the channel coordinates as we learned to expect in general in section 3. However, it shows a coherent sum over the orbitals $n_1\ell_1, n_2\ell_2$ of the stripped nucleons. This illustrates in shell model language the sensitivity of two-nucleon stripping to correlations between the two particles, to a sort of "collective" effect in the wave function. The interpretation in terms of individual properties of the two nucleons then is not so trivial as in single-nucleon stripping. For example, suppose there were only one possible way to select orbitals 1 and 2, and that the giant resonance structure for each of these orbitals individually were known. Then would a giant resonance for two-nucleon transfer be the product of the two individual giant resonances? Probably not. Properly correlated states will probably be found mainly at the low-energy end of the product spectrum, because of the nuclear attraction. Indeed, it may be that one or two states in which the correlation is especially excellent will be assembled from among a large collection of pairs of orbitals, and that these states will be found very low in the nuclear spectrum, far away from the centres of gravity of the associated single-particle states. Mottelson has suggested this explanation of data recently obtained by COHEN and MEAD, and by MEAD and MICHELETTI [31].

It is interesting to mention a few selection rules. These are easily seen as properties of the wave functions of the projectile nuclei.

Reaction	Quantum numbers of nuclear transition
(H^3, p)	S = 0, T = 1
(H^3, n)	S = 0, T = 1 or S = 1, T = 0
(He^3, n)	S = 0, T = 1
(He^3, p)	S = 0, T = 1 or S = 1, T = 0
(α, d)	S = 1, T = 0

8. EXCHANGE AND RECOIL EFFECTS

Procedures for doing DWB with finite range forces are related to the topics of this section. Successful calculational methods for this problem are nearing completion, through joint work of the Oak Ridge and University of Pittsburgh groups, and interesting results are emerging. However, I will not discuss these questions.

Particle exchange effects, and effects due to recoil of the massive nuclear core, cannot be treated with the same generality I have been following until now. Details of the internal structure of the target nucleus become important. However, I will consider a special case which is of considerable general interest. In particular, it will enable us to consider the so-called

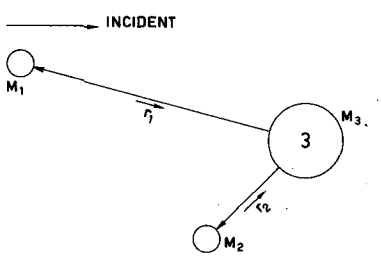

Fig. 20

Coordinate system used in the discussion of exchange and recoil effects

"heavy-particle stripping". The case we treat is illustrated in Fig. 20. Particle 1 is incident on a system composed of particle 2 bound to an inert core.

All three particles interact, and the Hamiltonian shall be

$$H = K + U_1(r_1) + U_2(r_2) + V(r_{12}). \qquad (8.1)$$

The internal structure of particles 1 and 2, which may be composite (particle 1 may be a deuteron), will be disregarded. Therefore certain kinds of processes are not considered; often these are the largest ones.

If particle 1 emerges from the reaction we have the "direct" amplitude, T_{fi} (DIR). If particle 2 emerges we have the "exchange amplitude", T_{fi} (EX).

If particles 1 and 2 are identical then (anti-) symmetrization is accomplished by taking an appropriate linear combination of the two amplitudes. However, the direct and exchange amplitudes can, in any case, be computed independently.

If the coordinates of the three nucleons with respect to an origin fixed in the laboratory are $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$, then we have

$$\vec{R} = (M_1 + M_2 + M_3)^{-1}(M_1\vec{x}_1 + M_2\vec{x}_2 + M_3\vec{x}_3) \qquad (8.2)$$

$$\vec{r}_1 = \vec{x}_1 - \vec{x}_3, \qquad \vec{r}_2 = \vec{x}_2 - \vec{x}_3,$$

where $\vec{R}$ is the coordinate of the centre of mass. Then the kinetic energy operator becomes

$$M_1^{-1}\nabla^2_{x_1} + M_2^{-1}\nabla^2_{x_2} + M_3^{-1}\nabla^2_{x_3} = (M_1 + M_2 + M_3)^{-1}\nabla^2_R + (M_1^{-1} + M_3^{-1})\nabla^2_{r_1}$$

$$+(M_2 + M_3^{-1})\nabla^2_{r_2} + 2M_3^{-1}(\nabla_{r_1}\cdot\nabla_{r_2}).$$

The internal part of the kinetic energy is denoted K, for use in Eq. (8.1),

$$\begin{aligned} K &= K_1 + K_2 + \Delta K, \\ K_1 &= -(\hbar^2/2)(M_1^{-1} + M_3^{-1})\nabla^2_{r_1} = -(\hbar^2/2\mu_1)\nabla^2_{r_1}, \\ K_2 &= -(\hbar^2/2)(M_2^{-1} + M_3^{-1})\nabla^2_{r_2} = -(\hbar^2/2\mu_2)\nabla^2_{r_2}, \\ \Delta K &= -\hbar^2/M_3(\nabla_{r_1}\quad\nabla_{r_2}). \end{aligned} \qquad (8.3)$$

Plane waves in the incident and emerging channels are

$$\begin{aligned} \overset{0}{\phi}_i &= \phi_i(2)\, e^{i\vec{k}_1\cdot\{\vec{r}_1 - (M_2/M_2 + M_3)\vec{r}_2\}} \\ \overset{0}{\phi}_f &= \phi_f(2)\, e^{i\vec{k}_2\cdot\{\vec{r}_1 - (M_2/M_2 + M_3)\vec{r}_2\}} \\ \psi &= \psi_f(1)\, e^{ik_2\cdot\{\vec{r}_2 - (M_1/M_1 + M_3)\vec{r}_1\}}. \end{aligned} \qquad (8.4)$$

where $\phi_i(2)$, $\psi_f(1)$ are wave functions for the bound systems of 2 and 3, or 1 and 3, respectively.

Now we may proceed by the usual formal methods of scattering theory to identify all terms of the transition amplitudes which are first order in either V or M_3^{-1}. Two Green's functions which are useful are:

$$G^{(+)} = (E - K_1 - K_2 - \Delta K - U_1 - U_2 - V + i\epsilon)^{-1}$$

$$G_0^{(+)} = (E - K_1 - K_2 - U_1 - U_2 + i\epsilon)^{-1} . \qquad (8.5)$$

The exact wave function of the system is

$$\underline{\Psi}^{(+)} = \{ 1 + G^{(+)} [U_1 + V] \} \overset{0}{\phi}_i . \qquad (8.6)$$

The two transition amplitudes are

$$T_{fi} (DIR) = \langle \overset{0}{\phi}_f | U_1 + V | \Psi^{(+)} \rangle \qquad (8.7a)$$

$$T_{fi} (EX) = \langle \overset{0}{\psi}_f | U_2 + V | \Psi^{(+)} \rangle . \qquad (8.7b)$$

The only really noteworthy aspect of Eqs (8.6), (8.7) is that in these equations the only appearance of the operator ΔK is in the Green's function $G^{(+)}$. It does not appear explicitly as an operator because the wave functions $\overset{0}{\phi}_i$, $\overset{0}{\phi}_f$, $\overset{0}{\psi}_f$ are eigenfunctions of problems in which ΔK is included. It is in this manner that the boundary conditions force the structure of the calculation.

The operator ΔK does appear explicitly if we iterate Eq. (8.6) once, using the Green's function $G_0^{(+)}$. Then

$$\Psi^{(+)} = \{ 1 + G^{(+)} [\Delta K + V] \} \{ 1 + G_0^{(+)} [U_1 - \Delta K] \} \overset{0}{\phi}_i . \qquad (8.8)$$

If only terms of $\Psi^{(+)}$ up to first order in V and ΔK are retained, then it reduces to

$$\Psi^{(+)} \approx \{ 1 + G_0^{(+)} U_1 \} \overset{0}{\phi}_i + G_0^{(+)} V \{ 1 + G_0^{(+)} U_1 \} \overset{0}{\phi}_i + G_0^{(+)} \Delta K G_0^{(+)} U_1 \overset{0}{\phi}_i . \qquad (8.9)$$

Now we insert (8.9) into T_{fi} (DIR), say, and keep only up to first order terms:

$$T_{fi} (DIR) \approx \langle \overset{0}{\phi}_f | V \{ 1 + G_0^{(+)} U_1 \} | \overset{0}{\phi}_i \rangle + \langle \overset{0}{\phi}_f | U_1 G_0^{(+)} \Delta K G_0^{(+)} U_1 | \overset{0}{\phi}_i \rangle$$

$$+ \langle \overset{0}{\phi}_f | U_1 \{ 1 + G_0^{(+)} U_1 \} + U_1 G_0^{(+)} V \{ 1 + G_0^{(+)} U_1 \} | \overset{0}{\phi}_i \rangle . \qquad (8.10)$$

If part of the operator of the last term is permitted to operate to the left, as usual, then this term combines nicely with the first term, and (8.10) reduces to

$$T_{fi}(\mathrm{DIR}) \approx \langle \overset{0}{\phi}_f | U_1 \{1 + G_0^{(+)} U_1\} | \overset{0}{\phi}_i \rangle + \langle \overset{0}{\phi}_f | U_1 G_0^{(+)} \Delta K G_0^{(+)} U_1 | \overset{0}{\phi}_i \rangle$$

$$+ \langle \{1 + G_0^{(-)} U_1\} | \overset{0}{\phi}_f | V | \{1 + G_0^{(+)} U_1\} \overset{0}{\phi}_i \rangle . \qquad (8.11)$$

The second and third terms of Eq. (8.11) may be simplified by dropping the recoil corrections in the boundary conditions, in $\overset{0}{\phi}_i$ and $\overset{0}{\phi}_f$. We define

$$\chi_i^{(+)}(1)\,\phi_i(2) \equiv \{1 + G_0^{(+)} U_1\}\,\phi_i(2)\, e^{i(\vec{k}_1 \cdot \vec{r}_1)}$$

$$\chi_f^{(-)}(1)\,\phi_f(2) \equiv \{1 + G_0^{(-)} U_1\}\,\phi_f(2)\, e^{i(\vec{k}_2 \cdot \vec{r}_1)}$$

$$\chi_f^{(-)}(2)\,\psi_f(1) \equiv \{1 + G_0^{(-)} U_2\}\,\psi_f(1)\, e^{i(\vec{k}_2 \cdot \vec{r}_2)} . \qquad (8.12)$$

The third of these definitions is given for later use. Then up to terms of first order

$$T_{fi}(\mathrm{DIR}) \approx \{\langle \overset{0}{\phi}_f | U_1 | \phi_i \chi_i^{(+)} \rangle + \langle \phi_f \chi_f^{(-)} | U_1 | \overset{0}{\phi}_i \rangle\}$$

$$+ \langle \phi_f e^{i(\vec{k}_2 \cdot \vec{r}_1)} | U_1 G_0^{(+)} \Delta K G_0^{(+)} U_1 | \phi_i e^{i(\vec{k}_1 \cdot \vec{k}_1)} \rangle + \langle \phi_f \chi_f^{(-)} | V | \phi_i \chi_i^{(+)} \rangle . \qquad (8.13)$$

The third term of Eq. (8.13) expresses the usual inelastic excitation of the initial nucleus, in DWB approximation, by collision of particles 1 and 2. The other two terms are recoil corrections. They express excitation of the target nucleus by collision of the incident projectile with the core. They obviously vanish if the core should be infinitely massive. The first term expresses recoil corrections in the boundary conditions, and the second expresses corrections in the kinetic energy operator. We may note that each of the matrix elements of the first term of Eq. (8.13) factors, to give the form of a matrix element for 1 multiplying a matrix element for 2. There is no need to pursue this here.

By manipulations similar to those above we find the first order parts of the exchange transition amplitude to be

$$T_{fi}(\mathrm{EX}) \approx \{\langle \overset{0}{\psi}_f | U_2 | \phi_i \chi_i^{(+)} \rangle + \langle \psi_f \chi_f^{(-)} | U_1 | \overset{0}{\phi}_i \rangle\}$$

$$+ \langle \psi_f e^{i(\vec{k}_2 \cdot \vec{r}_2)} | U_2 G_0^{(+)} \Delta K G_0^{(+)} U_1 | \phi_i e^{i(\vec{k}_1 \cdot \vec{r}_1)} \rangle + \langle \psi_f \chi_f^{(-)} | V | \chi_i^{(+)} \phi_i \rangle . \qquad (8.14)$$

The third term of Eq. (8.14) expresses the "knock-on" ejection of particle 2 by collision with particle 1. The other two terms are recoil corrections,

as before. Equations (8.13) and (8.14) are the first complete presentations of all these effects, of which I am aware.

The first term of Eq. (8.14) has achieved fame under the name of "heavy-particle stripping". It was first discussed by MADANSKY and OWEN [32] for the case of deuteron stripping. It was supposed to represent the physical idea that a (d, p) reaction might proceed by interaction of the incident deuteron (particle 1 here) with the core of the target nucleus, adhering to the core, and shaking loose a proton (particle 2 here) which was initially bound to the core. The name "heavy-particle stripping" refers to the idea that the incident deuteron has stripped the core from the target nucleus. Supposedly the proton emitted in such a process would emerge preferentially in the backward hemisphere. Because back-angle peaks are often seen in the laboratory the process has been very popular among experimentalists. Numerical evaluation of this term has normally been conducted by substituting plane waves in the matrix elements, and the results obtained in this way usually fit the above heuristic picture.

Unfortunately, all such calculations, nearly the entire literature on the subject, are wrong. It is now well-known [33] that the exact term vanishes if the core is infinitely massive, because of the orthogonality of wave functions belonging to different energies, one wave function bound, and one in the continuum. A projectile interacting with an infinitely massive core can only be scattered elastically. This property is obvious in Eq. (8.14). The use of plane waves destroys this orthogonality. The results so obtained are far less meaningful than even the usual plane wave results, and have no resemblance to the original, exact expressions.

Two further remarks may be made about these questions: One is that it is interesting to evaluate the terms which are exhibited in Eqs. (8.13), (8.14), to see what the recoil effects actually are. I will not pursue this here. The other remark is that correct DWB calculations of the ordinary, strong DI interaction processes (for example, ordinary Butler stripping) often yield large back-angle peaks. This fact is at last rather well known. Of course, it is possible that back-angle peaks found in an actual experiment are associated with exchange or recoil effects, but the mere appearance of such peaks in an experiment is no evidence for this at all.

For a (d, p) process, the physical analogy between heavy-particle stripping and normal Butler stripping is not false. Alteration of the mass ratios does not make the two processes qualitatively different. However, the mass ratios do determine which sorts of mathematical approximations may be used in the two different situations. The approximations used in Butler stripping treat rather accurately the kinetic energy of the (light) particle which is transferred. In the case of heavy particle stripping the kinetic energy of this (heavy) particle is dominated by the recoil terms which we have been discussing, and the approximations generally used have treated these terms very badly. The basic physical theories are alike, as Madansky and Owen suggested.

Between its last interaction with one partner and its first interaction with the other, the particle transferred in a stripping reaction propagates off the energy shell. Unless this were so the interaction at each vertex would be elastic, and no change of state at either vertex would be possible. Free two-body collisions yield only elastic scattering. Now in an interaction with a light particle a heavy body receives both little momentum and little energy, therefore it is not forced very far off the energy shell. These ideas again show that heavy-particle stripping is not very likely.

However, the ideas of the preceding paragraph do indicate another reaction mechanism for heavy-particle stripping, not having to do with the recoil kinetic energy of the core, but rather with its nuclear structure. Off-energy-shell propagation of the core can be assisted by the excitation of internal degrees of freedom of the core, neglected in all the preceding discussion here. In effect, the interactions of the core with its initial and final partners are converted into three-body collisions. I will not attempt to develop this idea.

Finally, we may go back to Eqs (8.13) and (8.14) and pay attention to the simple $V(r_{12})$ terms, rather than to the recoil terms. In the direct and exchange amplitudes these terms, are, respectively,

$$\langle \phi_f(2)\chi_f^{(-)}(1) \mid V(r_{12}) \mid \phi_i(2)\chi_i^{(+)}(1) \rangle ,$$

$$\langle \psi_f(1)\chi_f^{(-)}(2) \mid V(r_{12}) \mid \phi_i(2)\chi_i^{(+)}(1) \rangle .$$

The exchange amplitude is large only if there is simultaneous good overlap between $\chi_i^{(+)}$ and ψ_f, and between $\chi_f^{(-)}$ and ϕ_i. Except at very low energies each of these integrals involves large momentum transfer, and is expected to be small. In the direct amplitude, on the other hand, the product $\chi_f^{(-)*}(1)\chi_i^{(+)}(1)$ involves small momentum transfer, if we are at small scattering angles, and therefore the overlap integral is expected to be large. At large scattering angles the situation is less clear. There is then large momentum transfer in the direct amplitude, and poor overlap, but the exchange amplitude still is of the form of a product of two integrals, in each of which there is poor overlap. These remarks appear to be typical of exchange effects in DI reactions.

9. RELATION BETWEEN DI AND CN

"Unified theories" of nuclear reactions, after various formal differences, all seem to make contact with experiment in terms of some picture of competition for the incident flux, namely, of how the incident flux is divided between the DI and CN modes, preserving unitarity, consistency, and so forth. Some of the difficulties in these questions were mentioned in the Introduction.

Exact formal theories like the Kapur-Peierls or Wigner-Eisenbud theories are based on decomposition of the wave function in terms of a complete set of plausible "compound nucleus" eigenstates. These theories yield for the matrix which couples channels a form in which each matrix element is a sum over some sort of resonance energy denominators, multiplied by reduced widths. Under reasonable [34] assumptions of random signs for the reduced widths the various terms of the sum make statistically independent contributions to the cross-section. The statistical compound nucleus theories of WOLFENSTEIN [35] and of HAUSER and FESHBACH [36], collectively denoted as WHF, are thereby obtained. On the whole, the WHF theory is what we mean when we speak loosely of the "CN mode of reaction".

Randomness of sign of the reduced widths is supposed to be a consequence of dynamical independence of the many CN eigenstates into which a channel wave function is decomposed, these eigenstates involving many unrelated degrees of freedom. However, if a few degrees of freedom play a particularly important part in many nearby CN eigenstates, then the reduced widths for these states are correlated, and calculations based on random signs are not valid. It is at this point that statistical theories fail if DI modes are important.

We eliminate the correlation among the signs of the reduced widths by guessing a physical model for the DI part of the dynamics, and separating off that part of the wave function which exactly follows this model. Suppose the incident and emerging plane waves are Φ_i and Φ_f, in a commonly used notation, and that H_M is the Hamiltonian which governs the DI model. Suppose K is the part of the Hamiltonian of which Φ_i is an eigenstate. Then the wave function may be written in iterated form,

$$\Psi = \{ 1 + (E - H + i\epsilon)^{-1} (H - H_M) \} \phi_i^{(+)}, \tag{9.1a}$$

$$\phi_i^{(+)} = \{ 1 + (E - H_M + i\epsilon)^{-1} (H_M - K) \} \Phi_i , \tag{9.1b}$$

where $\phi_i^{(+)}$ is an eigenstate of H_M. In terms of Ψ the transition amplitude is computed, and the Gell-Mann, Goldberger transformation is introduced,

$$T_{fi} = \langle \Phi_f | H - K | \Psi \rangle = \langle \Phi_f | H_M - K | \phi_i^{(+)} \rangle + \langle \phi_f^{(-)} | (H - H_M) | \Psi \rangle . \tag{9.2}$$

Of course

$$\phi_f^{(-)} = \{ 1 + (E - H_M - i\epsilon)^{-1} (H_M - K) \} \Phi_f . \tag{9.3}$$

Equation (9.2) is essentially all there is to a unified theory. The first term is the transition amplitude predicted by the DI model, which has been treated at length in the remainder of these notes; the second term is everything else. The second term may be decomposed in CN eigenstates, and now if the separation in Eq. (9.2) has been done well, then the reduced widths found in this new decomposition will be statistically independent. When the cross-section is computed the fluctuations of sign of the second term then cause its interference with the first term to vanish, if an energy average is performed. The energy-averaged cross-section is

$$\bar{\sigma} = (M_i^* M_f^* / (2\pi h^2))^2 (k_f / k_i) \sum_{Av} \{ | \langle \Phi_f | H_M - K | \phi_i^{(+)} \rangle |^2$$

$$+ \overline{| \langle \phi_f^{(-)} | H - H_M | \Psi \rangle |^2} \} = \sigma (DI) + \sigma (CN). \tag{9.4}$$

The WHF theory is applied to the second term of Eq. (9.4). SANO et al., for example, give a detailed discussion [37].

The most obvious failing of the unified theory is that it does not provide

a method for choosing the model H_M for any given situation. It is only a structure in which the consequences of some otherwise chosen model can be consistently developed. This structure permits applying the usual approximations of the CN theory to a well-defined part of the problem. If such calculations are successful in fitting experiment, then we know that the model has been chosen correctly, that the correlation among reduced widths has been eliminated, that therefore the CN theory has been applied correctly. Such after-the-fact reasoning is not unusual in physics.

In principle, perfect consistency demands that the first term of Eq. (9.2) be treated by the method of coupled channels. This would mainly be necessary (section 6) at very low energies.

The essential physical fact of the unified theory, which determines its application to experiment, is that Eqs (9.2), (9.4) prescribe the values of the parameters to be used in the WHF theory of the CN part of the cross-section. The physical parameters of the WHF theory are the transmission coefficients T_c for the various channels, here just indicated schematically by the subscript c. A transmission coefficient is

T_c = (probability that ingoing flux in channel c forms a compound nucleus) (9.5)

Evidently if DI reactions were negligible then T_c would be found directly from the reflection coefficients η_ℓ of the optical model as $T_c = 1 - |\eta_\ell|^2$, where ℓ is the angular momentum implied in c. This is at present the normal way of computing T_c, superseding the so-called "barrier penetration coefficients" of Blatt and Weisskopf, although expressing the same physics. In any case, however the T_c are computed, the WHF theory gives a perfectly explicit formula for σ(CN) in terms of the T_c. For that part of σ(CN) corresponding to the transition from channel c with angular momentum ℓ, to channel c' with angular momentum ℓ', this formula is

$$\sigma(c', c) = (2\ell + 1)(\pi/k^2)T_c(E) \sum_J \{ T_{c'}(E')A_J(\ell s | \ell' s' | \Theta) / \sum_{c'', E''} T_{c''}(E'') \}, \quad (9.6)$$

where s, s' are the spins in channels c, c'. The quantity A_J depends on the angular momenta and on scattering angle, and gives the angular distributions. It is a linear combination of Clebsch-Gordan coefficients and spherical harmonics, and may be replaced in terms of Biedenharn-Blatt-Rose Z coefficients. The total angular momentum $J = |\vec{s} + \vec{\ell}|$ is conserved; this fact causes the A_J to be non-trivial, and leads to non-isotropic cross-sections. The factor standing before the sum in Eq. (9.6) is the cross-section for compound nucleus formation, ignoring angular momentum. The remaining factor of Eq. (9.6) expresses competition between the different open channels c'' available for decay of the compound nucleus, and is the fraction of the CN cross-section which goes into channel c'.

Now the unified theory cannot alter Eq. (9.6), which merely follows the CN part of the flux statistically, taking account of angular momentum conservation. However, it does prescribe a more accurate computation of the T_c, according to the definition of Eq. (9.5). We recognize that ingoing flux in channel c can either be reflected, or can go into CN excitation, or can make DI transitions to other channels, avoiding the CN. Therefore

$$T_{c(\ell)} = \{ 1 - |\eta_\ell|^2 - \Sigma_f |T_{fi}^{DI}(\ell)|^2 \}, \qquad (9.7)$$

where $T_{fi}^{DI}(\ell)$ is the DI transition amplitude to final state f, for incident orbital angular momentum ℓ. It is Eq. (9.7), ultimately, which expresses the competition for the incident flux, and unites the CN and DI theories.

In their paper, CHASE, WILETS and EDMONDS [22] discuss the possibility of using Eq.(9.7) to compute T_c. However, they reject this idea as being too laborious, because to get the $T_{c''}$ for all the channels, to use in Eq. (9.6), a complete set of DI calculations would have to be done for every channel. They therefore drop the DI term of Eq. (9.7), reverting to the older calculation of the T_c. I am not aware of any other paper in which a practical application of the unified theory of the T_c was even considered. It is not clear whether this matters. I do not know of any systematic survey of experiment to attempt to assess the relative importance of the two reaction modes.

The CN and DI excitation functions at low energy behave quite differently. Equation (9.6) shows the behaviour of the CN theory. It shows that the cross-section for exciting a given low final state is linear in the transmission coefficients, and therefore tends to increase, at first, as the T_c increase. As the energy becomes much higher, many final state channels c'' open, the denominator in Eq. (9.6) increases rapidly, and the cross-section to a given low final state drops. This pattern is familiar. Similar understanding of the DI excitation function is slightly confused by the failure of DWB approximation at very low energy. However, DWB is adequate to give qualitative understanding. The results are seen, say, in Eqs (3.7), (3.8). The cross-section is seen to be quadratic in the overlap of both the initial and final wave functions $\chi_i^{(+)}$, $\chi_f^{(-)}$ with the nuclear form factor F_L. Because the transmission factors T_c are roughly proportional to the (overlap)^2 of the channel wave functions with the nuclear interior, one may put the DWB result in terms of the T_c, and say that the DWB cross-section to a given final state is roughly quadratic in the transmission coefficients. At low energy, therefore, just above threshold for a given final state, and before many final state channels have begun to open, the CN cross-section increases more rapidly than does DI. As the energy becomes higher the DI cross-section becomes relatively more important, especially above the energy of the "Coulomb barrier". These effects were studied experimentally by TAKETANI and ALFORD[38], who used angular distribution and angular correlation in (p, p'γ) to identify the DI part of their reaction. They found that the DI part first became appreciable above the Coulomb barrier. Fig. 21 shows the ratio of σ(DI) to σ(REAC), as computed by Chase, Wilets and Edmonds, for (n, n') excitation of U^{238}. It is seen here that the DI part of the reaction cross-section is negligible at threshold, but then increases rapidly, and soon becomes a substantial part of the entire reaction cross-section.

Toward higher energy DI cross-sections tend to remain rather constant with energy. Competition as new channels open is not as strong an effect as in the CN process. Eventually cross-sections do start to fall with energy, as momenta become large, and overlap integrals become reduced. However, the overlap integrals tend to be dominated mainly by momentum transfer, and under circumstances in which this quantity is small the cross-sections continue to be large, and conversely. (An interesting example of the last remark is in (d, α) reactions, where, because of the large mass increase and energy release, the momentum transfer in excitations of low-lying levels

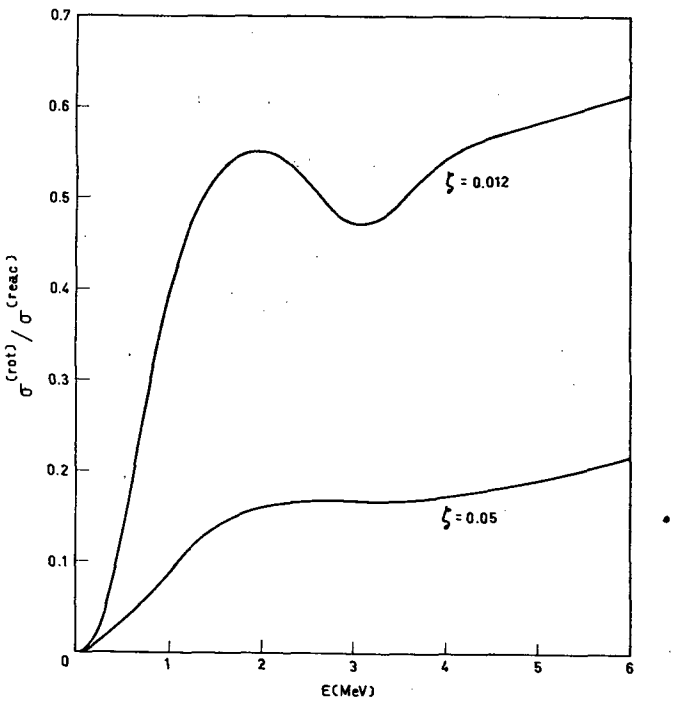

Fig. 21

Fraction of reaction cross-section which is contributed by the DI process, as a function of incident energy, for the reaction U^{238} (n, n'). Results are shown for two different values of the absorbtivity, ξ

is much greater than in excitations of much higher states. Undoubtedly this influences the α-particle spectrum.)

The excitation function in the case of strongly-absorbed projectiles is interesting. From the discussion given earlier we expect that the inelastic cross-section, at all bombarding energies, will be some given constant fraction of the nuclear area. However, this concerns the total cross-section. Because the angular distribution is progressively compressed toward smaller angles as the energy rises, the magnitude of the cross-section at $\Theta = 0^0$, say, rises as k^2. The formulas show this behaviour. It must continue over the entire range of energy in which strong absorption continues.

Finally, let me remark on interference between DI and CN. This effect appears if the density of states in the compound nucleus is not great enough so that the interference between the two terms of Eq. (9.2) can average to zero. In this case Eq. (9.4) is not correct. A review of some of the interference effects which then occur was given by YOSHIDA [39] at the Kingston conference. The effects are complicated, and not much progress has been made with them. Perhaps the only case in which simple analysis may be possible is that of the interference of the DI mode with a single, isolated CN resonance. Figs 22, 23, 24 show an early example of such resonances, found in the reaction C^{12} (d, p), at fairly low energy [40]. For the analysis of such resonances it is perfectly plausible that we need merely add to the

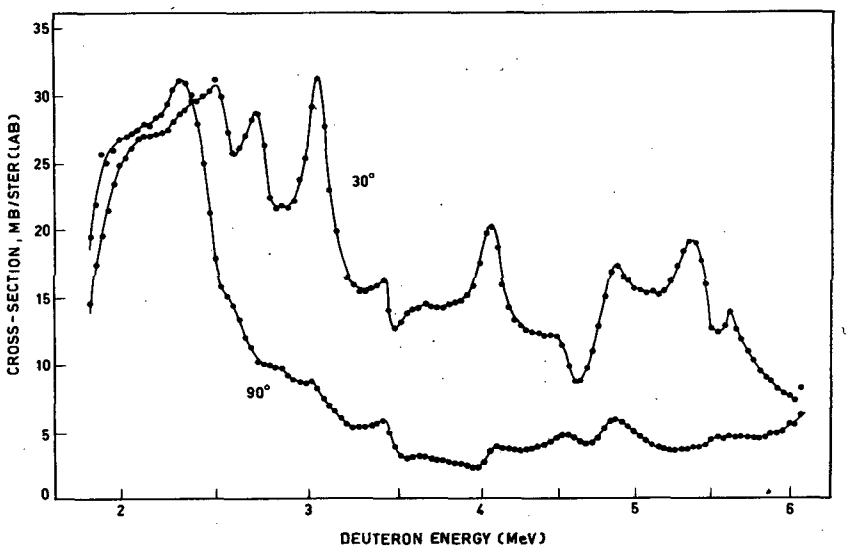

Fig. 22

Two excitation functions for the reaction $C^{12}(d, p)$

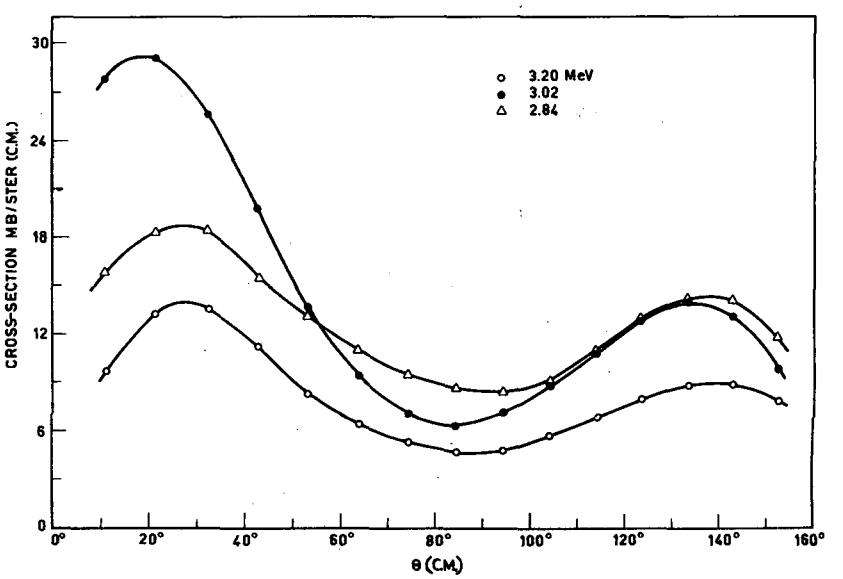

Fig. 23

Some angular distributions near the 3.0 MeV resonance

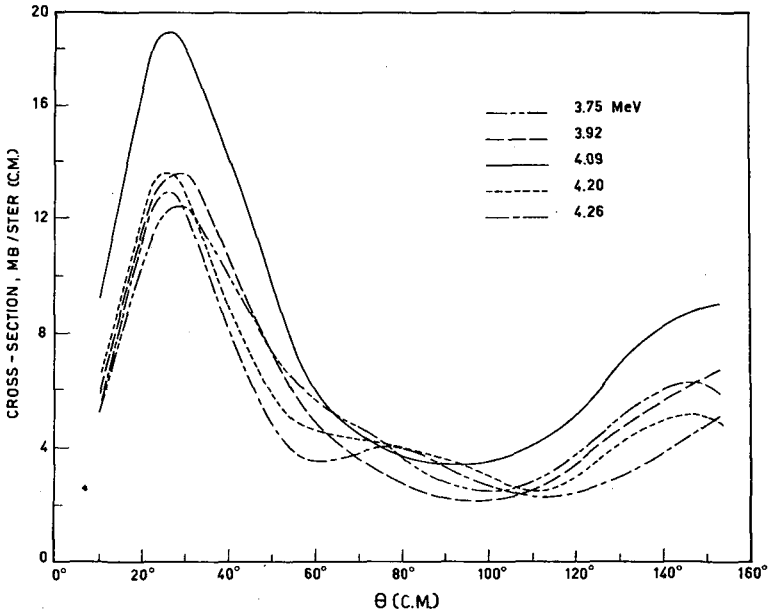

Fig. 24

Some angular distributions near the 4.0 MeV resonance

DI amplitude the Breit-Wigner amplitude for the resonance. The latter amplitude is the product of the Breit-Wigner S-matrix element

$$S_{cc'}^{(BW)} = i\Gamma_{\lambda c}^{\frac{1}{2}}\Gamma_{\lambda c'}^{\frac{1}{2}}/(E_\lambda + \Delta_\lambda - E - i(\Gamma_\lambda/2)), \qquad (9.8)$$

multiplied by the appropriate spherical harmonic and angular momentum coupling coefficients. The only parameters which must be varied in this analysis are the usual resonance widths and position, and aside from these parameters the analysis is unique. The early work by the Rice Institute group, and others, also used an arbitrary adjustable phase parameter. However, they used the DI amplitude predicted by plane-wave Born approximation, and this amplitude is purely real. The correct DWB amplitude has a phase which varies with scattering angle, and with its use the observed effects very likely can be fitted. Some resonance parameters can thereby be obtained. It may be worthwhile to perform this analysis.

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DISPERSION THEORY OF DIRECT NUCLEAR REACTIONS

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1. INTRODUCTION

1.1. Difficulties of the old theory

The main difficulty of nuclear theory is that nuclei contain many (i.e. more than two) but not too many particles. Therefore, the precise equations of motion (Schrödinger equation) become practically useless, and at the same time it is impossible to apply statistical methods with confidence. The latter circumstance is graphically expressed in direct nuclear reactions. The essence of these phenomena consists in that a particle hitting the target nucleus transfers its energy and momentum either to one nuclear nucleon or to a comparatively small group of nucleons. This fact would not by itself be surprising if at the same time we did not observe a directly opposite picture corresponding to the production of a compound nucleus, i.e. the statistical distribution among all degrees of freedom of the energy transferred to the nucleus. In macroscopic physics the co-existence of such processes is impossible since they would contradict the second law of thermodynamics. Such processes occur quite often in nuclear physics because of the inapplicability of the asymptotic laws of the theory of probabilities. Since statistical methods were obviously unsuited for the direct process theory, this led to the conviction that it was necessary to return to the Schrödinger equation for a system of many interacting particles. But the technique of solving such equations is still confined to perturbation theory and therefore it was the latter that was used to describe direct nuclear reactions despite the fact that the interaction between nucleons is strong and the application of perturbation theory to the interaction of free nucleons (to n-p or p-p scattering, for example) leads to results which strongly contradict experimental data. The results of the application of perturbation theory to direct nuclear reactions sometimes agree with experimental data and sometimes contradict them, but in either case they can hardly satisfy the investigator because it seems impossible to give the reasons for the agreement if it is not accidental. In short, the theory behaves like an unpredictable person.

A major success in the application of perturbation theory to direct processes in the Butler theory of deuteron stripping ((d, p), (d, n)) and pick-up ((p, d), (n, d)). The Butler theory [1] satisfactorily predicts the position of the first maximum (by the increase of the angle) in the angular distribution of reaction products as a function of the orbital momentum of the nucleon captured by the nucleus (stripping reaction) or picked up by an incident particle (pick-up reaction). This result permitted the use of stripping and pick-up reactions in nuclear spectroscopy. At the same time this led to the problem of understanding the true meaning of the Butler approximation. This problem was also essential because the Butler theory inadequately describes several other features of the stripping and pick-up reactions (such as the change of angular distribution

with the energy of incident particles, the relation of intensities at the maxima of angular distributions, absolute values of cross-sections and, sometimes, the relative probabilities for the excitation of different states of residual nuclei).

A new method in direct process theory [2] was offered not so long ago (in 1961). The method is based on fairly general properties of the reaction amplitudes and is free from the un-justified assumptions of the former theory, in particular the application of perturbation theory. This method makes it possible to obtain several new results and obtain a uniform description of a great variety of processes (such as direct reactions of the conventional type at low and medium energies, the transfer of nucleons in the bombardment of nuclei by multi-charged ions and the processes of fragmentation at high energies). At the same time the new approach explains, with surprising simplicity, the causes of the formerly enigmatic success of the Butler theory and indicates the limits of its applicability. The method referred to is known as dispersion theory or dispersion method. In the form used for the description of direct processes, the dispersion theory originated and developed in the physics of the strong interactions of elementary particles. The theory has replaced the Hamiltonian formalism of quantum field theory and contributed to a substantial advance in the solution of some problems.

The possibility of applying dispersion theory to the quantitative description of direct processes stems from the very structure of this theory in which any "compound" particle (a nucleus, for example) which actually exists in a free state is treated exactly as an "elementary" particle.

It is significant that certain sequences of dispersion theory are manifested in the properties of direct reactions no doubt more saliently than in the physics of the strong interactions of elementary particles. The "dispersion nature" of direct processes "sticks out" of experimental data so obviously that to grasp the essence of the dispersion approach it is worthwhile enumerating briefly the basic facts of direct nuclear reactions.

1.2. Distinctive feature of direct nuclear reactions

Direct reactions of the type A(x, y)B or A(x, yz)B differ from processes occurring through a compound nucleus by the following peculiarities:

(1) The energy spectrum of outgoing particles is not of the "vaporization" type: the number of particles with energies far exceeding the temperature of the respective compound nucleus is scores and indeed hundreds of times as large as the intensity predicted by the Maxwell distribution at these energies.

(2) The angular distribution of reaction products in the centre-of-mass system of colliding particles possesses a sharp anisotropy "forward-backward" with respect to the direction of motion of the incident particles. Sometimes, at low energies, the angular distribution has a maximum for angles larger than $\pi/2$. However, more often we observe an increasing cross-section in the transitions to small angles, i.e., to small momentum transfers.

(3) The "knocking-out" of complex particles from nuclei, i.e. reactions of the type (x, xd), (x, xt), (x, xα) etc. seem especially enigmatic. Heavier fragments (such as lithium nuclei, carbon nuclei and so on) are knocked out when the energies of the incident particles are high. What is surprising here is the fact that the kinematic picture is rather close to the collision of free particles, or at any rate it differs in no way from nucleon ejection processes,

the (x, xp) or (x, xn) reactions. It is tempting to assume that a kind of "molecular structure" is common to nuclei. It is difficult, however, to accept this hypothesis for a number of general considerations (for example, the energy of separation of such a "molecule" from a nucleus often proves to be larger than the binding energy of the "molecule" itself) and, which is the main thing, practically any particles from a nucleon up to comparatively complex nuclei are knocked out of the same nucleus as a result of direct reactions.

It should be added that direct reactions are rather universal: they are observed in a fairly wide energy range (from several MeV to at least several GeV) practically in all nuclei and with all those particles (from nucleons to multicharged ions) which can be used under modern experimental conditions to bombard the nucleus.

1.3. Analyticity of amplitudes and Feynman graphs

In contrast to the conventional theory of nuclear reactions, the dispersion method is concerned directly with the amplitudes of reactions, and not with wave functions.

The amplitude of the reaction

$$A + x \to B + y \tag{1.1}$$

is defined as a complex quantity M connected with the differential cross-section in the centre-of-mass system by the formula*

$$\frac{d\sigma}{d\Omega} = \frac{m_{xA} m_{yB}}{4\pi^2} \frac{p_y}{p_x} |M|^2 \tag{1.2}$$

Here we have

$$m_{xA} = m_x m_A / (m_x + m_A),\ m_{yB} = m_y m_B / (m_y + m_B) \tag{1.3}$$

where m_i are the particle masses in the reaction, p_i their momenta and $d\Omega$ is a solid angle element. In the case of the reaction (1.1) the amplitude M is a function of two variables, i.e. the square of the momentum transfer q^2 and the total kinetic energy E of the colliding particles in their c.m.s. (centre-mass-system)

$$q^2 = (\vec{p_x} - \vec{p_y})^2, \qquad E = E_x + E_A \tag{1.4}$$

The point of departure of dispersion theory is the proposition that $M(q^2, E)$ is an analytical function of the variables q^2 and E.**

Let us recall that the function f(z) of a complex variable z is called analytic in a certain region of z if throughout the region it is unique and differentiable an infinite number of times. These conditions prove to be so rigid that they lead to many consequences, forming a well-developed mathematical

* Throughout the following $\hbar = 1$ and $c = 1$. The transition to the usual system of units is quite simple: in the final formula the ratios p/m, E/m are replaced by p/mc and E/mc respectively, after which the common factor of the type m^α is complemented, depending on the dimensionality of the quantity calculated, by uniquely determined factors $\hbar^\beta c^\gamma$ (α, β and γ are positive or negative rational numbers).

** In non-relativistic physics this proposition should follow from the Schrödinger equation. Analyticity with respect to E is obvious since the equation itself and the boundary conditions analytically depend on E. Analyticity with respect to q^2 has been proved so far only for potential scattering (see, for example, [3]).

formalism. The latter (theory of analytical functions) along with what is known as the unitarity condition, which will be discussed below, form the basis of the dispersion theory of nuclear processes.

The analyticity of $M(q^2, E)$ means, in particular, that if the function $M(q^2, E)$ is known in a certain finite interval of variables q^2 and E, it can be extended uniquely outside this interval to any values of q^2 and E (including complex values) lying in the analyticity region. This fact directly leads to several important conclusions. First of all, we note that the amplitudes M depend not only on q^2 and E but also on the masses of the particles involved in the reaction. All six values (q^2, E and the masses of the particles A, x, B and y) are relatively invariant since it can readily be seen that q^2 and E can be expressed via the invariants $P_x \cdot P_y$ and $P_A P_x$ where P are the 4-momenta of the particles A, x and y. In the non-relativistic approximation ($E/m \ll 1$) we have, for example, when $Q = 0$

$$E = \frac{P_A \cdot P_x - m_A m_x}{m_A + m_x}, \quad \frac{q^2}{2} = \frac{(P_A P_x - m_A m_x)(m_x - m_y)^2}{(m_A + m_x)^2} - m_x m_y + P_x \cdot P_y \quad (1.5)$$

Since we have, besides,

$$P_i^2 = m_i^2, \quad (1.6)$$

and also (taking into account conservation laws)

$$\begin{aligned} P_A^2 &= P_A \cdot P_B + P_A \cdot P_y - P_A \cdot P_x \quad & P_y^2 &= P_A \cdot P_y + P_x \cdot P_y - P_B \cdot P_y \\ P_x^2 &= P_x \cdot P_B + P_x \cdot P_y - P_A \cdot P_x \quad & P_B^2 &= P_A \cdot P_B + P_x \cdot P_B - P_B \cdot P_y \end{aligned} \quad (1.7)$$

we can say that the amplitude M depends on the six relativistic scalar products. $P_A \cdot P_x$, $P_x \cdot P_y$, $P_A \cdot P_B$ and $P_B \cdot P_x$, $P_A \cdot P_y$, $P_B \cdot P_y$. Since we have assumed that M is an analytical function of two of them ($P_A \cdot P_x$ and $P_x \cdot P_y$) we have to admit that the dependence of the other four is also analytical inasmuch as all of these scalar products are physically sovereign. This means that the amplitude M is also an analytical function of the particle masses. Hence it follows that, due to the uniqueness of the analytical continuation, we have a "unique" amplitude, so to say, of the reaction for particles of any masses. It is important, however, that while changing the masses of the particles involved we should not change their other quantum numbers (number of baryons, electric charge, spin, etc.).

This conclusion makes it possible to study nuclear reactions with Feynman graphs. Suppose the particles have such masses that the decay

$$x \rightarrow a + y \quad (1.8)$$

is possible in reality, while the masses of the particles A, B and a are such that the synthesis

$$A + a \rightarrow B \quad (1.9)$$

is actually possible. The reactions (1.8) and (1.9) taken together and leading to the reaction (1.1) can be represented by the graph of Fig. 1.

The amplitude M_0 corresponding to graph 1* does not obviously vanish for the values of masses satisfying the inequality

* In references to particular graphs the number should be taken to refer to the figure of that number.

$$m_a < m_B - m_A, \quad m_a < m_x - m_y. \tag{1.10}$$

From the uniqueness of the analytical continuation it follows that an analytical function which vanishes in a certain interval of variables (for example, that of their values which do not satisfy the inequality (1.10)) vanishes throughout the analyticity region.

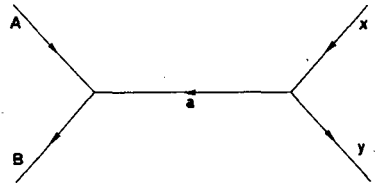

Fig. 1

Hence it follows that since M_o is an analytical function of masses, it must not vanish (except for certain points perhaps) in the region of the values of mass which do not satisfy the inequality (1.10), for example, in the region

$$m_a > m_x - m_y, \qquad m_a > m_B - m_A. \tag{1.11}$$

This means, for example, that, the amplitude of the stripping reaction A(d, p)B must necessarily contain a part which is the analytic continuation of graph 1 into the region of masses satisfying the inequality (1.11). This part of the amplitude may still be conveniently represented by graph 1, the decay (1.8) and synthesis (1.9) being regarded as virtual processes, i.e. processes which could occur in reality if the condition (1.10) were fulfilled.

Obviously, on the basis of the analyticity and conservation laws of the number of nucleons, electric charge, spin and other quantum characteristics of the particles involved we can draw more complicated graphs, for example, of the type of the triangle and rectangle graphs represented in Figs. 2 and 3.

1.4. Amplitude singularities and types of nuclear reactions

Though the number of possible graphs for the given reaction (1.1) is infinite, it seems plausible that direct processes are described by graphs with a small number of virtual particles since a distinctive feature of direct reactions is the transfer of energy and momentum to a few degrees of freedom of the system, i.e. to few nuclear particles. If this is the case, different graphs must contribute to different functional dependences on q^2 and E of the total amplitude because, for example, the dependence of the cross-section of direct processes (simplest graphs) on q^2 sharply differs from that of reactions occurring through a compound nucleus (combination of complicated graphs). It is clear, intuitively, that, say, the amplitudes M_o, M_Δ and $M_\square$ corresponding to graphs 1, 2 and 3 must, in general, be different functions of q^2 and E since different virtual reactions enter into different graphs. This

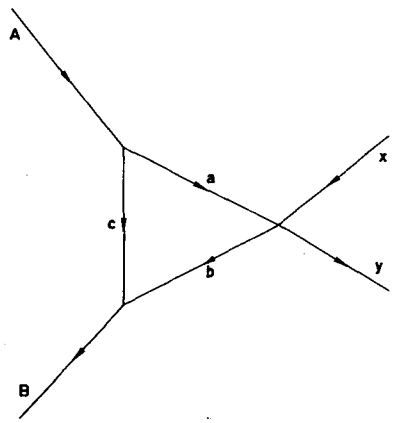

Fig. 2

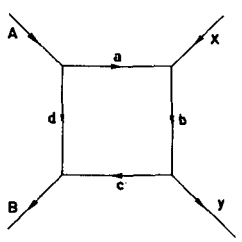

Fig. 3

means that the singularities of amplitudes corresponding to different graphs must, in general, be different (by their position and character) since two analytical functions with identical singularities may differ only by an additive constant and constant factor. An analytical function which has no singularities anywhere (not even at infinity) is a constant. It can be said, therefore, that any analytical function is almost determined by its singularities. We say "almost" because given the position and character of the singularities of an analytical function we can restore its behaviour near a certain singularity accurately to within a constant factor or term.

From all that has been said above it follows in particular that the amplitudes of direct reactions must have definite inherent singularities with respect to variables q^2 and E. In this sense direct reactions are no exception. In point of fact, all known types of nuclear processes are determined by the position and character of the amplitude singularities. Indication of the singularities imparts an accurate meaning to the commonly used but otherwise not very definite concepts of the "type of a reaction" or "mechanism of a reaction".

This point is illustrated by Table I in which the phenomena observed in the known nuclear reactions are listed versus the amplitude singularities, corresponding to them, over the variables q^2 and E.

TABLE I

PHENOMENA OBSERVED IN NUCLEAR REACTIONS AND CORRESPONDING AMPLITUDE SINGULARITIES

Phenomena observed	Amplitude singularities in E	Amplitude singularities in q^2
Energy resonances in cross-sections	Pole near physical region, but outside it for complex $E = E_0 - \frac{i\Gamma}{2}$	Pole $\lvert q^2 \rvert \to \infty$
Near-threshold anomalies ("cusps")	Branch points $E = \pm Q$	
Direct reactions	Branch points	1) Poles (Butler-French) 2) Branch points near physical region, but outside it

According to Table I, the poles and branch points with respect to q^2 lying near the physical region, i.e. at the finite values of q^2, correspond to direct processes. Comparison of direct processes with reactions occurring through the compound nucleus shows that "resonances" (poles), not with respect to the energy E but with respect to the variable q^2, are characteristic of direct processes, the "resonance" values of q^2 lying outside the physical region. For example, the Butler stripping theory corresponds to taking into account the amplitude poles with respect to q^2 situated on the real axis when $q^2 < 0$, i.e. outside the physical region. This situation is indicated in Fig. 4 which also displays the formal cause of the forward-backward angular anisotropy, characteristic of direct reactions; the decrease of cross-sections as q^2 increases is similar in nature to the decrease of total neutron cross-sections with the increase of energy in the case of a resonance level at a negative energy. Using the Butler factor, $1/W^2(qR)$, we eliminated in the curve shown in Fig. 4 the q-dependence of the cross-section due to the finiteness of the size of the nucleus (this factor corresponds in its physical meaning to the separation of the factor $\sqrt{E}$ out of the neutron width).

Concluding this preliminary survey of the main concepts of dispersion theory and direct processes, it is necessary, to avoid possible misunderstanding, to make the following point. The Feynman graphs first appeared in quantum field theory as a sequence of perturbation theory. The expression itself for the amplitude corresponding to a Feynman graph was derived from Lagrangian field theory formalism under the assumption that the interactions are small. Therefore a certain small interaction constant ($\alpha = e^2/\hbar c = 1/137$ in electrodynamics, for example) corresponded to each vertex of the graph. For this reason a graph with many vertices was regarded as a small quantity. Though the Feynman graphs we have discussed have much in common with the perturbation theory graphs, they are by no means identical to the latter in their meaning: the assumption on the smallness of interaction constants

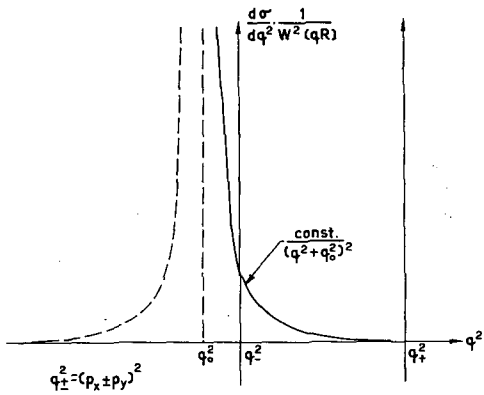

Fig. 4

will never be used when the expression for the amplitude corresponding to some graph or other is derived.

From the view point of the Lagrangian quantum field theory, the graphs used in dispersion theory are a sum of an infinite number of perturbation theory graphs. For example, the pole graph of dispersion theory is a sum of an infinite number of pole graphs of perturbation theory (having other singularities besides a pole) the part having only a pole singularity being isolated from this sum. By its structure this part of the sum exactly coincides with the simplest pole graph of perturbation theory (since it has no singularities besides a pole either) but the constants entering into the "dispersion" graph will be different and are not at all bound to be small. Making use of the terms of quantum field theory, we can say that the "dispersion" pole graphs correspond to the renormalization of interaction constants and the pole particle mass.

The connection described above between the "dispersion" and perturbation theory graphs accounts for the fact that sometimes (but by no means always) the calculation of a direct process amplitude by the formulae of perturbation theory may yield correct results (in the character of functional dependences). We shall see later that this is the case with the Butler deuteron stripping or pick-up theory, but the situation is altogether different in stripping or pick-up reactions for particles with higher binding energies (such as the (α, t), (p, α) reactions, etc.).

1.5. Immediate problems on hand

According to the above, the first problem of direct process dispersion theory is a calculation of the amplitudes corresponding to the simplest Feynman graphs. The problem essentially reduces to finding the singularities of the Feynman graphs. Starting from rather general quantum propositions and the analyticity of the amplitudes, we can not only establish the type of the singularities corresponding to a certain graph, but also find the location of these singularities if the masses of the particles involved (real and virtual) are known. In this respect the dispersion theory of direct processes is richer than the Breit-Wigner theory of resonance reactions which does not predict the position of the resonance level.

As was indicated above, the forward-backward angular anisotropy in the centre-of-mass system of direct reactions is one of the most important features of these reactions. Therefore, it is interesting to determine first of all the position and type of the direct-processes amplitude singularities with respect to the variable q^2. This will be dealt with in chapters 2, 3 and 4. Simultaneously, we shall obtain the formulae for the amplitudes of the simplest Feynman graphs expressing them through the amplitudes of virtual decays (syntheses) and those of virtual reactions (the reaction $a + x \to b + y$ in graph 2 may serve as an example). The problem of the effects connected with the finite size of the nucleus calls for a special study and will be dealt with in chapter 5.

Considerations of singularities with respect to the variable E leads us to take into account the interaction in the initial and final states. An infinite number of graphs possess the same singularities (branch points) in this case. Their summation leads to the Omnes-Muskhelishvili integral equation which permits an accurate solution expressing the effects of the interaction in the initial and final state directly through the amplitudes of scattering of the particles x on the nucleus A and particles y on the nucleus B.

In conclusion we shall consider some reactions of a more complicated type when the number of finite products is more than two (such as the knocking out of particles from nuclei and the capture by nuclei of slow μ^-, π^- and K^- mesons).

2. SINGULARITIES OF FEYNMAN GRAPHS

2.1. Certain mathematical theorems

We shall see in the following that the amplitudes belong to a class of analytical functions $F(z)$ which possess the following properties:

(1) On the real axis ($z = x$), $F(x)$ is real when $x < x_0$;
(2) $F(z)$ have no singularities anywhere in the complex plane of z except the real values $z \geq x_0$;
(3) When $|z| \to \infty$, $F(z)$ decreases more rapidly than $1/|z|$.

The amplitudes do not always satisfy the latter condition. When they do not we can consider instead of the amplitude M the function $\widetilde{M} = M/\varphi(z)$ where $\varphi(z)$ is any function increasing sufficiently rapidly when $|z| \to \infty$ and not interfering with the fulfilment of the first two conditions.

Let us now consider the sequences of the conditions (1) - (3). Using the well-known Cauchy formula we can write

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{F(z')}{z' - z} dz' \qquad (2.1)$$

(Γ is a close contour around the point z). According to condition (2), the contour Γ can be chosen as is shown in Fig. 5. When $R \to \infty$ the integral over the circle vanishes because of the condition (3). We obtain

$$F(z) = \frac{1}{2\pi i} \int_{x_0}^{\infty} \frac{F(x' + i\epsilon) - F(x' - i\epsilon)}{x' - z} dx'. \qquad (2.2)$$

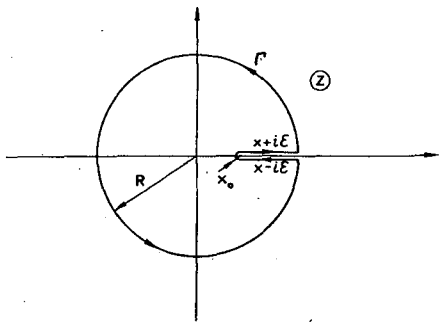

Fig. 5

Since according to condition (1) we have

$$F(x) = F^*(x) \qquad \text{when} \qquad x < x_0 \tag{2.3}$$

we find, using Eq. (2.2), that the function

$$f(x) = \frac{1}{2i} \{ F(x + i0) - F(x - i0) \} \tag{2.4}$$

is real when $x > x_0$. This means that $F(z)$ can be represented as

$$F(z) = \frac{1}{\pi} \int_{x_0}^{\infty} \frac{f(x')}{x' - z} \, dx' \tag{2.5}$$

where

$$f(x) = f^*(x) \qquad \text{when} \qquad x > x_0 \tag{2.6}$$

From Eqs. (2.5) and (2.6) we have

$$F(z^*) = F^*(z) \tag{2.7}$$

Let us show that

$$f(x) = \operatorname{Im} F(x + i0) \qquad \text{when} \qquad x > x_0 . \tag{2.8}$$

Indeed, after substituting the known formula

$$\frac{1}{x' - x \mp i\epsilon} = \frac{\mathcal{P}}{x' - x} \pm i\pi\delta(x' - x) \tag{2.9}$$

(the symbol $\mathcal{P}$ means that in the integration of the first term the main value of the integral is determined) into Eq. (2.5) when $z = x + i\epsilon$ and taking the limit $\epsilon \to +0$, we obtain

$$F(x + i0) = \frac{\mathcal{P}}{\pi} \int_{x_0}^{\infty} \frac{f(x')}{x' - x} \, dx' + if(x) . \tag{2.10}$$

Equation (2.8) is proved since the first term in (2.10) is real. From Eq.(2.7) (or directly with the aid of Eq. (2.9)) we then obtain

$$\mathrm{Im}\, F\,(x - i0) = -f(x) \tag{2.11}$$

Equations (2.8) and (2.11) lead us to the following important conclusion: ImF(z) jumps as we cross the real axis when $x > x_0$. This means that the point x_0 is a branch point of the function F(z) since from this point onward the function becomes ambiguous on the real axis $\mathrm{Im}\, F(x + i0) \neq \mathrm{Im}\, F\,(x - i0)$. In other words, the function F(z) is analytic in the plane z with a cut along the real axis from the point x_0 to ∞. To make F(z) unique on the real axis when $x > x_0$ we must stipulate which value of F(z) on the sides of the cut should be regarded as the value of F(x) when $x > x_0$. Such a condition is known as the choice of the sheet of a function. From the purely mathematical point of view all sheets are of equal importance. Physical considerations, however, make one of the sheets preferable. Note that the sheet is often given by the interval of variation of arg z. For example, if the value of F(z) is taken on the upper side of the cut, then

$$0 \leq \arg z < 2\pi. \tag{2.12}$$

The interval

$$2\pi \leq \arg z < 4\pi \tag{2.13}$$

corresponds to the lower side of the cut.

Concluding out short mathematical digression, let us note that if the function F(z) has poles on the real axis our conclusions practically do not change except for the fact that the pole terms should be added to the right-hand side of Eq. (2.5):

$$F(z) = \sum_i \frac{g_i}{z - x_i} + \frac{1}{\pi} \int_{x_0}^{\infty} \frac{f(x')}{x' - z}\, dx' \;;\; x_i < x_0. \tag{2.14}$$

In physical applications we always deal with the integrals of F(z) over the real axis. If the poles x_i lie in this case within the integration region, the rules of passing the poles should be added to (2.14) to impart a certain meaning to the integrals. We may agree, for example, to determine the integral of the pole term as the main value, or pass the poles, shifting the integration contour into the upper or lower half-plane. The former method (integration in the sense of the main value) is equivalent to the assumption that the phase of F(z) at the pole remains constant as compared with its value in the neighbourhood of the pole on the real axis. We shall see later that this case does not apply in dispersion theory. The displacement of the integration contour into the upper or lower half-plane is equivalent to that of the pole into the lower ($x_i \to x_i - i\epsilon$) or the upper ($x_i \to x_i + i\epsilon$) half-plane respectively.

The above propositions can be illustrated by two examples as follows.

2.2. Two examples

Example 1: n-p scattering amplitude

The n-p scattering amplitude F(E) for low energies can be represented in this well-known form

$$F(E) = \frac{1}{2} F_0(E) + \frac{\sqrt{3}}{2} F_1(E) \tag{2.15}$$

where

$$F_0(E) = i/(\sqrt{mE} - i\alpha_0) \qquad \alpha_0 < 0 \tag{2.16}$$

$$F_1(E) = i/(\sqrt{mE} - i\alpha_1) \qquad \alpha_1 > 0 \tag{2.17}$$

are the scattering amplitudes in the singlet and triplet states. It is clear from Eqs. (2.16) and (2.17) that F(E) has a branch point at E = 0 and is an analytical function in the complex E-plane with a cut from 0 to ∞, F(E) being real on the real axis when E < 0. Since the wave function in the singlet and triplet channel is of the asymptotic form

$$\psi(\vec{r}) = e^{i\vec{k}\vec{r}} + F(E)(e^{ikr}/r), \qquad k^2 = mE. \tag{2.18}$$

and must contain a divergent wave, we have

$$k = \sqrt{mE} > 0 \tag{2.19}$$

for real E > 0. This fixes the choice of the sheet

$$F(E) = F(E + i0). \tag{2.20}$$

Indeed, Eq. (2.19) means that

$$0 \leq \arg E < 2\pi. \tag{2.20a}$$

Hence we have

$$\mathrm{Im}\sqrt{E} > 0 \tag{2.20b}$$

which occurs on the upper side of the cut.

After rewriting Eqs. (2.16) and (2.17) as

$$F_{0,1}(E) = \frac{-\alpha_{0,1} + i\sqrt{mE}}{mE + \alpha_{0,1}^2} \tag{2.21}$$

we note that because of the difference in the signs of α_0 and α_1 the triplet amplitude has a pole with respect to E on the sheet (2.20a) while the singlet amplitude has none (since the residue at the pole vanishes in this case). On the second sheet, on the other hand ($\sqrt{mE} < 0$ when E > 0), it is the singlet amplitude that has a pole.

The rules of passing the poles should now obviously be given so that the singlet pole be passed below ($\alpha_0^2 \to \alpha_0^2 + i\epsilon$) and the triplet above ($\alpha_t^2 \to \alpha_t^2 - i\epsilon$). Indeed, the rules of passing the poles should be given only to attach a meaning to the integrals of E over F(E). If, for example, the sheet F(E + i0) is chosen, then according to the above it is only the triplet pole that is a "caution" and has to be passed while staying on the upper side of the cut. The situation is reversed on the sheet F(E - i0): in this case the singlet pole must be avoided by staying on the lower side of the cut.

Example 2: $F(z) = H_y(z) + i H_x(z)$

Consider a static magnetic field $\vec{H}$ as a function of a point of the plane $z = x + iy$, the component of the field perpendicular to the plane z being assumed to vanish. A field of this kind may be created by currents normal to the plane and situated, for example, as is shown in Fig. 6.

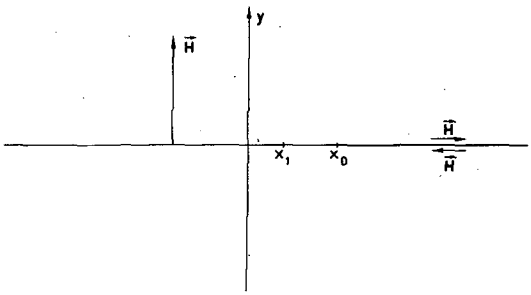

Fig. 6

The point x_1 in Fig. 6 designates the crossing of the plane z with a thin current-carrying conductor perpendicular to this plane, and the boldface line from the point x_0 to infinity represents the crossing of the plane z with a thin metal sheet, also perpendicular to it, with a current flowing in the same direction as in the conductor x_1. Obviously, x_1 is a pole of the function $F(z)$ and x_0 is a branch point of precisely the type we have considered (a jump of the imaginary part), $F(z)$ being real on the real axis ($H_x = 0$) when $x < x_0$ everywhere except for the point x_1. Let us consider $F(z)$ as an analytical function and decide on the choice of the sheet and the rule of passing the poles. On the real axis $F(z)$ is ambiguous when $x > x_0$ and we must agree on the choice of one of the two possible values. From physical considerations it is clear that this condition must be connected with the rule of passing a pole. Indeed, the sign of H_x at the point x_1 is indefinite; it depends on from which half plane we approach this point. Obviously, if we assume that H_x at the pole is directed in the same way as in the upper half plane, the same condition has to be accepted on the cut, i.e. the sheet with current, as well. This conditions the choice of the sheet $F(z)$ if the rule of passing the poles is given. Inversely, the choice of the sheet $F(z)$ conditions, by physical considerations, the rule of passing the poles. Another feature is noteworthy in this example: the use of the analyticity of $F(z)$ and the presence of the boundary condition allow us to find the field $\vec{H}$ if the currents are known. In other words, analyticity and boundary conditions on a cut can be used instead of the equations of electrodynamics. Indeed, using Eq. (2.14) and the boundary condition

$$H_x(x + i0) - H_x(x - i0) = \frac{4\pi}{c} j(x) \qquad (2.22)$$

where $j(x)$ is the linear current density, we immediately obtain

$$F(x) = \frac{2}{c} \frac{J_1}{x - x_1 - i\epsilon} + \frac{2}{c} \int_{x_0}^{\infty} \frac{j(x')\, dx'}{x' - x - i\epsilon} \qquad (2.23)$$

where J_1 is the current density in the conductor x_1.

Equation (2.23) shows very distinctly why the rule of passing a pole is so closely connected with the choice of the sheet: the fact is that an integral over a cut is, by its physical meaning, an integral over poles, as it were we regard a current-carrying metal sheet as a set of linear currents the intensity of which is $dJ(x) = j(x)\,dx$. The situation in dispersion theory is strikingly similar, the unitarity relation playing the role of the boundary condition (2.22).

2.3. Unitarity condition

The unitarity of the S-matrix is the second basic postulate of dispersion theory. In non-relativistic quantum physics this property is a consequence of the hermicity of the Hamiltonian. The S-matrix is connected with the matrix of the amplitudes M by the relation

$$S_{\alpha\beta} = \delta_{\alpha\beta} + i(2\pi)^4 M_{\alpha\beta}\ \delta(P_\alpha - P_\beta) \tag{2.24}$$

where

$$\delta(P_\alpha - P_\beta) = \delta(\vec{p}_\alpha - \vec{p}_\beta)\,\delta(E_\alpha - E_\beta - Q). \tag{2.25}$$

Here the subscripts α and β designate the initial ($\alpha = A + x$, for example) and final ($\beta = B + y$, for example) states of the system, and $\vec{p}_\alpha$, $\vec{p}_\beta$, E_α and E_β the total momenta ($\vec{p} = \vec{p}_X + \vec{p}_A$, for example) and kinetic energies ($E_\alpha = E_X + E_A$) before and after the reaction respectively. The S-matrix is dimensionless and in its physical meaning differs from the matrix M in that it also describes those cases when no interaction occurs (if we "switch off" all interactions the matrix M will vanish and the matrix S will be unity). In the matrix form Eq. (2.24) will be

$$S = 1 + i(2\pi)^4 T \tag{2.26}$$

where 1 is a unit matrix and

$$T_{\alpha\beta} = M_{\alpha\beta}\delta(P_\alpha - P_\beta). \tag{2.27}$$

By definition $|S_{\alpha\beta}|^2$ is the absolute probability for the reaction with the initial state α and final state β. Hence it follows that

$$\sum_\beta |S_{\alpha\beta}|^2 = 1 \tag{2.28}$$

since the sum of the probabilities of all possible reactions for a given initial state is always equal to unity. If we designate

$$S^*_{\alpha\beta} = S^+_{\beta\alpha} \tag{2.29}$$

we shall be able to re-write Eq. (2.28) as

$$\sum_\beta S_{\alpha\beta} S^+_{\beta\alpha} = (SS^+)_{\alpha\alpha} = 1. \tag{2.30}$$

If we generalize Eq. (2.30) and demand that it hold not only for the diagonal elements of the matrix SS^+ but for the entire matrix as well, we shall obtain the total unitarity condition

$$SS^+ = 1. \tag{2.31}$$

Besides Eq. (2.30), from Eq. (2.31) follow the orthogonality relations

$$\sum_{\beta} S_{\alpha\beta} S^{+}_{\beta\gamma} = \delta_{\alpha\gamma} \tag{2.32}$$

Substituting Eq. (2.26) into Eq. (2.31) we obtain

$$2A \equiv -i(T - T^{+}) = (2\pi)^4 T T^{+} \tag{2.33}$$

Going over to the matrix elements in Eq. (2.33) we can readily obtain

$$A_{\alpha\beta} = \frac{(2\pi)^4}{2} \sum_{\gamma} T_{\alpha\gamma} T^{+}_{\gamma\beta}. \tag{2.34}$$

The summation in Eq. (2.34) is performed over all possible "intermediate" states γ. Let us emphasize that the states which differ only in the direction or magnitude of particle momenta are regarded as different states and therefore the sum in Eq. (2.34) includes the integration over the phase volumes of the "intermediate" particles (i.e. particles forming the states "γ"). Let us designate by $A^{(n)}_{\alpha\beta}$ a part of the sum (2.34) containing the states γ in which there are n particles $a_1 \ldots a_n$. Then we have

$$A_{\alpha\beta} = \sum_{n} A^{(n)}_{\alpha\beta} \tag{2.35}$$

$$A^{(n)}_{\alpha\beta} = \frac{1}{2} \frac{V^n}{(2\pi)^{3n-4}} \cdot \sum_{a_1 \ldots a_n} \left\{ T_{\alpha\gamma} T^{+}_{\gamma\beta} \prod_{i=1}^{n} d^3 p_{a_i} \right\}. \tag{2.36}$$

Here V denotes the normalizing volume, and the summation is performed over all possible types of intermediate particles and over spin variables if there are non-zero-spin particles among the intermediate particles. Let us emphasize that since the matrix elements $T_{\alpha\gamma}$ and $T^{+}_{\gamma\beta}$ contain δ-functions (see Eq. (2.4)) of $P_\alpha - P_\gamma$ and $P_\gamma - P_\beta$, Eq. (2.13) contains only such intermediate states as are allowed by conservation laws. In other words, the relation (2.13) contains the amplitudes of real processes. Eqs (2.35) and (2.36) are of that form of the unitarity condition as is used practically in dispersion theory. We shall deal only with single-particle (n = 1) and two-particle (n = 2) intermediate states.

If we designate

$$2B \equiv T + T^{+} \tag{2.37}$$

then according to the definition of the matrix A (Eq. (2.33)) we can write that

$$T = B + iA. \tag{2.38}$$

The matrices B and A, as is clear from Eqs. (2.33) and (2.37), are hermitian

$$B = B^{+}, \qquad A = A^{+}. \tag{2.39}$$

The matrix A may therefore be referred to as the "imaginary part" of the matrix T and the matrix A connected with A by the relation

$$\mathcal{A}_{\alpha\beta} = A_{\alpha\beta}\delta(P_\alpha - P_\beta) \qquad (2.40)$$

as the imaginary part of the matrix of the amplitudes M. Although the non-diagonal matrix elements of A are complex, our designation is justified because in the matrix algebra the hermitian matrices are analogues of real numbers, and all relations between complex numbers are transferred onto the matrices with the substitution of the hermitian instead of complex conjugation. As we saw in section 2.1. an analytical function can be expressed by the integral of its imaginary part. This means that the amplitude $M_{\alpha\beta}$ can be expressed by $A_{\alpha\beta}$. Owing to the unitarity condition (2.36) $A_{\alpha\beta}$ may in turn be expressed by the amplitudes of other processes. This is precisely our aim because by expressing the amplitude of the reaction of interest to us by the amplitudes of "intermediate" (virtual) processes, we thereby represent this amplitude in the form of Feynman graphs, the number of intermediate particles n in the unitarity relation being obviously connected with the number of internal lines of the graph. For example, n = 1 corresponds to graph 1, and n = 2 to graphs 2 and 3. The proposition that the simplest Feynman graphs with singularities with respect to q^2 nearest to the physical region are essential for direct processes will mean that only one-particle and two-particle intermediate states γ are essential in the unitarity condition.

Thus the unitarity condition, despite its apparent triviality is very important in the theory, establishing as it does the connection between the amplitudes of different processes. This allows us in the final analysis to represent the amplitude in the form of Feynman graphs without resort to the summation of the series of perturbation theory which is inapplicable to nuclear interactions.

Let us now proceed to the programme we have mapped out.

2.4. Pole graphs

Let us consider the unitarity conditions under the assumption that the main contribution to $\mathcal{A}_{\alpha\beta}$ comes from the quantity $\mathcal{A}^{(1)}_{\alpha\beta}$ corresponding to taking into account the one-particle intermediate states. Having designated an intermediate particle by a we can write according to Eq. (2.36)

$$\mathcal{A}_{\alpha\beta} = \pi \int T_{\alpha\gamma}\, T^{+}_{\gamma\beta}\, d^3p_a. \qquad (2.41)$$

The summation over a is omitted in Eq. (2.41) for the sake of simplicity (this means that only one type of one-particle intermediate state is regarded as essential for a given reaction). We shall also regard a as a zero-spin particle (taking the spin into account will change nothing in the essence of the matter but will clog the formulae with Clebsch-Gordan coefficients) and the normalizing volume V as equal to 1. Let us now note that

$$d^3p_a \equiv 2m_a\,\delta(2m_a E_a - p_a^2)\, d^4P_a \qquad (2.42)$$

and replace $\mathcal{A}$ and T in Eq. (2.41) by their expressions through A and M (see sect. 2.3). We then obtain, having performed integration over d^4P_a,

$$A_{\alpha\beta} = 2\pi m_a\,\delta(2m_a E_a - p_a^2) M_{\alpha\gamma}\, M^{+}_{\gamma\beta}. \qquad (2.43)$$

Now $\vec{p}_a$ and E_a must be such that the energy and momentum conservation laws in the intermediate transition be fulfilled

$$P_\gamma = P_\alpha \tag{2.44a}$$

or

$$P_\gamma = P_\beta. \tag{2.44b}$$

If any of the equations (2.44) is not fulfilled, $A_{\alpha\beta}$ vanishes because of the δ-functions contained in $T_{\alpha\gamma}$ and $T^+_{\gamma\beta}$.

Let us introduce the notation

$$r = 2m_a E_a - p_a^2 \tag{2.45}$$

and recall that A(r) is the imaginary part of M(r) (see section 2.3). From Eq. (2.43) it follows that $A(r) \neq 0$ only at the point $r = 0$. In particular $A(r)=0$ when $r < 0$. Our purpose is to restore the amplitudes $M_{\alpha\beta}$ by $A_{\alpha\beta}$ with the aid of the theorems presented in section 2.1. We shall therefore define the amplitude so that

$$A_{\alpha\beta}(r) = \operatorname{Im} M_{\alpha\beta}(r + i0). \tag{2.46}$$

Hence it follows that

$$M_{\alpha\beta}(r) = M_{\alpha\beta}(r + i0) \tag{2.46b}$$

since the limit transition must not be defined differently for the real and imaginary part of the analytical function. Applying now Eqs (2.5) and (2.8) when $x_0 = r_0 < 0$ we obtain

$$M_{\alpha\beta}(r) = -2m_a \frac{M_{\alpha\gamma}(0)\, M^+_{\gamma\beta}(0)}{r + i\epsilon}. \tag{2.47}$$

Consider the reaction (1.1). *The following intermediate states are possible in this case:*

$$\gamma_1 = A + y + a_1, \quad \gamma_2 = B + x + a_2, \quad \gamma_3 = x + y + a_3, \quad \gamma_4 = a_4 \tag{2.48}$$

The intermediate processes that correspond to these intermediate states are as follows

$$\alpha \to \gamma_1 = A + x \to A + y + a_1, \qquad \gamma_1 \to \beta = A + a_1 + y \to B + y \tag{2.49}$$

$$\alpha \to \gamma_2 = A + x \to B + a_2 + x, \qquad \gamma_2 \to \beta = B + a_2 + x \to B + y \tag{2.50}$$

$$\alpha \to \gamma_3 = A + x \to x + y + a_3, \qquad \gamma_3 \to \beta = x + y + a_3 \to B + y \tag{2.51}$$

$$\alpha \to \gamma_4 = A + x \to a_4, \qquad \gamma_4 \to \beta = a_4 \to B + y. \tag{2.52}$$

The intermediate states and reactions are obtained by the following rules:

(a) Since the number of nucleons conserves, the appearance of an intermediate particle should be accompanied by the disappearance of an initial particle;

(b) The intermediate particle which has appeared in one intermediate transition must vanish in another transition;

(c) The disappearance of an intermediate particle must be accompanied by the appearance of a final particle.

The transitions (2.49) obviously correspond to the decay of the particle x and the synthesis of the nucleus B, i.e. the processes (1.8) and (1.9). Thus the amplitude $M_{\alpha\beta}$ corresponding to the intermediate states γ_1 is represented by graph 1. The transitions (2.50) designate the decay of the nucleus A and the synthesis of the particle y. The graph corresponding to this case is represented in Fig. 1a.

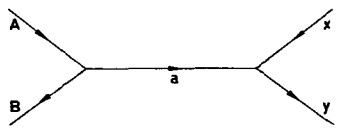

Fig. 1a

The intermediate reactions (2.51) correspond to the decay of the nucleus A and synthesis of the nucleus B. In this case the amplitude is represented by the graph of Fig. 1b.

Fig. 1b

Fig. 1c

Finally, the synthesis and decay of an intermediate particle occur in the transitions (2.52). Graph 1c corresponds to this case. Thus, the amplitude with a pole over the variable r corresponds to graphs with one internal line. These graphs are known therefore as pole graphs. Equation (2.47) is obtained on the basis of the unitarity relation (2.41) according to which $A_{\alpha\beta}$ does not vanish only when conditions (2.44) are fulfilled, i.e. when all intermediate processes may take place for the free particles A, x, B, y and a. This fact is reflected in Eq. (2.47) in that it contains the amplitudes of intermediate processes for the value of $r = 0$. This, according to Eq. (2.45), precisely means that the intermediate particle a is also free and consequently the amplitudes $M_{\alpha\gamma}$ and $M^{+}_{\gamma\beta}$ describe the real decays and syntheses enumerated above. We are interested in the reactions (1.1) between the stable particles A and x for which no decays are really possible. Using the analyticity principle, however, we can obtain from Eq. (2.47) the formula for the amplitudes of the reactions of interest to us by the analytical continuation over the masses, as was indicated in chapter 1. We can see that Eq. (2.47) permits such an analytical continuation since it is analytical over the variable r (containing the particle masses) and since $M_{\alpha\gamma}$ and $M^{+}_{\gamma\beta}$ are also, by the initial assumption (see section 1.3), analytical functions of the particle masses. Thus, Eq. (2.47) applies without change to the reactions with stable particles if we understand by $M_{\alpha\gamma}$ and $M^{+}_{\gamma\beta}$ the amplitudes of decays and syntheses (2.49) - (2.52) analytically continued into the stability region. To obtain the explicit form of $M_{\alpha\beta}$ as functions of q^2 and E we have only to express r through these variables. However, instead of q^2 and E it is convenient to use other variables containing the energies and momenta of the particles A, x, B and y. We introduce these new variables in section 2.5.

2.5. The variables s, t and u

For the following it is convenient to introduce these invariant (with respect to the Galilean transformations) variables

$$s = -(\vec{p}_x + \vec{p}_A)^2 + 2(m_A + m_x)(E_A + E_x) \tag{2.53}$$

$$t = -(\vec{p}_x - \vec{p}_y)^2 + 2(m_x - m_y)(E_x - E_y) \tag{2.54}$$

$$u = -(\vec{p}_A - \vec{p}_y)^2 + 2(m_A - m_y)(E_A - E_y). \tag{2.55}$$

Of these three variables only two are independent since s, t and u are connected by the relation:

$$s + t + u = -2m_B Q \tag{2.56}$$

where

$$Q = m_A + m_x - m_B - m_y. \tag{2.57}$$

The variable s is, obviously, simply expressed through the total kinetic energy E in the centre-of-mass system

$$s = 2(m_A + m_x)E. \tag{2.58}$$

The variable t is linearly connected with q^2 and E

$$t = -q^2 + \frac{2(m_x - m_y)}{m_A + m_x}[(m_y - m_x)E - m_B Q]. \tag{2.59}$$

The variable u is expressed through the sum of momenta $\vec{p}_x + \vec{p}_y = \vec{p}$ in the centre-of-mass system

$$u = -p^2 + \frac{2(m_A - m_y)}{m_A + m_x}[(m_y - m_A)E - m_B Q]. \tag{2.60}$$

Throughout the physical region the variables t and u are negative and the variable s is positive. The physical region boundaries t_m and u_m (on the side of the maximum values of t and u) are given by the formulae

$$t_m = \begin{cases} 0 & \text{if} \quad Q(m_x - m_y) \leq 0 \\ 2Q(m_y - m_x) & \text{if} \quad Q(m_x - m_y) > 0 \end{cases} \tag{2.61}$$

$$u_m = \begin{cases} 0 & \text{if} \quad Q(m_A - m_y) \leq 0 \\ 2Q(m_y - m_A) & \text{if} \quad Q(m_A - m_y) > 0 \end{cases} \tag{2.62}$$

The minimum value of s is given by the relation

$$s_{min} = \begin{cases} 0 & \text{if } Q \geq 0 \\ -2(m_A + m_x)Q & \text{if } Q < 0 \end{cases} \tag{2.63}$$

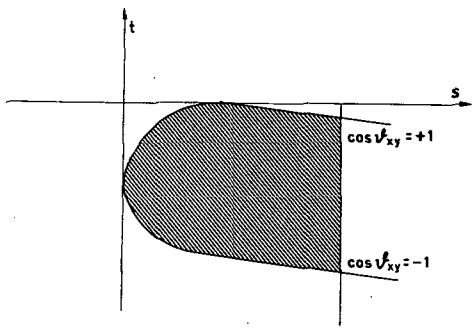

Fig. 7

Fig. 7 represents schematically the physical region of variation of t with a given value of the variable s.

The picture is similar for the variable u with the only difference that the minimum values of $|t|$ are reached when the particle y is discharged in the direction of the particle x (in the centre-of-mass-system) while the values of $|u|$ are lowest when the particles y are discharged against the direction of the particles x.

We shall see below that the variables s, t and u (which are the non-relativistic analogues of the Mandelstam variables) are also convenient in the sense that the amplitude singularities of the reactions (1.1) with respect to these variables are expressed exclusively through the masses of the real and virtual particles involved.

2.6. Amplitude poles with respect to the variables s, t, u

Let us express the variable r for different intermediate states (2.48) through the variables s, t and u. Using the conservation laws (2.44) we obtain

Graph 1: $$r = t - 2m_a \epsilon^x_{ay} \tag{2.64}$$

$$\epsilon^1_{23} = m_2 + m_3 - m_1 \tag{2.65}$$

Graph 1a: $$r = t - 2m_a \epsilon^y_{ax} \tag{2.66}$$

Graph 1b: $$r = u - 2m_a \epsilon^A_{ay} \tag{2.67}$$

Graph 1c: $$r = s + 2m_a \epsilon^a_{Ax}. \tag{2.68}$$

Substituting Eqs (2.64), (2.66), (2.67) and (2.68) into Eq. (2.47) we obtain the amplitudes of the pole graphs 1 to 1c. Thus we see that the amplitudes of graphs 1 and 1a (corresponding to stripping or pick-up reactions) have a pole with respect to the variable t for the "unphysical" value

$$t_0 = 2m_a \epsilon^x_{ay}, \; t_0 = 2m_a \epsilon^y_{ax}. \tag{2.69}$$

Hence it follows that the stripping and pick-up amplitudes will take on the physical region boundary when $t = t_m$ (see Eq. (2.61)). Since the smallest

value of q^2 ($\cos\theta_{xy} = +1$, see Fig. 7) corresponds to t_m, according to the above, this means that the angular distribution must be peaked forward. Note that, according to Eq. (2.69), t_0 is farther removed from the physical region boundary the larger the mass of the virtual particle a and binding energy ϵ at the right vertex. Obviously, for light particles x, t_0 is a minimum for the deuteron stripping and pick-up since in this case we have a minimum mass m_a equal to the nucleon mass and the binding energy is low. For the deuteron stripping and pick-up we have

$$t_0 = 4.47 \text{ MeV} \cdot \text{AMU}. \tag{2.70}$$

The proximity of t_0 to t_m for deuteron reactions precisely accounts for the success of the Butler theory corresponding to the pole approximation (taking into account the effect of the nuclear finite size, which will be discussed in chapter 5). In the case of multicharged ions, the pole approximation may describe the reactions of nucleon transfer, t_0 being sometimes smaller than for the deuteron stripping and pick-up reactions. For example, when $x = Be^9$ and $y = Be^8$

$$t_0 = 3.4 \text{ MeV} \cdot \text{AMU}. \tag{2.71}$$

In general, t_0 for the reactions on multicharged ions with neutron transfer is as a rule of the same order as or less than that for stripping with light particles (except for a deuteron).

Graph 1b, corresponding to the exchange stripping or heavy pick-up, has a pole with respect to the variable u. Its amplitude reaches a maximum at $u = u_m$ since the pole

$$u_0 = 2m_a \epsilon^A_{ay} \tag{2.72}$$

lies in the unphysical region at the value $u > 0$. Hence it follows that the angular distribution will be peaked backward.

From Eq. (2.60) it is clear, however, that the anisotropy of the angular distribution will in this case be manifest only at sufficiently low energies E (the energy E enters into Eq. (2.60) with a large coefficient $(m_y - m_A)$).

Finally, a pole over the variable s corresponds to graph 1c. For direct reactions this pole graph is not of much interest since the angular distribution of the reaction product it gives is isotropic. Besides, the pole with respect to s lies at negative values of this variable

$$s_0 = -2m_a \epsilon^a_{Ax} \tag{2.73}$$

and hence the amplitude will reach a maximum at $s = 0$ or $s = \mp 2(m_A + m_x)Q$, i.e. at low energies. For these reasons it is difficult to distinguish between the contribution from graph 1c and that from the compound nucleus.

The n-p scattering in the triplet state is an example of the physical process at low energies for which graph 1c is essential. In this case the particle a is a deuteron and

$$s_0 = 8.54 \text{ MeV} \cdot \text{AMU}. \tag{2.74}$$

It should be emphasized that graph 1c does not correspond to the production of a compound nucleus.

We have come to the pole graphs by considering one-particle intermediate states. These are not the only possible ones and two-particle states should be considered to show the legitimacy of discarding other intermediate states in the unitarity condition. The amplitudes $M^{(2)}$ corresponding to these states may be small for two reasons:

(a) The amplitudes of virtual processes are small;

(b) The singularities, corresponding to the two-particle intermediate states, lie considerably farther from the physical region boundary than the poles.

There appear no physical causes, generally speaking, which could ensure the fulfilment of condition (a) (this is discussed in more detail in chapter 5). The problems involved in point (b) will be dealt with in chapters 3 and 4.

3. SINGULARITIES OF FEYNMAN GRAPHS (cont'd)

3.1. Two-particle intermediate states

Consider the unitarity condition for two-particle intermediate states. According to Eq. (2.36) and putting V = 1, we obtain

$$\mathcal{A}^{(2)}_{\alpha\beta} = \frac{1}{8\pi^2} \int T_{\alpha\gamma} T^{+}_{\gamma\beta} \, d^3p_b \, d^3p_c \qquad (3.1)$$

where b and c are intermediate particles and the sign of the sum is discarded for the same reasons as in section 2.4. Replacing d^3p_c by Eq. (2.42) and performing the integration over $\vec{p}_c$ we obtain

$$\mathcal{A}^{(2)}_{\alpha\beta} = \frac{m_c}{4\pi^2} \int M_{\alpha\gamma} M^{+}_{\gamma\beta} \, \delta(2m_c E_c - p_c^2) d^3p_b. \qquad (3.2)$$

The integration over d^3p_b can be performed in the simplest way in the centre-of-mass system of the intermediate particles b and c in which $p_c = p_b$. Having performed the integration, we obtain

$$\mathcal{A}^{(2)}_{\alpha\beta} = \frac{m_{bc}}{8\pi^2} \, p \int M_{\alpha\gamma} M^{+}_{\gamma\beta} \, d\Omega_b. \qquad (3.3)$$

In Eq. (3.3) we have

$$p^2 \leq 2 \, m_{bc} w, \qquad w = E_b + E_c \qquad (3.4)$$

and $d\Omega_b$ is a solid-angle element of the particle b in the centre-of-mass system of intermediate particles. Just as in the case of one-particle intermediate states, Eq. (3.4) holds only when the conservation laws (2.44) in the intermediate reactions are fulfilled.

3.2. Graphs for two-particle intermediate states

Among possible two-particle intermediate states for the reaction (1.1) there are states of the type

$$\gamma_1 = b + c \, , \quad \gamma_2 = b + c + y. \qquad (3.5)$$

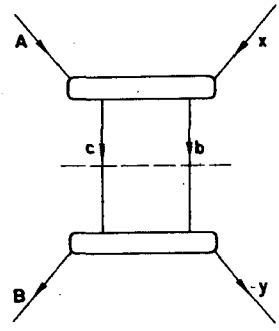

Fig.8

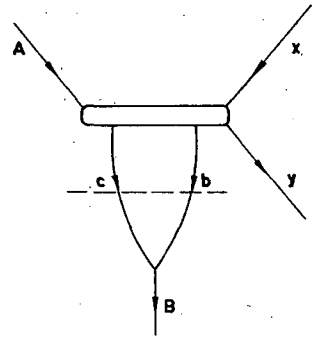

Fig.9

The graphs 8 and 9 correspond respectively to these states.
The intermediate reactions are divided in these graphs by dashed lines. The amplitude of each of these intermediate reactions depends on its kinematic variables. In general, this dependence will be the more complicated, the more complicated are the graphs describing the intermediate reactions. The simplest case will obviously be the one when both amplitudes $M_{\alpha\gamma}$ and $M^{+}_{\gamma\beta}$ are constants. As will be clear from the following, this case, however, is of no interest for direct process theory. Among the simplest graphs graph 2 is the most interesting (see chapter 1). This graph is obtained from graph 9 under the assumption that the first intermediate reaction is described by pole graph 10, and the amplitude $M^{+}_{\gamma\beta}$ of the second intermediate reaction is a constant. Note that graph 10 differs somewhat from the pole graphs considered in chapters 1 and 2. The difference is that there is neither synthesis nor decay of particles at the right vertex, but the intermediate reaction

$$a + x \rightarrow b + y. \qquad (3.6)$$

The general formula (2.47) for the pole graph amplitude holds, of course, for graph 10 as well since the derivation of this formula does not depend on the specific type of the intermediate processes at work at the vertices. The amplitude of the reaction (3.6) is a function of its kinematic variable. We shall assume, however, that it is a constant.

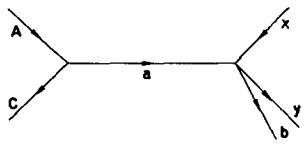

Fig. 10

To obtain the formula corresponding to the triangle in graph 2 we could employ the same procedure as in the derivation of Eq. (2.47), starting from the unitarity condition, Eq. (2.5), and then extending analytically the amplitude with respect to the masses into the stability region. This programme can be fulfilled comparatively simply for graphs of type 8 when, for example, scattering is the first or second intermediate reaction (i.e. c = A, b = x, or c = B, b = y). We shall make use of this in chapter 7 to take into account the interaction in the initial and final states. For graphs 2 the analytical continuation of the unitarity relation (3.3) and integral (2.5) over the masses proves to be more complicated and calls for relativistic treatment even when all kinetic energies of the particles involved are small. The fact is, as shown in chapter 1, that continuation over the masses actually means continuation over relativistic invariants, and such a procedure requires a knowledge of the analytical properties of the amplitudes of the intermediate reactions and the imaginary part of the amplitude precisely with respect to these relativistic variables. The simple case of pole graphs has been considered above and the example of graph 4, when scattering is one of the intermediate reactions, is an exception. In the latter case, it is actually only necessary to perform the analytical continuation over the non-relativistic variable s (i.e. over the energy of the colliding particles), which can be done in the framework of the non-relativistic approximation. The analytical properties of graph 2 on the basis of the unitarity relation were first studied by MANDELSTAM [4] We shall illustrate his method using a simple example in the appendix to chapter 4. Here we shall make use of a simpler, though less rigorous, method first offered by KARPLUS, SOMMERFIELD and WICHMANN [5] and later developed by LANDAU [6]. As related to the non-relativistic graphs of direct nuclear processes this method was treated in detail by BLOKHINTSEV, DOLINSKY and POPOV [7, 8].

3.3. Feynman integral

Equation (2.47), which we have obtained for the pole graph amplitude from the unitarity relation and analyticity principle, formally coincides accurately with the Feynman amplitude corresponding to taking into account the lowest approximation of perturbation theory. In perturbation theory the amplitudes $M_{\alpha\gamma}$ and $M^{+}_{\gamma\beta}$ are real interaction constants and are assumed to be small. In our case the latter is not necessary at all and the meaning of the amplitudes $M_{\alpha\gamma}$, $M^{+}_{\gamma\beta}$ on which we dwelt sufficiently in the previous lectures is quite different. Nevertheless, the formal coincidence of the amplitudes of Feynman perturbation theory with those of dispersion theory, corresponding to taking into account the intermediate states with a definite number of particles n, is not observed for the pole graphs alone but can be used for finding and investigating the analytical properties of more complicated graphs than pole graphs. We have mentioned the fact that the Feynman

amplitudes coincide with the dispersion theory amplitude because, first, the positions of the singularities of the Feynman integrals, which we propose to consider in this section, do not depend on the magnitude of the constants. Secondly, the increase of the order of approximation in perturbation theory involves the increase of the number of virtual particles, since a limited number of particles is produced or vanishes in any "elementary" interaction event. The latter circumstance makes it possible to establish easily the correspondence between the Feynman integral and the amplitude $M_{\alpha\gamma}^{(n)}$ corresponding to taking into account n-particle intermediate states in the unitarity relation.

The Feynman amplitude can be written in the well-known form of the following integral

$$M_{\alpha\beta} = G \int \frac{d^3p_1\, dE_1 \ldots . d^3p_\ell\, dE_\ell}{(p_1^2 - 2m_1E_1 - i\epsilon)\ldots(p_n^2 - 2m_nE_n - i\epsilon)}. \qquad (3.7)$$

In Eq. (3.7) G is a constant expressed through the products of the amplitudes of the virtual reactions at the vertices of the graphs, and P_i, E_i, and m_i (i = 1, 2, ... n) are the momenta, kinetic energy and masses of the virtual particles. The number n equals that of internal lines of a graph, and ℓ is, the number of independent 4-momenta of virtual particles, i.e. such 4-momenta as are not fixed by conservation laws for the given 4-momenta of the outer particles A, x, B and y. The conservation laws hold, just as in a pole graph, at each vertex. If we designate the number of vertices in a graph by v, then v, n and ℓ will be connected by the relation

$$\ell = n - v + 1$$

(v-ℓ bonds are imposed on n independent quantities: altogether we have v conservation laws, but one of them ensures the conservation of momentum and energy throughout the reaction and imposes no restrictions on the variables of the virtual particles). For a pole graph we have n = 1, v = 2 and ℓ = 0. For a triangle graph 2 n = 3, v = 3 and ℓ = 1. For a rectangle graph 3 n = 4, v = 4 and ℓ = 1. In general, according to Eq. (3.8), ℓ = 1 for graphs with a number of lines equal to that of vertices. Such graphs will be referred to as one-contour graphs. The one-contour graph amplitude is thus given by the integral ($n \geq 2$)

$$M_{\alpha\beta} = G \int \frac{d^3p_1\, dE_1}{(p_1^2 - 2m_1E_1 - i\epsilon)\ldots(p_n^2 - 2m_nE_n - i\epsilon)}. \qquad (3.9)$$

It is precisely the one-contour graphs (apart from the pole graphs considered above) that are of practical interest for direct nuclear reactions. In this chapter, therefore, we shall only deal with one-contour graphs. Note that the integral (3.9) converges at any value $n > 2$. When n = 2 the integral diverges. This divergence is only due to the assumption that the amplitudes of the virtual processes (entering into the constant G) are constant quantities, and thus does not involve any cardinal difficulties of the theory. The momenta and energies of the virtual particles 2, 3 ... are expressed through $\vec{p}_1$, E_1 and the momenta and energies of the outer particles.

Since the integral (3.9) contains only scalars (p^2 - 2mE and d^3pdE) invariant with respect to the transformations of the reference system, this

integral is an invariant quantity and consequently can depend only on the invariant kinematic variables s, t and u introduced in the preceding chapter. Thus the integral (3.9) is an analytical function of the variables s, t and u. Let us now turn to the singularities of the amplitude (3.9). For this purpose let us use the identity

$$\frac{1}{r_1 r_2 \ldots r_n} = (n-1)! \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_n \frac{\delta(\alpha_1 + \ldots + \alpha_n - 1)}{(\alpha_1 r_1 + \ldots + \alpha_n r_n)^n} \tag{3.10}$$

and re-write Eq. (3.9) as

$$M_{\alpha\beta} = G(n-1)! \int \delta(\sum_i \alpha_i - 1)\, d\alpha_1 \ldots d\alpha_n \int \frac{d^3 p_1\, dE_1}{(p_1^2 + \vec{A}\vec{p}_1 + BE_1 + C - i\epsilon)^n} \tag{3.11}$$

where $\vec{A}$, B and C are the linear forms of α_i and functions (quadratic and linear) of the momenta and energies of the outer particles as well as the masses of the virtual particles. In particular,

$$B = 2 \sum \lambda_i \alpha_i m_i \tag{3.12}$$

where

$$\lambda_i = \pm 1 \tag{3.13}$$

depending on whether the i-th line is directed along or against the direction (given by the direction of the line i = 1) of passing the contour of a graph. For example, for graph 2 $\lambda_a = 1$, $\lambda_b = 1$ and $\lambda_c = -1$ if we put a = 1. The vector $\vec{A}$ is a linear combination of the momenta of outer particles with the coefficients 0, $\pm 2\alpha_i$. Thus, for graph 2 we have

$$\vec{A} = 2\alpha_2 \vec{q} - \alpha_3 \vec{p}_A \tag{3.14}$$

where

$$\vec{q} = \vec{p}_x - \vec{p}_y.$$

Finally, the constant C is a linear combination of the squares of outer momenta and energies. Thus, for graph 2 we have

$$C = \alpha_2 q^2 + \alpha_3 p^2 - 2\alpha_2 m_b [Q' + E_x - E_y] + 2m_c \epsilon_{ac}^{A} \tag{3.15}$$

$$Q' = m_a + m_x - m_b - m_y.$$

Now we can perform the integration over $d^3 p_1$ and E_1, thus obtaining an integral explicitly depending on outer invariants as on parameters. The integration over E_1 can easily be performed with the aid of the formula

$$\int_{-\infty}^{+\infty} \frac{dE}{(BE + R - i\epsilon)^\nu} = \frac{2\pi i}{(\nu - 1) R^{\nu-1}} \delta(B). \tag{3.16}$$

Having then performed the integration over $\vec{p}_1$ we obtain the following final formula

$$M_{\alpha\beta} = \frac{i\pi^3 (2n-7)!!}{2^{n-3}} G \int \frac{\delta(\Sigma_i \alpha_i - 1)\,\delta(\Sigma_i \alpha_i \lambda_i m_i)\, d\alpha_1 \ldots d\alpha_n}{\{-\frac{1}{2}\sum_{ij} \alpha_i \alpha_j (r_{ij} + 2m_{ij} Q_{ij}) - i\epsilon\}^{n-5/2}}. \quad (3.17)$$

The invariant quantities r_{ij} are constructed in the following way. Let us contract to one-point all internal lines connecting the lines i, j (see Fig. 11).

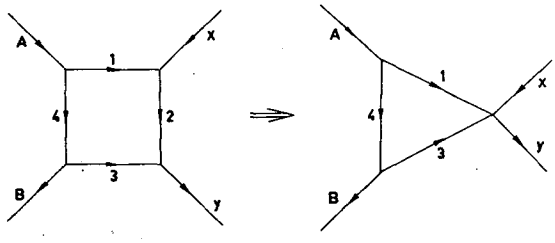

Fig. 11

Let us designate the algebraic sums of the outer momenta kinetic energies and masses adjacent to our generalized vertex by $\vec{p}_{ij}$, E_{ij} and m_{ij} respectively, $\vec{p}$, E and m being taken with + for the lines entering the vertex and with - otherwise. Then we shall have

$$r_{ij} = -p_{ij}^2 + 2m_{ij} E_{ij}. \quad (3.18)$$

The quantity Q_{ij} entering Eq. (3.17) is the energy yield at such a vertex (i.e. Q_{ij} equals the sum of the masses colliding at the vertex minus the sum of the masses going out of the vertex, the outer mass m_{ij} being always assumed entering the vertex).

Clearly, in the case of the reaction (1.1) ($n \leq 4$ for a one-contour graph) the variables r_{ij} can be expressed linearly through the variables s, t and u. It is much simpler to obtain the connection between r_{ij} and s, t and u in each specific case than write it out in the "general form" using cumbersome notations.

Taking into account all that has been said above, we can represent schematically the amplitude $M_{\alpha\beta}$ as follows

$$M_{\alpha\beta} = G' \int \frac{\delta(\Sigma\alpha - 1)\,\delta(\Sigma\lambda\alpha m) d\alpha_1 \ldots d\alpha_n}{(F_s s + F_t t + F_u u + R - i\epsilon)^{n-5/2}} \quad (3.19)$$

where F and R are the quadratic functions of α_i and linear functions of the masses of the (real and virtual) particles involved. Our problem is now to find the singularities of $M_{\alpha\beta}$ over the variables s, t or u, the singularities with respect to the variables t or u nearest to the physical region being of primary interest to us. These singularities are responsible for the angular anisotropy characteristic of direct reactions. They could be found without calculating the integral (3.17) [7].

In the case of the reaction (1.1) $n \leq 4$ of interest to us, the integral (3.17) is simply calculated and the singularities of $M_{\alpha\beta}$ can be found directly from the explicit form of the analytical function. This is precisely the procedure already adopted (chapter 4). However, it is worthwhile to indicate

first some general properties of the Feynman amplitude singularities and methods of investigating them.

3. 4. Certain properties of Feynman integral singularities

The basic idea of the method of finding the singularities without directly calculating the integral (3. 17) is as follows. If we deal with an analytical function F(z) given through the parametric dependence of a certain integral

$$F(z) = \int_a^b f(\alpha, z)\, d\alpha \tag{3.20}$$

it is clear that a singularity z_0 of the function F(z) means the divergence of the integral (3. 20) when $z = z_0$. This divergence may take place because when $z = z_0$, $f(\alpha, z_0)$ goes to infinity either at one of the integration limits or inside the interval (a, b). The latter case requires a special study since the divergence of $f(\alpha, z)$ inside the integration interval is a necessary but insufficient condition for the existence of a singularity.

If

$$f(\alpha, z) = \frac{1}{\varphi(\alpha, z)} \tag{3.21}$$

then for a singularity to exist it is necessary that $\varphi(\alpha, z)$ when $a < z < b$ have at least a second-order zero such that when passing it by changing the integration contour, this multiple zero becomes two zeroes lying on both sides of the integration contour (see Fig. 12).

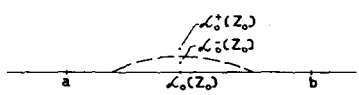

Fig. 12

The integral

$$F(z) = \int_a^b \frac{\alpha\, d\alpha}{(\alpha^2 - c^2 - i\epsilon)\left[(\alpha - \sqrt{z})^2 - d^2 - i\epsilon\right]} \tag{3.21a}$$

may serve as an example in such a case.

These requirements, and taking into account the specific form of the integral (3. 17), lead to the rules of finding the Feynman integral singularities. It can be shown [7] that the singularities of the amplitude (3. 17) are given by the following equation

$$\sum_{ij} (-1)^{i+j} D_{ij} = 0 \tag{3.22}$$

where D_{ij} is the minor of the element (i, j) of the determinant composed of the quantities

$$\eta_{ij} = -\frac{\lambda_i \lambda_j}{m_i m_j}(r_{ij} + 2m_{ij}\ Q_{ij}). \tag{3.23}$$

Equation (3.22) can, according to LANDAU [6], also be obtained from the condition of the compatibility of the requirements

$$\sum_{i=1}^{n} \alpha_i \lambda_i p_i = 0 \tag{3.24a}$$

$$r_i = 0 \tag{3.24b}$$

and the conservation laws at each vertex of a graph. Nor are these equations valid for one-contour graphs alone (in the general case the requirement (3.24a) must hold for each contour of a multicontour graph). The requirement (3.24) means that singularities occur at such values of s, t and u at which virtual particles are free. We have seen that in the particular case of a graph with one internal line the condition (3.24b) means the existence of a pole.

Besides the singularities given by Eq. (3.22) there are singularities corresponding to the case when the denominator in Eq. (3.17) for certain values of kinematic variables vanishes for all values of the variables α. These singularities are called trivial. In particular, the energy thresholds of a reaction are such singularities.

Finally, the following is also important. Let us write the integral (3.17) as

$$M_{\alpha\beta} = \int_0^1 d\alpha_k\, F'(\alpha_k, s, t, u) \tag{3.25}$$

where

$$F' = G' \int \frac{\delta(\Sigma\alpha - 1)\ \delta(\Sigma\alpha\lambda m)\ d\alpha_1 \ldots d\alpha_{k-1} d\alpha_{k+1} \ldots d\alpha_n}{(F_s s + F_t t + F_u u + R - i\epsilon)^{n-5/2}}. \tag{3.26}$$

Let us consider $F'(\alpha_k = 0) \equiv F_0(s, t, u)$. Then it is obvious that the singularities of $F_0(s, t, u)$ coincide with those of a graph having one internal line less, i.e. a graph derived from the initial one by the "contraction" of the k-th line. It then follows from Eq. (3.25) that this singularity of the "contracted" graph is also a singularity of the amplitude $M_{\alpha\beta}$ since the integrand diverges at the lower integration limit ($\alpha_k = 0$). Hence we reach the following important conclusion: the graph in question has also the singularities of all "contracted" graphs.

Let us now proceed to the singularities of the triangle graph amplitude.

4. TRIANGLE GRAPH

4.1. Triangle graph amplitude

In this chapter we shall consider graph 2, putting a = 1, b = 2 and c = 3. It can readily be seen that in this case the denominator of the integrand depends on the variable t only and consequently the entire amplitude $M_{\alpha\beta}$ is a function of one variable t. Indeed, we have for graph 2

$$r_{ac} = r_A = 0\,, \quad r_{bc} = r_B = 0 \tag{4.1}$$

since the particles A and B are free. The only non-zero invariant of r_{ij} is r_{ab}, and according to Eq. (3.18)

$$r_{ab} = t. \tag{4.2}$$

It will also be noted that for graph 2 we have

$$-\lambda_c = \lambda_a = \lambda_b = 1 \tag{4.3}$$

$$Q_{ab} = Q'\,, \quad Q_{bc} = -\epsilon^B_{bc}, \quad Q_{ab} = -\epsilon^A_{ac} \tag{4.4}$$

$$Q' = m_a + m_x - m_b - m_y. \tag{4.4a}$$

The integration in Eq. (3.17) over two of the three variables α (α_b and α_c, for example) eliminates the δ-functions after which there remains a single integral of the type

$$M_{\alpha\beta} = G' \int d\alpha_a \,(A\alpha_a^2 + B\alpha_a + C - i\epsilon)^{-\frac{1}{2}} \tag{4.5}$$

which can be calculated without difficulties. As a result, we obtain the following formula for the amplitude M_Δ of triangle graph 2

$$M_\Delta(t) = CF_\Delta(z) \tag{4.5a}$$

$$C = (ig/8\pi)\,(m_b m_c/\sqrt{m_A m_B})\,(M'/\sqrt{t_\Delta - t_1}) \tag{4.6}$$

$$g = -2M_A M_B^+ m_a \tag{4.6a}$$

where M_A, M_B and M' are the amplitudes of the virtual decay of the nucleus A, synthesis of the nucleus B and the reaction $a + x \to b + y$ respectively,

$$z = 1 + (t - t_\Delta/t_\Delta - t_1) \tag{4.7}$$

$$t_\Delta = 2m_a m_b\,[\,(\epsilon^B_{bc}/m_{bc})^{\frac{1}{2}} + (\epsilon^A_{ac}/m_{ac})^{\frac{1}{2}}]^2 - 2\,(m_x - m_y)Q' \tag{4.8}$$

$$t_1 = -2(m_x - m_y)Q \tag{4.9}$$

$$F_\Delta(z) = z^{-\frac{1}{2}} \ln(1 + z^{\frac{1}{2}}/1 - z^{\frac{1}{2}}). \tag{4.10}$$

Note that all amplitudes of virtual reactions are regarded here as constant quantities. It is clear from Eq. (4.10) that the function $F_\Delta(z)$ has a singularity at $z = 1$. According to Eq. (4.7) this means that t_Δ is a singularity with respect to the variable t of the amplitude $M_\Delta(t)$ of triangle graph 2: $t = t_\Delta$ the amplitude M_Δ goes to infinity. From Eq. (4.8) it follows that

$$t_\Delta > 0 \tag{4.11}$$

i.e. lies always in the unphysical region of the variable t. Equation (4.8)

also shows that t_Δ is the larger the greater the mass of virtual particles and binding energies at the vertices of a graph.

Note that Eq. (4. 8) for a triangle singularity can also be obtained from the general equation (3. 22) for triangle graph singularities, this graph being in this case a quadratic equation for the quantity η_{ab} connected with t by the relation

$$t = -2[m_a m_b \eta_{ab} + (m_x - m_y)Q'\,]. \qquad (4.12)$$

The quantities η_{ac} and η_{bc} are

$$\eta_{ac} = -\epsilon^A_{ac}/m_{ac}\,, \qquad \eta_{bc} = -\epsilon^B_{bc}/m_{bc}. \qquad (4.13)$$

The equation for η_{ab} and its solution are of the form

$$\eta^2_{ac} + \eta^2_{bc} + \eta^2_{ab} - 2\eta_{ac}\,\eta_{bc} - 2\eta_{ac}\,\eta_{ab} - 2\eta_{bc}\,\eta_{ab} = 0 \qquad (4.14)$$

$$\eta_{ab} = \eta_{ac} + \eta_{bc} \pm 2\sqrt{\eta_{ac}\,\eta_{bc}}. \qquad (4.15)$$

The negative sign of the root in Eq. (4. 15) corresponds to Eq. (4. 8). The point $z = 1$, and consequently $t = t_\Delta$, is a branch point. The rule of passing the poles, leading to the imaginary additional term $i\epsilon$ ($\epsilon > 0$) uniquely indicates the way the sheet of the function $F(z)$ should be chosen. Since α_a and α_b are positive, the denominator of Eq. (3. 17) can be re-written in such a form that the imaginary additional term would relate directly to the quantities r_{jk}

$$r_{jk} \to r_{jk} + i\epsilon\,. \qquad (4.16)$$

According to Eq. (4. 2), this means that

$$t \to t + i\epsilon. \qquad (4.17)$$

Hence it follows that we must take the values $M_\Delta(t)$ on the upper side of the cut, i. e. $M_\Delta(t + i0)$. The latter should be directed from the point t_Δ to $+\infty$ since all other possible branch points lie in the unphysical region when $t > 0$. The planes of the complex variables t and z are represented in Figs. 13 - 14. Cross-hatching shows the physical region. Note that the point $z = 0$

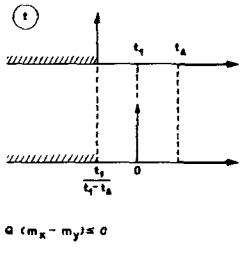

Fig. 13

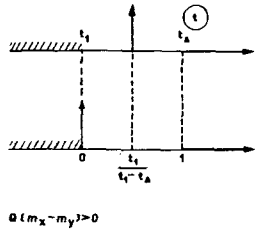

Fig. 14

and consequently $t = t_1$ is not a singularity of the amplitude. Indeed, from Eq. (4. 10) we directly obtain

$$F(0) = 2 \qquad (4.18)$$

and it can easily be seen that the value of $F(z)$ does not depend on the choice of the branch $\sqrt{z}$ ($F(z)$ does not change from the substitution $\sqrt{z} \to -\sqrt{z}$).

Summing up all that has been said above, $F(z)$ as an analytical function in the cut plane is given by the following formulae

$$F(z) = \begin{cases} (-z)^{\frac{1}{2}} 2 \arctan (-z)^{\frac{1}{2}} & -\infty < z < 0 \ (-\infty < t < t_1) \quad (4.19a) \\ z^{-\frac{1}{2}} \ln(1+z^{\frac{1}{2}}/1-z^{\frac{1}{2}}) & 0 < z < 1 \ (t_1 < t < t_\Delta) \quad (4.19b) \\ z^{-\frac{1}{2}}\{\ln(z^{\frac{1}{2}} + 1/z^{\frac{1}{2}} - 1) + i\pi\} & 1 < z < +\infty \ (t_\Delta < t < \infty) \quad (4.19c) \end{cases}$$

Thus in the physical region of the variation of t, $F(z)$ is of the form (4.19a) or (4.19b). From these formulae follows the important conclusion that far from a singularity $M_\Delta(t)$ decreases with the increase of $t - t_\Delta$ much more slowly than the pole amplitudes M_0:

$$|M_\Delta(t)| \sim |1/(t-t_\Delta)^{\frac{1}{2}}| \ , \ (t-t_\Delta)/(t_\Delta - t_1) >> 1 \qquad (4.20)$$

whereas under the same conditions

$$|M_0| \sim 1/(t - t_\Delta) \qquad (4.21)$$

This means that far removed triangle graphs can essentially affect the absolute values of the cross-sections of a reaction and consequently the calculation by these data of the reduced widths from the Butler formula.

Let us now compare the amplitude $M_\Delta(t)$ with the pole amplitude $M_C(t)$ which we shall write as

$$M_0(t) = (g'/t_0) F_0(z') \qquad (4.22)$$

where

$$F_0(z') = 1/(z'+1) \quad , \quad z' = -t/t_0 \qquad (4.23)$$

Then for the ratio of the amplitudes M_Δ and M_0 we have

$$\frac{M_\Delta}{M_0} = i(g/g')(m_c^2/m_A m_B)^{\frac{1}{2}}[m_b M' t_0/8\pi \ (t_\Delta - t_1)^{\frac{1}{2}}][F_\Delta(z)/F_0(z')] \qquad (4.24)$$

the amplitude M' being expressed, according to Eq. (1.2), through the cross-section for the virtual reaction (3.6) by the formula

$$|M'| = 2\pi\{(d\sigma/d\Omega)(1/m_{ax} m_{by})(p'_x/p'_y)\}^{\frac{1}{2}} \qquad (4.25)$$

where p'_x and p'_y are the momenta of the particles x and y in the centre-of-mass system of the particles a and x.

It can readily be seen that all dimensionless factors in Eq. (4.24) with the exception of the last one may, in general, equal 1 in order of magnitude. Indeed, with respect to the quantities g and g' this proposition is self-evident, the masses m_C, m_A and m_B are of the same order (or otherwise the particles m_a and m_b must be heavy and the singularity would, according to Eq. (4.8), lie far) so that

$$\sqrt{\frac{m_c^2}{m_A m_B}} \sim 1 \ .$$

The factor $(1/8\pi)(m_b M' t_0/(t_\Delta - t_1)^{\frac{1}{2}})$ is known least of all since it contains the amplitude M' of a virtual reaction. If, however, we proceed from the known cross-sections for real processes we shall see that this factor can also lie in the interval 10^{-2} to 1. Thus the ratio of the amplitudes of a triangle and pole graph is given in the final analysis by the ratio $F_\Delta(z)/F_0(z')$ and the latter in turn depends on how large $|z|$ and z' are as compared with 1, i.e. on the remoteness of t_Δ and t_0 from the physical region boundary. This is why the comparison of the positions of singularities of different graphs is the primary criterion to see how essential a certain mechanism of the reaction is. In the section below we cite data on the values of t_Δ for different reactions.

4.2. Position of triangle singularities for specific reactions

If we know the masses of the particles A, x, B and y and the virtual particles a, b and c, we can calculate t_Δ by Eq. (4.8). It has been pointed out above that to find t_Δ nearest to the physical region we should choose graphs of type 2 where the lowest binding energies at the vertices would correspond to the lightest particles a and b. Table II lists the results of such calculations for reactions on certain light nuclei. In the first column of the table the positions of the poles (t_0) calculated by Eqs. (2.64) - (2.68) are given for each reaction. The physical region boundary t_m with respect to the variable t (see Eq. (2.61)) is given in the third column for each reaction. The quantities listed in Table II are calculated for the ground state of the nuclei B and C. If the nucleus C is in the excited state C*, then $\epsilon^{B}_{C^*b} > \epsilon^{B}_{Cb}$ and according to Eq. (4.8) the singularity t^*_Δ is larger than t_Δ. The data given in Table II are quite illuminating. They reveal the cause of the success of the Butler theory for the case of deuteron stripping or pick-up corresponding

TABLE II

POLES AND NEAREST TRIANGLE SINGULARITIES WITH RESPECT TO t FOR CERTAIN DIRECT PROCESSES

Reaction	(d, n)			(α, t)			(He^3, α)		
Target	t_0	t_Δ	t_m	t_0	t_Δ	t_m	t_0	t_Δ	t_m
B^{10}	4.47	138	-12.9	39.6	173	0	41.1	53.3	0
C^{12}	4.47	225	0	39.6	260	-24.9	41.1	232	0
N^{14}	4.47	141	-10.2	39.6	176	0	41.1	97.8	0
O^{16}	4.47	162	0	39.6	197	0	41.1	193	0

Notation: t_0 is a pole, t_Δ-a triangle singularity, t_m - the physical region boundary; [t] = MeV . AMU

to the pole graph (taking into account the effects of the finite size of the nucleus, see chapter 5). Indeed, it is clear from Table II that in the case of deuteron reactions the nearest value of t_Δ is roughly 50 times as large as t_0. This occurs because t_0 is situated rather close to the physical region

due to a low deuteron binding energy. An altogether different situation prevails in the stripping reaction for particles different from a deuteron. For example, the reaction $B^{10}(He^3,\alpha)B^9$ is a pick-up reaction, but the triangle singularity is rather close to the pole. It might seem that the reaction $C^{12}(\alpha,t)N^{13}$ is exactly the same as the stripping reaction $C^{12}(d,n)N^{13}$ However, the nearest singularity is removed only 4.4 times as far from the physical region boundary as a pole. The fact that we have not just one, but several triangle graphs with closely situated singularities, is also of basic importance.

Table III lists the data, taken from [7], on the position of triangle singularities for the reactions $C^{12}(\alpha,t)N^{13}$, $C^{12}(He^3,\alpha)C^{11}$, with all values referring to the ground state of real and virtual particles. This example is typical: as a rule there are always at least 3 or 4 triangle graphs with closely situated singularities. If we take into account all possible graphs in which the virtual nucleus is excited with a not very high energy of excitation, the number of triangle graphs with closely situated singularities will increase even more (this circumstance is especially essential for medium and heavy nuclei).

TABLE III

TRIANGLE SINGULARITIES OF CERTAIN REACTIONS
($[t_\Delta]$ = MeV · AMU)

Reaction	Virtual particles			t_Δ
	a	b	c	
$C^{12}(He^3,\alpha)C^{11}$	d	p	B^{10}	232
	α	He^3	Be^8	332
	He^3	He^2	Be^9	528
	t	d	B^9	532
$C^{12}(\alpha,t)N^{13}$	p	He^2	B^{11}	260
	n	d	C^{11}	280
	d	He^3	B^{10}	608

In addition to the triangle graphs 2, the graphs represented in Fig. 15 have singularities with respect to momentum transfer.

Graph 15b refers to the pick-up of at least two nucleons. The amplitudes of these graphs and their singularities are obtained in exactly the same way as shown for graph 2. The formula for the singularities of graph 15b can be derived from Eq. (4.15) if we take the + sign before the root. The difference in the choice of signs is due to the fact that in the case of graph 15b $\lambda_a \lambda_b = -1$, and not +1, as in graph 2. Accordingly, the function $F_\Delta(z)$ is taken on the lower side of the cut.

Graph 15a is a perfect analogue of graph 2. The formulae for graph 15a can be derived from the formulae of the preceding section after the substitution

$$t \to t + 2(m_x - m_y)Q. \qquad (4.16)$$

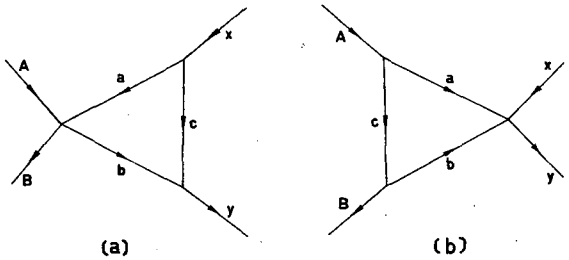

Fig. 15

The values t_Δ for graph 15 are the same in order of magnitude as for graph 2. For example t_Δ of graph 15a for the reaction $C^{12}(He^3, \alpha)C^{11}$ when a = p and b = c = d amounts to 277 MeV.AMU.

The main conclusion to be drawn from the analysis of the location of singularities is that the pole (i.e., Butler) mechanism of a reaction is not, in general, a predominating one except for the cases of deuteron stripping and pick-up.

4.3. Concluding remarks

The position of the singularities over t determines the degree of "peripherality" of a direct reaction. The nearer a singularity is to the physical region boundary, the more peripheral the graph. This follows from the fact that the mean momentum of virtual particles is equal in order of magnitude to $\sqrt{t_0}$ or $\sqrt{t_\Delta}$. Thus, if $\sqrt{t_\Delta} \ll k$ where k is the momentum of a virtual particle inside the potential well, the reaction occurs in the main outside the potential well, that is, on the periphery of the nucleus.

It has been pointed out above that the pole graphs with singularities with respect to t correspond to the Butler theory. Triangle graphs taking into account the finite size of the nucleus (see chapter 5) correspond to what is known as the "surface scattering" mechanism considered by AUSTERN, BUTLER and McMANUS [9].

The dispersion method:

(1) Imparts a physical meaning to calculations by perturbation theory;

(2) Permits the prediction of which cases can be expected to have the predominating contribution from which mechanism; and

(3) Is universal in the sense that in terms of this method all virtual particles (and not only those which can be described by a wave function, i.e. exist for durable times in the nucleus) are of equal status.

Using the graph technique and formulae for singularities we can indicate the most essential mechanism for each specific reaction.

It has been pointed out earlier that for direct process theory rectangle graphs of type 3 can be of practical interest besides the pole and triangle graphs. The amplitude and singularities of a rectangle graph require similar calculations to those for a triangle graph. Rectangle graphs may play a certain role in reactions of the type (n, He^3), (p, t) or in reactions on multicharged ions with a transfer of two nucleons.

The singularities of more complicated graphs, just as of triangle ones, become more distant as the binding energies at the vertices and the masses of virtual particles increase. The classes of allowable graphs essentially

decrease because of the non-relativistic approximation. The amplitude of the graphs satisfying the relation (n is the number of internal lines and ℓ is that of independent 4-momenta of virtual particles)

$$n < 5\,\ell/2 \tag{4.27}$$

are in general constants equal in order of magnitude to $\sqrt{\epsilon/m}$ or even $\sqrt{1/mR}$ where R is the nuclear radius. This circumstance is due to the fact in the case of (4.27) the Feynman integral converges only at the expense of the vertex form factors (i.e. the integral (3.7) diverges). Graph 16 is a case in point.

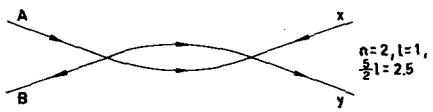

Fig.16

The graphs for which

$$n = 5\,\ell/2 \tag{4.28}$$

may in principle play a certain role in non-relativistic processes, since by order of magnitude they are proportional to $(\epsilon/m)^{\frac{1}{2}} |\ln(\epsilon/M)|$ (see [8]). Graph 17 is of this type.

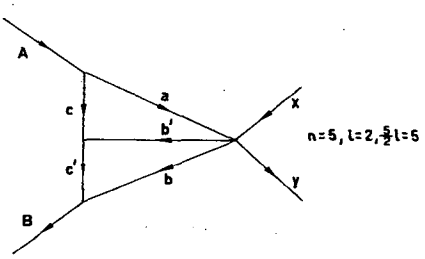

Fig.17

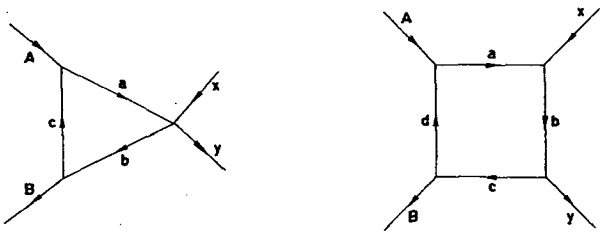

Fig.18

Finally, let us note that the one-contour graphs in which all $\lambda_i = +1$ (Fig. 18) are also relativistic, though for a different reason. The fact is that in this

case the non-relativistic Feynman integral (3. 17) vanishes because it is impossible to satisfy the condition

$$\sum_i \alpha_i m_i = 0, \quad \alpha_i > 0 \tag{4.29}$$

which arises owing to the δ-function $\delta(\sum_i \alpha_i \lambda_i m_i)$ when $\lambda_i = +1$.

APPENDIX TO CHAPTER 4

Derivation of triangle graph amplitude from unitarity condition

To illustrate the method we shall consider graph 19 in which the masses of two virtual particles (b and c) are equal to m. The masses of the initial particles (A and x) are equal to m_0 and the masses of the final particles (B and y) are equal to m_f.

The dashed line in Fig. 19 divides two intermediate reactions. We shall consider the amplitude of graph 19 as a function of the square, designated

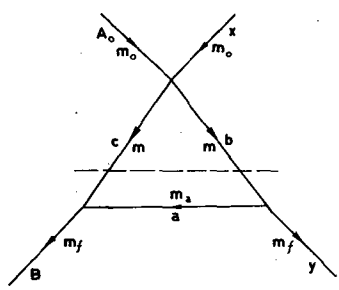

Fig. 19

by the letter S, of the sum of the 4-momenta of the particles A and x

$$S = (P_A + P_x)^2. \tag{4.30}$$

In the non-relativistic approximation we have

$$S = s + (m_A + m_x)^2 = s + 4m_0^2. \tag{4.30a}$$

In the centre-of-mass system the variable S equals the square of the sum of the total energies of the particles A and x:

$$S = W^2, \quad W = \epsilon_A + \epsilon_x, \quad \epsilon_x = m_x + E_x, \quad E_A = m_A + E_x. \tag{4.31}$$

The two-particle unitarity condition (3. 1) is calculated in the relativistic case just as in the non-relativistic one with the only difference that instead of the substitution d^3p_c by Eq. (2. 42) we should write down

$$d^3p_c = 2\epsilon_c \, \delta(P_c^2 - m^2) d^4P_c. \tag{4.32a}$$

Then from Eq. (3. 1) we can easily obtain a formula perfectly analogous to Eq. (3. 3) and passing into it in the non-relativistic approximation

$$A^{(2)}_{\alpha\beta} = \frac{p}{8\pi^2} \frac{\epsilon_b \epsilon_c}{W} \int M_{\alpha\gamma} M^{+}_{\gamma\beta} \, d\Omega_b. \tag{4.32}$$

Since $\epsilon_b = \epsilon_c = \epsilon$ because of the equality of the masses, we have

$$A^{(2)}_{\alpha\beta} = \frac{\epsilon^2}{W} \frac{p}{8\pi^2} \int M_{\alpha\gamma} \quad M^{+}_{\gamma\beta} \, d\Omega_b. \tag{4.33}$$

For the amplitude $M^{+}_{\gamma\beta}$ we can evidently write, according to graph 19,

$$M^{+}_{\gamma\beta} = \frac{g}{P_a^2 - m_a^2 + i\eta} \tag{4.34}$$

and from the conservation laws we obtain

$$P_a^2 = m_f^2 + m^2 - \frac{W^2}{2} + 2p_0 p_z \tag{4.35}$$

where p_0 is the momentum of the final particles in the centre-of-mass system

$$p_0 = \sqrt{\frac{S}{4} - m_f^2} \, . \tag{4.36}$$

p is the momentum of the intermediate particles in the centre-of-mass system

$$p = \sqrt{\frac{S}{4} - m^2} \tag{4.37}$$

and z is the cosine of the angle between the momenta of the particles b and y in the centre-of-mass system. Choosing the direction $\vec{p}_y$ as the polar axis, we can write

$$d\Omega_b = - dz \, d\varphi \, . \tag{4.38}$$

Substituting Eqs (4.38) and (4.34) into Eq. (4.35) and performing the integration over $d\varphi$ we obtain on the assumption that $M_{\alpha\gamma}$ is independent of z:

$$A^{(2)}_{\alpha\beta} = \frac{M_{\alpha\gamma}}{8\pi} \frac{\epsilon^2}{W p_0} \int_{-1}^{+1} \frac{dz}{z - \phi(S, m_f^2)} \tag{4.39}$$

where

$$\phi(S, m_f^2) = \frac{S - 4m^2 + 2(m_k^2 - m_f^2)}{\sqrt{(S - 4m^2)(S - 4m_f^2)}} \tag{4.40}$$

while

$$m_k^2 = m^2 + m_a^2. \tag{4.41}$$

The unitarity condition (4.39) has a physical meaning, as was pointed out in chapters 2 and 3, when all intermediate reactions are really possible and the entire reaction is possible as a whole. In this case $A^{(2)}_{\alpha\beta}$ is the imaginary part of the amplitude for the values $S > 4\,m^2$ and $S > 4\,m_f^2$ ($4\,m^2$ is the threshold of the first intermediate reaction $\alpha \to \gamma$, and $4\,m_f^2$ is that of the entire reaction $\alpha \to \beta$).

In other words, $A^{(2)}_{\alpha\beta}$ has the meaning of the imaginary part of the amplitude when

$$m_f < m. \tag{4.42}$$

The amplitude $M^{(2)}_{\alpha\beta}$ can then be restored by the general formula (2.5) which in the present case will be written as

$$M^{(2)}_{\alpha\gamma} = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{A^{(2)}_{\alpha\gamma}(S')}{S' - S} \, dS'. \tag{4.43}$$

If

$$m_f > m \tag{4.44}$$

the first intermediate reaction becomes possible earlier than the entire reaction as a whole and therefore the unitarity relation has no physical meaning when

$$4m^2 < S < 4m_f^2 \tag{4.45}$$

and consequently $A^{(2)}_{\alpha\beta}$ in this region is no imaginary part of the amplitude. Hence (4.43) ceases to be valid. We can, however, obtain $M^{(2)}_{\alpha\beta}$ in case (4.44) by considering Eq. (4.43) as an analytical function of m_f and performing the analytical continuation of (4.43) from the region (4.42) into the region (4.44). With this aim we should substitute Eq. (4.39) into (4.43) and consider the analytical continuation of the integral (4.43). It is clear that if in the movement from $m_f < m$ to $m_f > m$, $A^{(2)}_{\alpha\beta}$ as a function of $m_f{}^2$ remains regular and does not go to infinity under any $4m^2 < S < \infty$ the integral in (4.43) will have meaning and Eq. (4.43) remains valid. The problem thus reduces to the investigation of the analytical properties of $A^{(2)}_{\alpha\beta}$. From Eq. (4.39) it is clear that if $|\phi(S, m_f)| > 1$, then $A^{(2)}_{\alpha\beta}$ is regular. If, on the other hand,

$$|\phi(S, m_f)| = 1 \tag{4.46}$$

for certain m and S entering the integration region in the integral (4.43), then $A^{(2)}_{\alpha\beta}$ will be singular since the integrand in Eq. (4.39) goes to infinity at one of the integration limits. Let us find for given masses the point S_Δ at which condition (4.46) holds

$$S_\Delta = 4m^2 - \frac{(m_f^2 - m_k^2)^2}{m_a^2}. \tag{4.47}$$

From Eq. (4.47) it is clear that when

$$m_f^2 = m_k^2 \tag{4.48}$$

we have $S_\Delta = 4m^2$, i.e. equal to the lower integration limit in the integral (4.43). Consequently, in the case of (4.48) the integral (4.43) will have no meaning since the integrand is singular at the lower integration limit. From Eq. (4.47) it is clear furthermore that if

$$m_f^2 > m_K^2 \tag{4.48a}$$

then, according to Eq. (4.41) S_Δ again becomes smaller than $4m^2$, i.e. goes

beyond the integration region in Eq. (4.43). This does not mean, however, that under the condition (4.4a) Eq. (4.43) again becomes valid. Indeed, if m_f^2 is regarded as a complex variable, then with a small positive imaginary part of m_f^2, the point S_Δ will move with the change of m_f^2 as is shown in Fig. 20. It is clear from Fig. 20 that the trajectory of S_Δ obviates the point $4m^2$ on the right. It is easy to see from Eq. (4.39) that the function $A^{(2)}_{\alpha\beta}(S)$ is analytical in the plane S, with a cut along the curve shown in Fig. 20, for any values of m_f. But since the function at the cut is not unique, the integral

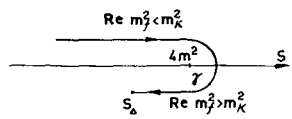

Fig. 20

(4.43) becomes meaningless since the integration contour in Eq. (4.43) crosses the cut. A definite meaning can be imparted to this integral by defining it as an integral over the contour Γ shown in Fig. 21.

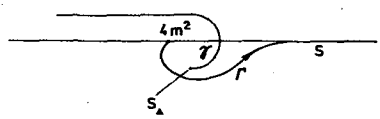

Fig. 21

If we now make the imaginary part of m_f^2 tend to zero, the line γ will pass onto the real axis and we shall have

$$\int_\Gamma \frac{A^{(2)}_{\alpha\beta}(S')}{S'-S}\,dS' \longrightarrow \int_{4m^2}^{S_\Delta} \frac{A^{(2)}_{\alpha\beta}(S+i\epsilon)-A^{(2)}_{\alpha\beta}(S'-i\epsilon)}{S'-S}\,dS' + \int_{4m^2}^{\infty} \frac{A^{(2)}_{\alpha\beta}(S')}{S'-S}\,dS'. \tag{4.49}$$

Thus

$$M^{(2)}_{\alpha\beta}(S) = \frac{1}{\pi}\int_{4m^2}^{S_\Delta} \frac{A^{(2)}_{\alpha\beta}(S'+i\epsilon)-A^{(2)}_{\alpha\beta}(S'-i\epsilon)}{S'-S}\,dS' + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{A^{(2)}_{\alpha\beta}(S')}{S'-S}\,dS'. \tag{4.50}$$

Let us now turn to the non-relativistic approximation, making $m^2 \to \infty$ and substituting S for the variable s by Eq. (4.30a)*. Then the second integral in Eq. (4.50) goes to zero and we obtain

$$M^{(2)}_{\alpha\beta}(s) = \frac{1}{\pi}\int_{S_\Delta}^{\infty} \frac{A^{(2)}_{\alpha\beta}(s'-i\epsilon)-A^{(2)}_{\alpha\beta}(s'+i\epsilon)}{s'-s}\,ds' \tag{4.51}$$

* Strictly speaking it is impossible to turn to non-relativistic approximation in this example, because the masses of the particles B and y are equal. But the scheme of the consideration and formulae (4.52)-(4.54) are correct.

where

$$s_\Delta = S_\Delta - 4m^2. \tag{4.52}$$

From Eq. (4.39) it follows that

$$A^*_{\alpha\beta}(S) = A_{\alpha\beta}(S^*) \tag{4.53}$$

and therefore we obtain finally

$$M^{(2)}_{\alpha\beta}(s) = -\frac{2i}{\pi}\int_{s_\Delta}^{\infty}\frac{\operatorname{Im} A^{(2)}_{\alpha\beta}(s')}{s' - s}\,ds'. \tag{4.54}$$

Thus in the case under discussion $M^{(2)}_{\alpha\beta}(s)$ is expressed with the aid of the dispersion integral of $\operatorname{Im} A^{(2)}_{\alpha\beta}(s)$ and not of $A^{(2)}_{\alpha\beta}(s)$ itself. As we have seen, this result follows from the fact that

$$m_f^2 < m^2 + m_a^2$$

i.e. the fact that the particles B and y possess small binding energy (as compared with the rest mass).

In conclusion it should be noted that Eq. (4.54) remains valid for unequal masses of intermediate particles, and if we replace a by t, it can be transferred unchanged onto the graphs with triangle singularities for t which we have considered.

It can readily be seen that Eq. (4.54), if we substitute $\operatorname{Im} A_{\alpha\beta}$ from Eq. (4.39) (calculated in the non-relativistic approximation), coincides with the result obtained from the direct calculation of the Feynman integral.

5. AMPLITUDE VERTEX PARTS OF DIRECT NUCLEAR PROCESSES

5.1. Definition of vertex part

Virtual decays and captures occur in the graphs we have considered. The amplitudes of these processes will be called vertex parts. We shall confine our treatment to such vertex parts where only one of the particles entering a vertex is virtual. Vertex parts of this kind are considered in detail in [10].

We shall begin with specifying the concept of a three-tail vertex part as shown in Fig. 22. If we deal with a "purely pole" graph, i.e. a graph without any singularity except the pole, the vertex part is expressed through the matrix element M^1_{23} of the decay of particle 1 into particles 2 and 3 and is a number known as the reduced vertex part

$$\gamma^1_{23} = (m_3/\pi)^{\frac{1}{2}} M^1_{23}. \tag{5.1}$$

The sum of the graphs shown in Fig. 22 has several singularities besides a pole (the black dots correspond to the reduced vertex part for the decay $1 \to 2 + 3$, crosses designate the reduced vertex part, the same for all terms of the sum, for the second vertex at which the line of particle 3 ends). These singularities are determined by the singularities of the graphs follow-

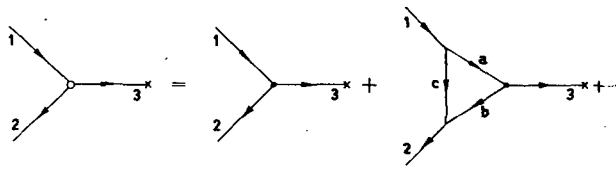

Fig. 22

ing the first terms making up a purely pole graph. Since the number of summed graphs is infinite in the sum as a whole there may in principle appear singularities which do not occur in any term of the series. An essentially singular point may for example be such a singularity (we shall see later on that this is perhaps the situation in the case of nuclear vertex parts). The amplitude corresponding to the sum of graphs in Fig. 22 can be represented as Eq. (2.47). This formula will give the vertex part Γ^{1}_{23} (subscripts and superscripts will be omitted) for the decay vertex $1 \to 2 + 3$. The vertex part Γ is a function of the masses of the particles and the invariant variable t_{12}

$$t_{12} = -(\vec{p_1} - \vec{p_2})^2 + 2(m_1 - m_2)(E_1 - E_2). \qquad (5.2)$$

If the particles 1 and 2 are free (i.e. $r_1 = 0$ and $r_2 = 0$) then in the rest system of particle 1 the variable t_{12} can be simply expressed through the square of momentum transfer $q^2 = (\vec{p_1} - \vec{p_2})^2$:

$$t_{12} = -(m_3/m_{23})q^2 \qquad (5.3)$$

where m_{23} is the reduced mass of particles 2 and 3.

From energy conservation law we have

$$E = m_1 - m_2 - m_3 \qquad (5.4)$$

where E is the total kinetic energy of particles 2 and 3 in their centre-of-mass system. If $m_1 < m_2 + m_3$ (particle 1 is stable) then E is negative and equals the absolute value of the binding energy of particles 2 and 3. Thus, if the masses of particles 2 and 3 are regarded as fixed, the vertex part will be a function of the momentum transfer q and energy E. In the next section we shall consider the analytical properties of $\Gamma(q, E)$ with respect to the variable q.

5.2. Vertex part in single-particle model

It will be shown that the calculation of vertex parts essentially reduces to the quantum description of the relative motion of particles 2 and 3. This is why a one-particle (optical) model can be used for calculating the vertex parts. In the one-particle model the wave function of the relative motion of particles 2 and 3 $\psi(\vec{r})$ where $\vec{r} = \vec{r_2} - \vec{r_3}$ ($\vec{r_2}$ and $\vec{r_3}$ are the vector radii of the centres of gravity of particles 2 and 3) corresponds to the "decaying" nucleus. The function $\psi(\vec{r})$ is finite at zero and decreases exponentially for sufficiently large r. If particles 2 and 3 are zero spin particles and the nucleus of 1 has spin ℓ, the radial part $\Phi_\ell(q)$ of the wave function Fourier component is given by the equation

$$\varphi_\ell(q) = (2/\pi)^{\frac{1}{2}}\, i^{-\ell} \int_0^\infty \psi_\ell(r)\, j_\ell(qr) r^2 dr, \qquad (5.5)$$

where $j_\ell(qr)$ is a spherical Bessel function and $\psi_\ell(r)$ is the radial part of the wave function. Since

$$\psi_\ell(r) \underset{r\to\infty}{=} C_\ell(e^{-\kappa r}/r) \qquad (5.6)$$

where

$$\kappa = \sqrt{2m\epsilon}\,, \qquad \epsilon = -E > 0 \qquad (5.7)$$

from Eq. (5.6) it follows that $\varphi_\ell(q)$ must be of the form

$$\varphi_\ell(q) = F_\ell(q)/(q^2 + \kappa^2), \qquad (5.8)$$

and

$$F_\ell(i\kappa) = (2/\pi)^{\frac{1}{2}} C_\ell. \qquad (5.8a)$$

The function $F_\ell(q)$ is analytic in the upper half-plane of the complex variable q. When $|q| \to \infty$, $F_\ell(q)$ may behave as $q^a\, e^{bq}$ where a and b are any complex numbers (a stronger-than-exponential growth is forbidden because of the convergence of the integral (5.5)). Thus $F_\ell(q)$ has no singularities in the final part of the upper half-plane of the variable q, but may have, in general, a pole or an essentially singular point at infinity. The function $F_\ell(q)$ coincides with the vertex part $\Gamma(q)$ accurately to within a constant factor. The simplest way to verify this is to consider in terms of perturbation theory the weak local interaction of some radiation with particle 3 (for example, the interaction with electromagnetic field leading to a nuclear photo-effect, the capture of orbital electrons or μ^--mesons, etc.). Obviously, the calculation taking into account Eq. (5.8) will lead to a formula with a pole denominator while the remaining dependence on q will be given by the function F(q) since the vertex part corresponding to the weak local interaction will be a constant. It is precisely such a formula that corresponds to the graph in the left-hand side of the equation in Fig. 22 (in this case the cross corresponds to the vertex part for the weak interaction calculated by perturbation theory and proportional to the first power of the interaction constant). On the strength of the above we arrive at the following formula for the vertex part $\Gamma(q)$:

$$\Gamma(q) = (2/\pi)^{\frac{1}{2}}\, i^{-\ell}(q^2 + \kappa^2) A_\ell \int_0^\infty \psi_\ell(r) j_\ell(qr) r^2 dr. \qquad (5.9)$$

The constant A_ℓ is expressed through the reduced vertex part γ, thus

$$A_\ell = (2/\pi)^{\frac{1}{2}}\, i^\ell \frac{\gamma}{\lim\{(q^2 + \kappa^2) \int_0^\infty \psi_\ell(r) j_\ell(qr) r^2 dr\}}. \qquad (5.9a)$$

The reduced vertex part can in turn be expressed by wave functions (see section 5.4). Equation (5.9) in combination with the formulae of section 5.4, thus completely determines the vertex part.

If $q \ll k$, where k is the wave number of a particle inside the nucleus with radius R, integration over the inner region (from 0 to R) will yield a small contribution to integral (5.9). This can easily be seen if we take a rectangular well as an example, when $\psi_\ell(kr)$ inside the nucleus equals $j_\ell(kr)$ accurately to within the normalizing factor. Since we have

$$\int_0^\infty j_\ell(kr) j_\ell(qr) r^2 dr = (\pi/2\ kq)\delta(k-q)$$

the integral over the inner region will be small* when $(k + q)R \gg 1$. In this case the lower limit of integration in Eq. (5.9) can be replaced by R_0 which leads to the Butler formula if the vertex part (5.9) is introduced into the amplitude of the deuteron stripping pole graph. It should be borne in mind that the quantity q in Eq. (5.9) is given as the momentum transfer in the system of the disintegrating (or forming) particle 1 by Eq. (5.3) and does not, in general, coincide with the momentum transfer in the c.m.s. of the total reaction. In particular, the quantity q depends on energy in the c.m.s. of the reaction (see Eqs (5.10) and (5.13) below). This circumstance is especially essential when the momenta transferred are small. When we have large q comparable with k we cannot neglect the contribution from the inner region** in Eq. (5.9).

The transparency of nuclear substance for the emitted (or absorbed) virtual particles is essential when calculating the vertex parts. For nucleons, for example, the nuclear potential is real for negative energies in the first approximation (the imaginary part of the potential corresponding to the mixing of one-particle configurations is small). The problem of taking into account the absorption of virtual particles in nuclear substance will be considered in section 5.4. Note that the smooth decrease of the potential on the nuclear surface (instead of a jump in the case of a rectangular well) may prove essential for the angular distribution curve. In terms of analytical properties the explanation is that in the case of a potential with a diffuse edge there arise branch points on the real axis (see below). As a result additional terms corresponding to these peculiarities appear in $\Gamma(q)$.

Equation (5.9) is universal in the sense that the vertex part it gives may enter into the amplitudes of widely different processes: usual low- and medium-energy nuclear reactions with a nucleon-deficient nucleus as incident particle, reactions on multi-charged ions (such as nucleon transfer reactions), high-energy reactions with the capture of slow π^- and K^- mesons by nuclei. In the case of a pole mechanism, the amplitude of the process will, as follows from Eq. (2.47), be proportional to the product of the vertex parts of the vertices which contain the lines corresponding to the target nucleus and incident particle. For the pole mechanism of deuteron stripping or pick-up we thus obtain the Butler formula since the deuteron vertex part, accurately to the zero neutron-proton range, is a constant. When, on the other hand, the radius of the incident particle is much larger than $1/\kappa$ (which is just the case in the nucleon transfer on multi-charged ions) the vertex part corresponding to the incident particle will also be a function of the mo-

* For example, when the potential well is 40 MeV deep, $R = 3.6 \times 10^{-13}$ cm, the nucleon binding energy is 6 MeV, $\ell = 0$ and $qR = 1$, the contribution of the integral over the inner region is about 7%.

** This is one of the main reasons why Butler's "surface stripping" theory describes fairly well the experimental angular distribution in the region of small angles, but not so well with experiment when the momenta transferred are large.

mentum transfer and this will strongly affect the angular distribution of the products of the reaction. It should be borne in mind, however, that the variables t_{12} differ from vertex to vertex. If, for example, the pole mechanism of the reaction A(x, y)B is such that A and B enter one vertex, while x and y another, we have

$$t_{AB} = -q^2 \pm 2m_3\{[\mu_{Ax} - \mu_{By}]E - \mu_{By}Q\} \tag{5.10}$$

$$t_{AB} = t_{xy} - 2m_3Q \tag{5.11}$$

where μ_{ij} is the ratio of the reduced mass of the particles i and j to the mass of the particle i. The sign of Eq. (5.10) is chosen depending on what occurs at the vertex AB: emission (upper sign) or absorption of a virtual particle. In these notations Eq. (2.47) will be of the form

$$M_{\alpha\beta} = 2\pi \frac{\Gamma_{AB}(q_{AB})\Gamma^*_{xy}(q_{xy})}{t_{AB} - 2m_3 \epsilon_{AB}} \tag{5.12}$$

where

$$q^2_{AB} = -\mu_{3B} t_{AB};\ q^2_{xy} = -\mu_{3y} t_{xy}. \tag{5.13}$$

It is clear from Eqs. (5.10) and (5.12) that when $\mu_{Ax} \neq \mu_{By}$ the vertex parts will depend not only on the momentum transfer q in the c.m.s. of the colliding particles, but also on the energy E. Since q_{AB} increases with E, the contribution of the integral over the inner region in Eq. (5.9) will change with the energy E. As a result the effective radius of the vertex determined from the comparison of the usual Butler formula with experimental data will also change with energy. An inaccuracy in the determination of the vertex radius will also affect the reduced vertex part (or reduced reaction width) since it is expressed through $\Gamma(i\kappa_{AB} R)$.

5.3. Vertex part singularities

To study the singularities of the vertex part (5.9) it is convenient to use the integral equation for $\Gamma(q)$ which can readily be obtained from the Schrödinger equation for $\psi_\ell(r)$. This integral equation is of the form

$$\Gamma(q) = -(2/\pi)^{3/2} \int \Delta_\ell(q, q', k) \frac{\Gamma(q')V(k)}{q'^2 + \kappa^2} q'^2 k^2 \, dq' \, dk. \tag{5.14}$$

Here

$$\Delta_\ell(q, q', k) = \int_0^\infty j_0(kr) j_\ell(q'r) j_\ell(qr) r^2 \, dr \tag{5.15}$$

and

$$V(k) = (2/\pi)^{\frac{1}{2}} \int_0^\infty U(r) j_0(kr) r^2 \, dr \tag{5.16}$$

where U(r) is the potential of the interaction of the particles 2 and 3. In investigating the singularities of the vertex part the specific value of ℓ is inessential and therefore we put $\ell = 0$ for the sake of simplicity. Then we have

$$\Delta_0(q, q', k) = \frac{\pi}{4} \frac{1}{q q' k} \tag{5.17}$$

if q, q′ and k can be sides of a triangle, and

$$\Delta_0(q, q', k) = 0 \tag{5.17a}$$

if this condition is not fulfilled.

Equation (5.14) can be re-written as

$$\Gamma(\sqrt{s}) = \text{const} + \int_0^\infty \Gamma(\sqrt{s'})\,\phi(s, s')ds' \tag{5.18}$$

where the kernel $\phi(s', s)$ is given by the formula

$$\phi(s', s) = \frac{\varphi(s', s)}{s' + \kappa^2} \tag{5.19a}$$

$$\varphi(s', s) = \frac{1}{4\sqrt{s}} \int_{(\sqrt{s} - \sqrt{s'})^2}^{(\sqrt{s} + \sqrt{s'})^2} V(\sqrt{s''})ds''. \tag{5.19b}$$

Here $\sqrt{s}$ is determined so that $\sqrt{s} > 0$ if s is real and positive.

Since the singularities of the solution of the integral equation are given by its kernel, Eq. (5.18) need not be solved to find the singularities of the vertex part; it is sufficient to study the singularities of the kernel (5.19) as a function of s and s′. It is clear from Eq. (5.19) that the analytical properties of $\phi(s', s)$ depend on the properties of the Fourier components V(k) of the potential V(r). Let us consider the potentials satisfying the following condition

$$\underset{r \to \infty}{U(r)} = r^{-1} \sum_i \alpha_i \exp(-\alpha_i r)$$

i.e. behaving asymptotically as the superposition of the Yukawa potentials. From Eq. (16) it follows then that V(k) is of the form

$$V(k) = \sum_i v_i(k)/(k^2 + \alpha_i^2) \tag{5.21}$$

where $v_i(k)$ has no singularity in the finite part of the upper half-plane of the complex variable k. At the same time when $|k| \to \infty$, $v_i(k)$ may behave like $k^a e^{bk}$ where a and b are any complex numbers. In other words, $v_i(k)$ and consequently V(k) may have an essential singularity at infinity. If there is no such singularity (i.e. if b = 0) the potential U(r) is of the form (5.20) for all r. The potentials employed in the theory of the nucleus are usually of a type similar to (5.21). Here are two examples of such potentials.

(a) A potential of the type of a rectangular well with diffuse edge

$$U(r) = \begin{cases} U_0 & r < R \\ U_0 R r^{-1} \exp\{-\alpha(r - R)\} & r > R \end{cases} \tag{5.22a}$$

$$V(k) = U_0 (\pi/2)^{\frac{1}{2}} v(k)/(k^2 + \kappa^2) \tag{5.22b}$$

$$v(k) = \frac{1}{k}\left\{\frac{(\alpha^2 + k^2)(\sin kR - kR\cos kR)}{k^2} + \alpha R \sin kR - kR\cos kR\right\}$$

(b) A potential similar to the Woods-Saxon potential

$$U(r) = \frac{U_0}{r[1 + \exp\alpha(r - R)]} \tag{5.23a}$$

$$V(K) = (v_0/k)(\pi/2)^{\frac{1}{2}} \{ -(\pi/\alpha)(\cos kR/\mathrm{sh}(\pi k/2) + 1/k + k \sum_{\nu=1}^{\infty} (-1)^{\nu} (\exp[-\alpha\nu R]/(\alpha^2\nu^2 + k^2))\}. \tag{5.23b}$$

(The last term in brackets has the same poles as the first one. Therefore when $e^{-\alpha R} \ll 1$, which holds for all nuclei except the lightest, the sum over ν can be omitted.) The factor $1/r$ absent in the Woods-Saxon potential has been introduced because of the requirement (5.20). Due to this factor V(k) has only simple poles, while the Fourier components of the Woods-Saxon potential would have multiple poles $1/[k^2 + (\alpha\nu)^2]$. Note that the pole multiplicity is not essential for further conclusions and the requirement (5.20) is imposed only for certainty. This requirement is natural if one proceeds from the physical considerations based on the field nature of nuclear interactions.

Equation (5.19) is a singular integral equation. From the theory of equations of this kind (see [11]), for example) it is known that the solution has a singularity for that value of S for which both the factors in Eq. (5.19a) have a common singularity with respect to s' or if the kernel is singular at the ends of the integration interval. In other words, the singularities of $\Gamma(\sqrt{s})$ coincide with the singularity of the function $\varphi(-\kappa^2, s)$. Substituting Eq. (5.21) into Eq. (5.19b) and assuming in accordance with the above that

$$\sqrt{s'} = +i\kappa$$

we obtain

$$\varphi(-\kappa^2, s) = 4^{-1} s^{-\frac{1}{2}} \sum_i \varphi_i(s) \tag{5.24}$$

where

$$\varphi_i(s) = \int_{(\sqrt{s} - i\kappa)^2}^{(\sqrt{s} + i\kappa)^2} \frac{v_i(\sqrt{s''})}{s'' + \alpha_i^2} ds''. \tag{5.24b}$$

Since $v_i(\sqrt{s''})$ has no singularities in the finite part of the plane of the complex variable $\sqrt{s''}$, $\varphi_i(s)$ has only a singularity at the point

$$(\sqrt{s_i} - i\kappa)^2 = -\alpha_i^2. \tag{5.25}$$

Taking into account the above definition of $\sqrt{s}$ we obtain

$$\sqrt{s_i} = i(\alpha_i + \kappa). \tag{5.26}$$

Thus the vertex part of $\Gamma(q)$ as a function of the variable q^2 has singularities at the points

$$q_i^2 = -(\alpha_i + \kappa)^2. \tag{5.27}$$

This means that the singularities of the vertex part in the finite part of the plane of the variable q^2 are determined by the asymptotics of the potential or, in other words, by the diffuseness of the potential well edge. If we have

a rectangular well, $\alpha_i = \infty$ there are no singularities at all for the finite values of the variable q^2. The results obtained are quite natural from the physical point of view. Indeed, the diffuseness of the potential well edge is characterized for real nuclei by the value $\alpha_i \approx 1.54 \times 10^{13}$ cm^{-1} [12]. If we put $\kappa = 5 \times 10^{12}$ cm^{-1} (this corresponds to $\epsilon = 5$ MeV for a nucleon) then we have $q_i^2 = -158$ MeV. AMU. Now, the singularities of triangle graphs are usually precisely of this order of magnitude, as we have seen in the preceding chapter. Thus the diffusion of the potential well edge phenomenologically takes into account the set of peripheral graphs represented in Fig. 22. These results also allow us to see where the diffuseness of the potential well edge may have a substantial effect on the angular distribution of direct reaction products. This obviously happens when the singularity of the graphs responsible for the main contribution to the direct process amplitude is close to Eq. (5. 27) or corresponds to larger negative values of q^2, i. e. if this singularity lies at the values $-q^2 \gtrsim 100$ to 200 MeV. AMU).

We have seen above that since we have $R \neq 0$ the Fourier components of the potentials used in nuclear physics have an essential singularity with respect to q at infinity. As a result, the vertex parts $\Gamma(q)$ also have an essential singularity with respect to q at infinity. This conclusion directly follows from the fact that with a complex q tending in absolute value to infinity the kernel (5. 19) increases exponentially for any real $s' > 0$ and in particular for $s' \to \infty$. This means that when $|q| \to \infty$ the kernel of the integral equation (5. 18) diverges at the upper limit. According to the theorem we have mentioned, it follows hence that the solution of the equation will also have an essential singularity at infinity. If we have a rectangular well there are no singularities with respect to q^2 in the finite part of the plane, and the vertex part is not a constant only because of an essential singularity. The diffuseness of the potential well edge does not eliminate the singularity connected with the nuclear radius since the integral

$$\int_0^\infty \frac{\Gamma(q')}{q'^2 + \kappa^2} q' dq'$$

converges because of $\psi_\ell(r)$ being finite at zero, and the kernel $\phi(s, s') = \mathcal{F}(q, q')$ for any finite q tends to zero when $q' \to \infty$. Therefore when $q - \kappa \gg \alpha_i$ in the integral

$$\int_0^\infty \frac{\Gamma(q')}{q'^2 + \kappa^2} q' dq' \int_{|q-q'|}^{|q+q'|} \frac{v_i(k)}{k^2 + \alpha_i^2} k dk.$$

α_i^2 can be neglected compared with k^2 and we obtain an integral equation which in the region of large q does not differ in any way from the case of a rectangular well provided $v_i(k)$ has an essential singularity of the type $A \cos kR + B \sin kR$.

In the nuclear model under study the diffuseness of the potential well edge thus leads in the vertex part to the singularities corresponding to the peripheral graphs of the type represented in Fig. 22. As to the radius of the nucleus, an essential singularity with respect to q at infinity corresponds to it. In a sense (in the framework of the model used) the latter conclusion answers the question put in [1] about the analytical properties of the direct

process amplitude due to the nuclear radius being finite. It is not clear whether the presence of an essential singularity results from the rough approximation of the properties of the nucleus by the potential well with a diffuse edge or such is the true situation despite the predictions of the dispersion theory of the elementary particles. All we can say now is that the search for the graphs whose momentum transfer singularities would correspond to the nuclear radius has not yet been successful. Investigations (undertaken by the author in collaboration with V. S. Popov) showed that as the Feynman graphs became more complicated their momentum transfer singularities receded into the region of larger negative values (to the best of our knowledge not a single example can be cited when a Feynman graph singularity would approach the physical region with the increase of inner lines). Barring the possibility of essential singularities in precise theory, the nuclear radius must be explained either by a singularity on the unphysical sheet or some sequence of the graphs which have eluded the theoreticians' attention. Going back to the model concepts, it should be noted that so far the complex nucleus has not been described satisfactorily with the aid of a potential in the form of the superposition of Yukawa potentials. This is why the fact that the vertex parts have an infinitely remote essential singularity with respect to the momentum transfer has to be reckoned with in the practical theory of direct nuclear reactions. This means that dispersion relations with respect to the momentum transfer can be written only after the preliminary division of the amplitudes into the suitably selected exponential factors. This device does not involve any intrinsic difficulties since the asymptotic behaviour of the vertex parts is known. Since the amplitudes corresponding to various graphs can, as we have seen in chapters 3 and 4, be obtained by direct calculation of the Feynman integral (3.17) the simplest approach to the estimation of the finite size of a nucleus is to substitute, in the integral of (3.17), the quantities given by Eq. (5.9) (see [13]) instead of the constant vertex parts.

5.4. Reduced vertex parts

The reduced vertex part γ is connected with the decay probability per time unit λ of particle 1 into particles 2 and 3. This decay may occur if $m_1 > m_2 + m_3$. From Eqs (2.24) and (5.1) it follows that

$$\gamma\gamma^* = (\lambda m_3/m_{23}^2)\sqrt{m_{23}/2E}. \qquad (5.28)$$

Our problem is to express λ, and consequently γ, through the wave functions of the single-particle model and continue analytically the formula thus obtained up to negative values of $E = -\epsilon$. The complex energy $W = E - i\lambda/2$ is known to correspond to the unstable particle in quantum mechanics. When λ are small, the following expression holds for λ [14]

$$\lambda = \left[(2m_{23}E)^{\frac{1}{2}}/m_{23}\right]\left[\chi^2_{lE}(R)/\int_0^R \chi^2_{lE}(r)\,dr\right], \qquad (5.29)$$

where

$$\chi_{lE}(r) = r\,\psi_{lE}(r) \qquad (5.29a)$$

and $\psi_{lE}(r)$ is the solution, finite at zero and corresponding to the real energy

E, of a radial equation. Since $\chi_{\ell E}(r)$ is real, λ is real too when $E > 0$. Substituting Eq. (5.29) into Eq. (5.28) we obtain

$$\gamma\gamma^* = (m_3/m_{23}^2)\left[\chi_{\ell E}^2(R)/\int_0^R \chi_{\ell E}^2(r)dr\right] \qquad (5.30)$$

We can now determine γ in the following manner:

$$\gamma = \chi_{\ell E}(R)\left[\sqrt{m_3}/m_{23}\right]\left[\int_0^R \chi_{\ell E}^2(r)\,dr\right]^{-\frac{1}{2}}. \qquad (5.31)$$

Equation (5.31) is an analytical function of E and can be extended to the negative values of $E = -\epsilon$. Equation (5.31) holds for the real potential. If the state of the "decaying" nucleus is not a purely single state, the system will not in general be described by the wave function $\psi_{\ell E}(r)$. The variation of γ due to the dissipation of single-particle states can be described phenomenologically taking into account the "absorption" of particle 3 (or 2) in nuclear matter. For positive energies this operation reduces to the introduction of the complex potential (optical model). Let us see how the expression for the constant λ of the decay of the state with complex energy $W = E - i\lambda/2$ will change. We consider $E>0$ so small that the imaginary part of the potential is small and the effect can be estimated by perturbation theory. Then the change of energy δW is given by the formula [15]

$$\delta W = -i\left(\int_0^\infty \chi_{\ell E}^2(r)U_{2E}(r)dr\Big/\int_0^\infty \chi_{\ell E}^2(r)\,dr\right) \qquad (5.32)$$

Here $U_{2E}(r)$ is the optical potential imaginary part

$$U_E(r) = -U_{1E}(r) - iU_{2E}(r)$$

with the sign reversed*.

Equation (5.32) should in general include the eigenfunctions $\chi_{\ell W}(r)$ corresponding to the complex level W. These functions are complex. It is clear, however, that since λ is small the imaginary part $\chi_{\ell W}^2(r)$ is also small (this quantity is an analytical function of W; the fact that when W is complex we have $\chi_{\ell W}(r) \to \infty$ at $r \to \infty$, is of no importance because of the determination of the integrals in Eq. (32)). Therefore $\chi_{\ell W}(r)$ can be replaced with real functions $\chi_{\ell E}(r)$. Under these conditions δW will be purely imaginary and we shall have

$$\delta\lambda = -2i\delta W = 2\left(\int_0^\infty \chi_{\ell E}^2(r)U_{2E}(r)dr\Big/\int_0^\infty \chi_{\ell E}^2(r)\,dr\right) \qquad (5.33)$$

Hence it is clear that even when $U_{2E}(r) \ll U_{1E}(r)$, i.e. when the absorption

* According to [15], the integral in Eq. (5.32) should be understood in the sense of taking the limit:

$$\int_0^\infty \chi_{\ell E}^2(r)\,dr = \lim_{\alpha\to 0}\int_0^\infty e^{-\alpha r^2}\chi_{\ell E}^2\,dr$$

in nuclear matter is low, λ may decrease appreciably if $U_{2E} \simeq \lambda$. To find how λ and hence γ change in case the state is bound we must continue Eq. (5.33) analytically up to the negative E. To do this we must construct an analytical function of E which would coincide with $U_{2E}(r)$ when $E > 0$. We assume for simplicity that the complex potential well is rectangular. It is known that the depth of the complex potential well of the interaction of particle 3 with nucleus 2 can be expressed through the forward scattering amplitude of particle 3 on the nucleons of nucleus 2:

$$U_E = -(2\pi n/\overline{m}_3)\, f_E(0). \tag{5.34}$$

Here n is the number of nucleons per unit volume, $\overline{m}_3$ is the reduced mass of the nucleon and particle 3, and $f_E(0)$ the amplitude of the scattering forward of particle 3 on a nucleon. Equation (5.34) does not take into account the decrease of the phase volume due to the restrictive operation of the Pauli principle. This effect will be taken into account later on, The amplitude $f_E(0)$ is an analytical function of the energy and can be represented as

$$f_E(0) = \sum_\ell (2\ell + 1) f_\ell(E). \tag{5.35}$$

If the energy $E > 0$ but less than the first threshold of a possible inelastic process in the collision between particle 3 and a nucleon, it follows from the unitarity of the S-matrix that

$$\mathrm{Im}\, f_\ell(E) \equiv (1/2i)(f_\ell - f_\ell^*) = \sqrt{2\overline{m}_3 E}\, f_\ell f_\ell^*. \tag{5.36}$$

Solving Eq. (5.36) with respect to f_ℓ^* we obtain

$$f_\ell^* = f_\ell/[1 + 2i(2\overline{m}_3 E)^{\frac{1}{2}} f_\ell] \tag{5.36a}$$

The first part of Eq. (5.36a) is an analytical function $\tilde{f}_\ell(E)$ of the energy coinciding with $f_\ell^*(E)$ when $E > 0$ (but less than the first threshold). This circumstance enables us to continue analytically Eq. (5.36) up to the negative values of E:

$$(1/2i)(f_\ell - \tilde{f}_\ell) = \sqrt{2\overline{m}_3 E}\, f_\ell \tilde{f}_\ell. \tag{5.37}$$

Equation (5.37) makes it possible to solve the problem of finding the analytical function $\tilde{U}_2(E)$ coinciding for sufficiently small $E > 0$ with the imaginary part of the optical potential U_{2E}:

$$\tilde{U}_2(E) = (\pi n i/\overline{m}_3)(2\overline{m}_3 E)^{\frac{1}{2}} \sum_\ell (2\ell + 1)\,[f_\ell^2/(1 + 2i(2\overline{m}_3 E)^{\frac{1}{2}} f_\ell)]. \tag{5.38}$$

Incidentally $\sqrt{2\overline{m}_3 E}$ when $E > 0$.

The case is especially simple when in the region of small values of $|E|$ any one term, for example the term $\ell = 0$, is essential in Eq. (5.38) (such a situation will practically always take place if in the negative energy region under study the "particle 3 + nucleon system" has no level with a higher value of angular momentum). In this case $\tilde{U}_2(E)$ can be expressed directly through the optical potential. Indeed, since in this case we have

$$f_E(0) = f_O(E) \tag{5.39}$$

then

$$f_0(E) = -(\overline{m}_3/2\pi n)\, U_E. \tag{5.40}$$

Hence we have

$$\widetilde{U}_2(E) = \frac{\sqrt{2\overline{m}_3 E}\; U_E^2}{1 - \frac{i\overline{m}_3\sqrt{2\overline{m}_3 E}}{\pi n}\, U_E} \cdot \frac{\overline{m}_3}{4\pi n}. \tag{5.41}$$

Proceeding to negative energies ($E = -\epsilon$) and assuming that

$$n = 3/(4\pi r_0^3)$$

we obtain

$$U_2(-\epsilon) = U[x/(1 + 4x)] \tag{5.41a}$$

where

$$x = \overline{\kappa}\,\overline{m}_3\; U\, r_0^3/3, \qquad \overline{\kappa} = \sqrt{2\overline{m}_3\,\epsilon}. \tag{5.42}$$

I have mentioned that Eqs. (5.34) and (5.38) - (5.40) do not take into account the restrictive effect of the Pauli principle. Taking the Pauli principle into account, we can write for the complex potential

$$U_E = (2\pi n/\overline{m}_3)\left\{\varphi_1(E)\,\mathrm{Re}\, f_E(0) + i\,\varphi_2(E)\,\mathrm{Im}\, f_E(0)\right\} \tag{5.34a}$$

where $\varphi_1(E)$ and $\varphi_2(E)$ are the functions taking into account the Pauli principle. The function $\varphi_1(E)$ changes rather slowly and can be put equal to unity in the first approximation for small $|E|$. The function $\varphi_2(E)$ for $|E|$ close to zero is less than unity. If the nucleus were an ideal nucleon Fermi gas (it is in this approximation that U_{2E} are calculated when $E > 0$ [16]), $\varphi_2(E)$ would vanish when $E < 0$. If, however, the diffuseness of the Fermi distribution surface is taken into account $\varphi_2(E)$ does not vanish either when $E = 0$ or when $E < 0$. Thus Eq. (5.38), taking into account the Pauli principle, must be rewritten as

$$\widetilde{U}_2(E) = -\frac{i\pi n\sqrt{2\overline{m}_3 E}}{\overline{m}_3}\,\varphi_2(E)\,\sum_{\ell}(2\ell+1)\,\frac{f_\ell^2}{1 + 2i\sqrt{2\overline{m}_3 E}\, f_\ell}.$$

Equations (5.41) and (5.41a) do not in general hold since Eq. (5.40) is not valid. If, however, the imaginary part $f_0(E)$ is small when $E < 0$ Eq. (5.40) applies approximately. In this case we obtain

$$\widetilde{U}_2(-\epsilon) = U[x/(1 + 4x)]\varphi_2(-\epsilon) \tag{5.41b}$$

where U is real under the assumption made. It is precisely this situation that takes place for neutrons, for example, when we have (see Eq. (2.16))

$$f_0(E) = i/(\sqrt{m_0 E} - i\alpha).$$

The function $\varphi_2(E)$ when $E \leqslant 0$ can be regarded as a constant and probably

equal in order of magnitude to $(\Delta E_f/E_f)^2$ where ΔE_f is the diffuseness of the Fermi distribution surface and E_f is a Fermi energy limit. If the diffuseness of the Fermi surface is due only to pair interactions, we have

$$\Delta E_f/E_f \approx 1/20.$$

Substituting Eq. (5.38a) or (5.41b) into Eq. (5.33) we obtain a formula taking into account the effect of the complexness of the potential on the reduced vertex part. Bearing in mind that for the bound state for a sufficiently large R we can put

$$\int_0^R \chi_{\ell E}^2 \, dr \approx \int_0^\infty \chi_{\ell E}^2 \, dr$$

and normalizing the last integral to unity, we obtain

$$\lambda = \frac{i\sqrt{2m\epsilon}}{m} \chi_{\ell\epsilon}^2 (R) - 2\tilde{U}_2(-\epsilon) \qquad (5.42a)$$

$$\gamma^2 = (m_3/m_{23}^2) \left\{ \chi_{\ell\epsilon}^2 (R) + i(2m_{23}/\epsilon)^{\frac{1}{2}} \tilde{U}_2(-\epsilon) \right\} \qquad (5.43)$$

where $\tilde{U}_2(-\epsilon)$ is given by Eq. (5.38a) or approximately by Eq. (5.41). The functions $\chi_{\ell\epsilon}(r)$ are normalized so that

$$\int_0^R \chi_{\ell\epsilon}^2 (r) \, dr = 1.$$

It was shown above that Eq. (5.41b) must be a fairly good approximation to Eq. (5.38a) for nucleons. It is not impossible that the same applies to other light particles whose nuclear scattering is described satisfactorily for low energies by the optical potential with a small imaginary part. Equation (5.41b) includes only one quantity which is not measured directly in experiment, that is the function $\varphi_2(E)$ at E close to 0. However, this quantity can be calculated for estimations in a similar way to that of SITENKO [8] for nucleons. In the case of nucleons, when $\epsilon = 5$ MeV, $r_0 = 1.4$ fermi, $U_1 = 40$ MeV and $U_2 = 0$, $\varphi_2(-\epsilon) \approx 2 \times 10^{-3}$ * we obtain by Eq. (5.41b) $\tilde{U}_2(-\epsilon) = 0.05$ MeV. It is significant that since $\tilde{U}_2(-\epsilon)$ is real in this example, according to Eq. (5.43), our taking into account the diffuseness of nucleon levels increases the absolute value of the reduced vertex part γ and makes the quantity γ complex.

On the basis of what we have said we can draw the following conclusions:

(1) The one-particle effect is essentially different for positive and negative energies: in the former case it decreases substantially λ and consequently $|\gamma|^2$ and in the second case $|\gamma|^2$ may even increase as compared with the purely single-particle model. Therefore the values of particle widths cannot in general be used for the decay of the compound nucleus in direct process theory.

(2) The non-single-particle effect tells more strongly on the quantity $|\gamma|^2$ the lower the binding energy is at the vertex. If the particle-nucleon

* This estimate is based on the smallness of the reduced neutron widths when E = 0.

scattering amplitude is real for negative energies (as is the case for nucleon-nucleon scattering) the non-single-particle effect on the quantity will be small.

Concluding our consideration of vertex parts, we should emphasize that the procedure described of an analytical continuation of the optical potential imaginary part is still rather imperfect. It can probably be substantially improved if the optical potential is expressed through the amplitudes of scattering of incident particles not on free nucleons but on nuclear quasi-particles. The modern Fermi liquid theory should be brought into play to solve the problem in question.

In the case of triangle graphs a knowledge of the virtual reaction amplitude is also essential as well as that of vertex parts. The amplitude corresponding to a triangle graph contains, as seen in chapter 4, the integral of the virtual reaction amplitude over the unphysical region of the variables. Unfortunately, the behaviour of the amplitudes of the reactions between nucleon-deficient nuclei in the unphysical region is insufficiently known, except for the case of nucleon-nucleon scattering at low energies.

6. KNOCK-OUT REACTION

6.1. Kinematic problems

We shall refer to all processes in which the number of final particles is more than two for knock-out reactions. The simplest of such processes are reactions of the type

$$A + x \longrightarrow B + y + z . \tag{6.1}$$

The theory of knock-out processes is far more complicated that the theory of reactions (1.1) considered in the previous lectures. The fact is that the knock-out reaction amplitude depends on a larger number of variables. Let N be the total number of particles involved in the reaction (i.e. the sum of the numbers of initial and final particles). The momenta and energies of these particles are connected by the conservation laws

$$P_A + P_x = P_B + P_y + P_z + \ldots + P_f \tag{6.2}$$

and the conditions

$$P_i^2 = m_i^2 \qquad (i = A,\ x,\ B,\ y \ldots f). \tag{6.3}$$

The number of the components of 4-momenta is 4N. Equations (6.2) and (6.3) establish N + 4 relations between them. Thus the number of independent components is 3N - 4. How many independent quantities, <u>invariant</u> for the transformation of the reference system, can be made up of these 3N - 4 independent components? Since the number of parameters of the reference system transformation is 6 (3 Euler angles and 3 components of the velocity of the new reference system) the number of the wanted independent invariants is clearly 3N - 10 since any invariant can be calculated in any reference system and in particular in one where any 6 of 3N - 4 independent components of 4-momenta have preset values.

For reaction (6. 1) for example, N = 5 and consequently the number of independent invariants on which the amplitude of the reaction depends is also 5. We can choose, for example,

$$s_{Ax},\ s_{Bz},\ s_{zy},\ t_{zx},\ t_{xy} \tag{6.4}$$

or

$$s_{Ax},\ s_{Bz},\ s_{zy},\ s_{By},\ t_{xy} \tag{6.5}$$

as such 5 invariants.

Here s_{Ax} and t_{xy} are the invariants s and t we have used before, t_{AB} can be obtained from t_{xy} by replacing x and y with A and B, and s_{Bz}, s_{By} and s_{yz} by replacing in s_{Ax} the particles x and A with the particles (B, z), (B, y) and (y, z) respectively. The sets of variables (6. 4) and (6. 5) are expressed through each other linearly. The invariants S_{ij} are expressed through the total kinetic energy of the particles i and j in their centre-of-mass system (see chapter 2). Note that the use of invariant variables for knock-out reactions is especially convenient.

6. 2. Movement of singularities

Thus the theory of knock-out reactions must deal with the analytical functions of at least five variables. The properties of such functions may be quite complicated because the position of singularities over some variables may depend on the values of other variables, and not only on the masses of particles, as was the case for the reactions (1. 1). Evidently, this will be the general situation for the amplitude of direct nuclear reactions corresponding to different Feynman graphs. This can readily be seen from the graph represented in Fig. 23.

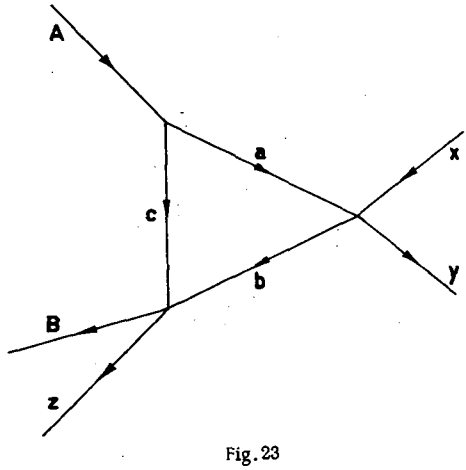

Fig. 23

This graph differs from graph 2 only in that the nucleus has been replaced by two nuclei B and z. The position of the triangle singularity t_Δ of graph 2 is given by Eq. (4. 8) which contains the quantity

$$\epsilon^{B}_{cb} = m_b + m_c - m_B.$$

Since $m_B = \sqrt{P_B^2}$, in the case of graph 23 the mass m_B is replaced by the quantity

$$m_B \longrightarrow \sqrt{(P_B + P_z)^2} = m_B + m_z + s_{Bz} / 2(m_B + m_z) \qquad (6.6)$$

and instead of the constant ϵ_{bc}^{B} Eq. (4.8) contains the variable

$$\epsilon_{bc}^{B+z} = Q_{bc} - s_{Bz} / 2(m_B + m_z). \qquad (6.7)$$

The second term in Eq. (6.7) is simply the kinetic energy in the centre-of-mass system of the particles B and z. This energy changes with the energy of the colliding particles A and x, the emission angles z with respect to x, etc. and consequently the singularity of graph 23 t_Δ will shift in the complex plane of the variable t, t_Δ also assuming complex values. Equation (4.8) for t_Δ holds in the case of graph 23; we have only to continue it analytically over the mass of the particle B, i.e. over ϵ_{bc}^{B}. For the analytical continuation of Eq. (4.8) over ϵ_{bc}^{B} we must choose the branch of $\sqrt{\epsilon_{bc}^{B}}$. This branch is so chosen for example, that for real $\epsilon_{bc}^{B} > 0$ we should have

$$\sqrt{\epsilon_{bc}^{B}} > 0. \qquad (6.8)$$

From Eq. (4.8) we can see that t_Δ may prove quite near to the physical region. For example, when $\epsilon_{bc}^{B+z} = 0$, t_Δ will be the same in order of magnitude as t_0 for pole graphs. The movement of t_Δ in the complex plane t_{xy} is shown for graph 23 in Fig. 24.

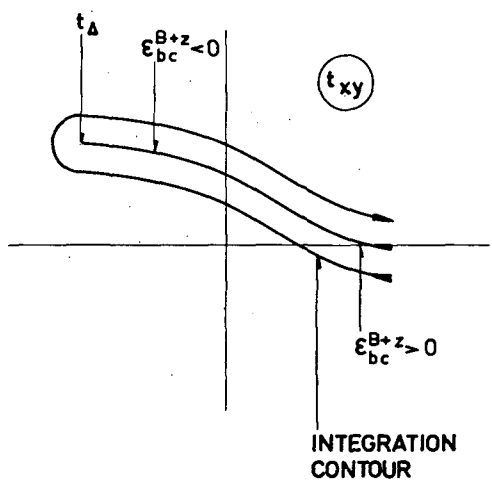

Fig. 24

The amplitude of graph 23 given by the formulae of chapter 4 is analytical in the plane t_{xy} cut along the line of the movement of t_Δ from a given value t_Δ to $+\infty$, M_Δ being taken (in accordance with the imaginary correction $i\epsilon$ for real ϵ_{bc}^{B+z}) on the upper side of the cut.

The following conclusions can be drawn from what has been said above. First, the relation between the contributions from different graphs to the

amplitude of the process (6. 1) depends essentially not only on the masses of the particles involved, but also on the energies of the outgoing particles so that different graphs may prove decisive in different regions of the energy spectrum of the products of the reaction. Secondly, in the case of the reactions (6. 1) and other knock-out reactions there is no general rule concerning the removal of singularities from the physical region as the graph becomes more complicated, for the positions of the singularities depend on the values of the kinematic invariants. A case in point is the triangle graph 23 whose complex singularities may lie nearer to the physical region than the pole with respect to t_{xy} of the graph shown in Fig. 25.

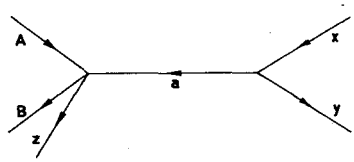

Fig. 25

Despite the above-mentioned difficulties, the treatment of reactions of the type (6. 1) on the basis of dispersion theory proves useful for one reason because in terms of this theory we can understand an otherwise mysterious effect like the "quasi-elastic knock-out" from nuclei complex particles like d, t, He^3, α and the capture of π^- - and K^--mesons by nuclear clusters in complex nuclei.

In our next section we consider such a reaction as an example.

6. 3. Knock-out of complex particles

It must be clear from what has been said before that in the theory under study the calculation of the knock-out of complex-particles (d, t, α etc.) does not necessitate the assumption that they exist in the nucleus for a long enough time compared with the collision time (in other words, there is no need to describe the state of these particles by wave functions). According to the formalism developed here, nuclei "consist of" deuterons and α-particles, in the same sense in which a neutron is said to consist of a proton and π^--meson. In other words, even a stable nucleus is a dynamic system virtually emitting and absorbing back all kinds of particles. These virtual particles make up the nuclear periphery just as the virtual π-mesons form the periphery of a nucleon.

This means that the "knock-out" of complex particles should be understood as the "knock-out" of a π-meson from a nucleon is understood. This

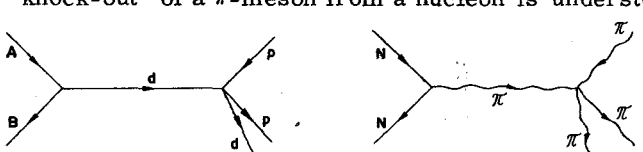

Fig. 26

circumstance is illustrated by Fig. 26 representing the pole graph of the reaction (p, pd) and the Chew and Low graph for the process $\pi + N \to N + 2\pi$.

The same applies to reactions like the capture of π^-- or K^--mesons by nuclear clusters. The direct production of deuterons and tritons in the capture of stopped π^--mesons by the light nuclei of photoemulsion (C^{12}, O^{16}) may furnish an example of the process in question. This reaction was observed in [18] and treated theoretically in [19].

The simplest pole graphs for this reaction on the C^{12} nucleus are shown in Fig. 27, the position of the pole t_0 over the variable t_{AB} being indicated for each graph.

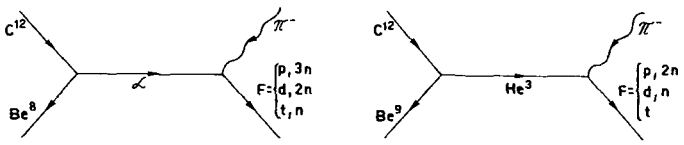

Fig. 27

The boundary of the physical region t over the variable t_{AB} is zero for all reactions except C^{12} (π^-, t) Be^9. In the latter case $t_m = -684$ MeV. AMU. Since according to Eq. (2.47) the square of the absolute value of the pole amplitude at the physical region boundary $\sim 1/(t_m + t_0)^2$, the yield of tritium by graph (27b) must be small compared with graph 27a. The yield of p and d by graph 27b must also be smaller than by graph 27a, though the remoteness of the pole in the case of 27b is somewhat compensated by the increase of the phase volume (a smaller number of particles in the final state). Note that an appreciable contribution to the reaction in question may also come from the triangle graphs since the nearest triangle singularity lies at $t_\Delta = 108$ MeV · AMU (Fig. 28). It is clear, however, that graph 27a is the

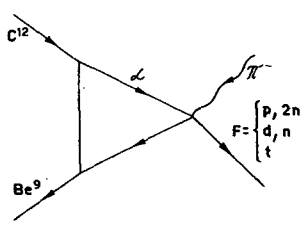

Fig. 28

main one and must account at least for a considerable part of the entire effect observed. According to graph 27a π^- is captured by an α-particle virtually emitted by the C^{12} nucleus. By Eq. (2.47) the amplitude of this process could be expressed by the amplitude of the capture of π^--mesons by a free α-particle. The latter amplitude can be regarded as a constant in the first approximation. For the sake of simplicity we shall regard the amplitude of the virtual α-decay of C^{12} also as a constant. Note that that the reduced vertex part of the α-decay of C^{12} will not enter the relative probability for the yield of p, d and t. Now, the dependence of the vertex part on t_{AB} will not strongly affect the energy spectrum of final particles (because there are many of them) and hence still less the relative yields (since these quantities are given by integrals over energy spectrum).

Starting from Eq. (2.47) we obtain the following expression for the differ-

ential probability of the capture of a π^--meson by the C^{12} nucleus for a certain unit of time

$$d\lambda = \frac{2m\alpha}{(2\pi)^{3n-5}} \cdot \frac{|\gamma|^2 \, |M_{\gamma\beta}|^2}{(t_{AB} - t_0)^2} \, \delta\left(\sum_i \vec{p}_i\right) \delta\left(\sum_i E_i - Q\right) \prod_{i=1}^{n} d^3 p_i. \qquad (6.9)$$

In Eq. (6.9) γ is the reduced vertex part of the α-decay of C^{12}, $\vec{p}_i$ and E_i are the momenta and kinetic energies of the final particles

$$Q = m_{C^{12}} + m_\pi - \sum_i^n m_i \qquad (6.10)$$

and $M_{\alpha\beta}$ is the amplitude of the capture of a π^--meson by an α-particle at rest. The amplitude $M_{\gamma\beta}$ is connected with the differential probability $d\lambda'$ for the capture of a pion at rest by an α-particle at rest with a formula similar to Eq. (6.9):

$$d\lambda' = \frac{|M_{\beta\gamma}|^2}{(2\pi)^{3n'-4}} \, \delta\left(\Sigma \vec{p}_i\right) \delta\left(\sum_i E_i - Q' - E_\pi\right) \prod_{i=1}^{n'} d^3 p_i \qquad (6.11)$$

$$Q' = m_\alpha + m_\pi - \sum_{i=1}^{n'} m_i. \qquad (6.12)$$

Note that the taking into account of the pole dependence on t_{AB} in Eq. (6.9) is a certain over-estimation of the accuracy since we have already neglected the dependence of the vertex part on t_{AB} which is at least as strong. It would be more consistent, therefore, to replace t_{AB} in Eq. (6.9) by its value t_m at the physical region boundary (it has been pointed out that $t_m = 0$ for graph 27a). However, we have conserved the pole dependence on t_{AB} in Eq. (6.9) in order to trace how strongly it distorts the energy spectrum of final particles as compared with the spectrum conditioned by the phase volume.

Using Eqs. (6.9) and (6.11) we can easily obtain the relation between the ratios of the yields of different particles in the capture of π-mesons by a carbon nucleus and α-particle

$$\lambda_F/\lambda_p = (\lambda'_F/\lambda'_p)(C_F/C_p) \qquad (6.13)$$

where

$$C_F = \int_a^{b_F} \sqrt{\frac{x - a}{x^3}} \, dx \qquad (6.14)$$

and

$$a = 0.05 \qquad (6.15)$$

$$b_p = 0.79, \qquad b_d = 0.81, \qquad b_t = 0.85. \qquad (6.16)$$

From these data it follows that according to graph 27a the relations

$$\lambda_F/\lambda_p = \lambda'_F/\lambda'_p \qquad (6.17)$$

must hold with a rather high degree of accuracy. Unfortunately, the available experimental data on the capture of π^--mesons by He^4 nuclei are quite inaccurate and contradictory. Those given in [20] seem to be the best and according to this investigation we have

$$\lambda'_d/\lambda'_p \simeq 1/3 \qquad \lambda'_t/\lambda'_p \approx 0.5 \text{ to } 0.7. \qquad (6.18)$$

However, these quantities are obtained with low statistical accuracy. The relation

$$\lambda'_t/(\lambda'_t + \lambda'_d + \lambda'_p) = 1/3 \qquad (6.19)$$

is measured in this investigation much better. Note that this figure sharply differs from the data of AMIRAJU and LEDERMAN [21] who suggest that the probability for the discharge of tritium in the capture of π^--mesons by an α-particle is 1/60.

In the investigation by VAISENBERG et al. [18], quoted before, dealing with the capture of π^- by C^{12} nuclei, the ratio $(\lambda_\alpha + \lambda_t)/\lambda_p$ is measured most confidently. This quantity is in good agreement with the data of (6.18). However, the value of λ_t/λ_p differs from (6.18) at least by a factor of two. Table IV compares the experimental data of [18] with the calculation for graph 27a in which the results of [20] on the capture of π^--mesons by He^4 nuclei are used.

TABLE IV

RELATIVE YIELD OF p, d AND t IN THE CAPTURE OF π^--MESONS IN C^{12} AND O^{16}

Final products	Experiment [18]	Theory [19]
p	1	1
d	0.75 ± 0.07	0.3 to 0.4
t	0.14 ± 0.10	0.5 to 0.7
d + t	0.91 ± 0.11	0.8 to 11

The experimental [18] and theoretical [19] spectra of deuterons emitted in the capture of π^--mesons by C^{12} and O^{16} nuclei are compared in Fig. 29. The solid line in Fig. 29 plots the curve corresponding to graph 27a, taking into account the pole dependence and constant vertex part for the decay of C^{12}. The dashed line represents the curve corresponding to the neglect of the pole dependence. The solid and dashed curves differ little, as is to be expected.

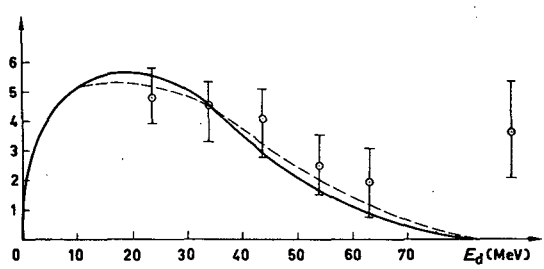

Fig. 29

Figure 29 also shows that the agreement between the theoretical and experimental results is satisfactory. This circumstance shows that graph 27a gives the main contribution to the probability for the reaction since the shape of the deuteron energy spectra differs for graphs 27b and 28. In particular, the pole dependence and the dependence of the vertex part on t_{AB} are more strongly pronounced in these spectra. Moreover, the curves corresponding to the phase volume differ since the number of final particles in graphs 27b and 28 is smaller than in graph 27a. The deuteron energy spectra corresponding to graph 27b and 28 are similar to those of tritium for graph 27a. A spectrum of the latter kind is shown in Fig. 30, a solid curve corresponding to the estimation of the pole dependence and a dashed one to the phase volume, as in Fig. 29. At present there are no experimental data on the energy spectrum of tritium produced in the capture of π^--mesons by C^{12} nuclei.

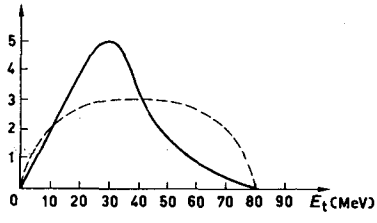

Fig. 30

In conclusion it should be emphasized that the experimental data on the capture of π^-- and K^--mesons by light and lightest nuclei are still meagre and the study of these reactions should be continued since they are of interest at least in two aspects. First, the study of such reactions will enable the validity of the general concepts on the nature of direct processes to be checked, and secondly, the measurement of the absolute probabilities for these processes when they are described by pole graphs of the type 27a and 27b enables the reduced vertex parts of virtual decays with emission of complex particles to be estimated experimentally. If the underlying concepts of the theory are correct, the values of reduced vertex parts thus obtained must agree with the experimental results of the investigation of other reactions and, in particular, that of the knock-out of complex particles (such as (p, pα), (p, pd) etc.) in a wide energy range of the initial beam.

7. INTERACTION IN INITIAL AND FINAL STATES

7.1. Distorted waves method

The distorted waves method is often used to take into account the interactions in the initial and final states of the reactions (1.1).

The method can be reduced essentially to the following. The amplitude of the reaction (1.1) is calculated by the perturbation theory formula

$$M_{\alpha\beta} = \int \psi_\beta^* H' \psi_\alpha dv \qquad (7.1)$$

where H′ is the Hamiltonian of the energy of the interaction causing the reaction and regarded as perturbation, and ψ_α and ψ_β are the wave functions of the initial and final states unperturbed by the interaction H′ but taking into account the distorting effect of the interactions of the particles x and A and the particles y and B on the motion of the centres of gravity of these particles. In other words we have

$$\psi_\alpha = \psi_A \psi_x \psi_{Ax} \tag{7.2}$$

$$\psi_\beta = \psi_B \psi_y \psi_{By} \tag{7.3}$$

where ψ_A, ψ_x, ψ_y and ψ_B are the inner wave functions of the particles A, x, y and B, and ψ_{Ax} and ψ_{By} are the wave functions of the relative motion of the centres of mass of the particles A and x, and B and y. In the method of distorted waves ψ_{Ax} and ψ_{By} are represented by wave functions describing scattering on a complex potential, i.e. the wave functions of the optical model satisfying the Schrödinger equation (see [12], for example)

$$-(1/2\,m_{Ax})\nabla^2\,\psi_{Ax}(\vec{r}_{Ax}) + U(\vec{r}_{Ax})\;\psi_{Ax}(\vec{r}_{Ax}) = E\,\psi_{Ax}(\vec{r}_{Ax}) \tag{7.4}$$

$$U(r_{Ax}) = -U_1(r_{Ax}) - i\,U_2(r_{Ax}) \tag{7.5}$$

where $U(r_{Ax})$ is the complex potential. The function ψ_{By} satisfies a similar equation.

This technique is unsatisfactory for the following reasons:

(1) The operator H′ is a Hamiltonian of a strong (nuclear) interaction to which perturbation theory is inapplicable;

(2) If this circumstance is ignored, Eq. (7.1) of perturbation theory can be derived under the assumption that the unperturbed wave functions ψ_α and ψ_β are orthogonal, which is not the case. Note that because of the complexity of the potential not even the functions ψ_{Ax} (or ψ_{By}) corresponding to different values of the total kinetic energy E of the particles A and x (or B and y) are orthogonal.

(3) It is known that the optical model satisfactorily predicts the scattering amplitude. However it is not yet clear how adequately it describes the wave function of relative motion near the nucleus.

Therefore, the method of distorted waves cannot be justified theoretically (or at least no one has succeeded in doing so) and it is used mainly because the conventional theory of nuclear reactions has been unable to suggest any other more rigorous and at the same time no less practical method. A comparison of experimental data with the predictions by the distorted waves method sometimes yields satisfactory results, while substantial discrepancies are observed at other times. The causes of these discrepancies are not clear mainly because the causes of the agreement, whenever it is observed, are not clear. The dispersion theory enables us to obviate these difficulties. The theory furnishes a theoretically substantiated and practical method of taking into account the interactions in the initial and final states, and this method proves even somewhat simpler than the distorted waves method as far as calculations are concerned. Another merit of the new method is that the effects of the interaction in the initial and final states are expressed directly through the experimentally observed phases of scattering of the particles x on the nucleus A and the particles y on the nucleus B. For

this reason the optical model is used in the new method as prescribed, so to speak, that is for finding the scattering phases, and not the wave functions near the nucleus.

7.2. Graphs taking into account the interactions in the initial and final states

Assuming that the singularities of the amplitude of the direct process (1.1.) with respect to the variable t (or u) are given by graphs of the type 1 and 1a (or 1b). Then the graphs shown in Fig. 31 will correspond to taking into account the interaction in the initial state.

In these graphs one of the virtual reactions which may recur an unlimited number of times is the scattering of the particle x on the nucleus A. Let us consider the first of the graphs of Fig. 31. This graph has no non-

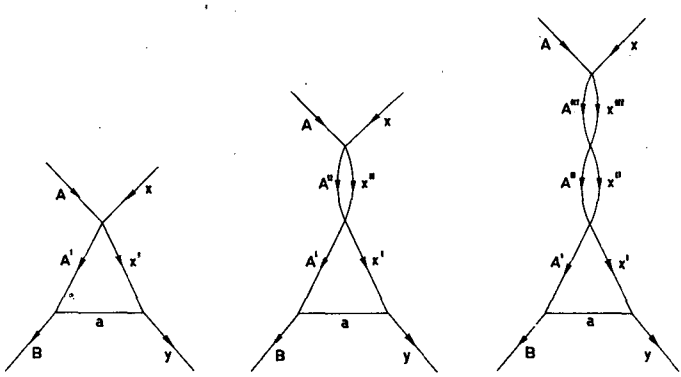

Fig. 31

relativistic singularities with respect to the variable t but has a singularity with respect to s when s = 0 since it is contained in a "contracted graph" shown in Fig. 31a.

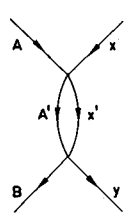

Fig. 31a

This can easily be seen if we turn to the general formula (3.17) for one-contour graphs. Indeed, in the case under study the denominator has only one quantity $r_{ij} + 2m_{ij} Q_{ij} \equiv s$. Therefore the denominator vanishes when s = 0 and consequently graph 31a has a trivial singularity with respect to s when s = 0. All other graphs of Fig. 31 have the same singularity since we can readily see from Eq. (3.7) that each of them is the product of Feynman integrals for one-contour graphs of the type shown in Fig. 31. The case is very similar for the interaction in the final state.

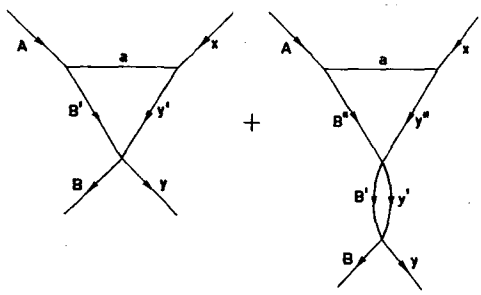

Fig. 32

In this case we shall have graphs 32, the singularities of which for s lie at $s = 0$ if $Q < 0$ (i.e. if we have a threshold reaction) and at $s = -Q$ if $Q > 0$, in other words, at $\epsilon = E + Q = 0$ where ϵ is the total kinetic energy of the particles B and y in the centre-of-mass system. Besides graphs 31 and 32 there may be graphs of the type shown in Fig. 33 which possess the singularities of graph 31 as well as those of graph 32. Since all graphs of Figs. 31, 32 and 33 possess the same singularities none of them can be

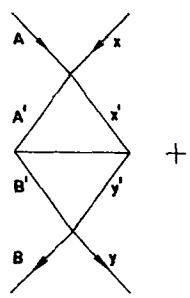

Fig. 33

neglected when finding the total amplitude unless the scattering amplitudes are small for some reason or other, which is not generally so in the case of nuclear scattering under consideration. Thus we face the need for finding the sum of an infinite series of graphs. The simplest way of solving this problem is not the direct summation of the Feynman integrals corresponding to graphs 31-33 but by using the conditions of unitarity and dispersion relation (2.5) for the variable s or, which is the same, for the energy E of the colliding particles in their centre-of-mass system. This leads us to a graph of the type shown in Fig. 8 in which scattering is one of the intermediate reactions. It was mentioned in chapter 3 that the consistent employment of the analyticity and unitarity principles is possible in this case without recourse to relativistic formulae.

7.3. Omnes-Muskhelishvili equation

Let us assume that the intermediate particles b and c in graph 8 are the particles A and x. The first intermediate process will then be the scattering of the particles x on the nucleus A and the second intermediate process re-

action (1.1) itself. Note that the matrices T and T^+ are interchangeable with respect to the unitarity (2.33). Accordingly, the unitarity relation (3.3) can be re-written as

$$A^{(2)}_{\alpha\beta} = (m_{Ax}\, p/8\pi^2) \int M^{+}_{\alpha\alpha'} M_{\alpha'\beta}\, d\Omega_{x'} \qquad (7.6)$$

where α' designates the intermediate states γ to emphasize the identity of the initial and intermediate particles. Let us now pass from the amplitude $M_{\alpha\alpha'}$ to the function $f_{\alpha\alpha'}$ usually referred to as the scattering amplitude.

$$M_{\alpha\alpha'} = (2\pi/m_{Ax}) f_{\alpha\alpha'}\,, \qquad (7.7)$$

its square of the absolute value being equal, according to Eq. (1.2), to the scattering differential cross-section. Equation (7.6) will then take the form

$$A^{(2)}_{\alpha\beta} = (p/4\pi) \int f^{*}_{\alpha\alpha} M_{\alpha'\beta}\, d\Omega_{x'} \qquad (7.8)$$

and

$$p^2 = 2m_{Ax} E \qquad (7.9)$$

where E is the total kinetic energy of the particles x and A in the centre-of-mass system. Quite similarly, we can take into account the intermediate states γ in which the particles b and c are y and B. Taking these states into account the two-particle relation of unity can be written as

$$A^{(2)}_{\alpha\beta} = (p/4\pi) \int f^{*}_{\alpha\alpha} M_{\alpha'\beta}\, d\Omega_{x'} + (k/4\pi) \int M_{\alpha\beta'} f^{*}_{\beta\beta'}\, d\Omega_{y'} \qquad (7.10)$$

where

$$k^2 = 2m_{By}\epsilon\,, \qquad \epsilon = E + Q \qquad (7.11)$$

and $f_{\beta\beta'}$ is the amplitude of scattering of the particles y on the nucleus B.

Let us now consider the analytical properties of the amplitude of the reaction $M^{(2)}_{\alpha\beta}$ with respect to the variable s or, which is the same, over the variable E; it is assumed at first that $Q < 0$. In this case, the second term in Eq. (7.10) vanishes when $E = -Q$ because of the factor k and remains zero when $E < -Q$ since the first intermediate reaction $\alpha \to \beta'$ is forbidden under this condition by the conservation laws. The first term in Eq. (7.10) vanishes when $E = 0$ because of the factor p and remains zero when $E < 0$ since the first (for this term) intermediate reaction $\alpha \to \alpha'$ (scattering of the particles x on the nucleus A) becomes impossible. Thus the imaginary part of the amplitude $M^{(2)}_{\alpha\beta}$ vanishes at $E \leq Q$. Condition (1) of section 2.1 is thereby satisfied. If we now assume that the other two conditions of section 2.1 are fulfilled we can, using the general formula (2.5), express the amplitude $M^{(2)}_{\alpha\beta}(t, E)$ through its imaginary part (7.10), $M^{(2)}_{\alpha\beta}(t, E)$ being analytical in the complex plane of E with a cut from 0 to ∞. Note first of all that in the present case we are interested in the value of $M_{\alpha\beta}(t, E)$ at the cut since the physical region of the variable E extends from 0 to ∞. Since the integrand f(x) in Eq. (2.5) is the imaginary part of the function $F(x + i0)$ and we identify $A_{\alpha\beta}(t, E)$ precisely with the imaginary part of the wanted amplitude $M_{\alpha\beta}(t, E)$, it is clear that the latter will be given as the value of the function yielded by Eq. (2.5) on the upper side of the cut. Apart from the two-particle terms of (7.10), the imaginary part $A_{\alpha\beta}(t, E)$ contains other terms $A^{(\omega)}_{\alpha\beta}(t, E)$ which have no amplitude of the reaction (1.1) itself. Thus we have

$$A_{\alpha\beta} = A^{(0)}_{\alpha\beta} + A^{(2)}_{\alpha\beta} \tag{7.12}$$

On the strength of what has been said we can, using Eq. (2.5), write

$$M(t, E) = M_0(t, E) + \frac{1}{4\pi^2} \int_0^\infty \frac{p(E')f^*(E')M(t, E')\,d\Omega_{x'}}{E' - E - i\eta}\,dE'$$

$$+ \frac{1}{4\pi^2} \int_0^\infty \frac{k(E')h^*(E')\,M(t', E')d\Omega_{y'}}{E' - E - i\eta}\,dE'. \tag{7.13}$$

In Eq. (7.13) the subscripts α, α', β and β' are dropped and the notation

$$f_{\beta\beta'}(\epsilon) = h(E) \tag{7.14}$$

is introduced and $M_0(t, E)$ designates those terms of the amplitude which correspond to the imaginary part $A^{(0)}_{\alpha\beta}$ and which vanish if $A^{(0)}_{\alpha\beta} \equiv 0$. Equation (7.13) is the integral equation with respect to the wanted amplitude $M(t, E)$ with the kernel

$$\frac{p(E')f^*(E') + k(E')h^*(E')}{E' - E - i\eta}. \tag{7.15}$$

We shall call Eq. (7.13) the Omnes-Muskhelishvili equation. Equations of the type (7.13) were first investigated mathematically by MUSKHELISHVILI [11], and OMNES [22] pointed to the importance of this equation in the dispersion theory of the strong interactions of elementary particles. Equation (7.13) was applied to the theory of direct nuclear reactions in [2].

We have said that Eq. (7.13) holds if the conditions formulated in section 21 are fulfilled. According to the condition (2), all branch points of the amplitude $M(t, E)$ as a function of E must lie on the real axis when $E \geq 0$. The properties of the solution of the integral equation are determined by the properties of the kernel (7.15) and the properties of the free term $M_0(t, E)$. If the kernel (7.15) can be integrated throughout the interval $0 < E' < \infty$ for all $E \neq 0$, condition (2) reduces to the requirement that there should be no branches which do not lie on the real axis when $E \geq 0$ in the free term $M_0(t, E)$. The physical meaning of this term is obvious since when $f \to 0$ and $h \to 0$, $M \to M_0$. The free term is thus the amplitude of the process without taking into account the interaction in the initial and final states. Hence it follows that $M_0(t, E)$ is given by one of the graphs or a sum of the graphs, considered in chapters 2 - 4, which have no branch points with respect to s of the type of the triangle singularities (4.52). We have seen that this condition is fulfilled for all graphs of interest for the theory of direct reactions and in particular for pole graphs 1 and 1b and triangle graphs 2.

Condition (3) of section 2.1 means, as applied to the case under study, that $f^*(E)$ and $h^*(E)$ must decrease sufficiently rapidly when $|E| \to \infty$. If this is not the case, then instead of $M(t, E)$ we should, as indicated in section 2.1, consider the function $\tilde{M}(t, E) = M(t, E)/\varphi(E)$ satisfying the requirement in question and write the dispersion relation (7.13) for this function. In the following we shall assume that condition (3) is fulfilled for the partial scattering amplitudes $f^*_l(E)$ and $h^*_l(E)$.

Note, that by definition the quantities p and k in eq. (7.13) are positive when $E > 0$, which corresponds to the branch

$$\mathrm{Im}\, p = \mathrm{Im}\sqrt{2m_{Ax}E} > 0. \qquad (7.16)$$

When writing (7.13) we proceeded from the assumption that $Q < 0$. If $Q > 0$ the lower limit of integration in the second integral of Eq. (7.13) should be replaced by $-Q$. Obviously, such an analytical continuation over the lower integration limit is possible since in the process we do not go beyond the physical region with respect to the variable ϵ (see Eq. (7.11)) which is the total kinetic energy of the particles B and y. For this reason the scattering amplitude $h(E)$ remains with $Q > 0$ regular in the interval $-Q \leq E \leq 0$.

Let us now proceed to the solution of Eq. (7.13). To demonstrate the essence of the method, we shall consider the solution of Eq. (7.13) with one integral term (i.e. taking into account the interaction only in the initial or only in the final state). Taking into account two integral terms does not change the essence of the matter whereas the formulae become somewhat more cumbersome.

As will be clear later on, Eq. (7.13) can be solved accurately without resort to the iteration procedure. If, on the other hand, we perform iterations over the free term $M_0(t, E)$ and represent the solution as the sum of several consecutive iterations, each term of this series will represent one of the graphs shown in Figs. 31-33.

Please note also that Eq. (7.13) can be represented graphically as in Fig. 34 in which a cross designates the free term, a rectangle the wanted amplitude and circles the scattering amplitudes f^* and h^*.

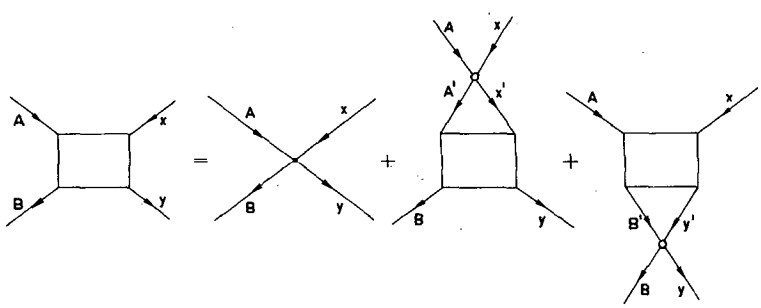

Fig. 34

7.4. Solution of the Omnes-Muskhelishvili equation

The amplitude $M(t, E)$ can be regarded as a function of the variables $M(z, E)$ where z is the cosine of the angle between the direction $\vec{p}_x$ and $\vec{p}_y$. The scattering amplitude f is in turn a function of the cosine of the scattering angle, i.e. the angle between the directions $\vec{p}_x$ and $\vec{p}_x$ where $\vec{p}'_x$ is the momentum of the particle in the intermediate state. The required amplitude of the process in the integrand depends on the variables $M(z', E')$ where z' is the cosine of the angle between $\vec{p}_x$ and $\vec{p}'_y$. Using the expansions

$$M(z, E) = \sum_l (2l + 1) M_l(E) P_l(z) \qquad (7.17)$$

$$f(z, E) = \sum_l (2l + 1) f_l(E) P_l(z) \qquad (7.18)$$

where $P_\ell(z)$ are the Legendre polynominals, using also the formula

$$z' = z_1 z - \sqrt{1 - z_1^2}\sqrt{1 - z^2}\cos(\varphi_1 - \varphi) \tag{7.19}$$

and applying then the well-known theorem for the addition of spherical functions, we can perform the integration over $d\Omega_{x'}$ after which Eq. (7.13) reduces to the equation for the partial amplitudes $M_\ell(E)$:

$$M_\ell(E) = M_{0\ell}(E) + \frac{1}{\pi}\int_0^\infty \frac{u_\ell^*(E')\,M_\ell(E')}{E' - E - i\eta}\,dE'. \tag{7.20}$$

Here we have

$$u_\ell(E) = p f_\ell(E) = e^{i\delta_\ell(E)} \sin\delta_\ell(E) \tag{7.21}$$

and $\delta_\ell(E)$ is the scattering phase for the ℓ-th partial wave. We shall drop the subscript ℓ since in Eq. (7.20) it is the same for all partial amplitudes.

Equation (7.20) can be solved in the following way. Consider the function

$$\phi^{\pm}(E) = (2\pi i)^{-1}\int_0^\infty \frac{M(E')\,u^*(E')}{E' - E \mp i\eta}\,dE'. \tag{7.22}$$

According to Eq. (2.9) we have

$$\phi^+ + \phi^- = (\pi i)^{-1}\mathcal{P}\int_0^\infty \frac{M(E')\,u^*(E')}{E' - E}\,dE' \tag{7.23}$$

$$\phi^+ - \phi^- = M(E)\;u^*(E). \tag{7.24}$$

Our problem now reduces to finding the functions $\phi^{\pm}$.

Let us represent Eq. (7.20) as

$$M(E) = M_0(E) + (\pi i)^{-1}\mathcal{P}\int_0^\infty \frac{M(E')u^*(E')}{E' - E}\,dE' + i u^*(E)\,M(E) \tag{7.25}$$

and substitute Eqs. (7.23) and (7.24) for the integral and the last term. We then obtain

$$\phi^+(1 - 2iu^*) - \phi^- = M_0 u^*. \tag{7.26}$$

Let us put further

$$\phi^{\pm}(E) = F^{\pm}(E)\;\rho^{\pm}(E) \tag{7.27}$$

and assume that the function $\rho^{\pm}$ satisfies the relation

$$(\rho^+/\rho^-)(1 - 2iu^*) = 1. \tag{7.28}$$

Substituting Eq. (7.27) into Eq. (7.26) and using Eq. (7.28), we obtain

$$F^+ - F^- = M_0 u^* / \rho^-. \tag{7.29}$$

Equation (7.29) holds for all $0 \leq E \leq \infty$. Obviously, we can regard $F^{\pm}$ as values of the function

$$F(E) = (2\pi i)^{-1} \int_0^\infty \frac{M_0(E')\, u^*(E')\, dE'}{(E' - E)\, \rho(E')} + R(E) \tag{7.29a}$$

on the upper and lower sides of the cut from 0 to ∞ in the complex plane E. In Eq. (7.29a) R(E) is an arbitrary function having no singularities in the finite part of the plane. The polynomial in E or an exponential function is such a function. But since we required at the outset that the solution M(E) should decrease sufficiently rapidly when $|E| \to +\infty$ (without this requirement Eq. (7.13) could not be written) we must put

$$R(E) = 0. \tag{7.29b}$$

From Eqs. (7.29a) and (7.29b) we thus have

$$F^{\pm} = (2\pi i)^{-1} \int \frac{M_0(E')\, u^*(E')\, dE'}{(E' - E \mp i\eta)\rho^-(E')}. \tag{7.30}$$

Now we have only to find the functions $\phi^{\pm}$ in order to obtain the functions $\rho^{\pm}$. For that purpose let us make use of Eq. (7.28). Taking the logarithm of this relation and taking Eq. (7.21) into account, we obtain

$$\ln \rho^+(E) - \ln \rho^-(E) = 2i\delta^*(E). \tag{7.31}$$

Using then the same device as for finding the functions $F^{\pm}(E)$ we obtain

$$\ln \rho^{\pm} = \pi^{-1} \int_0^\infty \frac{\delta^*(E')\, dE'}{E' - E \mp i\eta} \tag{7.32}$$

or

$$\rho^{\pm} = \exp\left\{ \pi^{-1} \int_0^\infty \frac{\delta^*(E')\, dE'}{E' - E \mp i\eta} \right\} \tag{7.33}$$

Equation (7.33) in fact culminates in the solution of Eq. (7.20). Bringing together all the results for $\rho^{\pm}$, $F^{\pm}$ and $\phi^{\pm}$ we obtain finally

$$M_\ell(E) = M_{0\ell}(E) + \pi^{-1} \rho_\ell^+(E) \int_0^\infty \frac{M_{0\ell}(E') e^{-i\delta_\ell^*(E')} \sin \delta_\ell^*(E')\, dE'}{\rho_\ell^-(E')(E' - E - i\eta)}. \tag{7.34}$$

The solution of $M_\ell(E)$ can be expressed only through the p.v.-integrals:

$$M_\ell(E) = M_{0\ell}(E) e^{i\delta_\ell^*(E)} \cos \delta_\ell^*(E) + \pi^{-1} \rho_\ell(E)\, e^{i\delta_\ell^*(E)}\, \mathcal{P} \int_0^\infty \frac{M_0(E') \sin \delta_\ell^*(E')}{\rho_\ell(E')(E' - E)} dE' \tag{7.35}$$

where

$$\rho_\ell(E) = \exp\left\{ \pi^{-1} \mathcal{P} \int_0^\infty \frac{\delta_\ell(E')}{E' - E} dE' \right\}. \tag{7.36}$$

Thus the solution of Eq. (7.20) is given by the quadratures (7.36) and (7.35). Note that Eq. (7.36) can be re-written for practical calculations as

$$\rho_\ell(E) = \exp\left\{\pi^{-1}(E + E_0)^n \mathcal{P}\int_0^\infty \frac{\delta^*_\ell(E')\, dE'}{(E' + E_0)(E' - E)}\right\} \qquad (7.36a)$$

where E_0 is an arbitrary positive number. Equation (7.36a), which can be obtained by dividing Eq. (7.31) by $(E + E_0)$, is more convenient in numerical calculations in the sense that the integrand decreases more rapidly and consequently the calculated integral converges more rapidly. The scattering phases $\delta^*(E)$ for nuclear reactions are the sum of Coulomb phases and purely nuclear scattering phases.* As a result the function $\rho(E)$ is the product

$$\rho(E) = \rho_N(E)\,\rho_C(E) \qquad (7.37)$$

of the two functions $\rho_N(E)$ and $\rho_c(E)$, each of which is given by Eq. (7.36) or Eq.(7.36a), ρ_N containing only the nuclear phase and ρ_c only the Coulomb phase.

It is shown in the investigation by KAMINSKY and ORLOW [23] that

$$\rho_{c\ell} = \left\{(\pi\gamma/\mathrm{sh}\pi\gamma)\prod_{n=1}^{n=\ell}(1 + \gamma^2 n^{-2})\right\}^{\frac{1}{2}}, \quad \ell \geq 1; \quad \rho_{co} = \left\{\pi\gamma/\mathrm{sh}\,\pi\gamma\right\}^{\frac{1}{2}} \qquad (7.38)$$

where **

$$\gamma = Z_A Z_x e^2 m_{Ax}/p \quad . \qquad (7.39)$$

As regards ρ_N, $E^{-\frac{1}{2}} \tan\delta_N(E)$ can be approximated for practical calculations by a rational function (see [23, 24])

$$\tan\delta^*_{N\ell}(E) = E^{\frac{1}{2}}\, Q_\ell(E)/P_\ell(E) \qquad (7.40)$$

where $Q_\ell(E)$ and $P_\ell(E)$ are polynominals. Then it is not difficult to show that $\rho_N(E)$ is given by the formula

$$\rho_N(E) = c\prod_k (E - z_2^k)^{\frac{1}{2}}/\prod_k (E - y_2^k)^{\frac{1}{2}}, \; c = \prod_k (E_0 + y_2^k)^{\frac{1}{2}}/\prod_k (E_0 + z_2^k)^{\frac{1}{2}} \qquad (7.41)$$

where y_k and z_k are the roots of the polynomials $P_\ell(x^2) \pm i x\, Q_\ell(x^2)$. Eq. (7.41) is obtained from Eq. (7.36a). Thus with the aid of Eqs. (7.37), (7.38) and (7.41) the solution of Eq. (7.20) reduces to one numerical quadrature.

The problems of the solution of the Omnes-Muskhelishvili equation as applied to nuclear reactions are considered in more detail in [23].

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* We mean here the nuclear phases in the case of Coulomb field. (The nuclear phases are not equal to the nuclear phases without Coulomb field.)

** When we have Coulomb field, the lower limit in the integral in Eq. (7.35) is equal to the value of Coulomb barrier energy.

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THE NUCLEAR SHELL MODEL AND ITS RELATION WITH OTHER NUCLEAR MODELS

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1. AN INTRODUCTION TO SHELL MODEL CALCULATIONS

The starting point of all versions of the shell model is the physical idea that the interaction between a given nucleon and all the others resembles that between a nucleon and a fixed field.

From this starting point one might attempt to construct a field which is self-consistent but this approach is not followed in most shell-model calculations because of the complications that arise. The more usual approach has been to use the idea of an average field to provide a complete set of single-particle wave functions. Then, if the parameters of the field (e.g. its size) are correctly chosen, we would expect to reach a good approximation to the nuclear-wave function by taking that configuration of single-particle wave functions which has lowest energy in this field. The wave functions could clearly be improved by allowing the mixing of excited configurations but this is rarely done because of the resulting complexity of the problem. Even in the lowest configuration there are in general many independent wave functions for a many-particle system which would all be degenerate in the average field. To find the nuclear energy levels and wave functions we must therefore build up the energy matrix in this degenerate set, using the inter-nucleon two-body forces, and then diagonalize this matrix.

If the detailed form of the nuclear forces was known we might regard such calculations as the first step towards an exact calculation in which higher configurations were included but every indication is that the convergence would be extremely slow. It is more usual to treat an energy calculation in the lowest configuration unashamedly as a model calculation and to attempt to deduce, by comparisons with experimental data in the many-particle nuclei, the nature of the effective nuclear forces required in that configuration. If the model is realistic then we should not expect these effective forces to change very much in going from one nucleus to its neighbour and since there are many more pieces of data than available parameters we may make significant predictions and thus test the model.

Even within this class of model calculations there are different philosophies. At one extreme is the Israel group, TALMI, DE-SHALIT and co-workers [5] who keep rigidly to the lowest j^k configuration. This has the great advantage that very few matrix elements of the Hamiltonian are involved and these may be deduced from a fit to the known spectra. It is, however, well known that such simple wave functions give poor agreement with transition rates and moments if the real operators for these processes are used. They must therefore try to extract the matrix elements of model moment operators also from the data.

If one takes a more general model, allowing mixing of the lowest configurations, it is no longer possible to deduce all the required matrix elements of the Hamiltonian as there are so many. One must then resort to a

definite assumption of a Hamiltonian with possibly a few parameters, such as range and exchange properties, to be chosen. Although such an approach, which is the one I usually take, is not designed to give close fits to the spectra, one finds reasonable agreement and, in addition, the moments and transition data are generally predicted correctly using the real operators, suggesting that the wave functions are a little nearer the truth than in a pure configuration.

In these lectures I shall describe some of the group theoretical techniques used in classifying states of a pure configuration and of mixed configurations and in calculating energy matrices. In some cases this will lead to a description of collective behaviour and to a connection with other nuclear models.

Before launching into a description of techniques it is important to glance at the physical basis of the shell model [1] and see precisely what mathematical problems this poses.

In all shell-model claculations the average field is taken to be spherical. To do otherwise would raise quite serious problems concerning the orientation angles of the field which would bring us into a discussion of the foundations of the collective model. I shall return to this point at the end of the course but at this stage the average field is always assumed spherical.

The shell structure reveals itself most clearly near the closed shells which occur whenever the number of nucleons is just sufficient to fill completely an energy level in the field, taking account of the Pauli Principle and filling systematically from the lowest level upwards. Closed shell nuclei would be expected to be rather more tightly bound than neighbouring nuclei and this should show up in a variety of ways, e.g. as a discontinuity in a plot of the binding energy differences between successive nuclei, as a peak in a plot of the excitation energies of the first excited states of even-even nuclei, etc. These features were observed to occur at the (magic) numbers 2, 8, 20, 28, 50, 82, 126 for either neutrons or protons.

It was found impossible to reproduce these numbers with a simple spin-independent field V(r) with any reasonable shape. We know, however, that the two-body nuclear force depends on the spin and it would not therefore be too surprising to find the need for a spin-dependence in the field. From invariance arguments this could only be of the spin-orbit kind ($s.\ell$) and the inclusion of such a term was found to produce the observed magic numbers in a very natural way. The exact radial shape of the field is unimportant for most calculations and for simplicity of the wave functions is generally taken as a harmonic oscillator (Fig. 1). Although this field is clearly wrong

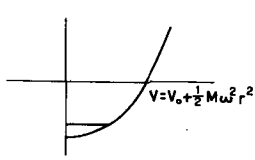


Fig. 1

Harmonic oscillator potential

at large distances this is not important for the well-bound levels used in shell-model calculations. In fact, a numerical calculation [2] of wave functions and radial integrals for the nuclei of mass 38 using a Saxon-Woods

field (Fig. 2) shows quite negligible difference from those calculated in an oscillator field.

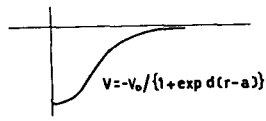


Fig. 2

Saxon-Woods potential

The single-particle functions ψ (nℓjm) are labelled by the orbital angular momentum ℓ, the total angular momentum $j = \ell \pm \frac{1}{2}$ and the radial quantum number n and are formed simply by vector coupling the spin and orbital angular momenta. For a single particle outside a closed shell (or a single hole) we may then predict the angular momentum of the nucleus to be $J = j$, simply that of the last particle. Likewise the magnetic and quadrupole moments will be just those of the single particle and can be simply calculated. The angular momenta and magnetic moments are in good agreement with the known data but there are substantial errors in the quadrupole moments. This is clearly a danger signal but it is not serious enough to make us abandon the model. I shall return to this question later.

When we have a number k of nucleons outside a closed shell we may couple them together in many ways and the particular coupling scheme in the ground state will depend on the nature of the nuclear force which splits this degeneracy. In the early days of the shell model a particularly simple coupling scheme was proposed, namely that each pair of neutrons (or protons) in a shell j coupled together to give zero angular momentum. Thus every even-even nucleus would have $J = 0$ while an odd nucleus would have $J = j$, the angular momentum of the single uncoupled nucleon. Furthermore, the magnetic and quadrupole moments would be just those of the last single particle, ignoring antisymmetry. Comparison with experiment shows that the angular momenta J are generally given correctly although in the region of $A \sim 180$ there is very little agreement. The magnetic moment predictions, the Schmidt lines, are in fair agreement. The quadrupole moments do not resemble the single-particle value at all.

To predict the angular momenta of excited states, the spectrum, and to improve the agreement in the moments mentioned above we must, in the first place, make a more careful study of the way in which nucleons in an orbit j couple together under the influence of a two-body force. Thus, one of the problems of technique which I shall briefly discuss is that of classifying functions and constructing an energy matrix in a configuration j^k.

In the second place we must consider the possibility of mixing between configurations. The physical importance of this will clearly depend on the nearness of the energies of orbits in the average field and the strength of interaction between them. Thus, in some nuclei we find very pure j-configurations while in others there is strong mixing. In light nuclei, $A < 40$, the spin-orbit force does not cause different oscillator levels to overlap in energy so that, here, the most important configuration mixing is between orbits of the same oscillator energy and in particular between the two $j(= \ell \pm \frac{1}{2})$ values with the same radial wave function. If the mixing between these two j-values is strong, then it might be simpler to work in the L-S coupling scheme with a pure configuration ℓ^k with respect to the orbital wave

functions. We cannot simultaneously diagonalize both the total spin S and the individual particle j-values. Since we know that for two-body central forces alone, S is a good quantum number, we see that L-S coupling would be a good approximation if the two-body central force dominated over the spin-orbit splitting and that, in this limit, the j-configurations would be strongly mixed. Another of our problems will therefore be to study L-S coupling in a configuration ℓ^k. The presence of both central forces and a spin-orbit splitting of comparable importance will result in an intermediate coupling scheme which can only be described numerically, in terms of a basis of functions in either L-S or j-j coupling. The choice of basis is then governed simply by personal convenience.

The first oscillator shell, filling when $4 < A < 16$, contains a single ℓ-value ($\ell = 1$). We should therefore expect to understand these nuclei on the basis of a pure p^k configuration although there may well be considerable mixing of the two j-values, the $p_{\frac{3}{2}}$ and $p_{\frac{1}{2}}$ orbits. Detailed intermediate coupling calculations have been made [3] in this shell with very satisfactory results.

In the next oscillator shell, $16 < A < 40$, there are two ℓ-values ($\ell = 2$ and 0) and we should be prepared for mixing of configurations here. The single-particle nucleus O^{17} confirms, by its even-parity spectrum (Fig. 3) and by the results of stripping, that these two ℓ-values are present and close

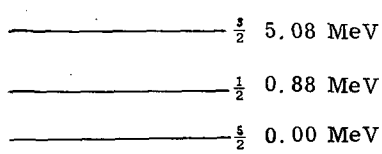


Fig. 3

Energy levels of O^{17}

in energy, the d state being split into $d_{\frac{5}{2}}$ and $d_{\frac{3}{2}}$ levels by a spin-orbit force. The mean position of the d state in the absence of any splitting is therefore just 1.1 MeV above the s state. In this shell one finds indeed that the d and s orbits are mixed the s state. In this shell one finds indeed that the d and s orbits are mixed but calculations [4] which take this into account, again give good agreement for spectra, transition probabilities and moments. Detailed intermediate coupling calculations have only been carried out for two or three nucleons in this ds-shell as the matrices become very large for a greater number.

Thus the general question of the mixing of orbital configurations is raised and in particular the mixing of those orbital configurations which lie in a single oscillator configuration. We shall discuss this problem in some detail and see how it produces rotational features which are in fact present even in these light nuclei.

We now leave the physical problems for a while to study some aspects of the Theory of Groups, which we shall find useful.

2. THE USE OF GROUP THEORY

I must assume that you are familiar with the elementary ideas of groups theory but I shall begin by reminding you, very briefly, of those concepts which I shall use most frequently.

A group is defined as a collection of operations (elements) which satisfy the simple conditions that (i) the product of any two operations gives another operation of the set, (ii) the unit operator is included in the set, (iii) to every element there is an inverse and (iv) the Associative Law holds, (ab)c= a(bc). One can most easily visualize a group of physical operations such as rotations about an axis through integral multiples of $\frac{\pi}{2}$, a group with four elements. Rotations about an axis through an arbitrary angle also form a group which has an infinite number of elements and is an example of a Continuous Group, as distinct from a Finite Group.

One may equally well construct a group from mathematical operations on a set of functions. The operations are then matrices which cause transformations among the functions. The link between the idea of a physical operation and a matrix transformation is easily made if we consider functions $f(x, y)$ of position in a plane. Then, if we rotate the x and y axes about the z-axis through an angle θ the position coordinates in this rotated frame are given simply by

$$x' = x \cos\theta + y \sin\theta$$

$$y' = - x \sin\theta + y \cos\theta.$$

This physical operation will have induced the transformation $R\, f(xy) = f(x'y') = f'(xy)$, where we have made substitutions for x' and y'. Thus the rotation has induced a transformation $f \rightarrow f'$ in the functions. If we express our functions in terms of some complete set then this transformation will assume matrix form, although in general the matrix will be of infinite order.

This brings us to the concept of representations and irreducible representations of a group. Suppose we find a function such that $R_\theta f = f$ for all operations R_θ of the group, then we shall call f a scalar. This is obviously a very special function, but in general we can try to form sets of functions f_α^i, where α labels the set and $i = 1, 2, \ldots, n$ labels the functions within a set, having the property $R_\theta f_\alpha^i = \Sigma M_{ij}^\alpha(\theta) f_\alpha^j$ for all operations of the group. The important feature is that the functions on the right-hand side belong to the same set as that on the left-hand side. Thus, each set α is invariant under the group, the scalar being a special case of a set with only one member. The matrices $M_{ij}^\alpha(\theta)$ are said to provide a representation of the group which is labelled by α and spanned or spread out by the set of n_α functions f_α^i. The representation has dimension n_α, the size of the matrices. These matrices then combine in the same way as the group operators themselves. We have

$$R_\varphi\, R_\theta\, f_\alpha^i = \sum_j M_{ij}^\alpha(\theta)\, R_\varphi\, f_\alpha^j = \sum_k \left(\sum_j M_{ij}^\alpha(\theta)\, M_{jk}^\alpha(\varphi) \right) f_\alpha^k$$

so that the matrix corresponding to a product of group operations is the product of the matrices corresponding to each operation. For a given representation α we have a correspondence between each group element and a matrix of dimension n_α.

It may be possible to divide one of these sets into parts, each of which is itself invariant under the group. This would reduce the representation to a number of representations of smaller dimension. If such a division is not possible we refer to the set and the representation as irreducible. These irreducible representations are of great importance and throughout this course wherever I use the word representation I shall invariably mean an

irreducible representation. We sometimes refer to the functions which span an irreducible representation as belonging to that representation. For example, the spherical harmonics $Y_m^\ell(\theta,\varphi)$ with $m = \ell, \ell-1 \ldots, -\ell$ span a $(2\ell + 1)$-dimensional irreducible representation, labelled by ℓ, of the group R_3 of rotations in three dimensions.

In quantum mechanics we are concerned with the eigenvalue problem $H\psi = E\psi$. The theory of groups is of immediate value here because if the Hamiltonian H is invariant under a group then there will be a degeneracy of all eigenfunctions ψ which belong to the same irreducible representation of the group. This follows directly from the property that an irreducible representation is spanned by taking all group operations on a single function of that representation. Further, if an operator is invariant with respect to a group it will have zero matrix elements between functions belonging to different irreducible representations. For these properties it is essential that we mean irreducible representations.

We make use of these properties in two slightly different ways. In the first place, if our Hamiltonian is precisely invariant with respect to a certain group, the representation labels of that group will be good quantum numbers for the system. In the second place, we may be able to find a group which, although it does not leave the Hamiltonian completely invariant, nevertheless does so in an approximate sense. For example, it may leave the most important part of the Hamiltonian invariant.

In nuclear structure we make use of the symmetric group (of all permutations of the particles), the rotation group in three dimensions and various Unitary Groups and their subgroups. The Hamiltonian is precisely invariant under permutation of all co-ordinates of the particles but the Pauli Principle restricts us to totally antisymmetric wave functions anyway, so we learn nothing more from this invariance. However, if the Hamiltonian is invariant under permutations of, say, the orbital co-ordinates alone, then the representation labels of the symmetric group applied to the orbital wave functions only will become good quantum numbers. With spin-dependent forces this invariance is not satisfied so that use of the symmetric group will fall into the second category above. In the same way, the Hamiltonian is necessarily invariant with respect to rotations of spin and orbital co-ordinates together, leading to the good quantum number J, the total angular momentum. Only for rather special Hamiltonians, however, do we have invariance with respect to rotations in, say, the orbital space alone. This happens for pure central forces and leads to the quantum number L, the orbital angular momentum, which is also the representation label for the group R_3 in orbital space only.

The main purpose of this course will be the study of other groups in this second category which provide, through their representation labels, a useful classification of the many-particle wave functions in shell-model configurations. As a preliminary to this I shall briefly describe the properties of the symmetric and unitary groups which I shall need.

2.1. The symmetric group S_k

The group of all permutations of k particles is called the symmetric group. It has k! elements and is denoted by S_k. You are no doubt familiar with the symmetrizing and antisymmetrizing operators

$$S = \sum_{\text{all } p} p$$

and

$$A = \sum_{\text{all } p} \epsilon_p \, p,$$

where p denotes a permutation and ϵ_p its parity. These operators have the properties

$$SF = F_s, \qquad AF = F_a,$$

where F is any function and F_s, F_a are, respectively, totally symmetric and antisymmetric functions.

By definition, a totally symmetric function satisfies

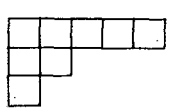
$$pF_s = F_s$$

for all p and a totally antisymmetric function satisfies

$$pF_a = \epsilon_p F_a.$$

Thus, each of these functions would span a one-dimensional irreducible representation of S_k. It is natural, therefore, to ask how the functions of more general symmetry may be organized into irreducible invariant sets.

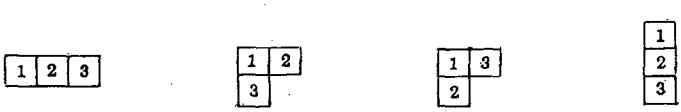
The answer is that there corresponds one such set to each "partition" $[f_1 \, f_2 \, f_3 \ldots]$ of k into integral parts f_i which satisfy $f_1 \geqslant f_2 \geqslant f_3$ etc. and of course $f_1 + f_2 + \ldots$ etc. $= k$. This partition will be denoted briefly by a symbol [f] and illustrated by a Young pattern, which is a shape formed from k squares of which f_1 have been put in the first row, f_2 in the second row and so on. Thus [521] is illustrated by



The number $n_{[f]}$ of independent functions in such a set may be shown to be equal to the number of standard tableaux, defined as any arrangement of the numbers 1, 2, ... k, one into each block in such a way that the numbers increase to the right along every row and increase downwards in each column.

The irreducible representations of S_k are thus labelled by partitions [f] and have dimension $n_{[f]}$. The choice of the functions which span an irreducible representation is clearly arbitrary within a linear combination. To each particular choice of functions there will correspond definite representation matrices. Apart from an overall normalization, therefore, the choice of representation matrices and basis functions is the same thing.

Consider three particles. The possible partitions are [3] [21] [111], with standard tableaux



The three representations labelled by these partitions thus have dimensions 1, 2 and 1 respectively. The first and last are totally symmetric and anti-

symmetric respectively, while the middle one is of mixed symmetry. Yamanouchi uses a more concise notation ($a_k\ a_{k-1} \ldots 2\ 1$) for the standard tableaux in which a_i denotes the row in which particle i occurs. In this notation, the four possible three-particle functions are denoted by (111), (211), (121), (321).

In shell-model calculations it is rarely necessary to construct these functions or matrices explicitly but to illustrate the behaviour of these functions of mixed symmetry I shall briefly describe the standard Young-Yamonouchi-Rutherford representation of S_k.

A very simple general rule may be given for the representation matrices. Since any permutation may be expressed as a product of adjacent transpositions $P_{n-1\,n}$ it is only necessary to give a rule for the representation matrices of this particular permutation. We use a symbol r to distinguish different standard tableaux of a given shape [f]. The standard matrix U_{rs} for the permutation $P_{n-1\,n}$ has matrix elements as follows:

(i) $U_{rr} = +1$ if n-1 and n are in the same row of the tableau r;

(ii) $U_{rr} = -1$ if n-1 and n are in the same column of the tableau r;

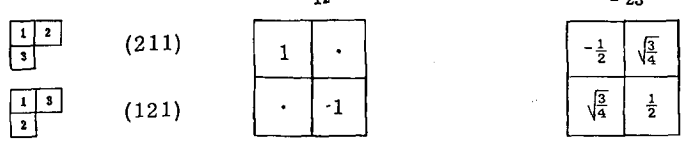
(iii) If n-1 and n are in neither the same row nor column then there will be a tableau s which differs from r only by the interchange of n-1 and n. In this case

$$U_{rr} = -\frac{1}{\rho_{rs}}, \qquad U_{ss} = \frac{1}{\rho_{rs}}, \qquad U_{rs} = U_{sr} = \sqrt{1-\frac{1}{\rho_{rs}^2}}$$

where ρ_{rs} is the "axial distance" from the number n-1 to n in the tableau r. This distance is defined by the number of steps in horizontal and vertical directions, counting steps to the left and downwards as positive;

(iv) All other elements are zero.

Thus the representation matrices for adjacent transpositions in [21] are



so that

$$P_{12}\,\varphi(211) = \varphi(211)$$

$$P_{12}\,\varphi(121) = -\varphi(121)$$

and

$$P_{23}\,\varphi(211) = -\tfrac{1}{2}\varphi(211) + \sqrt{\tfrac{3}{4}}\,\varphi(121)$$

$$P_{23}\,\varphi(121) = \sqrt{\tfrac{3}{4}}\,\varphi(211) + \tfrac{1}{2}\varphi(121).$$

The simplest possible example of functions of mixed symmetry would be

$$\varphi(211) = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3)$$

$$\varphi\,(121) = \frac{1}{\sqrt{2}}\,(x_1 - x_2).$$

The operators S and A may be generalized so that we may construct, not simply functions which are totally symmetric or antisymmetric, but which have mixed symmetry and correspond to a definite row r of a shape [f]. These are called *Young operators* but we shall not need them here.

Product representations

If we have a set of functions F which span a representation [f] of S_r and a set G which span a representation [g] of the same group then the set of all products will span a representation of S_k with dimension $n_{[f]} \times n_{[g]}$ which will in general be reducible. It is of considerable interest to know into which irreducible constituents this reduction will lead. This is the same problem as that of taking appropriate linear combinations of the products to form irreducible invariant sets under S_k.

This process is called the reduction of an Inner Product

$$[f] \times [g] = \sum_{[h]} a_{[h]}\ [h].$$

The integers $a_{[h]}$ denote the number of times that a particular representation [h] appears in the reduction and they may be determined by using the group characters. For example, we get one relation between these coefficients by using the characters of the unit operator which are just the dimensions of the representations. Thus

$$n_{[f]} \times n_{[g]} = \sum_{[h]} a_{[h]}\, n_{[h]}.$$

For example, we find $[21] \times [3] = [21]$
$[21] \times [21] = [3] + [21] + [111]$
$[21] \times [111] = [21]$.

Let us look at the second of these relations in some detail. For brevity, we denote the functions F(211) and F(121) by F_a and F_b respectively. Then this relation tells us that by taking suitable combinations of the four products F_aG_a, F_aG_b, F_bG_a, F_bG_b we may form functions of the symmetry types given. By using the representation matrices we find that

$\sqrt{\frac{1}{2}}\,(F_aG_a + F_bG_b)$ is totally symmetric [3]

$\sqrt{\frac{1}{2}}\,(F_aG_b - F_bG_a)$ is totally antisymmetric [111]

$\sqrt{\frac{1}{2}}\,(F_aG_a - F_bG_b)$ is of mixed symmetry [21] (211)

$\sqrt{\frac{1}{2}}\,(F_aG_b + F_bG_a)$ is of mixed symmetry [21] (121).

We meet this inner product when considering the product of functions of different coordinates of the same particles, for example isotopic spin and intrinsic spin in supermultiplet theory.

The Outer Product of two representations arises when we consider products of functions F of particles 1, 2, ... k_1, which span a representation [f] of S_{k_1} with functions G of particles $k_1 + 1$, $k_1 + 2$, ... $k_1 + k_2$ which span a representation [g] of S_{k_2}. If we now ask how these products transform under the group $S_{k_1 + k_2}$ we soon see that they do not in general span a representation of that larger group. They will do so, however, if we include with them

all products obtained by permuting the particle numbers between the F and G. The reduction of this outer product is denoted by

$$[f] \otimes [g] = \sum_{[h]} A_{[h]} [h]$$

and the integers $A_{[h]}$ follow from the rule of Littlewood which is as follows:

Add to the shape [f] a number g_1 of blocks containing the symbol α. Then add a number g_2 of blocks containing the symbol β etc. in such a way that, (i) at every stage the resultant shape is a standard one, (ii) no two identical symbols appear in the same column, (iii) reading all added symbols from the top right corner along the rows taken in order, the number of symbols β must not exceed the number of symbols α at any stage and the number of symbols γ must not exceed the number of symbols β, etc. If a shape [h] can be thus formed in a number $A_{[h]}$ of ways then the representation [h] will occur $A_{[h]}$ times in the reduction.

The dimension check for the outer product is

$$n_{[f]} \times n_{[g]} \times \frac{(k_1 + k_2)!}{k_1!\, k_2!} = \sum_{[h]} A_{[h]}\, n_{[h]}.$$

For example

$$[21] \otimes [21] = [42] + [411] + [33] + 2[321] + [3111] + [222] + [2211].$$

This process has an obvious application in the addition of k_2 particles to k_1.

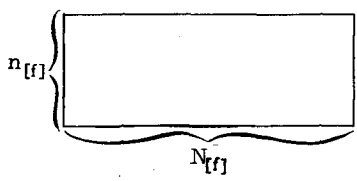
2.2. The unitary group

Consider the problem of putting a number k of particles into any of a set of n single-particle states φ_i. If we consider the single-particle states as components of a vector in n dimensions then the k-particle states ψ will be tensors of rank k in this n-dimensional space. (For example the $(2\ell + 1)$ states $\varphi(\ell m)$ of a single particle in an orbit ℓ would define a $(2\ell + 1)$ - dimensional vector space.) If we impose a linear transformation $\varphi'_m = U_{mn}\varphi_n$ on all the single-particle functions then this will induce a corresponding transformation among the set of all k-particle product functions ψ. The most general of such transformations which preserves normalization and orthogonality is a unitary transformation, $U^*_{tr} = U^{-1}$.

The functions ψ will clearly span a representation of the group U_n of all unitary transformations in the n-dimensional function space of each single particle, but, in general it is not irreducible. Let us now classify the functions ψ according to their permutation symmetry, i.e. by a label [f] of S_k. Since the unitary transformation is totally symmetric in the particle numbers, it cannot transform from one [f] to a different shape. Hence the functions of a particular symmetry type [f] by themselves span a representation of U_n which again is in general reducible. For, consider the functions of type [f] corresponding to a particular standard tableau (or row of the representation) r. These by themselves ([f] and r fixed) must span a representation of U_n again because of the symmetry of the unitary transformation. This representation may be proved irreducible. Because a function labelled by [f]r' may be found from one labelled by [f]r simply by a permutation, the symmetry of the unitary transformations leads to the conclusion that we obtain identical representations from r and r'. The irreducible represen-

tations of U_n are thus labelled by the partition [f] alone, the same symbol as was used for S_k.

One may visualize this situation in which the tensors of symmetry [f] provide irreducible representations of both S_k and U_n by placing all such functions in a rectangular array in which the rows are labelled by the Yamanouchi symbol r and all functions of symmetry [f]r are placed in the r^{th} row.



From the arguments given above, the functions in any particular row will span the irreducible representation [f] of U_n. The length of the rows is therefore just the dimension $N_{[f]}$ of this representation of U_n while the length of the columns is just the dimension $n_{[f]}$ of the representation [f] of S_k. The functions in each row may now be ordered in such a way that functions in a given column differ only in the particle numbering, i.e. they are identical with respect to the unitary transformation among the single-particle states. Thus each column will span the representation [f] of S_k while each row spans the representation [f] of U_n.

Let us now look at some examples to clarify this rather subtle link between the two groups.

(i) Consider n = 2, a two-dimensional vector space such as we meet with the intrinsic spin $m_s = \pm\frac{1}{2}$ or with the charge wave functions of a nucleon, representing neutron or proton. We denote the two single-particle states by φ_+, φ_-. The product wave functions for 2 and 3 particles are organized in Table I by their symmetries [f].

The simple normalization constants have not been included here and the particle numbering is implied by the ordering of products. Thus $\varphi_+\varphi_+\varphi_-$ means $\varphi_+(1)\varphi_+(2)\varphi_-(3)$.

TABLE I

k	[f]		$N_{[f]}$
1	[1]	$\varphi_+,\ \varphi_-$.	2
2	[2]	$\varphi_+\varphi_+,\ (\varphi_+\varphi_- + \varphi_-\varphi_+),\ \varphi_-\varphi_-$.	3
	[11]	$(\varphi_+\varphi_- - \varphi_-\varphi_+)$	1
3	[3]	$\varphi_+\varphi_+\varphi_+, (\varphi_+\varphi_+\varphi_- + \varphi_+\varphi_-\varphi_+ + \varphi_-\varphi_+\varphi_+), (\varphi_-\varphi_-\varphi_+ + \varphi_-\varphi_+\varphi_- + \varphi_+\varphi_-\varphi_-), \varphi_-\varphi_-\varphi_-$.	4
	[21] (211)	$(2\varphi_+\varphi_+\varphi_- - \varphi_+\varphi_-\varphi_+ - \varphi_-\varphi_+\varphi_+), (2\varphi_-\varphi_-\varphi_+ - \varphi_-\varphi_+\varphi_- - \varphi_+\varphi_-\varphi_-)$	2
	[21] (121)	$(\varphi_+\varphi_-\varphi_+ - \varphi_-\varphi_+\varphi_+), (\varphi_-\varphi_+\varphi_- - \varphi_+\varphi_-\varphi_-)$	2
	[111]		0

One can see here that, whereas the representation [2] of S_k is one-dimensional, the representation [2] of U_2 is three-dimensional. By chance, the label [21] gives two-dimensional representations in both groups.

(ii) Some entries are given below in Table II for n = 3 where, for conciseness, we have denoted the three single-particle states by x, y and z.

TABLE II

k	[f]		$N_{[f]}$
1	[1]	x, y, z .	3
2	[2]	xx, yy, zz, (xy +yx), (xz +zx), (yz +zy).	6
	[11]	(xy -yx), (xz -zx), (yz -zy).	3
3	[3]		10
	[21] (211)		8
	[21] (121)		8
	[111]	det (xyz)	1

The total number of products for k particles is n^k so that we have a relation

$$n^k = \sum_{[f]} n_{[f]} \; N_{[f]} ,$$

which may be verified in the examples.

A simple formula can be obtained for the dimension $N_{[f]}$ of the representation [f] of U_n

$$N_{[f]} = \prod_{1 \le i < j \le n} (f_i - f_j + j - i)/(j-i)$$

We have seen that the irreducible representations of U_n are labelled by any partition [f] of any number k but there is one limitation on the [f] if the representation is to be non-trivial. The number of rows of the shape [f] must not exceed n. This is obvious, because if we were to have more than n rows, the basis functions of the representation would necessarily be totally antisymmetric in a group of more than n particles, from the properties of the representation matrices of the symmetric group. But since our vector space has only n dimensions, i.e. there are only n single-particle states to choose from, it is impossible to form functions antisymmetric in more than n particles. This property shows up in the examples where no [111] representation could be formed for U_2. With U_3 this representation can just occur, being one-dimensional with a simple determinant as basis function. Likewise for U_2 the representation [11] is one-dimensional.

Product representations

We met an inner product with respect to the symmetric group when we had a product of two functions of the same particles.

Now we meet an inner product with respect to the unitary group when we consider products of two functions which transform according to definite

representations [f] and [g] under the same unitary group U_n. The reduction of this product representation

$$[f] \otimes [g] = \sum_{[h]} A_{[h]} [h]$$

of U_n has coefficients $A_{[h]}$ identical with those of the outer product reduction of S_k with the exception that any representations [h] with more than n rows must be ignored as trivial. For example, in the outer product reduction

$$[21] \otimes [21] = [42] + [411] + [33] + 2[321] + [3111] + [222] + [2211]$$

of S_6, the right-hand side simplifies to [42] + [33] when considered as an inner product reduction of U_2 and to [42] + [33] + [411] + 2[321] + [222] with U_3.

The dimension check

$$N_{[f]} \otimes N_{[g]} = \sum_{[h]} A_{[h]} N_{[h]}$$

may be applied to these reductions in U_n.

This inner product reduction is in fact present when we go from k = 2 to k = 3 by adding a particle, as in the tables. For example in U_3

$$[2] \otimes [1] = [3] + [21]$$
$$[11] \otimes [1] = [21] + [111]$$

with dimension checks $6 \times 3 = 10 + 8$ and $3 \times 3 = 8 + 1$, which may in fact be used to calculate the dimensions.

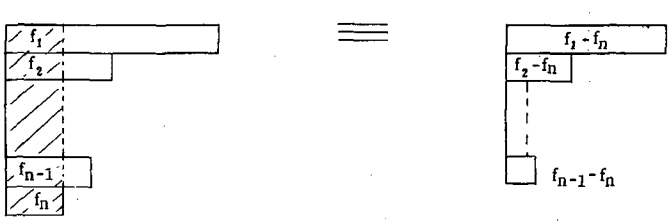
The special (or unimodular) unitary group

Since $UU^*_{tr} = I$ for a unitary matrix, it follows that

$$|\det U|^2 = 1 \text{ so that } \det U = e^{i\alpha}.$$

If we restrict the transformations of U_n to those for which det U = 1 we shall clearly have a sub-group of U_n, denoted by SU_n and called the unimodular (or special) unitary group.

In general an irreducible representation of a group G, although obviously being a representation of any sub-group H of G, will not be irreducible with respect to H. However, in this reduction of U_n to SU_n the irreducible representations [f] of U_n remain irreducible under SU_n. A simplification does occur nevertheless in that certain representations which were inequivalent under U_n become equivalent under SU_n. Precisely, $[f_1\, f_2\, f_3 \ldots\, f_n]$ becomes equivalent to $[f_1\text{-}f_n,\ f_2\text{-}f_n, \ldots f_{n-1}\text{-}f_n]$ the shape obtained by removing all complete columns of n blocks, i.e.



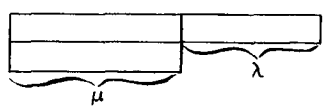
The proof of this result follows from the fact that the totally antisymmetric representation [111... 1], with a single column of n blocks, being one-dimensional must, for unimodular transformations, necessarily be just the identity representation [0] associating the number 1 with each transformation. If then we form the product representation

$$[f_1-1,\ f_2-1,\ \dots\ f_n-1] \otimes \overbrace{[111\dots\ 1]}^{n} = [f_1,\ f_2,\ f_3 \dots\ f_n]$$

in which only a single term occurs on the right-hand side we have shown the equivalence between $[f_1,\ f_2,\ \dots\ f_n]$ and the representation obtained by removing one full column. The process may be repeated until all full columns have been removed.

Thus, the irreducible representations of SU_n may be labelled by a shape of n-1 rows so that, in particular, SU_2 needs only one row, a single integer. It is in fact a well-known property that there is a correspondence between the groups SU_2 and R_3 (a homomorphism, 2-1, see [WEYL p. 144). Hence there should be a unique relation between the representation labels of these two groups. I shall assume that you are familiar with R_3 and in particular with the fact that rotations leave invariant the $(2J + 1)$-dimensional space spanned by the angular momentum eigenfunctions $\psi(JM)$ for fixed J and $M = J, J-1, \dots -J$. In group-theoretical language these functions span a $(2J + 1)$-dimensional irreducible representation of R_3 labelled by J. It is now easily seen, from dimension alone, that the representation $[f_1]$ of SU_2 with a single row will be the same as that of R_3 labelled by $J = f_1/2$.

The group SU_3, which we shall use in the later part of the course, needs two rows to label its representations. We use the notation $(\lambda\mu)$ where $\lambda = f_1 - f_2$, $\mu = f_2$.



3. CLASSIFICATION OF MANY-BODY WAVE FUNCTIONS

3.1. Multiplets and supermultiplets

We must form a wave function which is totally antisymmetric with respect to permutation of all co-ordinates. Such functions may be classified according to the irreducible representation [f] of S_k by which they transform with respect to permutation of one type of co-ordinate alone.

For example, in a j^k configuration we may divide the co-ordinates into those referring to the charge (or isotopic spin) space and to the spin-orbital (j) space. Then, to form an antisymmetric function from one φ which has symmetry [f] in the j-space demands a charge function χ of adjoint symmetry $[\tilde{f}]$, this shape being obtained from [f] by interchanging rows and columns. Then by a suitable phase convention [6] in the adjoint representation, the normalized antisymmetric wave function is simply

$$\Psi = \sqrt{\frac{1}{n_{[f]}}} \sum_r \chi_r\ [\tilde{f}]\Phi_r\ [f]. \qquad (3.1)$$

The convention for the adjoint representation is to define $\tilde{U}_{\tilde{r}\tilde{s}}\ (p) = \epsilon_p\ U_{rs}\ (p)$ which differs from the standard representation for the adjoint shape by a

different choice of phases and enables us to write Ψ without phase factors on the right-hand side of (3.1).

The antisymmetry of Ψ is simply shown, for

$$P\Psi = \sqrt{\frac{1}{n_{[f]}}} \sum_r \sum_s \sum_t \tilde{U}_{\tilde{r}\tilde{s}}\ (p)\ \chi_{\tilde{s}}\ U_{rt}\ (p)\ \varphi_t$$

$$= \sqrt{\frac{1}{n_{[f]}}}\ \epsilon_p \sum_r \sum_s \sum_t U_{rs}\ (p)\ U_{rt}\ (p)\ \chi_{\tilde{s}} \varphi_t$$

$$= \sqrt{\frac{1}{n_{[f]}}}\ \epsilon_p \sum_t \chi_{\tilde{t}} \varphi_t = \epsilon_p \Psi,$$

using the orthogonality of the matrices U.

This antisymmetrization leads therefore to a link between the S_k labels [f] and $[\tilde{f}]$ of the j-space and the charge space of the wave function. But now the link between the groups S_k and U_n means that the behaviour of the charge part of the wave function under U_2 is labelled by $[\tilde{f}]$. This immediately restricts $[\tilde{f}]$ to not more than two rows, leading to the corresponding restriction of not more than two columns in [f] for the spin-orbital space. Because of the SU_2 - R_3 correspondence and because the properties of R_3 are more familiar than those of SU_2 through the frequent use of angular momentum, the isotopic spin-label $T = (\tilde{f}_1 - \tilde{f}_2)/2$ is generally used.

These simple arguments have shown that in j^k, there is a unique correspondence [7] between the isotopic spin T and the spin-orbital space symmetry [f] and further that [f] has no more than two columns. Also [f] has no more than (2j + 1) rows from its significance as a representation label of U_{2j+1}. This conclusion is illustrated in Table III. We return later to the problem of further sub-classification of the spin-orbital states of given symmetry [f] in the j^k configuration.

TABLE III

k	[f]	$[\tilde{f}]$	T
1	[1]	[1]	$\frac{1}{2}$
2	[2]	[11]	0
	[11]	[2]	1
3	[21]	[21]	$\frac{1}{2}$
	[111]	[3]	$\frac{3}{2}$
4	[22]	[22]	0
	[211]	[31]	1
	[1111]	[4]	2

From the physical point of view, this classification is of great use because T is a good quantum number for charge independent forces. To the best of our knowledge, nuclear forces satisfy this condition, being thus invariant under transformations in the U_2 space. Apart from effects of the Coulomb force, which are generally small, this leads to [f] being a good quantum number.

If, instead of lumping the spin and orbital co-ordinates together, as above, we treat them separately, as we must if we intend to set up an L-S coupling scheme, the procedure may be simply generalized. The usual way [6] is to classify the antisymmetric functions by the symmetry [f] of their orbital parts alone. Hence, the combined charge and spin parts must have the adjoint symmetry $[\tilde{f}]$ and we may use the same antisymmetrizing procedure as before if we interpret φ as the orbital wave function and χ as the charge-spin function.

As there are only four possible charge-spin states of a single particle, it follows that the number of rows of $[\tilde{f}]$ and hence the number of columns of [f] must not exceed four.

To classify further by the symmetry in charge and spin spaces separately we must find which shapes [g] and [h] will satisfy the inner product reduction

$$[g] \times [h] = \sum_{[e]} a_{[e]} [e] \tag{3.2}$$

with $[\tilde{f}]$ occurring in the sum and with [g] and [h] having not more than two rows. This gives rise to a set of [g], [h] combinations and if $a_{[\tilde{f}]} > 1$ for some of these, then that combination will occur in the classification independently a number $a_{[f]}$ times.

Since both charge and spin spaces are two-dimensional we again have a unique correspondence $T = \frac{1}{2}(g_1 - g_2)$ and $S = \frac{1}{2}(h_1 - h_2)$. The set of TS combinations occurring for a given [f] is called a "Wigner Supermultiplet". Examples are given in Table IV. For these small values of k, the situation

TABLE IV

k	[f]	$[\tilde{f}]$	[g]	[h]	T	S
1	[1]	[1]	[1]	[1]	$\frac{1}{2}$	$\frac{1}{2}$
2	[2]	[11]	[2]	[11]	1	0
			[11]	[2]	0	1
	[11]	[2]	[2]	[2]	1	1
			[11]	[11]	0	0
3	[3]	[111]	[21]	[21]	$\frac{1}{2}$	$\frac{1}{2}$
	[21]	[21]	[3]	[21]	$\frac{3}{2}$	$\frac{1}{2}$
			[21]	[3]	$\frac{1}{2}$	$\frac{3}{2}$
			[21]	[21]	$\frac{1}{2}$	$\frac{1}{2}$
	[111]	[3]	[3]	[3]	$\frac{3}{2}$	$\frac{3}{2}$
			[21]	[21]	$\frac{1}{2}$	$\frac{1}{2}$

$a_{[\tilde{f}]} > 1$ does not arise so that the T and S labels are sufficient to specify the charge-spin state. This difficulty arises for the first time when k = 6 and [f] = [321] with [g] = [h] = [42]. In this case there will be two independent states with T = S = 1 and [f] = [321] which must be distinguished in some arbitrary way.

Although, by the previous arguments, T is still a good quantum number, this does not now imply that [f] will be good. Only if the nuclear Hamiltonian is also independent of spin will this be so, with the result that this more detailed classification is less fundamental. It is natural of course that, as we make the classification more detailed, the quantum numbers (representation labels) introduced will become less good.

3.2. The ℓ^k configuration

We now face the problem of classifying the orbital states of symmetry [f] in ℓ^k. There are of course a great many ways of doing this but the one of most obvious use is that which uses the total angular momentum L. This provides the L-S coupling scheme which is good for central forces. The problem of which L values occur for given symmetry [f] is the same as the problem of reducing the representation [f] of $U_{2\ell+1}$ when the group is restricted to R_3. The group R_3 is obviously a subgroup of $U_{2\ell+1}$ since the rotation matrices are a special case of the unitary matrices of $U_{2\ell+1}$.

This reduction is most easily accomplished [8] by using the known reduction of product representations in both groups. We illustrate for $\ell = 1$ and 2 for k = 1, 2 and 3 in Tables V and VI.

TABLE V

$\ell = 1$

k	[f]	L
1	[1]	P
2	[2]	SD
	[11]	P
3	[3]	PF
	[21]	PD
	[111]	S

TABLE VI

$\ell = 2$

k	[f]	L
1	[1]	D
2	[2]	SDG
	[11]	PF
3	[3]	SDFGI
	[21]	PD^2FGH
	[111]	PF

The result for two particles, namely that the even L are symmetric while the odd L are antisymmetric, follows immediately from the symmetry of the vector-coupling coefficients or by simply looking at the product functions for fixed M, starting with the maximum value. We then extend to three particles by studying the products

$$[2] \otimes [1] = [3] + [21] \text{ and } [11] \otimes [1] = [21] + [111],$$

together with the products in R_3 which are the usual angular momentum coupling rules. Thus for $\ell = 1$, we have

$$(S + D) \times P = P + (P + D + F) \text{ and } P \times P = S + P + D.$$

Now the totally antisymmetric function [111] is a scalar in U_3 and also must be scalar in R_3. Hence the reduction $[111] \rightarrow S$ is trivial and by subtraction in the products above this leads to $[21] \rightarrow P + D$, $[3] \rightarrow P + F$.

For higher shells, one must find the reduction of the totally antisymmetric tensors by looking at the possible M values for the possible determinants. Thus for the [111] states of d^3 we must choose any three of the m values of a single d-particle. This gives M = 3, 2, 1^2, 0^2, -1^2, -2, -3, showing the reduction [111] $\rightarrow$ P + F. From this, we may deduce the [21] and [3] reductions as above.

We see here that, even with three d-particles, there are two D states of symmetry [21] which we cannot as yet distinguish. For more particles this difficulty increases rapidly, indicating the need for some further subclassification of the states.

Classification by sub-groups

If we could find a sub-group G of $U_{2\ell+1}$ which contains R_3 then we could use the representation labels of G to further distinguish the wave functions.

A systematic way of searching for such groups was introduced by RACAH [9, 10] in the corresponding problem of atomic spectroscopy. It uses the infinitesimal operators of the group.

In the theory of continuous groups, which have an infinite number of elements, it is shown that any element may be generated from a basic set of infinitesimally small transformations. The operators of this basic set are denoted by X_σ so that an arbitrary infinitesimal transformation may be written $S_a = 1 + \sum_\sigma \delta_a^\sigma X_\sigma$.

It is further shown that the group conditions on the operations S_a impose a condition that the infinitesimal operators X_σ must commute among themselves, i.e.

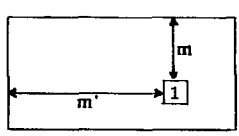
$$[X_\sigma, X_\rho] = \sum_\tau c_{\sigma\rho}^\tau X_\tau \qquad (3.3)$$

and that this condition is sufficient that the set of operators X_σ should be the infinitesimal operators (or generators) of a group.

For the unitary group, these infinitesimal operators are skew Hermitian for if we write $U = 1 + \epsilon A$ with ϵ small then the unitary condition gives $1 = U_{tr}^* U = (1 + \epsilon A_{tr}^*)(1 + \epsilon A) = 1 + \epsilon (A_{tr}^* + A) + \ldots$ so that $A_{tr}^* = -A$. We may write $A = iH$ with H a Hermitian operator.

For the group U_n there will be n^2 such operators while the group SU_n has $n^2 - 1$ operators, the latter being traceless. These are just the numbers of independent $n \times n$ matrices satisfying the Hermitian and the Hermitian traceless conditions.

Let us now be specific and study the operators of the group $U_{2\ell+1}$ in the space of the single-particle functions $\varphi(\ell m)$. There are many ways of organizing these $(2\ell+1)^2$ operators or matrices. The simplest is to define matrices

$E_{mm'} =$ 

with 1 in the mm' position and zero elsewhere. Then

$$(\ell m'' \mid E_{mm'} \mid \ell m'') = \delta_{m'm''} \delta_{mm''} \qquad (3.4)$$

and the commutation relations

$$[E_{mm'},\ E_{m''m'''}] = \delta_{m'm''} E_{mm'''} - \delta_{mm'''} E_{m''m'} \tag{3.5}$$

are satisfied.

Alternatively, we might classify the operators by their behaviour under rotations, i.e. as irreducible tensor operators in Racah's language. Let us denote them by u^r_q and the Wigner-Eckart theorem gives their matrix elements as

$$(\ell m \mid u^r_q \mid \ell m') = (\ell r m' q \mid \ell m) \frac{(\ell \| u^r \| \ell)}{\sqrt{2\ell+1}} . \tag{3.6}$$

The factor $(\ell \| u^r \| \ell)$ will enter only as a single number multiplying all matrices and may therefore be defined to be unity without loss of generality.

The only possible r values are $r = 0, 1, \ldots 2\ell$ with $q = r,\ r-1, \ldots -r$; otherwise the operator is zero. One easily sees that there are just $(2\ell+1)^2$ of these operators, as there are of the $E_{mm'}$.

The two choices of basis for the operators are related by

$$u^r_q = \frac{1}{\sqrt{2\ell+1}} \sum_{m,m'} (\ell r\, m' q \mid \ell m)\, E_{mm'} \tag{3.7}$$

from which the commutation relations

$$[u^r_q,\ u^{r'}_{q'}] = \sum_{r'',q} \left\{ (-)^{r+r'+r''} - 1 \right\} \sqrt{2r''+1}, \tag{3.8}$$

$$\times\ W(rr'\ell\ell\,;\, r''\ell)\ (rr'qq' \mid r''q'') u^{r''}_{q''}$$

may be deduced.

To find a sub-group of $U_{2\ell+1}$ we must find a sub-set of these operators which is closed under commutation. For this purpose the basis u^r_q is most convenient. We see at once that $r'' = 0$ can never appear in the commutation for this would imply $r = r'$ whereupon the curly bracket factor vanishes. Thus all u^r_q with the exception of u^0_0 generate a group, which is simply $SU_{2\ell+1}$. Further if we put $r = r' = 1$ then only $r'' = 1$ occurs. Hence the three operators u^1_q generate a group, which is just R_3. This must be so because we know the infinitesimal operators of R_3 to be the three angular momentum operators, i.e. the three components of a vector, and u^1_q is the only vector at our disposal.

So far we have learnt nothing new from this operator approach. However, our search for a sub-group G containing R_3 is now reduced to a search for a sub-set of the u^r_q containing the u^1_q. If we put $r' = 1$ then only $r'' = r$ appears in the commutation relation but with $q'' \neq q$ in general. From this it follows that, if we include with the u^1_q any component of another tensor operator then all components of that operator must be included. The problem thus reduces to one of finding which values of r may be included with the u^1_q to give a closed set under commutation.

The set of u^r_q with odd r satisfy this condition since if r and r' are odd, the curly bracket vanishes unless r'' is also odd. The group generated by this set is $R_{2\ell+1}$ of orthogonal transformations in $(2\ell+1)$ dimensions, or, in

other words, real unitary transformations. It leaves the two-body scalar product (in the usual R_3 sense) invariant

$$\Phi\ (L = 0) = \frac{1}{\sqrt{2\ell+1}} \sum_m (-)^m \varphi_m\ (1)\, \varphi_{-m}(2).$$

The irreducible representations [f] of $U_{2\ell+1}$ reduce on restriction to the sub-group $R_{2\ell+1}$. Rules for this reduction are given by Littlewood following contraction of the tensors with respect to the scalar product Φ. The irreducible representations of $R_{2\ell+1}$ are then labelled by a set of ℓ integers $(\sigma_1 \sigma_2, \ldots \sigma_\ell)$ denoting the symmetry of the remaining tensor after maximum contraction.

In the p-shell, this leads to no additional classification since $R_{2\ell+1} = R_3$ for $\ell = 1$, but the labels [f] were almost sufficient for a complete separation of functions in that shell.

In the d-shell we may now use [8] the chain of groups $U_5 \rightarrow R_5 \rightarrow R_3$ with representation labels [f] $(\sigma_1 \sigma_2)$L as shown in Table VII. The problem of the two unseparated D-states of [21] symmetry in d^3 is now resolved, one transforming like a single particle (10) under R_5, the other transforming in the less simple fashion (21).

TABLE VII

k	[f]	$\sigma_1\sigma_2$	L
1	[1]	(10)	D
2	[2]	(00)	S
		(20)	DG
	[11]	(11)	PF
3	[3]	(10)	D
		(30)	SFGI
	[21]	(10)	D
		(21)	PDFGH
	[111]	(11)	PF

3.3. The j^k configuration

An almost identical procedure may be used [7] with the configuration j^k, starting from the operators u^r_q with r = 0, 1, 2, ... 2j, which generate the group U_{2j+1}. Again, the removal of u^0_0 restricts the group to SU_{2j+1} and the operators u^1_q describe R_3. Again, the odd values of r provide a set of u^r_q closed under commutation which therefore generate a sub-group of U_{2j+1} containing R_3 and leaving the two-body scalar product (in the R_3 sense)

$$\Phi\ (J = 0) = \frac{1}{\sqrt{2j+1}} \sum_m (-)^{j-m} \varphi_m\ (1) \varphi_{-m}\ (2)$$

invariant.

This invariant is however an antisymmetric second-rank tensor in the $(2j+1)$ dimensional space whereas in discussing the ℓ^k configuration, the scalar product $\Phi(L = 0)$ was a symmetric tensor. This follows directly from the fact that whereas ℓ was integral, j is half-integral. The group which preserves this antisymmetric form is called the Symplectic Group, denoted by Sp_{2j+1}, its irreducible representations being labelled by $j + \frac{1}{2}$ integers $(\rho_1 \rho_2 \ldots \rho_{j+\frac{1}{2}})$. The reduction is again obtained by contraction, with respect to the antisymmetric scalar product and Table VIII shows the resulting classification in the $j = 5/2$ shell using this chain of groups $U_{2j+1} \rightarrow Sp_{2j+1} \rightarrow R_3$.

TABLE VIII

k	[f]	T	$(\rho_1\rho_2\rho_3)$	J
1	[1]	$\frac{1}{2}$	(100)	$\frac{5}{2}$
2	[2]	0	(200)	1, 3, 5
	[11]	1	(000)	0
			(110)	2, 4
3	[21]	$\frac{1}{2}$	(100)	$\frac{5}{2}$
			(210)	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, (\frac{1}{2})^2$
				$\frac{9}{2}, \frac{11}{2}, \frac{13}{2}$
	[111]	$\frac{3}{2}$	(100)	$\frac{5}{2}$
			(111)	$\frac{3}{2}, \frac{9}{2}$

Both here and in the ℓ^k configuration we use the word seniority to describe the rank v of the tensor remaining after the maximum contraction. Thus

$$v = \sum_{i=1}^{j+\frac{1}{2}} \rho_i \text{ or } \sum_{i=1}^{\ell} \sigma_i .$$

Notice that even with three particles this chain of groups fails to separate the two $J = 7/2$ states and again this situation gets worse as the particle number increases. In such cases some arbitrary method must be used to classify them.

For higher values of ℓ and j it is possible to find further intermediate groups to help in the classification but this process never leads to a complete separation of all functions by the representation labels.

We must now ask whether this classification has any useful physical meaning. Assuming that we may ignore configuration mixing we want to see if the new labels approximately diagonalize a nuclear two-body force. Alternatively, we may ask for what forces are the labels good quantum numbers within the configuration. An answer to this second question may be given quite generally.

To each of the groups we are considering there is a quadratic form [10, 11] in the infinitesimal operators, called a Casimir Operator, which has the property of commuting with all the group operators. In other words it is scalar under the group and therefore has the properties that it is diagonal in the irreducible representations and has the same eigenvalue for all

functions in a given representation, that value being simply a function of the representation labels. For example, in the group R_3 the Casimir Operator is J^2 with eigenvalues $J(J+1)$ for all M of given J and no coupling between different J.

For k particles, the Casimir Operator of Sp_{2j+1} is

$$G = \sum_{r(odd)} (2r + 1)\ (U^r \cdot U^r) \qquad (3.9)$$

where $U^r_q = \sum_{i=1}^{k} u^r_q(i)$ is the infinitesimal operator in the k-particle system. Then

$$G = \sum_{r(odd)} (2r+1)\left\{ \sum_{i=1}^{k} \left(u^r(i) \cdot u^r(i)\right) + 2 \sum_{i<j}^{k} \left(u^r(i) \cdot u^r(j)\right) \right\}$$

expressed as a single-particle term which has the same value for all states of the configuration and a two-body term. Now, if the nuclear force is such as to have the same two-body matrix elements as this two-body operator $\sum_{r(odd)} (2r+1)\ (u^r(i) \cdot u^r(j))$, then in the k particle system the matrix elements of the nuclear force will, apart from a constant, be those of the Casimir Operator G. For such a force, then, the representation labels will be good quantum numbers.

The two-body operator is essentially the Casimir Operator for two-particles and its spectrum has all angular momenta degenerate except the J(or L) = 0 state. This is just the spectrum given by the "pairing Hamiltonian" [12], in a pure

(other J) ═══════

J = 0 ———————

configuration defined through its matrix elements

$$(m\,m' \mid H \mid \tilde{m}\,\tilde{m}') = B\ \delta(\tilde{m}, -\tilde{m}')\ \delta(m, -m')\ (-)^{m+\tilde{m}}$$

In other words, a pairing force leads to the $R_{2\ell+1}$ (or Sp_{2j+1}) classification within a pure ℓ^k (or j^k) configuration.

This is still rather a formal type of force but a short-range force has a spectrum not very different from this. For example, a δ-force has the following energies in a two-particle system, normalizing the zero spin energy

L (or J)	d^2	f^2	$(7/2)^2$
0	1	1	1
2	0	0.27	0.25
4	0	0.18	0.12
6	-	0.24	0.06

to unity. One sees that the d^2 case gives, accidentally, the precise degeneracy required and even the other examples do not deviate very far. Another

way of putting this result is that the two single-particle wave functions overlap most strongly in the J (or L) = 0 state. Then, provided our forces do not have too long a range, we might expect these classifications to be useful, the low seniority levels lying lowest. In an even-even nucleus this predicts a lowest state of seniority zero and therefore J = 0 while in an odd nucleus it predicts seniority 1 and thus J = j (or L = ℓ). In j-j coupling the predictions are just those of the early single-particle model in which nucleons were paired off to zero spin and provide a justification for that model. In some cases, however, the magnetic moments etc. in states of seniority 1 differ significantly from the single-particle values due to the antisymmetrization.

Fractional parentage coefficients

Having defined all the states of the configuration by their group theoretical properties it is in principle possible to construct them explicitly. We naturally avoid this for k > 2 and use the method of fractional parentage to calculate matrix elements of one- or two-body operators.

If ψ is an antisymmetric function of particles 1, 2, 3, ... k,
$\bar{\psi}$ " " " " 1, 2, 3, ... k-1,
φ is a single-particle " " k,

then we may write

$$\psi = \sum_{\bar{\psi}} (\bar{\psi}|\}\psi)\,\{\bar{\psi}, \varphi\} \qquad (3.10)$$

where the symbol $\{\bar{\psi},\varphi\}$ denotes vector coupling of $\bar{\psi}$ and φ to the appropriate quantum numbers in ψ. (Either TJ or TS and L depending on the coupling scheme.)

This expansion is clearly possible since the sum is carried over <u>all</u> antisymmetric states $\bar{\psi}$ of the first k-1 particles and ψ being antisymmetric in all particles must necessarily be so in the first k-1. The numbers $(\bar{\psi}|\}\psi)$ are called the "fractional parentage coefficients".

To evaluate the matrix of a one-body operator $Q = \sum_{i=1}^{k} Q(i)$ in a k-particle system

$$(\psi'|\,Q\,|\psi) = k\,(\psi'|\,Q(k)\,|\psi)$$

$$= k \sum_{\bar{\psi}} (\bar{\psi}|\}\psi')(\bar{\psi}|\}\psi)\Big(\{\bar{\psi}\varphi\}'\Big|\,Q(k)\,\Big|\{\bar{\psi}\varphi\}\Big), \qquad (3.11)$$

using the facts that ψ and ψ' are antisymmetric and that $\bar{\psi}$ does not contain particle k. For a particular coupling scheme and a definite tensor operator form for Q we may then reduce the remaining matrix element to one in the last particle only, together with Wigner and Racah functions.

For a two-body scalar operator $H = \sum_{i<j}^{k} H_{ij}$ we find

$$(\psi'|\,H|\psi) = \frac{k(k-1)}{2}\,(\psi'|\,H_{12}|\,\psi)$$

$$= \tfrac{1}{2}k(k-1)\sum_{\bar{\psi}}\sum_{\bar{\psi}'}(\bar{\psi}'|\}\psi')(\bar{\psi}|\}\psi)(\bar{\psi}'|\,H_{12}|\,\bar{\psi}).$$

But if we define $\bar{H} = \sum_{i<j}^{k-1} H_{ij}$ we have $(\bar{\psi}'|\,\bar{H}|\bar{\psi}) = \tfrac{1}{2}(k-1)(k-2)(\bar{\psi}'|\,H_{12}|\,\bar{\psi})$

$$(\psi'|\,H|\,\psi) = \left(\frac{k}{k-2}\right)\sum_{\bar{\psi}}\sum_{\bar{\psi}'}(\bar{\psi}'|\}\psi')(\bar{\psi}|\}\psi)(\bar{\psi}'|\,\bar{H}|\,\bar{\psi}) \qquad (3.12)$$

which expresses the matrix for the k-particle system in terms of that for the (k-1)-particle system. Alternatively, it is possible to define fractional parentage coefficients for the removal of two (or even more) particles. If $\tilde{\psi}$ is an antisymmetric function of particles 1, 2, 3, ... k-2, and θ is an antisymmetric function of particles k-1, k, then we may expand

$$\psi = \sum_{\tilde{\psi},\theta} (\tilde{\psi}, \theta | \} \psi) \{\tilde{\psi}, \theta\} \qquad (3.13)$$

giving

$$(\psi' | H | \psi) = \frac{k(k-1)}{2} \sum_{\tilde{\psi},\theta,\theta'} (\tilde{\psi}, \theta | \} \psi') (\tilde{\psi}, \theta | \} \psi) (\theta' | H_{k-1k} | \theta)$$

reducing the k-particle matrix to the two-body matrix. Generally, this procedure is preferable for a two-body force although the two-particle coefficients $(\tilde{\psi}, \theta | \} \psi)$ are more difficult to calculate.

I shall not enter into a description [6, 10, 13, 14] of the calculation of these coefficients, which makes use of the various group operators by which the states are defined. I might simply remark that they may be factorized. For example, in L-S coupling

$$(\tilde{\psi} | \} \psi) = \sqrt{\frac{n_{(\tilde{f})}}{n_{[f]}}} (\ell^{k-1} [\tilde{f}] \bar{\gamma} L | \} \ell^k [f] \bar{\gamma} L) ([\tilde{f}] \bar{T} \bar{S} | \} [\bar{f}] T S),$$

while in j-j coupling the J and T parts separate. Thus the TS factor may be used in any orbital configuration. We include γ here to denote other quantum numbers such as the R_5 labels. To calculate the separate factors one uses the group operators of $U_{2\ell+1}$, $R_{2\ell+1}$ and R_3 taking advantage of the fact that states in the same irreducible representation of a group may be linked by the operators of that group whereas states of different representations may not. Usually the coefficients for the state with greatest L-value are trivial and a chain calculation with the group operators may be started from there. Any states not separated by the group may be defined at this stage by an arbitrary choice of an otherwise indeterminate fractional parentage coefficient. For most systems amenable to hand calculation, these coefficients have been determined.

We now go on to study the mixing of configurations and in particular a coupling scheme which generates rotational features [17, 18]. We shall develop a method [19] for calculating matrix elements which avoids the fractional parentage method.

3.4. Mixing of configurations

For a long time (1952) there has been strong evidence for the existence of rotational features in heavy nuclei, particularly in the region (A ~ 180) in which quadrupole moments many times larger than the single-particle value were observed. In this region the shell model has many levels close together and would not be expected to work well at all without extensive mixing of configurations. In light nuclei, however, the shell model has worked very well, not only at closed shells but throughout the lp-shell and as far

into the ds-shell as calculations have been carried, namely the nuclei of mass 19.

More recently (1956) rotational spectra were observed in much lighter nuclei, for example in Mg^{24} which is quite close to the region in which successful shell-model calculations were made. The question was therefore posed whether such rotational features could come out from a shell-model calculation. Of course, the shell model provides a complete set of states in which any kind of motion could be constructed. The crucial point is to see if such features emerge with just that mixing of configurations normally considered in shell-model calculations for light nuclei, namely the configurations with the same energy in the oscillator field. If rotations were to appear only after a substantial mixing of very high configurations, this would indicate a breakdown of the shell model. Thus, either the shell model breaks down between F^{19} and Mg^{24} or the rotations will emerge in a natural way within the shell-model framework. In view of the shell model successes in the p-shell the first of these possibilities seemed unlikely and the co-existence of the shell model and the rotational model descriptions were clearly demonstrated when the Nilsson model was applied to F^{19} even though it was necessary to introduce a substantial mixing of rotational bands. Thus, for example, in Mg^{24} it would seem likely that a detailed shell-model calculation allowing mixing of configurations d^8, d^7s; d^6s^2 ... etc. would lead to a simple rotational spectrum when the matrices, of order 100 or so, were diagonalized. There should obviously be a simple way of reaching such a simple result.

We know that it is necessary to use intermediate coupling in light nuclei, resulting from the inclusion of both central and spin-orbit forces. However, it seems clear that the latter, being a single-body force, is not the essential ingredient for producing rotations. We shall therefore work in an L-S coupling scheme, including spin-orbit forces later. Even in the pure L-S coupling limit we shall be able to make comparison with experiment for the even-even nuclei. One expects to find such nuclei fairly close to L-S coupling because they have S = 0 in the lowest levels, giving no zero-order contribution from the spin-orbit force. In the α-particle nuclei, which have N = Z (even) this approximation should be especially good because the two lowest supermultiplets are well separated. Such remarks are confirmed in the p-shell nuclei Be^8 and C^{12}.

If rotational features do emerge in the ds-shell it would be natural to expect them also in the p-shell where the L-S coupling energies were calculated in 1937 by FEENBERG and PHILLIPS [15]. In fact, these energies are proportional to L(L + 1) for states of given [f] and the L-values are precisely those of rotational bands although cut-off at rather small values of L. For example, in the [4] states of Be^8 the values of L are 0, 2 and 4; a K=0 band, cut off at L = 4.

This L(L + 1) behaviour may be seen in an elementary way as a consequence of the smallness of the ℓ-value, although we shall show later that it has much wider significance. In this simple shell there are only three orbital states of two particles L = 0, 1 and 2 and we can find three two-body operators

$$I = \sum_{i<j} I_{ij} \quad , \text{ the two-body unit operator}$$

$$M = \sum_{i<j} P_{ij} \quad , \text{ the Majorana (permutation) operator}$$

$$\mathcal{L} = \sum_{i<j} (\ell_i \cdot \ell_j) = \tfrac{1}{2} (L^2 - \sum_i \ell_i^2)$$

which may be immediately evaluated in a system of k particles. The first two are independent of L and the third has eigenvalues $\frac{1}{2}\{L(L+1) - n\ell(\ell+1)\}$. Hence, any two-body force may be written as a sum of these operators and must therefore have energies proportional to L(L+1) within a given [f]. The two operators, other than the unit operator, are essentially the Casimir operators of the groups U_3 and R_3 by which the p-shell states are labelled.

Thus we see that rotational features are present in the p-shell in L-S coupling although they are not particularly noticeable in the spectra because of the low cut-off, the effect of spin-orbit forces and the wide spacing of the levels, for example in C^{12} and Be^8. A similar study of the pure d-shell reveals no such features but we know that the ld and 2s levels are close in energy and mix strongly. It is therefore possible that rotations will again appear when these configurations are mixed. Let us therefore return to the group theoretical method to see if we can find ways of classifying states of mixed configurations which are physically significant in the hope that one of these will have rotational properties.

Consider the mixing of orbits ℓ_a and ℓ_b. The single-particle functions will now span a space of $(2\ell_a + 1)+(2\ell_b + 1)$ dimensions and the full group of unitary transformations in these functions is now $U_{2\ell_a + 2\ell_b + 2}$. To study sub-groups of this as we did for the pure configurations, we construct the set of infinitesimal operators, which now need more labels to distinguish the ℓ-value as well as the m-value. Thus we introduce the operators $u^r_q(\ell\ell')$ defined by

$$(\ell'''\| u^r \| \ell'') = \delta(\ell\ell''')\,\delta(\ell'\ell'') \tag{3.14}$$

as a simple generalization from the pure configuration, their commutation relations being

$$[u^r_q(\ell\ell'),\ u^s_p(hh')] = \sum_{r,v} \sqrt{(2r+1)}\,(t\,s\,q\,p\,|\,r\,v) \times \left\{(-)^{t+s-r}\,\delta(\ell'h)\,W(t\,s\,\ell h';\,r\ell')\,u^r_v(\ell h') - \delta(\ell h')\,W(t\,s\,\ell' h;\,r\,h')\,u^r_v(h\ell')\right\}. \tag{3.15}$$

Again, we define the many-particle operator

$$U^r_q(\ell\ell') = \sum_{i=1}^{k} u^r_q(\ell\ell';\,i)$$

where $u^r_q(\ell\ell';\,i)$ is the single-particle operator defined above for particle i. The U^r_q have the same commutation relations as the u^r_q and describe the group of simultaneous transformations of all particles.

We now want to find a sub-set of these operators which describe a sub-group containing R_3. There are now two vector operators but it is easily seen that the operators of R_3, the angular momentum operators, are given by $L_q = \Sigma_\ell \sqrt{\ell(\ell+1)(2\ell+1)}\,U_q(\ell\ell)$ with ℓ taking the values ℓ_a and ℓ_b.

In general, the set of operators $u^r_q(\ell_a\ell_b) + (-)^r u^r_q(\ell_b\ell_a)$ for all r, with the $U^r_q(\ell_a\ell_a)$ and $U^r_q(\ell_b\ell_b)$ for odd r satisfy the group conditions (3.3) of being closed under commutation. They describe the group $R_{2\ell_a + 2\ell_b + 2}$ and provide a generalization of seniority to a mixed configuration. This classification would diagonalize a simple pairing force but does not seem much use for nuclear forces as we shall see in an example.

Consider now the mixing of the ld and 2s orbits which is of great interest from a physical point of view for the nuclei with $16 < A < 40$. Then the full

group is U_6 and the sub-group just mentioned is R_6. The classification then follows by the methods of 3.2, reducing representations as the group is restricted by stages from $U_6 \rightarrow R_6 \rightarrow R_3$ as illustrated in Table IX.

TABLE IX

k	[f]	$(\sigma_1\sigma_2\sigma_3)$	L
1	[1]	(100)	SD
2	[2]	(000)	S
		(200)	SD^2G
	[11]	(110)	PDF
3	[3]	(100)	SD
		(300)	$S^2D^2FG^2I$
	[21]	(100)	SD
		(210)	$P^2D^3F^2G^2H$
	[111]	(111)	P^2F^2

The two symmetric S states of two particles are separated into one of seniority zero which is explicitly $\sqrt{\frac{1}{6}}\psi\{(s^2)S\} + \sqrt{\frac{5}{6}}\psi\{(d^2)S\}$ in terms of the pure configurations. This is in fact a good approximation to the lowest S-state when a short-range force is diagonalized but the R_6 group has failed to separate the two D-states which mix strongly with a two-body force and are well separated in energy. We therefore abandon this coupling scheme.

There is, however, an alternative sub-group for this particular case of the ds-shell, described by the eight operators $u_q^1(dd)$ and $-\sqrt{7}\,u_q^2(dd) + 2\{u_q^2(ds) + u_q^2(sd)\}$. From the number of operators and their commutation relations this group may be identified with SU_3, the special unitary group in three dimensions. This same sub-group also appears when mixing between any set of degenerate oscillator levels is considered, i.e. (f p) (gds) etc. In fact this group is intimately connected with the degeneracies of the oscillator and, as we shall see in the next chapter, gives rise to rotational motions.

4. THE GROUP U_3 AND THE APPEARANCE OF ROTATIONS

It was shown by JAUCH and HILL [16] that a spherical oscillator Hamiltonian in n-dimensions commutes with the operators of a group U_n so that the eigenvalues are labelled by the irreducible representations of U_n and there is a degeneracy of functions belonging to the same representation. Thus U_3 is the symmetry group of the three-dimensional oscillator, the familiar oscillator number N being simply the representation label [NOO] of U_3. Only the totally symmetric tensors can be formed with a single particle so that the oscillator wave functions of energy $(N + \frac{3}{2})\hbar\omega$ are the symmetric tensors of rank N in the U_3 space, which reduce on restriction to the group R_3 to the familiar ℓ-values.

The operators of this symmetry group U_3 may be expressed either in Cartesian form or in spherical tensor form. If we define creation and destruction operators

$$a_x^+ = (x - ib^2 p_x)/b\sqrt{2} \qquad \text{etc.}, \tag{4.1}$$
$$a_x = (x + ib^2 p_x)/b\sqrt{2}$$

for oscillator quanta, then the nine components

$$A_{ij} = \tfrac{1}{2}(a_i^+ a_j + a_j a_i^+) \tag{4.2}$$

of a second-rank Cartesian tensor commute with the oscillator Hamiltonian

$$H_o = \tfrac{1}{2} b^2 p^2 + \tfrac{1}{2}(r^2/b^2) = A_{xx} + A_{yy} + A_{zz}. \tag{4.3}$$

Here i and j run over the indices x, y and z. From the familiar commutation relations

$$[a_i, a_j] = [a_i^+, a_j^+] = 0, [a_i, a_j^+] = \delta_{ij}$$

we deduce

$$[A_{ij}, A_{k\ell}] = \delta_{jk} A_{i\ell} - \delta_{i\ell} A_{kj}.$$

The eight combinations of A_{ij} with zero trace are closed under commutation and describe the group SU_3. In spherical tensor form we have a scalar H_o, a vector

$$A_{xy} - A_{yx} = i L_z, \; A_{yz} - A_{zy} = iL_x, \; A_{zx} - A_{xz} = iL_y$$

and a second-degree tensor which we call Q_q with components

$$\begin{aligned} Q_o &= 2A_{zz} - A_{xx} - A_{yy}, \\ Q_1 + Q_{-1} &= -i\sqrt{6}(A_{yz} + A_{zy}), \\ Q_1 - Q_{-1} &= -\sqrt{6}(A_{zx} + A_{xz}), \\ Q_2 + Q_{-2} &= \sqrt{6}(A_{xx} - A_{yy}), \\ Q_2 - Q_{-2} &= i\sqrt{6}(A_{xy} + A_{yx}). \end{aligned} \tag{4.4}$$

Explicitly, the vector $\vec{L}$ is just the angular momentum operator $\vec{L} = (\vec{r} \times \vec{p})$ while

$$Q_q = \sqrt{\frac{4\pi}{5}} \{r^2 Y_q^2(\theta_r, \varphi_r) + p^2 b^4 Y_q^2(\theta_p, \varphi_p)\}/b^2 \tag{4.5}$$

where θ_r, φ_r are the polar angles of position and θ_r, φ_p the polar angles of momentum. Thus, for example,

$$Q_o = \{(2z^2 - x^2 - y^2) + b^4(2p_z^2 - p_x^2 - p_y^2)\}/2b^2. \tag{4.6}$$

The existence of this symmetry group explains why the degenerate orbits of any oscillator level N may be classified by the group U_3. Here we shall concentrate on the ds-shell (N = 2), first deriving the classification, then confirming that it is physically useful and finally developing a simple projection form for the wave functions which enables the energy matrix to be calculated in a simple way and exhibiting rotational properties.

4.1. The U_3 classification

The six single-particle functions $s_0 d_2 d_1 \ldots d_{-2}$ span a six-dimensional vector space, the functions of k-particles being tensors of rank k. In the usual way, a classification of these functions by their permutation symmetry [f] leads at the same time to a classification by the irreducible representations [f] of U_6. If we now restrict the group operations $U_6 \rightarrow SU_3 \rightarrow R_3$ we shall introduce the representation labels L of R_3 and $(\lambda\mu)$ of SU_3, see section 2.2. In Table X we show the $U_6 \rightarrow SU_3$ reduction for $k \leqslant 3$, obtained by studying the

TABLE X

k	[f]	$(\lambda\mu)$
1	[1]	(20)
2	[2]	(40)(02)
	[11]	(21)
3	[3]	(60)(22)(00)
	[21]	(41)(22)(11)
	[111]	(30)(03)

reduction of product representations in both groups as in section 3.2. We can look on a single particle as a vector in six dimensions or as a symmetric second-rank tensor in 3 dimensions. With two particles we have a tensor of rank 2 in 6 dimensions and of rank 4 in 3 dimensions. The reduction above tells us which symmetries $(\lambda\mu)$ may occur with respect to the 3-dimensional space when the symmetry [f] of the function with respect to the 6-dimensional space is specified.

The reduction from $SU_3 \rightarrow R_3$ may be obtained in a similar manner and one finds a general rule. For this stage of the problem we may forget about the U_6 group entirely and concentrate on the L-values contained in the tensors of symmetry $(\lambda\mu)$ in a three-dimensional space. This is just like filling a p-shell with particles. By finding the number of ways of constructing totally symmetric $(\lambda\ 0)$ states of a number λ of p-particles with fixed M we find simply that $(\lambda\ 0)$ contains values

$$L = \lambda,\ \lambda-2,\ \lambda-4,\ \ldots\ .\quad 1 \text{ or } 0,$$

depending on whether λ is odd or even. By using the product reduction

$$(\lambda + 1,0) \times (1,0) = (\lambda + 2,0) + (\lambda,\ 1)$$

and knowing the L-values in the symmetric representation ($\mu = 0$), we may deduce that $(\lambda 1)$ contains values

$$L = \lambda + 1,\ \lambda,\ \lambda-1,\ \ldots\ 2, 1.$$

This procedure obviously continues to higher values of μ and we find that $(\lambda 2)$ contains values

$$L = (\lambda+2),\ (\lambda+1),\ (\lambda)^2,\ (\lambda-1),\ (\lambda-2)^2, \ldots \begin{cases} 2^2, 0 \\ 3^2, 2, 1 \end{cases}$$

according as λ is even or odd.

A general rule emerges, that the L-values are just those of a series of rotational K-bands with

$$K = \mu,\ \mu-2,\ \ldots\ 1 \text{ or } 0 \quad \text{and with} \quad L = K,\ K+1, \ldots K+\lambda \tag{4.7}$$

for each of these values of K except K = 0 when $L = \lambda, \lambda-2, \ldots 1$ or 0. We shall find a simple reason for this result in section 4.5.

This classification is reasonably complete in the sense that multiplicities in $(\lambda\mu)$ do not occur until we have five particles and we shall always find these multiplicities occur quite high in energy and are not physically interesting. We shall find a precise way of defining a label K to distinguish states with the same $(\lambda\mu)$ and L.

4.2. Comparison with shell-model eigenfunctions

We must now ask if the classification has any physical meaning. The two-particle states are

$$\psi\{(40)S\} = \sqrt{\frac{5}{9}}\,\psi\{(s^2)S\} + \sqrt{\frac{4}{9}}\,\psi\{(d^2)S\}$$

$$\psi\{(40)D\} = \sqrt{\frac{7}{9}}\,\psi\{(ds)D\} - \sqrt{\frac{2}{9}}\,\psi\{(d^2)D\}$$

with the orthogonal combinations belonging to (02). The G state must come entirely from d^2. These coefficients may be found by forming D and S states from this known G-state with the help of the U_3 group operators Q_q. The states formed in this way must belong to the same representation (40) as the G-state.

The (40)S-state is quite close to the lowest S-state found by diagonalizing a two-body force and in fact has an overlap of 83% with the (00) S state in the R_6 scheme. However, the U_3 scheme has also mixed the two D-states and we find that the (40) D-state also is close to that found by diagonalization.

A more stringent test was applied by making a comparison for the [4] states of four particles, which we would expect to describe the low levels of Ne^{20}. With this symmetry the possible $(\lambda\mu)$ values are (80), (42), (04), (20) so that there are four S-states, five D-states etc. In the shell-model calculation the energy matrix was set up in a pure configuration scheme with again four S-states etc. coming from configurations d^4, d^3, d^2s^2, ds^3 and s^4. On diagonalization it was found that the lowest S-state contained 92% of the S-state of (80), the corresponding figures for the D, G and I states being 99%, 92% and 99%, respectively. The configuration mixing was very large, for example the S-state contained percentages 22, 10, 41 and 27 respectively in the configurations d^4, d^3s, d^2s^2 and s^4.

4.3. The quadrupole force

We must now ask the two related questions: Why are the U_3 wave functions with the greatest value of λ good approximations to the shell-model eigenfunctions; and for which two-body force will the U_3 wave functions be exact eigenfunctions?

The second question is simply answered by introducing the Casimir operator of this group as we did for the seniority scheme. This operator must be scalar in R_3 and there are only two quadratics of this kind which we can form, (L.L) and (Q.Q). The Casimir operator is that combination of these two which also commutes with the group operators Q_q. One finds

$$G = \tfrac{1}{4}\{3(L.L) + (Q.Q)\} \tag{4.8}$$

with a convenient normalization. A general formula for the eigenvalues of Casimir operators has been given by RACAH [20] which in SU is

$$\langle G \rangle = g(\lambda\mu) = (\lambda^2 + \lambda\mu + \mu^2) + 3(\lambda+\mu). \tag{4.9}$$

(In fact, this formula may be derived very simply using a Cartesian form for G and operating on the leading state of the Cartesian basis described in 4.4).

Thus the operator G will be diagonal in the U_3 scheme with the same eigenvalue for all states of a representation $(\lambda\mu)$. But if these states are classified by their angular momentum L then (L.L) will be diagonal so that (Q.Q) is also diagonal with the eigenvalues

$$\langle (Q.Q) \rangle = 4g(\lambda\mu) - 3L(L+1). \tag{4.10}$$

Not only is (Q.Q) diagonal but the energies for given $(\lambda\mu)$ follow the rotation sequence L(L+1). Because of the equivalence between the operators r and p in the harmonic oscillator we have from (4.5) that for each particle i

$$Q_q(i) \equiv \sqrt{\frac{16\pi}{5}}\, r^2(i)\, Y_q^2\big(\theta_r(i)\varphi_r(i)\big)$$

within an oscillator shell. Thus also within a shell

$$\begin{aligned}(Q.Q) &\equiv \frac{16\pi}{5} \sum_{ij} r^2(i) r^2(j) \big(Y^2(i) \cdot Y^2(j)\big) + \text{single-particle terms} \\ &\equiv 8 \sum_{i<j} r^2(i) r^2(j) P_2(\cos\theta_{ij}) \quad + \text{single-particle terms.}\end{aligned} \tag{4.11}$$

Hence, apart from the single-particle terms, an attractive quadrupole force with the particular radial dependence $r^2(i)r^2(j)$ will give rise to the U_3 wave functions, with the greatest values of g $(\lambda\mu)$ lying lowest, and a rotational spectrum.

To answer the first of our two questions we must now try to relate this quadrupole interaction to the nuclear force. It is of course trivial that any force may be expanded into multipoles, that within an oscillator configuration these can only be of even order and that the monopole term is relatively unimportant for the splitting of levels. The precise radial form in (4.11) comes out from a fourth power force

$$r_{ij}^4 = (r_i^2 + r_j^2 - 2r_i r_j \cos\theta_{ij})^2$$

$$= r_i^4 + r_j^4 + \frac{8}{3} r_i^2 r_j^2 + \frac{4}{3} r_i^2 r_j^2 P_2(\cos\theta_{ij}) - 4(r_i^2 + r_j^2) r_i r_j \cos\theta_{ij}. \qquad (4.12)$$

Of these terms, the last vanishes in a configuration because it has odd parity in each particle while the first three reduce to constants or single-particle terms, leaving the quadrupole force of (4.11).

The significance of the r_{ij}^4 term may be argued if we remember that we are only evaluating the energy matrix in the low configuration where the wave functions fall off sharply for large distances. Thus, if we were to try and approximate the nuclear force numerically by a sum of small powers in r_{ij} the first two terms $(a + br_{ij}^2)$ would not cause any splitting, being just equivalent to a central oscillator field. The next term is r_{ij}^4 which would need a negative (attractive) coefficient to slow down the increase caused by the r_{ij}^2 term as r_{ij} increases.

Only two radial integrals enter in the p-shell so that a force of any shape may be exactly replaced by a sum of powers up to r_{ij}^4. This is the reason why any force gives an L(L+1) spectrum in the p shell.

We shall not pursue this question or make any further use of the quadrupole force. Rather we shall make use of the U_3 functions as a useful basis in which to carry out calculations. To this end we now study the structure of the wave functions.

4.4. A Cartesian basis

We chose to classify the states within a representation $(\lambda\mu)$ by their L value for the obvious reason that L is a good quantum number for central forces. We now define an equivalent basis, in which L is not diagonal, from which we shall find a simple way of constructing the functions with definite L by an angular momentum projection.

The three operators A_{xx}, A_{yy} and A_{zz} of (4.2) commute and are just the one-dimensional oscillators in x, y and z. We then construct a "Cartesian basis" for the k-particle functions in which all these operators are diagonal, with values denoted by N_x, N_y and N_z. Since all states have the same value kN for the sum $H_0 = A_{xx} + A_{yy} + A_{zz}$ we consider the remaining two operators $A_{xx} - A_{yy}$ and $2A_{zz} - A_{xx} - A_{yy} = Q_0$ with values denoted by $\nu = N_x - N_y$ and

$$\epsilon = 2N_z - N_x - N_y = 3N_z - kN. \qquad (4.13)$$

Thus, ν and ϵ describe the distribution of quanta between the three directions and since the N_x etc. are integral, ν changes in units of 2 and ϵ in units of 3.

We now observe that the operators

$$w_0 = \tfrac{1}{2}(A_{xx} - A_{yy}), \quad w_{+1} = -\sqrt{\tfrac{1}{2}}\, A_{xy} \text{ and } w_{-1} = \sqrt{\tfrac{1}{2}}\, A_{yx} \qquad (4.14)$$

are closed under commutation, thus describing a group. In fact, with the normalizations given above their commutation relations are just those of the angular momenta, i.e. the operators of an SU_2 (or R_3) group. This is, of course, simply the symmetry group of the two-dimensional oscillator in x and y alone. We may therefore introduce in this Cartesian basis a label Λ

which will behave like an angular momentum and refer to an irreducible representation of this group SU_2. Since Q_0 commutes with w_0, w_{+1} and w_{-1} we may use the label ϵ at the same time.

In analogy with angular momenta, the operator

$$(w_0^2 - w_{+1}w_{-1} - w_{-1}w_{+1}) = \tfrac{1}{4}(A_{xx} - A_{yy})^2 + \tfrac{1}{2}(A_{xy}A_{yx} + A_{yx}A_{xy}) = \frac{1}{12}(Q_2Q_{-2} + Q_{-2}Q_2 + 3L_0^2)$$

has eigenvalues $\Lambda(\Lambda+1)$ which will be $(2\Lambda+1)$-fold degenerate, the degeneracy being labelled by the values $\Lambda, \Lambda-1, \ldots -\Lambda$ of w_0, which we may diagonalize like the z-component of an angular momentum. For each Λ therefore, ν takes on values $2\Lambda, 2\Lambda-2, \ldots -2\Lambda$.

We use a notation $\varphi(\epsilon\,\Lambda\,\nu)$ for the Cartesian states of a representation $(\lambda\mu)$ and we shall see that these three labels $\Lambda\,\nu\,\epsilon$ are complete in the sense that there never occur two states in a given $(\lambda\mu)$ with the same values for Λ, ν and ϵ. This follows when we use the reduction $SU_3 \to SU_2 \times U_1$ to deduce which $\Lambda\,\nu\,\epsilon$ occur in given $(\lambda\mu)$, the operators of this sub-group being w_0, $w_{\pm1}$ and Q_0. The procedure is to remove from the Young pattern $(\lambda\mu)$ a symmetric tensor representing that part of the original tensor in the z-component of the space. The remaining pattern describes the symmetry in x and y components leading to Λ as half the difference of the rows. Thus

$$\begin{array}{|c|c|}\hline \mu & \lambda \\ \hline \mu \\ \cline{1-1}\end{array} \to [\mu] \times [\lambda+\mu] + [\mu+1,0] \times [\lambda+\mu-1] + [\mu,1] \times [\lambda+\mu-1] + \ldots.$$

$$\ldots. + [\lambda+\mu-1,\mu] \times [1] + [\lambda+\mu,\mu-1] \times [1] + [\lambda+\mu,\mu] \times [0], \qquad (4.15)$$

giving the combinations in Table XI. Any of these terms which gives a non-standard tableau, i.e. has the first row shorter than the last, or has a negative number, must of course be ignored. This means that in the table under the heading 2Λ we get only the first entries if $\mu = 0$, the first two entries if $\mu = 1$ and so on.

TABLE XI

ϵ	2Λ
$2\lambda+\mu$	μ
$2\lambda+\mu-3$	$\mu+1,\ \mu-1$
$2\lambda+\mu-6$	$\mu+2,\ \mu,\ \mu-2$
. . .	. . . .
. . .	. . . .
$-\lambda-2\mu-3$	$\lambda+1,\ \lambda-1$
$-\lambda-2\mu$	λ

4.5. A projection integral

The "leading function" (or function of maximum weight) of a representation $(\lambda\mu)$ is defined as having the maximum value $(2\lambda+\mu)$ for ϵ and, for this

value of ϵ, the maximum value μ for ν. Thus, the representation labels appear as quantum numbers of the leading state. From (4.13) this gives $\lambda = N_z - N_x$, $\mu = N_x - N_y$ in terms of the Cartesian quantum numbers of the leading state. We see now that the high values of λ and μ, which were found to be lowest in energy, have the greatest deformation in their leading state, with μ measuring the departure from axial symmetry.

The leading function plays an important role and for brevity we denote it by φ, omitting the labels $\epsilon \Lambda \nu$. From its very definition, φ has the properties

$$A_{zx}\varphi = A_{zy}\varphi = A_{xy}\varphi = 0 \tag{4.16}$$

since for example it is impossible to put more quanta in the z-direction, as this would increase ϵ and we have defined φ to have the largest possible ϵ in the representation. Hence, from the relations (4.4), we have also

$$A_{xz}\varphi = -iL_y\varphi, \quad A_{yz}\varphi = iL_x\varphi, \quad A_{yx}\varphi = -iL_z\varphi \tag{4.17}$$

and of course $(A_{xx} - A_{yy})\varphi = \mu\varphi$, $(2A_{zz} - A_{xx} - A_{yy})\varphi = \epsilon\varphi$.

Thus, of the eight group operators of SU_3, three give zero, two are diagonal and three are equivalent to angular momentum operators, when acting on φ. This means that any group operation on φ is equivalent to a function of the angular momentum operators on φ and hence that any function in the representation may be constructed from the leading state by a function of angular momentum operators.

This is a crucial property because it means that if we expand

$$\varphi = \sum_K b_K \chi_K = \sum_K b_K \sum_L c(K,L)\psi(LK) = \sum_{K,L} a(K,L)\,\psi(LK) \tag{4.18}$$

where $a(K,L) = b_K c(K,L)$, then the representation is spanned by all functions $F(L_q)\,\psi(LK)$. But an angular momentum operation can only change the projection K of $\psi(LK)$ within the limits L to -L. This set of basic functions may therefore be denoted by $\psi(KLM)$ where M is the value of L_z and K is included to show from which term in the series (4.18) that particular M-value was formed. It is necessary to include K in the labelling of $\psi(KLM)$ because it is not generally true that, in the notation of (4.18),

$$(L_{+1})^{K'-K}\psi(LK) \propto \psi(LK').$$

We have thus introduced, in a precise manner, the additional label K which is sufficient, together with L and M, to classify completely the "angular momentum basis" $\psi(KLM)$ of the representation $(\lambda\mu)$.

The coefficients a(K,L) are calculated in the appendix and found to be non-zero for $K = \mu, \mu-2, \ldots 1$ or 0 and $L = K, K+1, \ldots \mu+\lambda$ except when $K = 0$, in which case $L = \mu+\lambda, \mu+\lambda-2, \ldots 1$ or 0.

The range of K-values here is just that used in (4.7) to enumerate the states but the range of L-values is slightly greater, rising to $\mu+\lambda$ rather than $K+\lambda$. When $\mu > 1$ this means that the set $\psi(KLM)$ is slightly over-complete in the high values of L. Furthermore, the functions $\psi(KLM)$ are not orthogonal with respect to K. We calculate the overlaps

$$(K'LM \mid KLM)$$

in the appendix and they are found to be small unless L approaches the "cut-off" in the rotational band at $L = \lambda + \mu$. Thus, in the representation (84) we find $(02|22) = 0.034$, $(04|24) = 0.139$, $(24|44) = 0.043$ and $(04|44) = 0.002$. Hence, although we may clearly overcome these difficulties of non-orthogonality and over-completeness, they are significant only for the high L-values of little physical interest.

The process of extracting $\psi(KLM)$ from φ is simply one of angular momentum projection, for if we denote by $\psi_\Omega(L'K')$ the function $\psi(L'K')$ referred to a set of axes at an orientation which we denote simply as Ω then

$$\psi_\Omega(L'K') = \sum_{M'} D^{L'*}_{M'K'}(\Omega)\,\psi(L'M') \tag{4.19}$$

so that

$$\frac{(2L+1)}{a(KL)}\int D^{L}_{MK}(\Omega)\varphi_\Omega\,d\Omega \approx \sum_{K'L'}\frac{a(K'L')}{a(KL)}(2L+1)\int D^{L}_{MK}(\Omega)\psi_\Omega(K'L'K')\,d\Omega$$

$$= \sum_{K'L'}\frac{a(K'L')}{a(KL)}(2L+1)\sum_{M'}\psi(K'L'M')\int D^{L'*}_{M'K'}(\Omega)\,D^{L}_{MK}(\Omega)\,d\Omega = \psi(KLM).$$

Here, Ω is short for the three Euler angles of rotation and we have made use of the orthogonality of the D-functions. Thus, all functions $\psi(KLM)$ in the angular momentum basis may be expressed as a projection integral

$$\psi(KLM) = \frac{2L+1}{a(KL)}\int D^{L}_{MK}(\Omega)\varphi_\Omega\,d\Omega \tag{4.20}$$

from just the single leading function φ in the Cartesian basis.

A formula for a(KL) is derived in the appendix from which also the overlaps may be calculated. In the limit $\lambda \gg L$, the overlaps vanish and for fixed K, $a(KL) \propto \sqrt{2L+1}$.

This is the limit of very strong deformations and if we were to suppose that the function φ_Ω could be separated $\varphi_\Omega \approx \delta(\Omega-\alpha)\psi_{int}(r)$ into a function of internal coordinates and a delta function of orientation angles α for the strongly deformed distribution φ, then (4.20) reduces to the simple rotational model form

$$D^{L}_{MK}(\alpha)\;\psi_{int}(r'). \tag{4.21}$$

Thus, a link with the rotational model is established but in (4.20) there are none of the difficulties of separation of collective coordinates inherent in the rotational model.

Furthermore, there is no problem of redundant states in the projection (4.20) because, having projected from the leading state φ of $(\lambda\mu)$ we disregard all other Cartesian functions in $(\lambda\mu)$, knowing that they can produce no new states by projection. To get other states we must move on to a different representation $(\lambda'\mu')$ and project from its leading state.

4.6. Quadrupole moments

The integral form (4.20) simplifies the calculation of matrix elements. Consider the group operator Q_q. We have

$$Q_q \psi(KLM) = \frac{(2L+1)}{a(KL)} \int D^L_{MK}(\Omega)\, Q_q \varphi_\Omega \, d\Omega \qquad (4.22)$$

but

$$Q_q = \sum_{q'} D^2_{qq'}(\Omega)\, Q_{q'}(\Omega)$$

transforming into the rotated frame. Now, using the reduction

$$D^L_{MK}(\Omega)\, D^2_{qq'}(\Omega) = \sum_{L'} (L2Mq \mid L'M+q)(L2Kq' \mid L'K+q') D^{L'}_{M+q,K+q'}(\Omega)$$

we have

$$Q_q \psi(KLM) = \frac{(2L+1)}{a(KL)} \sum_{L'} \sum_{q'} (L2Mq \mid L'M+q)(L2Kq' \mid L'K+q) \int D^{L'}_{M+q,K+q'}(\Omega) \left\{ Q_{q'} \varphi \right\}_\Omega d\Omega$$

where both $Q_{q'}$ and φ are now referred to the rotated frame. If we now use

$$Q_0\varphi = (2\lambda+\mu)\varphi, \quad (Q_2+Q_{-2})\varphi = \sqrt{6}\mu\varphi, \quad (Q_2-Q_{-2})\varphi = \sqrt{6}\, L_0\varphi, \quad Q_{\pm1}\varphi = \mp\sqrt{3}\, L_{\pm1}\varphi$$

from (4.4) and (4.17) and evaluate the products $L_q\varphi$ by expanding φ according to (4.18), we arrive at the formula

$$\begin{aligned} Q_q\psi(KLM) = \sum_{L'} \left(\frac{2L+1}{2L'+1}\right) (L2Mq \mid L'M+q) \Bigg[& \frac{a(KL')}{a(KL)} (L2KO \mid L'K) \\ & \times \{2\lambda + \mu + \tfrac{1}{2}(L'(L'+1) - L(L+1)+6)\} \psi(KLM) \\ & + \frac{a(K+2,L')}{a(K,L)} (L2K2 \mid L'K+2)(\mu+K+2)\psi(K+2LM) \\ & + \frac{a(K-2,L')}{a(K,L)} (L2K-2 \mid L'K-2)(\mu-K+2)\psi(K-2LM) \Bigg]. \end{aligned} \qquad (4.23)$$

Notice here that when $K = 1$, the $K-2$ term gives a diagonal contribution. The non-orthogonality of K precludes referring to the coefficients on the right hand side of (4.23) as matrix elements.

For large λ, (4.22) gives approximately

$$(KL' \,||\, Q \,||\, KL) = \sqrt{2L+1}\ .\ 2\lambda\, (L2KO \mid L'K)$$

which is precisely the same dependence on K and L as in the rotational model with a product wave function (4.21). The intrinsic quadrupole moment of that model is replaced by λ. As particles are added in the first half of a shell, the greatest value of λ, which in examples we have seen to describe the lowest levels, increases with the number of particles.

The operator Q_q is the quadrupole moment of the mass so that this increases with the number of particles instead of remaining near the single particle value. For $T = 0$ nuclei this mass moment is also the quadrupole moment of the charge.

4.7. Central-force matrix elements

In this section, we develop a method for calculating the matrix elements of a two-body central force in L-S coupled, antisymmetric states, classified

according to the SU_3 group, for k particles in an oscillator shell N. Such a state is denoted by

$$\Psi(TS[f](\lambda\mu)KL\ M_T M_S M)$$

where the antisymmetrization has been carried out in the usual way by summing over products of orbital functions of symmetry [f] and charge-spin functions of adjoint symmetry $[\tilde{f}]$. We make use of the projection formula

$$\Psi(TS[f](\lambda\mu)KLM_TM_SM) = \frac{(2L+1)}{a(\lambda\mu KL)}\int D^L_{MK}(\Omega)\Phi_\Omega(TS[f](\lambda\mu)M_TM_S)\,d\Omega, \qquad (4.24)$$

which is simply (4.20) with all the charge-spin labels included on both sides. They play no part in the projection which refers only to the orbital co-ordinates. With central forces, TSM_T and M_S are good quantum numbers and for brevity we henceforth omit these symbols from our equations. We further simplify the notation by writing γ for $[f](\lambda\mu)$. Let H be the Hamiltonian, containing only central forces. Then

$$H\Psi(\gamma KLM) = \frac{(2L+1)}{a(\lambda\mu KL)}\int D^L_{MK}(\Omega)\,H_\Omega\Phi_\Omega(\gamma)\,d\Omega \qquad (4.25)$$

since H is invariant under rotations.

Now expand $H\Phi$ in the complete set of Cartesian functions

$$H\Phi(\gamma) = \sum_{\gamma'\epsilon\Lambda\nu} h(\gamma,\gamma'\epsilon\Lambda\nu)\ \Phi(\gamma'\epsilon\Lambda\nu) \qquad (4.26)$$

with

$$h(\gamma,\gamma'\epsilon\Lambda\nu) = (\Phi(\gamma'\epsilon\Lambda\nu)|\ H\ |\Phi(\gamma)).$$

Since $\Phi(\gamma'\epsilon\Lambda\nu)$ is of the same representation as its leading state $\Phi(\gamma')$, we must be able to express it as some function of the group operators acting on $\Phi(\gamma')$. But in section 4.5 we showed that any group operation on a leading state Φ was equivalent to some function of the angular momentum operators. Hence we may write

$$\Phi(\gamma'\epsilon\Lambda\nu) = F(\lambda'\mu'\epsilon\Lambda\nu\,;\,\underset{\sim}{L})\Phi(\gamma'), \qquad (4.27)$$

where $\underset{\sim}{L}$ is an abbreviation for the three components of the angular momentum operators, so that (4.26) becomes

$$H\Phi(\gamma) = \sum_{\gamma'} G(\gamma\gamma';\underset{\sim}{L})\,\Phi(\gamma') \qquad (4.28)$$

where we have defined

$$G(\gamma\gamma';\underset{\sim}{L}) = \sum_{\epsilon\Lambda\nu} h(\gamma\gamma'\epsilon\Lambda\nu)\,F(\lambda'\mu'\epsilon\Lambda\nu;\underset{\sim}{L}). \qquad (4.29)$$

The operator on the right-hand side of (4.28) may be readily evaluated by expanding $\Phi(\gamma')$ into its angular momentum components as in (4.18). Thus, defining the matrix elements

$$g(\gamma\gamma', L'K'K'') = (L'K''|\ G(\gamma\gamma';\underset{\sim}{L})\ |\ L'K') \qquad (4.30)$$

(4.28) becomes

$$H\Phi(\gamma) = \sum_{\gamma' K'L'} G(\gamma\gamma';\ \underset{\sim}{L})\,\Psi(\gamma' K'L' K')\,a(\lambda'\mu' K'L')$$

$$= \sum_{\gamma' K'L' K''} g(\gamma\gamma', L'K'K'')\,a(\lambda'\mu' K'L')\,\Psi(\gamma' K'L' K''). \qquad (4.31)$$

This result is equally true for functions referred to the rotated frame so that we may add the suffix Ω to Φ and Ψ in (4.31). Inserting (4.31) into (4.25) gives

$$H\Psi(\gamma KLM) = (2L+1) \sum_{\gamma', K', K'L'} \frac{a(\lambda'\mu' K'L')}{a(\lambda\mu KL)}\, g(\gamma\gamma'; L'K''K') \int D^{L}_{MK}(\Omega)\,\Psi_{\Omega}(\gamma' K'L' K'')\,d\Omega.$$

Transforming Ψ_{Ω} back to the original axes

$$\Psi_{\Omega}(\gamma' K'L' K'') = \sum_{M'} D^{L'*}_{M'K''}(\Omega)\,\Psi(\gamma KLM)$$

and using the orthogonality of the rotation matrices, we obtain

$$H\Psi(\gamma KLM) = \sum_{\gamma'} \sum_{K'} \frac{a(\lambda'\mu' K'L)}{a(\lambda\mu KL)}\, g(\gamma\gamma'; LK'K)\,\Psi(\gamma' K'LM). \qquad (4.32)$$

By defining $a(-KL) = (-)^{\lambda+\mu+L}\,a(KL)$ we ensured that $\Psi(\gamma\text{-}KLM) = \Psi(\gamma KLM)$ so that only positive K need be discussed. However, the sum over K' in (4.32) includes both positive and negative values which may be collected together, giving

$$H\Psi(\gamma KLM) = \sum_{\gamma' K' \geq 0} \left\{ \frac{a(\lambda'\mu' K'L)\,g(\gamma\gamma', LK'K) + a(\lambda'\mu'\text{-}K'L)\,g(\gamma\gamma', L\text{-}K'K)}{a(\lambda\mu KL)} \right\} \Psi(\gamma' K'LM)$$

$$= \sum_{\gamma'} \sum_{K' \geq 0} \frac{a(\lambda'\mu' K'L)}{a(\lambda\mu KL)}\, g(\gamma\gamma', LK'K) + (-)^{\lambda+\mu+L}\, g(\gamma\gamma', L\text{-}K'K)\Psi(\gamma' K'LM) \qquad (4.33)$$

$$= \sum_{\gamma'} \sum_{K' \geq 0} \frac{a(\lambda'\mu' K'L)}{a(\lambda\mu KL)}\, \tilde{g}(\gamma\gamma', LK'K)\Psi(\gamma' K'LM)$$

where we have defined, for $K' > 0$,

$$\tilde{g}(\gamma\gamma', LK'K) = g(\gamma\gamma', LK'K) + (-)^{\lambda+\mu+L}\, g(\gamma\gamma', L\text{-}K'K). \qquad (4.34)$$

Because functions with the same γ but different K are not orthogonal, the matrix $\tilde{g}$ is not symmetric.

One may, however, use the known symmetry of the matrix $(\Psi(\gamma K'L'M)\,|H|\,\Psi(\gamma KLM))$, together with (4.33) and a knowledge of the overlaps, to relate the elements of $\tilde{g}$ above the diagonal to those below.

We are, of course, interested in the eigenvalue problem $H\tilde{\Psi}_n = E_n\,\tilde{\Psi}_n$ and if we write

$$\tilde{\Psi}_n = \sum_{\gamma} \sum_{K \geq 0} \alpha_n(\gamma KL)\,a(\lambda\mu KL)\,\Psi(\gamma KLM) \qquad (4.35)$$

and substitute into (4.33) we get the system of equations

$$\sum_{\gamma} \sum_{K\geq 0} \tilde{g}(\gamma\gamma', LK'K)\alpha_n(\gamma KL) = E_n\alpha_n(\gamma' K'L) \qquad (4.36)$$

for the energies E_n and the coefficients $\alpha_n(\gamma KL)$. Although (4.36) is exact, for central forces, we approximate in practice by taking a finite set of γ. The E_n and α_n then emerge as the latent roots and vectors of the non-symmetric matrix $\tilde{g}(\gamma\gamma', LK'K)$. In deriving (4.36) from (4.33) we have assumed that the functions $\Psi(\gamma KLM)$ are linearly independent and, although this is generaly valid, it is violated when $L > \lambda + 1$. States with such high values of L generally lie high in the spectrum and are therefore of little physical interest. They may of course be dealt with if necessary by using the known overlaps to reduce the sum in (4.33) and (4.36) to one over a linearly independent set.

4.8. Discussion of the central-force formula

Having given this brief and rather formal derivation of (4.36) we now return to give a little more detail of the steps involved in determining the matrix $\tilde{g}$. The main steps are

(i) Calculation of the matrix elements $h(\gamma\gamma', \epsilon\Lambda\nu)$ of (4.26);
(ii) Derivation of the operators $F(\lambda'\mu', \epsilon\Lambda\nu\,;\, \underset{\sim}{L})$ of (4.27);
(iii) Construction of the operators $G(\gamma\gamma'; \underset{\sim}{L})$ of (4.29);
(iv) Calculation of the matrix elements $g(\gamma\gamma', L'K'K'')$ of (4.30);
(v) Construction of $\tilde{g}$ from g using (4.34).

At first sight, step (i) might appear just as difficult as the original problem of evaluating matrix elements $(\Psi'|\, H\,|\Psi)$, calling for the use of complicated fractional parentage coefficients. There is, however, an important difference. The functions Φ, being in a Cartesian representation, involve no orbital vector coupling, and especially for those γ lying lowest in the spectrum, have a very simple structure in terms of single-particle Cartesian states. We therefore carry out this step by expressing the Φ in terms of Slater determinants and then evaluating $h(\gamma\gamma', \epsilon\Lambda\nu)$ directly. The symmetry of H under rotations and parity inversion which implies invariance under the separate parity inversions of the Cartesian axes, leads to selection rules in ϵ and ν. Since these "Cartesian parities" of the wave functions Φ are given by the parities of the total Cartesian oscillator numbers N_x, N_y and N_z and since these numbers are related to ϵ and ν by the equations

$$N_z = \tfrac{1}{3}(\epsilon + kN)$$

$$N_x = \tfrac{1}{6}(2kN + 3\nu - \epsilon)$$

$$N_y = \tfrac{1}{6}(2kN - 3\nu - \epsilon)$$

it follows that the matrix elements $h(\gamma, \gamma', \epsilon\Lambda\nu)$ will be non-zero only if ϵ changes by units of 6 and $(3\nu+\epsilon)$ by units of 12. A further selection rule puts an upper limit on such changes. In a shell with single particle oscillator number N, the maximum change that any two-body operator can produce in any of the numbers N_x, N_y or N_z is 2N. Hence the maximum changes in ϵ and ν are 6N and 4N respectively. This greatly reduces the number of possible matrix elements when N is small.

Step (ii) depends entirely on the properties of the group SU_3. Defining p, q and r by the relations $3(p+q) = 2\lambda + \mu - \epsilon$, $p - q = 2\Lambda - \mu$ and $2r = 2\Lambda - \nu$, and using the methods of RACAH [10], it may be shown that, in terms of Cartesian operators

$$F(\lambda\mu\epsilon\Lambda\nu;\underset{\sim}{L}) \equiv N(pqr)A^r_{yx}(A_{yx}A_{xz} - A_{yz}(A_{xx} - A_{yy} + 1)^q)\, A^p_{xz} \qquad (4.37)$$

where

$$N(pqr) = \left\{\frac{\lambda!\mu!(\lambda+\mu+1)!p!q!r!(\mu+p+1)!}{(\lambda-p)!(\mu-q)!(\lambda+\mu+1-q)!(\mu+p-q-r)!(\mu+p-q+1)!}\right\}^{-\frac{1}{2}}.$$

The equivalences of (4.17) together with the commutation relations then enable us to derive $F(\lambda\mu\epsilon\Lambda\nu;\underset{\sim}{L})$ in terms of angular momentum operators alone. For the most important cases, $(p+q) \leqslant 2$, we may write

$$F(\lambda\mu\epsilon\Lambda\nu;\underset{\sim}{L}) = \tfrac{1}{2}N\sum_i L_0^{2i}\{A_i + B_i L^2 + C_i(L^2_{+1} + L^2_{-1}) + D_i L_0(L^2_{+1} - L^2_{-1})\} \qquad (4.38)$$

and the coefficients A_i, B_i, C_i and D_i have been given in Table XII as functions of λ and μ.

Steps (iii) and (v) need no discussion and step (iv) is also elementary, involving just the matrix elements of angular momentum operators

$$(LK' \mid L_0 \mid LK) = K\delta(K', K) \qquad (4.39)$$

$$(LK' \mid L_{\pm 1} \mid LK) = \mp\{(L \mp K)(L \pm K + 1)/2\}^{\frac{1}{2}}\, \delta(K', K \pm 1).$$

We are, of course, most interested in the matrix elements of $\mathcal{G}^r$ which are diagonal in γ and we now discuss the significance of the contribution to such matrix elements from the different values of $\epsilon\Lambda\nu$ in the sum (4.29). Considering them in order of decreasing ν and ϵ, the first contribution contains the diagonal element of H in the leading state and the operator $F = 1$. In practice, we find this to be the largest contribution and, because F is just the unit operator, it leads to a degeneracy between levels of different K and L, though of course separating levels with different γ. The contributions from maximum ϵ but less than maximum ν lead to F-operators containing powers of L_0^2 only. These give rise to a separation of bands, labelled by K, but still retain a degeneracy between states with the same K but different L. When $\epsilon = \epsilon_{max} - 6$, the next highest value of ϵ to give non-zero contributions, the F-operators contain terms in L^2, L^2_{+1} and L^2_{-1} which lead to a rotational spectrum in L together with a mixing of states with different K. In practice we have found that the bands are sufficiently separated, by the L_0^2 term, that the K-mixing is generally small. Lower values of ϵ lead to terms like L^4 and hence to departures from the pure rotational spectrum, but we have found their contribution to be small for the low L-values of physical interest.

5. APPLICATION TO NUCLEI IN THE ds-SHELL

The coupling scheme described here has nothing new to add to the p-shell nuclei $4 < A < 16$ which have been satisfactorily described [3] in the shell model. From a mathematical point of view it is applicable to any oscillator configuration, even to those which are not lowest in energy for a par-

TABLE XII

LOWEST REPRESENTATIONS FOR EVEN-EVEN NUCLEI IN THE ds-SHELL

Configuration	Parity	O^{16}	O^{18}	Ne^{20}	Ne^{22}	Mg^{24}	Mg^{26}	Si^{28}	Si^{30}	S^{32}	S^{34}	A^{36}	A^{38}	Ca^{40}
$s^4p^{12}(d,s)^{A-16}$	+	(00)	(40)	(80)	(82)	(84)	(10, 2)	(12, 0)	(2, 10)	(4, 8)	(28)	(08)	(04)	(00)
$s^4p^{11}(d,s)^{A-15}$	-	(21)	(61)	(82)	(84)	(94)	(11, 2)	(1, 12)	(3, 10)	(39)	(19)	(07)	(03)	-
$s^4p^{12}(d,s)^{A-17}(f,p)$	-	-	(50)	(90)	(11, 1)	(11, 3)	(12, 3)	(14, 1)	(4, 11)	(69)	(68)	(48)	(36)	(32)

ticular number of particles. From a physical point of view, however, the spin-orbit force, and the trend towards an average field more square than the oscillator, combine to cause a mixing of oscillator configurations. For this reason, applications of the method have been confined so far to the ds-shell where this mixing does not seem to occur to an appreciable extent.

5.1. The spectrum of Mg^{24}

Only a few detailed applications of the method [19, 22] have been made so far, the most interesting being to Mg^{24}. The experimental spectrum shown in Fig. 4 contains a K = 0 band, followed at about 4 MeV by a k = 2 band and at 6.43 MeV by a second J = 0 level. Working in L-S coupling for this "α-particle" nucleus the lowest partition [44] contains the U_3 representation (84) as its lowest. We would therefore expect to find bands with K = 0, 2 and 4 but as the energy dependence on K is roughly K^2, the K = 4 band is probably very high in energy, about 15 MeV. The spectrum resulting from the use of the same two-body force as was used in the intermediate coupling calculations for A = 19, is shown in Fig. 1, when only this single representation was used. Although the rotational pattern is undoubtedly given, the absolute energies are too small by a factor of 2. The admixing of higher representations and the inclusion of spin-orbit forces increases these energies without much change in relative positions. Initial calculations indicate that the mixing is as high as 15% but that the mixing of K is only 1% in the lowest L = 2 state, increasing somewhat for greater values of L. The second J = 0 level at 6.4 MeV must belong to a higher representation, probably (46), although at this energy it is possible that higher configurations are entering.

The γ-decay scheme of Mg^{24} is shown in Fig. 5, all transitions being essentially E2 in character. The branching ratio of the second J = 2 level demands a K-mixing of about 1% which is of the same order of magnitude as that calculated here but as the decay is very sensitive to this small component of the wave function, it is difficult to calculate the ratio reliably.

5.2. Even parity levels of even-even nuclei

Experience has shown that the even-even nuclei in the p-shell and early ds-shell are reasonably well described in L-S coupling, although this approximation deteriorates towards the end of a shell. For an even-even nucleus in L-S coupling the lowest states have S = 0 and therefore J = L, while in the particular cases of Mg^{24} and Ne^{20} one finds that the lowest levels in the L-S limit are well described by the lowest representation $(\lambda\mu)$. One is therefore tempted to survey the entire ds-shell and suggest, on the basis of the values [17] of $(\lambda\mu)$, which bands are expected to lie low in each nucleus.

In the first row of Table XIII we give the representations $(\lambda\mu)$ expected to lie lowest in each nucleus, the possible K-values then forming a sequence decreasing from K = min (λ,μ) by units of two.

If we now assume that the ordering of bands within a representation is the same as that found in detailed calculations for Ne^{20} and Mg^{24}; namely that the low values of K occur lowest in energy, then our first conclusion is that all nuclei considered have a K = 0 band lowest. Although this is in agreement with the known data it is a conclusion in common with almost any model. However, we may go further than this. The nuclei O^{18}, Ne^{20}, Si^{28}, A^{36}, A^{38} have min $(\lambda,\mu) = 0$ in their lowest representation which there-

fore consists of a single K = 0 band, while the remaining nuclei Ne^{22}, Mg^{24}, Mg^{26}, Si^{30}, S^{32}, S^{34} contain K = 2 and in some cases K = 4 bands in addition to a K = 0 band in their lowest representation. We should therefore expect to find the second 2^+ level above the second 0^+ level in the first group of nuclei, both levels coming from a higher representation. In contrast, the second group of nuclei may very well have their second 2^+ level below the second 0^+ level, the former being the first member of a K = 2 band from the lowest representation and the latter the lowest level of a higher representation. Only a detailed calculation of the competition between an excited band (K = 2) of the lowest representation and the lowest band of the first excited representation can determine this ordering in the second group of nuclei.

The experimental data on these nuclei is unfortunately rather sparse but confirms the picture drawn above. Almost nothing is known of the spectra of A^{36} and A^{38} but in the remaining three nuclei of the first group, the second 2^+ level is in fact found above the second 0^+ level. In four of the six nuclei of the second group the second 2^+ is found below the second 0^+.

5.3. Odd parity levels of even-even nuclei

According to the shell model, the low odd-parity levels arise from configurations in which either a p-particle has been excited into the ds-shell or a ds-particle has been excited into the fp-shell. The U_3 classification for these excited configurations may be derived by the same methods and, assuming that the representations with high λ and μ are lowest also in the excited configurations, we have listed in rows 2 and 3 of Table XIII the representations expected to lie lowest in each configuration. It is seen that at the beginning of the shell, the largest $(\lambda\mu)$ values are obtained by exciting a p-particle, while in the rest of the shell the excitation of a ds-particle gives the largest values and is therefore expected to give the lowest energy levels. In considering excited configurations one must always remove spurious states involving excitation of the motion of the centre of mass, but the representations listed, having the highest values for λ and μ are entirely free of such spuriousness.

The methods of section 4.7 may equally well be applied to the calculation of spectra in these excited configurations but such a programme has only just been started. However, the indication from the table is that $K = 1^-$ bands will appear low in the spectra of odd parity levels of these even-even nuclei in the first half of the shell. The single exception to this rule is Ne^{20}, which has the two representations (90) and (82) competing for the lowest odd parity levels. The (82) representation contains a K = 2 band a K = 0 band with even J and a preliminary calculation indicates that this K = 0 band is above the K = 2 band. Although this may seem strange against the background of the even parity systematics, in which the K = 0 band is invariably found below the K = 2 band of the same representation, we must remember that the K = 0 band of the (82) representation has even J but odd parity. Such a band cannot arise in the extreme model of rigid body rotations and it is therefore not surprising to find it raised in energy. In Ne^{20} the lowest bands to be expected for the odd parity levels are the K = 2 from (82) and the K = 0, with odd J, from (90). Recent experimental work [23] suggests that just such K = 2 and K = 0 bands are in fact present.

We mentioned above that $K = 1^-$ bands are expected low in the other even-even nuclei considered in the first half of the shell. The spectrum of

TABLE XIII

THE OPERATORS

$$F(\lambda\mu\epsilon\Lambda\nu;L) = \tfrac{1}{2}N\Sigma_i L_0^{2i}\{A_i + B_i L^2 + C_i (L_{+1}^2 + L_{-1}^2) + D_i L_0 (L_{+1}^2 - L_{-1}^2)\}$$

where $3(p+q) = 2\lambda+\mu-\epsilon$, $(p-q) = 2\Lambda-\mu$, $2r = 2\Lambda-\nu$.

rqp	i	A_i	B_i	C_i	D_i
200	0	2μ			
	1	-2			
400	0	$6\mu(\mu-2)$			
	1	$-4(3\mu-4)$			
	2	2			
002	0	2λ	-1	1	
	1	1			
202	0	$2\lambda(\mu+2)$	$-(\mu+2)$	$(\mu+2)$	
	1	$\mu+2-2\lambda$	1	-1	
	2	-1			
402	0	$6\lambda\mu(\mu+2)$	$-3\mu(\mu+2)$	$3\mu(\mu+2)$	
	1	$3\mu^2+6\mu-12\lambda\mu-8\lambda$	$2(3\mu+2)$	$-2(3\mu+2)$	
	2	$2(\lambda-3\mu-2)$	-1	1	
	3	1			
111	0				$\mu+2$
	1	$-(2\lambda+\mu+2)$	1	-1	
	2	-1			
311	0				$(\mu+2)(3\mu-2)$
	1	$(2\lambda+\mu+2)(2-3\mu)$	$(3\mu-2)$	$(2-3\mu)$	$-(\mu+2)$
	2	$2(\lambda-\mu+2)$	-1	1	
	3	1			
020	0	$2\mu^2(\lambda+\mu+1)$	$-\mu^2$	$-\mu(\mu+2)$	$2(\mu+1)$
	1	$-(2\lambda+2\mu-\mu^2+2)$	1	-1	
	2	-1			
220	0	$2\mu^2(\mu-2)(\lambda+\mu+1)$	$-\mu^2(\mu-2)$	$-\mu(\mu-2)(\mu+2)$	$2(\mu+1)(\mu-2)$
	1	$-(2\lambda+\mu)(\mu-1)(\mu+2)$ $-5\mu^2+4$	$(\mu-1)(\mu+2)$	$(\mu^2+\mu+2)$	$-2(\mu+1)$
	2	$2\lambda+\mu+4-\mu^2$	-1	1	
	3	1			

levels in that band may, however, deviate markedly from the J(J+1) rule. The quadratic terms in the operator $\underset{\sim}{L}$ which play an important part in determining the spectrum, give rise not only to the L(L+1) term, but also to coupling between the parts of the wave function with K = + 1 and K = -1. It

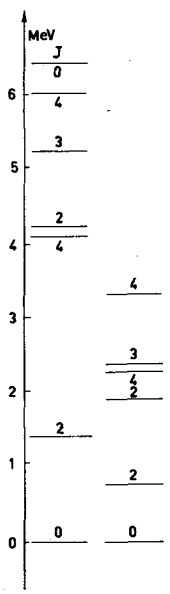


Fig. 4

Experimental and calculated spectrum of Mg^{24}

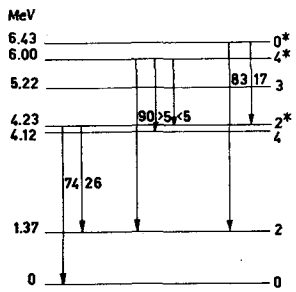


Fig. 5

The γ-decay scheme of Mg^{24}

then follows that their contribution to the energies depends on J(=L) through the expression $J(J+1)\{1+B(-1)^J\}$. This situation is similar to the familiar decoupling of $K = \frac{1}{2}$ bands in odd nuclei and we have plotted the resulting spectrum as a function of B in Fig. 6. Without making detailed calculations for B, one might see whether this formula may be fitted to the known data

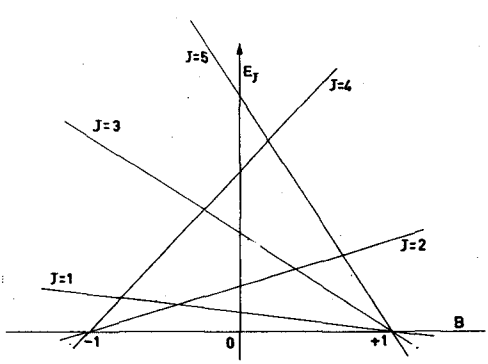


Fig. 6

Spectrum for a decoupled K=1 band

in a purely empirical way. Unfortunately, one needs to know the positions of four levels to do this and in none of the nuclei in question is this information yet available. It is also possible that mixing of rotational bands will obscure any test of the formula. In O^{16} the lowest representation (21) for the odd parity levels has only three levels J = 1, 2 and 3 and it is trivial that these may be fitted to the formula above, demanding B = 1.5. The fact that earlier calculations [24] reproduced the experimental pattern of levels in this nucleus implies that calculation would also reproduce essentially this value for B.

5.4. Odd nuclei

In odd nuclei, since $S \neq 0$, the spin-orbit force must be taken into account. If it is very large we shall have pure j-j coupling and no rotational features. If it is very small, we get a small splitting of the levels of given orbital angular momentum L according to $J = L + S, \ldots |L - S|$, again with no rotational spectrum. We now investigate the coupling of S to the orbital angular momentum of states classified by the group U_3. Use K_L to denote the quantity previously denoted by K and consider

$$\Psi([f]\,TS(\lambda\mu)K_L LJM) = \sum_{M_S M_L} \Psi([f]\,TS(\lambda\mu)K_L LM_S M_L) C^{JM}_{SM_S LM_L}$$

$$= \sum_{M_S M_L} C^{JM}_{SM_S LM_L} \frac{(2L+1)}{a(K_L L)} \int D^{L}_{MK_L}(\Omega)\, \Phi'_{\Omega}([f]\,TS(\lambda\mu)M_S)\, d\Omega \qquad (5.1)$$

where we have used the usual vector coupling and Φ'_{Ω} denotes the antisymmetric function in T, S and Cartesian co-ordinates but in which only the orbital co-ordinates are referred to the rotated frame. If we denote by Φ_{Ω} the corresponding function in which the spin co-ordinates are also referred to this frame then

$$\Phi'_{\Omega}(M_S) = \sum_{K_S} D^{S}_{M_S K_S}(\Omega)\, \Phi_{\Omega}(K_S) \qquad (5.2)$$

where we have, for brevity, dropped the symbols $[f]\,TS(\lambda\,\mu)$. Thus

$$\Psi(K_L LJM) = \sum_{K'_S} \sum_{M_S M_L} (SLM_S M_L \,|\, JM) \frac{(2L+1)}{a(K_L L)} \int D^{L}_{M_L K_L}(\Omega)\, D^{S}_{M_S K_S}(\Omega) \Phi_{\Omega}(K_s)\, d\Omega.$$

Combining the two D-functions and making use of an orthogonality relation in the Wigner coefficients we find that

$$\Psi(K_L LJM) = \sum_{K(K_S)} \frac{(2L+1)}{a(K_L L)} (SLK_S K_L \,|\, JK) \int D^{J}_{MK}(\Omega)\, \Phi_{\Omega}(K_S)\, d\Omega \qquad (5.3)$$

where we have defined $K = K_L + K_S$.

Thus, a function of definite L is expressed as a sum over functions projected with a definite (total) K. If we now invert using another orthogonality relation for the Wigner coefficients we have

$$(2J+1) \int D^{J}_{MK}(\Omega) \Phi_{\Omega}(K_S)\, d\Omega = \sum_{L} a(K_L L)(SLK_S K_L \,|\, JK) \Psi(K_L LJM)$$

or

$$\Psi(K_L KJM) = \frac{1}{a(K_L, SKJ)} \sum_L a(K_L L)\,(SLK_S K_L | JK)\,\Psi(K_L LJM)$$

where we have defined

$$\Psi(K_L KJM) = \frac{(2J+1)}{a(K_L, SKJ)} \int D^J_{MK}(\Omega)\,\Phi_\Omega(K_S)\,d\Omega. \qquad (5.4)$$

Here, $a(K_L, SKJ)$ is defined as the normalization coefficient, related to the previous coefficients by

$$a(K_L, SKJ)\,a(K_L, SK'J)\,(K_L KJM\,|K_L K'JM)$$

$$= \sum_L (SLK_S K_L | JK)(SLK'_S K_L | JK')\,a^2(K_L L).$$

Alternatively, these coefficients may be calculated directly by the method used in the appendix for the $a(K_L L)$. These new functions $\Psi(K_L KJM)$ are now neither orthogonal in K_L or K, but nearly so for large $\lambda \gg L$.

What physical significance does this new quantum number K possess, implying a mixing of L but not of S or [f] ? The lowest partition [f] for an odd nucleus contains a single value of $S = \frac{1}{2}$ but many values of L. If, therefore, we neglect mixing of [f] but allow mixing of these different L-values with a spin-orbit force it is possible that such a coupling scheme could emerge. We shall see that in fact it does, provided that the spin-orbit force is large compared with the energy difference of different L-values, but small compared with the energy difference between different $(\lambda\mu)$ or K_L so that these labels are preserved.

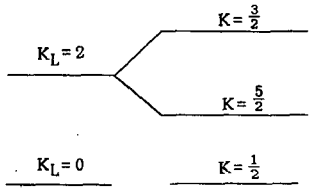
This result may be seen by considering an intermediate coupling Hamiltonian.

$$H = \sum_i \xi\,(s_i.\ell_i) + \sum_{i<j} V(r_{ij}).$$

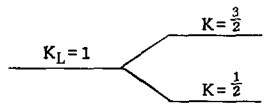
We again evaluate $H\Psi$ by the methods of 4.7 with the leading function $\Phi(K_S)$ now also involving spin co-ordinates. Let us now investigate the form of the most important equivalent operators which arise with this Hamiltonian. They will depend on L and S, but as the integral (5.4) does not involve L it is most simple to write L = J-S to obtain an operator in J and S. As in section 4.8, the central force gives the form $\alpha_0+\alpha_1 L_0^2+\alpha_2 L^2+\ldots$. The constant term will separate different [f] and $(\lambda\mu)$, while the term in α separates different K_L in a representation, without removing the degeneracy of different $K = K_L + K_S$ for fixed K_L. If we now write $L^2 = (J - S)^2 = J^2+S^2 -2(J.S)$ we shall get a constant $S(S+1)$ from S^2 and a $J(J+1)$ splitting from J^2. The term (J.S), however, will cause mixing of different K and the same K_L, tending to destroy the coupling scheme defined by K in (5.4). This is of course to be expected, since we know that the central force alone preserves L and must necessarily mix values of K. But if we now consider the spin-orbit part of H we see that it will cause a splitting of different K with the same K_L through the term $s_0\ell_0$. If this is sufficiently large, it will reduce the effect of (J.S) in mixing K and thus tend towards the wave functions (5.4) labelled by K and containing rotational spectra in J. The remaining terms $s_{+1}\ell_{-1}$ etc. will cause the band-mixing and $K = \frac{1}{2}$ decoupling familiar in the rotational model.

Calculations for odd-nuclei are still in an early stage, but there is some encouragement from Mg^{25} where rotational bands $K = \frac{1}{2}^+$, $\frac{5}{2}^+$ and $\frac{1}{2}^+$ have

been observed. In the U_3-model, the greatest symmetry [441] of the nine particles contains representations (66) and (93) low in energy and close together. The spin-orbit force causes the $K = \frac{5}{2}$ of (66) and the $K = \frac{1}{2}$ of (93)



(66)



(93)

to be depressed. It seems likely, therefore, that the three bands observed belong to these two representations. The configurational compositions of the leading functions of these two representations are simply $\varphi_0^4 \varphi_+^4 \varphi_2$ and $\varphi_0^4 \varphi_+^4 \varphi_-$ respectively.

6. CONCLUDING REMARKS

We have discussed some of the physical and mathematical problems encountered in setting up energy matrices in shell model configurations. In particular, we have seen how rotational features exist within a degenerate oscillator level, but the closed oscillator shells have not been excited. The very presence of the collective motion in the particles outside closed shells must induce some polarization of the closed shell core which we have not yet taken into account. So far as the energies are concerned, this effect may possibly be taken into account by some renormalization of the strength of the two-body force among the outside particles.

The most noticeable effect of such a quadrupole polarization will of course be in the quadrupole moments. The picture of weak coupling [25] to quadrupole oscillations of the core leads to an additional contribution to the quadrupole moment which is proportional to the mass quadrupole moment of the outside particles. A correction of this kind is very simple to evaluate in the U_3 wave functions and it is quite possible that less specific assumptions about the type of core excitation lead to a correction of the same kind. The quadrupole moment and electric quadrupole operator is therefore

$$e \sum_{i=1}^{k} \tfrac{1}{2}(1-\tau_0(i))\, Y_q^2(i) r^2(i) + \alpha e \sum_{i=1}^{k} Y_q^2(i) r^2(i) \qquad (6.1)$$

where αe is the additional "effective charge" on all outside particles due to the polarization. It must be remembered that this effective charge is specific to quadrupole effects.

In O^{17} and F^{17}, the E2 lifetimes for the $\frac{1}{2}^+ \to \frac{5}{2}^+$ decay of the first excited states are measured, $2.55 \pm 0.13 \times (10)^{-10}$ and $4.45 \pm 0.22\ (10)^{-10}$ respectively. These are both of the order of the single-particle value, even though O^{17} has a single neutron outside the closed shell. Calculations from (6.1) involve the wave function size parameter b which should be given a value of about 1.8 f to fit the nuclear radius. From the two lifetimes quoted

we may deduce both α and b, finding $\alpha = 0.50$ and b = 1.82 f, in good agreement with the radius estimate for b. If we now calculate the quadrupole moment of O^{17} we find $-3.26\ e(10)^{-26}\ cm^2$, close to the experimental figure of $-2.65\ e(10)^{-26}\ cm^2$.

In the nuclei of mass 18 and 19, the intermediate coupling wave functions [4] were used to calculate a number of E2 lifetimes using the same values for α and b as in the nuclei of mass 17. The results, shown in Table XIV, again show consistent agreement with experiment.

TABLE XIV

Nucleus	$J_i \to J_f$	E_i (MeV)	E_f (MeV)	τ(sec) cal.	τ(sec) exp.
F^{18}	$3 \to 1$	0.94	0	$5\,(10)^{-11}$	$<2\,(10)^{-10}$
	$5 \to 3$	1.13	0.94	$1.7(10)^{-7}$	$2\,(10)^{-7}$
F^{19}	$\frac{5}{2} \to \frac{1}{2}$	0.198	0	$0.97(10)^{-7}$	$1.25(10)^{-7}$
Ne^{19}	$\frac{5}{2} \to \frac{1}{2}$	0.241	0	$1.85(10)^{-8}$	$1.8\,(10)^{-8}$

Such detailed intermediate coupling wave functions are not available for heavier nuclei, but if we use the L-S coupled U_3 wave functions for Ne^{20} and Mg^{24} we calculate lifetimes $9.6\ (10)^{-13}$ and $0.71\ (10)^{-13}$ for the $2 \to O$ decay of the first excited states in these nuclei. The measured values $5.6\ (10)^{-13}$ and $0.76\ (10)^{-13}$ are in promising agreement indicating that the same effective charge of 0.5 e is operative even when there are eight particles outside the shell. The core excitation and the outside particles contribute about equally to the matrix element.

For magnetic moments, M1 transitions and β-decay, agreement is generally reached in the lowest oscillator configuration. This is to be expected since, in perturbation theory, there would be no first-order corrections to these processes in contrast with the quadrupole operators.

The previous remarks about core polarization lead one to consider the possibility of using a non-spherical average field for the particles (see the notes of S.T. BELYAEV [12]). This would provide a natural way of departing from the lowest configuration in a spherical field and would immediately produce quadrupole effects of the required size. It does, however, bring with it difficulties concerning the stability of the deformations in the field, the calculation of inertial parameters of the field and other questions of internal consistency which do not arise in the spherical field.

All applications of the U_3 coupling scheme in this course have been made to the ds-shell. In higher oscillator shells the classification of states and evaluation of matrix elements would proceed in just the same way. For example, in the fp-shell, a single particle would transform like a tensor of rank three in the U_3 space, labelled by (30). Experimentally, the single-particle spectrum in Ca^{41} has an $f_{\frac{7}{2}}$ level lowest and well separated from the other levels of the fp-shell.

Because of this, the nuclei at the beginning of the fp-shell appear to be well understood on the basis of the pure $f_{\frac{7}{2}}$ shell. In other words, we might here expect the spin-orbit force to disturb the U_3 coupling. However, most

calculations in the $f_{\frac{7}{2}}$ shell have been made for the nuclei with neutrons alone outside the shell, i.e. the states with high isotopic spin. These are the very states which have the least orbital symmetry and in which therefore the central force is weakest, so that they are the least favourable for the U_3 scheme. There is not very much experimental information on the nuclei with smaller values of T but it will be very surprising if they do not show the same collective efforts as in the ds-shell, though perhaps in a less pure form. For heavier nuclei, the intrusion of levels from higher oscillator configurations, for example the $g_{\frac{7}{2}}$ level, may make this model based on the oscillator degeneracies too unrealistic.

APPENDIX

To calculate the coefficients a(KL) of (4.18) and (4.20) we introduce the projection operator

$$P(KLM) = (2L+1)\int d\Omega D^{L}_{MK}(\Omega)\, R(\Omega),$$

where $R(\Omega)$ is a rotation operator defined by

$$R(\Omega)\,\Phi = \Phi_{\Omega}.$$

Using (4.18) and (4.20), the matrix element of this operator in a leading state Φ is

$$A(KLK') = (\Phi \mid P(KLK') \mid \Phi)$$

$$= a(K,L)\; a^{*}(K',L)\; (\psi(K'LK') \mid \psi(KLK')), \qquad (A.1)$$

remembering that $\psi(K'LM)$ and $\psi(KLM)$ are not in general orthogonal. A knowledge of these matrix elements would lead to values for $|a(KL)|^2$ and for the overlaps $(K'LM \mid KLM)$.

Although in our applications, the Φ are made up of a number of particles in any oscillator shell N, the a(KL) and A(KLK') are purely group concepts and depend only on the representation labels $(\lambda\mu)$. (This is like the angular momentum functions $\Phi(JM)$ of a system of particles behaving just like angular momentum functions of a single particle with that same J and M). To calculate A(KLK') therefore we may choose the simplest possible form for a function Φ so long as it is the leading state of a representation $(\lambda\mu)$. Such a simple form is obtained by considering a system of particles each with one oscillator quantum, or in other words, a system of vectors in the SU_3 space. Then, a function of $(2\mu+\lambda)$ particles of permutation symmetry $[f] \equiv [\lambda+\mu,\mu]$ will by definition, belong to a representation $(\lambda\mu)$ of SU_3.

The leading state of such a representation would have $(\lambda+\mu)$ quanta in the z direction and μ in the x direction, being given explicitly by

$$\Phi = \Phi(12)\,\Phi(34)\,.....\,\Phi(2\mu-1, 2\mu)\,\Phi_z(2\mu+1)\,...\Phi_z(2\mu+\lambda) \qquad (A.2)$$

where

$$\Phi(12) = \sqrt{\tfrac{1}{2}}\,(\Phi_z(1)\,\Phi_x(2) - \Phi_z(2)\,\Phi_x(1))$$

and $\Phi_x(1)$ denotes a state of particle 1 with one quantum in the x-direction and none in the y or z directions. We now evaluate the operation $R(\Omega)\Phi$ by considering the rotation on each factor in (A.2). Furthermore, since Φ in (A.1) is a product of factors referring to different particles, the integrals over particle co-ordinates implied in the matrix element $(\Phi \mid R(\Omega) \mid \Phi)$ may be evaluated separately. Hence

$$A(KLK') = (2L+1)\int D^{L}_{K'K}(\Omega)\,(\Phi \mid R(\Omega) \mid \Phi)^{\mu}\,(\Phi_z \mid R(\Omega) \mid \Phi_z)^{\lambda}\, d\Omega.$$

Since Φ_z is the z component of a vector and Φ the y component of a vector product, the matrix elements for the general Euler angle rotation α,β,γ are

$$(\Phi_z \mid R(\Omega) \mid \Phi_{\bar{z}}) = \cos\beta$$

$$(\Phi \mid R(\Omega) \mid \Phi) = \cos\alpha \cos\gamma - \cos\beta \sin\alpha \sin\gamma .$$

Inserting the explicit form for $d\Omega$ and writing $D^L_{K'K}(\Omega) = e^{iK'\alpha} e^{iK\gamma} d^L_{K'K}(\beta)$ with the function $d^L_{K'K}(\beta)$ given by ROSE [21] we obtain

$$A(KLK') = \frac{(2L+1)}{8\pi^2} \int_0^{2\pi} d\alpha\, e^{iK'\alpha} \int_0^{2\pi} d\gamma\, e^{iK\gamma} \int_{-1}^{1} d\cos\beta$$

$$\times \{\cos\alpha \cos\gamma - \cos\beta \sin\alpha \sin\gamma\}^{\mu} \cos^{\lambda}\beta\, d^L_{K'K}(\beta) .$$

Finally the binomial expansion is introduced:

$$A(KLK') = \frac{(2L+1)}{8\pi^2} \sum_n \frac{\mu!(-1)^n}{n!(\mu-n)!} \left\{ \int_0^{2\pi} d\alpha\, e^{iK'\alpha} \sin^n\alpha \cos^{\mu-n}\alpha \right\}$$

$$\times \left\{ \int_0^{2\pi} d\gamma\, e^{iK\gamma} \sin^n\gamma \cos^{\mu-n}\gamma \right\} \int_{-1}^{1} d\cos\beta \cos^{\gamma+n}\beta\, d^L_{K'K}(\beta) . \qquad (A.3)$$

For particular values of L, K and K' it is simple to insert the explicit form for the function $d^L_{K'K}(\beta)$, taken from ROSE [21], and carry out the resulting elementary integrals.

If we set K' = K in (A.1) we have simply $A(KLK) = |a(K,L)|^2$ from which (A.3) enables us to deduce $|a(K,L)|$. The phase is arbitrary insofar as we are defining the phases of the $\psi(KLM)$ and we choose $a(K,L)$ to be real and positive for $K \geq 0$. In order to ensure the convenient identity $\psi(-KLM) = \psi(KLM)$, the relation $A(KL-K) = (-1)^{\lambda+\mu+L} A(KLK)$ from (A.3) implies the phase relation $a(-KL) = (-1)^{\lambda+\mu+L} a(KL)$ for $K \geq 0$.

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ELECTROMAGNETIC PROPERTIES OF ATOMIC NUCLEI

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INTRODUCTION

The study of specific physical systems, especially on the quantum level, is mostly done through their interactions with other, simpler, physical systems. Among the latter, the electromagnetic field still occupies the most prominent place. Its relatively simple properties and the certainty with which they are known make the electromagnetic field one of the best probes of physical systems.

The formal interpretation of such measurements is carried out through that part of the combined system Hamiltonian which describes the interaction between the physical system and the electromagnetic field. This interaction can be written in the form

$$H_{int} = -\int j_\mu(x)\, A^\mu(x)\, dx, \qquad (1)$$

where $A^\mu(x)$ is the vector potential of the electromagnetic field and $j_\mu(x)$ is the four vector representing the current operator of the specific physical system.

It is obvious from (1) that, depending on the specific experiment in question, all we can hope to get from electromagnetic experiments is a couple of matrix elements of $j_\mu(x)$ averaged over the distribution of $A^\mu(x)$. For instance, if we have a system of non-interacting point particles, each having a current operator $j_\mu^{(k)}(x_k)$, then

$$j_\mu(x) = \Sigma_k\, \delta(x - x^{(k)})\, j_\mu^{(k)}(x_k) \qquad (2)$$

and an electromagnetic experiment will measure a matrix element

$$\langle \Psi_f(x_1, \ldots, x_A) \mid \Sigma_k\, \delta(x - x^{(k)})\, j\mu^{(k)}(x_k) \mid \Psi_i(x_1, \ldots, x_A) \rangle \qquad (3)$$

averaged over the space-time distribution of $A^\mu(x)$.

Choosing $A^\mu(x)$ in different forms we could, in principle, make a complete study of the x dependence of the matrix element (3). However, to study the detailed behaviour of (3) over a distance of order R we have to choose an electromagnetic field with Fourier components of order $\hbar/R$ and therefore with an energy of order $c\hbar/R$. Measuring R in electron radii ($\sim 2.8 \times 10^{13}$ cm):

$$R = r(e^2/mc^2) \qquad (r\text{-a pure number})$$

we find for the energy associated with the photons of the required electromagnetic field that

$$E = (137/r)mc^2 = (70/r)\text{MeV}.$$

Hence, to find details over distances of the order of 1/5 nuclear radius ($r \approx 1/10$) electromagnetic fields with photons of about 700 MeV are required. In most nuclear experiments, where the energies involved are of the order of 10 MeV or less, one cannot, therefore, expect to find detailed features of the nuclear structure, and only properties of the nuclear wave function, averaged over the whole nucleus, have a chance of being studied.

This is not the only limitation on the variety of data which is available in practice. If, for simplicity, we confine ourselves for the moment to systems of non-interacting particles, then we can make use of a well-known decomposition of the three-dimensional delta function to derive other limitations.

We have:

$$\delta(\vec{r} - \vec{r}^{(k)}) = \frac{1}{2\pi r r^{(k)}} \delta(r - r^{(k)}) \Sigma_{\ell m} \frac{2\ell + 1}{2} Y^*_{\ell m}(\Omega)\, Y_{\ell m}(\Omega^{(k)}) \qquad (4)$$

where Ω and $\Omega^{(k)}$ are the angular coordinates of the vectors $\vec{r}$ and $\vec{r}^{(k)}$. The integration over d^3x in (1) will now pick from (4) only those values of ℓ which correspond to the angular momenta present in the electromagnetic field. This process will therefore also leave only a few contributions in the matrix element (3) coming from a limited number of multipoles of the physical systems. Stated another way, because of conservation of total angular momentum we can derive information pertaining to only few multipole moments of the physical system, namely those which correspond to the angular momenta present in the probing electromagnetic field.

In addition to having relatively limited information from experiment because of the experimental limitations on the energy and the angular momentum of the available electromagnetic fields, we are faced with another difficulty of a theoretical origin. In order to say something about the system under consideration we ought to be able to compare the experimental values for (3) (or averages thereof) with the "calculated" values. Both Ψ and the operator $j_\mu(x)$ are necessary to know the latter and both are generally not precisely known.

1. THE NUCLEAR WAVE FUNCTION

Let us briefly review the problems connected with the evaluation of Ψ and $j_\mu(x)$ in the nuclear case.

Strictly speaking, Ψ should be an eigenfunction of a Hamiltonian which describes both the nucleons and the mesons, since the average number of mesons around a nucleon is not small and the packing of several nucleons together may affect the meson cloud around each one of them in an appreciable way. Thus the Hamiltonian which determines Ψ should be taken as

$$H = H_{nuc} + H_{mes} + H_{int}. \qquad (5)$$

Here H_{nuc} is the free nucleons' Hamiltonian, H_{mes} that of the mesons written in second quantization to allow for the nonconstancy of the number of mesons and H_{int} is the interaction Hamiltonian in which the nucleons serve as sources for the meson field in H_{mes}.

If we are interested in the correlations induced among the nucleons as a result of their interaction with the meson field then we can replace $H_{mes} + H_{int}$, in the non-relativistic limit, by a nucleon-nucleon potential. In the one-pion exchange approximation one then gets for the pseudo-scalar meson with pseudo-vector-coupling (i. e. $H_{int} = G\Psi(x)\underset{\rightarrow}{\tau}\gamma_5\gamma_\mu\Psi(x)\partial^\mu\underset{\rightarrow}{\phi}(x)\rangle$ the well-known result for the two nucleon potential:

$$V(|r_1 - r_2|) = -(4\pi)^{-1}(G/2M)^2(\underset{\rightarrow}{\tau}_1 . \underset{\rightarrow}{\tau}_2)(\vec{\sigma}_1 . \nabla_x)(\vec{\sigma}_2 . \nabla_x)(\exp\{-\mu x\}/\mu x)$$

$$x = |r_1 - r_2|$$

$$\hbar = c = 1.$$

μ.... meson's mass

M... nucleon's mass

It has been demonstrated experimentally that for the high angular momentum phase shift ($\ell > 4$) this potential reproduces the date very well. For the lower ℓ values, however, the situation is still unclear.

If we go to systems with more than two particles and try to eliminate the mesons there, we see that again it can be done in some approximation, but new additional potentials - 3-and more-body potentials - show up. Thus, inasmuch as we are interested in the energies and phase shifts in the A-nucleon problem, we could, in principle, proceed to find eigenfunctions of (5) which refer to the coordinates of both nucleons and mesons. Equivalently, at least to some approximations, we could instead solve a Schrödinger equation of an A-nucleon problem with no meson coordinates, but then we have to introduce 2-3-.... and up to A-body forces.

The problem of deriving a useful wave-function for a system of nucleons, either with or without explicit reference to the mesons, would have been hopelessly complicated had it not been for the Pauli principle. As was shown by BRUECKNER et al. [1] the effect of the Pauli principle in a system of many nucleons is to reduce drastically the effects of the 2-body correlations at separations of about 1 fermi. Thus the Pauli principle reduces the importance of 3 and more-body clusters and increases thereby the relative importance of the 2-body forces. I am not aware of any quantitative estimate of the reduction of the effects of many-body forces in nuclear matter but the relatively short healing distance, the 2-body interaction in the nuclear wave function and the low nuclear density make it very plausible to assume that we can neglect the 3-and more-body forces completely. This is one of the most commonly accepted approximations for the construction of Ψ.

2. *THE CURRENT OPERATOR*

Ψ in itself still does not give us all the information required for the evaluation of the electromagnetic properties of nuclei. We still have to know the current operator $j_\mu(x)$. Strictly speaking, we could have got the structure of the current operator from expression (5) for the total Hamiltonian, or better still from the corresponding Lagrangian. One then obtains the usual current of the nucleons $-e\,\bar{\psi}\gamma_\mu\psi$, that of the mesons, i. e. $(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$ etc. If the meson coordinates are eliminated in favour of 2-, 3-,.., A-body interaction between the nucleons, then an appropriate modification is required

in the form of the current operator. The interaction of the electromagnetic field with the charged mesons exchanged among the nucleons should be incorporated in a function involving the nucleons' coordinates if the meson coordinates are eliminated from the problem.

We shall not go into the survey or the details of such calculations here. It is sufficient for our present purposes to note that for a given meson theory there exists, in principle, the possibility of deriving an approximate current operator involving the nucleon coordinates (including spins and isospins), and that normally such a current operator can be broken up into the following sum:

$$j_\mu (x_1, \ldots, x_A) = \sum_k j_\mu^{(1)}(x_k) + \sum_{i<j} j_\mu^{(2)}(x_i - x_j) + \ldots \qquad (6)$$

Here $j_\mu^{(1)}(x)$ is a single-particle current operator, $j_\mu^{(2)}(x_i - x_j)$ - a current operator which depends in an irreducible way on the separation, or correlation, between pairs of nucleons, etc.

In the absence of any interaction between the nucleons and the mesonic field, the lowest states will correspond to states with no mesons and the current operator for these states will reduce to the familiar, single-particle form $j_\mu = -e\bar{\psi}\gamma_\mu\psi$. In the presence of a meson-nucleon interaction two things happen: first, the single-particle form will be changed and secondly, new terms will appear which depend on the coordinates of more than one particle. The first effect will convert the "bare" nucleon into a "dressed" one, and will also introduce "quenching" effects depending on the density of the system. It will thus lead to an "effective single-particle current" whose structure may be similar to that of the system with no meson, but whose "constants" may be different and density dependent. The second effect introduces the so-called "exchange currents", or better still "interaction currents", which then give rise to the well-known interaction moments.

To summarize, we see that an ultimate theory would start from a Hamiltonian containing the nucleons and the mesons, figure out the current operator, and then use eigenstates of the above Hamiltonian to figure out expectation values of the electromagnetic interaction. So long as this is not practical, we may have to satisfy ourselves with a theory in which the meson coordinates are eliminated and the current operator properly modified. Inasmuch as the underlying meson theory is not fully understood we may have to be satisfied with some sort of guesses for the forces among the nucleons and the structure of the current operator. One is, of course, guided by the very extensive analysis of the two-nucleon potential obtained from scattering studies, but at present there is still no clear evidence as to the role played by 3-or more-body forces.

Even assuming a full knowledge of the equivalent internucleon forces, it is still very doubtful whether a straightforward solution of the many-nucleon problem could be achieved at the present stage. Most probably an approximation method in the form of one model or another will have to be used. Anticipating such a possibility, it may therefore be of some interest to see what specific features of the various models play an important role in the interpretation of electromagnetic interactions, and to check to what extent empirical evidence sheds some light on these features. One may, of course, ask very specific questions like: Is the single particle picture with harmonic oscillator wave-functions a good picture for the magnetic moment of the nucleus or not? However, with no good theoretical backing for such questions it is doubtful whether this is a fruitful line to follow. A better

approach seems, at present, to be that of asking less specific questions which may give an answer to questions like: What is the limit on the observed interaction currents in nuclei? or: Can one ascribe an effective charge to a neutron in the shell model? etc.

It is assumed in what follows that $\Psi(x_1, \ldots, x_A)$ is prescribed by one model or another and that the general structure of $j_\mu(x)$ is given. We shall try to see what features of Ψ and $j_\mu(x)$ can be tested using the electromagnetic interactions and to what extent such features are borne out by experiment. We shall confine ourselves only to static moments and transitions between bound states and shall not discuss the very important questions of nucleon capture and photo-reactions.

3. THE CENTRAL FIELD APPROXIMATION

As mentioned above, we shall take as our starting point a Hamiltonian with two-body forces only:

$$H' = \sum_i T_i + \sum_{i<j} V'_{ij} \,. \tag{7}$$

This Hamiltonian describes a nucleus moving freely in space. Most models prefer to describe a nucleus tied to a certain point in space. In order to do so without disturbing the structure of the nucleus we tie it at its centre of mass; we shall find it convenient to do so with harmonic force, and we are thus led to study the following Hamiltonian, whose intrinsic spectrum is identical with that of (7):

$$H = \sum_i T_i + \sum_{i<j} V'_{ij} + \omega^2 (A^{-1} \sum_i \vec{r}_i)^2, \tag{8}$$

where ω is an arbitrary constant. We have the identity

$$(A^{-1} \sum_i \vec{r}_i)^2 = A^{-1} \sum_i r_i^2 - A^{-2} \sum_{i<j} (\vec{r}_i - \vec{r}_j)^2 \tag{9}$$

which shows that a harmonic force on the centre of mass is equivalent to a harmonic force on each of the particles plus a mutual two-body harmonic repulsion. Using (9) we obtain from (8)

$$H = \sum_i T_i + \alpha \sum_i r_i^2 + \sum_{i<j} V_{ij} \,, \tag{10}$$

where $\alpha = \omega^2 / A$ and

$$V_{ij} = V'_{ij} - (\omega^2 / A^2) (\vec{r}_i - \vec{r}_j)^2. \tag{11}$$

The internal spectrum of (10) is identical with that of (7), the only difference between them being that of the centre-of-mass motion.

To derive approximate solutions of (10) one can use a Hartree-Fock self-consistent field approach. One then uses an auxiliary Hamiltonian $H_{s.c.}$ given by

$$H_{s.c.} = \sum_i T_i + \alpha \sum_i r_i^2 + \sum_i U(r_i), \tag{12}$$

whose eigenfunctions are the antisymmetric wave functions $\Phi_k(x_1, \ldots, x_A)$, i. e.

$$H_{s.c.}\,\Phi_k = E_k\,\Phi_k .$$

The energy in the g. s. is then given by

$$\begin{aligned} E^{(0)}_{H.F.} &= \langle \Phi_o \Big| \sum_i T_i + \alpha \sum_i r_i^2 + \sum_{i<j} V_{ij} \Big| \Phi_o \rangle \\ &= \langle \Phi_o \Big| \sum_i T_i + \alpha \sum_i r_i^2 + \frac{1}{2} \sum_i U(r_i) \Big| \Phi_o \rangle \qquad (13) \\ &= \frac{1}{2}\, E_o + \frac{1}{2} \langle \Phi_o \Big| \sum_i T_i + \alpha \sum_i r_i^2 \Big| \Phi_o \rangle \end{aligned}$$

and is stationary with respect to variation in the single particle orbits.

As a matter of fact the above procedure of deriving a self-consistent field can be carried out for any value of ω, the spring constant of the force holding the centre of mass. One can use this freedom in the choice of ω to improve the approximation even further. No study of this possibility has so far been carried out in detail.

4. GENERALIZED MOMENTS

As mentioned in the introduction, the measurement of the properties of a system of several particles generally yields information only on a limited number of its moments. Thus if a test particle p interacts with the system α through an interaction $H_{int}(\alpha, \rho)$ and if, as a result of this interaction, the test particle goes from $\varphi_i(\rho) \to \varphi_f(\rho)$ and the system from $\Psi_i(\alpha) \to \Psi_f(\alpha)$, then what we measure experimentally is the matrix element

$$\left| \langle \Psi_f(\alpha)\ \varphi_f(\rho) \Big| H_{int}(\alpha, \rho) \Big| \Psi_i(\alpha)\ \varphi_i(\rho) \rangle \right|^2 . \qquad (14)$$

$H_{int}(\alpha, \rho)$, being a scalar, can be decomposed into a sum of products of irreducible tensors:

$$H_{int}(\alpha, \rho) = \sum_k f_k(r_\alpha\, r_p)\ \underline{T}^{(k)}(\alpha) \cdot \underline{T}^{(k)}(\rho) . \qquad (15)$$

Here r_α stands for the intrinsic coordinates of the system relative to a direction α fixed in it in any manner; r_p is the distance of p from the origin. Because of the triangular conditions we see that if $\Psi_i(\alpha)$ and $\Psi_f(\alpha)$ have definite angular momenta J_i and J_f (as is generally the case), then the measurement with the test particle can only teach us about the few moments in Ψ_i and Ψ_f with $|J_i - J_f| \leqslant k \leqslant J_i + J_f$. Thus, assuming H_{int}, $\varphi_i(\rho)$ and $\varphi_f(\rho)$ are known, then the measurement will at most yield a number for

$$\langle \Psi_f(\alpha) \Big| f_k(r_\alpha, r)\ \underline{T}^{(k)}(\alpha) \Big| \Psi_i(\alpha) \rangle , \qquad (16)$$

where $|J_i - J_f| \leqslant k \leqslant J_i + J_f$, and r obtains any arbitrary value. In many cases

we shall obtain only the average of (16) over some distribution of r, and only $k = |J_i - J_f|$ will actually be determined with any appreciable certainty.

Expression (16) is a generalized moment of the system and applies to electromagnetic moments as well as to moments derived from other measurements. Indeed, with a big enough choice of test particles the variety of "radial" functions $f_k(r_\alpha, r)$ can be such as to give a fairly detailed picture of $\Psi_i(\alpha)$ and $\Psi_f(\alpha)$. In practice, however, the choice is not that big and consequently the information obtained is not too detailed. The best-known moments are those derived from the electromagnetic interaction; their derivation is well known and we shall not list them here; then come those derived from β-decay, where, again, the functional form is quite well known; finally there are the moments derived from the nuclear interactions, where the functional form is less certain.

Although our main purpose is to discuss the information which can be obtained from the electromagnetic moments, it is interesting first to look a little into those derived from other sources, since a comparison of the various moments is likely to be of some importance. Let us first consider a simple, rather classic example [2], namely that of the nucleus $_{19}K^{40}_{21}$. This nucleus can be looked upon as $_{19}K^{39}_{20}$ plus a test particle in the form of the 21st neutron. From $_{20}Ca^{41}_{21}$ we know that the 21st neutron moves in an f7/2 orbit, and since $_{19}K^{39}_{20}$ has J = 3/2 there are altogether 4 possible states of the test particle with respect to K^{39}. An easy algebra shows that if T^k are the different moments of K^{39} resulting from the interaction with the test particle and if the interaction can be treated in first-order perturbation, then the energies of the different states in K^{40} are given by

$$E(K^{40}, J) = \text{Const} \sum (-1)^{j+J_c+J} \begin{Bmatrix} J_c & j & J \\ j & J_c & k \end{Bmatrix} T^{(k)} \qquad (17)$$

where j = 7/2 (the test particle's orbit), J_c = 3/2 the angular momentum of K^{39} and k runs over all allowed values, i. e. k = 0, 1, 2 and 3. This relation can be inverted to give $T^{(k)}$ in terms of $E(K^{40}, J)$.
Using the observed energies

E (2) = 797 keV, E (3) = 29 keV, E (4) = 0, E (5) = 885 keV,

one obtains

$T^{(1)}$ = +1660 keV $T^{(2)}$ = 4860 keV $T^{(3)}$ = - 945 keV.

The same test particle, the 21st neutron, also probes another nucleus, namely $_{17}Cl^{37}_{20}$. In Cl^{38} one again knows the four states with J = 2, 3, 4 and 5:

E (2) = 0, E (3) = 762 keV, E (4) = 1310 keV, E (5) = 670 keV;

the resulting moments turn out to be:

$T^{(1)}$ = + 1500 keV $T^{(2)}$ = - 4920 keV $T^{(3)}$ = -400 keV (18)

It is interesting to note the similarity between the two sets of moments (17) and (18). (The apparent big difference between the two values of $T^{(3)}$ should not be taken too seriously since these numbers come out as the differences between two big, experimentally determined, numbers; relatively small errors in these energies may result in big effects on $T^{(3)}$.) According to the shell model, $_{17}Cl^{37}_{20}$ is related to $_{19}K^{39}_{20}$ via the particle-hole conjugation (for protons in the $d_{3/2}$ orbit). Under such conjugations odd tensors remain unchanged and even tensors (with $k \neq 0$) change their sign but not their magnitude. That this actually happens in the analysis of the energy levels lends,

of course, great support to the shell model in this region of the periodic table.

It should, however, be remembered that all this result says is that if $_{17}Cl^{37}_{20}$ is described by the state $a^{+}_{3/2\,3/2}\,|\,{}_{16}S^{36}_{20}\rangle$ then it is consistent to describe $_{19}K^{39}_{20}$ by the state of $a^{+}_{3/2\text{-}3/2}\,a^{+}_{3/2\text{-}1/2}\,a^{+}_{3/2\,1/2}\,|\,{}_{16}S^{36}_{20}\rangle$. Here $_{16}S^{36}_{20}$ is the ground state of S^{36} and a^{+}_{jm} is an operator which, when operating on S^{36}, creates a proton in a state of angular momentum j and z-projection m, and it is assumed that the interaction of the $f_{7/2}$ neutron with these protons involves the operators a_{jm} through the combination $a^{+}_{jm}\,a_{jm'}$.

5. EXAMPLE - THE PROTON $f_{7/2}$ SHELL

A somewhat more complicated example of the same nature is obtained if we consider all nuclei with 28 neutrons and with $20 \leqslant Z \leqslant 28$. We shall again use the shell model for their analysis but it is obvious that a similar analysis could be carried out using any other model which attempts to describe these nuclei.

According to the shell model the 28 neutrons form an "inert" closed shell and the protons all fill the $f_{7/2}$ shell. We can consider one of these protons as the test particle which probes the structure of the wave function of all the rest of them. Antisymmetry complicates the picture somewhat and formally it is best to proceed as follows:

Consider the antisymmetric wave functions of n particles in the j-orbit, and call them $|\,j^n\,\alpha\;JM\,\rangle$ where α is any other quantum number required to describe these states in addition to J and M. The space spanned by the functions $|\,j^{n-2}\alpha'\;J'M'\,\rangle . |\,j^2\,J''M''\,\rangle$ is bigger than that spanned by $|\,j^n\,\alpha JM\,\rangle$ since the former are antisymmetric only in the first n-2 particles and the last 2 particles separately and not necessarily in all of them. Hence it is always possible to find coefficients $C^{\alpha JM}_{\alpha'JM',\,J''M''}$ such that

$$|j^n\;\alpha\;JM\rangle = \Sigma\, C^{\alpha JM}_{\alpha'J'M',\,J''M''}\,|j^{n-2}\;\alpha'J'M'\rangle|j^2\;J''M''\rangle .$$

It is easy to see that the M, M' and M'' dependence of C is just that of the appropriate Clebsch-Gordan coefficient so that we have

$$|\,j^n\alpha\;JM\rangle = \sum_{\alpha'J'J''}(\alpha'\,J';\,J''\,|\}\;\alpha\;J)\,|\,j^{n-2}\,(\alpha'\,J')\;j^2\,(J'')JM\rangle \qquad (19)$$

where $(\alpha'J';\,J''\,|\}\;\alpha\;J)$ is a "fractional parentage coefficient".

In many cases of practical interest the quantum numbers α and α' are superfluous, and we see that the fractional parentage coefficients in such cases are uniquely determined by J' J'' and J. If we introduce a two-body interaction between nucleons in such cases we find for the energies

$$E(j^n\,J) = \langle\,j^n\;J\,|\sum_{i<j} V_{ij}\;|\,j^n\;J\,\rangle = \frac{n(n-1)}{2}\langle\,j^n\;J\,|\,V_{n-1,\,n}\,|\,j^n\;J\,\rangle$$

$$= \frac{n(n-1)}{2}\sum_{J''}\left[\sum_{J'}(J'J''\,|\}\;J)^2\right]\langle\,j^2\,J''\,|\,V_{12}\,|\,j^2\,J''\rangle . \qquad (20)$$

There are $(2j+1)/2$ allowed values of J'', and for $n > 2$ there are generally many more allowed values of J. Equation (20) tells us that it should be possi-

ble to express all the energies $E(j^n J)$ in terms of the few quantities $< j^2 J'' | V | j^2 J'' >$ with the help of the interaction-independent coefficients $(J' J'' | \} J)$.

The quantities $< j^2 J'' | V | j^2 J'' >$, which represent the average two-body interactions in the n-particle system, can in their turn be expressed in terms of the interaction moments $T^{(k)}$ by the relation

$$\langle j^2 J | V | j^2 J \rangle = \sum_k (-1)^{2j+J} \begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} T^{(k)}. \qquad (21)$$

It is thus possible to determine moments of an interaction between equivalent nucleons.

In the $f_{7/2}$ shell it is not quite true that the α-quantum numbers are superfluous. In fact, in the $f^4_{7/2}$-configuration there are two states with $J = 2$ and two with $J = 4$ and further quantum numbers are required for their characterization. To introduce this new quantum number, the seniority [3], we proceed as follows:

An n-particle antisymmetric state can be obtained from an (n-2)-particle antisymmetric state in a simple way [4]. We construct first the state $| j^{n-2} (J_1) j^2 (J_2) J >$ which is antisymmetric in the first n-2 particles and the last two but not with respect to their exchange. We now antisymmetrize this function with respect to the interchange of particles n-1 and n with any of the particles $1, \ldots, n-1$. We than obtain either zero or an antisymmetric function of n particles in the shell j coupled to J. If we start from $J_2 = 0$ (and therefore $J_1 = J$), the state $| j^n J >$ obtained in this way is said to have the same seniority as the state $| j^{n-2} J >$ we started with.

We see immediately that in the configuration j^n states can have the seniorities $v = n, n-2, n-4, \ldots$. A state $| j^n J >$ is said to have the seniority v=n if a pair coupled to zero cannot be extracted from it, i. e. if we have $(j^{n-2} (J_2) j^2 (0) J | \} j^n J) = 0$.

It is easy to see that a tensor operator of an odd degree is diagonal with respect to the seniority. In fact one has for the reduced matrix elements:

$$(j^n v J || T^{(k)} || j^n v' J') = (j^v v J || T^{(k)} || j^v v J') \delta_{vv'} \text{ if k is odd.} \qquad (22)$$

One also finds that for even tensors

$$(j^n v J || T^{(k)} || j^n v J') = \frac{2j + 1 - 2n}{2j + 1 - 2v} (j^v v J || T^{(k)} || j^v v J') \text{ if k is even.} \qquad (23)$$

From (23) it follows that matrix elements of even tensors, diagonal in the seniority, vanish in the middle of a shell. Consider now the $f_{7/2}$ shell. For 2 or 3 particle configurations the total angular momentum defines the states uniquely. For 4 particles we are in the middle of a shell and hence:

$$\left((7/2)^4 \, v=4 | \underline{T}^{(k)} (i) \, \underline{T}^{(k)} (j) | (7/2)^4 \, v=2\right)$$

$$= \sum \text{const} \left((7/2)^4 \, v=4 | T^{(k)} (i) | (7/2)^4 \, v''\right) \left((7/2)^4 \, v'' | T^{(k)} (j) | (7/2)^4 \, v=2\right) = 0.$$

In fact v'' could be 4, in which case for even k the first factor vanishes (be-

cause of (23)), whereas for odd k the second factor will vanish because of (22). Similar results follow for the only other possible choice $v'' = 2$. Hence any 2-body interaction is diagonal with respect to the seniority in the $f_{7/2}$ shell [4].

TABLE I [6]

CALCULATED AND MEASURED ENERGY LEVELS FOR ELEMENTS WITH N = 28

		Energy			Remarks
		Exp.	Calc (a)	Calc (b)	
$_{22}Ti^{50}_{28}$	2+	1.56	1.56	1.51	
	4+	2.76	2.76	2.67	
	6+	3.00	3.00	2.98	
$_{23}V^{51}_{28}$	$5/2^-$	0.32	0.32	0.32	E(2) = 1.41
	$3/2^-$	0.93	1.01	1.30	E(4) = 2.44
	$11/2^-$	1.61	1.58	1.69	E(6) = 2.81
	$9/2^-$	1.81	1.69	1.97	
	$15/2^-$	2.70	2.75	2.92	
$_{24}Cr^{52}_{28}$	2+	1.43	1.45	1.50	v = 2 E(2) = 1.46
	4+	2.37	2.36	2.41	v = 4 E(4) = 2.78
	4+	2.77	2.77	2.67	v = 2 E(6) = 3.12
	6+	3.11	3.12	2.98	v = 2
	2+	3.61	3.61	3.51	v = 4
	5+	3.83	3.82	3.69	v = 4
	8+	?	5.22	4.95	v = 4
$_{25}Mn^{53}_{28}$	$5/2^-$	0.37		0.32	
	$3/2^-$	1.27		1.30	
	$11/2^-$	?		1.69	
	$9/2^-$			1.97	
	$15/2^-$			2.92	
$_{26}Fe^{54}_{28}$	2+	1.41	1.41	1.50	
	4+	2.55	2.55	2.67	
	6+	2.97	2.97	2.98	
					least sq. fit: E(2) = 1.50 E(4) = 2.67 E(6) = 2.98

For corresponding diagrams see Figs. 1-5

This last result is not true for 3-body interactions. Hence, the existence of seniority admixtures in a pure $f_{7/2}$ configuration can be taken as an indication for a residual 3-body force [5]. We shall therefore dwell a little longer on the analysis of this shell.

Table I and the accompanying decay schemes (Figs. 1-5) show the sort of fit that is obtained by choosing: (a) a least square fitted set of moments for each value of n; and (b) one set of least square fitted moments for all the $f_{7/2}$ proton shell [6].

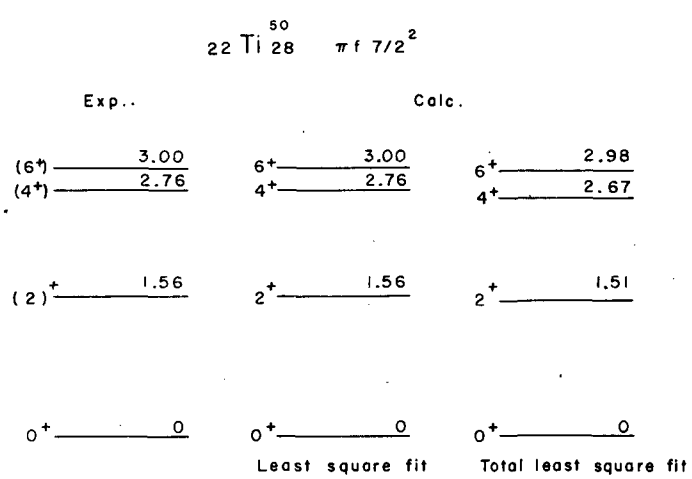

Fig. 1

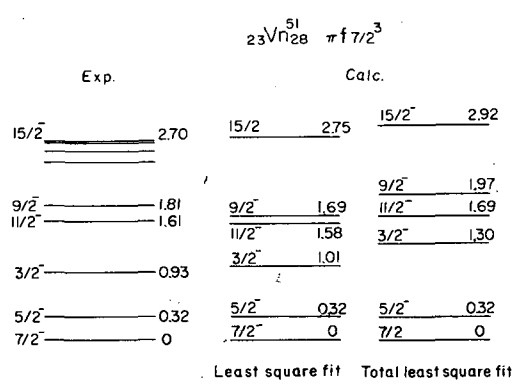

Fig. 2

The consistency of the results lends great support to the description of all these nuclei in terms of the proton configuration $(f_{7/2})^n$ but it should be stressed again that all we have shown is the existence of a relation of the type (20) between the energies in the configurations j^n and $(2j + 1)/2$ pa-

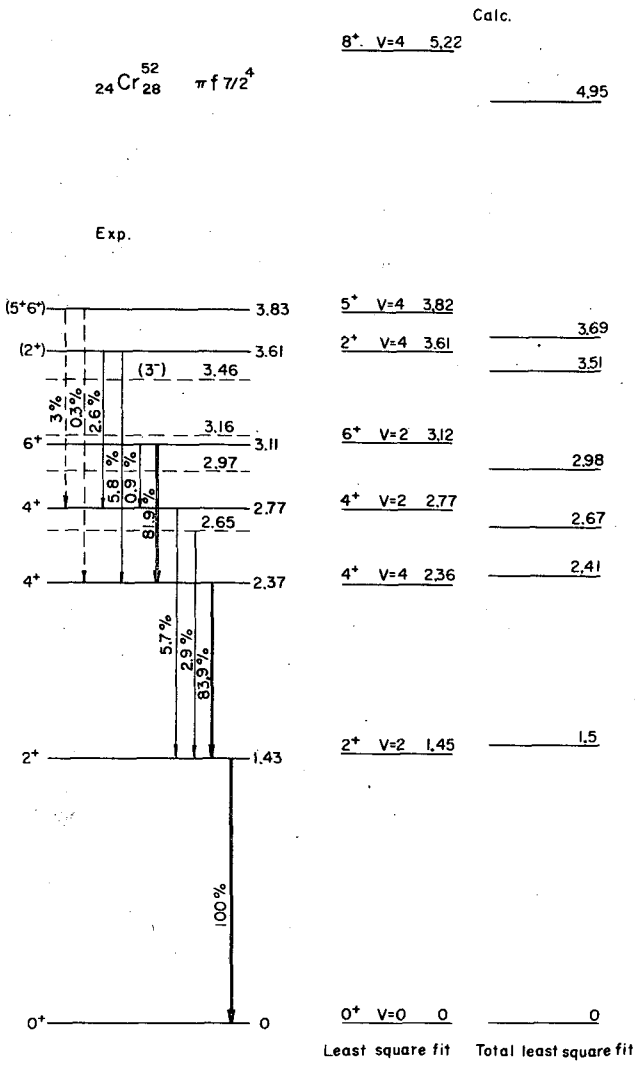

Fig. 3

rameters $< j^2\ J''\ |V|j^2\ J'' >$. Imagine, indeed, that we consider a second order correction to (20); it can include a term like

$$\delta\ E(j^n\ J) = (1/\Delta E) < j^n\ J\ |\ \Sigma\ V_{ij}\ |\ j^{n-2}\ j_o^2\ J > < j^{n-2}\ j_o^2\ J\ |\ \Sigma V_{ij}\ |\ j^n\ J >$$

in which a pair from the j-orbit is excited into the j_o orbit. Taking into account the antisymmetry we see easily that

$$\delta\ E(j^n\ J) = [n(n-1)/2] \sum_{J''} [\sum_{J'} (J'J''\ |\ }\ J)^2\ ]\ \frac{1}{\Delta E}\ |\ <j^2\ J''\ |\ V_{12}\ |\ j_o^2\ J''>\ |^2$$

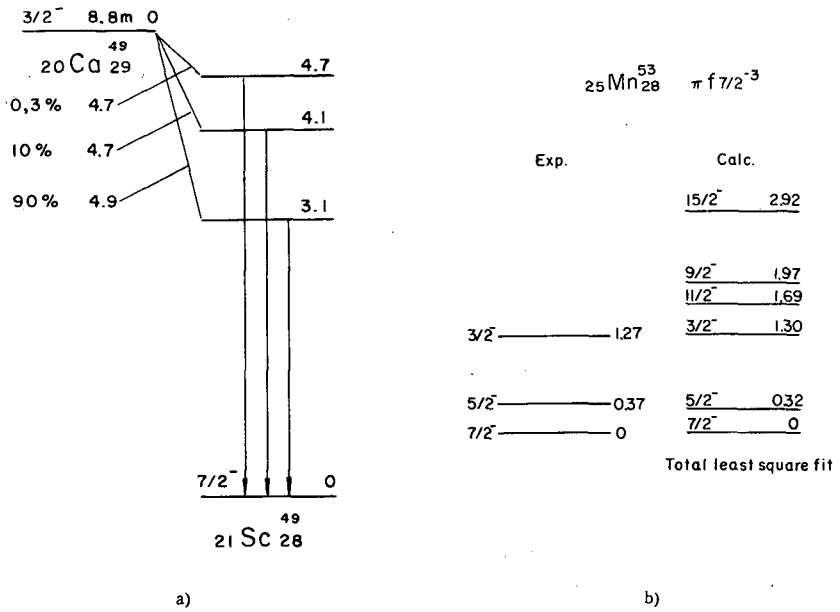

Fig. 4

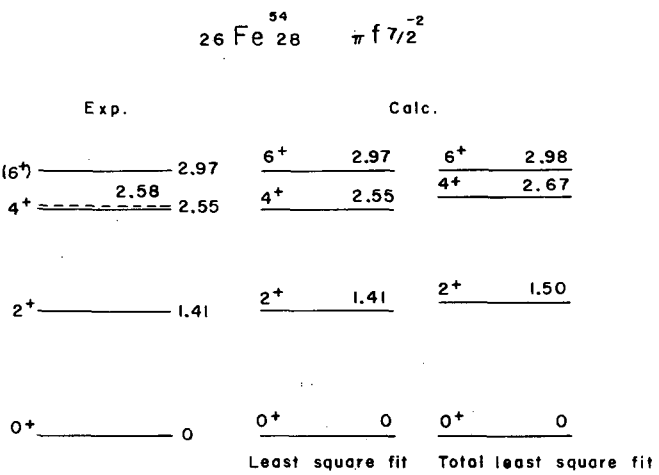

Fig. 5

and thus

$$E(j^n J) + \delta E(j^n J) = [n(n-1)/2] \sum_{J''} [\sum_{J'} (J'J'' \mid) J)^2] \left[< j^2 J'' \mid V_{12} \mid j^2 J'' > + \frac{| < j^2 J'' \mid V_{12} \mid j_0^2 J'' > |^2}{\Delta E} \right].$$

We therefore obtain again an expression similar to (20) correct to second order. We can easily extend this analysis to higher order corrections as well, convincing ourselves that there is a whole set of diagrams in the perturbation expansion, the so-called ladder diagrams, whose inclusion would not alter the relation between energies in the j^n configuration and those of j^2. It is for this reason that a shell model analysis of experimental data can only give us information about an "effective" two-body interaction which can be used to account for the observed data using a simple first-order perturbation theory; it cannot tell us much about the "real" interaction. All the moments of the interaction that we derive are, of course, moments of the effective interaction only.

Although it is difficult at the present stage to relate these effective interactions in finite nuclei to the original nuclear interaction, the generalized moments, as derived from experiment, can give us some interesting information about the effective interaction. Thus, consider the matrix element

$$\langle j^2 J | V | j^2 J \rangle = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} (jm_1 jm_2 \mid JM)(jm_1' jm_2' \mid JM) \tag{24}$$

$$\times \int \varphi^*_{jm_1}(r_1\theta_1\varphi_1)\,\varphi^*_{jm_2}(r_2\theta_2\varphi_2)\,V(r_1, r_2, \cos\omega_{12})\,\varphi_{jm_1'}(r_1\theta_1\varphi_1)\,\varphi_{jm_2'}(r_2\theta_2\varphi_2)\,d^3r_1 d^3r_2 .$$

Since V depends only on the angle between $\vec{r}_1$ and $\vec{r}_2$ we can carry out the integration over one direction and obtain for (24) the expression [7]

$$\langle j^2 J | V | j^2 J \rangle = \tfrac{1}{2} \int_{-1}^{1} A^J_{jj}(\cos\omega)\,\Omega(\cos\omega)\,d(\cos\omega), \tag{25}$$

where

$$\Omega(\cos\omega) = \int R_j^2(r_1^2)\,R_j^2(r_2^2)\,V(r_1, r_2, \cos\omega)\,dr_1\,dr_2 \tag{25a}$$

and

$$A^J_{jj}(\cos\omega) = (-1)^{J+1}(2j+1)^2 \sum_k (2k+1) \begin{pmatrix} j & k & j \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} P_k(\cos\omega). \tag{25b}$$

Noting the physical meaning of $\Omega(x)$ we see that we can approximate it by

$$\Omega(x) = \begin{cases} x^{2n} & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

where n is large for short-range forces and small for long-range forces. Using that fact that

$$\int_0^1 x^{2n} P_k(x)\,dx = 0, \text{if } n < k,$$

we see that for long-range forces multipoles with high k will vanish and that for short-range forces we should expect non-vanishing high multipoles. In the above example of $(f_{7/2})^n$ the range of the interaction could actually be estimated in this way to be around 1.5 fermis.

6. CORRECTIONS TO MULTIPOLE MOMENTS

Coming back to the electromagnetic moments, we still have to determine the form of their operators. We have already mentioned the difficulties

connected with the interaction, or exchange, currents, and the necessity of properly defining the wave functions with which we propose to calculate expectation values. In this connection there is a well-known theorem - the Siegert theorem - whose general proof was given by SACHS and AUSTERN [8]. This theorem states that electric multipole moment operators are independent, in their form, of the interaction currents. The reason for this is that such moments are actually determined by the density alone, i.e. they are the moments of $\Psi^* \Psi$. Whereas Ψ may be, and generally is, quite sensitive to the nature of the inter-nucleon force, once Ψ is given, the operation which leads from it to the value of the moments is independent of the dynamics of the system. The magnetic moments, on the other hand, depend on the decomposition into irreducible tensors of the current operator. The latter is defined so as to satisfy the continuity equation $(\partial\rho/\partial t) + \operatorname{div} \vec{j} = 0$. The operator of $\partial\rho/\partial t$, however, depends on the Hamiltonian (it is the commutator of ρ with H) and velocity dependent terms in H, for instance, will affect $\partial\rho/\partial t$ strongly. Such effects have to be "compensated" by div $\vec{j}$, and hence the intimate connection between $\vec{j}$, or its magnetic moments, and the nature of the Hamiltonian.

Since the electric multipole operators are so well known we can, in principle, use them to check the extent to which our basic approximation, that of neglecting the meson coordinates, is actually valid. To this end one has to know the wave-function of a given nucleon fairly well and then check whether any given electric moment can be obtained from it using the conventional operators. To the extent that the meson cloud around each nucleon is greatly modified by the presence of other nucleons, this will not be the case and we shall then conclude that a proper description of nuclei should include mesonic degrees of freedom explicitly.

Because of their simple and well-understood forms, the electric multipole moments also serve as very sensitive tests for the adequacy of a given wave-function to describe appropriate nuclear states. Whereas with the magnetic moments some discrepancies between "theory" and experiment may be disposed of by uncertainties in the operators, such discrepancies in the case of electric multipoles indicate a failure of the wave-function.

On the other hand we ought to be careful not to throw away wave-functions which fail to describe some electric moments properly. These operators may have the property of "magnifying" the effects of small corrections in the wave-functions and may present a much too sensitive test for the adequacy of a given wave-function for other purposes.

Similarly one should exercise great care in interpreting agreement of calculated moments with experimental ones as an indication of the validity of the wave-function for other purposes. Thus the simple-minded operator for the magnetic moment operator for protons only or for neutrons only is diagonal in LS-coupling. Real wave-functions may therefore deviate appreciably from LS-coupling before any effects in the expectation values of μ are noticed.

To give an example of the "magnification" effects of some operators, consider the operator $Q = \Sigma Y_{2K}(\theta_1 \varphi_1)$ in the state

$$| j_1^{n_1}(0)\, j_2^{n_2}(J_2)\, J \rangle \quad (n_1 = 2j_1 + 1 \text{ is even; } n_2 \text{ -odd}).$$

If the actual wave function contains a first-order admixture then we consider

$$|\psi\rangle = |j_1^{n_1}(0) j_2^{n_2}(J_2) J\rangle$$

$$+\sum (\Delta E)^{-1} \langle j_1^{n_1-1}(j_1) j_3 J_1' j_2^{n_2}(J_2) J | \sum V_{ij} | j_1^{n_1}(0) j_2^{n_2}(J_2) J\rangle | j_1^{n_1-1}(j_1) j_3 J_1' j_2^{n_2}(J_2) J\rangle. \quad (26)$$

The expectation value to first order in the admixture is given by (we are interested only in reduced matrix elements)

$$\langle \psi \| Q \| \psi \rangle = \langle j_2^{n_2} J_2 \| Q \| j_2^{n_2} J_2 \rangle$$

$$+\sum (2/\Delta E)[(2J_2+1)/5]^{\frac{1}{2}} \langle j_1^{n_1}(0) \| Q \| j_1^{n_1-1}(j_1) j_3 J_1' \rangle \langle j_1^{n_1-1}(j_1) j_3 (J_1') j_2^{n_2}(J_2) J | \sum V_{ik} | j_1^{n_1}(0) j_2^{n_2}(J_2) J \rangle.$$

Since Q is a tensor of the second order we have non-vanishing contributions only from $J_1' = 2$; the interaction will then also contribute only through its quadrupole-quadrupole part. Assuming this part is simply $(\Sigma Q(i))\cdot(\Sigma Q(k))$ we shall find for the reduced matrix

$$\langle \psi \| Q \| \psi \rangle = \langle j_2^{n_2} J_2 \| Q \| j_2^{n_2} J_2 \rangle$$

$$+\sum (2/\Delta E)[(2J_2+1)/5]^{\frac{1}{2}} \langle j_1^{n_1}(0) \| Q \| j_1^{n_1-1}(j_1) j_3 J_1' \rangle (-1)^{2J_2} \langle j_1^{n_1-1}(j_1) j_3 J_1' \| Q \| j_1^{n_1}(0) \rangle$$

$$\times \langle j_2^{n_2} J_2 \| Q \| j_2^{n_2} J_2 \rangle \begin{Bmatrix} J_1' & J_2 & J_2 \\ J_2 & 0 & 2 \end{Bmatrix}$$

$$= \langle j_2^{n_2} J_2 \| Q \| j_2^{n_2} J_2 \rangle \left[1 + (2/5) \sum (1/\Delta E) | \langle j_1^{n_1}(0) \| Q \| j_1^{n_1-1}(j_1) j_3\, 2 \rangle |^2 \right]. \quad (27)$$

Thus we see that in this particular case all sorts of admixtures of the type $|\, j_1^{n_1-1}(j_1)\; j_3\, J_1 \rangle$ to the closed shell $j_1^{n_1}(0)$ add up with the same phase in their contribution to the enhancement of the "zero-order" matrix element $\langle\, j_2^{n_2}\; J_2 \|\, Q \|\, j_2^{n_2}\; J_2 \,\rangle$. As shown by the example above this happens only because we are calculating an expectation value of an operator which coincides with a part of the interaction. Thus through the special nature of the residual interaction V_{ik}, the expectation values of one multipole or another may become very sensitive to small modifications in the wave-function.

Equation (27) also indicates why the concept of an "effective charge" may be of some use in analysing electromagnetic properties of nuclei. If the quadrupole operator is $e\sum_i Y_{2n}(\theta_i\, \varphi_i)$ then Eq. (27) shows that the effect of the polarization of the closed shells is to modify it into $e' \sum_i Y_{2n}(\theta_i\, \varphi_i)$ where

$$e' = e \left[1 + (2/5) \sum (1/\Delta E) \langle |j_1^{n_1}(0) \| Q \| j_1^{n_1-1}(j_1) j_3 2 \rangle |^2 \right]. \quad (28)$$

The important thing to note is that e' is independent of the open shells (j_2) or the number of particles in them (n_2) or the total angular momentum of the state considered. As long as it is consistent to include the polarization of the closed shells only to terms linear in the admixture of the excited states, and provided the interaction V_{ik} has the simple feature we described above, the quadrupole operators remain unchanged in their form. In other words,

we can continue to take matrix elements with the zero-order wave-function and include the above first-order corrections through a modification of e. Furthermore we notice that it is possible for the first-order corrections in ψ to be relatively unimportant for the evaluation of some nuclear properties and still make e' significantly different from e through the addition of numerous small effects. Finally we notice that the concept of an "effective charge" may be valid for one multipole and not for another, and that even if it is valid for a number of multipoles there is no reason for it being the same for different multipoles.

The situation is not so simple for more realistic interactions and one obtains effective charges which depend, at least formally, also on some details of the zero-order wave-function. Numerical estimates, however, show that their dependence is not too strong and that the above rather qualitative considerations do have also some quantitative value. Characteristically, one obtains in actual cases, for quadrupole transitions, $e'/e \approx 1.5\text{-}2$, with a similar value obtained also for the effective charge associated with transitions between states of different configurations.

The fact that first-order polarizations of the closed shells affect some electric moments only to the extent of renormalizing their charge explains also why one can expect that even selection rules should hold with zero-order wave-functions when normally we can anticipate not so pure wave-functions. In the case of $_{24}Cr^{52}_{28}$, (Fig. 6), for instance, we may have a clear manifestation of such a situation. We saw that in the middle of a shell all matrix elements of even operators between states of the same seniority vanish (e.g. (23)).

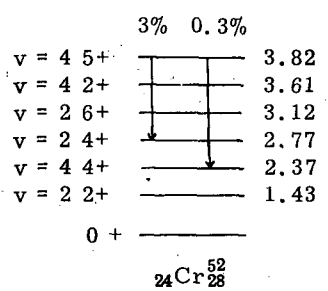

Fig. 6

Therefore E2 transitions should be slow between states of the same seniority and fast (i.e. normal) between states of different seniorities in $_{24}Cr^{52}_{28}$ (half-filled proton shell and a filled neutron shell). The seniority and spin assignments in the figure above are supported by energy considerations, and it is indeed observed that the more energetic transition is ten times slower than the $(5+\ v=4) \rightarrow (4+v=4)$ transition. The E2 components in each still await an accurate determination, but there is good reason to believe that the M1 component is strongly hindered (see below), so that the observed "anomalous" behaviour may be the seniority selection rule on E2 transitions in the middle of a shell.

7. MAGNETIC DIPOLE

Siegert's theorem, as mentioned previously, does not say anything about the magnetic multipoles. We are less certain of their structure and they are

generally not as useful as the electric multipoles in analysing nuclear structure. An exceptional case is that of the magnetic dipole moment and we shall now proceed to the study of some of its general properties.

The magnetic moment operator is a vector operator and can be decomposed into a sum of single-particle operators, two-particle operators, etc.

$$\vec{\mu} = \Sigma \vec{\mu}_i + \sum_{i<j} \vec{\mu}_{ij} + \dots \quad .$$

To the extent that we can neglect the two and more-body operators, it is possible to make some statements about the matrix elements of $\vec{\mu}$ without further specification of its form. Indeed, Lande's formula states that

$$\langle jm|\vec{\mu}|jm'\rangle = \alpha_j \langle jm|\vec{j}|jm'\rangle$$

where α_j is independent of m, m' or the specific component of $\vec{\mu}$ considered. It therefore follows that

$$\langle j^n JM|\Sigma \vec{\mu}_i|j^n J'M'\rangle = \alpha_j \langle j^n JM|\vec{J}|j^n J'M'\rangle = 0 \quad \text{if } J \neq J', \qquad (29)$$

Thus magnetic dipole transitions vanish between any two states of the same configurations in jj-coupling [4a] . A similar theorem holds for the M1 transitions between two multiplets in LS coupling if $\vec{\mu}$ can be decomposed into two parts, one diagonal in S and the other diagonal in L.

The above theorem is a very powerful tool for the study of purity of nuclear configurations since it asserts so little about the structure of $\vec{\mu}$. Indeed, if one believes one has a pure configuration j^n then the existence of M1 transitions between its states requires the existence of $\vec{\mu}_{ij}$ in $\vec{\mu}$. This, however, can be checked independently since we have

$$(j^n J \| \sum_{i<j} \vec{\mu}_{ij} \| j^n J')$$

$$= \tfrac{1}{2} n(n+1) \Sigma (-1)^{J_2+J_{n-2}+J+J'} \sqrt{(2J+1)(2J'+1)} (j^{n-2}(J_{n-2}) j^2(J_2) | \} j^n J) \qquad (30)$$

$$\times (j^{n-2}(J_{n-2}) j^2(J_2) | \} j^n J') \times \begin{Bmatrix} J_2 & J & J_{n-2} \\ J' & J_2 & 1 \end{Bmatrix} (J_2 \| \vec{\mu}_{12} \| J_2).$$

We note that since $\vec{\mu}$ is a vector operator and the j^2 configuration has got only states with even values of J_2, the only terms which appear in (30) are $(J_2 \| \mu_{12} \| J_2')$ in which $J_2 = J_2'$; thus all M1 transitions within the configuration j^n, as well as all the static magnetic moments in this configuration, should be given in terms of the $(2j+1)/2$ "parameters"

$$(J_2 \| \vec{\mu}_{12} \| J_2) \text{ with } J_2 = 2, 4, 6, \dots\dots, 2j-1.$$

If observed M1 transitions between states of the same configuration fail to satisfy relation (30) where $(J_2 \| \vec{\mu}_{12} \| J_2)$ are derived from the static moments, then one is forced to conclude that the configuration is not pure (or, more precisely, that it cannot be described in terms of zero-order wave-functions with effective magnetic moment operators of the type $\Sigma \vec{\mu}_i + \sum_{i<j} \vec{\mu}_{ij}$). There are no nuclei on which enough data is available so far to test these ideas in full. However, by combining energy considerations and indications from

E2 transitions it is possible to put some limits on the size of $< \Sigma \vec{\mu}_{ij} >$ relative to $< \Sigma \vec{\mu}_i >$. At present these limits are not interesting since considerably smaller limits can be obtained by studying neutron capture on protons [9].

8. CORE EXCITATIONS IN ODD-A NUCLEI

In order to analyse the data on magnetic moments and M1 transitions further we should specify the operator $\vec{\mu}_i$ in more detail. Before we do this, however, it may be interesting to see how we can use the empirical fact that to a good approximation $\vec{\mu} = \Sigma \vec{\mu}_i$ in order to clarify some possible coupling schemes.

Given an odd-even nucleus, in which the odd nucleon is, for simplicity, in an $S_{1/2}$ orbit, one can ask whether it is a good approximation to consider some excited states as consisting of the odd-particle in the same $S_{1/2}$ orbit coupled to an excited state of the even-even "core" [10]. Formally, if J_c stands for the total angular momentum and other quantum numbers of the even-even core, and j describes the odd nucleon, we believe the ground state to be described by $|\, J_c = 0, j; J = j\rangle$ and we ask for the validity of the description by $|\, J_c = 2, j; J'\rangle$ of some specific excited states. In the absence of any residual interaction among the nucleons in the nucleus, the above description of states is perfectly legitimate if the "core" does not include nucleons in the j-orbit. When the interaction is switched on, the validity of such a coupling scheme depends on the degeneracy of states with total angular momentum J in the lowest group of degenerate states, and on the strength of the residual interaction relative to the spacings between groups of degenerate states.

If we assume that in the absence of residual interaction the lowest group of degenerate states of the core consists of the observed low-lying excited states, then we know that for this group $J_c = 0, 2, 4$ etc. If $j = 1/2$, the degeneracy of a state with a given J in the lowest group of degenerate states is therefore determined by the degeneracy of the state $J = J \pm 1/2$ (whichever is even); these degeneracies are rather low and, moreover, as the interaction is switched on, most of it is "absorbed" in splitting the degenerate states with the same J_c and one does not expect much admixture between the two states $|\alpha\, J_c\, j\rangle$ and $|\alpha'\, J_c\, j\rangle$. Thus we can use nuclei with $j = 1/2$ to study the extent to which the residual interaction is small or big compared to the separation between two different groups of degenerate states (in the zero-order approximation).

To be more specific, we may be confronted with a nucleus with an observed energy spectrum given in Fig. 7; we are asked to decide between two

$5/2^+$ ----------

$3/2^+$ ----------

$1/2^+$ ----------

Fig. 7

different interpretations:

(a) the "single particle" interpretation which asserts, essentially, that

$$|1/2^+\rangle = |0\ s\,1/2\ 1/2\rangle,\quad |3/2^+\rangle = |0\ d\,3/2\ 3/2\rangle$$

$$\text{and}\quad |5/2^+\rangle = |0\ d\,5/2\ 5/2\rangle;$$

(b) the "core excitation" interpretation where

$$|1/2^+\rangle = |0\,s\,1/2\;1/2\rangle, |3/2^+\rangle = |2\,s\,1/2\;3/2\rangle$$

$$\text{and } |5/2^+\rangle = |2\,s\,1/2\;5/2\rangle$$

Apart from energy considerations into which we shall not enter here, one can use electromagnetic properties to decide between the two alternatives. Taking the more reliable electric multipole moments first, we can write the quadrupole operator, for instance, in the form $Q = Q^{(c)} + Q^{(p)}$ where $Q^{(p)}$ operates on the odd particle and $Q^{(c)}$ on all the rest of the particles in the core. A simple inspection reveals that if we consider the E2 transitions $3/2^+ \rightarrow 1/2^+$ and $5/2^+ \rightarrow 1/2^+$, then, after correction for the energy dependence, the reduced transition rates are equal to each other under both alternatives. We may try to infer something from the absolute rate, but we know already that first-order polarization effects are absorbed in the single-particle description, and we have no a priori reliable information on the structure of $Q^{(c)}$.

If we look at the magnetic dipole data the situation is different. The single-particle interpretation does not give us any specific "predictions" unless we specify the operator $\vec{\mu}_i$ more fully; the core excitation interpretation, on the other hand, gives very specific relations between the magnetic moments of the states involved and the M1 transition $5/2^+ \rightarrow 3/2^+$. If we write $\vec{\mu} = \vec{\mu}^{(c)} + \vec{\mu}^{(p)}$ then it is easy to see that the three static moments and the M1 rate are all given in terms of specific combinations of $\langle 2 \| \mu^{(c)} \| 2 \rangle$ and $\langle \frac{1}{2} \| \mu^{(p)} \| \frac{1}{2} \rangle$. Thus there are two independent relations between these four quantities which can be checked experimentally. It is, of course, not impossible that the same relations will be satisfied "accidentally" also by the single-particle interpretation; however, the study of several available cases shows that this is very improbable. At present there seems to be an appreciable number of cases in which "core excitation" gives a fairly good explanation of the available data, and the analysis of the electromagnetic properties of these nuclei plays a crucial role in enabling us to make specific statements without involving too detailed a theory about the structure of the electromagnetic multipole operators.

9. GENERAL THEOREMS ON MAGNETIC MOMENTS

Coming back to the magnetic dipole operator, we know that for a single nucleon its structure is particularly simple, namely

$$\vec{\mu}_{sp} = g_\ell \vec{\ell} + g_s \vec{s}. \tag{31}$$

Apart from the complications due to exchange currents, which we shall disregard, this operator may change its form if we want to incorporate in it, rather than the wave function, first-order polarization effects. We saw above that if we are considering first-order polarization contributions to an operator of degree k, then it has the form of the zero-order matrix element of a k-th multipole in the residual interaction. It is for this reason that corrections to the quadrupole moment, induced by an interaction which contained $Q^{(i)}$, $Q^{(k)}$, looked like an effective charge. In the case of a magnetic moment we have to look for the dipole-dipole part in the interaction where both

dipoles have positive parity. For velocity independent interactions such terms could be of the form $\Sigma\, f(r_i)\vec{\sigma}_i$ or $\Sigma\, f(r_i)[Y_2(\theta_i \varphi_i) \times \vec{\sigma}_i]^{(1)}$. Thus such corrections to the magnetic dipole operator will affect g_s and not g_ℓ; in other words, we expect the magnetic dipole operator to be of the form

$$\vec{\mu} = g_\ell \vec{\ell} + [g_s(r) \times \vec{s}]^{(1)}, \tag{32}$$

where $g_s(r)$ may be a scalar or a second-rank tensor. The former will be the case if the interaction is of the type $\vec{\sigma}_1 \cdot \vec{\sigma}_2\, V_{12}$, and the latter in the case of tensor forces.

Although it is difficult to make definite statements about $g_s(r)$ without a detailed knowledge of the residual interaction, we can predict in which direction it will affect the free-nucleon's g-factor. If we consider an odd proton with spin up, say, polarizing the closed shells, then the following holds: this proton with spin up likes to see other protons with their spin down (Pauli principle); it likes, however, to see other neutrons with the spin up (the ^3S state of the deuteron is more bound than the ^1S); since the intrinsic moment of the neutron is opposite to that of a proton we see that in both cases the net effect of the polarization is to decrease the intrinsic angular momentum of the polarizing proton. Thus $g_s(r)$ will be a function whose expectation values, taken with zero-order wave functions, will be less than the free particle's g-factor. This is probably one of the important contributions to the "quenching" of intrinsic magnetic moments.

The fact that g_ℓ is not affected by first-order polarization is borne out by the general "slope" in plots of μ versus j for odd-proton and odd-neutron nuclei (Schmidt diagrams); the quenching of g_s has also been known for some time. Furthermore we see that apart from the contributions of tensor forces, the modified structure of $\vec{\mu}$ is still such as to make even the $\Delta\ell$ selection rule apply to magnetic dipole transitions; violations of this rule can come through terms of the type $[Y_2 \times \vec{\sigma}]^{(1)}$ and are therefore expected to be slower than normal allowed M1 transitions (i.e. between spin-orbit doublets).

To the extent that interaction moments can be neglected there are some interesting theorems about the moments of conjugate nuclei. To formulate them it is convenient to stay with the simple operators $\vec{\mu} = g_e \vec{\ell} + g_\ell \vec{s}$ and use the full "complicated" wave function Ψ. An M1 amplitude in a system containing protons and neutrons is then given by

$$\begin{aligned} A = \langle J'\, T' M_T' | \Sigma [(1+\tau_{i3})/2]\, \vec{\ell}_i + g_{sp} \Sigma [(1+\tau_{i3})/2]\, \vec{s}_i \\ + g_{sn} \Sigma [(1-\tau_{i3})/2]\, \vec{s}_i \,|\, JTM_T \rangle = \langle J'T'M_T' | \tfrac{1}{2}(\Sigma \vec{\ell}_i + \vec{s}_i) \\ + (g_{sp} + g_{sn} - 1)\tfrac{1}{2}\Sigma s_i + \tfrac{1}{2}\Sigma\, \tau_{i3}\,(\vec{\ell}_i + (g_{sp} - g_{sn})\, \vec{s}_i \,|\, JTM_T \rangle . \end{aligned} \tag{33}$$

In (33) we have explicitly indicated the isospin dependence of the states since we want to consider this part specifically. Take first static moments where $J = J'$ $T = T'$ and $M_T = M_T'$. Using the Wigner-Eckart theorem in isospin space we have

$$\begin{aligned} \vec{\mu}(TM_T) = \tfrac{1}{2} \langle J | \vec{J} + (g_{sp} + g_{sn} - 1) \Sigma \vec{s}_i | J \rangle (-1)^{T-M_T} \begin{pmatrix} T & 0 & T \\ -M_T & 0 & M_T \end{pmatrix} (T \| 1 \| T) \\ + \tfrac{1}{2} \langle J | \Sigma_i (T \| \tau_i \| T)(\vec{J}_i + (g_{sp} - g_{sn} - 1)\vec{s}_i \,|\, J \rangle (-1)^{T-M_T} \begin{pmatrix} T & 1 & T \\ -M_T & 0 & M_T \end{pmatrix}. \end{aligned}$$

We can now use the orthogonality of the Clebsch-Gordan coefficients in order to obtain

$$\Sigma\,(-1)^{T-M_T}\begin{pmatrix} T & 0 & T \\ -M_T & 0 & M_T \end{pmatrix}\vec{\mu}\,(TM_T) = (1/2T+1)^{\frac{1}{2}}\sum_{M_T}\vec{\mu}\,(TM_T) =$$

$$= \frac{1}{2}\left\langle J \,\middle|\, \vec{J} + (g_{sp} + g_{sn} - 1)\,\Sigma\,\vec{s}_i \,\middle|\, J \right\rangle (T\|\,1\,\|\,T)$$

$$\text{or}\quad \sum_{M_T}\vec{\mu}\,(TM_T) = \frac{1}{2}\,(2T+1)\left\langle J\middle|\,\vec{J} + (g_{sp} + g_{sn} - 1)\,\Sigma\,\vec{s}_i\,\middle|\, J\right\rangle .$$

Since the anomalous part g_{sp} and g_{sn} are equal in magnitude and opposite in their signs we see that $\sum_{M_T}\vec{\mu}(TM_T)$ is independent of the anomalous part of the intrinsic moments and, to the extent that the quenching is charge symmetric, this sum should be independent of quenching too. For self-conjugate nuclei, where $M_T = 0$ we find, since $\begin{pmatrix} T & 1 & T \\ 0 & 0 & 0 \end{pmatrix} = 0$, that

$$\vec{\mu}\,(T, M_T \neq 0) = \tfrac{1}{2}\left\langle J\middle|\,\vec{J} + (g_{sp} + g_{sn} - 1)\,\Sigma\,\vec{s}_i\,\middle|\, J\right\rangle$$

and again $\vec{\mu}$ is independent of the anomalous moment or charge-symmetric modifications of it.

For M1 transitions between two states of equal T in a self-conjugate nucleus we find [11]

$$A = [\,1/2(2T+1)^{\frac{1}{2}}]\,(g_{sp} + g_{sn} - 1)\left\langle J'\middle|\,\Sigma\,\vec{s}_i\,\middle|\, J\right\rangle \approx 0.9/(2T+1)^{\frac{1}{2}}\left\langle J\middle|\,\Sigma\,\vec{s}_i\,\middle|\, J\right\rangle .$$

This should be compared with a "normal" amplitude which is of order $1/2\,(g_{sp} - g_{sn}) \approx 4.7$. Thus M1 transitions in self-conjugate nuclei, taken between states of the same T, should be slowed down by a factor of about $(4.7/0.9)^2 \approx 30$.

We see that M1 transitions are very often hindered through one mechanism or another. This can actually be expected since the magnetic dipole operator is so closely connected with a constant of the motion, namely the total angular momentum. Really fast M1 transition can occur in odd-odd nuclei, where they are affected by the factor $g_{sp} - g_{sn}$; such transitions have actually been measured and analysed.

10. COMPARISON OF DIFFERENT MOMENTS

Before finishing this rather sketchy survey of some of the properties of nuclear multipole moments it may be interesting to mention also the possibility of comparing moments determined by electromagnetic methods with moments determined by other methods.

We have discussed earlier the moments of K^{39} and Cl^{37} as determined by their nuclear interaction with the 21st $f_{7/2}$ -neutron. The moments derived there are actually products of the moments of these nuclei with the corresponding moment of the 21st neutron. To get rid of the latter we can take ratios of the moments and obtain

$$[T^{(1)}(K^{39}) \;/\; T^{(1)}(Cl^{37})] = 1.1; [T^{(2)}(K^{39}) \;/\; T^{(2)}(Cl^{37})] = -1.0;$$

$$[T^{(3)}(K^{39}) \;/\; T^{(3)}(Cl^{37})] = 2.23.$$

If we believe the observed value of μ to be due to a modification of g_s, then we can use the measured moments of K^{39} and Cl^{37} to obtain

$$(\langle g_s\, \sigma \rangle K^{39} / \langle g_s\, \sigma \rangle Cl^{37}) = 1.26$$

and similarly

$$(Q(K^{39}) \;/\; Q(Cl^{37})) = -1.1 \pm 0.3.$$

No data is available on the electromagnetic octopole moments. Although there is certainly no equality between the ratios as derived from electromagnetic measurements and those from nuclear interactions, and although a priori there is no reason to expect such equality, it is interesting to note the similarity of the ratios. If this is found to be the case in further studies as well, it may indicate the rather weak dependence of the interaction moments on the radial coordinates.

A more interesting comparison of moments is between those derived from β - decay and those from electromagnetic interactions. Here one knows (Meyer-Jensen) that in jj-coupling the Gamow-Teller matrix element for β -decay between mirror nuclei is given by

$$M_{GT}^2 = (J+1/J)\left|2 \left\langle \Sigma\, \sigma_z^{(i)}\, t_z^{(i)} \right\rangle_{M=J}\right|^2.$$

For the magnetic dipole moment we derived the expression (33). In diagonal elements in jj-coupling we can replace

$$\vec{\ell}_i \rightarrow \frac{\langle (\vec{\ell}_i\, \vec{j}) \rangle}{j(j+1)}\, \vec{j}_i = \frac{j(j+1) + \ell(\ell+1) - 3/4}{2j(j+1)}\, \vec{j}_i$$

$$\vec{s}_i \rightarrow \frac{\langle (\vec{s}_i\, \vec{j}) \rangle}{j(j+1)}\, \vec{j}_i = \frac{j(j+1) + 3/4 - \ell(\ell+1)}{2j(j+1)}\, \vec{j}_i.$$

Hence we can write

$$\mu = \frac{J}{2}\left[1 + \frac{j(j+1) + 3/4 - \ell(\ell+1)}{2j(j+1)}\,(g_{sp} + g_{sn} - 1)\right]$$

$$+ \tfrac{1}{2}\left[\frac{j(j+1) + \ell(\ell+1) - 3/4}{j(j+1) + 3/4 - \ell(\ell+1)} + (g_{sp} - g_{sn})\right]\left\langle \Sigma\, s_z^{(i)}\, t_z^{(i)} \right\rangle_{M=J}.$$

We see that both the magnetic moments and the Gamow-Teller β -decay are determined by the same matrix element $\langle \Sigma\, s_z^{(i)}\, t_z^{(i)} \rangle$, and the consistency can be checked once we assume that g_{sp} and g_{sn} are known provided jj-coupling is valid. The agreement is quite good.

Another example can be taken from 1st forbidden β -decay [12]. The operator giving rise to a unique forbidden β-decay is $(1/\ell)(\vec{\sigma}\cdot \mathrm{grad})\, Y_{2K}(r)$;

however, the same operator gives rise also to the M2 electromagnetic radiation. An analysis of some experimental data involving $d_{3/2} \rightarrow f_{7/2}$ transitions gives:

γ - decay		β - decay	
Nuc.	$\|M\|^2$	Nuc.	$\|M\|^2$
$_{18}A^{39}_{21}$(1.52 MeV)	5.2×10^{-6}	$_{17}Cl^{39}_{22} \rightarrow {}_{18}A^{39}_{21}$	3.3×10^{-6}
$_{19}K^{41}_{21}$(1.29 MeV)	3.7×10^{-6}	$_{18}A^{41}_{23} \rightarrow {}_{19}K^{41}_{22}$	2.2×10^{-6}

$[\,|M|^2$ in units $(\hbar/m_e c)^2\,]$.

The observed values of $|M|^2$ are about a factor 7-8 smaller than calculated with single-particle wave-functions. It is interesting to note the consistency between the observed values of $|M|^2$ as derived from β-decay and from γ-decay. This may indicate that whatever renormalization there is of the M2 operator is due to nuclear effects rather than field-theoretic effects which should be so different in these two cases.

We have tried to give a brief survey of the sort of information that can be obtained on nuclear wave-functions and the nuclear moment operators from the study of observed moments. Obviously any assertion about the wave-function reflects itself on the moment-operator and the interrelation between them, though formally very simple, has not been studied yet as carefully as it should. It is still an open question to what extent the residual interaction can be renormalized to allow the use of simple-functions and what is the structure induced on the multipole moment operators as a result of such transformations. It is possible that when this relation is better understood we shall also be able to explain the many regularities observed in the empirical data on nucleon moments which are still a mystery to us.

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EFFECT OF SUPERCONDUCTING PAIRING CORRELATIONS ON NUCLEAR PROPERTIES

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INTRODUCTION

The development of nuclear physics has shown that the independent particle model, along with residual interactions between nucleons, describes the nuclear structure most correctly, at least at small excitation energies. The mathematical methods developed in constructing the superfluidity [1] and superconductivity [2] theories have proved to be essential when analysing residual nucleon interactions. In view of the fact that these methods are general and noting the similarity between the nuclear matter properties and the electronic structure of metals, BOGOLYUBOV [3] pointed out that nuclear matter can be superfluid. BOHR, MOTTELSON and PINES [4] noticed that the nuclear excitation spectra and the spectra superconducting states of metals are alike. They considered it reasonable to apply the methods used in the theory of superconductivity to study the properties of the finite nuclei.

These mathematical methods were employed in detail by BELYAEV [6] in the theory of an atomic nucleus, when he made use of the "principle of compensation of dangerous graphs", as well as by the author of these lectures, [5, 7] who used the variational principle suggested by Bogolyubov.

Investigation of the pairing correlations between nucleons of the superconducting type described in a number of papers allowed an interpretation of some nuclear properties which could not be accounted for in the framework of the model of independent particles, i. e. the energies of internal excitation states of even-even nuclei, the density of single particle levels of odd A nuclei, the momenta of inertia, equilibrium shape of the nuclei and some other properties.

This paper is devoted to the study of the effect which the pairing correlations of the superconducting type produce on the atomic nuclei properties.

1. NEW VARIATIONAL PRINCIPLE AS A GENERALIZATION OF THE HARTREE-FOCK METHOD

In considering the nuclear properties as a many-body system we shall make use of the variational method. The Hartree-Fock variational principle is one of the main methods of studying the many body problem. It has proved to be especially important in investigating atomic and molecular spectra and is widely used for studying nuclear matter properties. However, according to Hartree-Fock's method, the energy minimum is sought on a class of quasi-independent wave functions of individual particles and the pairing and more complicated correlations between the particles are disregarded.

BOGOLYUBOV has suggested a new variational principle [8] which is a natural generalization of the Hartree-Fock one. In his method the energy minimum is looked for over a wider class of functions than with that of Hartree-Fock, i. e. the wave functions of pairs are taken into consideration as well as the quasi-independent wave functions of individual particles.

The taking into account of pairing correlations in the many-body problem has led to the construction of the superconductivity theory and allowed the investigation of some important properties of the atomic nuclei. Let us deal with the formulation of this variational principle.

Consider a system of interacting nucleons in a nucleus described by a Hamiltonian of the form

$$H = \sum_{f,f'} T(f,f') a_f^+ a_{f'} + \tfrac{1}{2} \sum_{f_1,f_2,f_1',f_2'} K(f_1, f_2; f_2', f_1') a_{f_1}^+ a_{f_2}^+ a_{f_2'} a_{f_1'} \qquad (1.1)$$

where f is a system of indices characterizing the nucleon state. The operators a_f, a_f^+ satisfy the commutation relations

$$\begin{aligned} a_{f_1}^+ a_{f_2} + a_{f_2} a_{f_1}^+ &= \delta_{f_1,f_2} \\ a_{f_1} a_{f_2} + a_{f_2} a_{f_1} &= 0 \end{aligned} \qquad (1.2)$$

$$T(f,f') = E(f,f') - \lambda \delta_{ff'} . \qquad (1.3)$$

The real functions $E(f,f')$ and $K(f_1,f_2,f_2',f_1')$ satisfy the following relations

$$E(f,f') = E(f',f)$$

$$K(f_1,f_2;f_2',f_1') = -K(f_1,f_2;f_1',f_2') = K(f_1',f_2';f_2,f_1). \qquad (1.4)$$

Let us perform a linear canonical transformation of the Fermi amplitudes

$$a_f = \sum_\nu \left\{ u_{f\nu} \alpha_\nu + v_{f\nu} \alpha_\nu^+ \right\}, \qquad (1.5)$$

so that it does not violate their commutation properties; C-numbers $u_{f\nu}$ and $v_{f\nu}$ have to satisfy the following relations

$$\begin{aligned} \xi(f,f') &= \sum_\nu \left\{ u_{f\nu} u_{f'\nu}^* + v_{f\nu} v_{f'\nu}^* \right\} - \delta_{ff'} = 0 \\ \eta(f,f') &= \sum_\nu \left\{ u_{f\nu} v_{f'\nu} + v_{f\nu} u_{f'\nu} \right\} = 0 \\ \eta^*(f,f') &= \sum_\nu \left\{ u_{f\nu}^* v_{f'\nu}^* + v_{f\nu}^* u_{f'\nu}^* \right\} = 0, \end{aligned} \qquad (1.6)$$

where

$$\xi(f_1,f_2) = \xi^*(f_2,f_1), \qquad \eta(f_1,f_2) = \eta(f_2,f_1)$$

$$\alpha_\nu = \sum_f \left\{ u_{f\nu}^* a_f + v_{f\nu} a_f^+ \right\}. \qquad (1.7)$$

We shall define the state Ψ which is to be considered as a new vacuum

$$\alpha_\nu \Psi = 0. \qquad (1.8)$$

We find further as the average value of H over this state

$$\langle H \rangle \equiv \bar{H} = \sum_{f,f'} T(f,f') \sum_\nu v_{f\nu}^* v_{f'\nu} + \sum_{f_1,f_2,f_1',f_2'} K(f_1,f_2;f_2',f_1')$$

$$\times\Big\{\sum_{\nu} v^{*}_{f_1\nu}\ v_{f'_1\nu}\ \sum_{\nu'} v^{*}_{f_2\nu'}\ v_{f'_2\nu'}$$
$$+\ \tfrac{1}{2}\sum_{\nu} u^{*}_{f_1\nu}\ v^{*}_{f_2\nu}\ \sum_{\nu'} u_{f'_1\nu'}\ v_{f'_2\nu'}\Big\}\ . \qquad (1.9)$$

We determine $u_{f\nu}$ and $v_{f\nu}$ from the minimum condition of $\bar{H}$ which is written as

$$\delta\left\{\bar{H} + \sum_{f_1,f_2}[\lambda(f_2,f_1)\xi(f_1,f_2) + \mu^{*}(f_2,f_1)\eta(f_1,f_2) + \mu(f_2,f_1)\eta^{*}(f_1,f_2)]\right\} = 0, \qquad (1.10)$$

where $\lambda(f_1,f_2)$, $\mu(f_1,f_2)$ are the Lagrange factors, and the variations $\delta u_{f\nu}$, $\delta u^{*}_{f\nu}$, $\delta v_{f\nu}$ and $\delta v^{*}_{f\nu}$ are treated as independent. The chemical potential, which also plays the role of the Lagrange factors, is determined from the condition of the conservation of the number of particles n on the average

$$n = \sum_{f} \langle a^{+}_{f} a_{f}\rangle = \sum_{f,\nu} v^{*}_{f\nu}\ v_{f\nu}. \qquad (1.11)$$

Thus the new variational principle has been formulated: the functions $u_{f\nu}$ and $v_{f\nu}$ satisfy the stationary equations, Ψ is the wave function of the ground state, and $\bar{H}$ is regarded as the energy of the atomic nucleus ground state.

Further, following [9] we get the equations for determining $u_{f\nu}$ and $v_{f\nu}$ in an explicit form. From (1.9) we find

$$(\delta\bar{H}/\delta u^{*}_{f\omega}) + \sum_{f''}\left\{\lambda(f,f'')\,u_{f''\omega} + \mu(f,f'')\,v^{*}_{f''\omega}\right\} = 0,$$
$$(\delta\bar{H}/\delta v^{*}_{f\omega}) + \sum_{f''}\left\{\lambda(f,f'')\,v_{f''\omega} + \mu(f'',f)\,u^{*}_{f''\omega}\right\} = 0,$$

and two equations of their complex-conjugates.

We form the following expressions

$$A(f,f') \equiv \sum_{\omega}\left\{ v_{f'\omega}\,(\delta\bar{H}/\delta u^{*}_{f\omega}) + u_{f'\omega}\,(\delta\bar{H}/\delta v^{*}_{f\omega})\right\} + \mu(f,f') + \mu(f',f) = 0$$
$$B(f,f') \equiv \sum_{\omega}\left\{ u^{*}_{f'\omega}\,(\delta\bar{H}/\delta u^{*}_{f\omega}) + v^{*}_{f'\omega}\,(\delta\bar{H}/\delta v^{*}_{f\omega})\right\} + \lambda(f,f') = 0$$

eliminate the Lagrange factors and get the main equations in the form

$$\mathcal{A}(f,f') = A(f,f') - A(f',f) = 0$$
$$\mathcal{B}(f,f') = B(f,f') - B^{*}(f',f) = 0. \qquad (1.12)$$

It is worthwhile noting that the functions $\mathcal{A}(f,f')$ and $\mathcal{B}(f,f')$ are not independent, but related through

$$\sum_{f,f'}\Big\{u_{f\nu}v_{f'\nu'}\ \mathcal{A}^{*}(f',f) + u^{*}_{f'\nu'}\ v^{*}_{f\nu}\ \mathcal{A}(f,f')$$
$$+(u_{f\nu}u^{*}_{f'\nu'} - v^{*}_{f'\nu}\ v_{f\nu'})\,\mathcal{B}(f,f')\Big\} = 0. \qquad (1.13)$$

Therefore, if $\mathcal{A}(f,f') = 0$, then from (1.13) it follows that $\mathcal{B}(f,f') = 0$ and only the first one of these equations can be considered.

We introduce the functions

$$\phi(f_1, f_2) = \langle a_{f_1} a_{f_2} \rangle = \sum_\nu u_{f_1\nu} v_{f_2\nu}$$

$$F(f_1, f_2) = \langle a^+_{f_1} a_{f_2} \rangle = \sum_\nu{}' v^*_{f_1\nu} v_{f_2\nu}, \qquad (1.14)$$

where

$$\phi(f_1, f_2) = -\phi(f_2, f_1); \quad F^*(f_2, f_1) = F(f_1, f_2). \qquad (1.14')$$

We rewrite the expression for $\overline{H}$ as

$$\overline{H} = \sum_{f, f'}{}' T(f, f') F(f, f')$$

$$+ \sum_{f_1, f_2, f_1', f_2'} K(f_1, f_2, f_2', f_1') \{F(f_1, f_1') F(f_2, f_2') + \tfrac{1}{2}\phi^*(f_1, f_2) \phi(f_1', f_2')\} \qquad (1.9')$$

and find an explicit form $A(f, f')$

$$-A(f, f') = \sum_{f''} \{\phi(f', f'') \xi(f, f'') - \phi(f, f'') \xi(f', f'')\}$$

$$+ \sum_{f_1, f_2, f''} \phi(f_1, f_2) \{K(f, f''; f_2, f_1) F(f'', f') - K(f', f''; f_2, f_1) F(f'', f)\} -$$

$$- \sum_{f_1, f_2} \phi(f_1, f_2) K(f, f'; f_2, f_1) = 0 \qquad (1.15)$$

where

$$\xi(f, f') = T(f, f') + 2 \sum_{f_1, f_2} K(f, f_1; f_2, f') F(f_1, f_2) \qquad (1.16)$$

$$n = \sum_f F(f, f). \qquad (1.11')$$

Equations (1.15) and (1.11') can be treated as those for defining $\phi(f_1, f_2)$ and $F(f_1, f_2)$ without passing to $u_{f\nu}$ and $v_{f\nu}$. The functions $\phi(f_1, f_2)$ and $F(f_1, f_2)$ are not independent but related by

$$F(f_1, f_2) = \sum_{f''} \{F(f_1, f'') F(f'', f_2) + \phi^*(f'', f_1) \phi(f'', f_2)\},$$

$$\sum_{f''} \{F(f, f'') \phi(f'', f') + F(f', f'') \phi(f'', f)\} = 0. \qquad (1.17)$$

Note that rigorous mathematical reasons for using the functions of the type $\phi(f_1, f_2)$ were given by BOGOLYUBOV [10]. He introduced the notion of the quasi-averages and pointed out that in order to make use of any form of the perturbation theory for studying the statistical equilibrium states it is necessary, first of all, to switch the degeneration off or, which is the same, one should deal not with the functions constructed from the usual averages which satisfy all the selection rules, but with the functions constructed from the quasi-averages which do not satisfy some of these rules.

In the particular case $v_{f\nu} = v_f \delta_{f\nu}$ and $u_{f\nu} = u_f \delta_{f\bar{\nu}}$ one can easily find the basic equations in the theory of super-conductivity obtained earlier by the method of compensating "dangerous" graphs. Note that the new variational principle and the method compensating dangerous graphs are equivalent.

It is necessary for solving the stationary equations (1.10) of the variational principle to give the minimum of the energy in the system of inter-

acting nucleons that the second variation be positive for solution (1. 10). This condition can be conveniently written as

$$\sum_{f,\omega} \{ E_1 \delta v^*_{f\omega} \, \delta v_{f\omega} + E_2 \, \delta u^*_{f\omega} \, \delta u_{f\omega} \} > 0. \qquad (1.18)$$

This is fulfilled for positive eigenvalues of E_1 and E_2 provided that E_1, E_2, $\delta u_{f\nu}$, $\delta v_{f\nu}$ are determined from the equation for the eigenvalues.

Bogolubov's variational principle is a generalization of the well-known Hartree-Fock. In this connection among the solutions in Bogolubov's method there are always the solutions inherent to the Hartree-Fock method. Let us formulate the Hartree-Fock method. For this purpose we divide the system of quantum numbers ν into two parts F and G, $\nu \in F$ designates inside the Fermi sphere, and $\nu \in G$ outside the Fermi sphere. We choose $u_{f\nu}$ and $v_{f\nu}$ as follows:

$$\begin{aligned} u_{f\nu} &= 0, \quad v_{f\nu} = \omega_{f\nu} \quad \text{if } \nu \in F \\ u_{f\nu} &= \omega_{f\nu}, \quad v_{f\nu} = 0 \quad \text{if } \nu \in G \end{aligned} \qquad (1.19)$$

It can be easily seen that (1. 6) reduces to the normalization condition of the functions $\omega_{f\nu}$. Then $\phi(f, f') = 0$ and

$$F(f, f') = \sum_{\nu \in F} \omega^*_{f\nu} \, \omega_{f'\nu}$$

The corresponding condition of stationarity is written as

$$\delta \left\{ \overline{H}_F + \sum_{f,f'} \lambda'(f, f') \xi(f', f) \right\} = 0, \qquad (1.20)$$

where

$$\overline{H}_F = \sum_{f,f'} T(f, f') F(f, f') + \sum_{f_1, f_2, f'_1, f'_2} K(f_1, f_2; f'_2, f'_1) F(f_1, f'_1) F(f_2, f'_2) \qquad (1.21)$$

is the mean energy of the nuclear ground state in the independent-particle model.

As long as there are Hartree-Fock solutions among those of Bogolubov's method, then it is of interest to formulate the conditions under which the Hartree-Fock method does not yield the energy minimum of the interaction particle system. In this case the energy minimum should be sought in a wider class of solutions taking the pairing particle correlations into account as well. Thus, the absence of the energy minimum of the class of the functions in the Hartree-Fock method can be regarded as a condition for the existence of pairing correlations [11].

2. PAIRING NUCLEON CORRELATIONS IN ATOMIC NUCLEI

2. 1. Basic approximation

The interactions between nucleons in a nucleus can be roughly divided into long-range and short-range parts. The long-range part is responsible for the creation of the average nuclear field upon which the independent particle model is based. The short-range part leads mainly to the formation of pairing correlations between nucleons of the superconducting type.

Making use of the formulae obtained earlier we distinguish explicitly the self-consistent field and the interaction leading to the pairing nucleon correlations. Out of a set of quantum numbers f we separate $\sigma = \pm 1$ so that the states having different signs σ would be conjugate under time reversal.

We consider the approximation [12]

$$F(f, f') = F(f)\delta(f - f'). \tag{2.1}$$

It follows from the relation

$$F(f, f') = F^*(f', f) \tag{1.14'}$$

that $F(f) = F^*(f)$. Further, from (1.17) it is clear that

$$\phi(f, f') = \phi(f)\delta(f + f'), \tag{2.2}$$

$$F(f) = F(f)^2 + \phi^*(f)\phi(f). \tag{2.2'}$$

The average energy in this approximation assumes the form

$$\overline{H} = \sum_f \left\{ T(f) + \sum_{f'} K(f, f'; f', f)F(f') \right\} F(f) + \\ + \tfrac{1}{2} \sum_{f, f'} K(f, -f; -f', f')\phi^*(f)\phi(f'), \tag{2.3}$$

and the basic equation is written as

$$2\xi(f)\phi(f) + (1 - 2F(f))\sum_{f'} K(f, -f; -f', f')\phi(f') = 0 \tag{2.4}$$

where

$$\xi(f) = T(f) + \sum_{f'} K(f, f'; f', f)F(f').$$

Thus, in approximation (2.1) from the interaction of the most general type we take explicitly the self-consistent field and the interaction between nucleon pairs in the states conjugated under time reversal.

In approximation (2.1) there are no collective effects. We shall not consider in the following the interactions leading to collective effects nor take into account the connection between internal motion and rotation. Although these effects are essential in some cases, they will not noticeably affect the properties of the atomic nucleus to be studied. Moreover, the investigations based on approximation (2.1) may yield additional information as to when these effects must by taken into account.

Only one of all the types of residual interactions in approximation (2.1) $K(f, -f; -f', f')$, is chosen. This implies that strong correlations between nucleons occur only when they are in the states with the same energy and quantum numbers except σ. This is so because the nature of the residual short-range interaction is such that it leads to a much stronger interaction in the state with zero angular momentum than in other states. The binding energies of the last neutron in light nuclei indicate that the correlations in the states $(f, -f)$ are strong, while in other states the role of the residual interactions is negligible. Indeed [13], when the odd neutron and the odd proton are in the states with identical quantum numbers, as in Na^{22}, Al^{26}, P^{30}, Cl^{34} and K^{38}, then the binding energy of the last neutron is of the order

of 11-12 MeV, i. e. the same as for nuclei with an even number of neutrons of Ne^{22}, Mg^{26}, Al^{27}, P^{31}, S^{34}, Cl^{35} and others where two outer neutrons are paired. On the other hand, when the odd neutron and proton are in different quantum states, as in Na^{24}, Al^{28}, P^{32}, Cl^{36}, K^{40} and K^{42}, the binding energy of the last neutron is of the order of 7-8 MeV, i. e. the same as for the nuclei of Ne^{21}, Mg^{25}, Si^{31}, S^{33} and others in which the last neutron does not take part either in pair or in quadruple correlations. Hence, strong correlations between nucleons in the states with different quantum numbers are absent.

The problem is formulated as follows [5, 7]. Consider the residual nucleon interactions in the independent particle model with the Hamiltonian

$$H = \sum_{s\sigma} \left\{ E(s) - \lambda \right\} a^+_{s\sigma} a_{s\sigma} - \sum_{s,s'} G(s+, s-; s'-, s'+) a^+_{s+} a^+_{s-} a_{s'-} a_{s'+} . \qquad (2.5)$$

The nucleon state is described by a set of quantum numbers $(s\sigma)$, determined by the form of the average field; $a^+_{s\sigma}$, $a_{s\sigma}$ are the creation and absorption operators of a nucleon, $E(s)$ single-particle energies in the state s of the average field. The function $G(s_1\sigma_1, s_2\sigma_2; s'_2\sigma'_2, s'_1\sigma'_1)$ is real and satisfies (1.4). λ is the parameter, which plays the role of the chemical potential and which is determined from the condition

$$n = \sum_{s\sigma} \langle a^+_{s\sigma} a_{s\sigma} \rangle . \qquad (2.6)$$

It requires that the number of nucleons n should be conserved in the mean. Putting $F(s) = v_s^2$ and $\phi(s) = u_s v_s$, then (2. 2) becomes

$$u_s^2 + v_s^2 = 1 . \qquad (2.7)$$

Equation (2. 4) will thus take the form

$$2\xi_s u_s v_s + (u_s^2 - v_s^2) \sum_{s'} G(s+, s-; s'-, s'+) u_{s'} v_{s'} = 0 \qquad (2.8)$$

where

$$\xi_s = E(s) - \lambda - G(s+, s-; s-, s+) v_s^2 .$$

Equation (2. 8) allows a trivial solution

$$\begin{aligned} u_s &= 1 - \theta_F(s) \\ v_s &= \theta_F(s) \end{aligned} \qquad (2.9)$$

which corresponds to the normal state of the system. The function $\theta_F(s) = 1$ if $E(s) < E_F$ and $\theta_F(s) = 0$ if $E(s) > E_F$, where E_F is the energy of the Fermi surface or the energy of the last filled orbital (further denoted by K) in the independent particle model.

Let us introduce the correlation function

$$C_s = \sum_{s'} G(s+, s-; s'-, s'+) u_{s'} v_{s'} \qquad (2.10)$$

connected with u_s and v_s by

$$u_s^2 = \tfrac{1}{2}\left(1+\frac{E(s)-\lambda}{\epsilon_s}\right), \qquad v_s^2 = \tfrac{1}{2}\left(1-\frac{E(s)-\lambda}{\epsilon_s}\right)$$

$$u_s v_s = C_s/2\epsilon_s, \qquad \epsilon_s = \sqrt{C_s^2 + (E(s)-\lambda)^2}.$$

As a result we get the following system of basic equations [5, 7]:

$$C_s = \tfrac{1}{2}\sum_{s'} G(s+, s-; s'-, s'+) C_{s'} / \sqrt{C_{s'}^2 + (E(s') - \lambda)^2} \qquad (2.11)$$

$$n = \sum_s \left\{1 - [E(s) - \lambda / [\sqrt{C_s^2 + (E(s) - \lambda)^2}]\right\}. \qquad (2.12)$$

These equations have been obtained independently by BELYAEV [6], who used the method of "dangerous" graph compensation. Note that the superfluid state with $C \neq 0$ is energetically more favourable if compared with the normal state (2.9).

2.2. Condition for existence of neutron-proton pairing correlations

Let us investigate under what condition the neutron-proton pairing correlations of the superconducting type are absent and whether it is possible to consider such residual interactions in the proton and neutron systems independently [12].

The Hamiltonian of the residual proton and neutron interactions is written in the general form

$$H = \sum_{s\sigma\tau} \{E(s) - \lambda\tau\} a^+_{s\sigma\tau}\, a_{s\sigma\tau} - \sum_{\substack{s,s'\\ \tau,\tau'}} G_{\tau\tau'}(s,+,\tau,s,-,\tau';s',-,\tau',s',+,\tau) a^+_{s'+\tau} a^+_{s-\tau'} a_{s'-\tau'} a_{s'+\tau'} \qquad (2.13)$$

where $a_{s\sigma p} \equiv a_{s\sigma}$ is the proton operator and $a_{s\sigma n} \equiv b_{s\sigma}$ is the neutron one. In order to find the condition we make use of that part of (2.13) which corresponds to n-p interactions. In order to obtain a closed system of equations we introduce an auxiliary Hamiltonian. In this case the auxiliary Hamiltonian has the form

$$H_0 = \sum_{s\sigma} \left\{(E(s) - \lambda_p) a^+_{s\sigma} a_{s\sigma} + (E(s) - \lambda_n) b^+_{s\sigma} b_{s\sigma}\right\} - \sum_{s,s'} G_{pn}(s+, s-; s'-, s'+) \times \left\{A^*(s) b_{s'-} a_{s'+} + A(s') a^+_{s'+} b^+_{s+} - A^*(s) A(s)\right\}. \qquad (2.14)$$

where $A(s) = \langle b_{s-}, a_{s+}\rangle$.

The investigation is being made with the aid of the advanced and retarded Green's functions [14]. Let us give the main formulae. Let A(t) and B(t') be the operators in Heisenberg's representation. Then the retarded and advanced Green's functions are written down as

$$G_r(t-t') \equiv \ll A(t)B(t') \gg_r = -i\theta(t-t')\langle[A(t), B(t')]\rangle,$$

$$G_a(t-t') \equiv \ll A(t)B(t') \gg_a = i\theta(t'-t)\langle[A(t), B(t')]\rangle, \qquad (2.15)$$

where

$$[A, B] = AB + BA, \qquad \theta(t) = \int_0^t \delta(t')dt',$$

and the equations for the Green's functions are obtained in the form

$$i(\partial/\partial t)\langle\langle A(t)B(t')\rangle\rangle = \delta(t - t')\langle [A(t), B(t')]\rangle + \langle\langle\{A(t)H - HA(t)\}B(t')\rangle\rangle. \tag{2.16}$$

We write the correlation function in the spectral representation

$$\langle A(t)B(t')\rangle = \int_0^\infty I(\omega)e^{-i\omega(t-t')}\,d\omega \tag{2.17}$$

$$\langle B(t')A(t)\rangle = \int_{-\infty}^{\infty} I(\omega)e^{-i\omega(t-t')}\,d\omega. \tag{2.17'}$$

Let us go over to the Fourier transform in time of the Green's function

$$\langle\langle A(t), B(t')\rangle\rangle = \int_{-\infty}^{\infty} \langle\langle A|B\rangle\rangle_E\, e^{-iE(t-t')}\,dE \tag{2.18}$$

and to its spectral representation

$$\langle\langle A|B\rangle\rangle_{E\pm i\epsilon} = (1/2\pi)\int_{-\infty}^{\infty} \frac{I(\omega)\,d\omega}{E - \omega \pm i\epsilon} \tag{2.19}$$

where

$$iI(E) = \langle\langle A|B\rangle\rangle_{E-i\epsilon} - \langle\langle A|B\rangle\rangle_{E+i\epsilon}. \tag{2.20}$$

We make use of the auxiliary Hamiltonian (2.14) and get the following system of equations for the Green's functions

$$\{E - (E(s) - \lambda_n)\}\langle\langle b^+_{s-} \mid a^+_{s+}\rangle\rangle = -C(s)\langle\langle a_{s+} \mid a^+_{s+}\rangle\rangle \tag{2.21}$$

$$\{E - (E(s) - \lambda_p)\}\langle\langle a_{s+} \mid a^+_{s+}\rangle\rangle = (1/2\pi) - C(s)\langle\langle b^+_{s-} \mid a^+_{s+}\rangle\rangle$$

and others. Where

$$C(s) = \sum_{s'} G(s+, s-; s', s+)A(s').$$

The solutions of these equations are found in the form

$$\langle\langle a_{s+} \mid a^+_{s+}\rangle\rangle_E = [E+E(s) - \lambda_n]/2\pi[(E+\Delta)^2 - \epsilon(s)^2] \tag{2.22}$$

$$\langle\langle b^+_{s-} \mid b^+_{s+}\rangle\rangle_E = -C(s)/2\pi[(E+\Delta)^2 - \epsilon(s)^2]$$

where

$$\epsilon(s) = \sqrt{C(s)^2 + \{E(s) - \tfrac{1}{2}(\lambda_n + \lambda_p)\}^2}, \qquad \Delta = \tfrac{1}{2}(\lambda_p - \lambda_n).$$

We rewrite (2.22) as

$$\ll a_{s+} \mid a^{+}_{s+} \gg_E = (1/4\pi)\left\{\frac{1-[E(s)-\frac{1}{2}(\lambda_p+\lambda_n)]/\epsilon(s)}{E+\Delta+\epsilon(s)} + \frac{1+[E(s)-\frac{1}{2}(\lambda_p+\lambda_n)]/\epsilon(s)}{E+\Delta-\epsilon(s)}\right\}$$

$$\ll b_{s-} \mid a^{+}_{s+} \gg_E = -(1/2\pi)[C(s)/2\epsilon(s)]\left\{[1/(E+\Delta-\epsilon(s)] - [1/(E+\Delta+\epsilon(s)]\right\}$$

and find

$$iI_1 = (i/2)\{(1-[E(s)-\tfrac{1}{2}(\lambda_p+\lambda_n)]/[\epsilon(s)])\delta(E+\Delta+\epsilon(s))$$
$$+(1+[E(s)-\tfrac{1}{2}(\lambda_p+\lambda_n)]/[\epsilon(s)])\delta(E+\Delta-\epsilon(s))\},$$

$$iI_2 = -iC(s)/2\epsilon(s)\{\delta((E+\Delta-\epsilon(s)) - \delta(E+\Delta+\epsilon(s))\}.$$

By using (2.17) we get

$$\langle a_{s+}\, a^{+}_{s+}\rangle = \begin{cases} \frac{1}{2}\{1+[E(s)-\frac{1}{2}(\lambda_p+\lambda_n)]/\epsilon(s)\} & \text{if } \epsilon(s)>\Delta \\ 0 & \text{if } \epsilon(s)<\Delta, \Delta>0 \\ 1 & \text{if } \epsilon(s)<|\Delta|,\ \Delta<0 \end{cases} \tag{2.23}$$

$$\langle b^{+}_{s-}\, a^{+}_{s+}\rangle = \begin{cases} -C(s)/2\epsilon(s) & \text{if } \epsilon(s)>\Delta \\ 0 & \text{if } \epsilon(s)<\Delta \end{cases} \tag{2.24}$$

provided that C(s) satisfies the equation

$$C(s) = \tfrac{1}{2}\sum_{s'} G_{pn}(s+, s-; s'-, s'+)\frac{C(s')}{\sqrt{C(s')^2 + \{E(s') - \frac{1}{2}(\lambda_p+\lambda_n)\}^2}}.$$

Thus, the condition for the existence of neutron-proton correlations is as follows

$$|\lambda_p - \lambda_n| < 2C, \tag{2.25}$$

i.e. the difference in the chemical potentials of the neutron and proton systems must be less than the magnitude of the gap 2C. The condition of disappearance of n-p correlations of the superconducting type may be formulated as

$$|\lambda_p - \lambda_n| > 2\epsilon(f), \tag{2.26}$$

i.e. the difference in the chemical potentials must be greater than the excitation energy of the even system.

As long as the neutron potential well in the medium and heavy nuclei is about 5-10 MeV deeper than that of the proton and the magnitude of $2\epsilon(f)$ does not, as a rule, exceed 2-3 MeV, condition (2.26) may be considered as fulfilled. Thus, in medium and heavy nuclei the neutron proton pairing correlations of the superconducting type are absent and it is possible to study the superfluid properties for the neutron and proton systems separately. In light nuclei, condition (2.25) is fulfilled and, as shown in [5], they have neutron-proton pairing correlations of the superconducting type along with the proton-proton and neutron-neutron pairing correlations. In light nuclei

the pairing correlations are less important if compared with those in medium and heavy nuclei.

In light nuclei the quadruple correlations between nucleons are essential. It was shown [13], based upon the effective Hamiltonian which describes the interaction between nucleon pairs, that if the interaction between pairs is attractive, the formation of quadruple correlations is energetically favourable. The account of the pairing and quadruple correlations explains some regularities in the binding energies of the last neutron in light nuclei.

2.3. Superfluid properties of medium and heavy nuclei

Let us come back to the study of the interacting nucleon system with Hamiltonian (2.5), i.e. to the non-trivial solution of Eq. (2.11) describing the superfluid state of this system. The state of the Fermi particle system with energy less than in the system with successfully filled energy levels up to E_F (the Fermi surface energy) is called the superfluid state.

Consider the system consisting of an even number of neutrons (protons). We solve the equations

$$\alpha_{s\sigma}\Psi = 0 \tag{1.8}$$

and find the wave function in the form

$$\Psi = \prod_s (u_s + v_s a^+_{s+} a^+_{s-})\Psi_0, \tag{2.27}$$

where $a_{s\sigma}\Psi_0 = 0$. The ground state energy of the system is obtained as

$$\mathcal{E} = \sum_s \left\{2E(s)v_s^2 - C_s^2/2\epsilon_s\right\}, \tag{2.28}$$

while the equations for C and λ are of the form (2.11-12). The lowest excited states of the even system will be those with one broken pair, i.e. with two quasi-particles on the average field orbitals. The wave functions of such two quasi-particle states are written as

$$\alpha^+_{s_1\sigma_1}\ \alpha^+_{s_2\sigma_2}\ \Psi \approx a^+_{s_1-\sigma_1}\ a^+_{s_2-\sigma_2} \prod_{s\neq s_1, s_2} (u_s + v_s a^+_{s+} a^+_{s-})\Psi_0, \tag{2.29}$$

if $s_1 \neq s_2$. The energy difference between the excited and ground states is

$$\left\langle \alpha_{s_1\sigma_1}\ \alpha_{s_2\sigma_2}\ H\alpha^+_{s_2\sigma_2}\ \alpha^+_{s_1\sigma_1} \right\rangle - \langle H \rangle = \epsilon_{s_1} + \epsilon_{s_2}. \tag{2.30}$$

Thus, the excited states of the even system are separated from the ground state by an energy gap greater than 2C.

Consider the system consisting of an odd number of neutrons (protons). The equations for finding the correlation function C_s and the chemical potential λ in this case take the form:

$$C_s = \tfrac{1}{2}\sum_{s'\neq s_1} G(s+, s-; s'-, s'+)\left[C_{s'}/\sqrt{C^2_{s'} + \left\{E(s') - \lambda\right\}^2}\right] \tag{2.31}$$

$$n = 1 + \sum_{s\neq s_1}\left\{1 - \frac{E(s) - \lambda}{\sqrt{C_s^2 + \left\{E(s) - \lambda\right\}^2}}\right\}, \tag{2.32}$$

if the quasi-particle is on the level s_1 of the average field. The wave function and the energy of the system are written as

$$\alpha^{+}_{s_1}, \sigma_1 \Psi = a^{+}_{s_1} - \sigma_1 \prod_{s \neq s_1} (u_s + v_s a^{+}_{s+} a^{+}_{s-}) \Psi_0 \tag{2.33}$$

$$\mathcal{E}(s_1) = E(s_1) + \sum_{s \neq s_1} \left\{ 2E(s) v_s^2 - (1/2)(C_s^2 / \epsilon_s^2) \right\}. \tag{2.34}$$

The ground state of the odd system is that with one quasi-particle on the last filled level K of the medium field in the independent-particle model. In the particle excited states the quasi-particles are on the levels K+1, K+2 etc., while in the hole excited states the quasi-particles are on the average field levels K-1, K-2 etc. The energy difference between the excited and ground states of the odd system is

$$\mathcal{E}(s) - \mathcal{E}(K) = \epsilon_s - \epsilon_K = \sqrt{C_s^2 + \{E(s) - \lambda\}^2} - \sqrt{C_s^2 + \{E_F - \lambda\}^2}. \tag{2.35}$$

It is seen from this that the pairing correlations lead to a qualitative difference in the excitation spectra or the even-even and odd A-nuclei.

Note that the results obtained depend in no way upon a concrete set of quantum numbers, and therefore any form of the independent particle model can be employed. Therefore, this method can be used both in studying the properties of spherical nuclei and of deformed ones, involving axially non-symmetrical nuclei. We discuss some specific feature of the forces leading to the pairing correlations between nucleons of the superconducting type, i.e. the behaviour of the function $G(s+, s-; s'-, s'+)$. It is well known that the nature of the short-range forces is such that it leads to considerably stronger interaction in the states with total zero angular momentum than in other two-particle states. On the other hand, in medium and heavy nuclei strong correlations are observed only between those nucleons which are in the S-states with respect to each other. In this connection it is customary to believe that the residual interactions in question, which lead to the pairing correlations between the nucleons of the superconducting type, are short-range ones and may be represented as

$$G \sim \delta(\vec{r}_1 - \vec{r}_2).$$

This means that in the momentum space G is constant. Therefore in the shell and Nilsson models one may approximately consider $G(s+, s-; s'-, s'+)$ to be independent either of s' or s, i.e.

$$G = \text{const}. \tag{2.36}$$

We investigate the case when G = const. The correlation function here is constant

$$C = \sum_s G u_s v_s$$

and the basic equations (2.11) and (2.12) take the form

$$1 = (G/2) \sum_s [1/\sqrt{C^2 + \{E(s) - \lambda\}^2}], \tag{2.37'}$$

$$n = \sum_s \left\{ 1 - [E(s) - \lambda / \sqrt{C^2 + \{E(s) - \lambda\}^2}] \right\}. \tag{2.37''}$$

The energy of the system consisting of an even number of particles can be written as

$$\mathcal{E} = \sum_s 2E(s)v_s^2 - C^2/G. \qquad (2.38)$$

The pairing interaction constants for the neutron system G_N and for the proton system G_Z may be found according to the formulae

$$\begin{aligned} P_Z(Z,N) &= \tfrac{1}{2}\{2\mathcal{E}(Z-1,N) - \mathcal{E}(Z,N) - \mathcal{E}(Z-2,N)\} \\ P_N(Z,N) &= \tfrac{1}{2}\{2\mathcal{E}(Z,N-1) - \mathcal{E}(Z,N) - \mathcal{E}(Z,N-2)\}, \end{aligned} \qquad (2.39)$$

by using the experimental values of the pairing energies found from the mass difference of the atomic nuclei.

Thus, for the given values of G_N, G_Z and single-particle levels of the average field E(s) the correlation function C and the chemical potentials λ are found unambiguously - from Eqs (2.37) and (2.37'). This method of determining the basic characteristics of the superfluid state is essentially different from the approach employed by some authors [17], when the values of C are determined from the pairing energy, while λ is assumed to be equal to the energy of the Fermi surface E_F. The advantage of the above-mentioned method for finding C and λ over that applied in [17] consists in the following. First, in our case it becomes possible to determine C and λ for excited states taking into account the variation of C and λ with change in deformation of the nucleus etc. This cannot be done in the case of [17].

Secondly, our calculations are more accurate, unambiguous and reliable since we have at our disposal a single pairing interaction constant G which changes slowly and monotonously from nucleus to nucleus while the correlation function C changes abruptly depending on the specific behaviour of the energy levels of the self-consistent field. Moreover, C equals the pairing energy only roughly, whereas the deviation of λ from E_f is in some cases rather essential.

Thus, the account of the nucleon pairing correlations allowed the interpretation of the experimental data on the mass difference of even and odd nuclei as well as the energy gap in even-even nuclei. Some investigations permit the values to be obtained for the momenta of inertia of the deformed nuclei which are in agreement with experiment. Based upon these investigations, the conclusion can be drawn that the residual interactions both between neutrons and protons are attractive, and the ground states of medium and heavy nuclei are the superfluid ones. These superfluid properties of the ground and excited states should be taken into account in studying the nuclear structure.

3. SUPERFLUID NUCLEAR MODEL

3.1. Formulation of the model

The nuclear model in which the residual nucleon interactions leading to the superconducting pairing correlations are taken into consideration is called the superfluid model. The basic assumptions of the model are formulated so that it would be possible to make quantitative investigations of the properties of the ground and excited states of atomic nuclei. The very name "superfluid" model designates that this method of studying nuclear properties has a restricted character. The superfluid nuclear model [16] develops further the independent particle models and provides such a formu-

lation of the original method for studying pairing correlations [5-7] which is valid not only for a qualitative explanation of the atomic properties but also for quantitative studies of the properties of concrete nuclei.

Using the average field of the independent particle model, the superfluid nuclear model takes into account the short-range part of nucleon-nucleon interactions in a nucleus leading to pairing correlations under the following assumptions:

(1) The residual interactions both between neutrons and between protons are described by a Hamiltonian of the form:

$$H = \sum_{s\sigma} \{ E_0(s) - \lambda \} a^+_{s\sigma} a_{s\sigma} - G \sum_{s,s'} a^+_{s+} a^+_{s-} a_{s'-} a_{s'+} \qquad (3.1)$$

(2) The basic equations of the problem are found by means of the Bogolyubov variational principle [8], both for the ground and excited states of a nucleus, provided that the systems of equations which characterize the properties of the ground and excited states are obtained for them. The effect of the excitation on the superfluid properties is referred to as the "blocking effect".

(3) The mathematical method of solving the problem leads to the conservation of the number of particles on the average

$$n = \sum_{s\sigma} \langle a^+_{s\sigma} a_{s\sigma} \rangle . \qquad (3.2)$$

However, the calculations are made for quite definite nuclei.

The basic assumptions of the superfluid nuclear model are different from those of the original method of considering the pairing correlations [5-7]. These differences consist in the following: (a) The superfluid nuclear model takes into account the change in the superfluid nuclear properties in the transition from the ground to the excited state; (b) The conservation of the number of particles on the average, as long as in [5-7] the number of particles is not conserved even on the average. These differences are most essential for strongly deformed nuclei.

We find the equations for the characteristics of the superfluid state, the wave functions and the energies of the ground and excited states of the even and odd system. Let us perform a linear canonical transformation

$$a_{s\sigma} = u_s \alpha_{s-\sigma} + \sigma v_s \alpha^+_{s\sigma} , \qquad (3.3)$$

provided that $u_s^2 + v_s^2 = 1$. Now it is possible to get the mean value of the operator of the energy $\overline{H}$ by the state Ψ defined as $\alpha_{s\sigma} \Psi = 0$, i.e.

$$\overline{H} = 2\sum_s \{ E_0(s) - \lambda \} v_s^2 - G \sum_{s,s'} u_s v_s u_{s'} v_{s'} - G \sum_s v_s^4 .$$

As long as the term $G \sum_s v_s^4$ makes a contribution to the self-consistent field, we carry out the renormalization

$$E(s) = E_0(s) - (G/2) v_s^2 \qquad (3.4)$$

and get

$$\overline{H} = 2\sum_s \{ E(s) - \lambda \} v_s^2 - G \sum_{s,s'} u_s v_s u_{s'} v_{s'} . \qquad (3.5)$$

We determine u_s, v_s from the condition that $\overline{H}$ should be a minimum. As a result we have

$$2\{E(s) - \lambda\} u_s v_s - G(u_s^2 - v_s^2)\sum_{s'} u_{s'} v_{s'} = 0.$$

We introduce the correlation function *

$$C = G\sum_{s'} u_{s'} v_{s'} \tag{3.6}$$

and determine

$$u_s^2 = (1/2)\{1 + [E(s) - \lambda / \epsilon(s)]\}, \quad v_s^2 = (1/2)\{1 - [E(s) - \lambda / \epsilon(s)]\}$$

$$\epsilon(s) = \sqrt{C^2 + \{E(s) - \lambda\}^2}.$$

The wave function, the equations for determining C and the energy of the ground state of the even system are obtained as follows:

$$\Psi = \prod_s (u_s + v_s a_{s+}^+ a_{s-}^+)\Psi_0, \tag{3.7}$$

$$(2/G) = \sum_s (1/\sqrt{C^2 + \{E(s) - \lambda\}^2}), \tag{3.8}$$

$$n = \sum_s \{1 - [E(s) - \lambda] / [\sqrt{C^2 + \{E(s) - \lambda\}^2}]\} \tag{3.9}$$

$$\mathcal{E} = \sum_s E(s)\{1 - [E(s) - \lambda] / [\sqrt{C^2 + \{E(s) - \lambda\}^2}]\} - (C^2/G) \tag{3.10}$$

where $a_{s\sigma}\Psi = 0$. The wave functions, the energy and the basic equations for the two-quasi-particle excited states of the even system are found to be

$$\Psi(f_1, f_2) = a_{f_1\sigma_1}^+ a_{f_2\sigma_2}^+ \prod_{\substack{s\neq f_1, f_2 \\ f_1 \neq f_2}} (u_s(f_1, f_2) + v_s(f_1, f_2) a_{s+}^+ a_{s-}^+)\Psi_0 \tag{3.11}$$

$$\Psi(f_1, f_2) = [u_{f_1}(f_1, f_1) a_{f_1+}^+ a_{f_1-}^+ - v_{f_1}(f_1, f_1)] \prod_{s\neq f_1} (u_s(f_1, f_1) + v_s(f_1, f_1) a_{s+}^+ a_{s-}^+)\Psi_0 \tag{3.11'}$$

$$\mathcal{E}(f_1, f_2) = E(f_1) + E(f_2) + \sum_{s\neq f_1, f_2} E(s) v_s(f_1, f_2)^2 - C(f_1, f_2)^2/G \tag{3.12}$$

$$(2/G) = \sum_{s\neq f_1, f_2} (1/\sqrt{C(f_1, f_2)^2 + \{E(s) - \lambda(f_1, f_2)\}^2}), \tag{3.13}$$

$$n = 2 + \sum_{s\neq f_1, f_2} \left\{1 - \frac{E(s) - \lambda(f_1, f_2)}{\sqrt{C(f_1, f_2)^2 + \{E(s) - \lambda(f_1, f_2)\}^2}}\right\}. \tag{3.14}$$

The state with one quasi-particle on the K-level is the ground state of the system consisting of an odd number of particles. As excited states of the odd system, we consider both the single-quasi-particle and three-quasi-particle states. For the single-quasi-particle ground and excited states we get the wave function

* The correlation function denoted here by C is often designated in the literature by Δ.

$$\Psi(f_i) = a^+_{f_{0i}} \prod_{s \neq f_i} (u_s(f_i) + v_s(f_i) a^+_{s+} a^+_{s-}) \Psi_0 , \qquad (3.15)$$

the energy

$$\mathcal{E}(f_i) = E(f_i) + \sum_{s \neq f_i} E(s) v_s(f_i)^2 - [C(f_i)^2 / G] \qquad (3.16)$$

and the basic equations

$$(2/G) = \sum_{s \neq f_i} [1/\sqrt{C(f_i)^2 + \{E(s) - \lambda(f_i)\}^2}] , \qquad (3.17)$$

$$n = 1 + \sum_{s \neq f_i} \left\{ 1 - \frac{E(s) - \lambda(f_i)}{\sqrt{C(f_i)^2 + \{E(s) - \lambda(f_i)\}^2}} \right\} . \qquad (3.18)$$

In order to determine the main superfluid characteristics of the above-mentioned states, i.e. the correlation functions C and the chemical potentials λ, we solve, as in [18], the corresponding systems of equations by means of an electronic computer. As the average field levels, we can take slightly corrected energy levels of NILSSON's scheme [19].

3.2. Pairing energy and single-particle levels of odd A-nuclei

The pairing interaction constants of the neutron G_N and proton G_Z systems have been calculated from the experimental data on the mass difference of nuclei. In doing this, the following formula was used

$$P_N = \tfrac{1}{2} \{ 2\mathcal{E}(Z, N-1) - \mathcal{E}(Z, N) - \mathcal{E}(Z, N-2) \} \qquad (3.19)$$

or the stricter one

$$P_N = \tfrac{1}{4} \{ 3\mathcal{E}(Z, N-1) + \mathcal{E}(Z, N+1) - 3\mathcal{E}(Z, N) - \mathcal{E}(Z, N-2) \} \qquad (3.19')$$

where the corresponding experimental data were available. In finding the pairing interaction constants G_N and G_Z the calculated values of the pairing energies were compared thoroughly with the experimental data [20, 21]. The results of the analysis are plotted for example in Fig. 1 for the range $225 \leq A \leq 255$, where the experimental values of the pairing energies P_Z are shown by the dashed lines, and the calculated values for P_Z at $G_Z = 29/A$ MeV by the solid ones.

From a comparison of the calculated pairing energies, with the experimental data in both regions of strongly deformed nuclei $154 \leq A \leq 188$ and $225 \leq A \leq 255$, the following values for the pairing interaction constants have been obtained

$$G_N = \frac{26 - 27}{A} \text{ MeV}$$

$$G_Z = \frac{28 - 29}{A} \text{ MeV} \qquad (3.20)$$

in summing over 36 average field levels.

The summation in the equations for finding C and λ is made over 36 average field levels. We investigate as to how strongly the results of the calculations depend on the cut-off in these equations. To this end, we calcu-

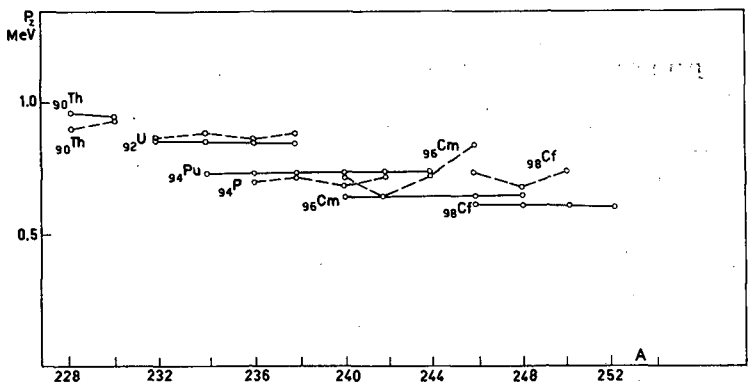

Fig. 1

The comparison of calculated values of the pairing energies P_Z (solid lines) and experimental values (dashed lines), for $225 \leq A \leq 255$.

late the correlation functions C and C(K) (C(K) is the correlation function of the ground state of the system with the odd number of nucleons) for Z = 94 and Z = 93 as well as the pairing energy P_Z for Z = 94 when the summation is performed over 17 average field levels below the K-state * and above K different cut-offs are made, namely at K+3, K+6, K+9, K+12, K+15 and K+18.

The results of the calculation are presented in Table I. For the same value of $G_Z = 0.0185\ \hbar\omega_0$ (which corresponds to the real nuclear forces in summing over 36 levels) the correlation function C depends on the cut-off strongly. It increases when the summation region becomes wider. However, at the same time the value of the pairing energy P_Z changes just as strongly. The ratio C(K)/C depends on the cut-off weakly if it is made high enough. In the calculations according to the superfluid nuclear model the value of G_Z is found by comparing the computed values of P_Z with the experimentally obtained values for the pairing energies. At the same time the values of G_Z depend on the cut-off. Therefore, to clarify the role of the cut-off, it is necessary to make the calculations at such different values of G_Z that for each cut-off one should get the same magnitude of the pairing energy. The results of such calculations are given in the lower part of Table I. We did not succeed in choosing such G_Z that for all the cut-offs it would be possible to obtain exactly the same magnitude of the pairing energies. However, the changes in P_Z for K+6 and higher are insignificant. It is seen from Table I, that if more than six levels are summed above K, then the correlation function C, the ratio C(K)/C and, therefore, all the superfluid properties are practically independent of the cut-off for the same magnitude of the pairing energy.

Thus, the main characteristics of the superfluid states do not practically depend on the cut-off if it is made at energies higher than 3-5 MeV both above and below K. When the summation in these equations is restricted, there is no necessity for introducing the cut-off constant since when account is taken of this cut-off the pairing interaction constant G is as if renormalized.

* We denote by K the last filled single-particle level of the average field, by K-1, K-2 etc. - the hole states, and by K+1, K+2 etc. - the particle states.

TABLE I

INVESTIGATION INTO THE ROLE OF THE CUT-OFF

	K+3	K+6	K+9	K+12	K+15	K+18
$G_Z(\hbar\omega_0^0)$	0.0185	0.0185	0.0185	0.0185	0.0185	0.0185
$P_Z(\hbar\omega_0^0)$	0.037	0.055	0.070	0.081	0.091	0.102
$C(\hbar\omega_0^0)$	0.059	0.077	0.091	0.103	0.114	0.125
C(K)/C	0.64	0.75	0.79	0.81	0.83	0.83
$G_Z(\hbar\omega_0^0)$	0.026	0.023	0.021	0.020	0.019	0.0185
$P_Z(\hbar\omega_0^0)$	0.088	0.1025	0.101	0.103	0.094	0.102
$C(\hbar\omega_0^0)$	0.123	0.130	0.128	0.127	0.123	0.125
C(K)/C	0.88	0.85	0.84	0.84	0.83	0.83

We have obtained that in the regions $154 \leq A \leq 188$ and $225 \leq A \leq 255$ the pairing interaction constants G_N and G_Z change in going from a nucleus to a nucleus and from one region of deformed nuclei to another with good accuracy as A^{-1}. We have obtained the only value of $G_N A$ for all the neutron systems and the only value of $G_Z A$ for all the proton systems in both groups of strongly deformed nuclei. This means that the region of the effective interaction restricted by the accepted cut-off is chosen correctly. This implies also that the assumption on the independence of G (s+, s-; s' -, s' +) both of s and s' is reasonable. Thus, the comparison of theory with experiment has shown that one of the main assumptions, G = const, is fulfilled with a high accuracy.

Now we consider the influence of superfluidity on the behaviour of the single-particle levels of the odd A nuclei. As is well known, the spin and parity of the state of the odd nucleus is determined by the spin and the parity of that average field level on which the quasi-particle is located. This results from the great role the pairing correlations play in a nucleus, and follows directly from the superfluid nuclear model. When pairing interaction is involved the behaviour of the average field levels changes in the following way:

(1) The pairing correlations, as a rule, do not alter the spin of the ground state of the system;

(2) The excitation energy of the system decreases rather quickly with G, although the compression of the single-particle levels does not occur uniformly;

(3) Hole and particle excited states behave differently with increasing G. However, the sequence of hole (particle) levels with respect to each other remains unchanged.

Figure 2 shows, by the example of N = 105, the influence of the pairing correlations on the single-particle levels of the average field. The ground state energy is put to zero, below $\mathcal{E}(K) = 0$ are plotted the energies of the hole excited states, and above $\mathcal{E}(K) = 0$ the energies of the particle excited states.

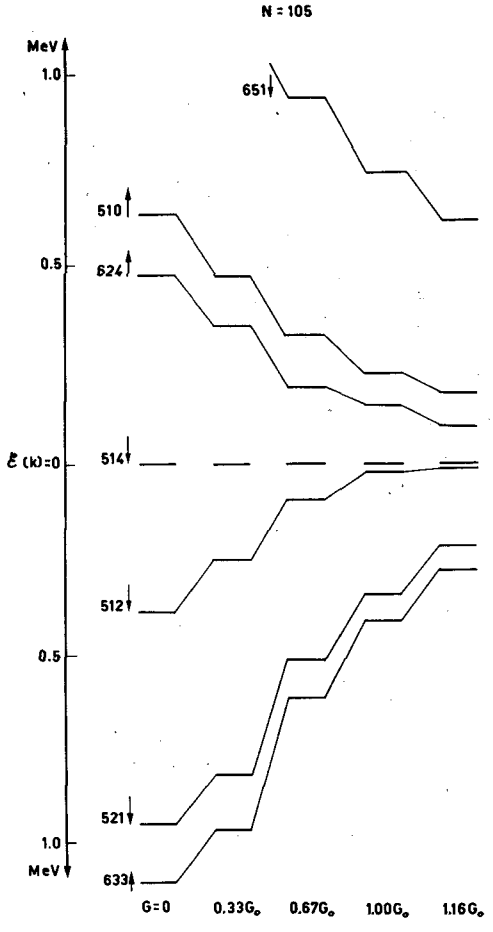

Fig. 2

The influence of the pairing correlations (for N = 105) in the single-particle levels of the average field.

On the left are demonstrated the characteristics of the states *, identical states are connected by lines.

In [18] the single-particle levels of many odd nuclei were computed, and the calculated values of the excitation energies are in better agreement with the experimental data than those given in the Nilsson schemes. However, as long as the found levels of the odd nuclei depend strongly on the behaviour of the average field levels, the main emphasis is laid upon the investigations of the single-particle level density. It was shown that both in the rare-earth and in the transuranic regions the density of the calculated low-energy single-

* The notation, as in [22], is based on the asymptotic quantum numbers. N is the total number of oscillating quanta, n_z is the number of oscillating quanta along the axis perpendicular to the symmetry axis, Λ is the component of the particle orbital angular momentum along the symmetry axis, Σ is the projection of the particle spin on this axis, $K = \Lambda \pm \Sigma$, π is the parity. The state is written as $K^{\pi}[Nn_z\Lambda]$ or, in short $Nn_z\Lambda\uparrow$, if $K = \Lambda + \Sigma$ and $Nn_z\Lambda\downarrow$; if $K = \Lambda - \Sigma$, $\hbar\omega_0 = 41\, A^{-\frac{1}{3}}$ MeV.

particle levels agrees with the experimental data and is twice as large as that of the levels in Nilsson's scheme. It should be noted that the increase in the density of the levels is accounted for by the superfluid properties of the ground and excited states and cannot be obtained by changing the behaviour of the single-particle levels in the independent particle model.

The single-particle levels of the odd nuclei yield information on the energy levels of the average field, which is necessary for calculating the energies of the even-even nuclei, for analysing the β-transition rates etc. Therefore, experiments aimed at finding them are of great interest.

3.3. Investigation of the accuracy of the calculations and the choice of parameters

Here we shall consider the question as to what restricts the accuracy of the calculations by the superfluid nuclear model and whether it is possible to study, on its basis, not only the general regularities in the behaviour of nuclei, but also specific features of each nucleus. The errors in the calculations are due to the following. First, we do not know the position of the average field levels accurately enough, as well as their fluctuation in passing from a nucleus to a nucleus. Secondly, the mathematical methods employed for solving the problem are approximate.

Let us now deal with the first of these two circumstances. It is well known that the superfluid properties of the system depend strongly on the behaviour of some levels of the average field in the vicinity of the K level and on the magnitude of the pairing interaction constant G. Preliminary calculations were based on slightly corrected levels of Nilsson's scheme which fails to yield the right sequence of energy levels in all cases. Nor does it give the necessary distance between them. Therefore, the accuracy of the calculations is restricted mainly to a rough description of the behaviour of the average field levels. Thus, the calculations made earlier have some shortcomings which are inherent, first of all, in Nilsson's scheme itself, and, secondly, which are due to insufficiently known parameters of the equilibrium deformations. The third shortcoming is that the change in the average field of the proton (neutron) system is not taken into account by changing the number of neutrons (protons) etc.

Therefore in [20] a modified scheme of calculations was suggested which was based on the experimental data concerning the single-particle levels of the odd A nuclei and on pairing energies. In the main, use was made of Nilsson's scheme, but the behaviour of some levels near each K level was corrected and the pairing interaction constant G was chosen so that one might obtain the single-particle spectra of the odd A-nuclei and the pairing energies which would be consistent with the experimental data. In the course of fulfilling this programme with an electronic computer, the equations were solved for finding C and λ, and the energies $\mathcal{E}$ and $\mathcal{E}$ (K) were calculated for different parameters E(s) and G. The best values of these parameters were chosen. As long as the position of some levels near K is fixed in accordance with the experimental data and the behaviour of the other levels does not practically affect the properties of the system, the calculations which do not involve the wave functions, are independent of a concrete form of the average field potential.

The application of such a calculation scheme has led to much better agreement between the calculated and experimental values for the energies of the excited states of even-even strongly-deformed nuclei if we compare

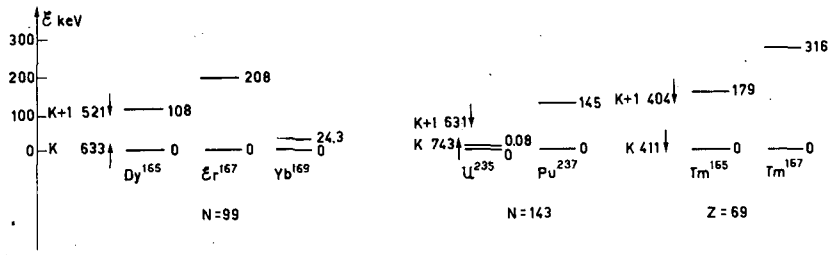

Fig. 3

The energy levels of the nuclei with N = 99, N = 143 and Z = 69.

them with earlier calculations based on Nilsson's scheme. However (in order to clear up the validity of the basic assumptions of the superfluid nuclear model without introducing a large number of parameters), we deliberately neglected the fluctuations of the average field levels in going from a nucleus to a nucleus. These fluctuations are displayed in the behaviour of the single-quasi-particle levels of the odd A-nuclei. In some cases such fluctuations are considerable, which is seen, for instance, from Fig. 3, where the changes in the behaviour of the levels of the nuclei with N = 99, N = 143 and Z = 69 are given. Further on, the accuracy of the calculations of the excited state energies for even-even nuclei may be increased. For this purpose for each even-even nucleus one should take its own set of the average field levels. The levels of the neighbouring odd nuclei, calculated according to this latter assembly, will be in good agreement with the corresponding experimental data. However, no experimental data necessary for such calculations are thus far available.

Let us proceed now to the second cause of inaccuracy in our calculations; that is, for solving the problem we investigate the accuracy of the approximate method employed. We estimate roughly the accuracy of the mathematical method. With this aim, we find the ratio $\Delta n/2\Omega$ of the average quadratic fluctuation of the number of particles Δn to that of the considered states 2Ω of the average field, where

$$(\Delta n)^2 = \sum_s [C^2/(C^2 + \{E(s) - \lambda\}^2)] . \qquad (3.21)$$

As long as the functions characteristic of the superfluid properties are most effective in the energy range which is (3-4) C above and below the K level, we calculate $\Delta n/2\Omega$ for this range. The ratio $\Delta n/2\Omega$ is found to be 0.08 for the energy half-interval 3C, 0.06 for 4C and 0.05 for the half-interval 5C. In an extremely small half-interval 2C the ratio $\Delta n/2\Omega$ is 0.12. For all the states of the odd system and for the excited states of the even one, the average quadratic fluctuation of the number of particles is always less than Δn for the ground state of the even system. In other words, according to these estimates the errors in the method amount to 5% and at any rate they do not exceed 10%.

Still more interesting and fruitful is the investigation of the accuracy of the method with the model treated by PAWLIKOWSKI and RYBARSKA [23]. They considered the interactions described by Hamiltonian (3.1) of n particles located on the Ω twice degenerate equidistant levels. This problem was solved

exactly with an electronic computer for the case n = 6, $\Omega = 5$ for G equal to $0.5\,\Delta E$, $0.8\,\Delta E$, ΔE and $1.25\,\Delta E$, where $E_{i+1} - E_i = \Delta E$. An exact solution of this problem is compared with approximate solutions: (a) Without taking the blocking effect into account, as in the original method for treating the pairing correlations; and (b) Taking into account the blocking effect, as in the superfluid nuclear model.

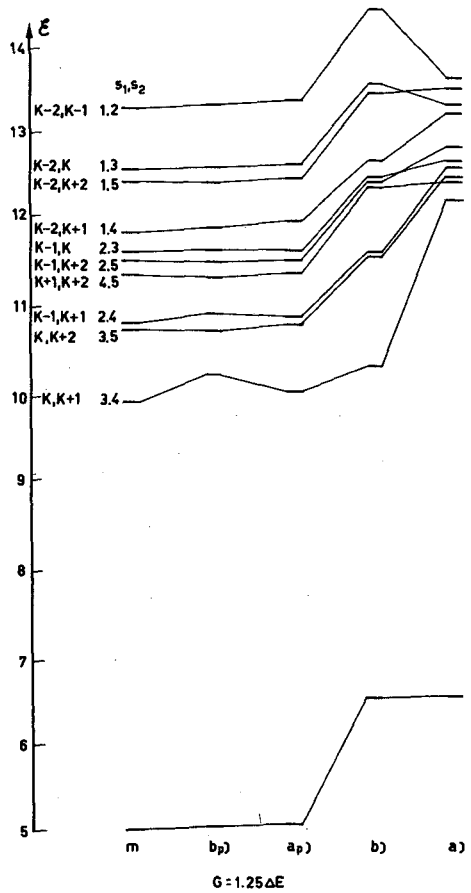

Fig. 4

Energies of the ground and excited states calculated by the exact method (m) and by approximate ones (a, b) for $G = 1.25\,\Delta E$.

A comparison is given in Fig. 4 of the energies of the ground and excited states calculated by the exact method (denoted by m) and by approximate ones (denoted by a and b) for $G = 1.25\,\Delta E$. The ground state energy of the system in the case of an exact solution for all the values of G is less than that obtained in the approximate method, which does not hold for excited states. Because of this, the excitation energies obtained in approximations (a) and (b) are on the average somewhat smaller than those in the case of an exact solution. Note that in the exact method the first excited state

(K, K+1) is somewhat lower than the magnitude of the gap 2C shown by the dashed line.

It can be seen from Fig. 4 that the method (b), corresponding to the approximation of the superfluid nuclear model, described the sequence of the excited states correctly, as well as their behaviour, although the magnitudes of the excitation energies are noticeably different from the values obtained in the exact solution. In the case of method (a) the errors have opposite signs, the sequence of levels changes if compared with the exact solution.

It should be noted that in the model considered, a much poorer accuracy is obtained in the solutions of the approximate methods compared with the calculations in case of strongly deformed nuclei. This is so because of too small values of n and Ω as well as due to relatively large magnitude of G.

It is well-known that in the calculations by the superfluid nuclear model the number of particles is conserced on the average, and the wave functions of the system of particles involve the admixtures of the states with N-2, N+2 etc. As mentioned in [24, 25], the accuracy in calculating the excitation energies becomes higher if, instead of the wave functions (3.7), their normalized projections are used for the subspace of the states of n particles. In Fig. 4 the exact calculations of the excitation energies are compared with the approximate ones by the methods (a) and (b) with the projected wave functions denoted by ap) and bp). In this case very good agreement is obtained with the exact solution, especially for large values of G.

Let us compare the densities of the particle distributions in the ground and excited states calculated by the exact and approximate methods. For this purpose, we present in Table II the density distribution of the number of particles in the ground state if the system for the values G equal to $0.5\Delta E$, ΔE and $1.25\,\Delta E$ in the case of the exact solution $\overline{N}_s$, the solution with projected wave functions $\langle \phi_0 | N_s | \phi_0 \rangle$ and for the approximate solution v_s^2. One can see from the Table that v_s^2 describes the density distribution of the number of particles rather well. The ratio $v_s^2/\overline{N}_s$ varies within 0.91-1.25. The calculations with projected wave functions fail to lead to a noticeable improvement in the approximations (a) and (b).

Table III lists the density distributions of particle pairs in the two-quasiparticle states with $G \approx 1.25\,\Delta E$. It can be seen that the calculations by method (b) describe the distribution of particle pairs in the excited states correctly. It is worthwhile noting that in the state (K, K+1) (method (b)) the pairing correlations are suppressed more strongly than in the exact method while the contrary occurs for the remaining states. The calculations with the projected wave functions lead to a noticeable weakening of the superfluidity if compared with the exact solution. In the calculations (method (a)) the number of particles is not conserved on the average and the density of pair distribution is different from that in the exact solution.

The investigation of the density distribution of the number of particles in the ground and excited states has shown that the calculations by method (b) are in good agreement with exact solutions. With increasing Ω the accuracy of the calculations by method (b) may only become higher. It follows that the accuracy of the real calculations of v_s^2 and u_s^2 will be better. The superfluid corrections to the α- and β-transition rates are composed of the products and of the sums of u_s and v_s and therefore, they can be computed with a high accuracy. The conclusion to be drawn from the previous analysis is that no further improvement in the accuracy of the mathematical approxi-

TABLE II

DISTRIBUTION OF PAIR DENSITY IN THE GROUND STATE

s	$G = 1.25\Delta E$			$G = 1\,\Delta E$			$G = 0.5\Delta E$		
	$\bar{N}_s$	$\langle\Phi_0\|N_s\|\Phi_0\rangle$	v_s^2	$\bar{N}_s$	$\langle\Phi_0\|N_s\|\Phi_0\rangle$	v_s^2	N_s	$\langle\Phi_0\|N_s\|\Phi_0\rangle$	v_s^2
1. K-2	0.877	0.889	0.853	0.911	0.927	0.893	0.979	0.995	0.983
2. K-1	0.810	0.812	0.767	0.853	0.861	0.810	0.959	0.987	0.958
3. K	0.675	0.661	0.631	0.716	0.698	0.649	0.888	0.932	0.807
4. K+1	0.398	0.410	0.455	0.338	0.353	0.419	0.126	0.072	0.208
5. K+2	0.240	0.228	0.294	0.182	0.160	0.230	0.048	0.013	0.044

TABLE III

DISTRIBUTION OF PAIR DENSITY IN THE TWO-QUASI-PARTICLE EXCITED STATES FOR G = 1.25 Δ E

K_1, K_2 s_1, s_2	s_1	Exact method	Method (b) with the projection	Method (b)
K, K+1	1	0.970	0.999	0.986
3,4	2	0.953	0.998	0.966
	5	0.077	0.003	0.048
K, K+2	1	0.948	0.978	0.928
3,5	2	0.906	0.943	0.834
	4	0.146	0.079	0.238
K-1, K+1	1	0.968	0.992	0.967
2,4	3	0.910	0.957	0.840
	5	0.122	0.051	0.193
K+1, K+2	1	0.902	0.921	0.868
4,5	2	0.793	0.789	0.711
	3	0.305	0.290	0.421
K-1, K	1	0.966	0.985	0.965
2,3	4	0.794	0.804	0.685
	5	0.240	0.211	0.350

mation is required for the study of the α-and β-transition rates if the fluctuations in the average field levels are not taken into account in detail.

As long as the calculations by the superfluid nuclear model are based on the experimental data on the pairing energies, we compare the energies of the ground and excited states in the exact method and in (b) for the same values of the pairing energies P.

The energies of the ground and excited states calculated exactly for $P = P_0$, $G = \Delta E(m)$, and using the method (b) for $G = \Delta E$ and for $P = P_0$ are shown in Fig. 5. If in the exact solution $G = \Delta E$, $P = P_0$ then in method (b) the pairing energy assumes the value $P = P_0$ at $G = 1.09\,\Delta E$. If we carry out the calculations for the same value of the pairing energy $P = P_0$ by method (b) and compare them with those made by the exact method, then the errors in this case will decrease by a factor of two compared with the errors in the calculations done for the same value of $G = \Delta E$. This can be seen from Fig. 5. Since the calculations made according to the superfluid nuclear model yield the correct sequence of levels of the even-even nuclei energies and are based on the experimental data on pairing energies, then the accuracy of the calculations becomes effectively higher.

Let us investigate the influence of the fluctuation of the average field levels on the spectrum of the excited states of the even system. With this aim in view, we give the exact solutions of the problem for the case (m1) in Fig. 5 when the $s = 3$ level (which is the K-level) is lowered by $0.5\,\Delta E$,

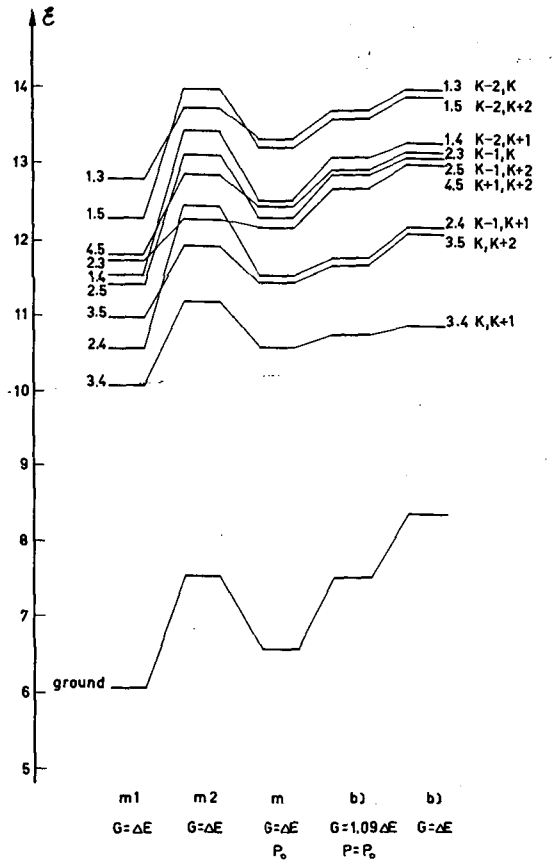

Fig. 5

Energies of the ground and excited states calculated exactly for $P = \mathcal{P}_0$, $G = \Delta E(m)$ and using the method (b) for $G = \Delta E$ and $P = P_0$.

and for the case (m2) when the s = 3 level is made 0.5 ΔE higher. It can be seen from Fig. 5, that the change in the configuration of the only level of the average field leads to a noticeable change in the energies of both the ground and excited states, and in some cases to a change in the sequence of the excited states of the system.

The investigation we have done confirms our main conclusion that the accuracy of the calculations according to the superfluid nuclear model is mainly restricted by a poor knowledge of the behaviour of the average field levels and their fluctuations, but not by the mathematical method employed.

Within the framework of the superfluid nuclear model, the basic characteristics of the superfluid states of strongly deformed nuclei have been calculated, according to which the energies of the two-quasi-particle levels of the even-even nuclei, the relative values of $\log ft_r$ for β-transitions in even and odd nuclei, the hindrance factors F in α-decays etc. have been computed. Let us deal with the choice of the parameters used in these calculations in the regions $154 \leq A \leq 188$ and $225 \leq A \leq 255$. The calculations are

based on the experimental data on the single-quasi-particle levels of the odd nuclei and pairing energies. As average field levels, we took the corrected levels of Nilsson's scheme in order to obtain the spectra of the single-quasi-particle levels of the odd nuclei which would be consistent with experiment. All the nuclei in the region $154 \leq A \leq 188$ were divided into two groups. [20], $154 \leq A \leq 174$ and $174 \leq A \leq 188$. In each group only one assembly of the single-particle levels of the average field is chosen both in the proton and in the neutron system. Note that the difference in the behaviour of some levels of the first group from the corresponding levels of the second one is not something unnatural because the equilibrium deformations have different values. In the mass region $225 \leq A \leq 255$ we could deal with only one assembly of the energies for the proton system and one assembly for the neutron system [21].

The behaviour of the calculated single-quasi-particle levels of the odd nuclei in the region $154 \leq A \leq 188$ is described in [20] and in the region $225 \leq A \leq 255$ in [21]. The calculated energies of the excited states are, for the most part, in agreement with the experimental data. However, we did not take into account the change in the average field in passing from one nucleus to another. Therefore, we cannot explain their behaviour like that shown in Fig. 3.

In order to determine the behaviour of the average field levels in the regions $154 \leq A \leq 188$ and $225 \leq A \leq 255$ for the proton and neutron systems we used 20 parameters characteristic of Nilsson's scheme itself, as well as of its modifications. It has been found from a comparison of the calculations with the experimental values of the pairing energies that in passing from one nucleus to another the neutron G_N and the proton G_Z pairing interaction constants change according to $1/A$ and their values in both regions of the strongly deformed nuclei are

$$G_N = \frac{26 - 27}{A} \quad \text{MeV}$$

$$G_Z = \frac{28 - 29}{A} \quad \text{MeV}.$$

Altogether 22 parameters have been used in the calculations. These parameters were found by 58 values of the pairing energies and by 205 characteristics of the ground and excited states of the odd nuclei. Thus, twenty-two free parameters are fixed so as to explain roughly 263 experimental facts on the single-particle levels of the odd A nuclei and on the pairing energies.

Note that for the given system of the average field levels and for the fixed magnitude of the interaction pairing constants G_N and G_Z the calculations based on the superfluid nuclear model are completely unambiguous. Such a formulation of the problem in which one and the same assembly of the single particle levels is used to calculate the properties of some nuclei has nothing to do with any fitting of the results obtained in the calculations with the corresponding experimental data. Furthermore, not a single new parameter is introduced in calculating the characteristics of the even nuclei. Therefore, the comparison of the calculated energy levels of the even-even nuclei with the corresponding experimental data is very important from the point of view of checking the validity of the basic assumptions of the superfluid nuclear model.

3.4. The blocking effect

The influence of the unpaired particles on the superfluid properties of the system in each state of the atomic nucleus is often referred to as the blocking effect. The superfluid state of the atomic nucleus results from the nucleon-nucleon interactions described by Hamiltonian (3.1). Therefore, if a nucleon is populated on any twice-degenerate level of the average field, then in view of the Pauli principle this level cannot be occupied by a pair. Due to the peculiarity of interaction (3.1) the average field levels, which the quasi-particles (unpaired nucleons) populate, are blocked with respect to the pairs, that is, these levels are not taken into consideration automatically in the calculations. Thus, the blocking effect implies that in calculating the superfluid properties of some states by means of the variational principle, the average field levels populated with quasi-particles are disregarded. This leads, for instance, to a change in the magnitude of the correlation functions $C(s_1, s_2)$ and the chemical potentials λ (s_1, s_2) compared with the states on which either there are no quasi-particles or they are situated on other levels of the average field. The blocking effect is very essential in the region of strongly-deformed nuclei, where the density of the average field levels is not great.

Note that besides the forces leading to the blocking effect, there are forces which play quite a definite role in a nucleus. However, we have neglected them. The case is that the forces leading to the blocking effect are different for the ground and excited states while the other forces produce more or less the same effect both on the ground and excited states. For this reason, they can be included in the average field.

The superfluid nuclear model is, in the main, that of independent quasi-particles. It takes into account only that part of the interaction of quasi-particles between each other leading to the blocking effect.

Note that the interaction of quasi-particles may lead to the appearance of collective effects, particularly for the levels 2^+, which look like the γ vibrations and which are responsible for an additional decreasing of the energies of these excited states.

The influence of the blocking effect is displayed in going from the ground state of the system, consisting of an even number of quasi-particles, to the single-quasi-particle states of the odd system. The correlation functions of ground states C(K) of the odd system are less than the magnitude C of the corresponding even system. This is clearly seen from Tables IV and V. The increase in the moments of inertia of the odd A nuclei compared with the even ones is an experimental proof of the existence of the blocking effect. The calculations of the moments of inertia of the odd nuclei are in good agreement with the experimental data, provided the blocking effect is taken into account. If one adheres strictly to Eqs. (3.17) and (3.18) $C(K \pm i)$ must be somewhat larger for the excited states than the correlation function $C(K)$ of the ground state of the odd system. However, this effect is rather small and is likely to be within the accuracy of the calculations we are making.

In the case of an even system the superfluidity of the system strongly decreases in passing from the ground to the two-quasi-particle excited states because of the blocking effect. To show the importance of the blocking effect we give ratios $C(K_1, K_2)/C$ in Tables IV and V for the neutron systems at a value of G which corresponds to the actual nuclear forces. The role of the blocking effect is clearly demonstrated in the exact solution of the model obtained by PAWLIKOWSKI and RYBARSKA [23] . The blocking effect leads

TABLE IV

VALUES OF THE CORRELATION FUNCTIONS FOR NEUTRON SYSTEMS

N	92	94	96	98	100	102	104	106	108	110
C(K, K+1)/C	0.43	0.47	0	0	0	0.01	0.01	0.23	0.24	0.50
C(K-1, K+1)/C	0.61	0.54	0.16	0.06	0.56	0.01	0.01	0.46	0.58	0.65
C(K, K+2)/C	0.55	0.54	0.64	0	0.10	0.23	0.13	0.53	0.50	0.57
C(K, K)/C	0.53	0.53	0.28	0.71	0.37	0.36	0.69	0.62	0.62	0.56
C(K-1, K)/C	0.66	0.58	0.40	0.72	0.59	0.44	0.74	0.67	0.71	0.69
C(K+1, K+2)/C	0.63	0.61	0.69	0.77	0.36	0.63	0.74	0.67	0.65	0.61
C(K-2, K-1)/C	0.75	0.68	0.54	0.76	0.75	0.61	0.79	0.75	0.81	0.81
C(K+1, K+4)/C	0.73	0.73	0.71	0.84	0.54	0.68	0.78	0.72	0.68	0.64
C(K+2, K+3)/C	0.71	0.75	0.84	0.82	0.62	0.78	0.81	0.76	0.73	0.68
C ($\hbar\omega_0^0$)	0.131	0.129	0.121	0.109	0.111	0.112	0.103	0.127	0.131	0.135
C(K)/C *	0.81	0.82	0.80	0.65	0.57	0.67	0.56	0.54	0.60	0.68

* for (N-1)/N

TABLE V

VALUES OF THE CORRELATION FUNCTIONS FOR NEUTRON SYSTEMS

N	138	140	142	144	146	148	150	152	154
C(K, K+1)/C	0.53	0.40	0.01	0	0	0	0	0.51	0.66
C(K-1, K+1)/C	0.61	0.44	0.41	0.35	0.25	0.41	0.55	0.81	0.69
C(K, K+2)/C	0.65	0.62	0.50	0.43	0.33	0.31	0.43	0.54	0.67
C(K, K)/C	0.54	0.57	0.57	0.49	0.58	0.58	0.64	0.59	0.64
C(K-1, K)/C	0.62	0.59	0.62	0.59	0.63	0.69	0.73	0.73	0.69
C(K+1, K+2)/C	0.65	0.69	0.70	0.64	0.65	0.62	0.62	0.64	0.68
C(K-2, K-1)/C	0.70	0.59	0.69	0.72	0.75	0.81	0.85	0.85	0.81
C(K+1, K+4)/C	0.75	0.76	0.78	0.72	0.72	0.65	0.63	0.66	0.69
C(K+2, K+3)/C	0.79	0.81	0.81	0.79	0.76	0.73	0.69	0.66	0.69
C $(h\omega_0^0)$	0.119	0.112	0.104	0.099	0.097	0.099	0.107	0.117	0.126
C(K)/C *	0.84	0.81	0.75	0.69	0.63	0.60	0.63	0.72	0.80

* for (N-1)/N

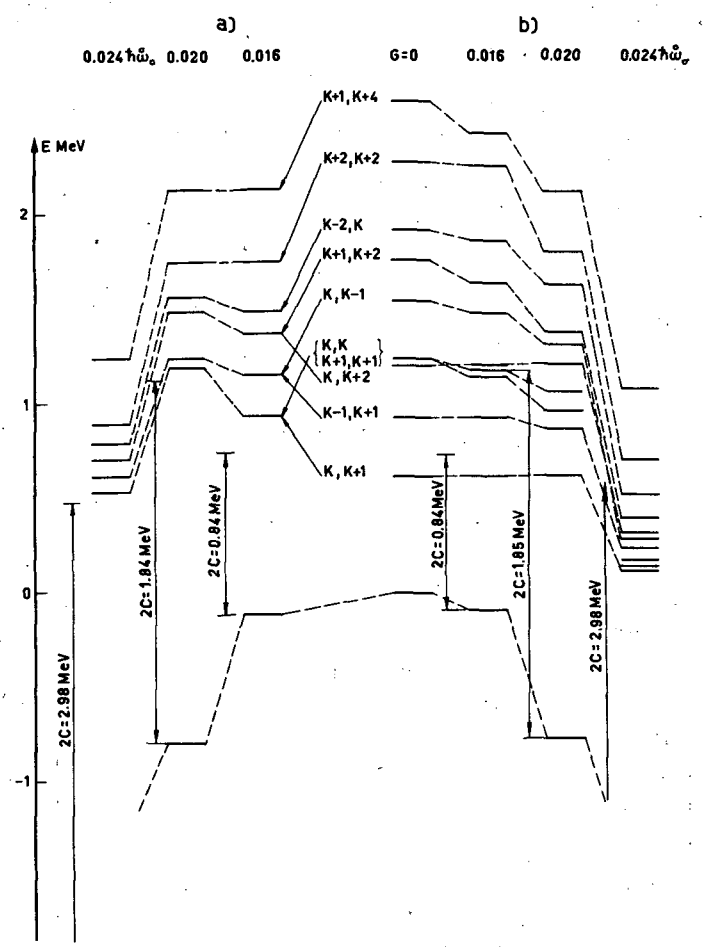

Fig. 6

Energies of the ground and excited states
a) calculated by the formula $\epsilon(s_1) + \epsilon(s_2)$
b) calculated according to the superfluid nuclear model with electronic computor.

to a noticeable decrease in the spread of the particle density in the two-quasi-particle excited states compared with the ground state of the system. It is worthwhile noting that the calculations by the superfluid nuclear model (method (b)) for the equidistant average field levels in all the excited states, except (K, K+1), yield somewhat less suppression of the superfluidity than by the exact method.

Let us consider the behaviour of the ground and two-quasi-particle excited states of the system consisting of an even number of particles (for instance, N = 106) as G increases. With this in mind, Fig. 6 case (b) shows the energies of the ground and excited states calculated according to the superfluid nuclear model with the aid of an electronic computer. Case (a) gives the energies of the excited states calculated by the formula

$$\epsilon(s_1) + \epsilon(s_2) \qquad (3.22)$$

according to the original formulation of the pairing correlations. The behaviour of the two-quasi-particle excited state energies as a function of G in case (b) is very different for small G from their behaviour in case (a), where a non-physical increase in the energy of some lower states is observed with G up to $G = 0.020\ \hbar\mathring{\omega}_0$ corresponding to the real nuclear forces. In case (b) the energies of both the ground and excited states decrease monotonously with G. This difference in the behaviour of the spectrum in case (b) compared with case (a) is due to the blocking effect.

This is displayed most distinctly in the falling of the energy of the (K, K+1) state lower than the magnitude of the gap 2C, while according to (3.22) the energy of this state must be more than 2C. Therefore, the comparison of theory with experiment, as far as the energy of the K, K+1) state is con-

TABLE VI

ENERGY OF STATE (K, K+1)
(MeV)

Nuclei	System	$K\pi$	Gap 2C	$\epsilon(K) + \epsilon(K+1)$	Energy	
					calcul.	observed
Cm^{244}	neutron	6+	1.30	1.36	0.92	1.042
W^{184}	proton	2-	1.61	1.87	1.3	1.150
W^{182}	proton	2-	1.61	1.88	1.3	1.290
	proton (Σ=1)	7-				1.961
	neutron	4-	1.89	1.97	1.5	1.554
	neutron (Σ=1)	5-				1.810
Hf^{180}	proton	8-	1.66	1.68	1.0	1.142
Hf^{178}	proton	8-	1.66	1.67	1.0	1.148
	neutron	8-	1.85	1.94	1.5	1.480
Yb^{172}	proton	3+	1.80	1.97	1.4	1.664
	proton (Σ=1)	4+				2.075
	neutron	3+	1.65	1.70	1.3	1.174
	neutron (Σ=1)	2+				1.468
Er^{168}	proton (Σ=1)	3-	1.82	1.90	1.3	1.543
	neutron (Σ=1)	3-	1.64	1.66	1.1	1.095
Er^{166}	neutron	6-	1.63	1.94	1.6	1.785
	neutron (Σ=1)	1-				1.826
Dy^{162}	neutron	5-	1.83	1.88	1.3	1.485
Dy^{160}	proton	2-	1.90	2.01	1.4	1.260
Gd^{156}	proton	4+	2.0	2.02	1.45	1.511
	proton (Σ=1)	1+				1.966
	neutron	1-	2.0	2.02	1.5	1.240
	neutron (Σ=1)	4-				2.042

cerned, may give evidence for the importance of the blocking effect in strongly deformed nuclei. It was shown in [18] that in the excited (K, K+1) state of the even system (i.e. in the state where one quasi-particle is on the K level, and the other on the next higher K+1 level) the superfluidity decreased considerably. This is connected with the fact that the correlated pairs cannot occupy the K and K+1 levels because of the Pauli principle. Therefore, in the states which the pairs can populate there appears a large gap for the strongly deformed nuclei. If the number of states below the gap is equal to the number of particles, it is energetically unfavourable for the pairs to populate the K+2 and higher levels, and the superfluidity in the (K, K+1) state then becomes considerably less. The calculated values of the energy of the (K, K+1) state for a number of nuclei are noticeably smaller than the magnitude 2C and agree well with the corresponding experimental data, as is seen from Table VI, which gives the value of the gap 2C, $\epsilon(K) + \epsilon(K+1)$, calculated by the superfluid model and the experimental energy levels (K, K+1) for some nuclei.

The investigations of the exact method have shown that the (K, K+1) level energy is below the magnitude of the gap 2C for G large enough. The calculations made according to the superfluid model yield somewhat underestimated values for the (K, K+1) state energies if we compare them with the exact solution of the problem, and therefore with the experimental data, which is confirmed in Table VI. These investigations showed also that in the approximate method in the (K, K+1) state the pairing correlations are more strongly suppressed than in the exact solution. This leads to the fact that the approximate calculations give somewhat underestimated values of the correlations to β-transitions to these states.

It can be seen from Table VI that we have failed to take into account certain forces. They act between the quasi-particles and lead to the spin splitting of the $K_1 \pm K_2$ state energies. The agreement of the calculations made on the superfluid nuclear model with the experimental data, as far as the depression of the (K, K+1) state energy below the gap is concerned, gives evidence for the importance of the blocking effect provided the magnitude of the spin splitting is not more than 700 keV.

The change in the superfluid properties of the system in the transition from the ground to excited states of the even system will undoubtedly affect the magnitude of the moments of inertia of the ground and excited states calculated according to the superfluid model of a nucleus. The moment of inertia for the ground state depends upon the superfluid properties both of the ground and excited states, i.e. upon the characteristics of the whole system. The moment of inertia of the system in an excited state is dependent on the superfluid properties both of the given and other states. Therefore, a sharp decrease in the magnitude of the correlation function $C(s_1, s_2)$ for the given excited state, e.g. for the (K, K+1) state, will not necessarily lead to the same considerable change in the magnitude of the moment of inertia.

The superfluid properties of the strongly deformed nuclei depend very much upon the magnitude of the pairing interaction constant G. If G were half the value corresponding to the nuclear forces in heavy nuclei, then the pairing correlations would practically be absent. If, on the other hand, G were twice as much, many features of the nuclei would alter considerably, and the shell structure would, at the least, be strongly masked. The differences in the superfluidity of the two-quasi-particle excited states relative

to each other and to the ground state of the system are essential in the region of strongly deformed nuclei and are outside the errors of the method.

Thus, the specific features of the superfluid model of a nucleus are important for the values of G, which correspond to the residual nuclear forces, and when the behaviour of the single particle levels of the average field is like that of the strongly deformed nuclei.

4. EFFECT OF PAIRING CORRELATIONS ON THE PROBABILITIES OF α, β AND γ-TRANSITIONS

The pairing correlations of nucleons of the superconductive type strongly affect the properties of the ground and excited states of atomic nuclei. They play, therefore, a great role in the α, β and γ-transitions in nuclei and in stripping reactions.

We shall investigate the influence of the pairing correlations on the probabilities of α, β and γ-transitions in strongly deformed nuclei within the framework of the superfluid model of the nucleus. In performing numerical calculations we shall use the data obtained in [20, 21] which characterize the properties of the ground and excited states of strongly deformed nuclei.

4.1. Effect of pairing correlations on the α-decay rates

We shall formulate the α-decay theory in the framework of the superfluid nuclear model and investigate the influence of pairing correlations of the superconductive type on the absolute probabilities of α-decays and especially on the hindrance factors F[27].

The matrix element of the α-decay of the parent nucleus with the wave function $\Psi = \Psi(Z)\Psi(N)$ represented as the product of the wave functions of the neutron and proton systems to the daughter one with $\Psi' = \Psi(N-2)\Psi(Z-2)$ is given by

$$M = \Psi^*(N-2)\Psi^*(Z-2)A\Psi(Z)\Psi(N), \tag{4.1}$$

the operator A being represented as follows

$$A = \tfrac{1}{4} \sum_{\substack{\nu,\nu';\omega,\omega' \\ \tau,\tau';\sigma,\sigma'}} W_{\tau,\tau';\sigma,\sigma}(p\nu, \nu' \mid n\omega, \omega') a_{\nu\tau} a_{\nu'\tau'} b_{\omega\sigma} b_{\omega'\sigma'}, \tag{4.2}$$

where $\tau = \pm 1$, $\sigma = \pm 1$ and the summation $\nu, \nu'(\omega, \omega')$ is carried out over the single-particle proton (or neutron) levels of the average field. The function W describes both the penetration of the α-particle through the potential barrier and the probability of its formation.

Let us find the matrix element of the α-decay of an even-even nucleus between the ground states. Making use of the wave functions (3.7) and taking into account the pairing correlations of nucleons, we get

$$M = \sum_{\nu\omega} W_{+-;+-}(p\nu, \nu \mid n\omega, \omega) u_\nu(Z-2) v_\nu(Z) \prod_{s \neq \nu} \left(u_s(Z-2) u_s(Z) + v_s(Z-2) v_s(Z)\right) \times$$
$$\times u_\omega(N-2) v_\omega(N) \prod_{s' \neq \omega} \left(u_{s'}(N-2) u_{s'}(N) + v_{s'}(N-2) v_{s'}(N)\right). \tag{4.3}$$

When the pairing correlations are absent (4.3), takes the form

$$M = W_{+-;+-}(p\nu = K(Z), \nu = K(Z) \mid n\omega = K(N), \omega = K(N)), \tag{4.4}$$

where we denote by K(Z) the last filled orbital of the system consisting of Z protons when the pairing correlations are absent. From (4.4) it is seen that if there are no pairing correlations then the α-particle can be formed only from two neutrons and two protons that occupy the last filled orbitals. Since the probability of formation of the α-particle in the nucleus is proportional to the overlap integral of the corresponding wave functions, then it must change essentially in the transition from one nucleus to another due to the change in the quantum numbers of the K-level, which is not observed experimentally. The effect of the pairing correlations leads to the α-particles being formed with a noticeable probability from pairs occupying many states both higher and lower than the K-state. This means that the α-decay involves an averaging of participation of many nucleon levels near the Fermi surface. This leads one, first, to the increase of the α-decay probability and second, to smoothing out the fluctuations in the probability of the α-particle formotion in the transition from nucleus to nucleus.

In order to distinguish between the effects of the pairing correlations of nucleons and those of other factors in α-decays the following approximation is considered. The diagonal part of W is independent both of the quantum numbers of protons and neutrons, i.e.

$$W_{+-;+-}(p\nu, \nu \mid n\omega, \omega) = W(p \mid n), \tag{4.5}$$

$$W_{+-;\sigma_1\sigma_2}(p\nu, \nu \mid n\omega_1, \omega_2) = W_{\sigma_1\sigma_2}(p \mid n\omega_1, \omega_2). \tag{4.5'}$$

Apparently, in evaluating the effect of the pairing correlations on the α-decay such an average treatment is correct.

The matrix element (4.3) in the approximation (4.5) is

$$M = W(p \mid n) R_N^{\frac{1}{2}} R_Z^{\frac{1}{2}}, \tag{4.6}$$

$$R_Z^{\frac{1}{2}} = \sum_\nu u_\nu(Z-2) v_\nu(Z) \prod_{s \neq \nu} (u_s(Z-2)u_s(Z) + v_s(Z-2)v_s(Z)). \tag{4.6'}$$

The product $R_N R_Z$ describes partially the change of the α-decay probabilities in the transition from one nucleus to another. If we make use of the characteristics of the superfluid states obtained in [21] then, e.g. for the α-decay of Cm^{244} to the ground state of Pu^{240}, we get $R_N = 38$, $R_Z = 45$, $R_N R_Z = 1700$. Calculations show that for nuclei in the region $230 \leq A \leq 254$ the quantities $R_N R_Z$ are within the limits $1500 < R_N R_Z < 3000$, i.e. the pairing correlations increase the probability of α-decay from the ground to the ground state of an even-even nucleus by more than three orders of magnitude.

From experimental data on α-decay, overestimated values of the radii of the nuclei were systematically obtained earlier, $R = r_0 A^{\frac{1}{3}}$ and $r_0 = 1.5$ f.

At the same time from other experiments $r_0 \approx (1.1 - 1.2)$f; thus, for instance, Igo obtained $r_0 = 1.17$ f from the data on α-particle scattering on heavy nuclei. If the nucleon pairing correlations in nuclei are taken into account, then from the data on α-decay, r_0 is found to be 1.17 f, i.e. the same radius as that obtained by Igo. Note that the deviation of the radii of nuclei in the region $230 \leq A \leq 254$ from $r_0 = 1.17$ f does not exceed 4%.

Note that the corrections R_N and R_Z depend on the number of the levels summed up. The expressions under the sum in $R_N^{\frac{1}{2}}$, when $|E(s) - \lambda| \gg C$ behaves as $C/|E(s) - \lambda|$, i.e. the logarithmic divergence taken place under the assumption (4.5). In our calculations the summation is carried out over 36 levels of the average field, the quantities C and λ are independent of the

cut-off. The corrections $R_N^{\frac{1}{2}}$ and $R_Z^{\frac{1}{2}}$ depend on this cut-off. In our calculations the outermost terms give the contribution to $R_N^{\frac{1}{2}}$ and $R_Z^{\frac{1}{2}}$ of the order of 1%; the account of the subsequent terms therefore introduces no significant changes. In the calculations of MANG and RASMUSSEN [28] the summation was performed only over 10 levels of the average field, which is evidently insufficient, and the blocking effect was not taken into account. They obtained for the α-transition $Cm^{244} \rightarrow Pu^{244}$ $R_N R_Z = 320$, while in our calculations $R_N R_Z = 1700$.

We find the matrix element of the α-transition to the two-quasi-particle excited states of an even-even nucleus. Thus, for the α-decay to a neutron state with quasi-particle occupying the orbitals f_1 and f_2 $(f_1 \neq f_2)$ we obtain

$$M(f_1, f_2) = W_{\sigma_1, \sigma_2}(p|\, nf_1, f_2) R_Z^{\frac{1}{2}} R_N(f_1, f_2)^{\frac{1}{2}}, \tag{4.7}$$

$$R_N(f_1, f_2) = v_{f_1}(N)^2 v_{f_2}(N)^2 \prod_{s \neq f_1, f_2} \left(u_s(N-2, f_1, f_2) u_s(N) + v_s(N-2, f_1, f_2) v_s(N)\right)^2 \tag{4.7'}$$

$R_N(f_1, f_2)$ being less than unity. According to the superfluid nuclear model the α particle is in this case formed only from neutrons being in the states f_1 and f_2. The α-decay rate is proportional to the neutron density of $v_{f_1}^2$, $v_{f_2}^2$ in these states of the parent. The hindrance factor is then of the form

$$F = \left(W(p|\, n) / W_{\sigma_1, \sigma_2}(p|\, nf_1, f_2)\right)^2 R_N / R_N(f_1, f_2). \tag{4.8}$$

From the superfluid nuclear model it follows that the probabilities of α-transitions to the two-quasi-particle states of even-even nuclei decrease by a factor $R_N / R_N(f_1, f_2)$ compared with the α-decay to the ground state. So, for the α-decay of Cm^{244} to the two-quasi-particle states of Pu^{240} with energy up to 2 MeV the ratio $R_N / R_N(f_1, f_2)$ takes values in the interval 150 - 500. Note that the product in (4.7') changes in the limits 0.3 - 0.9.

VOLKOV and VOROS [29] have calculated the hindrance factors for the α-transitions to β-vibration levels of even-even nuclei. They have obtained the values of F in the interval 10 - 80 which agree satisfactorily with the corresponding experimental data.

We consider a favoured α-decay of odd nuclei in which the quasi-particle occupies one and the same orbital in the parent and daughter. When the odd neutron is on the orbital f the matrix element is given by

$$M(f) = W(p|\, n) R_Z^{\frac{1}{2}} \cdot R_{N+1}(f)^{\frac{1}{2}}, \tag{4.9}$$

$$R_{N+1}(f)^{\frac{1}{2}} = \sum_{\omega \neq f} u_\omega(N-1, f) v_\omega(N+1, f) \prod_{s \neq \omega, f} \left(u_s(N-1, f) U_s(N+1, f) + v_s(N-1, f) v_s(N+1, f)\right). \tag{4.9'}$$

The hindrance factor

$$F = \left[M(N)^2 + M(N+2)^2\right] / 2M(N+1, f)^2 = \left[R_N + R_{N+2}\right] / 2R_{N+1}(f)$$

for the favoured α-decays in the region $230 \leq A \leq 254$ changes in the limits

$$1.2 < \left[R_N + R_{N+2}\right] / 2R_{N+1}(f) < 3. \tag{4.10}$$

The comparison of the experimental values of F with the calculated ones,

taking into account the fact that besides the α-particles emitted with $\ell = 0$ a fraction of them is emitted with $\ell = 2$ and $\ell = 4$, is given in Table VII. From Table VII it is seen that the theory is in satisfactory agreement with the experimental data.

TABLE VII

FAVOURED ALPHA-DECAYS

State	α-decays	F	
		experiment	calculated
9/2 - [734]	$Cf^{249} \rightarrow Cm^{245}$	1.8	1.8
7/2 + [624]	$Cm^{245} \rightarrow Pu^{241}$	2.2	2.0
5/2 + [622]	$Cm^{243} \rightarrow Pu^{239}$	1.5	1.8
1/2 + [631]	$Cm^{241} \rightarrow Pu^{237}$	2.7	2.1
1/2 + [631]	$Pu^{239} \rightarrow U^{235}$	2.5	2.1
3/2 - [521]	$Bk^{245} \rightarrow Am^{241}$	1.7	1.7
5/2 - [523]	$Am^{243} \rightarrow Np^{239}$	1.1	1.4
5/2 - [523]	$Am^{241} \rightarrow Np^{237}$	1.3	1.4
5/2 - [523]	$Am^{239} \rightarrow Np^{235}$	2.3	1.7
5/2 + [642]	$Np^{237} \rightarrow Pa^{233}$	3.8	1.4

We consider unfavoured α-decays in which the quasi-particle passes from one state to another. When the neutron passes from the state f_2 to the state f_1 ($f_1 \neq f_2$) the matrix element is

$$M(f_1, f_2) \approx W_{\sigma_1 - \sigma_2}(p \mid nf_1, f_2) R_Z^{\frac{1}{2}} R_{N+1}(f_1, f_2)^{\frac{1}{2}}, \qquad (4.11)$$

$$R_{N+1}(f_1, f_2) = u_{f_2}(N-1, f_1)^2 v_{f_1}(N+1, f_2)^2 \prod_{s \neq f_1 f_2} \left(u_s(N-1, f_1) u_s(N+1, f_2) + v_s(N-1, f_1) v_s(N+1, f_2)\right)^2. \qquad (4.11')$$

the product in (4.11') takes the values in the limits 0.65-0.95. When the α decay of an odd N-nucleus is unfavoured, the α-particles is readily formed out of proton pairs that occupy the orbitals near the Fermi surface and of neutrons occupying the states f_1, and f_2. Therefore the unfavoured α-decays are strongly hindered compared with the favoured ones. The hindrance factor F is

$$F = \left[W(p \mid n) / W_{\sigma_1 - \sigma_2}(p \mid n f_1, f_2)\right]^2 (R_N + R_{N+2}) / 2R_{N+1}. \qquad (4.12)$$

The ratio $(R_N + R_{N+2}) / 2R_{N+1}$ for α-transitions to the ground and hole states of strongly deformed nuclei lies between 50 and 130. This same ratio is 200 - 800 for α-decays to the particle K+2 state and it will exceed 10^3 when α-decays take place to the K+3 and higher states. The unfavoured α-decays to the particle excited states are more hindered compared to the transition taking place to the hole states. Since the pairing correlations contribute

considerably to F, then instead of the systematization of F depending on the quantum numbers of the states f_1 and f_2 it is necessary to make the systematization of $(W(p \mid n)/W_{\sigma_1 \sigma_2}(p \mid n f_1, f_2)^2$.

The unfavoured α-decays yield information on the non-diagonal parts of W, which can be used to calculate the hindrance factor for α-decays both of odd-odd nuclei, and takes place to the two-quasi-particle states of even-even nuclei.

We calculate the hindrance factors F for the α-decay of a 152-year isomer Am^{242m} with $K\pi = 5^-$ and configuration $p\,523\downarrow + n622\uparrow$. This is the α-decay of an odd-odd nucleus for which the assignment of spin and particles of the various levels are available [31]. The results of calculation of F and comparison with experiment are given in Table VIII. In the three left-hand columns we give the characteristics of the states of the daughter nucleus of Np^{238} and in the fourth and fifth column we give the experimental and calculated values of the hindrance factor F.

TABLE VIII

HINDRANCE FACTORS FOR THE ALPHA-DECAY Am^{242m} with $K\pi = 5^-$ and configuration $p523\downarrow + n622\uparrow$

State of daughter nuclei			Hindrance factor F	
$K\pi$	configuration	energy keV	observed	calculated
5^-	$p523\downarrow + n622\uparrow$	337	1.8	2.7
3^-	$p523\downarrow + n631\downarrow$	135	3×10^3	5.6×10^3
6^+	$p523\downarrow + n743\uparrow$	462	2×10^2	2.5×10^2
2^+	$p642\uparrow - n631\downarrow$	0	$> 6.2 \times 10^4$	3×10^6

In [31] a favoured α-decay of an odd-odd nuclei was first found. The data on this transition are listed in the first line of Table VIII. The calculated value of F is 1.5 times larger than the experimental one. This difference may be caused by the quasi-particle interaction not being taken into account. In the second and third lines we write favoured-unfavoured α-decays, i.e. decays favoured for the proton system and unfavoured for the neutron one. The results of calculation depend on the experimental values of F for the α-decay of Cm^{243} to Pu^{239}. The agreement between theory and experiment is satisfactory. In the last line we give the unfavoured α-decay of Am^{242m} to the ground state of Np^{238}. To estimate F we introduce the assumption that

$$\frac{W(p \mid n)}{W_{\tau_1 \tau_2, \sigma_1 \sigma_2}(p\nu_1 \nu_2 \mid n\omega_1, \omega_2)} = \frac{V(p)}{V_{\tau_1 \tau_2}(p\nu_1 \nu_2)} \cdot \frac{V(n)}{V_{\sigma_1 \sigma_2}(n\omega_1, \omega_2)}, \qquad (4.13)$$

i.e. F for unfavoured α-decay of odd-odd nuclei is equal to the product of the hindrance coefficients in the proton and neutron systems. To determine F_Z an average value is taken from the corresponding branches of the α-decays

of Am^{243} to Np^{239} and Am^{241} to Np^{237}. Since the lowest limit of F is also given in [31] there is no contradiction between the calculated and the experimental values of F.

Thus, the pairing correlations strongly affect the absolute probabilities of α-decays to the ground states of even-even nuclei and those of the favoured decays in odd nuclei as well, especially the values of the hindrance factors in the unfavoured α-decays and α-transitions to the two-quasi-particle levels of even-even nuclei. The account of pairing correlations leads to an improved agreement between theory and corresponding experimental data.

4.2. Superfluid corrections and additional classification of β-transition probabilities

We formulate [20] general rules for constructing the corrections to β-transitions due to the superfluidity of the ground and excited states. Besides keeping Alaga's selection rule classification of the probabilities for β-decay of strongly-deformed nuclei we introduce an additional selection rule. The role of the superfluid corrections is investigated by analysing log ft for the β-transitions between identical pairs of the single-particle states in different nuclei.

The matrix element describing the β-decay of a complex nucleus is written symbolically as

$$M \sim \Psi^{*}_{2n_N} \Psi^{*}_{2n_{Z+1}}(s_2) \sum_{\nu,\nu'} \langle \nu | \Gamma | \nu' \rangle \; a^{+}_{\nu} b_{\nu'} \Psi'_{2n_Z} \Psi'_{2n_{N+1}}(s_1) = \langle s_2 | \Gamma | s_1 \rangle R^{\frac{1}{2}}. \tag{4.14}$$

Here $\langle s_2 | \Gamma | s_1 \rangle$ is the single-particle matrix element of the transition and $R^{\frac{1}{2}} = (\Psi^{*}_{2n_N}\Psi'_{2n_N})(\Psi^{*}_{2n_Z}\Psi'_{2n_Z})$, where Ψ_N is the wave function of the N-particle system. The values ft characterizing the β-decay are obtained in the form

$$ft = \frac{\text{const}}{|\langle s_2 | \Gamma | s_1 \rangle|^2 R^{-1}}, \tag{4.15}$$

R being represented as $R = R_Z R_N$.

The quantities R_N and R_Z describe the change in the proton and neutron configurations of the nucleus associated with the β-transition. The proton and neutron systems will be considered independently of one another. We find R_i (i.e. R_N or R_Z)for β-decays with the participation of any number of quasi-particles in the initial and final states for the exclusion of those cases when there are two quasi-particles on one and the same level. We write R_i in the form

$$R_i = \gamma \prod_{s \neq f_1 \ldots f_k} (u_s u_s' + v_s v_s')^2 \tag{4.16}$$

with the functions u_s, v_s referring to the initial and u_s', v_s' to the final states. In the product $\prod_{s \neq f_1 \ldots f_k} (u_s u_s' + v_s v_s')^2$ there are no factors corresponding to the levels in which there are quasi-particles. The more alike the superfluid properties of the initial and final states, the closer this product approaches to unity. Further, if the number of paired particles in the initial and final states is the same, as e.g. in β-decay $^{108+1}_{72}Hf^{181} \rightarrow {}^{108}_{72+1}Ta^{183}$ then $\gamma = u_f^2$. If the number of paired nucleons varies in the course of the decay, as in $^{110}_{72+1}Ta^{183} \rightarrow {}^{108+1}_{74}W^{183}$ then $\gamma = v_f^2$, f being referred to the level on which a quasi-particle either disappeared or appeared. The functions u_f^2 or v_f^2 in (4.16) characterize

the superfluid properties of the system with a smaller number of quasi-particles. Thus, for instance, in the β-decay of the odd system into the ground state of the even one v_f^2 and u_f^2 are referred to the even system while in the β-decay of the single-particle odd state into the two-quasi-particle excited state, v_f^2 and u_f^2 are referred to the odd system, etc.

Consider the case when the pairing interaction constant G tends to zero, i.e. when the superfluid model passes into the independent-particle model. The correction then takes one of the two values $R_i = 1$ or $R_i = 0$. When $R_i = 1$ this corresponds to the case when the β-decay occurs without any change in the configuration of all the nucleons except one, whereas in the case $R_i = 0$ the β-decay is accompanied by a change in the configuration of more than one nucleon in the independent particle model. The particle transitions are those transitions in which a quasi-particle either disappears or appears on the single-particle levels f whose energy is higher than λ referred to the system with the smaller number of quasi-particles. For the hole transitions the energies of the single-particle levels f are lower than λ.

Let us make an additional (in comparison with Alaga's selection rules) classification of the β-decay probabilities of the strongly deformed complex nuclei, i.e. we divide all the β-transitions into three groups:

group I $\quad R_i(G = 0) = 1 \quad 0 < R_i(G \neq 0) < 1$

group II $\quad R_i(G = 0) = 0 \quad 0 < R_i(G \neq 0) < 1$

group III $\quad R_i(G = 0) = 0 \qquad R_i(G \neq 0) = 0$

The first group includes:

(a) Those β-decays whose initial and final states are the ground states of the system;

(b) The particle transitions when the number of pairs remains unaltered;

(c) The hole transitions when the number of pairs changes by unity.

The second group includes:

(a) The hole transitions when the number of particle pairs does not change;

(b) Particle transitions when the number of particle pairs changes by unity.

For the β-decay referred to in the second group the superfluid model allows non-zero transition probabilities, while these transitions are strictly forbidden in the independent particle model. It is worth noting that the corrections R_i calculated by the superfluid nuclear model, which are referred to in the first and second groups and which are associated with the β-transitions to the low-excited nuclear states ($\lesssim 0.3$ MeV), are equal to each other within an order of magnitude; in the transitions to the strongly excited states (1 MeV and higher) they differ greatly.

The analysis of experimental data shows that there are more than 20 already established β-transitions referred to in the second group. The observation of the β-transitions referred to in the second group shows the advantage of the superfluid nuclear model compared to any independent particle and provides further evidence for the presence of short-range pairing interactions.

While the first and second groups incorporate those β-decays in which only one quasi-particle in the proton (neutron) system disappears or appears

and the configuration of the remaining particles is left unaltered, the third group includes:

(a) The transitions in which the number of quasi-particles of the proton (neutron) system changes by more than unity;

(b) The transitions in which, besides the change in the number of quasi-particles by unity, the configuration of other quasi-particles changes.

The superfluid nuclear model is a model of independent quasi-particles. Therefore, the transitions associated with the changes of configuration of the quasi-particles are strictly forbidden. It would be of interest to have experimental data on the degree of forbiddenness of the transitions referred to the third group (which is called the F-forbiddenness) in the strongly deformed nuclei. To this end the probability should be found experimentally for β-decay of the single-quasi-particle state of the odd system into such a two-quasi-particle excited state of the even systems so that all three quasi-particles would be on different single-particle levels of the average field.

In [20, 21] the superfluid corrections are calculated for β-transitions between single-quasi-particle states in odd nuclei. The results of calculations can be found useful in analysing experimental data.

We must clear up the rule played by superfluid corrections and to do this we shall consider the β-transitions in odd nuclei. To eliminate the influence of the average field as much as possible we shall analyse the values of log ft for β-transitions between the pairs of identical single-particle states in different nuclei. In such an approach, of course, the influence of the single-particle matrix element $\langle s_1 \mid \Gamma \mid s_2 \rangle$ on the relative values of log ft_r is not entirely excluded since the average field changes slightly in the transition from one nucleus to another.

TABLE IX

BETA-TRANSITIONS OF ODD A-NUCLEI

State	Transition	State	Class.	R	log ft_e	log ft_r
7/2 + [404]	$Ta^{175} \rightarrow Hf^{175}$	7/2 - [503]	1u I I	0.41	6.2	6.2
	$Ta^{177} \rightarrow Hf^{177}$		1u I I	0.40	6.4	6.2
	$Ta^{183} \rightarrow W^{183*}$		1u I II	0.11	6.9	6.8
	$Ta^{185} \rightarrow W^{185}$		1u I II	0.19	6.5	6.5
5/2 - [532]	$Tb^{159} \leftarrow Gd^{159}$	3/2 - [521]	ah II I	0.07	6.7	6.7
	$Ho^{161} \leftarrow Er^{161}$		ah I I	0.53	~5.6	5.8
3/2 - [521]	$Bk^{245} \rightarrow Cm^{245}$	1/2 + [631]	1u I II	0.04	~7.0	7.0
	$Am^{241} \leftarrow Cm^{241}$		1u II I	0.14	≈7.3	6.5
	$Np^{237} \leftarrow U^{237}$		1u I I	0 43	6.0	6.0

The results of the investigations into the probabilities of β-transitions in odd nuclei are given in tables similar to Table IX where the experimental data and the results of calculations are systematized. Initial and final nuclei are written in the three left-hand columns of these tables, the additional

classification is given in the fourth column: 1. are transitions belonging to the first group, 2. are transitions belonging to the second group; they are written first for the proton system and then for the neutron one. The fifth column contains R. The experimental values of log ft_e are given in the sixth column. We calculate log ft_r normalizing them for the first from the given set of transitions between identical single-particle states in various nuclei. The values of log ft_r are given in the last column of the table. It can be seen from the table that the corrections for the superfluidity are essential for β-transitions belonging to the second group where they enable us to explain changes in the value of log ft_e for β-transitions between identical states in various nuclei.

Let us consider β-transitions in even nuclei. The relative probabilities of β-transitions in even nuclei can be calculated, making use of data on β-decay in odd nuclei between the same single-particle states, only in those cases when the selection rules are the same and there is no K or Λ-forbiddenness in even nuclei. The experimental data (in tables similar to Table X, [12, 21, 32]) on β-decays of even nuclei are summed up and compared with the corresponding calculations. Table X gives: all transitions between the states p 523↑ and n 523↓, the β-transitions in an odd nucleus, which were used for determining single-particle matrix elements, are also given. It can be seen from Table X that the calculated values of log ft_r are in satisfactory agreement with experimental data. This agreement is one more indication of the correctness of the two-quasi-particle aspect of the states of odd-odd nuclei and also of the excited states of even-even nuclei.

TABLE X

au BETA-TRANSITIONS

$S_z = 7/2\,-[523] \rightleftarrows S_n = 5/2\,-[523]$

State	beta-transition	State	R	log ft_e	log ft_c
7/2-	$Ho^{167} \rightarrow Er^{167}$	5/2-	0.52	4.8	4.8
7-	$Ho^{166} \rightarrow Er^{166}$	6-	0.38	5	4.9
0-	$Ho^{166} \rightarrow Er^{166}$	1-	0.38	5.2	4.8
1+	$Ho^{166} \leftarrow Dy^{166}$	ground	0.44	4.9	4.7
1+	$Ho^{162} \rightarrow Dy^{162}$	ground	0.25	4.7	5.3
6-	$Ho^{162} \rightarrow Dy^{162}$	5-	0.20	4.6	5.0
5+	$Ho^{160} \rightarrow Dy^{160}$	4+	0.38	4.8	4.9
1+	$Ho^{164} \rightarrow Dy^{164}$	ground	0.35	5.1	5.3
1+	$Ho^{164} \rightarrow Er^{164}$	ground	0.20	5.4	5.5
1+	$Tm^{164} \rightarrow Er^{164}$	ground	0.30	4.9	5.3
1+	$Tm^{166} \leftarrow Yb^{166}$	ground	0.29	4.6	4.9

Similar corrections must be included in the cross-sections of several nuclear reactions, e.g. the cross-section of the stripping reaction (dp) is

proportional to R_N. If the final state of the nucleus consists of an odd number of neutrons, then the correction has the form

$$u_f^2 \prod_{s \neq f} (u_s u_s(f) + v_s v_s(f))^2 , \tag{4.17}$$

and the particle excitation states of the final nucleus are more probable than the hole ones. If the final states of nuclei consist of an even number of neutrons then the correction, in the transition to the ground state, is

$$v_K^2 \prod_{s \neq K} (u_s u_s(K) + v_s v_s(K))^2 , \tag{4.18}$$

and takes the following form for the transition to the two-quasi-particle state of the final nucleus

$$u_f(K)^2 \prod_{s=K,f} (u_s(K) u_s(K, f) + v_s(K) v_s(K, f))^2 . \tag{4.18'}$$

4.3. Superfluid corrections to the probabilities of γ-transitions

We can calculate the change in the properties of a nucleus in electromagnetic transitions on the basis of the superfluid nuclear model. The importance of the role of the superfluid corrections R_γ to the probabilities of γ-transitions in strongly deformed odd A-nuclei has been demonstrated [16]. We shall formulate general rules for constructing the superfluid corrections to γ-transitions similar to those for β-transitions. The superfluid correction to the electromagnetic transition is represented in the form

$$R_\gamma = J \prod_{s \neq f_1 \ldots f_n} (u_s u_s' + v_s v_s')^2 \tag{4.19}$$

with the functions u_s, v_s referring to the initial and u_s', v_s' to the final states. In the product

$$\prod_{s \neq f_1 \ldots f_n} (u_s u_s' + v_s v_s')^2$$

there are no factors corresponding to those levels of the average field in which there are quasi-particles. If in the γ-transition a quasi-particle passes from the level f_1 to the level f_2 and the total number of quasi-particles remains constant, then

$$J = \left(u_{f_1}(f_2, f_3 \ldots f_n) u_{f_2}(f_1, f_3 \ldots f_n) - \eta v_{f_1}(f_2, f_3 \ldots f_n) v_{f_2}(f_1, f_3 \ldots f_n)\right). \tag{4.20}$$

If in the electromagnetic transition quasi-particles disappear (or appear) on the levels f_1 and f_2 and the total number of quasi-particles changes by 2, then

$$J = \left(u_{f_1}(f_3 \ldots f_n) v_{f_2}(f_3 \ldots f_n) + \eta v_{f_1}(f_3 \ldots f_n) u_{f_2}(f_3 \ldots f_n)\right)^2 \tag{4.21}$$

where $\eta = 1$ for electrical transitions and $\eta = -1$ for magnetic ones. The functions u_f and v_f in (4.20) and (4.21) refer to those single-particle levels on which a quasi-particle appears or disappears in the γ transition and which belong to those systems which have no quasi-particles on those levels. For example, in (4.21) u_f and v_f refer to a system with a smaller number of quasi-particles.

There are two possibilities when G = 0: R = 1 and R_γ = 0. The transitions for which R_γ = 1 and G = 0 include the hole-hole and particle-particle transitions, when the number of quasi-particles is constant and transitions in which quasi-particles disappear, one in the hole state with $E(f_1) < \lambda$ and another in the particle one with $E(f_2) > \lambda$ in the case when the number of quasi-particles changes by 2. In transitions for which R_γ = 0 when G = 0 we refer to the hole-particle transitions when the number of quasi-particles is constant and those in which the two quasi-particles disappeared either from levels higher than the K-level or from those lower than the K-level.

The values of the superfluid corrections vary in larger limits than the corrections to the β-decay probabilities. R_γ takes the values from 10^{-3} to 1. In the cases when in (4.20) and (4.21) we have the differences of two values, the accuracy of the calculations can be found to be insufficient.

When investigating electromagnetic transitions in strongly deformed nuclei we use, in general, the same approach as in investigating probabilities of β-transitions. In other words, the γ-transitions are systematized between identical states in different odd A-nuclei; the probabilities of γ-transitions in even A-nuclei are calculated using the experimental data on odd A-nuclei. The investigation of the electromagnetic transitions in strongly deformed nuclei enables us to find admixtures to the given state which consist both of states with other values of projections of angular momenta of the symmetry axis and of states with a different number of quasi-particles.

5. PROPERTIES OF THE GROUND AND EXCITED STATES OF STRONGLY DEFORMED NUCLEI

5.1. The nature of the ground and excited states of odd A-nuclei

The superfluid nuclear model yields a single-quasi-particle aspect of the ground and some excited states and a three-quasi-particle aspect for a number of higher excited states. The analysis of experimental data on the levels of the odd strongly deformed nuclei carried out by MOTTELSON and NILSSON [22] has shown that the spins and the partities of these states are unambiguously comparable with the corresponding characteristics of Nilsson's scheme and the values of log ft_e for β-transitions are classified according to the selection rules based on the asymptotic quantum numbers as follows:

$$\begin{aligned} &4.5 < \log ft_e < 5.0 \text{ au} \\ &6.0 < \log ft_e < 7.5 \text{ ah} \\ &5.5 < \log ft_e < 7.5 \text{ 1u} \\ &7.5 < \log ft_e < 8.5 \text{ 1h}. \end{aligned} \tag{5.1}$$

From this analysis follows the single-quasi-particle aspect of the ground and low excited states of odd nuclei.

The pairing correlations of a superconductive type essentially affect the probabilities of β-decays and lead one to the necessary systematization of the values of log $[ft_e R\eta]$ instead of log ft_e. The systematization is of the form

$$\begin{aligned} &4.0 < \log [ft_e R\eta] < 4.7 \text{ au} \\ &5.5 < \log [ft_e R\eta] < 6.5 \text{ ah} \\ &5.5 < \log [ft_e R\eta] < 6.5 \text{ 1u}. \end{aligned} \tag{5.2}$$

The distribution of the values of log $ft_e R\eta$ for the whole experimental data on β-decays of odd nuclei, given in Figs. 7a and 8b, show that the systematization (5.2) is rather good. It can be seen from Fig. 7a that there are two groups of allowed transitions au and ah. The clear separation between them

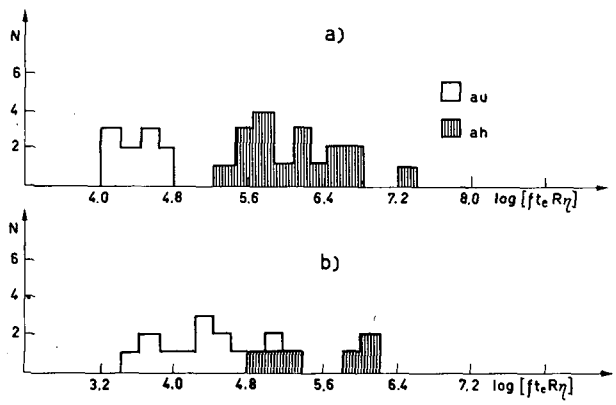

Fig. 7

Allowed beta-transitions
(a) odd nuclei, (b) even nuclei.

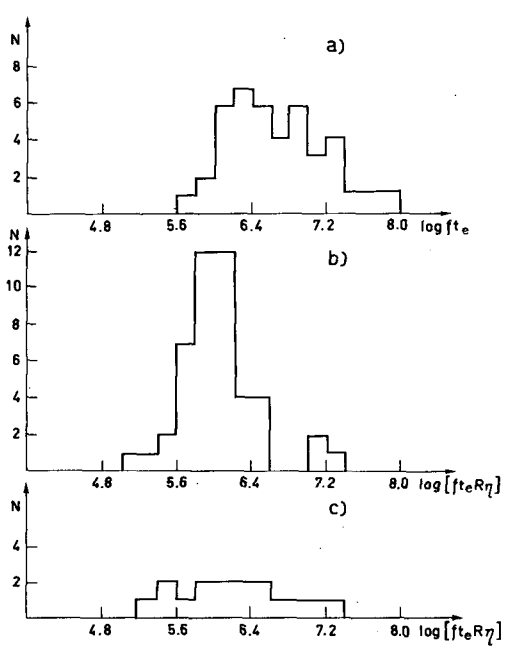

Fig. 8

1u beta-transitions
(a), (b) odd nuclei, (c) even nuclei.

testifies to the fact that the selection rules based on the asymptotic quantum numbers are executable. Figures 8a and 8b give histograms for the first forbidden unhindered lu β-transitions. From these histograms it can be seen that, in passing from the log ft_e to the log $ft_e R\eta$ classification, the regions of the values of the latter become narrower compared with the first ones and move on the side of smaller values. Comparing these histograms we see what an important role is played by the superfluid corrections to the β-transition probabilities. Note that all the three values of log $ft_e R\eta \approx 7.2$ (Fig. 8b) are referred to transitions between states $402\uparrow$ and $512\downarrow$ and a number of β-transitions in the transuranic region with log $ft_e R\eta \leqslant 5.6$ is badly determined experimentally. The dispersion of the values of log $ft_e R\eta$ is due both to the fluctuation in the average field levels and the inaccuracy of experimental data. The probabilities of the hindered β-transitions (ah and 1 h) are more sensitive to the fluctuations in the average field, compared with unhindered ones.

As long as the basic assumptions on the superfluid nuclear model are true the three-quasi-particle levels in odd A-nuclei should appear. The three-quasi-particle states must be of the two types: the first type (3n) and (3p) when all the three quasi-particles are either neutron or proton ones, the second type (2n, p) and (2p, n) when one quasi-particle is proton and two quasi-particles are neutron ones or, on the contrary, two quasi-particles are proton and one quasi-particle is neutron.

The three-quasi-particle states such as (3n) and (3p) must be at energies 1.5 MeV and higher. Only in Dy^{161} may the states such as (3n) be at energies of the order of 1 MeV. The β-transitions from the ground states of the even system of a parent nucleus to the states 3n and 3p are F-forbidden and it is rather difficult to observe them in β-transitions. Experimentally such states can be found either by the Coulomb excitation method or by studying γ-spectra in the transitions from high excited states. An observation of the states of such a type and a determination of the degree of the F-forbiddenness for β-transitions imposed upon them is of very great interest from the point of view of the model in question.

Recently it was reported [33] that a 155-day isomeric state of Lu^{177m} was discovered which has an energy of 1.2 MeV and spin 17/2 - 21/2. This state must be three-quasi-particle. There must be also observed in Lu^{177m} the three-quasi-particle states of type (3p) and of type (2np) with large spins, as long as there are orbits with great angular momenta near the Fermi surface. The analysis we made has shown that the state (3p) (K-1, K, K+1) with $K_\pi = 17/2$ - and the configuration $411\downarrow + 404\downarrow + 514\uparrow$ is likely to be the isomeric state Lu^{177m} which was found. According to our calculations, neglecting the spin splitting, its energy is 1.4 MeV. At the same time the state 17/2 - will be the lowest state of the multiplet. In this case Lu^{177m} will undergo a β-decay on the levels of Hf^{177} 17/2 +, 15/2+ with the configuration $p404\downarrow + p514\uparrow \pm n510\uparrow$; 17/2+ with the configuration $p404\downarrow + p514\uparrow + n521\downarrow$ and others. Further the γ transitions occur through the states 11/2 - ($p404\downarrow - p514\uparrow - n624\uparrow$) and 11/2+ ($p404\downarrow + p514\uparrow - n512\uparrow$) to the single quasi-particle levels of Hf^{177}. Possible configurations of the Lu^{177m} isomer given in [33] are unlikely, for they will lead either only to the isomeric state Hf^{177} or to a very short lifetime of Lu^{177m}.

The three-quasi-particle states of the type (2n, p) and (2p, n) with comparatively small spins must be well filled in β-decays. Once we disregard the interaction of three quasi-particles between them the probabilities of

β-transitions to these states must be the same as for transitions to the excited states of even-even nuclei. Let us look in which nuclei it is the easiest to observe experimentally the levels of such a type. Since the lowest states (2n, p) and (2p, n) are in the region 1-1.5 MeV then it is necessary that the energy Q released in β-transitions to the ground state would be sufficiently high and the β-transition rates to these states would be not strongly hindered, i.e. au, ah and 1u, so that they could be found at a low decay energy. Table XI gives some β-transitions to the states (2n, p) and (2p, n) which satisfy these requirements. The second column of the table gives the configuration of the state of a parent nucleus and the fourth gives the configuration of the three-quasi-particle states of a daughter nucleus. Here n, p denote the neutron and proton quasi-particles. The values of the energies of the states (2n, p) and (2p, n) given in the fifth column are roughly calculated without taking into account the interaction between quasi-particles. The seventh column gives the classification of the corresponding β-transitions and the eighth column the energy Q for β-transitions to the ground states.

The β-decay $Er^{161} \rightarrow Ho^{161}$, where the au transition with log ft_r = 4.8 is possible according to our calculations, is very favourable for finding levels of the type (2n, p). It is quite possible that the states in Ho^{161} with the energies 1.700 MeV and 1.830 MeV observed in [30] should be three-quasi-particle states with K_π 5/2- and 1/2-.

The analysis of β-decay given in Table XI shows that the three-quasi-particle states lead in a number of cases to another interpretation of the levels 7/2- and 9/2- in the region 1.-1.5 MeV which have been observed in several nuclei. In W^{181}, Hf^{177} and Hf^{175} the assignment of the levels 7/2- and 9/2 as the single-quasi-particle states 7/2 - [503] and 9/2 - [505] does not give rise to doubt while in Yb^{169} such a treatment seems to be unlikely. In Yb^{169} the three-quasi-particle states 9/2- and 7/2- with energy of the order of 1.5 - 1.6 MeV and with the 1u β-transitions from Lu^{169} must be observed. Therefore it is more correct to treat the states 7/2- with the energy 1.465 MeV and 9/2- with the energy 1.452 in Yb^{169} as three-quasi-particle ones, since for treating them as 7/2- [503] and 9/2 - [505] the energy values found experimentally are very low.

It can be seen from Table XI that there is a number of favourable possibilities for the experimental observation of three-quasi-particle states of the type (2n, p) and (2p, n). One of the most suitable criteria for finding these levels may be allowed unhindered au β-transitions to these levels in those cases when there are no such transitions to single-quasi-particles levels. In Table XI we give, as an axample, a number of β-decays to the levels (2n, p) and (2p, n) in order to draw the experimentalist's attention to these decays. The existence of the levels (2n, p) and (2p, n) follows immediately from the superfluid nuclear model and their absence would at least be strange.

An investigation of very high excited states of odd nuclei is of great interest from the point of view of the clarification: up to what excitation energies the single, three, five and so on quasi-particle aspect of the odd nuclei excited states is conserved.

5.2. Two-quasi-particle aspect of the excited states of even-even nuclei

In [21, 32] the two-quasi-particle levels of even-even nuclei were calculated on the basis of the superfluid nuclear model and compared with the experimental data. This comparison is one of the most important checks

TABLE XI

THREE-QUASI-PARTICLE STATES (2n, p) and (2p, n)

Parent nucleus	State	Daughter nucleus	Three-quasi-particle state	E MeV	Kπ	Class. β-decay log ft	Q MeV
$^{93}_{68}Er^{161}$	n521 ↑	$^{94}_{67}Ho^{161}$	n521 ↑ K, n523 ↓ K+2, p523 ↑ K	1.6-1.7	15/2-		> 2.0
					9/2-	2Λ (au)	
					5/2-	au 4.8	
					1/2-	au 4.8	
			n521 ↑ K, n521 ↑ K, p411 ↓ K+1	≈1.8-2.0	1/2-	1u	
			n521 ↑ K, n642 ↑ K+1, p411 ↓ K+1	≈1.8-2.0	9/2-7/2-	2Λ (ah)	
					3/2-	aΛ (ah)	
					1/2-	ah	
$^{98}_{71}Lu^{169}$	p404 ↓	$^{99}_{70}Yb^{169}$	p404 ↓ K+1, p523 ↑ K-1, n633 ↑ K	≈ 2	7/2-	1u	1.970
					21/2-		
			p404 ↓ K+1, p411 ↓ K, n521 ↓ K+1	≈1.5-1.7	9/2-	1u	
					7/2-	1u	
					5/2-	1u	
	p514 ↑		p514 ↑ K+1, p411 ↓ K, n521 ↓ K+1	≈1.5-1.7	11/2+	1u	
					9/2+	1u	
					7/2+	1u	

TABLE XI (cont'd)

Parent nucleus	State	Daughter nucleus	Three-quasi-particle state	E MeV	Kπ	Class. β-decay log ft	Q MeV
$^{100}_{73}Ta^{173}$	p404 ↓	$^{101}_{72}Hf^{173}$	p404 ↓ K+1, p411 ↓ K, n521 ↓ K	1.7	9/2-	1u	2.800
					7/2-	1u	
					5/2-	1u	
			p404 ↓ K+1, p514 ↑ K, n514 ↓ K+2	1.5-2.2	23/2+		
					9/2+	au	

of the correctness of the model under consideration, since all parameters which are necessary for these calculations are fixed on the basis of the experimental data on the single-particle levels of odd nuclear masses and the pairing energies. It is quite clear that the physical nature of the atomic nucleus is considerably more complicated than the model considered, therefore we are satisfied that we have obtained the correct general picture of the levels of even-even nuclei in which the most outstanding features of the ground and excited states are reflected.

The superfluid nuclear model is a model of independent quasi-particles. In this model the levels of even-even nuclei are described as the two-quasi-particle excited states. Thus, the two-quasi-particle aspect of the excited states of deformed even-even nuclei, used earlier by Gallagher for analysing β-decay of odd-odd nuclei and treating the levels of even-even nuclei, follows naturally from the superfluid nuclear model.

Before comparing the calculations with the experiment it is necessary to analyse and systematize the experimental data available. Such an analysis of all experimental data on the internal levels of strongly deformed nuclei in the region $150 < A < 190$ is made in [32].

In [32] the spectra of all even-even nuclei are calculated in the region $156 \leq A \leq 182$ and in [21] in the region $225 \leq A \leq 255$ and also log ft_r for a number of β-transitions. The results of calculation are given in tables. Table XII, which contains the two-quasi-particle levels of W^{182}, may serve as an example of these.

In Table XII we first give the neutron levels and then the proton ones. The configurations of the excited states are written in the first column. The second one contains $K\pi$, the state with anti-parallel spins, i.e. with $\Sigma = 0$ which according to Gallagher's rule must have a lower energy, being written first, and below the state with $\Sigma = 1$. The energies of these levels calculated according to the considered model are given in the third column. The fourth column contains experimentally measured energies of those levels whose spins are well established and which have quite definite configurations. In the right-hand side of the table are given β-transitions from odd-odd nuclei whose configurations are written at the head of the corresponding-columns. Note that according to the Gallagher-Moszkowski coupling rule the state with parallel neutron and proton spins possesses the lowest energy of the odd-odd nuclei doublet. For the β-decay of an odd-odd nucleus to each level of an even-even nucleus there corresponds the β-transition classification, provided $\Delta I \leq 2$, and in the right one the observable values of log ft_e and in brackets the calculated values of log ft_r.

Thus, tables of this type not only predict the energies of the two-quasi-particle levels in even-even nuclei but show at what rate these levels will be populated in the β-decay of the given odd-odd nucleus. It should be noted that the calculated values of the energies and of log ft depend only on the accepted scheme of single-particle levels and the pairing- energies, while the β-transitions depend on the assumptions about the configurations of odd-odd nuclei. The decay schemes can change if the experiment will show that an odd-odd nucleus has some other spin or parity, or it should assign some other configuration for a given $K\pi$.

The comparison of experimental data on the even-even nuclei energies with theory shows that there is a satisfactory agreement between calculated and observed energies of the levels and that the general conception of the two-quasi-particle excitations given by the model is found to be correct.

TABLE XII

TWO-QUASI-PARTICLE LEVELS OF $^{108}_{74}W^{182}$

State	$K\pi$	Energy MeV		$^{109}_{73}Ta^{182}$ p K-1	3- n K+1	$^{107}_{75}Re^{182}$ p K+1	7+ n K	$^{107}_{75}Re^{182}$ p K+1	2+ n K
		Calculated	Experiment	Class.	log ft	Class.	log ft	Class.	log ft
Neutron levels									
K, K+1	4-	1.5	1.554	ah	6.8 (6.9)			$1^{x}h$	> 7
	5-		(1.810)			$1^{x}h$	>6.8		
K, K K+1, K+1	0+	~1.6						2	
K, K+2	6-	1.9	1.830			1 u	6.6 (7.3)		
	3-		2.024	aF				1 u	6.4 (7.3)
K-1, K+1	4+	1.9		1 u					
	3+			1Λ(1u)				aF	
K-2, K	2-	2.3	2.185	aF				1 u	6.2 (6.7)
	7-					1 u	6.1 (6.7)		
K-2 = 512 ↑, K-1 = 514 ↓, K = 624 ↑, K+1 = 510 ↑, K+2 = 512 ↓									

TABLE XII (cont'd)

State	$K\pi$	Energy MeV		$^{109}_{73}Ta^{182}$ p K-1	3- n K+1	$^{107}_{75}Re^{182}$ p K+1	7+ n K	$^{107}_{75}Re^{182}$ p K+1	2+ n K
		Calculated	Experiment	Class.	log ft	Class.	log ft	Class.	log ft
Proton levels									
K, K+1	2-	1.3	1.290	aF	8.2			1 u	6.3 (6.4)
	7-		(1.979)			1 u	6.5 (6.4)		
K-1, K+1	6+	1.4	(1.757)			ah	(6.7)		
	1+			$1^{x}h$				ah	(6.5)
K, K K+1, K+1	0+	~1.8							
K-1, K+2	4+	2.0		1Λ(1u)					
	3+			1 u					
K-1, K	8-	2.0				1F			
	1-							1F	
K, K+2	4-	2.0		aF					
	5-								
K-1 = 404 ↓, K = 514 ↑, K+1 = 402 ↑, K+2 = 400 ↑.									

However, it is known that the situation is more complicated in a number of cases.

Note that the calculations in which the test set of single-particle levels of the average field is determined by using the experimental data on odd A-nuclei agrees better with the experimental data than with calculations in which the single-particle levels of the Nilsson scheme were used, which is easily seen from the example of the levels of Yb^{172} calculated in [32] and [18].

The comparison of the calculated energy levels of even-even nuclei with experimental data shows that the overwhelming majority of the calculated lowest two-quasi-particle levels, which are to be populated rapidly for appropriate β-transitions, is discovered experimentally. The task is to find experimentally all levels obtained from the calculations (or, to prove that some levels are absent). Thus, we should go over from the check of the validity of the main foundations of the model to an investigation of the total set of the levels of even-even nuclei and find deviations from the simple picture given by the superfluid nuclear model. If the assumed scheme of the single-particle levels of the average field is true then the following levels, rapidly populated in β-decays for example, must be observed: the proton level 1+ with $\epsilon > 1.4$ MeV in W^{182}, which must be populated in the β-decay of the 13 h Re^{182} with $\log ft_r = 6.5$ proton level 1- with $\epsilon \sim 1.3$ MeV in Hf^{178} from 1u β-transition of the 9.3 min Ta^{178}, etc.

From the calculations made on the basis of the superfluid nuclear model it follows that the energies of the levels (K, K+1) must, as a rule, be lower than the formal gaps 2C. This phenomenon follows immediately from the blocking effect. We summed up all the available data on these levels in table VI in order to clarify by comparing theory with experiment what role the blocking effect plays in a nucleus. The first column of Table VI gives the nucleus considered, the second one shows the system, proton or neutron, to which the given level refers by marking the states with $\Sigma = 1$ (the others have $\Sigma = 0$). The values of $K\pi$, the gap 2C, $\epsilon(K) + \epsilon(K+1)$, the energy calculated assuming blocking and the experimental energy are also given. Note that if the distance between the single-particle levels of the average field K and K+1 is large, then the energy of the system of the state (K, K+1) must be higher or of the order of the gap 2C, which occurs, for example, in the case of the neutron levels in Er^{166}.

It can be seen from Table VI that the measured and calculated values of the energies of the states (K, K+1) agree satisfactorily. Almost all the experimentally observed energies of the states (K, K+1) lie lower than the gap 2C.

It follows from a comparison of the results of calculating the behaviour of the (K, K+1) state energy with the experimental data that the blocking effect in strongly deformed nuclei plays an important role provided that the mean energy of the spin splitting does not exceed 700 keV.

The two-quasi-particle aspect of a number of excited states of even-even nuclei is proved by the analysis of experimental data on the β-transition probabilities (see histograms in Figs 7 and 8). From these histograms it follows that the regions of the values of $\log ft_e R\eta$ for β-transitions in even nuclei are approximately the same as in odd nuclei. A larger dispersion of the values of $\log ft_e R\eta$ is related both to the interactions of quasi-particles and the fluctuations in the average field levels, which we have not taken into account, and to the insufficient accuracy and reliability of the experimental data available.

In [32] a spin splitting has been found for some states whose energies (according to Gallagher's rule) in the states with antiparallel spins ($\Sigma = 0$) are somewhat lower than those in the states with parallel spins ($\Sigma = 1$). The spin splitting follows from the quasi-particle interaction. It points out that it is necessary to introduce additional terms into the Hamiltonian. However, the experimental data available on spin splitting are very poor and it would be desirable to increase the amount of such material. For example, two levels 4- should be observed in Er^{168} : the neutron level with the energy lower than 1.1 MeV and the proton one with the energy lower than 1.5 MeV. The β-decay to these levels from the state 3+ of Tm^{168} is Λ- forbidden and classified as $1\Lambda(1u)$.

It is very interesting to find those states of even-even nuclei into which β-decay is F-forbidden. These levels can be observed in the γ-transitions from the higher excited states.

Only a part of the residual forces acting between nucleons in a nucleus is taken into account (and, moreover, only approximately) in the superfluid nuclear model. Therefore it is interesting to investigate how strongly the residual interactions not taken into account affect the properties of the ground and excited states of strongly deformed nuclei. The investigation of the effect of pairing correlations on β-transition probabilities has shown [20] that β-transitions belonging to the third group (and called F-forbidden) are strictly forbidden in the superfluid nuclear model. An experimental determination of the degree of F-forbiddenness of the β-transitions is quite important from the point of view of clearing up the role of the residual forces not taken into account as well as clarifying whether the formulation of the properties of the ground and excited states of strongly deformed nuclei following from the superfluid nuclear model is true and exact.

As is known, the third group includes:

(a) β-decays with the change of the number of quasi-particles in the proton (neutron) systems by more than unity,

(b) β-decays where, besides the change of the number of quasi-particles in the proton (neutron) systems by unity, the configuration of other quasi-particles changes.

It is quite possible that the degree of forbiddenness of β-transitions will be different in this case. In case (a) the β-decay will be possible only where there are admixtures of the states with another number of quasi-particles. Such admixtures can appear, e.g. in the ground state of the system due to interactions leading to collective excitations. Such a forbiddenness is called the F_q-forbiddenness. The degree of the F_q-forbiddenness can be determined, for example, from the β-decay of the three-quasi-particle state to the ground state of the system. The experimental determination of the degree of the F_q-forbiddenness is necessary for investigating the microscopic structure of the collective states. Note that the F_q-forbiddenness must appear in α-decays and γ-transition too.

In case (b) the β-decay will proceed when the admixture of the states with other configurations is present, therefore it must have the same order of magnitude as K and Λ-forbiddennesses. The forbiddennesses of such a type are calles F_K-forbiddennesses. However, we notice that in some cases it must significantly differ from the Λ-forbiddenness.

The analysis of experimental data has shown that there does not exist any strictly fixed F_K-forbidden β-transition. Table XIII gives a number of transitions which are most convenient in determining the degree of the

TABLE XIII

BETA-TRANSITIONS OF THE TYPE F_k

Odd-odd nuclei			Even-even nuclei			Energy in MeV	Class. of β-decay	Σ
$^{99}_{63}Eu^{156}$	1-	p413↓ - n521↑	$^{92}_{64}Gd^{156}$	p	1 - 532 ↑ - 411 ↑	~1.7	aF	0
$^{97}_{69}Tm^{166}$	2+	p411↓ - n642↑	$^{98}_{69}Er^{166}$	n	1 - 523 ↓ - 633 ↑	1.828	1F	0
				n	2 + 523 ↓ - 521 ↓	~1.7	aF	0
					3 + 523 ↓ + 521 ↓		aF	1
$^{101}_{71}Lu^{172}$	4-	p404↓ + n521↓	$^{102}_{70}Yb^{172}$	p	3 + 411 ↓ + 402 ↑	~1.7	1F	0
					2 + 411 ↓ - 402 ↑		1 F	1
				p	5 - 411 ↓ + 514 ↑	~1.8	a F	0
					4 - 411 ↓ - 514 ↑		a F	1
$^{109}_{73}Ta^{182}$	3-	p404↓ - n510↑	$^{108}_{74}W^{182}$	p	2 - 514 ↑ - 402 ↑	1.290	a F	0
$^{147}_{93}Np^{240}$	1+	p642↑ - n624↓	$^{146}_{94}Pu^{240}$	n	2 + 631 ↓ - 622 ↑	~1	a F	1
				n	1 - 743 ↑ - 622 ↑	~1.3	1 F	0

F_K -forbiddenness. For example, the β-decay of Ta^{182} 3 - with the configuration p 404↓- n510↑ to the proton state 2- of W^{182} with the configuration 514↑ - 402↑ is F_K -forbidden. The energy of this state, which is well populated in the β-decay of 13 h Re^{182} is equal to 1.289 MeV, is in a good agreement with the calculated one, E = 1.3 MeV. According to the data available log ft_e = 8.2 for this aF_K β-transition. However, the values of the spin of 112 d Ta^{182} , the configuration of this state and the values of log ft_e = 8.2 are not quite reliable. If we assume that the treatment is correct, then the F_K -forbiddenness will lead to the β-decay rate being hindered by about a factor of 100. Apart from the transitions given in Table XIII a large number of the F_K -forbidden β-transitions are given in [12, 21, 32] .

Among comparatively high excited states of even-even nuclei the four-quasi-particle states must be observed in addition to the two-quasi-particle ones. Therefore two types of such states: the first type (4n) and (4p) when all the four quasi-particles are proton or neutron ones, the second type (2n, 2p) when two quasi-particles are proton and two others are neutron. The β-transitions to four-quasi-particle states such as (4n) and (4p) are F_q-forbidden and such states should be filled in γ-transitions from high excited states. The pairing correlations of a superconductive type will be absent in the majority of these states. The states (K-1, K, K+1, K+2) have the lowest energy. We evaluate this energy without taking into account the interaction between quasi-particles. For example, in W^{182} such a state (4n) has an energy of about 3 MeV, spins 10, 9, 7, 6, 3, 2, 1, 0 and a negative parity, and the state (4p) has an energy higher than 3 MeV, spins 11, 10, 6, 5, 4, 3, 2, 1 and a negative parity. The energies of a number of these states can be somewhat depressed because

of the interaction between quasi-particles. The four-quasi-particle states with other distributions of quasi-particles over the average field orbitals are somewhat higher. At excitation energies higher than 3 MeV the density of the even-even nuclear levels must increase strongly owing to the four-quasi-particle states.

The superfluid properties of the four-quasi-particle states (2n, 2p) are close to those of the corresponding two-quasi-particle states. These states should be well filled in the β-decays. In a number of cases such states can be found experimentally in the appropriate β-transition. The four quasi-particle levels of both types must be observed in all even-even strongly-deformed nuclei.

There must be six and more quasi-particle states among higher excited states of even-even nuclei, although it is not clear up to what energies such a treatment of the excited states will remain true in its general features. It is possible that neutron-spectroscopy experiments can answer this question.

Thus, the agreement of the calculations carried out on the basis of the superfluid nuclear model with the experimental data on the energies of the excited states of even-even nuclei and the probabilities of β-transitions provide evidence for the correctness of the initial foundations and the sufficient accuracy of the approximation of the model under consideration and, consequently, the two-quasi-particle aspect of this model can serve as a good basis for analysing the levels of even-even nuclei.

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SOME ASPECTS OF COLLECTIVE PROPERTIES OF NUCLEI

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1. INTRODUCTION AND BASIC EQUATIONS

1.1. Introduction

Certain questions connected with the collective properties of nuclei will be considered in this paper. There are two different complementary approaches in the consideration of collective motions. One method - we shall call it the phenomenological one - consists of directly introducing collective degrees of freedom and the corresponding collective Hamiltonian, H_{coll}, with some phenomenological parameters. The aim of such a theory is, first, to solve the Schrödinger equation with the Hamiltonian H_{coll}, and secondly to calculate the parameters in H_{coll} under certain assumptions or using certain models. The Davydov-Filippov model can serve as an example of such an approach to the investigation of collective properties of nuclei. There it is supposed that H_{coll} describes a non-axial rotator and the parameters of the Hamiltonian are then chosen so as to obtain the best fit to experimental data.

The second possible method of investigating collective excitations of nuclei - let us call it the microscopic one - works right from the beginning with a many-body Hamiltonian $H = H_0 + H_{int}$. Then such excitations in a system of nucleons, the nature of which is collective, are studied. In this method no additional collective degrees of freedom, and hence no additional phenomenological parameters, are introduced a priori. In principle, everything is determined by the nucleon-nucleon interaction H_{int}. As a matter of fact, the interaction H_{int} must practically be introduced phenomenologically, since it has not been possible to derive it from first principles so far. The actual difference is in the "degree of phenomenologicality", i.e. in the number of parameters introduced into the theory. One needs only one or two parameters, which in principle determine the various properties of all nuclei, to determine H_{int}, whereas in the above-mentioned example of a non-axial rotator, two parameters are introduced for each nucleus (the value of the first 2 + level and the degree of non-axiality γ).

We shall deal mainly with the microscopic approach and hence the first question is that of a reasonable choice of the nucleon-nucleon interaction H_{int}. A certain form of the interaction induces certain correlations in the motion of nucleons. Thus we can draw conclusions as to H_{int}, having analysed the character of the correlations between nucleons in nuclei.

The success of the shell model indicates that a considerable part of the nucleon-nucleon interaction can be taken into account by a self-consistent field. A self-consistent field is the result of the correlation between a large number of nucleons, i.e. a long-range correlation in this sense (although it is not induced by long-range forces). Obviously some sort of residual interaction between the nucleons remains after the self-consistent field has been separated. The question naturally arises: which part of the residual inter-

action is most important? Since the self-consistent field only takes long-range correlations into account, it is only natural to suppose that the residual interaction, on the contrary, causes short-range correlations (between a few nucleons) in the limit-pairing correlations.

COOPER [1] has noticed that bound states between particles become possible near to the Fermi surface ("Cooper pairs") in macroscopic Fermi systems even in the case of weak attraction. This leads to the non-stability of the ground state of an ideal Fermi gas and to the origin of completely new properties in the system (the superconductivity of metals). Does anything analogous to Cooper pairs exist in nuclei? A positive experimental indication of this is the effect of the pairing of nucleons on one and the same level which shows itself in the mass difference between neighbouring even and odd nuclei. Thus it is natural to suppose that, after the self-consistent field has been separated, the basic residual interaction is such that it causes pairing correlations of the Cooper type *.

1.2. Methods for the consideration of pairing correlation

Various methods for computing pairing correlation in Fermi-systems have been developed recently. They are to a great extent equivalent, though they may be convenient to a greater or smaller degree depending on the problem under consideration. The physical idea of these methods is as follows.

Since the forming of bound pairs is "of advantage", constructing the new ground state of a system from Cooper pairs is quite natural. Let the creation operator of a pair in the bound state be

$$A^+ = \sum_{\nu\nu'} \varphi_{\nu\nu'}\, a_\nu^+\, a_{\nu'}^+ , \qquad (1.1)$$

where a_ν^+ is the creation operator of one particle in state ν and $\varphi_{\nu\nu'}$ is the wave function of the bound state of the pair. It will be quite natural, therefore, to seek the ground-state function of the system with 2N particles as

$$\Psi_0^N = (A^+)^N |0\rangle . \qquad (1.2)$$

However, in practice it is impossible to make use of such a function. If N is large, the superposition of the (1.2) functions with different N may be used instead of (1.2), for example in the form

$$\Psi_0 = e^{\sigma A^+} |0\rangle , \qquad (1.3)$$

where the parameter σ may be chosen from the condition

$$\langle \Psi_0, \hat{N} \Psi_0 \rangle = N \langle \Psi_0, \Psi_0 \rangle . \qquad (1.4)$$

Further simplification of function (1.3) is due to the assumption about pair operator (1.1). If we assume that

$$\varphi_{\nu\nu'} = \tfrac{1}{2}\, \delta_{\nu'\tilde{\nu}}\, \varphi_\nu \quad ; \quad (\varphi_{\tilde{\nu}} = -\varphi_\nu) \qquad (1.5)$$

* We are considering medium and heavy nuclei. In light nuclei correlations of the α-particle type between four nucleons may be important.

(let us call states ν and $\tilde{\nu}$ conjugated) then (1.3) may be written as

$$\Psi_0 = \exp(\sigma \sum_{|\nu|} \varphi_\nu a_\nu^+ a_{\tilde{\nu}}^+) |0\rangle = \prod_{|\nu|} (1 + \sigma \varphi_\nu a_\nu^+ a_{\tilde{\nu}}^+) |0\rangle, \quad (1.6)$$

where $|\nu|$ is the pair of states ν, $\tilde{\nu}$. A function of the (1.6) type was introduced in the paper by BARDEEN, COOPER and SCHRIEFFER on the theory of superconductivity [2]. In this case ν and $\tilde{\nu}$ correspond to states of electrons with opposite momenta and spins, and condition (1.5) means that the total momentum and spin of the bound pair is zero. A parameter $\sigma \varphi_\nu$ was defined in [2] minimizing the expectation value of the Hamiltonian of the system in state (1.6).

BOGOLYUBOV proposed quite a general and simple method for consideration of the pairing correlation [3]. The idea of this method is based on the quasi-particle concept and involves the explicit introduction of quasi-particle operators by means of canonical transformation.

A few words about the physical meaning of quasi-particles. The low-lying excited states of many-particle systems can be approximately described as a set of independent elementary excitations or quasi-particles. The ground state is then considered to be a vacuum and the excitations of the system are induced by the creation of one or several quasi-particles. Such quasi-particles in an ideal Fermi gas are holes inside the Fermi sea and particles outside it.

The form of quasi-particles in the system with pairing may be established from that of function (1.6). Let us act on (1.6) with annihilation operator a_ν:

$$a_\nu \Psi_0 = \sigma \varphi_\nu a_{\tilde{\nu}}^+ \prod_{|\nu'| \neq \nu} (1 + \sigma \varphi_{\nu'} a_{\nu'}^+ a_{\tilde{\nu}'}^+) |0\rangle = \sigma \varphi_\nu a_{\tilde{\nu}}^+ \Psi_0,$$

from which one may see that

$$(a_\nu - \sigma \varphi_\nu a_{\tilde{\nu}}^+) \Psi_0 = 0$$

and, consequently, Ψ_0 is vacuum for a combination of operators of the type $\alpha_\nu \sim a_\nu - \sigma \varphi_\nu a_{\tilde{\nu}}^+$. Thus, quasi-particle operators should be sought as

$$\alpha_\nu = u_\nu a_\nu - v_\nu a_{\tilde{\nu}}^+, \quad (1.7)$$

where one must assume $u_\nu^2 + v_\nu^2 = 1$ in order that transformation (1.7) be canonical. The equation for u_ν, v_ν will be obtained from the minimum energy condition for the ground state Ψ_0 (defined as vacuum in respect to α_ν, i.e. satisfying equations $\alpha_\nu \Psi_0 = 0$); this is equivalent to the requirement banning quasi-particle pair creation from vacuum. The method of canonical transformation allows us to abandon condition (1.5) and consider pairs of a more general type. With this instead of (1.7) it follows that

$$\alpha_\nu = \sum_f (u_{\nu f} a_f - v_{\nu f} a_f^+). \quad (1.8)$$

The calculation of average values with a function of the (1.6) type is equivalent to the independent avarage of pairs of operators, such as

$$\langle a_1^+ a_2^+ a_{2'} a_{1'} \rangle = \langle a_1^+ a_{1'} \rangle \langle a_2^+ a_{2'} \rangle - \langle a_1^+ a_{2'} \rangle \langle a_2^+ a_{1'} \rangle + \langle a_1^+ a_2^+ \rangle \langle a_{2'} a_{1'} \rangle \quad (1.9)$$

and "diagonal" expressions only

$$\langle a_1^+ a_1 \rangle = v_1^2; \quad \langle a_1^+ a_{\tilde{1}}^+ \rangle = \langle a_{\tilde{1}} a_1 \rangle = u_1 v_1 \tag{1.10}$$

being different from zero. The decomposition of averages (1.9), but without the last term, is characteristic of the Hartree-Fock method for the self-consistent field. The last term in (1.9) accounts for the Cooper pairing effect. It is easy to make sure that if a more general transformation (1.8) is used instead of (1.7), relation (1.9) still holds for averages in the α vacuum state, but apart from the diagonal values (1.10) the more general averages of any pairs of operators are different from zero.

Under the general assumption that the ground state of the system is vacuum for quasi-particles between the averages of pairs of operators there exist the relations

$$\begin{aligned} \sum_3 (\langle a_1^+ a_3 \rangle \langle a_3^+ a_2 \rangle + \langle a_1^+ a_3^+ \rangle \langle a_3 a_2 \rangle) &= \langle a_1^+ a_2 \rangle \\ \sum_3 (\langle a_1^+ a_3 \rangle \langle a_2^+ a_3^+ \rangle + \langle a_2^+ a_3 \rangle \langle a_1^+ a_3^+ \rangle) &= 0 \end{aligned} \tag{1.11}$$

The variation of the Hamiltonian average value results in the equations

$$\begin{aligned} \mathcal{A}_{12} &\equiv \langle [a_1 a_2; H] \rangle = 0 \\ \mathcal{B}_{12} &\equiv \langle [a_1 a_2; H] \rangle = 0 \end{aligned} \tag{1.12}$$

which, together with (1.11), determine $\langle a_1^+ a_2 \rangle$ and $\langle a_1 a_2 \rangle$. Equations (1.11), and (1.12) (formulated by BOGOLYUBOV [4]) are a generalization of the Hartree-Fock method for systems with Cooper pairing. We shall use this method for considering some collective properties of nuclei.

1.3. Basic equations of generalized Hartree-Fock method

Let us write down the Hamiltonian of the nucleon system in secondary-quantized representation as

$$\begin{aligned} H = &\sum_{12} (\mathcal{E}_{12} - \lambda \delta_{12}) a_1^+ a_2 \\ &+ \tfrac{1}{2} \sum_{122'1'} \langle 12 | G | 2'1' \rangle a_1^+ a_2^+ a_{2'} a_{1'} \end{aligned} \tag{1.13}$$

where $\mathcal{E}_{12}$ is the matrix of the single-particle Hamiltonian, and λ the chemical potential of the system. Let single-particle states, transient into each other by time-reversal, be designated by $(1, \tilde{1})$ or $(\nu, \tilde{\nu})$. Then, in addition to the general properties of symmetry, we have in particular

$$\mathcal{E}_{12} = \mathcal{E}_{\tilde{2}\tilde{1}}; \quad \langle 12 | G | 2'1' \rangle = \langle \tilde{1}\tilde{2} | G | \tilde{2}'\tilde{1}' \rangle^*. \tag{1.14}$$

Substituting (1.13) into the first of equations (1.12) (the second one presents nothing new) we obtain

$$\mathcal{A}_{12} = \sum_3 \{\epsilon(13)\langle a_3 a_2\rangle + \epsilon(23)\langle a_1 a_3\rangle\} + \Delta(12) \qquad (1.15)$$
$$- \sum_3 \{\Delta(13)\langle a_3^+ a_2\rangle + \Delta(32)\ \langle a_3^+ a_1\rangle\} = 0$$

where

$$\epsilon(12) = \mathcal{E}_{12} - \lambda\delta_{12} + \sum_{1'2'} \overline{\langle 1\tilde{1}'|G|2'\tilde{2}\rangle}\langle a_{1'}^+ a_{2'}\rangle = \epsilon^*(21) \qquad (1.16)$$

$$\Delta(12) = -\sum_{1'2'}\langle 12|G|2'1'\rangle\langle a_{2'} a_{1'}\rangle = -\Delta(21) \qquad (1.17)$$
$$\overline{\langle 1\tilde{1}'|G|2'\tilde{2}\rangle} = \langle 1\tilde{1}'|G|2'\tilde{2}\rangle - \langle 1\tilde{1}'|G|2\tilde{2}'\rangle.$$

It should be noted that the interaction enters this equation only through the Cooper pairing value Δ and single-particle Hamiltonian ϵ, where it defines the additional self-consistent field.

If we formally assume ϵ and Δ to be diagonal, i.e.

$$\epsilon(12) = \epsilon_1\delta_{12}; \qquad \Delta(12) = \Delta_1\delta_{2\tilde{1}} \qquad (1.18)$$

then equations (1.11), (1.15) will be satisfied for diagonal averages

$$\rho_{12} = \langle a_2^+ a_1\rangle = \tfrac{1}{2}\left(1 - \frac{\epsilon_1}{E_1}\right)\delta_{12}; \qquad (1.19)$$
$$\langle a_2 a_1\rangle = \frac{\Delta_1}{2E_1}\delta_{1\tilde{2}}$$

where

$$E_1 = \sqrt{\epsilon_1^2 + \Delta_1^2}. \qquad (1.19')$$

The quantity $\epsilon(12)$ (1.16) is the Hermitian matrix of the single-particle Hamiltonian with the additional self-consistent field, and may be diagonalized by suitable choice of the single-particle states. Physically, this means finding single-particle eigenfunctions in the total self-consistent field. Equation (1.17), however, does not lead to the diagonal expression for $\Delta(12)$, as matrix elements of the type $\langle 1\tilde{2}|G|\tilde{1}'1'\rangle$ generally speaking, are different from zero*. Nevertheless, under certain physically reasonable assumptions the condition (1.18) for $\Delta(12)$ appears to be fulfilled. This is valid, in particular, in the model of the self-consistent potential with a flat bottom, when Δ may be considered constant in space coordinate representation [6]. In any case the basic Cooper pairing effect is determined just by this value $\Delta(1\tilde{1})$, and the non-diagonal parts $\Delta(12)$, if necessary, may be considered as perturbation. For the diagonal part of Δ from (1.17) and (1.19) we obtain

$$\Delta_1 = -\sum_2 \langle 1\tilde{1}|G|\tilde{2}2\rangle(\Delta_2/2E_2). \qquad (1.19'')$$

* In contrast to the case of an infinite system where they are zero because of the conservation of momentum.

For simplicity we shall in the following mostly consider Δ to be a constant, or to be more exact *

$$\Delta_1 = -\gamma_1 \Delta \quad \text{where} \quad \gamma_{\tilde{1}} = -\gamma_1 ; \quad |\gamma_1| = 1. \tag{1.19''}$$

1.4. Perturbation theory

Let the weak perturbation yielding the additional term V_{12} in the single-particle Hamiltonian ϵ_{12} be enforced on the system.

Let designations for corrections of first order be introduced

$$\rho_{12}^{(\pm)} = \tfrac{1}{2}\left(\langle a_2^+ a_1 \rangle^{(1)} \pm \langle a_{\tilde{1}}^+ a_{\tilde{2}} \rangle^{(1)}\right) = \rho_{21}^{(\pm)*} = \pm \rho_{\tilde{2}\tilde{1}}^{(\pm)}$$
$$h_{12}^{(\pm)} = \tfrac{1}{2}\left(\langle a_{\tilde{2}} a_1 \rangle^{(1)} \pm \langle a_2^+ a_{\tilde{1}} \rangle^{(1)}\right) = -h_{\tilde{2}\tilde{1}}^{(\pm)} = \pm h_{21}^{(\pm)*} \tag{1.20}$$

and respectively

$$\epsilon_{12}^{(\pm)} = U_{12}^{(\pm)} + V_{12}^{(\pm)} = \pm \sum_{1'2'} \langle 1\tilde{1}' | G | \tilde{2}'2 \rangle \rho_{1'2'}^{(\pm)} + \tfrac{1}{2}(V_{12} \pm V_{\tilde{2}\tilde{1}}) = \epsilon_{21}^{(\pm)*} = \pm \epsilon_{\tilde{2}\tilde{1}}^{(\pm)} \tag{1.21}$$
$$\Delta_{12}^{(\pm)} = -\sum \langle 1\tilde{2} | G | \tilde{2}'1' \rangle h_{1'2'}^{(\pm)} = -\Delta_{\tilde{2}\tilde{1}}^{(\pm)} = \pm \Delta_{21}^{(\pm)*}.$$

Equations (1.11) for the first-order terms in these notations take the form

$$[(\epsilon_2/E_2) + (\epsilon_1/E_1)]\rho_{12}^{(\pm)} - [(\Delta_2/E_2) \pm (\Delta_1/E_1)] h_{12}^{(\pm)} = 0$$
$$[(\Delta_2/E_2) \pm (\Delta_1/E_1)]\rho_{12}^{(\pm)} + [(\epsilon_2/E_2) - (\epsilon_1/E_1)] h_{12}^{(\pm)} = 0, \tag{1.22}$$

the compatibility of which is ensured by equality (1.19). Equations (1.22) yield the simple relation between $\rho^{(\pm)}$ and $h^{(\pm)}$. It is convenient, therefore, to introduce the new variables, satisfying (1.22) identically.

Let us introduce the values

$$\xi_{12}^{(\pm)} = u_1 u_2 \mp v_1 v_2 ; \quad \eta_{12}^{(\pm)} = u_1 v_2 \pm v_1 u_2 \tag{1.23}$$

where u, v are defined by the relations

$$u_1^2 - v_1^2 = \epsilon_1/E_1 \quad ; \quad 2u_1 v_1 = \Delta_1/E_1 \tag{1.23'}$$
$$u_{\tilde{1}} = u_1 \quad ; \quad v_{\tilde{1}} = -v_1.$$

* Strictly speaking, Δ_1 is a matrix in spin space of the form

$$\Delta_1(\sigma\sigma') = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}. \tag{1.19a}$$

Taking into account that the spin changes its direction under time reversal, i.e.

$$\varphi_{\tilde{1}}(\vec{r}, \sigma) = \varphi_1^*(\vec{r}, -\sigma) \tag{1.19b}$$

it is possible to eliminate the spin coordinate $\sigma = \pm 1$, replacing (1.19b) by

$$\varphi_{\tilde{1}}(\vec{r}) = \gamma_1 \varphi_1^*(\vec{r}) \tag{1.19c}$$

where, for example, $\gamma_1 = 1$ but $\gamma_{\tilde{1}} = 1$. As the non-zero components of $\Delta_1(\sigma\sigma')$ also pass into each other under time reversal we may use (1.19 ''') instead of matrix representation (1.19a).

Then it is clear * that because of (1.22) the quantities $\rho^{(\pm)}$ and $h^{(\pm)}$ can be expressed in the form

$$\begin{aligned} \rho_{12}^{(\pm)} &= \eta_{12}^{(\pm)} \; Z_{12}^{(\pm)} \\ h_{12}^{(\pm)} &= \xi_{12}^{(\pm)} \; Z_{12}^{(\pm)} . \end{aligned} \tag{1.24}$$

The values $Z_{12}^{(\pm)}$ are reasonably chosen as new unknown quantities. It should be noted that the symmetry properties resulting from (1.20) hold for $Z^{(\pm)}$

$$Z_{12}^{(\pm)} = - Z_{\bar{2}\bar{1}}^{(\pm)} = Z_{21}^{(\pm)*}. \tag{1.24'}$$

Separating terms of first order in equation (1.15) and making simple transformations we obtain

$$(\mathcal{A}_{\bar{2}1} \pm \mathcal{A}_{\bar{1}2}^{*})^{(1)} / 2\, \xi_{12}^{(\pm)} = E_{12} \; Z_{12}^{(\pm)} - \xi_{12}^{(\pm)} \; \Delta_{12}^{(\pm)} + \eta_{12}^{(\pm)} \; \epsilon_{12}^{(\pm)} = 0 \tag{1.25}$$

$$E_{12} = E_1 + E_2 .$$

Equation (1.25) defines corrections of the first order for averages from pairs of operators (1.20).

Non-diagonal corrections of first order enter the expression for the ground-state energy in quadratic combinations. Therefore, diagonal corrections to averages $\langle a^{+}a \rangle$ and $\langle aa \rangle$ of the second order should be considered simultaneously. Equations (1.11) have a kinematic nature, as they do not contain the intrinsic characteristic of the system (interaction). Using (1.11) one may establish some general relation between first-and second-order approximations in (1.11). Let us separate second-order terms in (1.11)

$$\begin{aligned} &(1 - 2\langle a_1^{+}a_1 \rangle^{0}) \langle a_1^{+}a_1 \rangle^{(2)} - \langle a_1^{+}a_{\bar{1}}^{+} \rangle^{0} (\langle a_{\bar{1}}\, a_1 \rangle^{(2)} + \langle a_1^{+}a_{\bar{1}}^{+} \rangle^{(2)}) \\ &= \sum_2 (\langle a_1^{+}a_2 \rangle^{(1)} \langle a_2^{+}a_1 \rangle^{(1)} + \langle a_1^{+}a_{\bar{2}}^{+} \rangle^{(1)} \langle a_{\bar{2}}\, a_1 \rangle^{(1)}) \end{aligned} \tag{1.26}$$

$$\langle a_1^{+}a_{\bar{1}}^{+} \rangle^{0} (\langle a_1^{+}a_1 \rangle^{(2)} - \langle a_{\bar{1}}^{+}a_{\bar{1}} \rangle^{(2)}) = \sum_2 (\langle a_{\bar{1}}^{+}a_{\bar{2}} \rangle^{(1)} \langle a_1^{+}a_{\bar{2}}^{+} \rangle^{(1)} - \langle a_1^{+}a_2 \rangle^{(1)} \langle a_2^{+}a_{\bar{1}}^{+} \rangle^{(1)}).$$

Substituting in (1.26) the values for averages of zero and first order and expressing the latter through $Z^{(\pm)}$ according to (1.20) and (1.24), we finally obtain the desired relations

$$W_1 \equiv \tfrac{1}{2} \epsilon_1 (\langle a_1^{+}a_1 \rangle^{(2)} + \langle a_{\bar{1}}^{+}a_{\bar{1}} \rangle^{(2)}) - \tfrac{1}{2} \Delta_1 (\langle a_{\bar{1}}\, a_1 \rangle^{(2)} + \langle a_{\bar{1}}^{+}a_{\bar{1}}^{+} \rangle^{(2)}) \tag{1.27}$$

$$\begin{aligned} &= E_1 \sum_2 (| Z_{12}^{(+)} |^2 + | Z_{12}^{(-)} |^2) \\ \tfrac{1}{2} (\langle a_1^{+}a_1 \rangle^{(2)} - \langle a_{\bar{1}}^{+}a_{\bar{1}} \rangle^{(2)}) &= \sum_2 (Z_{12}^{(-)} \, Z_{12}^{(+)*} + Z_{12}^{(-)*} \, Z_{12}^{(+)}). \end{aligned} \tag{1.28}$$

* Relations useful in the calculations should be noted:

$$\eta_{12}^{(\pm)^2} = 1 - \xi_{12}^{(\pm)^2} = \tfrac{1}{2} [1 - (\epsilon_1 \epsilon_2 / E_1 E_2) \pm (\Delta_1 \Delta_2 / E_1 E_2)]; \eta_{12}^{(+)} \eta_{12}^{(-)} = \tfrac{1}{2} [(\epsilon_1 / E_1) - (\epsilon_2 / E_2)]$$

$$\xi_{12}^{(+)} \, \xi_{12}^{(-)} = \tfrac{1}{2} [(\epsilon_1 / E_1) + (\epsilon_2 / E_2)]; \; \xi_{12}^{(\pm)} \eta_{12}^{(\pm)} = \frac{1}{2 E_1 E_2} (\epsilon_1 \Delta_2 \pm \epsilon_2 \Delta_1); \; \xi_{12}^{(+)} \eta_{12}^{(\pm)} = \tfrac{1}{2} [(\Delta_2 / E_2) \pm (\Delta_1 / E_1)]$$

Let us now find the correction to the ground-state energy. Using decomposition condition (1.9), we have from (1.13)

$$\langle H\rangle = \sum_{12}(\varepsilon_{12} - \lambda\delta_{12})\langle a_1^+ a_2\rangle + \tfrac{1}{2}\sum\langle \overline{1\,\tilde{1}'|G|\tilde{2}'2}\rangle\langle a_1^+ a_2\rangle\langle a_{\tilde{1}}^+ a_{\tilde{2}'}\rangle + \tfrac{1}{2}\sum\langle 1\,\tilde{2}|G|\tilde{2}'1'\rangle\langle a_1^+ a_{\tilde{2}}^+\rangle\langle a_{\tilde{2}'} a_{1'}\rangle \tag{1.29}$$

The first-order correction to $\langle H\rangle$ is lacking, as zero approximation is defined from the extremum of $\langle H\rangle$. (It can easily be verified using (1.22)). Second-order terms in (1.29) containing products of $\langle\ \rangle^{(1)}\langle\ \rangle^{(1)}$ can easily be reduced to the form

$$\tfrac{1}{2}\sum\langle \overline{1\,\tilde{1}'|G|\tilde{2}'2}\rangle(\rho_{21}^{(+)}\rho_{1'2'}^{(+)} - \rho_{21}^{(-)}\rho_{1'2'}^{(-)}) + \tfrac{1}{2}\sum\langle 1\,\tilde{2}|G|\tilde{2}'1'\rangle(h_{21}^{(+)}h_{1'2'}^{(+)} - h_{21}^{(-)}h_{1'2'}^{(-)}). \tag{1.30}$$

Second-order terms with diagonal averages in (1.29) may be reduced to $\sum_1 W_1$, where W_1 is defined in (1.27). Expressing (1.30) by means of (1.29) through $Z^{(\pm)}$ and making use of (1.27) we get

$$\langle H\rangle^{(2)} = \tfrac{1}{2}\sum E_{12}(|Z_{12}^{(+)}|^2 + |Z_{12}^{(-)}|^2) + \tfrac{1}{2}\sum\langle \overline{1\,\tilde{1}'|G|\tilde{2}'2}\rangle(\eta_{12}^{(+)}\eta_{1'2'}^{(+)}Z_{21}^{(+)}Z_{1'2'}^{(+)} - \eta_{21}^{(-)}\eta_{1'2'}^{(-)}Z_{21}^{(-)}Z_{1'2'}^{(-)}) + \tfrac{1}{2}\sum\langle 1\tilde{2}|G|\tilde{2}'1'\rangle(\xi_{21}^{(+)}\xi_{1'2'}^{(+)}Z_{21}^{(+)}Z_{1'2'}^{(+)} - \xi_{21}^{(-)}\xi_{1'2'}^{(-)}Z_{21}^{(-)}Z_{1'2'}^{(-)}). \tag{1.30'}$$

Making use of the definition for $\epsilon^{(\pm)}$ and $\Delta^{(\pm)}$ (1.21) we rewrite (1.30') as follows

$$\langle H\rangle^{(2)} = \tfrac{1}{2}\sum_{12} Z_{21}^{(+)}\{E_{12}Z_{12}^{(+)} + \eta_{12}^{(+)}(\epsilon_{12}^{(+)} - V_{12}^{(+)}) - \xi_{12}^{(+)}\Delta_{12}^{(+)}\} - \tfrac{1}{2}\sum_{12} Z_{21}^{(-)}\{E_{12}Z_{12}^{(-)} + \eta_{12}^{(-)}(\epsilon_{12}^{(-)} - V_{12}^{(-)}) - \xi_{12}^{(-)}\Delta_{12}^{(-)}\},$$

from which in view of the equations for $Z^{(\pm)}$ (1.25) we finally get

$$\langle H\rangle^{(2)} = -\tfrac{1}{2}\sum\eta_{12}^{(+)}V_{12}^{(+)}Z_{21}^{(+)} + \tfrac{1}{2}\sum\eta_{12}^{(-)}V_{12}^{(-)}Z_{21}^{(-)}. \tag{1.31}$$

Expression (1.31) defines the correction to the average value of the basic Hamiltonian H. For the perturbating Hamiltonian we obtain in a similar approximation

$$\langle V\rangle^{(2)} = \sum_{12} V_{12}\langle a_1^+ a_2\rangle^{(1)} \tag{1.32}$$

which, after simple calculation, takes the form of

$$\langle V\rangle^{(2)} = \sum_{12}\eta_{12}^{(+)}V_{12}^{(+)}Z_{12}^{(+)} - \sum_{12}\eta_{12}^{(-)}V_{12}^{(-)}Z_{21}^{(-)}. \tag{1.32'}$$

From (1.32') and (1.31) it is seen that

$$\langle H\rangle^{(2)} = -\tfrac{1}{2}\langle V\rangle^{(2)}.$$

1.5. Time-dependent self-consistent field

Hitherto we have been considering equations defining the time-independent ground state of the system. In describing collective excitations it will be useful to introduce the concept of a time-dependent self-consistent field. By collective excitations existing in the system is meant the possibility of separating a special degree of freedom connected with the self-consistent field whose excitation does not interfere with individual particle degrees of freedom. In other words, such quantities as single-particle states, as well as quasi-particles and their occupation numbers ("vacuum" for the ground state), retain their meaning for the excited states of the self-consistent field. In fact, the classification of excitations into collective and single-particle appears to be possible quite approximately, and their interference determines the process of relaxation and damping of collective excitations. In the time-dependent self-consistent field method, however, these effects are neglected.

Instead of (1.12) in the case of a time-dependent self-consistent field the main equations take the form of ($\hbar = 1$)

$$\begin{aligned} i\frac{\partial}{\partial t}\langle a_1\, a_2\rangle &= \langle [a_1\, a_2;\, H]\rangle \equiv A_{12} \\ i\frac{\partial}{\partial t}\langle a_1^+ a_2\rangle &= \langle a_1^+ a_2;\, H]\rangle \equiv B_{12}. \end{aligned} \tag{1.33}$$

If a time-dependent part of the self-consistent field may be considered as perturbation, then equations (1.33) are linearized. In view of (1.20) and (1.24) we obtain in this case

$$i\frac{\partial}{\partial t} Z_{12}^{(\mp)} = E_{12}\, Z_{12}^{(\pm)} - \xi_{12}^{(\pm)} \Delta_{12}^{(\pm)} + \eta_{12}^{(\pm)} \epsilon_{12}^{(\pm)} \tag{1.34}$$

from which for Fourier components $Z_{12}^{(\pm)}(\omega)$ we find

$$\omega\, Z_{12}^{(\mp)}(\omega) = E_{12} Z_{12}^{(\pm)}(\omega) - \xi_{12}^{(\pm)} \Delta_{12}^{(\pm)}(\omega) + \eta_{12}^{(\pm)} \epsilon_{12}^{(\pm)}(\omega). \tag{1.35}$$

Here, according to (1.20) and (1.21),

$$\begin{aligned} Z_{12}^{(\pm)}(\omega) &= -\, Z_{\tilde{2}\tilde{1}}^{(\pm)}(\omega) = \pm\, Z_{21}^{(\pm)*}(-\omega) \\ \Delta_{12}^{(\pm)}(\omega) &= -\, \Sigma \langle 1\tilde{2}|G|\tilde{2}'1'\rangle \xi_{1'2'}^{(\pm)} Z_{1'2'}^{(\pm)}(\omega) = -\, \Delta_{\tilde{2}\tilde{1}}^{(\pm)}(\omega) = \pm\, \Delta_{21}^{(\pm)*}(-\omega) \\ \epsilon_{12}^{(\pm)}(\omega) &= \pm\, \Sigma \overline{\langle 1\tilde{1}'|G|\,2'\tilde{2}\rangle}\, \eta_{1'2'}^{(\pm)} Z_{1'2'}^{(\pm)}(\omega) + V_{12}^{(\pm)}(\omega) = \epsilon_{21}^{(\pm)*}(-\omega) = \pm\, \epsilon_{\tilde{2}\tilde{1}}^{(\pm)}(\omega). \end{aligned} \tag{1.36}$$

It should be noted that equations (1.35) under transformation $1 \rightarrow 2$ and $\omega \rightarrow -\omega$ become complex conjugated equations. Thus it follows that the dispersion equation for (1.35) defines the frequency ω except for the sign.

After writing the basic equations let us consider certain problems connected with the collective properties of nuclei.

2. THE EQUILIBRIUM SHAPE OF NUCLEI

2.1. Formulation of the problem

Numerous models have been suggested recently to describe the excitations of those nuclei where the spectrum essentially differs from that of ro-

tation (spherical oscillating nucleus [12], non-axial rotator [13], model with γ-unstable potential [14] and others [15]). Although all these models make use of the essentially different concepts concerning the equilibrium shape of the nuclei, their predictions are quite close and experiment does not at present offer any possibility of choice in favour of any definite model. In such a situation the estimation of the equilibrium shape of the nucleus from its intrinsic structure is of special interest.

Deformation in nuclei results from the effect of polarization by nucleons in the upper unfilled shells, hence the problem of the equilibrium shape of the nucleus is reduced mainly to the definition of the equilibrium configuration of these nucleons. If we temporarily neglect the polarization of the closed-shell core and consider external nucleons to be in a spherical potential well, then their configuration is determined by the nature of their interaction. As was stated above, the main role in this case is played by the two parts of interaction: Cooper-pairing and the self-consistent field. When the closed shell is polarized, the external nucleons are affected not only by their own self-consistent field, but also by the additional field of the closed-shell core. But as these fields possess an identical symmetry and are proportional to each other, the closed-shell polarization effect is equivalent to the increase of the eigen-self-consistent field of the external nucleons, and its calculation is reduced to the renormalization of their interaction constant.

In the deformed nucleus the grouping into external nucleons and closed-shell core is rather conditional. Let us assume that the region "close to the Fermi surface" lies in the energy interval $\lesssim \hbar v/R$ (v - velocity at Fermi surface) and contains $\sim A^{2/3}$ one-particle states. In the slightly deformed nucleus $\hbar v/R$ coincides with the intershell distance. In this case, let the external nucleons be particles in the upper unfilled shell. It should be noted that the double inequality $\Delta \ll \hbar v/R \ll \mathcal{E}_F$ is valid. Thus, leaving the question of interaction renormalization aside, we shall consider only $\sim A^{2/3}$ nucleons close to the Fermi surface taking their Cooper pairing and the self-consistent field into consideration.

2.2. Choice of nucleon-nucleon interaction

Let us assume that the spherically symmetrical part of the self-consistent field is already taken into consideration in the original single-particle Hamiltonian ϵ, and the residual interaction results only in the deviation from spherical symmetry. Since the most essential role in nuclei is played by quadrupole anisotropy we may assume the additional self-consistent field to be of quadrupole symmetry, i.e.

$$\sum_{12'} \overline{\langle 11' | G | 2'2 \rangle} \langle a_1^+ a_2 \rangle = -\kappa \sum_{\mu} Q_\mu^* \langle 1 | q_\mu | 2 \rangle, \qquad (2.1)$$

where Q_μ and single-particle operator q_μ possess the property of spherical tensor operators of the second rank. The right-hand side of Eq. (2.1) can be easily interpreted as an interaction between a single-particle moment q_μ and the total moment Q_μ of the other particles. We can imply such an interaction to be a reflection of nucleon-nucleon "quadrupole interaction" of the form

$$G_Q(\vec{r}_1, \vec{r}_2) = -\kappa \sum_{\mu} q_\mu^*(\vec{r}_2)\, q_\mu(\vec{r}_1). \qquad (2.1')$$

The real interaction entering the left-hand side of (2.1) is not reduced only to (2.1'). In the general case $G(\vec{r}_1, \vec{r}_2)$ can be expressed as a series of scalar products of spherical tensor operators of different ranks. Equation (2.1) implies in fact that density matrix $\langle a_1^+ a_{2'} \rangle$, which possesses quadrupole symmetry, picks out the only quadrupole term from this sum*

In view of interaction (2.1) the one-particle Hamiltonian (1.16) takes the form

$$\epsilon(12) = \mathcal{E}_{12} - \lambda\delta_{12} - \kappa \sum_{\mu} Q_{\mu}^{*} \langle 1 | q_{\mu} | 2 \rangle, \qquad (2.2)$$

where the total quadrupole moment of the nucleons Q_μ is defined by

$$Q_{\mu} = \sum_{\nu\nu'} \langle \nu | q_{\mu} | \nu' \rangle \langle a_{\nu}^{+} a_{\nu'} \rangle = (-1)^{\mu} Q_{-\mu}^{*}. \qquad (2.3)$$

Let us choose as basic function of secondary quantization eigenfunctions $|\nu\rangle$ of Hamiltonian (2.2) at the fixed value of parameter Q_μ

$$\tilde{\epsilon} | \nu \rangle \equiv (\hat{\mathcal{E}} - \lambda - \kappa \sum_{\mu} Q_{\mu}^{*} \hat{q}_{\mu}) | \nu \rangle = \epsilon_{\nu}(Q) | \nu \rangle. \qquad (2.4)$$

The first of conditions (1.18) is now fulfilled. Let us assume that the second condition is also valid and put (see (1.19'''))

$$\Delta(12) = \gamma_1 \Delta \delta_{2\tilde{1}}, \qquad (2.5)$$

where Δ is constant. The interaction yielding the main contribution to Cooper pairing and resulting in (2.5) may be represented as

$$\langle 1\tilde{2} | G | \tilde{2}'1' \rangle = - G \gamma_1 \gamma_{1'} \delta_{12} \delta_{1'2'}. \qquad (2.6)$$

A set of matrix elements (2.6) is usually called pairing interaction.**

2.3. Equations defining the equilibrium deformation

In fulfilling conditions (2.4) and (2.5), equations (1.19) define average values of pairs of operators by the three parameters: Δ, chemical potential λ and quadrupole moment Q_μ. The equation for λ is found from the expression for the number of particles N:

$$N = \mathrm{Sp}(\hat{\rho}) = \tfrac{1}{2} \sum_{\nu} (1 - \epsilon_{\nu}/E_{\nu}). \qquad (2.7)$$

For the definition of Δ and Q_μ we get from (1.19), (2.3) and (2.5)

$$G \sum_{\nu} (1/2E_{\nu}) = 1 \qquad (2.8)$$

$$Q_{\mu} = \mathrm{Sp}(\hat{\rho}\hat{q}_{\mu}) = \tfrac{1}{2} \sum_{\nu} \langle \nu | q_{\mu} | \nu \rangle (1 - \epsilon_{\nu}/E_{\nu}). \qquad (2.9)$$

* Strictly speaking, the quadrupole term from the expansion of $G(\vec{r}_1, \vec{r}_2)$ in spherical tensor operators has the form of (2.1') but with κ depending on $|\vec{r}_1|$, $|\vec{r}_2|$. We shall neglect the inessential dependence and take κ as constant. We may also include the radial factor r^2 into q_μ and identify it with the single-particle quadrupole moment (redefining κ).

** Parameter G must be reasonably considered as phenomenological and taken from experiments. In this case its values for the deformed and spherical nuclei (in the same A-region) appear to be very close. Therefore, one may consider G independent of Q.

Knowing the averages of pairs of operators, the ground-state energy can easily be found. As a result of using the above formulae we obtain

$$W = \langle H \rangle + \lambda N = \sum_{\nu} \rho_{\nu\nu} (\epsilon_{\nu} + \lambda) + \tfrac{1}{2} \kappa \sum_{\mu} Q_{\mu}^{*} Q_{\mu} - \Delta^2 / 2G. \qquad (2.10)$$

We shall make an important remark. If Eq. (2.9) is temporarily not taken into consideration, parameters Q_{μ} remain free. In this case equations (2.4), (2.7) and (2.8) coincide with analogous equations in the problem treating nucleons in the external potential well with deformation parameters Q_{μ}. In such a problem one may consider the ground-state energy $W(Q_{\mu})$ and investigate its dependence on Q_{μ}. If expression (2.10) is used for $W(Q_{\mu})$ it can easily be seen that the positions of the extrema will be defined by equation (2.9). Indeed, differentiating (2.10) over Q_{μ} and assuming λ and Δ to be functions of Q_{μ} defined by equations (2.8) and (2.9), we get

$$(\partial W / \partial Q_{\mu}^{*}) = \kappa (Q_{\mu} - \mathrm{Sp}(\hat{\rho} \hat{q}_{\mu})); \qquad (2.11)$$

therefore, equation $\partial W / \partial Q_{\mu}^{*} = 0$ coincides with (2.9). The latter, in this case, may be treated as condition of consistency: only such deformations are physically allowable which coincide with the density deformation of the nucleons. It should be stressed that the function $W(Q)$, strictly speaking, has physical meaning only in the extremum points. In particular, the second derivatives from $W(Q_{\mu})$

$$(\partial^2 W / \partial Q_{\mu}^{*} \partial Q_{\mu'}) = \kappa (\delta_{\mu\mu'} - \frac{\partial}{\partial Q_{\mu'}} \mathrm{Sp}(\hat{\rho} \hat{q}_{\mu})) = \kappa (1-R)_{\mu\mu'} \qquad (2.12)$$

need not be identified with the physical tensor of the quadrupole restoring force; the latter may be calculated from the variation of energy of the system in the external field.* As a result, we find

$$K_{\mu\mu'} = \kappa \{R^{-1}(1-R)\}_{\mu\mu'} \qquad (2.13)$$

which differs from (2.12). In order to investigate the stability of extremum states it is quite sufficient to determine the sign of K only, so the difference between (2.12) and (2.13) is not essential for us.

2.4. Some general conclusions

Let us proceed to the investigation of the equation for Q_{μ}. First of all, it should be noted that Eq. (2.9) defines components Q_{μ} only with an accuracy up to the random orientation of axis. The nuclear shape is characterized by the two invariants

$$\sum_{\mu} Q_{\mu}^{*} Q_{\mu} = \beta^2 \qquad (2.14)$$

$$= -(2/7)^{1/2} \sum_{(\mu)} (2\mu' \, 2\mu'' | 2\mu) \, Q_{\mu} \, Q_{\mu'}^{*} \, Q_{\mu''}^{*} = \beta^2 \cos 3\gamma,$$

where $(2\mu' \, 2\mu'' | 2\mu)$ are Clebsch-Gordan coefficients. In analogy with (2.14) let us introduce scalar products of the operator of the single-particle quadrupole moment $\hat{q}_{\mu}$ and the combination Q_{μ}^{*}/β

* In [5] the restoring force was estimated by the Lagrange multiplier method, which yields a similar result.

$$\hat{q} = \sum_{\mu} \hat{q}_{\mu} Q_{\mu}^{*} / \beta \qquad (2.14')$$

$$\hat{r} = -(2/7)^{1/2} \sum_{(\mu)} \hat{q}_{\mu} (2\mu' 2\mu'' | 2\mu) Q_{\mu'}^{*} Q_{\mu''}^{*} / \beta^2 .$$

Then from (2.9) we obtain for parameters β and γ the following equations:

$$\beta = \mathrm{Sp}(\hat{\rho}\hat{q}) \qquad (2.15)$$

$$\beta \cos 3\gamma = \mathrm{Sp}(\hat{\rho}\hat{r}). \qquad (2.15')$$

If one chooses for coordinate axis the main axis of the ellipsoid Q_{μ} ($Q_0 = \beta\cos\gamma$; $Q_1 = Q_{-1} = 0$; $Q_2 = Q_{-2} = (1/\sqrt{2})\beta\sin\gamma$), then

$$\langle\nu|\hat{q}|\nu\rangle = \langle\nu|q_0|\nu\rangle \cos\gamma + \sqrt{2}\langle\nu|q_2|\nu\rangle \sin\gamma \qquad (2.16)$$

$$\langle\nu|\hat{r}|\nu\rangle = \langle\nu|q_0|\nu\rangle \cos 2\gamma - \sqrt{2}\langle\nu|q_2|\nu'\rangle \sin 2\gamma.$$

One may draw several conclusions from the form of equations (2.9) and (2.15). First, there always exists the solution corresponding to sphere ($\beta = 0$). In fact, in spherical nucleus density matrix ρ is independent of the projection of single-particle angular momentum, and $\sum_{m}\langle\nu|q_{\mu}|\nu\rangle = 0$, therefore the right sides of equations (2.15) or (2.9) when $\beta = 0$ ($Q_{\mu} = 0$) become zero. It should also be noted that if the first equation of (2.16) has a solution for β when $\gamma = 0$ or $\gamma = \pi/3$ then the second equation is also satisfied since when $\gamma = 0, \pi/3$ both equations (2.15) coincide (this is seen from (2.16)). This means that there may exist solutions with $\gamma = 0$ or $\pi/3$ corresponding to axially symmetric deformation of the nucleus.

2.5. The region of small deformations

The right sides of equation (2.15) are the quadrupole moments of nuclear density in the external well with deformation (β, γ). In order to determine the nature of the dependence of $\mathrm{Sp}(\hat{\rho}\hat{q})$ on β and γ it is useful to consider the limiting regions of large and small deformations. The ratio of shell level splitting $\kappa Q\langle\nu|q|\nu\rangle \sim \kappa\beta q$ and pairing energy Δ may serve as a useful criterion for the deformation value. The deformation parameter usually defined, $\delta R/R$, is related to β by the equation $\kappa\beta q \simeq \epsilon_F \delta R/R$, where ϵ_F - is the Fermi energy and q is an average absolute value of $\langle\nu|q|\nu\rangle$. By large and small deformations we mean the cases $\delta R/R \ll \Delta/\epsilon_F$ and $\delta R/R \gg \Delta/\epsilon_F$ respectively. (Let us recall that for the real strongly deformed nuclei $\delta R/R \approx A^{-1/3}$, i.e. the absolute deformation value $\delta R/R$ is always small). Also, noting that $\beta \sim A^{2/3}$ when $\delta R/R \approx A^{-1/3}$ we get the estimate $\kappa \approx \epsilon_F / q^2 A$.

In the region of small deformations the expansion of the right side of (2.9) commences with the linear terms Q_{μ} and in view of the invariance has the following form

$$\mathrm{Sp}(\hat{\rho}\hat{q}_{\mu}) = a Q_{\mu} - \tfrac{1}{2} b (7/2)^{\frac{1}{2}} \sum_{\mu'\mu''} (2\mu' 2\mu'' | 2\mu) Q_{\mu'} Q_{\mu''} + \ldots . \qquad (2.17)$$

where a and b are coefficients independent of Q_{μ}. Hence, for the right side of equation (2.15) we obtain

$$\mathrm{Sp}(\hat{\rho}\hat{q}) = a\beta + \frac{1}{2} b \beta^2 \cos 3\gamma + \ldots . \qquad (2.17')$$

and for a and b we have

$$a = \left[\frac{\partial}{\partial \beta} \mathrm{Sp}(\hat{\rho}\hat{q}) \right]_{\beta=0}; \quad b = \left[1/\cos 3\gamma \right] \left[\frac{\partial^2}{\partial \beta^2} \mathrm{Sp}(\hat{\rho}\hat{q}) \right]_{\beta=0}. \qquad (2.17'')$$

The calculation of (2.17) is simplified if we take into consideration that according to (2.4) and (2.14')

$$\partial \epsilon_\nu / \partial \beta = \langle \nu | \partial \hat{\epsilon} / \partial \beta | \nu \rangle = -\kappa \langle \nu | \hat{q} | \nu \rangle - \partial \lambda / \partial \beta \qquad (2.18)$$

and derivatives $\partial \Delta / \partial \beta$ and $\partial \lambda / \partial \beta$ are zero when $\beta = 0$. As a result of simple calculations we get

$$a = a_1 + a_2 = \kappa \sum_\nu (\Delta^2 / 2E_\nu^3)\, q^2_{\nu\nu} + \sum_\nu \rho_{\nu\nu} (\partial q_{\nu\nu} / \partial \beta) \qquad (2.19)$$

$$b \cos 3\gamma = \frac{3}{2} \kappa^2 \sum_\nu (\Delta^2 \epsilon_\nu / E_\nu^5)\, q^2_{\nu\nu} + \frac{3}{2} \kappa \sum_\nu (\Delta^2 / E_\nu^3) q_{\nu\nu} (\partial q_{\nu\nu} / \partial \beta) \qquad (2.19')$$
$$+ \sum_\nu \rho_{\nu\nu} (\partial^2 q_{\nu\nu} / \partial \beta^2)$$

where the expressions in the right sides are taken when $\beta = 0$. For derivatives from $q_{\nu\nu} = \langle \nu | \hat{q} | \nu \rangle$ entering (2.19) the exact formulae is easily obtained (differentiating equation (2.4)):

$$\partial q_{\nu\nu} / \partial \beta = 2 \kappa \sum_{\nu'}{}' [\, |q_{\nu\nu'}|^2 / (\epsilon_{\nu'} - \epsilon_\nu)]. \qquad (2.20)$$

Let us estimate both terms in (2.19). The value $\Delta^2 / 2E_\nu^3$ notably differs from zero in the region $\Delta \sim \mathcal{E}_F A^{-2/3}$ wide inside which lie $\sim A^{-1/3}$ single-particle states. This yields the estimate for the first term in (2.19), $a_1 \sim (\kappa q^2 / \mathcal{E}_F) A \sim 1$. On the other hand from (2.20) follows $\partial q_{\nu\nu'} / \partial \beta \sim \kappa q^2 / \mathcal{E}_F$, so the second term in (2.19) in the case of summation over $A^{2/3}$ states at the Fermi surface can be estimated as $a_2 \sim a_1 A^{-1/3} \ll a_1$. Analogous estimates show that the two latter terms in (2.19") are a factor $A^{-1/3}$ smaller than the former one. Thus, for particles at the Fermi surface one may conclude*

$$a \approx \kappa \sum_\nu (\Delta^2 / 2E_\nu^3) q^2_{\nu\nu}; \quad b \cos 3\gamma \approx \frac{3}{2} \kappa^2 \sum_\nu (\Delta^2 \epsilon_\nu / E_\nu^5)\, q^3_{\nu\nu}. \qquad (2.21)$$

Equations (2.15), (2.17), beside the spherical solution $\beta = \beta_0 = 0$, also possess the formal solution

$$\beta = \beta_1 = 2(1-a)/b \cos 3\gamma \qquad (2.22)$$

which, with a closely approaching unity, lies in the region of small β considered. To determine the stability of these solutions we shall find coefficients of the restoring force with respect to the change of variables β and γ. From (2.13) and (2.17) for small Q_μ ($|1-a| \ll 1$) it would be easy to obtain

$$K_{\mu\mu'} = \kappa(1-a)\, \delta_{\mu\mu'} + \kappa b (7/2)^{\frac{1}{2}} \sum_{\mu_1} (2_\mu\, 2_{\mu_1} | 2_{\mu'}) Q_{\mu_1}. \qquad (2.23)$$

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* If summing is made in a_2 over all states, then a_2 and a_1 will be of the same order. Term a_2 in this case results in the renormalization of quadrupole interaction constant κ.

The transition to variables β and γ is carried out by means of formulae (following from (2.14))

$$\partial Q_\mu/\partial\beta = Q_\mu/\beta;\ \partial Q_\mu/\partial\gamma = Q_\mu \mathrm{ctg}\, 3\gamma + (7/2)^{\frac{1}{2}}(1/\beta \sin 3\gamma) \sum_{\mu'\mu''} (2_\mu\, 2_{\mu''} | 2_{\mu'}) Q_{\mu'} Q_{\mu''} \tag{2.24}$$

Substituting (2.23) and (2.24) in equality

$$K_{xy} = \sum_{\mu\mu'} K_{\mu\mu'} (\partial Q_\mu/\partial x)(\partial Q^*_{\mu'}/\partial y);\ (x, y = \beta, \gamma), \tag{2.24'}$$

we obtain

$$K_{\beta\beta} = \kappa(1-a) - \kappa b \beta \cos 3\gamma$$

$$(1/\beta^2)K_{\gamma\gamma} = \kappa(1-a) + \kappa b \beta \cos 3\gamma. \tag{2.25}$$

For the spherical state ($\beta = 0$) the restoring force regarding variables β and $\beta\gamma$ is equal, as one may expect from the isotropy, and the stability condition takes the form of $a < 1$. As is clear from (2.21), the a value is widely dependent on the upper shell filling. It reaches maximum near the middle of the shell ("maximum filling") and is zero for the closed shell (where $\Delta = 0$). Thus, with a small number of particles (or holes) in the upper unfilled shell the spherical state is stable up to a critical filling for which $a = a_c = 1$. With a further increase of the filling the spherical nucleus becomes unstable.

The coefficients of elasticity in the stationary point (2.22) are equal to

$$K_{\beta\beta}(\beta_1) = \kappa(a-1)$$

$$K_{\gamma\gamma}(\beta_1) = -3\kappa(a-1)\beta_1^2 \tag{2.26}$$

from which it is clear that with sign $(a-1)$ either $K_{\beta\beta}$ or $K_{\gamma\gamma}$ is negative. Thus, the state with $\beta = \beta_1$ (2.22) is always unstable.

2.6. The region of large deformations ($\delta R/R \gg \Delta/\mathcal{E}_F$)

It should be noted that calculation of $\mathrm{Sp}(\hat\rho\hat q_\mu)$ in this limiting case is equivalent to the problem of the quadrupole moment of non-interacting particles ($\Delta \to 0$) in a weakly deformed well ($\delta R/R \ll 1$). In the quasi-classical picture, when the distribution of particles in coordinate space is described by the average density, defined by the shape of the potential well only, the $\mathrm{Sp}(\hat\rho\hat q_\mu)$ value is proportional to Q_μ and does not depend on the characteristic of the filled single-particle levels. Density fluctuations, connected with particles at the Fermi surface and, consequently, dependent on their quantum numbers, result in the additional contribution to the quadrupole moment. This part $\mathrm{Sp}(\hat\rho\hat q_\mu)$ is very sensitive to the sequence of levels at the Fermi surface, which is mainly defined by the shape of the deformation, parameter γ, and much less by the value of deformation β. Thus, a change of γ from 0 to $\pi/3$ inverts the order of levels, while small variations of β, in fact, do not change it. As a result of the above, the right side of (2.15) in the region of large deformations may be shown as follows

$$\mathrm{Sp}(\hat\rho\hat q_\mu) \approx B(\gamma) + C\beta. \tag{2.27}$$

Here the term $B(\gamma)$ is related to the particle density fluctuations at the Fermi surface. It is sensitive to the shell occupation and, therefore, unmonotonously dependant on the number of nucleons A. The last term in (2.27) is of a quasi-classic nature, so one may assert that the coefficient C is smoothly and monotonously dependant on A.

Expression (2.27) may be approached in another way. One may easily be convinced that the tensor $\mathrm{Sp}(\hat{\rho}\hat{q}_\mu)$ is the derivative over $-\kappa Q_\mu^*$ from the scalar function

$$U = \sum_\nu \rho_{\nu\nu}(\epsilon_\nu + \lambda) - \Delta^2/2b. \qquad (2.28)$$

(λ and Δ are functions of Q_μ, defined from Eqs (2.7) and (2.8)). The value of (2.28) is the nucleon energy in the external well in view of their pairing interaction. This energy depends only on invariant quantities β and γ, containing under large deformations the linear and quadratic in β terms

$$U \approx B(\gamma)\beta + \tfrac{1}{2} C\beta^2 \qquad (2.29)$$

where the first term is the nucleon energy in the upper unfilled shells and the last term is related mainly to the closed-shell particles. The derivative from (2.29) in β yields (2.27).

From (2.27) one may see that the value C is equal to $(\partial/\partial\beta)\,\mathrm{Sp}(\hat{\rho}\hat{q})$, i.e. is analogous to (2.17") in the region of large deformations. Simple analysis shows that in differentiating $\mathrm{Sp}(\hat{\rho}\hat{q})$ over β, the quantities λ and Δ may be considered to be constant* so that the expression for C formally concides with (2.19). In view of (2.20) we obtain

$$C = (\partial/\partial\beta)\,\mathrm{Sp}(\hat{\rho}\hat{q}) = \kappa \sum_\nu (\Delta^2/2E_\nu^3)\, q_{\nu\nu}^2 + \kappa \sum_{\nu\nu'}{}' [(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu)] \left| q_{\nu\nu'} \right|^2. \qquad (2.30)$$

It should be noted that $(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu) \to \Delta^2/2E_\nu^3$ when $\epsilon_\nu \to \epsilon_{\nu'}$, so the first term in (2.30) is the sum of diagonal terms, omitted in the second sum. Joining both terms we obtain**

$$C = (\partial/\partial\beta)\,\mathrm{Sp}(\hat{\rho}\hat{q}) = \kappa \sum_{\nu\nu'} [(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu)] \left| q_{\nu\nu} \right|^2. \qquad (2.31)$$

The quantity $(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu)$ as a function of ϵ_ν with $(\epsilon_{\nu'} - \epsilon_\nu)$ fixed has a sharp maximum at the Fermi surface ($\epsilon = 0$), the width of which is determined by the larger of the values $(\Delta, |\epsilon_{\nu'} - \epsilon_\nu|)$. The area under the curve is equal to unity. We can consider the matrix element $q_{\nu\nu'}$ to be a smoothly varying function of ϵ_ν for fixed values of $(\epsilon_{\nu'} - \epsilon_\nu)$. This makes it possible to perform the substitution

$$(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu) \to \delta(\epsilon_\nu)$$

---

* This is because the variations of Δ and λ under deformation are small compared with the shift of single-particle levels $\partial\epsilon/\partial\beta \sim \kappa q$. Evaluations show that $\partial\lambda/\partial\beta < \kappa q\, A^{-1/3}$ and $\partial\Delta/\partial\beta \sim \kappa q A^{-1}$. An exception is an oscillator model where $\partial\lambda/\partial\beta \sim \partial\epsilon/\partial\beta$. Taking the terms $\partial\lambda/\partial\beta$ into account results in the compensation of the first term in (2.30) for this model.

** Let us be more exact. The unification of both terms is possible if we take all nucleons into consideration. The second term in (2.30) may be neglected for nucleons at the Fermi surface. Doing this, the formulae in (2.34) will yield not the total deformation, but only that part of it which is associated with the external nucleons.

in (2.31) (see analogous calculations in[6]). After this the sum over ν' in (2.31) can be performed and, introducing the density of particles in the energetic and coordinate space (on the Fermi surface)

$$\rho(\mathcal{E}_F, \vec{r}) = \sum_\nu \varphi^*_\nu(\vec{r})\delta(\epsilon_\nu)\varphi_\nu(\vec{r}),$$

we find

$$C = \kappa \int d\vec{r}\, \hat{q}^2 \rho(\mathcal{E}_F; \vec{r}) = \kappa \rho(\mathcal{E}_F)\overline{q^2}. \qquad (2.32)$$

Here the average from $\hat{q}^2$ with an accuracy up to $\sim \delta R/R$ values does not differ from the average over spherical distribution, so one may assume that $\overline{q^2}$ is independent of the deformation. The value C, as was expected, does not depend on the deformation and is a smooth function of the number of nucleons A. The quantity C may be connected with the quasi-classical coefficient of elasticity. And, indeed, from (2.27) we have (see (2.12)) $R_{\mu\mu'} \approx C\delta_{\mu\mu'}$ from which, in view of (2.13) and (2.24'') we find

$$K_{\beta\beta} \approx \kappa[(1/C)-1]. \qquad (2.33)$$

From (2.33) it follows that for the stable system the inequality $C < 1$ should be fulfilled.

The solution of equation (2.15) in the region of large deformations in view of (2.27) takes the form

$$\beta = \beta_2 = B(\gamma)/(1-C). \qquad (2.34)$$

The quantitative calculation of the equilibrium deformation (2.34) cannot be carried out in the general form, as the function $B(\gamma)$ is sensitive to the model of single-particle levels. We shall discuss formula (2.33) later in connection with the vibrations of deformed nuclei.

2.7. Phase transition from spherical to deformed nuclei

Let us consider the nature of transition from spherical nuclei to the deformed ones qualitatively. It would be convenient for the analysis to write down equation (2.15) as follows

$$1/a = \Lambda(\beta) \qquad (2.35)$$

where the function $\Lambda(\beta)$ in the region of low deformations according to (2.17') is of the form

$$\Lambda(\beta) = 1 + (b/2a)\,\beta \cos 3\gamma + \ldots . \qquad (\beta \ll \Delta/\kappa q) \qquad (2.36)$$

and under large deformations (see (2.27)) is equal to

$$\Lambda(\beta) \approx C\, a^{-1} + B(\gamma)\, a^{-1}\beta^{-1} \qquad (\beta \ll \Delta/\kappa q). \qquad (2.37)$$

As one sees from (2.36) the behaviour of $\Lambda(\beta)$ in the region of small β depends on the sign of the expression $(b/a)\cos 3\gamma$. According to (2.21) the value a

is always positive and b may change its sign with shell filling.* The extreme expressions (2.36) and (2.37) permit qualitative presentation of the function $\Lambda(\beta)$ behaviour, also in the intermediate region.

Figure 1 plots function $\Lambda(\beta)$ for cases of positive (I) and negative (II) values b cos 3γ. Solutions of equations (2.35), representing extremum values β by fixed values of γ (extremum points of function $W(\beta)$ (2.10))are given by the intersection points of the curve $\Lambda = \Lambda(\beta)$ with horizontal straight line $\Lambda = 1/a$. Moreover, one must take into account the solution $\beta = 0$, excluded from (2.35). As was stated the $\beta = 0$ value corresponds to the minimum of $W(\beta)$ when $a < 1$ ($\Lambda > 1$ in Fig. 1) and to maximum when $a > 1$ ($\Lambda < 1$).

From Fig. 1 one can easily see the full picture of the distribution of the extremum points for $W(\beta)$.

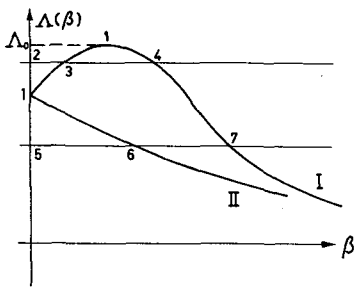

Fig. 1

I. b cos $3\gamma > 0$ (curve I. Fig. 1). In this case one should distinguish three different regions of shell filling (values a):

(1) $1/a > \Lambda_0 = \max \Lambda(\beta)$. One minimum when $\beta = 0$.
(2) $\Lambda_0 > 1/a > 1$. Three extremums: minimum when $\beta = 0$ (2) -maximum (3) -minimum (4).
(3) $1 > 1/a$. Maximum when $\beta = 0$ (5) and minimum for $\beta \neq 0$ (7).

II. b cos $3\gamma < 0$ (curve II, Fig. 1). Two regions of filling

(1) $1/a > 1$. Minimum when $\beta = 0$.
(2) $1 > 1/a$. Maximum when $\beta = 0$ (5) and minimum when $\beta \neq 0$ (6).

Thus, in a region of sufficiently small a in both cases $W(\beta)$ possesses the actual minimum when $\beta = 0$, which corresponds to the stable spherical state. On the other hand, in a region of sufficiently large a there exists only the minimum when $\beta \neq 0$.

Let us consider now the case of transition from spherical nuclei to deformed ones with the increase of filling (decrease $1/a$). When b cos $3\gamma < 0$ transition occurs when $1/a = 1$ and deformation beyond the transition point smoothly rises from zero ("phase transition of second kind"). In the case of b cos $3\gamma > 0$ (curve I) with increase of filling (lowering of horizontal line in Fig. 1) the function $W(\beta)$ acquires a second minimum for $\beta \neq 0$ (4), at first, besides the minimum in the point $\beta = 0$, which then gradually decreases. The transition to the deformed state will occur at the moment when both minima

* For a single j-level b is negative in filling the first half of the shell and is positive for the second half. In the oscillator model we have reversed behaviour. This is associated with the different directions of the level density gradient in these models. Real nuclei in this respect are nearer to the oscillator model.

become equal. In this case deformation proceeds from zero to some finite value ("phase transition of first kind").

Hitherto we have considered the extrema of $W(\beta)$ at fixed γ. Now we shall make use of the stability criterion with respect to the variation of parameter γ. As one sees from (2.26) in the region of small β the sign of the restoring force coefficient $K_{\gamma\gamma}$ is defined by the (1-a) value. Thus solutions for small β corresponding to curve II in Fig. 1, for which $1/a < 1$, appear to be γ-unstable. Therefore the transition to the deformed nucleus may occur only according to curve I. Thus the transition from spherical nuclei to deformed ones occurs with the jump in deformation and is, therefore, a phase transition of the first kind.

3. DO γ-DEFORMED NUCLEI EXIST?

3.1. Calculations in harmonic oscillator model

As stated above, the estimation of the equilibrium shape of the nucleus in the region of large deformations in the general form appears to be impossible.

Let us make use of the oscillator model for the single-particle potential. In this model the matrix elements of the single-particle quadrupole moment q_μ are diagonal inside one shell, and Eq. (2.4) is satisfied for the states $|\nu\rangle = |n_x n_y n_z\rangle$ (n_i - oscillator quantum numbers). State $\tilde{\nu}$ differs only by the spin direction.

For single-particle energies $\epsilon_\nu(\beta, \gamma)$ from (2.4) it follows that

$$\epsilon_\nu = \epsilon - \kappa\beta\,(q_\nu \cos\gamma + s_\nu \sin\gamma) \tag{3.1}$$

where $\epsilon = \hbar\omega n - \lambda$, and matrix elements q_ν and s_ν are equal* to

$$q_\nu = n_z/n - 1/3\,;\; s_\nu = (n_x - n_y)/\sqrt{3}n;\; n = \sum_i n_i. \tag{3.2}$$

Further calculations will be carried out with quasi-classical accuracy neglecting values of order $1/n$. Then summing over $(n_x n_y n_z)$ within the shell $n = \text{const}$ may be replaced by integrating over q_ν and s_ν

$$\sum_\nu \to 2\sqrt{3}\Omega \iint dq\,ds \tag{3.3}$$

where $2\Omega = n^2$ is the number of states in the shell under consideration. The region of integration in plane (q, s) is an equilateral triangle with the vertic in the points $(2/3; 0)$ and $(-1/3; \pm 3^{-\frac{1}{2}})$ (Fig. 2).

Let us first consider equation (2.7) for the chemical potential λ, i.e. for parameter ϵ in (3.1). In the region of large deformations ($\beta \gg \Delta/\kappa$) we may replace the density matrix $\rho_{\nu\nu}$ by the stepping function

$$\rho_{\nu\nu} = \begin{cases} 1 & \epsilon_\nu < 0 \\ 0 & \epsilon_\nu > 0 \end{cases} \tag{3.4}$$

and integrate over the region $\epsilon_\nu(q, s) < 0$. The equation defining the boundary $\epsilon_\nu(q, s) = 0$ determines the straight line which, when $0 < \gamma < \pi/3$, possesses

* We define q_ν, s_ν and consequently Q as dimensionless quantities. Dimension factors are included in parameter κ.

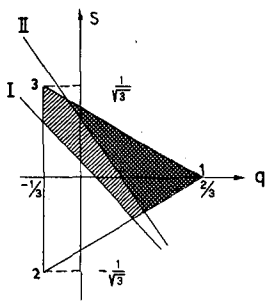

Fig. 2

negative slope to the axis q. When the line passes through vertex 1 the integration region vanishes. This is in accord with the unoccupied shell and corresponds to $\epsilon = \epsilon_{max} = (2/3)\kappa\beta\cos\gamma$. When the shell is fully occupied ($\epsilon_\nu(q, s) = 0$ passes through vertex 2) then $\epsilon = \epsilon_{min} = (2/3)\kappa\beta\cos[\gamma + (2\pi/3)]$. Thus, parameter ϵ varies when the shell is filled in the limits

$$= (2/3)\kappa\beta\cos[\gamma + (2\pi/3)] \leqslant \epsilon \leqslant (2/3)\kappa\beta\cos\gamma \qquad (3.5)$$

Let us introduce the new parameter δ instead of ϵ, putting

$$\epsilon = (2/3)\kappa\beta\cos(\gamma + 2\delta), \qquad (3.6)$$

then the full variation of ϵ is attained when $0 \leqslant \delta \leqslant \pi/3$.

Note that the region of integration over (q, s) depends on the position of the straight line $\epsilon_\nu(q, s) = 0$ about vertex 3 (the shaded regions in Fig. 2). It can easily be seen that the condition of the straight line $\epsilon_\nu(q, s) = 0$ passing through vertex 3 may be written in the form $\gamma + \delta = \pi/3$. Thus, the computation of integrals over (q, s) should be carried out separately for $\gamma + \delta < \pi/3$ and $\gamma + \delta > \pi/3$. Simple calculations give in the case $\gamma + \delta < \pi/3$

$$\sum_\nu \rho_{\nu\nu} = 2\sqrt{3}\,\Omega \iint\limits_{\epsilon_\nu < 0} dqds = 6\,\Omega \frac{\sin\gamma}{\sin 3\gamma} \frac{\sin^2\delta \sin^2(\gamma + \delta)}{\sin^4(\pi/3)}. \qquad (3.7)$$

It is convenient to introduce for the shell occupation characteristic the quantity

$$\chi = 1 - N/\Omega, \qquad (3.8)$$

which at $0 < N < 2\Omega$ varies within the limits $-1 < \chi < 1$. The equation (2.7), in view of (3.7), leads to the following relation between δ and χ

$$1 - \chi = 6 \frac{\sin\gamma}{\sin 3\gamma} \frac{\sin^2\delta \sin^2(\gamma + \delta)}{\sin^4(\pi/3)}; \qquad (\gamma + \delta < \frac{\pi}{3}) \qquad (3.9)$$

Analogous calculations for $\gamma + \delta > \pi/3$ result in a similar formula, but with substitution in (3.9)

$$\chi \to -\chi$$

$$\gamma \to \bar{\gamma} \equiv (\pi/3) - \gamma; \quad \delta \to \bar{\delta} \equiv (\pi/3) - \delta. \qquad (3.10)$$

The right-hand part of Eq. (2.15) is calculated in a similar way

$$\sum_{\nu} \rho_{\nu\nu}(q_{\nu}\cos\gamma + s_{\nu}\sin\gamma) \to 2\sqrt{3}\,\Omega \iint\limits_{\epsilon_{\nu}<0} dqds\ (q\cos\gamma + s\sin\gamma).$$

As a result, we get for $\gamma + \delta < \pi/3$

$$\beta = \frac{3}{2}\Omega\,\frac{\sin\gamma}{\sin 3\gamma}\,\frac{\sin^2\delta\,\sin^2(\gamma+\delta)}{\sin^4(\pi/3)}\left[\frac{\cos\delta\,\cos(\gamma+\delta)}{\cos^2(\pi/3)} - \frac{\sin\delta\,\sin(\gamma+\delta)}{\sin^2(\pi/3)}\right] \quad (3.11)$$

$$(\gamma + \delta < \frac{\pi}{3}).$$

Equation (3.9) defines parameter δ as a function of the occupation factor χ. Eliminating δ from (3.11) we get*

$$\beta = \Omega(1-\chi)\left[\cos\gamma - \sqrt{\frac{\sin 3\gamma}{6\,\sin\gamma}(1-\chi)}\right];\ (\gamma+\delta \leqslant \pi/3) \quad (3.12)$$

yielding the extremum value β at fixed γ*. The definition of the region in which (3.12) is valid after the exclusion of δ is as follows

$$\tan\gamma \leq \sqrt{3}\,[(1+\chi)/(3-\chi)]\ ;\ (\gamma+\delta\leqslant\pi/3). \quad (3.13)$$

In the region $\gamma + \delta > \pi/3$ the equation for β analogous to (3.12) is of the form (after substituting (3.10))

$$\beta = \Omega(1+\chi)\left[\cos\bar{\gamma} - \sqrt{\frac{\sin 3\bar{\gamma}}{6\,\sin\bar{\gamma}}(1+\chi)}\right]\ ;\ (\gamma+\delta>\pi/3). \quad (3.14)$$

It can easily be seen that expressions (3.12) and (3.14) coincide at the boundary $(\gamma + \delta = \pi/3)$. Note that equation (3.13) for the boundary of the two regions is invariant with respect to the replacement $\gamma \to \bar{\gamma}$; $\chi \to -\chi$.

Substituting the extremum values $\beta(\gamma)$ (3.12) or (3.14) in (2.10) we get the ground-state energy W as a function of parameter γ

$$W(\gamma) = \hbar\omega n N - \tfrac{1}{2}\kappa\beta^2(\gamma) - \tfrac{1}{2}\Delta^2/G. \quad (3.15)$$

In the region of large deformations considered the pairing energy Δ^2/G may be neglected compared with the deformation energy $\kappa\beta^2$, so that the absolute minimum of $W(\gamma)$ is defined by the maximum $\beta(\gamma)$. The investigation of function $\beta(\gamma)$ defined by Eqs.(3.12) and (3.4) is not very difficult. Let us state here the results.

The values $\gamma = 0$ and $\gamma = \pi/3$ are always extreme and one has to distinguish three regions of shell occupation:

(1) $-1 < \chi < -1/8$; $\gamma = 0$ - minimum;
$\gamma = \pi/3$ - maximum.
(2) $-1/8 < \chi < 1/8$; $\gamma = 0$ and $\gamma = \pi/3$ - maximum,

the absolute maximum is in $\gamma = \pi/3$ when $\chi < 0$ and in $\gamma = 0$ when $\chi > 0$. For

* When comparing the right side of (3.12) and general expression (2.27) we see that the term C_{β}, in this case, is lacking. This is due to the fact that we consider nucleons on one shell only. Taking into account matrix elements of q_{μ} connecting shells n, n ± 2 will lead to the appearance of this term and to the renormalization of equilibrium deformation β (see 2.34).

$\chi = 0$ (half-filled shell) both maxima become equal. There exists an intermediate minimum for $\gamma = \gamma(-\chi)$.

(3) $1/8 < \chi < 1$; $\gamma = 0$ - maximum; $\gamma = \pi/3$ - minimum.

The form of function $W(\gamma)$ in the above three regions of χ is illustrated in Fig. 3.

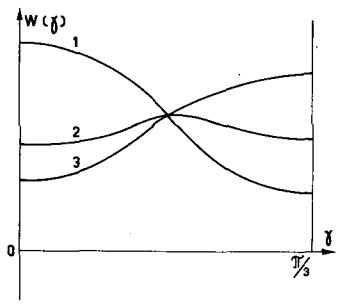

Fig. 3

Thus, the equilibrium form is always axially symmetrical. In filling the first half of the shell ($\chi > 0$) a prolate shape is advantageous ($\gamma = 0$), and in filling the second half ($\chi < 0$) it is oblate ($\gamma = \pi/3$).

3.2. Improved oscillator model

The total symmetry with respect to transformation from prolate to oblate shape ($\gamma \to \bar{\gamma}$) with simultaneous substitution of particles by holes ($\chi \to -\chi$) is a specific feature of the spherical harmonic oscillator model due to the additional degeneracy of single-particle levels in the orbital angular momentum. As is well known, good approximation to the real level scheme is attained by the inclusion of the term $D\ell^2$ yielding splitting over ℓ to the oscillator potential as well as spin-orbit interaction $C(\vec{\ell}.\vec{s})$ [7]. We shall consider these correction terms as perturbation. Such a consideration is valid, in any case, for the region of large deformations when level splitting caused by deformation is higher than that caused by terms $D\ell^2$ and $C(\vec{\ell}.\vec{s})$.

The correction of the first order to the ground state energy caused by the perturbation potential is given by the expression

$$W^{(1)} = \mathrm{Sp}\,(\hat{\rho}\hat{V}) = \sum_{\nu} \rho_{\nu\nu} \langle \nu | \hat{V} | \nu \rangle . \qquad (3.16)$$

Let us consider term $D\ell^2$ for which the diagonal matrix element is given by

$$\begin{aligned} D\langle n_x\, n_y\, n_z | \ell^2 | n_x\, n_y\, n_z \rangle &= 2D(n_x n_y + n_y n_z + n_z n_x + n) \\ &= (3/2)\, Dn^2 \,[4/9 - (q_\nu^2 + s_\nu^2)] \end{aligned} \qquad (3.17)$$

where q and s have been defined in (3.2). Substituting (3.17) into (3.16) we find

$$W^{(1)} = (3/2)\, Dn^2 \sum_{\nu} \rho_{\nu\nu} \,[(4/9) - (q_\nu^2 + s_\nu^2)] - (1/4)\, Dn^4 . \qquad (3.18)$$

For the larger symmetry of the formulas we have subtracted from $W^{(1)}$ the value $(1/4)Dn^4 = \frac{1}{2}\{W^{(1)}(N = 2\Omega) - W^{(1)}(N = 0)\}$. Using (3.4) and proceeding in

(3.18) to the integration, excluding δ by means of (3.9) we obtain in the region $\gamma + \delta < \pi/3$

$$W^{(1)} = -D\Omega^2 + \sqrt{\frac{32}{3}}\, D\Omega^2 (1-\chi)^{3/2} \sqrt{\frac{\sin\gamma}{\sin 3\gamma}} \times \left\{ \cos\gamma - \frac{3}{8} \left(1 + 2\,\frac{\sin\gamma}{\sin 3\gamma}\right) \sqrt{\frac{\sin 3\gamma}{6 \sin\gamma}\,(1-\chi)} \right\} \tag{3.19}$$

and when $\gamma + \delta \geq \pi/3$:

$$W^{(1)} = D\Omega^2 - \sqrt{\frac{32}{3}}\, D\Omega^2 (1+\chi)^{3/2} \sqrt{\frac{\sin\gamma}{\sin 3\bar{\gamma}}} \times \left\{ \cos\bar{\gamma} - \frac{3}{8} \left(1 + 2\,\frac{\sin\bar{\gamma}}{\sin 3\bar{\gamma}}\right) \sqrt{\frac{\sin 3\bar{\gamma}}{6 \sin\bar{\gamma}}\,(1+\chi)} \right\}; \tag{3.20}$$

from (3.19) and (3.20) follows the relation

$$W^{(1)}(\chi, \gamma) = -W^{(1)}(-\chi, \bar{\gamma}) \tag{3.21}$$

which confirms the fact that the term $D\ell^2$ eliminates the symmetry with respect to the replacement $\gamma \to \bar{\gamma}$; $\chi \to -\chi$.

Function $W^{(1)}(\gamma)$ defined by (3.19) and (3.20) has the following specific features (parameter $D < 0$ [7]):

(1) $-1 < \chi < -\frac{1}{2}$. Minima when $\gamma = 0$ and $\pi/3$ intermediate maximum for $\gamma \neq 0, \pi/3$.

(2) $-\frac{1}{2} < \chi < \frac{1}{2}$. Minimum when $\gamma = 0$, maximum when $\gamma = \pi/3$.

(3) $\frac{1}{2} < \chi < 1$. Maxima when $\gamma = 0$ and $\gamma = \pi/3$ intermediate minimum for $\gamma \neq 0, \pi/3$ (see Fig. 4).

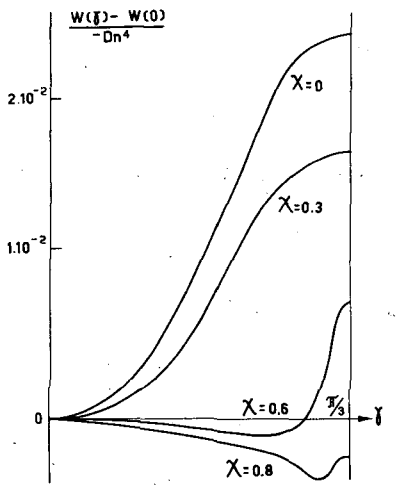

Fig. 4

It should be pointed out here that except for the narrow regions $\sim 0,75 < |\chi| < 1$ it holds everywhere that $W^{(1)}(\pi/3) > W^{(1)}(0)$.

As follows from the above analysis, the term $D\ell^2$ eliminates the symmetry between prolate and oblate deformations and expands the stability region of prolate nuclei. The minimum $W^{(1)}(\gamma)$ at the intermediate values of γ, as may be seen from the plots in Fig. 4, is hardly marked. It becomes notable only in the very beginning of shell filling ($\chi \approx 0.8 - 0.9$) where the value $W^{(1)}$ is itself small. Therefore, the term $D\ell^2$ may lead to the equilibrium non-axial deformation only when the ratio $|D|/\kappa$ is large enough. (At $\chi = 0.8$ it is necessary that $|D|/\kappa > 0.3$). According to the Nilsson model we put $|D| \approx 0.025\ \hbar\omega$. The parameter of quadrupole interaction estimated in [8] is $\kappa = 250\ A^{-1}$ MeV*. Then for the parameter ratio we find $|D|/\kappa \approx 0.1$ (for A = 125), which is too small for the appearance of equilibrium γ-deformation.

Thus, the main effect caused by the term $D\ell^2$ is the elimination of symmetry with respect to the oblate and prolate shapes. Figure 5 illustrates the energetical difference between the oblate and prolate states for various parameter values $|D|/\kappa$. As one may see from the plots, when $|D|/\kappa \approx 0.3$ the region of oblate deformations practically disappears.

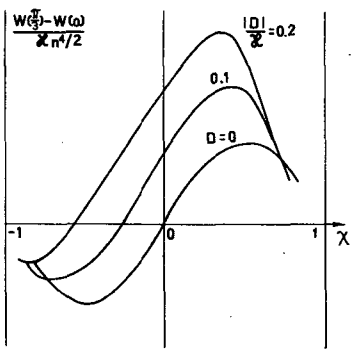

Fig. 5

Spin-orbit interaction $c(\vec{\ell}\cdot\vec{s})$ essentially changes the order of single-particle levels, but its contribution to the ground-state energy in the first order of perturbation theory is zero.

3.3. System with two types of nucleons

Till now we have been concerned with the system consisting of nucleons of a single type. Let us extend our results to the system of neutrons and protons. Cooper pairing occurs independently in neutron and proton subsystems. Quadrupole interaction should be considered identical for any pair of nucleons.** Then it can easily be seen that the basic formulae require only significant modifications. Equation (2.4) is valid if by Q_μ we mean the total quadrupole moment of neutrons and protons. Equations (2.7) and (2.8) are

* Our value κ differs from the parameter X in paper [8] by the factor 9/4, namely $\kappa = (9/4)X$.

** For the discussion of the possible difference in quadrupole interaction constants see [9].

now valid separately for neutrons and protons and define $\Delta_{n(p)}$ and $\lambda_{n(p)}$ respectively

$$\sum_{\nu} \rho^{n(p)}_{\nu\nu} = \tfrac{1}{2} \sum_{\nu_{n(p)}} \left(1 - \frac{\epsilon_\nu}{(\epsilon^2_\nu - \Delta^2_{n(p)})^{\frac{1}{2}}}\right) \equiv N_{n(p)}$$
$$G_{n(p)} \sum_{\nu_{n(p)}}^{\frac{1}{2}} \frac{1}{(\epsilon^2_\nu - \Delta^2_{n(p)})^{\frac{1}{2}}} = 1 \qquad (3.22)$$

The equation for Q_μ instead of (2.9) takes the form

$$Q_\mu = Q^n_\mu + Q^p_\mu = \sum_{\nu_p} \rho^n_{\nu\nu} \langle \nu | q_\mu | \nu \rangle + \sum_{\nu_p} \rho^p_{\nu\nu} \langle \nu | q_\mu | \nu \rangle \qquad (3.24)$$

and presently for the ground-state energy (2.10) we have

$$W(Q) = \sum_{\nu_n} \rho^n_{\nu\nu} (\epsilon_\nu + \lambda^n) + \sum_{\nu_p} \rho^p_{\nu\nu} (\epsilon_\nu + \lambda^p) + \tfrac{1}{2}\kappa \sum_\mu Q^*_\mu Q_\mu \qquad (3.25)$$
$$- \Delta^2_n / 2G_n - \Delta^2_p / 2G_p.$$

The treatment of the problem of nuclei transition from the spherical to the deformed shape has not essentially changed if we assume that the parameters (a, b, $B(\gamma)$, C) entering (2.36) and (2.37) are determined as some values averaged over neutrons and protons. Let us proceed, therefore, to the problem of equilibrium shape for the two-component nucleus in the region of large deformations (oscillator model).

As is directly seen from the structure of Eq. (3.24), equilibrium deformation β is determined by the expression

$$\beta(\gamma) = \beta_n(\gamma, \chi_n) + \beta_p(\gamma, \chi_p) \qquad (3.26)$$

where the function $\beta(\gamma, \chi)$ is given by the right sides of (3.12) and (3.14). The ground-state energy as a function of γ now takes the form of (cf. (3.15))

$$W(\gamma) = (\hbar\omega n N)_n + (\hbar\omega n N)_p - \tfrac{1}{2}\kappa\beta^2(\gamma) - \Delta^2_n / 2G_n - \Delta^2_p / 2G_p \qquad (3.27)$$

and (when neglecting pairing) is defined by the behaviour of $\beta(\gamma)$ (3.26). (We have not yet considered corrections to the oscillator model). The functions $\beta_{n(p)}(\gamma, \chi_{n(p)})$ (3.12), (3.14) in the first half of the shell ($\chi > 0$) are monotonously decreasing functions of γ, and in the second half monotonously increasing ones (except the narrow region of non-monotony near the middle of the shell). If neutrons and protons occupy the same halves of the shells then the sum of functions (3.26) will possess the same properties and, consequently, the equilibrium shape will be axially symmetrical. But if neutrons and protons occupy different halves of the shells ($\chi_n \chi_p < 0$) then function (3.26) may have a maximum for intermediate values of γ, thus leading to the non-axial equilibrium shape. We shall not carry out the cumbersome analysis of the two-parameter function (3.26) and find its maximum, restricting ourselves to estimating the parameter region only, at which there exists an intermediate maximum of $\beta(\gamma)$ (i.e. minimum $W(\gamma)$).

The function $W(\gamma)$ undoubtedly possesses a minimum for $0 < \gamma < \pi/3$ if the points $\gamma = 0$ and $\gamma = \pi/3$ (always extreme) correspond to maxima of $W(\gamma)$. Let us find, therefore, the region of χ_n and χ_p, for which $W(\gamma)$ possesses a maximum when $\gamma = 0, \pi/3$. These regions, whose boundaries are defined by means of (3.27), (3.12) and (3.14) from the equations

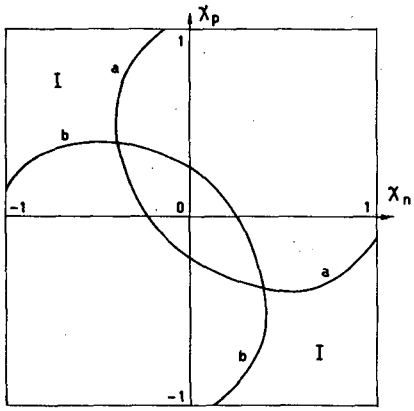

Fig. 6

$$[(1/\gamma)(dW/d\gamma)]_{0=\gamma} = 0 \; ; \left[(1/\bar{\gamma})(dW/d\bar{\gamma})\right]_{\bar{\gamma}=0} = 0$$

are illustrated in Fig. 6. As can be seen from the plots, there in fact exist such values of χ_n, χ_p for which the function $W(\gamma)$ possesses a maximum both for $\gamma = 0$ and $\gamma = \bar{\gamma}/3$ (region I limited by curves a, b). Such a case takes place when one sort of nucleon occupies the first third of the shell ($1 > \chi > 0.4$ or $0 < N/2\Omega < 0.3$), and the second occupies the last third ($-0.4 > \chi > -1$ or $1 > N/2\Omega > 0.7$). It should be noted that such an occupation of neutron and proton shells is not likely to occur in the known region of the deformed nuclei. The taking into account of term $D\ell^2$ leads to an inessential variation of the region of χ_n, χ_p with an intermediate minimum of $W(\gamma)$. Thus, in the two-component system, too, the appearance of an axially non-symmetrical equilibrium shape is in practice hardly probable.

It is clear that we cannot expect a death sentence for the γ-deformation from the qualitative analysis given above. It would be desirable to perform quantitative calculations using the real scheme of one-particle levels for those regions where the appearance of stable γ-deformation is most probable.

4. THE ROTATION OF DEFORMED NUCLEI. MOMENT OF INERTIA

4.1. Formulation of the problem

Let us proceed now to the consideration of collective excitations of nuclei, i.e. to the problem of the time-dependent self-consistent field. First, we shall consider the case of rotation of the deformed nuclei. This question refers in fact to the problem of finding the time-dependent self-consistent field, but the process of its solutions involves great difficulties. The so-called cranking-model has been generally used (or other equivalent methods) and results in the apparently correct value of the moment of inertia in case of rotation. Rotation of the quantum system leads to the additional energy

$$E_J^{rot} = \frac{\hbar^2 J(J+1)}{2\mathcal{J}} \qquad (4.1)$$

where J is the quantum number of the angular momentum and $\mathcal{J}$ the moment of inertia of the system. The separation of terms of the type of (4.1) from the Hamiltonian of the nucleon system, using a reasonable approximation, is a very complicated problem. The main complication is that perturbation theory must be constructed so that the rule for angular momenta addition (algebraically non-additive preservation law) is fulfilled. This difficulty may be overcome in the following way. Let us consider a rotational state with very large J, then the rotation is quasi-classical, and we may speak of the definite direction of the rotation axis (x) and instead of (4.1) we can write, assuming $J \approx J_x$,

$$E_J^{rot} \approx \hbar^2 J_x^2 / 2\mathcal{J} = \tfrac{1}{2}\mathcal{J}\,\Omega_x^2 \tag{4.2}$$

where $\Omega_x = \hbar J_x / \mathcal{J}$ is the angular velocity of rotation. Thus at large J the problem is reduced to the finding of the energy of state with a definite angular velocity. But since the moment of inertia is a coefficient independent of J, and having found it from (4.2), we simultaneously also define (4.1). The substitution of (4.1) by (4.2) is the main idea of the cranking model. We make use of an equivalent method.

We shall look for the lowest state of the system at a fixed average value of the angular momentum about the axis of rotation (perpendicular to the nucleus symmetry axis). For this purpose we shall add the term

$$H_\Omega = -\Omega \hat{J}_x = -\Omega \sum_{12} \langle 1 | j_x | 2 \rangle a_1^+ a_2 . \tag{4.3}$$

to the Hamiltonian, where $\hat{J}_x$ is the operator of angular momentum and Ω is the Lagrangian multiplier which will be determined later from the condition $\langle \hat{J}_x \rangle = J_x$.

4.2. General expression for moment of inertia

Let us consider (4.3) as perturbation. Then the problem is reduced to the calculation of the correction to the ground state of the system under external perturbation. Since the angular momentum operator changes its sign under time reversal, i.e.

$$\langle 1 | j_x | 2 \rangle = -\langle \tilde{2} | j_x | \tilde{1} \rangle, \tag{4.4}$$

then, according to the classification of perturbation (1.21), we have in this case

$$V_{12}^{(-)} = -\Omega \langle 1 | j_x | 2 \rangle ; \; V_{12}^{(+)} = 0; \tag{4.5}$$

therefore, we may put $Z_{12}^{(+)} = 0$. To define $Z^{(-)}$ we have from (1.25), (1.2) and (4.5)

$$E_{12} Z_{12}^{(-)} - \xi_{12}^{(-)} \Delta_{12}^{(-)} + \eta_{12}^{(-)} U_{12}^{(-)} = \Omega\, \eta_{12}^{(-)} \langle 1 | j_x | 2 \rangle . \tag{4.6}$$

The correction to the ground-state energy according to (1.31) is, in this case, equal to

$$\langle H \rangle^{(2)} = \tfrac{1}{2} \Omega^2 \sum_{12} \eta_{12}^{(-)} \langle 1 | j_x | 2 \rangle \, (Z_{12}^{(-)*} / \Omega). \tag{4.7}$$

The parameter Ω is found from the equation $\langle \hat{J}_x \rangle = J_x$, i.e. in view of (4.3) and (1.32)

$$\langle H_\Omega \rangle = -\Omega^2 \sum_{12} \eta_{12}^{(-)} \langle 1|j_x|2\rangle (Z_{12}^{(-)*}/\Omega) = -\Omega J_x. \tag{4.8}$$

From (4.7) and (4.8) it follows (see (4.2)) that

$$\langle H \rangle^{(2)} = \tfrac{1}{2} J_x^2 / \mathcal{J}, \tag{4.9}$$

where the moment of inertia $\mathcal{J}$ is determined by the following expression

$$\mathcal{J} = \sum_{12} \eta_{12}^{(-)} \langle 2|j_x|1\rangle (Z_{12}^{(-)}/\Omega). \tag{4.10}$$

For analysis, it would be convenient to represent $\mathcal{J}$ in separate terms. Using (4.6) we write

$$\mathcal{J} = \mathcal{J}^{(1)} + \mathcal{J}^{(2)} + \mathcal{J}^{(3)} \tag{4.11}$$

where

$$\mathcal{J}^{(1)} = \sum_{12} |\langle 1|j_x|2\rangle|^2 (\eta_{12}^{(-)2}/E_{12}) \tag{4.12}$$

$$\mathcal{J}^{(2)} = \sum_{12} \langle 2|j_x|1\rangle (\Delta_{12}^{(-)}/\Omega)(\xi_{12}^{(-)}\eta_{12}^{(-)}/E_{12}) \tag{4.13}$$

$$\mathcal{J}^{(3)} = -\sum_{12} \langle 2|j_x|1\rangle (U_{12}^{(-)}/\Omega)(\eta_{12}^{(-)2}/E_{12}). \tag{4.14}$$

The consideration of the rotation by means of the adiabatic perturbation theory (the usual form of the cranking model) gives only the first term in the moment of inertia [5]. This method is equivalent to taking into account the diagonal corrections to the average values only, i.e. $\langle a_1^+ a_1 \rangle^{(2)}$ and $\langle a_1^+ a_2^+ \rangle^{(2)}$ which may be done within the framework of the usual Bogolyubov u, v- transformation. The consideration of non-diagonal averages results in two additional terms. The term $\mathcal{J}^{(2)}$ takes account of the effect of rotation on Cooper pairing* (this is expressed by the appearance of $\Delta_{12}^{(-)}$). The last term, as follows from its structure, describes the variation of the self-consistent field under rotation. Note that so far we have obtained the explicit expression only for the main term $\mathcal{J}^{(1)}$ (4.12). Expressions (4.13) and (4.14) contain the function $Z_{12}^{(-)}$, which must be found in order to solve integral equation (4.6).

4.3. Estimation of the basic term $\mathcal{J}^{(1)}$

For quantitative calculations of the moment of inertia for real even-even nuclei we may restrict ourselves to the term $\mathcal{J}^{(1)}$. Using the explicit expression for $\eta^{(-)2}$ (see second footnote of 1.4) we have from (4.12) (considering $\Delta^{(0)}$ to be constant)

$$\mathcal{J}^{(1)} = \sum_{12} |\langle 1|j_x|2\rangle|^2 [E_1 E_2 - \epsilon_1 \epsilon_2 - \Delta^2]/\, 2E_1 E_2 (E_1 + E_2). \tag{4.15}$$

To estimate (4.15) let us use the method proposed in [6]. At the fixed index 1

* The term $\mathcal{J}^{(2)}$ was first obtained by MIGDAL, who used the method of Green's functions [6].

in the sum of (4.15) the second index 2 takes just the few values permitted by the selection rules for the matrix element of j_x. As can easily be checked the last factor in the sum at fixed difference $\epsilon_1 - \epsilon_2$ has an abrupt maximum over variable ϵ_1 whose width is Δ (or $|\epsilon_1 - \epsilon_2|$ if $|\epsilon_1 - \epsilon_2| > \Delta$) at the Fermi surface and

$$\int d\epsilon_1 [E_1 E_2 - \epsilon_1 \epsilon_2 - \Delta^2] / 2E_1 E_2 (E_1 + E_2) = 1 - g[(\epsilon_1 - \epsilon_2)/2\Delta] \qquad (4.16)$$

where

$$g(x) = \frac{\operatorname{arsh} x}{(1+x^2)^{\frac{1}{2}}}. \qquad (4.17)$$

If there is a sufficient number of levels in the region Δ (or $|\epsilon_1 - \epsilon_2|$) (this condition is practically valid for nuclei) one may make the following replacement in (4.15)

$$\eta_{12}^{(-)2} = [E_1 E_2 - \epsilon_1 \epsilon_2 - \Delta^2] / 2E_1 E_2 (E_1 + E_2) \rightarrow \left[1 - g\left(\frac{\epsilon_1 - \epsilon_2}{2\Delta}\right)\right] \delta(\epsilon_1). \qquad (4.18)$$

As a result we get

$$\mathcal{J}^{(1)} = \sum_{12} |\langle 1 | j_x | 2 \rangle|^2 \left[1 - g\left(\frac{\epsilon_1 - \epsilon_2}{2\Delta}\right)\right] \delta(\epsilon_1). \qquad (4.19)$$

In the model of the oscillator potential the matrix element $\langle 1 | j_x | 2 \rangle$ for the two types of transitions is different from zero:

(1) Transitions inside a single oscillator shell ("near transitions") for which

$$d' = |\epsilon_1 - \epsilon_2| = \hbar |\omega_\perp - \omega_z| \approx \hbar \omega \, \delta R/R \qquad (4.20)$$

(here ω_z, $\omega_\perp$ are oscillator frequencies in different directions, and $\delta R/R$ is the deformation of the potential);

(2) Transitions across the shell ("distant transitions") for which

$$d'' = |\epsilon_1 - \epsilon_2| \approx 2\hbar\omega. \qquad (4.20')$$

The following inequalities are then valid for near and distant transitions

$$d'/2\Delta \lesssim 1; \quad d''/2\Delta \gg 1. \qquad (4.20'')$$

The matrix elements of transitions are inversely proportional to the corresponding values of d. Taking account of all the above, we get the following estimation for (4.19)

$$\mathcal{J}^{(1)} \approx \mathcal{J}_0 [1 - g(d'/2\Delta)], \qquad (4.21)$$

where the rigid body moment of inertia is denoted by $\mathcal{J}_0$, which may be formally obtained from (4.15) or (4.19) in the limit $\Delta \rightarrow 0$. As follows from (4.21), Cooper pairing considerably decreases the moment of inertia. Although the above evaluations refer to the oscillator potential, qualitatively, they are also valid for other models.

A number of authors [10, 11] have carried out detailed calculations of the moment of inertia using formula (4.15) on the basis of the semi-empirical

Nilsson level scheme [7]. The results for all nuclei are in good agreement with the experimental values of $\mathcal{J}$ (deviations, as a rule, are less than 10%).

4.4. Hydrodynamical limit and gauge invariance

From a purely theoretical viewpoint one should not be restricted only to the term $\mathcal{J}^{(1)}$ in the moment of inertia in spite of the good agreement with experiment. In fact, in the limits of a very strong pairing strength ($\Delta \gg d', d''$), when the size of the Cooper pair $r_0 \sim \hbar v/\Delta$ (v - velocity at Fermi surface) becomes much less than the nuclear radius, the hydrodynamical equations of an ideal liquid should be valid, resulting in a moment of inertia

$$\mathcal{J}_{\text{irrot}} \approx \mathcal{J}_0 \, (\delta R/R)^2 . \tag{4.22}$$

Expression (4.15) or (4.19) in this limit becomes zero. The main role in this case belongs to the term $\mathcal{J}^{(2)}$ which yields (4.22)*.

Note that if interaction between nucleons is approximated by the pairing interaction (2.6), then $\Delta_{12}^{(-)} = 0$ and the term $\mathcal{J}^{(2)}$ becomes zero. Thus, when using model interaction (2.6), the transition to hydrodynamics at $\Delta \to 0$ is lacking. This is associated with the fact that interaction (2.6) is gauge non-invariant. For gauge-invariant interaction the "diagonal" matrix elements $\langle 1\tilde{1}|G|\tilde{1}'1'\rangle$ are connected with the "non-diagonal" $\langle 1\tilde{2}|G|\tilde{2}'1'\rangle$, so that the latter cannot be randomly considered to be equal to zero, as in (2.6). Let us consider, for instance, central short-range forces

$$G(\vec{r}_1 \vec{r}_2) = -G\delta(\vec{r}_1 - \vec{r}_2). \tag{4.23}$$

In this case equation (1.19'') for Δ can be transformed to

$$G \sum_1 \frac{\varphi_1^*(\vec{r})\,\varphi_1(\vec{r})}{2E_1} = 1. \tag{4.24}$$

For an arbitrary function of the space coordinates $\phi(\vec{r})$, in view of (4.23) and (4.24), one may easily get the following equation

$$\phi_{12}\,\gamma_2 = -\sum_{1'2'} \langle 1\tilde{2}|G|\tilde{2}'1'\rangle (E_{\tilde{1}'\tilde{2}'}/4E_{1'}E_{2'}) \phi_{1'2'}\gamma_{2'} , \tag{4.25}$$

yielding the relation between the different matrix elements of the interaction. (The value $\gamma_2 = -\gamma_{\tilde{2}}$ has been defined in (1.19'')).

The connection between the gauge-invariance of the interaction and the possibility of passing over to hydrodynamics has a simple physical meaning. In the hydrodynamic description macroscopically small parts of the liquid move relatively to each other in such a way that their intrinsic structure stays the same. In particular, the interaction between particles does not change. In other words a local (in the macroscopic sense) Galileian invariance exists in the liquid.

In a Galileian transformation the single-particle wave function transforms according to

$$\Psi(\vec{r}) \to \Psi(\vec{r})\, e^{i\vec{p}\cdot\vec{r}}, \tag{4.25'}$$

* This was shown by MIGDAL [6] who also obtained the explicit expression for $\mathcal{J}^{(2)}$ in an oscillator model.

where $\vec{P}/m$ is the relative velocity of the two frames of reference. If we have local Galileian invariance in the system, then $\vec{P}$ is a function of the space coordinates and hence the transformation (4.25') actually coincides with the gauge transformation

$$\Psi(\vec{r}) \to \Psi(\vec{r})\, e^{i\chi(\vec{r})} \qquad (4.25'')$$

where $\chi(\vec{r})$ is an arbitrary function of the space coordinates. Thus the interaction matrix element $\langle 1\tilde{2}|G|\tilde{2}'1'\rangle$ in a hydrodynamical system must be invariant under transformations (4.25").

The model Hamiltonian (2.6) is rather poor in the sense that it only describes the interaction of pairs at rest.* When the pair moves as a whole the interaction is switched off. Naturally it is not possible to pass over to the hydrodynamic limit for an arbitrarily large interaction of such a type.

4.5. Estimation of the term $\mathcal{J}^{(2)}$

Let us transform the integral equation (4.6) making use of (4.25). For simplicity let us neglect the variation of the self-consistent field under rotation and assume $U_{12}^{(-)} = 0$. Then, using the definition of $\Delta^{(-)}$ (1.36), we get from (4.6)

$$\Delta_{12}^{(-)} + \sum_{1'2'} \langle 1\tilde{2}|G|\tilde{2}'1'\rangle\, (\xi_{1'2'}^{(-)2}/E_{1'2'})\Delta_{1'2'}^{(-)} \qquad (4.26)$$

$$= -\Omega \sum_{1'2'} \langle 1\tilde{2}|G|\tilde{2}'1'\rangle\, (\xi_{1'2'}^{(-)}\eta_{1'2'}^{(-)}/E_{1'2'})\, \langle 1'|j_x|\tilde{2}\rangle.$$

In the case of central interaction the value $\Delta^{(-)}$ is a function of the space coordinates, so that for $\Delta_{12}^{(-)}$ ($\to \gamma_2\, \phi_{12}$) the equality (4.25) is valid. As a result, equation (4.26) may be reduced to the form

$$\sum_{1'2'} \langle 1\tilde{2}|G|\tilde{2}'1'\rangle \left\{ \left(\frac{E_{1'2'}}{4E_1E_{2'}} - \frac{\xi_{1'2'}^{(-)2}}{E_{1'2'}} \right) \Delta_{1'2'}^{(-)} - \Omega \frac{\xi_{1'2'}^{(-)}\eta_{1'2'}^{(-)}}{E_{1'2'}} \langle 1'|j_x|2'\rangle \right\} = 0$$

from which, after substituting $\xi^{(-)}, \eta^{(-)}$ and using (4.23), we obtain

$$\sum_{12} \varphi_2^*(\vec{r})\, \varphi_1(\vec{r}) [1/E_1E_2(E_1+E_2)] \{ (\epsilon_1-\epsilon_2)^2 \gamma_2 \Delta_{12}^{(-)} - 2\Omega\Delta\, (\epsilon_1-\epsilon_2) \langle 1|j_x|2\rangle \} = 0. \qquad (4.27)$$

Note that $i(\epsilon_1-\epsilon_2)\langle 1|j_x|2\rangle \equiv j_{12}^x$ is the matrix element of a function of the space coordinates $j_{12}^x = y(\partial u/\partial z) - z(\partial u/\partial y)$. Therefore, multiplying (4.27) by $j_{12}^{x*}(\vec{r})$ and integrating over $\vec{r}$, we obtain

$$\sum_{12} [1/E_1E_2(E_1+E_2)] \{ (\epsilon_1-\epsilon_2)^2 j_{21}^x\, i\Delta_{12}^{(-)}\gamma_2 - 2\Omega\Delta |j_{12}^x|^2 \} = 0. \qquad (4.28)$$

From (4.28) it is clear that equation (4.28) is satisfied if we assume that

$$\Delta_{12}^{(-)}\gamma_2 = -i\Omega\, j_{12}^x\, D/2\Delta \qquad (4.29)$$

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* In a macroscopic system the above statement follows directly from the form of the interaction, analogous to (2.6), namely $\langle p - p|G|-p'\, p'\rangle$.

where

$$D = \sum_{12} \frac{|j^x_{12}|^2}{E_1 E_2 (E_1 + E_2)} \Big/ \sum_{12} \frac{|j^x_{12}|^2}{E_1 E_2 (E_1 + E_2)} \left(\frac{\epsilon_1 - \epsilon_2}{2\Delta}\right)^2. \tag{4.30}$$

From (4.13) we then find

$$\mathcal{J}^{(2)} = \left(\sum_{12} \frac{|j^x_{12}|^2}{2E_1 E_2 (E_1 + E_2)}\right)^2 \Big/ \sum_{12} \frac{|j^x_{12}|^2}{2E_1 E_2 (E_1 + E_2)} \left(\frac{\epsilon_1 - \epsilon_2}{2\Delta}\right)^2. \tag{4.31}$$

Let us now assume the self-consistent potential of the deformed nucleus to be of the form

$$U = U(r_\delta^2) \tag{4.32}$$

where

$$r_\delta^2 = (x^2 + y^2)/(1 - \tfrac{2}{3}\delta R/R) + z^2/[1 + (4/3)(\delta R/R)]. \tag{4.33}$$

Then for the function j^x we obtain (accurate up to terms linear in $\delta R/R$)

$$j^x = y(\partial U/\partial z) - z(\partial U/\partial y) = -4yzU'(r^2)(\delta R/R). \tag{4.34}$$

Let us note also that in the case when $\Delta \gg d', d''$ one may substitute in (4.31)

$$1/E_1 E_2 (E_1 + E_2) \to (1/\Delta^2)\delta(\epsilon_1). \tag{4.35}$$

With the aid of (4.34) and (4.35) it is easy to see that the right side of (4.31) tends to the finite limit when $\Delta \to \infty$ and $\mathcal{J}^{(2)} \sim (\delta R/R)^2$ as it should be in the hydrodynamic limit.

5. VIBRATIONS OF DEFORMED NUCLEI

5.1. Model with quadrupole interaction

Let us apply the equations of the time-dependent self-consistent field to the consideration of the eigenvibrations of the deformed nuclei. We shall restrict ourselves to the consideration of quadrupole-type vibrations of nuclei with axially symmetrical deformations. Two types of vibrations are known to exist in this case. One type conserves the axial symmetry, β-vibrations, and in the second type the axial symmetry is disturbed, γ-vibrations. To solve the system of homogeneous, integral equations (1.35) we make a simplifying assumption as to the interaction matrix elements. To start with, we consider only the quadrupole interaction (2.1'), which naturally plays a decisive role in vibrations of the type considered. Then in (1.35) one must put

$$\Delta^{(\pm)}_{12} = \epsilon^{(-)}_{12} = 0, \tag{5.1}$$

$$\epsilon^{(+)}_{12} = -\kappa \sum_\mu q^*_\mu(12)\, Q_\mu,$$

where

$$Q_\mu = \sum_{12} q_\mu(12)\ \eta_{12}^{(+)} Z_{12}^{(+)}. \tag{5.2}$$

After substituting (5.1) into (1.35) we get

$$\omega Z_{12}^{(-)} = \sum_{12} Z_{12}^{(+)} - \kappa \eta_{(12)}^{(+)} \sum_\mu q_\mu^*(12)\, Q_\mu \tag{5.3}$$

$$\omega Z_{12}^{(+)} = E_{12} Z_{12}^{(-)}$$

or after eliminating $Z^{(-)}$

$$(E_{12}^2 - \omega^2) Z_{12}^{(+)} = \kappa \eta_{12}^{(+)} E_{12} \sum_\mu q_\mu^*(12)\, Q_\mu. \tag{5.4}$$

From (5.4) and (5.2) we obtain the system of algebraic equations for Q_μ

$$Q_\mu = \kappa \sum_{12} [E_{12} \eta_{12}^{(+)2} / (E_{12}^2 - \omega^2)]\, q_\mu(12) \sum_{\mu'} q_{\mu'}^*(12)\, Q_{\mu'}. \tag{5.5}$$

The factor $E_{12}\eta_{12}^{(+)2}/(E_{12}^2 - \omega^2)$ in the sum in (5.5) is a positive function with maximum at the Fermi surface while $q_\mu(12)$ is the alternating function. Hence, at not very large deformations ($\delta R/R \ll 1$) one may consider that the main term on the right-hand side of (5.5) is a term with $\mu' = \mu$. In this approximation we have

$$Q_\mu = \kappa \sum_{12} [E_{12} \eta_{12}^{(+)2} / (E_{12}^2 - \omega^2)]\, |q_\mu(12)|^2\, Q_\mu, \tag{5.6}$$

from which we get the following dispersion equation for eigenfrequency ω

$$\kappa \sum_{12} |q_\mu(12)|^2 [E_{12} \eta_{12}^{(+)2} / (E_{12}^2 - \omega^2)] = 1. \tag{5.7}$$

It is convenient to transform Eq. (5.7) somewhat. We introduce the notation

$$\sigma = \kappa \sum_{12} |q_\mu(12)|^2\ (\eta_{12}^{(+)2} / E_{12}) \tag{5.8}$$

$$B(\omega) = (\kappa/\sigma) \sum_{12} |q_\mu(12)|^2\ (\eta_{12}^{(+)2}/E_{12})\ [4\Delta^2/(E_{12}^2 - \omega^2)]. \tag{5.9}$$

Then Eq. (5.7) can be rewritten in the form

$$(\omega/2\Delta)^2 = [1/\sigma - 1]/B(\omega). \tag{5.10}$$

The right side of (5.10) has a simple physical meaning. The quantity $(1/\sigma - 1)$ is the restoring force coefficient (cf.(2.13)) and $B(\omega)$ the inertial parameter for the vibrations in question.

We shall use the same procedure in considering σ and $B(\sigma)$ as when calculating the momentum of inertia. If we keep the difference $d = \epsilon_1 - \epsilon_2$ fixed, the quantity $\eta_{12}^{(+)2}/E_{12}$ as a function of ϵ_1 has a sharp maximum of the width Δ (or d if $d > \Delta$) at the Fermi surface. Assuming that $|q_\mu(12)|^2$ changes little in this interval*, and taking into account that the integral of $\eta_{12}^{(+)2}/E_{12}$ over ϵ_1 is equal to unity, we may make a replacement in (5.8):

$$\eta_{12}^{(+)2}/E_{12} \to \delta(\epsilon_1) . \tag{5.11}$$

The same arguments allow an analogous substitution in (5.9)

$$\frac{4\Delta^2 \eta_{12}^{(+)2}}{E_{12}(E_{12}^2 - \omega^2)} \to \frac{1 - g(\ell)}{\ell^2} \delta(\epsilon_1) \tag{5.12}$$

where $g(x)$ has been defined in (4.17) and

$$\ell = \sqrt{(d^2 - \omega^2)/4\Delta^2} . \tag{5.13}$$

The quantity ℓ becomes purely imaginary for $\omega^2 > d^2 = (\epsilon_1 - \epsilon_2)^2$. For imaginary arguments the function $g(x)$ has the form

$$g(ix) = \begin{cases} \dfrac{\arcsin x}{x\sqrt{1-x^2}} ; & (x^2 < 1) \\ -\dfrac{\operatorname{arch} x}{x\sqrt{1-x^2}} ; & (x^2 > 1) . \end{cases} \tag{5.14}$$

It can be seen from (5.14) that $g(ix)$ has a singularity when x tends to unity from below.

Substituting (5.13) and (5.14) in (5.8) and (5.9) we obtain

$$\sigma = \kappa \sum_{12} \delta(\epsilon_1) |q_\mu(12)|^2 \tag{5.15}$$

$$B(\omega) = (\kappa/\sigma) \sum \delta(\epsilon_1) |q_\mu(12)|^2 (1 - g(\ell))/\ell^2 . \tag{5.16}$$

The quantity (5.16) can be expressed as

$$B(\omega) = [(1 - g(\ell))/\ell^2] \tag{5.17}$$

* Otherwise it is necessary to consider values of $|q_\mu(12)|$ averaged over the interval Δ (or d).

where the symbol [ ] means averaging with the weight $\kappa\delta(\epsilon_1)|q_\mu(12)|^2$, i.e.

$$[X] = (\kappa/\sigma) \sum_{12} \delta(\epsilon_1)|q_\mu(12)|^2 X(\epsilon_1 - \epsilon_2). \tag{5.18}$$

The operator of the single-particle quadrupole moment q has two types of matrix elements: the first, with $d < 2\Delta$, connects states with near-lying energies for q_0 diagonal; the second connects distant states with the transition energy $d'' \gg 2\Delta$. The weight function in (5.18) has consequently two peaks. Therefore when estimating the average (5.18) of a smooth function $X(\epsilon_1 - \epsilon_2)$ it can be assumed that

$$[X] = (\sigma'/\sigma) \approx X(d') + (\sigma''/\sigma) X(d'') \tag{5.19}$$

where d' and d'' are mean values of $\epsilon_1 - \epsilon_2$ for near and distant transitions and σ', σ'' are the corresponding contributions of these transitions to (5.15).

Using (5.17) and (5.19) we can transform Eq. (5.10) to the form

$$F(\omega) \equiv (\omega/2\Delta)^2 \left\{ (\sigma'/\sigma) \frac{1 - g(\ell')}{\ell'^2} + (\sigma''/\sigma) \frac{1 - g(\ell'')}{\ell''^2} \right\} = \sigma^{-1} - 1. \tag{5.20}$$

The left side of (5.20), F, as a function of ω is shown schematically in Fig. 7. The solutions of the dispersion equation (5.20) are obtained as the intersec-

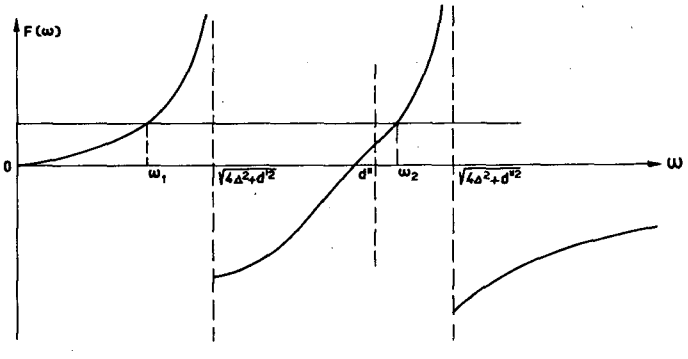

Fig. 7

tion points of the curve $F = F(\omega)$ with the horizontal line $F = \sigma^{-1} - 1$. It follows from the graph that there are two solutions

$$0 < \omega_1 < \sqrt{4\Delta^2 + d'^2}, \qquad \sqrt{4\Delta^2 + d'^2} \ll \omega_2 < \sqrt{4\Delta^2 + d''^2}. \tag{5.21}$$

Now we obtain these solutions directly from equation (5.10). Let us first consider the low-energy solution ω_1. Since $\omega_1 \ll d''$ we have $\ell'' \approx d''/2\Delta \gg 1$

and hence we can neglect the contribution of distant transitions in the sum in (5.16), containing the factor $(1 - g(\ell))/\ell^2$. If we now introduce the renormalized constant of the quadrupole interaction

$$\kappa_{\text{eff}} = \kappa/(1 - \sigma'') = \kappa/[1 - \kappa \sum_{12}'' \delta(\epsilon_1) |q_\mu(12)|^2] \tag{5.22}$$

the dispersion equation (5.10) can be written as

$$(\omega/2\Delta)^2 = \frac{1 - \kappa_{\text{eff}} \sum_{12}' \delta(\epsilon_1) |q_\mu(12)|^2}{\kappa_{\text{eff}} \sum_{12}' \delta(\epsilon_1) |q_\mu(12)|^2 (1-g(\ell))/\ell^2}, \tag{5.23}$$

i.e. it contains sums only over near transitions. Therefore mainly nucleons at the Fermi surface are involved in such excitations. The polarization of the other nucleons (described by distant transitions) only leads to a renormalization of the quadrupole interaction between the nucleons near the Fermi surface.

If the vibrations are adiabatic, i.e. if $\omega \ll 2\Delta$ (or, more exactly, if $|\ell'| \ll 1$), the expansion $g(\ell') \approx 1 - \frac{2}{3}\ell'^2$ can be used and (5.23) then becomes

$$(\omega/2\Delta)^2 = (3/2)\frac{1 - \kappa_{\text{eff}} \sum_{12}' \delta(\epsilon_1) |q_\mu(12)|^2}{\kappa_{\text{eff}} \sum_{12}' \delta(\epsilon_1) |q_\mu(12)|^2}. \tag{5.24}$$

Note that the energies of the β and γ-vibrations (related to q_0 and q_2 respectively) do not differ in the adiabatic approximation, since the sums in (5.24) can easily be shown to be independent of μ. The more precise formula (5.23) gives a difference between ω_β and ω_γ and the inequality $\omega_\beta < \omega_\gamma$ always holds. This follows from the fact that for β-vibrations $q_0(12) \sim \delta_{12}$, i.e. $d' = 0$, whereas for γ vibrations $d' \neq 0$.*

Let us now consider the second solution (5.21). In this case $\omega_2 \sim d'' \gg 2\Delta$, i.e. $|\ell'| \sim 1$ and $|\ell'| \approx \omega_2/2\Delta \gg 1$. Therefore the contribution of the near transitions to the function $\sigma B(\omega)$ (5.16) is negligible. The solution ω_2 of Eq. (5.10) lies in the region

$$d'' < \omega_2 < \sqrt{4\Delta^2 + d''^2} \tag{5.25}$$

since it is just in this region that the function $(\omega/2\Delta)^2 B(\omega)$ rapidly increases from the value $(2\sigma''/3\sigma)(d''/2\Delta)^2$ to infinity (see Fig. 7).

Thus the second solution, lying in the region (5.25) describes vibrations of a large number of the nucleons in the nucleus, since it is substantially

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* Similar results have been obtained in [16, 17]. γ-transitions have also been studied in [9]. Comparisons with experimental data are given in [9] and [17].

determined by distant transitions. The energies of these vibrations exceed the single-particle excitation energy 2Δ and therefore the existence of real excitations of this type is rather doubtful.

5.2. Effect of the gauge-invariant pairing interaction

We have investigated eigenoscillations of deformed nuclei, considering, in fact, only two parts of the interaction: quadrupole interaction (2.1′) and pairing interaction (2.6). As we have seen in the case of rotation, the use of gauge non-invariant interaction (2.6) results in the loss of the term $\mathcal{J}^2$ in the moment of inertia. So it is quite natural in the case of vibrations, too, to consider the more realistic interaction (4.23) instead of (2.6).

It is convenient to proceed from equations (1.35) to the system of equations for $\Delta^{(\pm)}$ and $\epsilon^{(\pm)}$. Making use of (1.36) it is easy to get

$$\Delta_{12}^{(\pm)} + \sum_{1'2'} \frac{\langle 1\tilde{2}|G|\tilde{2}'1'\rangle}{E_{1'2'}^2 - \omega^2} \{ E\xi^{(\pm)2}\Delta^{(\pm)} - E\xi^{(\pm)}\eta^{(\pm)}\epsilon^{(\pm)} + \omega\xi^{(\mp)}\xi^{(\pm)}\Delta^{(\mp)} - \omega\xi^{(\pm)}\eta^{(\mp)}\epsilon^{(\mp)} \}_{1'2'} = 0$$

$$\mp\epsilon_{12}^{(\pm)} + \sum_{1'2'} \frac{\overline{\langle 1\tilde{1}'|G|\tilde{2}'2\rangle}}{E_{1'2'}^2 - \omega^2} \{ E\eta^{(\pm)}\xi^{(\pm)}\Delta^{(\pm)} - E\eta^{(\pm)2}\epsilon^{(\pm)} + \omega\eta^{(\pm)}\xi^{(\mp)}\Delta^{(\mp)} - \omega\eta^{(\pm)}\eta^{(\mp)}\epsilon^{(\mp)} \}_{1'2'} = 0 . \tag{5.26}$$

For simplification of equations (5.17) note that $\xi_{12}^{(+)}$ at fixed difference $\epsilon_1 - \epsilon_2$ is an odd function of $\epsilon_1 + \epsilon_2$, i.e. $\xi_{12}^{(+)}$ changes sign at the Fermi surface, while $\xi_{12}^{(-)}$, $\eta_{12}^{(\pm)}$, E_{12} are even functions of $\epsilon_1 - \epsilon_2$. Thus, one may neglect in (5.17) terms containing $\xi_{12}^{(+)}$ linearly. In this case the equation for $\Delta^{(+)}$ is separated. Further, note that in the case of quadrupole vibrations the perturbations $\epsilon^{(\pm)}$ of the self-consistent field exhibit the same symmetry and it is natural to assume that $\epsilon^{(\pm)} \sim q_\mu$. On the other hand q_μ and $\epsilon^{(-)}$ have different parity with respect to time-reversal and therefore $\epsilon_{12}^{(-)} = 0$ for vibrations with quadrupole symmetry.* Taking the above statements into account we obtain from (5.26) a system of equations for $\Delta^{(-)}$ and $\epsilon^{(+)}$

$$\Delta_{12}^{(-)} + \sum_{1'2'} \frac{\langle 1\tilde{2}|G|\tilde{2}'1'\rangle}{E_{1'2'}^2 - \omega^2} \{ E\xi^{(-)2}\Delta^{(-)} - \omega\xi^{(-)}\eta^{(+)}\epsilon^{(+)} \}_{1'2'} = 0,$$

$$\epsilon_{12}^{(+)} + \sum_{1'2'} \frac{\overline{\langle 1\tilde{1}'|G|\tilde{2}'2\rangle}}{E_{1'2'}^2 - \omega^2} \{ E\eta^{(+)2}\epsilon^{(+)} - \omega\eta^{(+)}\xi^{(-)}\Delta^{(-)} \}_{1'2'} = 0 . \tag{5.27}$$

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* This condition is fulfilled automatically for the quadrupole interaction (2.1′). Note that the quadrupole part of the spin-spin interaction $V(1,2) \sim (\vec{\sigma} Y^{(2)})_1^{(2)} (\vec{\sigma} Y^{(2)})_2^{(2)}$ (see e.g. lectures by Prof. de-Shalit) leads to $\epsilon_{12}^{(+)} = 0$. However, such an interaction is probably of no importance.

Let us now make use of the gauge invariance of the interaction. Assuming $\Delta^{(-)}$ to be a function of the space coordinates and using relation (4.25) we transform the first of equations (5.27) to

$$\sum_{12} \frac{\varphi_2^*(\vec{r})\varphi_1(\vec{r})}{E_{12}^2 - \omega^2} \gamma_2 \left\{ E_{12}\left(\xi_{12}^{(-)2} - \frac{E_{12}^2 - \omega^2}{4 E_1 E_2}\right)\Delta_{12}^{(-)} - \omega \xi_{12}^{(-)} \eta_{12}^{(+)} \epsilon_{12}^{(+)} \right\} = 0 . \qquad (5.28)$$

It is only natural to suppose that $\Delta^{(-)}$ and $\epsilon^{(+)}$ have the same coordinate dependence. Let us put

$$\begin{aligned} \epsilon_{12}^{(+)} &= x q_\mu(12) \\ \Delta_{12}^{(-)} &= y q_\mu(12). \end{aligned} \qquad (5.29)$$

Multiplying (5.28) by $q_\mu(\vec{r})$ and integrating over $\vec{r}$ we obtain after some calculations

$$\sum_{12} |q_\mu(12)|^2 \frac{\Delta^2 E_{12}}{E_1 E_2 (E_{12}^2 - \omega^2)} \{ [((\epsilon_1 - \epsilon_2)/2\Delta)^2 - (\omega/2\Delta)^2] y + (\omega/2\Delta) x \} = 0. \qquad (5.30)$$

Let us now consider the second of equations (5.27). Substituting the quadrupole interaction (2.1') and dropping the small terms (cf. the transformation of (5.5) to (5.6)) we obtain

$$x - \sum_{12} \kappa |q_\mu(12)|^2 \frac{E_{12}}{E_{12}^2 - \omega^2} \{ \eta_{12}^{(+)2} x - (\omega\Delta/2E_1E_2) y \} = 0. \qquad (5.31)$$

Performing identical transformations we can bring (5.30) and (5.31) to the form

$$\begin{aligned} (\omega/2\Delta)\, A(\omega)\, x + C(\omega)\, y &= 0 \\ [(1/\sigma) - 1 - (\omega/2\Delta)^2 B(\omega)] x + (\omega/2\Delta) A(\omega) y &= 0 \end{aligned} \qquad (5.32)$$

where

$$A(\omega) = (\kappa/\sigma) \sum_{12} |q_\mu(12)|^2 (\Delta^2 E_{12}/E_1 E_2 (E_{12}^2 - \omega^2)) \qquad (5.33)$$

$$C(\omega) = (\kappa/\sigma) \sum_{12} |q_\mu(12)|^2 (\Delta^2 E_{12}/E_1 E_2 (E_{12}^2 - \omega^2)) \{ ((\epsilon_1 - \epsilon_2)/2\Delta)^2 - (\omega/2\Delta)^2 \}$$

and the quantities σ and $B(\omega)$ have been defined in (5.8) and (5.9). Demanding that the system (5.32) of algebraical equations have a non-trivial solution, we obtain the dispersion equation

$$(\omega/2\Delta)^2 = [(1/\sigma)-1]/[B(\omega) + A^{2*}(\omega)/C(\omega)]. \qquad (5.35)$$

In contrast to (5.10) the right-hand side of (5.35) contains an additional term in the denominator. This means that the consideration of the gauge-invariant pairing interaction leads to an additional term in the inertial parameter (in full analogy to the case of rotations).

Expressions similar to (5.17) can easily be obtained for $A(\omega)$ and $C(\omega)$

$$A(\omega) = [g(\ell)]\,; \quad C(\omega) = [\ell^2 g(\ell)]. \qquad (5.36)$$

For the first type of vibrations (ω_1 in (5.21)) when $|\ell'| < 1 \ll |\ell''|$ the contribution of distant transitions can be neglected when calculating $A(\omega)$. Contrary to this, the distant transitions give the main contribution to $C(\omega)$ so that in the adiabatic case ($|\ell'| \ll 1$) near transitions can be completely neglected.

There is a marked quantitative difference between the solutions of (5.35) and (5.10) although qualitatively they are the same. Indeed, in the adiabatical approximation the energies of β and γ-vibrations obtained from (5.35) are equal and if non-adiabatic corrections are taken into account $\omega_\beta < \omega_\gamma$, similarly as for equation (5.10). The actual difference between these equations appears when passing over to the hydrodynamical limit. For $\Delta \gg d', d''$ the inertial parameter $B(\omega)/4\Delta^2$ tends to zero, whereas the limit of the quantity $A^2(\omega)/C(\omega)4\Delta^2$ is finite

$$\frac{A^2(\omega)}{4\Delta^2 C(\omega)} \xrightarrow[\Delta\to\infty]{} \frac{\Sigma_{12}\delta(\epsilon_1)|q_\mu(12)|^2}{\sum\limits_{12}\delta(\epsilon_1)|q_\mu(12)|^2(\epsilon_1-\epsilon_2)^2} \qquad (5.37)$$

Thus, by taking into account the gauge-invariant pairing interaction, similarly as in the case of rotations, it is possible to pass over to the hydrodynamical limit.*

5.3. Critical analysis of the results

It can be seen from (5.35) that the vibrational energy depends essentially on the value of the restoring force coefficient ($\sigma^{-1} - 1$). Using the approximation (5.15) for the value σ, the restoring force coefficient is defined by

$$K = (\sigma^{-1} - 1) = (1 - \kappa \sum_{12}\delta(\epsilon_1)|q_\mu(12)|^2/\kappa \sum_{12}\delta(\epsilon_1)|q_\mu(12)|^2 \qquad (5.38)$$

* A similar result has just been obtained by Zaretaky and Urin (private communication).

and is easily seen not to depend on μ. In other word, the restoring force coefficients for β and γ-vibrations are equal to each other. This result is rather surprising from the physical point of view.

Relation (5.38) seems even more doubtful if the restoring forces for spherical and deformed nuclei are compared. The quantity σ'', determined by distant transitions is essentially the same for spherical and deformed nuclei (for β-vibrations cf. (2.19) and (2.30)). The quantity σ', describing the contribution of near transitions is of the same order of magnitude as σ'' for deformed nuclei. For spherical nuclei σ' is sensitive to the filling of the upper shell. In particular, for magic nuclei we have $\sigma' = 0$. Thus we have for the comparison of magic and deformed nuclei

$$\sigma''_{mag} \approx \sigma''_{def} \approx \sigma'_{def} \; ; \qquad \sigma'_{mag} = 0. \tag{5.39}$$

It follows from (5.39) that the restoring force coefficients K (5.38) are of the same order of magnitude for magic and deformed nuclei and differ only by a numerical factor. However this result is known to be wrong. The elasticity of magic nuclei is a volume effect, whereas the elasticity of deformed nuclei is determined by the properties of the surface (we mean the hydrodynamical part defined by (5.38)). Hence the correct ratio is

$$K_{def}/K_{mag} \sim A^{-\frac{1}{3}} . \tag{5.40}$$

Relation (5.40) cannot be fulfilled for any arbitrary value of κ in (5.38) and therefore it is necessary to presume that (5.40) must be considered as some sort of restriction on the choice of the parameter κ. What is the physical meaning of this restriction?

The parameter κ determines the strength of the quadrupole interaction, i.e. the value of the additional self-consistent field with quadrupole anisotropy. The spherically symmetrical part of the self-consistent field has been included directly in the single-particle Hamiltonian $\hat{\epsilon}$. Thus the two parts of the self-consistent field have been taken into account in two different ways. However, the self-consistent field in the nucleus is unique and uniform. In particular, the dividing of the field in a deformed nucleus into the spherical and anisotropic parts is very subjective and it is most certainly not correct to choose the magnitude of the anisotropic part arbitrarily. Thus it is evident from the physical point of view that the value κ must obey a certain condition for the self-consistent field to be uniform. This condition can be very simply formulated in the hydrodynamic limit, where the restoring force coefficient must be determined by surface tension. In other words, the main, volume-dependent, part of the restoring force coefficient K in the hydrodynamic limit must be equal to zero. The hydrodynamic expression for K is given just by (5.38). The condition of the homogeneity of the nuclear self-consistent field can then be formulated as (see 2.32)

$$\kappa \sum_{12} \delta(\epsilon_1) \left| q_\mu(12) \right|^2 \approx \kappa \rho(\mathcal{E}_F) \overline{q^2} = 1 . \tag{5.41}$$

Thus in (5.35) the main term (in the quasi-classical approximation) in the restoring force coefficient is equal to zero and the quantity σ must be calculated up to the next-order corrections to the quasi-classical approximation. These corrections naturally differ for β-and γ-vibrations and the main difference between the energies of β-and γ-vibrations is due to just this factor (since the small difference in the inertial parameters is now of no importance). The correlations to the quasi-classical approximation depend on the single-particle level-scheme and therefore a difference between the β-and γ-vibrational energies can be expected to change non-monotonously as a function of the atomic number A. Existing experimental data seem to confirm this qualitative statement.

6. STRUCTURE OF THE NUCLEON-NUCLEON INTERACTION. VIBRATIONS OF SPHERICAL NUCLEI

6.1. Expansion of the interaction potential in pairing states with a definite angular momentum

Considering various problems connected with the collective properties of nuclei we extracted certain parts of the nucleon-nucleon interaction, those most important in each special case. We divided the interaction matrix elements into two classes, one contributing to the self-consistent field and the other determining the Cooper pairing. Here we shall consider the structure of the nucleon-nucleon interaction in more detail and shall establish the connection between the two types of matrix elements.

We shall first consider the interaction to be described by some potential V. For simplicity we restrict ourselves to central forces. Then we separate the angular dependence of the interaction potential and express it with the help of spherical tensor operators

$$V(\vec{r}_a - \vec{r}_b) = \sum_K v_K(r_a r_b)\,(T^K(a)\,T^K(b))\,. \tag{6.1}$$

The matrix elements of tensor operators T^K between single-particle states $|1\rangle \equiv |n_1 \ell_1 j_1 m_1\rangle$ are equal to

$$\langle 1|\,T^K_\mu|\,1'\rangle \equiv [1/(2K+1)^{\frac{1}{2}}]\,(1||\,T^K||\,1')\,C^{K\mu}_{1\bar{1}'}, \tag{6.2}$$

where for brevity we have introduced

$$C^{K\mu}_{1\bar{1}'} = (-1)^{j_1 - m_1}\,(j_1 m_1 j_1' - m_1' \,|\, K\mu) \tag{6.3}$$

and $(1||\,T^K||\,1')$ is the reduced matrix element, not depending on the magnetic quantum numbers. Taking (6.2) into account the matrix element of the potential (6.1) can be written as

$$\langle 12|\,V|\,2'1'\rangle = \sum_k F_K(12;2'1')\,C^{K\mu}_{22'}\,C^{K\mu}_{1'\bar{1}'}. \tag{6.4}$$

where

$$F_K(12;2'1') = F^{(K)}(12;2'1')\,(2\|T^K\|2')\,(1'\|T^K\|1) \tag{6.5}$$

and $F^{(K)}$ is a radial Slater integral

$$F^{(K)}(12;2'1') = [1/(2K+1)]\int R_1(a)R_2(b)v_K(r_a r_b)R_{2'}(b)R_{1'}(a)r_a^2 r_b^2 dr_a dr_b. \tag{6.6}$$

Using the expansion (6.4) the interaction Hamiltonian becomes

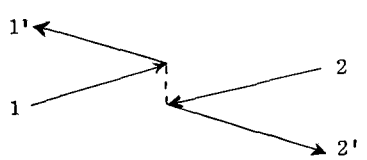
$$H_{int} = \sum_{122'1'} \langle 12|V|2'1'\rangle a_1^+ a_2^+ a_{2'} a_{1'}$$

$$\approx \sum_K \sum_{122'1'} F_K(12;2'1')\,(C^{K\mu}_{2\bar{2}'}\, a_2^+ a_{2'})\,(C^{K\mu}_{1'\bar{1}}\, a_1^+ a_{1'}). \tag{6.7}$$

Formula (6.7) represents H_{int} as a product of pair operators

$$\sum_{(m)} C^{K\mu}_{22'} a_2^+ a_{2'} \tag{6.8}$$

where the pairs consist of a particle and a hole and are characterized by the angular momentum K. The coefficients F_K in this case determine the strength of the interaction between pairs. Putting outgoing or ingoing lines respectively into correspondence with the operators a or a^+ we can represent each term in (6.7) by a Feynman graph



(6.9)

The lines on the same side of the dotted line correspond to operators bound into pairs.

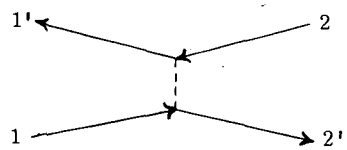
Besides (6.4) three more expansions of the matrix elements are possible, with different grouping into pairs

$$\langle 12|V|2'1'\rangle = \sum_K \bar{F}_K(12;2'1')\, C^{K\mu}_{2\bar{1}'}\, C^{K\mu}_{2'\bar{1}} \tag{6.10}$$

$$= \sum_K f_K(12;2'1')\, C^{K\mu}_{12} C^{K\mu}_{1'2'} \tag{6.11}$$

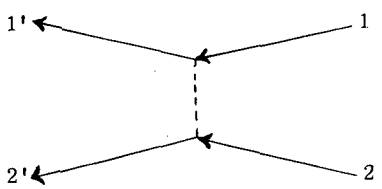
$$= \sum_K \bar{f}_K(12;2'1')\, C^{K\mu}_{12} C^{K\mu}_{2'1'} \tag{6.12}$$

(here $C_{12}^{K\mu}$ is the Clebsch-Gordan coefficient $(j_1 m_1 j_2 m_2 \mid K\mu)$. The expansion (6.10), in which the angular momentum of a particle-hole pair is fixed similarly as in (6.4), corresponds to the Feynman graph



(6.13)

In (6.11) and (6.12) two ingoing or two outgoing lines are connected into pairs (particle-particle or hole-hole pair). These expansions correspond to the graph



(6.14)

and can be reduced to each other by a simple permutation of the indices in the Clebsch-Gordan coefficient, so that:

$$\bar{f}_K(12;2'1') = (-1)^{j_1 - j_2 + K - 1} f_K(12;2'1'). \tag{6.15}$$

Using Racah algebra we can connect the coefficients of the various expansions. Thus, in addition to (6.15) we obtain

$$f_K(12;2'1') = \sum_L (-1)^{j_{2'} - j_2 + L} (2L+1) W(Kj_2' j_1 L; j_1' j_2) F(12;2'1') \tag{6.16}$$

$$\overline{F_K}(12;2'1') = \sum_L (2L+1) W(Kj_1' j_2' L; j_2 j_1) F(12;2'1') \tag{6.17}$$

where the W are Racah coefficients. The formulae (6.15 - 6.17) express all the coefficients of the expansions (6.10 - 6.12) with the help of F_K. Let us consider the structure of coefficients F_K (6.5).

6.2. Connection between the expansion coefficients in the asymptotic approximation

The radial part of the single-particle wave-functions depends weakly on the nuclear quantum numbers. The same is true for the radial matrix element $F^{(K)}$ (6.6). In addition, the dependence of $F^{(K)}$ on K is also weak,

because of the short-range character of the forces. (For a δ-function forces $F^{(K)}$ do not depend on K at all). Thus the main dependence in (6.5) on the quantum numbers is due to the reduced matrix elements, the explicit expression for which is

$$(1\|T^K\|2) = (-1)^{j_2-\frac{1}{2}}\left[\frac{(2j_1+1)(2j_2+1)(2\ell_1+1)(2\ell_2+1)}{2K+1}\right]^{\frac{1}{2}} W(\ell_1 j_1 \ell_2 j_2; \tfrac{1}{2}K)(\ell_1 0\, \ell_2 0|\, K0) \tag{6.18}$$

Note that the Clebsch-Gordan coefficient $(\ell_1 0 \ell_2 0|\, K0)$ in (6.18) contains a factor $(-1)^{(\ell_1+k+\ell_2)/2}$ and is thus an alternating quantity for arbitrary values of ℓ_1 and ℓ_2. The collective effects that we are studying are described by coherent quantities, containing sums over wide regions of the levels ℓ_1 and ℓ_2. Hence the main contribution* to collective effects comes from the diagonal quantities $(j\ell\|\, T^K\|\, j\,\ell)$. In addition, the main contributions to the sums over states $|\, j\ell\,\rangle$ come from levels with high angular momenta. Thus, it is possible to use an asymptotic expression, valid for $2j+1 \gg 1$

$$(\ell j\|\, T^K\|\, \ell j) \approx \theta(K)\,(-1)^{\frac{K}{2}}\frac{K!}{2^K\left(\frac{K}{2}!\right)^2}\left((2j+1)^2-K^2\right)^{\frac{1}{2}} \tag{6.19}$$

where $\theta(K)$ is equal to unity for even K and to zero for odd K. Putting (6.19) into (6.5) we obtain (for $K \ll 2j+1$)

$$F_K(12;21) \approx F^{(K)}\,\theta(K)\sqrt{(2j_1+1)(2j_2+1)}\left[K/2\left(\frac{K}{2}!\right)\right]^2 \tag{6.20}$$

where $F^{(K)} = F^{(K)}(12;21)$. The quantity (6.20) describes coherent particle-hole interaction. From (6.16) and (6.17) we obtain for the coherent parts of the other coefficient

$$f_K(11;22) = \sum_L (-1)^{j_2-j_1+L}(2L+1)W(Kj_2j_1L;j_2j_1)F_L(11;22)$$

$$\tilde{F}_K(12;12) = \sum_L (2L+1)W(Kj_2j_1L;j_2j_1)\,F_L(12;12). \tag{6.21}$$

From the symmetry properties of $(1\|\, T\|\, 2)$ it is easy to obtain the relation

$$F_K(11;22) = (-1)^{j_1-j_2}F_K(12;12) \tag{6.22}$$

making it possible to connect the right-hand sides of (6.21). As a result we obtain

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* A consideration of the general case leads to a complication of the formulas without changing the essence of the matter and so we restrict ourselves to a consideration of diagonal terms only.

$$\widetilde{F}_K(12;12) = (-1)^{\ell_1-\ell_2} f_K(11;22) \tag{6.23}$$

and further it is only necessary to consider one of the relations (6.21). The main contribution to the sum (6.21) comes from terms with large L. This enables us to use the asymptotic expression for $F_L(12;12)$. For large values of ℓ, j, L it is easy to obtain

$$(1\|T\|2) \approx (-1)^{(\ell_1+L+\ell_2)/2}\theta(\ell_1+L+\ell_2)2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\times\begin{cases}[\tan(\alpha_{12}^L/2)]^{\frac{1}{2}}\\ \pm[\cot(\alpha_{12}^L/2)]^{\frac{1}{2}}\end{cases} \tag{6.24}$$

where the first expression is to be taken for $\Delta j - \Delta\ell \equiv (j_1 - j_2) - (\ell_1 - \ell_2) = 0$ and the second for $\Delta j - \Delta\ell = \pm 1$. The quantity $\alpha_{12}^L (0 \le \alpha_{12}^L \le \pi)$, having the meaning of the "angle" between j_1 and j_2 is defined as

$$\cos\alpha_{12}^L = \frac{L(L+1) - j_1(j_1+1) - j_2(j_2-1)}{2\left(j_1(j_1+1)\,j_2(j_2+1)\right)^{\frac{1}{2}}}\,. \tag{6.25}$$

Putting (6.24) into (6.5) we find

$$F(12;12) \approx \widetilde{F}\quad \theta(\ell_1+\ell_2+L)2\pi^{-1}\quad \begin{cases}\tan\dfrac{\alpha_{12}^L}{2}\\ \cot\dfrac{\alpha_{12}^L}{2}\end{cases}$$

$$\widetilde{F}^{(L)} = F^{(L)}(11;22)\,. \tag{6.26}$$

correspondingly for even and odd values of $\Delta j - \Delta\ell$. To calculate the sum in (6.21) we can also use the asymptotic form for the Racah coefficients

$$W(K j_2 j_1 L; j_2 j_1) \approx \frac{(-1)^K}{\sqrt{(2j_1+1)(2j_2+1)}}\,P_K(\cos\alpha_{12}^L) \tag{6.27}$$

and then replace the summation over L by an integration according to

$$\Sigma\,\theta(\ell_1+L+\ell_2)(2L+1)\ldots \to \tfrac{1}{4}(2j_1+1)(2j_2+1)\int d\cos\alpha_{12}^L \ldots,. \tag{6.28}$$

Using (6.26), (6.27) and (6.28) we obtain from (6.21)

$$\widetilde{F}_K(12;12) = (-1)^K\sqrt{(2j_1+1)(2j_2+1)}\,\frac{1}{2\pi}\int_0^\pi \widetilde{F}^{(L)}\,P_K(\cos\alpha^L)\begin{pmatrix}\tan\dfrac{\alpha^L}{2}\\ \cot\dfrac{\alpha^L}{2}\end{pmatrix} d\cos\alpha^L. \tag{6.29}$$

If we neglect the weak dependence of $\tilde{F}^{(L)}$ on L the integral in (6.29) can be taken in the general form. For even K we have

$$I_K \equiv \frac{1}{\pi}\int_0^{\pi} P_K(\cos\alpha)\cot\frac{\alpha}{2}\,d\cos\alpha = \left[\frac{K!}{2^K\left(\frac{K}{2}!\right)^2}\right]. \qquad (6.30)$$

From (6.29) and (6.30) we obtain

$$\tilde{F}_K(12;12) = \tfrac{1}{2}\sqrt{(2j_1+1)(2j_2+1)}\, I_K \tilde{F}. \qquad (6.31)$$

Comparing (6.31) and (6.20) and taking into account (6.23) we find for even K the following relations between the expansion coefficients:

$$\tilde{F}_K(12;12) = (-1)^{\ell_1-\ell_2} f_K(11;22) = \tfrac{1}{2} F_K(12;21). \qquad (6.32)$$

The particle-particle interaction in a state with angular momentum K (taking exchange effects into account) is determined by the quantity (see (6.11), (6.12) and (6.15))

$$\Gamma_{pp}^{(0)}(1;2;K) = \tfrac{1}{2}\left[f_K(11;22) - \bar{f}_K(11;22)\right] = \theta(K) f_K(11;22). \qquad (6.33)$$

For a particle-hole pair the corresponding quantity is defined by a sum of the graphs (6.9) and (6.13)

$$\Gamma_{ph}^{(0)}(1;2;K) = F_K(12;21) - \bar{F}_K(12;12). \qquad (6.34)$$

Combining (6.32), (6.33) and (6.34) we finally obtain

$$(-1)^{\ell_1-\ell_2}\Gamma_{pp}^{(0)}(1;2;K) = \Gamma_{ph}^{(0)}(1;2;K) = \tfrac{1}{2} F_K(12;21). \qquad (6.35)$$

6.3. Short summary and generalization of the results

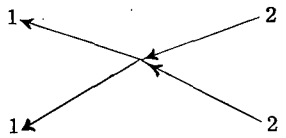
The above calculations were based on two assumptions about the nucleon-nucleon interaction

A. The interaction is described by some potential.

B. The forces are short-range.

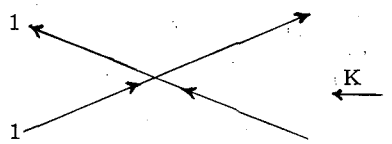
In such a case

(1) The first-order vertex parts for two particles with the angular moment K

$\Gamma_{pp}(11;22;K) =$ 

(6.36)

and those for a particle and a hole

$\Gamma_{ph}(11;22;K) =$ 

(6.37)

are connected by relation (6.35).

(2) The quantities (6.36) and (6.37) for large single-particle angular momenta become separable (see (6.20))

$$\Gamma(11;22;K) = \Gamma(K)\sqrt{2j_1 + 1}\sqrt{2j_2 + 1}. \tag{6.38}$$

Note that when more general non-diagonal quantities $\Gamma(11';22';K)$ are considered (see footnote to 6.2), we would obtain analogous results. In particular, instead of (6.38) we can obtain

$$\Gamma(11';22';K) = \Gamma(K)\sqrt[4]{(2j_1 + 1)(2j_1' + 1)}\sqrt[4]{(2j_2 + 1)(2j_2' + 1)}. \tag{6.39}$$

(3) The quantities (6.36) and (6.37) are determined by only one parameter ($F^{(K)} \approx$ const. in (6.20)) in the asymptotic approximation. Note that if we include spin-spin interaction in the potential analogous results can be obtained, but naturally there will be one more parameter.

The following question obviously arises: which of the results obtained are only consequences of the assumptions A and B and which hold in more general cases.

In the generalized Hartree-Fock method which we are considering, the explicit expression for the Hamiltonian $H = H_0 + H_{int}$ is used and the matrix elements of the interaction H_{int} enter directly into the final results. Thus formally this method uses some kind of perturbation theory. However, it can be shown that the final results do not change if we consider the nucleon-nucleon interaction more consistently. In such a case the interaction matrix elements $\langle 12|G|2'1'\rangle$ are replaced by some effective interaction, which is an infinite sum of different Feynman graphs *[18].

In general the quantities (6.36) and (6.37), i.e. the effective interactions of two particles or of a particle and a hole are determined by different sets

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* The most convenient method investigating pairing correlations in a system for such purpose is that of GREEN's functions [18, 19].

of Feynman graphs. Hence relation (6.35) is not satisfied for these quantities. On the other hand the dependence of the effective interaction Γ on the quantum numbers (formula (6.39)) is determined by kinematic factors. Thus (6.39) can be expected to hold also in the general case, but the parameters $\Gamma(K)$ will differ for particle-particle and particle-hole pairs.

The classification of vertex parts as particle-particle and particle-hole ones corresponds to our original division of matrix elements into two classes. It is easy to understand that particle-particle vertex parts cause the Cooper pairing * and that the particle-hole interaction contributes to the self-consistent field. In particular, the quadrupole interaction is determined by the quantity Γ_{ph} (K = 2).

6.4. Vibrations of spherical nuclei

We shall use the results on the structure of the effective interaction in investigating the vibrations of spherical nuclei.

According to the above, we now replace the interaction matrix elements by the effective interaction according to

$$\begin{aligned} \langle 1\tilde{2}\,|\,G\,|\,\tilde{2}'1'\rangle &\longrightarrow \sum_K \Gamma_{pp}(12;2'1';K)\, C^{K\mu}_{1\tilde{2}}\, C^{K\mu}_{1'\tilde{2}'}, \\ \overline{\langle 1\tilde{1}'\,|\,G\,|\,\tilde{2}'2\rangle} &\longrightarrow \sum_K \Gamma_{ph}(12;2'1';K)\, C^{K\mu}_{1\tilde{2}}\, C^{K\mu}_{1'\tilde{2}'}. \end{aligned} \tag{6.40}$$

Putting (6.40) into (1.35) we obtain the following equation for the component $Z^{(\pm)}_{12}(\omega, K)$ with the angular momentum K

$$\begin{aligned} \omega Z^{(\mp)}_{12}(\omega, K) = E_{12} Z^{(\pm)}_{12}(\omega, K) + \xi^{(\pm)}_{12} \sum_{1'2'} \Gamma_{pp}(12;2'1';K) \xi^{(\pm)}_{1'2'} Z^{(\pm)}_{1'2'}(\omega, K) \\ \pm \eta_{12} \sum_{1'2'} \Gamma_{ph}(12;2'1')\, \eta^{(\pm)}_{1'2'} Z^{(\pm)}_{1'2'}. \end{aligned} \tag{6.41}$$

We now assume that Γ_{pp} and Γ_{ph} are separable according to (6.39)

$$\begin{aligned} \Gamma_{pp}(12;2'1';K) &= -\Gamma_{pp}(K)\, g_{12}\, g_{1'2'} \\ \Gamma_{ph}(12;2'1';K) &= -\Gamma_{ph}(K)\, g_{12} g_{1'2'} \\ g_{12} &= \{(2j_1 + 1)(2j_2 + 1)\}^{\frac{1}{2}}. \end{aligned} \tag{6.42}$$

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* Cooper pairs in spherical nuclei can be shown to be characterized by the total angular momentum J = 0. Thus the ground state of a system with Cooper pairing is an analogy (and a generalization) of a state with seniority v = 0 in j^n configuration.

Putting (6.42) into (6.41) we obtain, after transformation, the following system of algebraical equations

$$\varphi = \Gamma_{pp} \sum_{12} [\, g^2 \xi^{(+)2} E/(E^2 - \omega^2)]_{12} \varphi + \Gamma_{ph} \sum_{12} [(g^2 \xi^{(+)} \eta^{(+)} E/(E^2 - \omega^2)]_{12} \chi$$

$$+ \omega \Gamma_{pp} \sum_{12} [(g^2 \xi^{(+)} \xi^{(-)} /(E^2 - \omega^2)]_{12} \psi$$

$$\chi = \Gamma_{pp} \sum_{12} [\, g^2 \xi^{(+)} \eta^{(+)} E/(E^2 - \omega^2)]_{12} \varphi + \Gamma_{ph} \sum_{12} [(g^2 \eta^{(+)2} E/(E^2 - \omega^2)]_{12} \chi \tag{6.43}$$

$$+ \omega \Gamma_{pp} \sum_{12} [(g^2 \eta^{(+)} \xi^{(-)} /(E^2 - \omega^2)]_{12} \psi$$

$$\psi = \Gamma_{pp} \sum_{12} [\, g^2 \xi^{(-)2} E/(E^2 - \omega^2)]_{12} \psi + \omega \Gamma_{pp} \sum_{12} [(g^2 \xi^{(-)} \xi^{(+)} /(E^2 - \omega^2)]_{12}$$

$$+ \omega \Gamma_{pp} \sum_{12} [(g^2 \xi^{(-)} \eta^{(+)} /(E^2 - \omega^2)]_{12} \chi.$$

It is necessary when solving system (6.43) to make certain assumptions about the single-particle level scheme. To make a qualitative analysis of the solutions we shall consider the case of a shell consisting of a system of degenerate states. Then $\xi_{12}^{(+)}$, $\eta_{12}^{(\pm)}$ and E_{12} are constant and the system (6.43) can be greatly simplified. Taking into account that $\xi_{12}^{(-)} = 1, \eta_{12}^{(-)} = 0$ 0 and $\xi^{(+)2} + \eta^{(+)2} = 1$ we obtain after simple calculations the following dispersion equation

$$(E^2 - \omega^2)^2 \{ (E^2 - \overline{\Gamma}_{pp}(K))^2 - \omega^2 - (E - \overline{\Gamma}_{pp}(K)) (\overline{\Gamma}_{ph}(K) - \overline{\Gamma}_{pp}(K)) \eta^{(+)2} \} = 0 \tag{6.44}$$

where

$$\overline{\Gamma}(K) = \Gamma(K) \sum_{12} g_{12}^2 . \tag{6.45}$$

From (6.44) we obtain the vibration energies

$$\omega_K = (E - \overline{\Gamma}_{pp}(K)) \sqrt{1 - \frac{\overline{\Gamma}_{ph}(K) - \overline{\Gamma}_{pp}(K)}{E - \overline{\Gamma}_{pp}(K)} \eta^{(+)2}} \tag{6.46}$$

It follows from (6.46) that if the constants $\overline{\Gamma}_{ph}$ and $\overline{\Gamma}_{pp}$ are close to each other, (i.e. if (6.35) is approximately valid), then

$$\omega_K \approx E - \overline{\Gamma}_{pp}(K).$$

Such a spectrum has a simple physical interpretation. The quantity E is the binding energy of a Cooper pair. In an "ideal break-up" of such a pair,

when the particles are infinitely separated after the break-up, the excitation energy of the system is just equal to E. Actually the excitation energy is smaller by the value of the interaction between the particles in the final state (in our case $\overline{\Gamma}_{pp}$ (K))*. Thus for $\overline{\Gamma}_{ph} \approx \overline{\Gamma}_{pp}$ the spectrum is in fact of a single-particle nature.

Collective properties arise if the condition

$$[\overline{\Gamma}_{ph}(K) - \overline{\Gamma}_{pp}(K)] / [E - \overline{\Gamma}_{pp}(K)] \gtrsim 1 \qquad (6.47)$$

is fulfilled. Then the excitation energy can become considerably smaller than the single-particle one. Moreover, in such a case there is a dependence on the shell filling ($\eta^{(+)^2}$)**, and this clearly indicates the collective nature of the excitations.

The quantities $\Gamma(K)$ can be expected to decrease rapidly as K increases. So for a δ-functional interaction (see (6.35) and (6.20)) $\Gamma(2) = \frac{1}{4}\Gamma(0)$) Therefore condition (6.47) can actually be fulfilled for K = 2 provided that

$$\overline{\Gamma}_{ph}(2) \gg \overline{\Gamma}_{pp}(2) . \qquad (6.48)$$

It can now be said with assurance that quadrupole vibrations of the type considered do exist in real nuclei, i.e. that the condition (6.47) is fulfilled for K = 2. If (6.48) is considered to be valid the quantity $\Gamma_{pp}(2)$ can be neglected. Then the equations for quadrupole vibrations will only contain two parameters: Γ_{pp} (0) (=E) and Γ_{ph} (2), i.e. the interaction of two particles with the angular momentum K = 0 and that of a particle and a hole with K = 2. Note that the model Hamiltonian containing these two types of interaction is just that considered in these lectures. These qualitative arguments give an insight into the reason for the success of quantitative calculations of the spectra of spherical nuclei, based on the simple Hamiltonian with Cooper pairing and quadrupole interaction [8].

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* Note that since $E = \overline{\Gamma}_{pp}(K = 0)$ equation (6.46) for K = 0 gives $\omega = 0$. This is quite natural since Cooper pairs with K = 0 were already taken into account in the ground state.

** The value $\eta^{(+)^2}$ in this case changes from zero for magic nuclei to unity for nuclei with a half-filled shell.

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WEAK INTERACTIONS AMONGST NUCLEONS AND LEPTONS

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1. INTRODUCTION

In this series of lectures we are concerned with weak interactions amongst nucleons and leptons. To be more specific, we shall study the effects due to the three following 4-fermion interactions: beta-interaction, muon decay interaction and muon capture interaction. These concern the following processes with elementary particles:

beta-interaction

$$\begin{aligned} & n \to p + e^- + \bar{\nu} \\ \text{or} \quad & p \to n + e^+ + \nu \\ \text{or} \quad & e^- + p \to n + \nu \end{aligned} \tag{1.1}$$

muon decay

$$\mu^\pm \to e^\pm + \nu + \bar{\nu}. \tag{1.2}$$

muon capture

$$\mu^- + p \to n + \nu. \tag{1.3}$$

These processes may be studied either for free elementary particles or for nucleons contained in atomic nuclei. We limit ourselves mainly to those 4-fermion interactions for which the strangeness is conserved and we shall not consider weak interactions which do not conserve strangeness and which describe the leptonic or non-leptonic decay of K-mesons and hyperons (except for a few remarks in section 8).

Even with this limitation the subject is so extensive that a subjective, somewhat arbitrary choice of topics is inevitable. Emphasis will be laid on those phenomena which seem of the most fundamental importance either for elementary particle interactions or for nuclear structure. We shall confine ourselves mostly to allowed transitions. It will be emphasized and explained in some detail that the study of muon capture in complex nuclei deals with an extensive field of "muonic nuclear physics", which has only just started to be studied. The present situation is reviewed generally, while detailed calculations are carried out for some selected examples.

The appendices give some notations and details of calculations. I am much indebted to my co-worker Mr. H. P. C. Rood for his help in preparing these notes.

2. GENERALITIES ON FOUR-FERMION INTERACTIONS; THE TWO-COMPONENT NEUTRINO THEORY

In these lectures we shall use the following Hamiltonian for β-interaction

$$\mathcal{H}^{\beta} = \frac{1}{\sqrt{2}} \Sigma_i \, (\bar{\psi}_p \Gamma_i \psi_n) \, [g_i \, (\bar{\psi}_e \Gamma_i \psi_\nu) + g_i' \, (\bar{\psi}_e \Gamma_i \gamma_5 \psi_\nu)] \quad + \quad \text{h.c.} \tag{2.1}$$

$$(i = S, V, T, A, P)$$

$$\left.\begin{aligned} \Gamma_S &= 1 \\ \Gamma_V &= \gamma_\mu \\ \Gamma_T &= \tfrac{i}{2} (\gamma_\lambda \gamma_\mu - \gamma_\mu \gamma_\lambda) \\ \Gamma_A &= i \gamma_\mu \gamma_5 \\ \Gamma_P &= \gamma_5 \end{aligned}\right\} \tag{2.2}$$

The adjoint $\bar{\psi}$ is related to the hermitian conjugate ψ^* by

$$\bar{\psi} = \psi^* \gamma_4 \tag{2.3}$$

(h.c. means hermitian conjugate).

Of course $\mathcal{H}_\beta$ may be written alternatively with the ψ^*'s as

$$\mathcal{H}_\beta = \frac{1}{\sqrt{2}} \Sigma_i (\psi_p^* \, \Omega_i \psi_n) \, [\, g_i (\psi_e^* \Omega_i \psi_\nu) + g_i' (\psi_e^* \Omega_i \gamma_5 \psi_\nu)] \; + \; \text{h.c.} \tag{2.4}$$

$$\Omega_i = \gamma_4 \Gamma_i \, . \tag{2.5}$$

The interaction Hamiltonian (2.1) contains 10 complex constants g_i and g_i' and thus 20 arbitrary parameters*. With this Hamiltonian one does not in general have invariance for space reflection, charge conjugation or time reversal. The "classical" Hamiltonian, for which these invariances exist, is obtained by putting $g_i' = 0$ and assuming that the g_i are real in (2.1); this means that 5 parameters remain. We shall now summarize a number of points concerning these transformations. We indicate by P the transformation for space reflection, C the transformation for charge conjugation, T the transformation for time reversal.

We may consider these transformations either as transformations of the state vectors, e.g.

$$|A\rangle' = P^{-1} |A\rangle, \tag{2.5}$$

or as corresponding transformations of the operators, e.g.

$$H_i' = P H_i P^{-1} . \tag{2.6}$$

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* In (2.1) and (2.4) we have written Hamiltonian densities (indicated by script $\mathcal{H}$), where the ψ's should be taken in the same point; the total Hamiltonian $H = \int d^3 x \, \mathcal{H}(\vec{x})$ is indicated by printed H.

P and C are unitary operators, T is an anti-unitary operator; this means that T can be written as

$$T = U_T K \tag{2.7}$$

where U_T is a unitary operator and K is the operator for complex conjugation. In this case (2.6) becomes

$$H_i' = T\, H_i^*\, T^{-1}. \tag{2.7a}$$

We designate as proper Lorentz transformations the Lorentz transformations without reflections; P and T are improper Lorentz transformations. The theory of elementary particles with strong and electromagnetic interactions is invariant for C, P and T transformations (as well as for products of these transformations). At least this is in agreement with all available experimental evidence.

Experiments have shown that no invariance exists for all three operations C, P and T for weak interactions. We now mention the following important theorem of great generality:

2.1. CPT Theorem

If a local Lagrangian theory (which may contain derivative couplings of finite order) is invariant under proper Lorentz transformations, it is also invariant under the product CPT (and its permutations PCT etc.) although the theory may not be separately invariant under each of the operators C, P or T.

This theorem goes back to SCHWINGER [1], PAULI [2] and LÜDERS [3, 4]. An elegant proof was given by JOST [5]. It implies that, if we have a Hamiltonian which is not invariant for P, it cannot be invariant both for C and for T.

2.2. Transformation for space reflection P

We may denote the Hamiltonian (2.1) concisely as

$$H_\beta = \frac{1}{\sqrt{2}} \Sigma_i (g_i H_i + g_i' H_i') + \text{h.c.} \tag{2.8}$$

For the P-transformation the separate terms H_i and H_i' behave according to

$$\begin{aligned} P\, H_i\, P^{-1} &= H_i \\ P\, H_i'\, P^{-1} &= -H_i'. \end{aligned} \tag{2.9}$$

Hence the entire Hamiltonian transforms as

$$P\, H_\beta\, P^{-1} = \frac{1}{\sqrt{2}} \Sigma_i (g_i H_i - g_i' H_i') + \text{h.c.} \tag{2.10}$$

It follows that invariance for space reflection requires that

$$\text{all } g_i' = 0. \tag{2.11}$$

The observed quantities can be expressed as the sum of the absolute squares of certain matrix elements

$$\Sigma|M|^2 = \Sigma_{ij}\,[\,(Q_{ij}\,g_i^{*}g_j + \text{c.c.}) + (Q'_{ij}\,g_i'^{*}g_j' + \text{c.c.}) + (R_{ij}\,g_i^{*}g_j' + \text{c.c.})]. \tag{2.12}$$

Q'_{ij}, Q_{ij} and R_{ij} are certain functions of the measured momenta and spins. Under space reflection P the polar vectors change sign: $\vec{r}\to-\vec{r}$, $\vec{p}\to-\vec{p}$, while axial vectors, e.g. angular momenta such as a spin $\vec{s}$, do not change sign: $\vec{s}\to\vec{s}$. When invariance exists for P, expressions for a transition probability may only contain scalars such as $\vec{p_1}\cdot\vec{p_2}$; $\vec{s_1}\cdot\vec{s_2}$, etc.

When no invariance for P exists, pseudoscalars may occur, e.g. it can be shown that R_{ij} should be a pseudoscalar; hence it could contain terms such as $(\vec{p_1}\times\vec{p_2})\cdot\vec{p_3}$ or $\vec{s}\cdot\vec{p}$.

2.3. Transformation for time reversal T

It can be shown that the separate terms of (2.8) transform under simply as

$$T\,H_i\,T^{-1} = H_i. \tag{2.13}$$

As a consequence, the entire Hamiltonian transforms under this anti-unitary transformation as

$$T\,H_\beta\,T^{-1} = \frac{1}{\sqrt{2}}\Sigma_i\,(g_i^{*}H_i + g_i'^{*}H_i') + \text{h.c.} \tag{2.14}$$

It follows that invariance under time reversal requires that:

$$\text{all } g_i \text{ and } g_i' \text{ are real.} \tag{2.15}$$

Under time reversal both momentum vectors and angular momentum vectors change their sign $\vec{p}\to-\vec{p}$, $\vec{s}\to-\vec{s}$.

In view of the observables which may occur, we have to distinguish between two cases (cf. e.g. [16]).

A. No strong interactions exist between the decay products in the final states.

B. Strong interactions exist between the decay products in the final state.

In case A, a transition probability may be calculated in the "Born-approximation". By "strong interactions" (in case B.) we mean here electromagnetic interactions; the most usual cases are β-decays taking the Coulomb field of the nucleus into account. We may now formulate some conclusions for cases A and B in the following way. We distinguish even and odd operators with respect to time reversal; for example,

even 0_+ $\quad \vec{p_1}\cdot\vec{p_2},\ \vec{s}\cdot\vec{p},\ \vec{s_1}\cdot\vec{s_2}$ (2.16)

odd 0_- $\quad \vec{s}\cdot(\vec{p_1}\times\vec{p_2}),\ \vec{p_1}\cdot(\vec{p_2}\times\vec{p_3}),\ \vec{s_1}\cdot(\vec{s_2}\times\vec{s_3})$. (2.17)

It can then be shown that the coupling constants in the expectation values of 0_+ and 0_- occur in the following combinations in case A (ψ denotes a final state)

$$\langle \psi | 0_+ | \psi \rangle = \Sigma_{ij} (0_+)_{ij} (g_i^* g_j + g_i g_j^*) \tag{2.18}$$

and

$$\langle \psi | 0_- | \psi \rangle = \Sigma_{ij} (0_-)_{ij} (g_i^* g_j - g_i g_j^*). \tag{2.19}$$

Hence we see that if any observables of the form 0_- exist in this case, it means that H_β is not invariant under time reversal. In case B the conclusions are changed and become more complicated because of the final state interactions. Let $|B\rangle$ be a final state (stationary state) and $|B^{out}\rangle$ be the outgoing wave part of it; let the final state interactions cause a phase shift η_B, hence a phase factor $e^{i\eta_B}$ in $|B^{out}\rangle$. We then conclude that the coupling constants must occur in the following combinations in the expectation values of the even and odd operators 0_+ and 0_-:

$$\langle \psi | 0_+ | \psi \rangle = \Sigma_{ij} (0_+)_{iB,jB'} \{ (g_i^* g_j + g_i g_j^*) \cos(\eta_B - \eta_{B'}) + i (g_i^* g_j - g_i g_j^*) \sin(\eta_B - \eta_{B'}) \} \tag{2.20}$$

$$\langle \psi | 0_- | \psi \rangle = \Sigma_{ij} (0_-)_{iB,jB'} \{ (g_i^* g_j - g_i g_j^*) \cos(\eta_B - \eta_{B'}) + i (g_i^* g_j + g_i g_j^*) \sin(\eta_B - \eta_{B'}) \} . \tag{2.21}$$

It is seen that these expressions reduce to (2.18) and (2.19) for $\eta_B = \eta_{B'} = 0$. One concludes from (2.20) and (2.21) that time-reversal invariance can also be checked by means of the second term in (2.20) if final state interactions exist *.

2.4. Transformation for charge conjugation C

Considering the Hamiltonian (2.8) and taking into account that C is equivalent to PT according to the CPT theorem, we conclude, using (2.9) and (2.14),

$$C H_\beta C^{-1} = \frac{1}{\sqrt{2}} \Sigma_i (g_i^* H_i - g_i'^* H_i') + \text{h.c.} \tag{2.22}$$

Comparing (2.8) and (2.22) it follows that the requirements for invariance under charge conjugation are

$$\begin{aligned} &g_i \quad \text{real}, \\ &g_i' \quad \text{imaginary}. \end{aligned} \tag{2.23}$$

Charge conjugation or particle-anti-particle conjugation has the property that it changes the sign of all charges; hence also the sign of an electric current is changed. However, momentum or angular momentum vectors such as $\vec{p}$ and $\vec{s}$ do not change under C.

* In (2.20) and (2.21) only the expressions with g_i were given, while the analogous expressions with the g_i' were omitted for the sake of simplicity; g_i and g_i' behave differently for P, but not for T.

2.5. The Pauli-Pursey transformations and the conditions for invariance under C, P and T

The conditions for invariance under C, P and T formulated above were obtained from the requirement that the form of the Hamiltonian should be invariant under these transformations.

However, PAULI and PURSEY [6, 7] have drawn attention to the fact that certain transformations exist which leave all physical results unaltered (cf. also [8, 9, 10, 11]) although the form of the Hamiltonian is changed. Simple examples of such transformations are

(a) a simple phase factor for a fermion field

$$\psi' = e^{i\varphi}\psi , \tag{2.24}$$

(b) a factor $e^{i\alpha\gamma_5}$; for a neutrino field (if $m_\nu = 0$),

$$\psi_\nu' = e^{i\alpha\gamma_5}\ \psi_\nu . \tag{2.25}$$

Such transformations correspond to certain transformations of the coupling constants and hence to transformations of the Hamiltonian. Physical results for a transition probability should contain the coupling constants only in combinations which are invariant for such transformations. Suppose a Hamiltonian H_β' gives identical physical results with another Hamiltonian H_β satisfying one of the preceding conditions of invariance under C, P or T, because it is related to the original Hamiltonian H_β by a Pauli-Pursey transformation. One can then say that H_β' also satisfies an invariance requirement for invariance under C, P or T; however, these conditions are less restrictive than (2.11), (2.15) or (2.23). The latter conditions are sufficient conditions but they are not necessary for the invariance of the physical results.

Pursey has given the following characterization of the transformations which leave the physical results unaltered:

The average over given sets of initial and final states of the squared modulus of the S-matrix element for any process should be invariant for any unitary transformation, which leaves the observable actions of the initial and final states invariant. The consequences of these principles were worked out for an interaction, which is even more general than the one formulated in (2.1).

$$\mathcal{H}_\beta = \frac{1}{\sqrt{2}}\,\Sigma_i\,(\bar{\psi}_p\Gamma_i\ \psi_n)\,\{\bar{\psi}_e\,\Gamma_i\ [\,(g_i + g_i'\gamma_5)\psi_\nu + (f_i + f_i'\gamma_5)\,\gamma_5\psi_\nu^c]\} + \text{h. c.} \tag{2.26}$$

This Hamiltonian provides the possibility of a simultaneous emission of neutrinos as well as of antineutrinos; f_i and f_i' are coupling constants introduced here in addition to g_i and g_i'.

$\psi_\nu^c = C\bar{\psi}$ gives the charge conjugate field (C here is a Dirac matrix, not to be confused with the charge conjugation operator C).

The interaction (2.26) may alternatively be written as

$$\mathcal{H}_\beta = \frac{1}{\sqrt{2}}\,\Sigma_i\,(\bar{\psi}_p\Gamma_i\ \psi_n)\,\{\bar{\psi}_e\,\Gamma_i\ [g_i^R\psi_\nu^R + g_i^L\ \psi_\nu^L + f_i^R\ \gamma_5\psi_\nu^{cR} + f_i^L\ \gamma_5\ \psi_\nu^{cL}]\} + \text{h. c.} \tag{2.27}$$

if we put

$$\begin{aligned} \psi_\nu^L &= \tfrac{1}{2}(1+\gamma_5)\psi_\nu \qquad g_i^L = g_i + g_i' \qquad f_i^L = f_i + f_i' \\ \psi_\nu^R &= \tfrac{1}{2}(1-\gamma_5)\psi_\nu \qquad g_i^R = g_i - g_i' \qquad f_i^R = f_i - f_i' \,. \end{aligned} \tag{2.28}$$

R and L correspond to positive and negative helicity. If we now introduce two-component quantities ξ_i and η_i according to

$$\xi_i = \begin{pmatrix} g_i^L \\ f_i^L \end{pmatrix} \qquad \eta_i = \begin{pmatrix} g_i^R \\ f_i^R \end{pmatrix} , \tag{2.29}$$

it can be shown that the most general transformations admitted by the above-mentioned principle are

$$\hat{\xi}_i = e^{i(\varphi+\alpha)} A\xi_i \quad , \quad \hat{\eta}_i = e^{i(\varphi-\alpha)} A\eta_i \tag{2.30}$$

where A is a 2×2 unitary, unimodular matrix

$$A = \begin{pmatrix} a & -b^* \\ b & -a^* \end{pmatrix} , \quad |A| = 1 . \tag{2.31}$$

These transformations form a 5-parameter group. For further details we refer to the references given before.

It may be useful to mention that the greater generality of (2.26) in comparison with (2.1) shows up only in double processes such as in double beta decay. However, no observed phenomena seem to require this greater generality; (2.1) is sufficient for the explanation of all observations and we shall no longer make use of (2.26).

2.6. The two-component neutrino theory

We write the Dirac equation in the form

$$(E + \vec{\alpha}.\vec{p} + \beta m)\psi = 0 \quad (\hbar = c = 1). \tag{2.32}$$

Putting

$$\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \qquad (\chi \text{ and } \varphi \text{ are 2-component spinors}) ,$$

the equation may be written in the following form with the representation of the Dirac matrices which we use

$$\left.\begin{aligned} E\varphi + \vec{\sigma}.\vec{p}\chi - m\varphi &= 0 \\ E\chi + \vec{\sigma}.\vec{p}\varphi + m\chi &= 0 \end{aligned}\right\} . \tag{2.33}$$

If we apply this to neutrinos, for which we assume the mass to be zero ($m_\nu = 0$), this becomes

$$\left.\begin{aligned} E\varphi + (\vec{\sigma}.\vec{p})\chi &= 0 \\ E\chi + (\vec{\sigma}.\vec{p})\varphi &= 0 \end{aligned}\right\} . \tag{2.34}$$

We now introduce

$$\psi_+ = \tfrac{1}{2}\begin{pmatrix}\varphi + \chi\\ \varphi + \chi\end{pmatrix}$$
$$\psi_- = \tfrac{1}{2}\begin{pmatrix}-(\varphi - \chi)\\ \varphi - \chi\end{pmatrix} \qquad (2.35)$$

In our representation we have

$$\gamma_4 = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \qquad \gamma_5 = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} \qquad (2.36)$$

Hence we see that we can write (2.35) as

$$\left.\begin{aligned}\psi_+ &= \tfrac{1}{2}(1 + \gamma_5)\psi\\ \psi_- &= \tfrac{1}{2}(1 - \gamma_5)\psi\end{aligned}\right\} \qquad (2.37)$$

from which we see that

$$\begin{aligned}\gamma_5\psi_+ &= \psi_+ \qquad (\text{or} \quad (1 - \gamma_5)\psi_+ = 0)\\ \gamma_5\psi_- &= -\psi_- \qquad (\text{or} \quad (1 + \gamma_5)\psi_- = 0).\end{aligned} \qquad (2.38)$$

Hence we see that ψ_+ and ψ_- are eigenfunctions of γ_5 with eigenvalues +1 and -1.

We see from (2.34) and (2.35) that we may write

$$\psi_+ = \tfrac{1}{2}\begin{pmatrix}(1 - \vec{\sigma}.\vec{\nu})\varphi\\ (1 - \vec{\sigma}.\vec{\nu})\varphi\end{pmatrix} \qquad \psi_- = \tfrac{1}{2}\begin{pmatrix}-(1 + \vec{\sigma}.\vec{\nu})\varphi\\ (1 + \vec{\sigma}.\vec{\nu})\varphi\end{pmatrix} \qquad (2.39)$$

with

$$\vec{\nu} = \frac{\vec{p_\nu}}{|\vec{p_\nu}|} = \frac{\vec{p_\nu}}{E_\nu} \; . \qquad (2.40)$$

(2.34) may thus be written as

$$\begin{aligned}E\psi_+ &= -(\vec{\sigma}.\vec{p})\psi_+\\ E\psi_- &= +(\vec{\sigma}.\vec{p})\psi_- \; .\end{aligned} \qquad (2.41)$$

We may now say that ψ_+ and ψ_- are each fields with only two (independent) components. ψ_+ corresponds to a particle with spin anti-parallel to the momentum; ψ_- corresponds to a particle with spin parallel to the momentum.

According to (2.41) ψ_+ and ψ_- satisfy 2-component equations; these equations are invariant for proper Lorentz transformations.

If we consider the transformation for space reflection, which for Dirac fields is specified by

$$\psi^P = \gamma_4 \psi \ ,$$

this provides for ψ_+ and ψ_-

$$(\psi_+)^P = \gamma_4 \psi_+ = \tfrac{1}{2} \gamma_4 \ (1 + \gamma_5) \psi \ = \tfrac{1}{2} \ (1 - \gamma_5) \psi^P$$

$$(\psi_-)^P = \gamma_4 \psi_- = \tfrac{1}{2} \gamma_4 \ (1 - \gamma_5) \psi \ = \tfrac{1}{2} \ (1 + \gamma_5) \psi^P .$$

Hence we see that the chirality is changed by the space reflection; the equations (2.41) are not separately invariant for space reflection. However, if we have a theory which is non-invariant for space reflection, there is no objection to the introduction of such fields.

In accordance with the convention for neutrino and antineutrino (and the helicities found in nature), we call the particle associated to ψ_+ a neutrino (negative helicity) and the particle associated to ψ_- an antineutrino (positive helicity). Space reflection changes a neutrino into an antinuetrino. Only for particles with zero rest mass is a helicity possible which is invariant for proper Lorentz transformations. For particles with mass different from zero one can always find a proper Lorentz transformation which transforms the particle to a rest system. After that one can transform in such a way that the helicity of the particle obtains opposite sign.

The use of a two-component neutrino theory in connection with weak interaction phenomena was proposed by LEE and YANG [12], LANDAU [13] and SALAM [14].

If we consider the Hamiltonian (2.1) it is clear that it contains coupling to two-component neutrinos as a special case, namely for $g_i' = \pm g_i$. Which sign is correct (if the two-component neutrino theory works at all) has to be determined from experiment.

We have reviewed the two-component neutrino theory here only very briefly; for further details we refer to the original papers. Concerning the possibilities of 2-component theories for particles with mass we refer to a paper by CASE [15].

3. BETA-RADIOACTIVITY

3.1. Introduction

In this section we want to discuss the most important phenomena which can be observed in beta-radioactivity. We first discuss the "classical" phenomena, which do not show non-conservation of parity. We confine ourselves to allowed transitions and neglect the influence of the Coulomb field of the nucleus. The calculations for beta-radioactivity and muon capture are analogous in many respects. In both cases the nucleons in the nuclei have non-relativistic energies (kinetic energies of the order of 20 MeV, corresponding to $v/c \approx 0.20$, their rest mass being 930 MeV). The simplifications resulting from this fact can be introduced in a variety of ways: (a) neglect of the "relativistically small" nuclear matrix elements or use of non-relativistic approximations for these matrix elements either using

a limiting form of the Dirac equation for low energies or the Foldy-Wouthuysen transformation; (b) reduction of the general Hamiltonian to an effective Hamiltonian which must be applied to 2-component spinors (instead of 4-component Dirac spinors).

We shall use (a) for β-radioactivity and (b) for muon capture, thus demonstrating the somewhat different, although essentially equivalent, procedures.

3.2. "Classical" phenomena of beta radioactivity

As a first example of calculating some transition probabilities we derive the transition probability for allowed beta transitions providing the shape of the beta spectrum and the electron-neutrino directional correlation. We can start from the expression for a transition probability derived in first order in the time-dependent perturbation theory

$$P_{i\to f}(\alpha)\,d\alpha = \frac{2\pi}{\hbar} \sum_f \left| \int \psi_f^* H_{int} \psi_i \, d\tau \right|^2 \rho(E,\alpha)\,d\alpha \qquad (3.1)$$

where

$\rho(E,\alpha)\,d\alpha = (dn/dE)$ = number of states per energy interval of the final states for an energy which is about equal to the energy of the initial state and for values of the parameter(s) α between α and $\alpha + d\alpha$,

ψ_i, ψ_f initial and final state wave functions of the entire system,

α (continuously varying) parameter(s), on which the final state of the system depends,

Σ_f summation (or integration) for those parameters of the final state which are not observed.

Applying this to $\beta^{\pm}$-emission of atomic nuclei we can write this out (as will be shown below) in the following form

$$P(E,\vec{p},\vec{q})\,dE\,d\omega_e\,d\omega_\nu = \frac{1}{(2\pi)^5} \Sigma_e \Sigma_\nu \left| \langle f | H_\beta | i \rangle \right|^2 pE\,q^2\,dE\,d\omega_e\,d\omega_\nu \quad . \qquad (3.2)$$

We shall use units such that

$$\hbar = 1, \quad c = 1, \quad m_e = 1 .$$

Further, the symbols have the following meanings:

E = energy of the electron;
E_ν = energy of the neutrino;
$\vec{p}$ = momentum of the electron;
$\vec{q}$ = momentum of the neutrino;
E_0 = maximum energy of the electrons.

We have:

$$q = E_\nu = E_0 - E .$$

$d\omega_e$ = differential solid angle for the direction of $\vec{p}$,
$d\omega_\nu$ = differential solid angle for the direction of $\vec{q}$,
Σ_e = sum over both states of polarization of the electron,
Σ_ν = sum over both states of polarization of the neutrino, (we take a 4-component neutrino here).

In order to derive (3.2) from (3.1) we have to evaluate $\rho(E,\alpha)$. The parameters α from (3.1) are specified here by $\alpha = (E, \vec{p}, \vec{q})$. The total energy E from (3.1) should not be confused with the electron energy E from (3.2). The latter energy determines how the total energy E_o is distributed over the electron and the neutrino. In order to determine $\rho(E,\alpha)$ we take for ψ_e and ψ_ν plane wave solutions normalized to one particle per unit volume. If we consider one particle in the unit volume there is one eigenstate for a volume $h^3 = (2\pi)^3\hbar^3$ in momentum space. Hence we see that the number of states with electron momentum between p and p+dp and neutrino momentum between q and q+dq, the directions of $\vec{p}$ and $\vec{q}$ being contained in the solid angles $d\omega_e$ and $d\omega_\nu$, is given by

$$dn = \frac{p^2 dp d\omega_e}{h^3} \cdot \frac{q^2 dq d\omega_\nu}{h^3},$$

or using units such that $\hbar = 1$:

$$dn = \frac{p^2 dp d\omega_e \, q^2 dq d\omega_\nu}{(2\pi)^6}. \tag{3.3}$$

We introduce the energy E instead of the momentum p of the electron, using

$$p^2 = E^2 - 1 \; ; \qquad pdp = EdE \; ;$$

we obtain

$$\rho(E,\alpha)\, d\alpha = \frac{dn}{dE_{tot}} = \frac{pEq^2 \, d\omega_e \, d\omega_\nu dE}{(2\pi)^6}. \tag{3.4}$$

We note that three of the four differentials dE, dq, $d\omega_e$, $d\omega_\nu$ correspond to $d\alpha$, while one differential (either dE or dq) corresponds to dE_{tot}. We substitute (3.4) into (3.1) to obtain (3.2). We have still to explain how one obtains specific expressions for the matrix element contained in (3.2).

In this sub-section we shall take an interaction density as an arbitrary linear combination of the scalar (S), vector (V), tensor (T), axial vector (A) and pseudoscalar (P) interaction, which we write as

$$\mathcal{H}_\beta = \sum_{k=1}^{5} g_k^\beta \, \mathcal{H}_k + \text{hermitian conjugate} \tag{3.5}$$

(k = 1, 2 ... 5 or k = S, V, T, A, P).

g_S^β, g_V^β, g_T^β, g_A^β, g_P^β are the coupling constants in β-radioactivity of the different interactions of which $\mathcal{H}_\beta$ according to (3.5) is a mixture (we shall omit the superscript β when no confusion can arise). When separating the "large" and "small" terms (for non-relativistic nucleon velocities) we can write the expressions for

$$\mathcal{H}_k = (\Psi_f^* \Omega_k \tau^{(+)} \Psi_i)(\psi_e^* \Omega_k \psi_\nu) \qquad \text{as:}$$

$$\mathcal{M}_S = (\Psi_f^* \beta \tau^{(+)} \Psi_i)(\psi_e^* \beta \psi_\nu) \qquad \text{(a)}$$

$$\mathcal{M}_V = (\Psi_f^* \tau^{(+)} \Psi_i)(\psi_e^* \psi_\nu) - (\Psi_f^* \vec{\alpha} \vec{\tau}^{(+)} \Psi_i)(\psi_e^* \vec{\alpha} \psi_\nu) \qquad \text{(b)}$$

$$\mathcal{M}_T = (\Psi_f^* \beta \vec{\sigma} \vec{\tau}^{(+)} \Psi_i) \cdot (\psi_e^* \beta \vec{\sigma} \psi_\nu) + (\Psi_f^* \beta \vec{\alpha} \vec{\tau}^{(+)} \Psi_i) \cdot (\psi_e^* \beta \vec{\alpha} \beta_\nu) \qquad \text{(c)} \qquad (3.6)$$

$$\mathcal{M}_A = (\Psi_f^* \vec{\sigma} \vec{\tau}^{(+)} \Psi_i) \cdot (\psi_e^* \vec{\sigma} \psi_\nu) - (\Psi_f^* \gamma_5 \tau^{(+)} \Psi_i)(\psi_e^* \gamma_5 \psi_\nu) \qquad \text{(d)}$$

$$\mathcal{M}_P = (\Psi_f^* \beta \gamma_5 \tau^{(+)} \Psi_i)(\psi_e^* \beta \gamma_5 \psi_\nu) \qquad \text{(e)}$$

"large" "small"

ψ_e = electron wave function,
ψ_ν = neutrino wave function,
Ψ_i = wave function of nucleus in initial state,
Ψ_f = wave function of nucleus in final state,
$\tau^{(+)}$ = the transition operator for the transformation of a neutron into a proton (cf. App. IV for this notation).

The expression (3.5) specifies a Hamiltonian density $\mathcal{H}_\beta(\vec{x})$: energy density - energy pro unit volume; the four wave functions must be taken at the same place. The total Hamiltonian (or strictly speaking its matrix element) is obtained by integration over all space

$$H_\beta = \int d^3\vec{x}\, \mathcal{M}_\beta(\vec{x}). \qquad (3.7)$$

The matrix element for β-emission occurring in (3.2) may now be written more explicitly as

$$|\langle f| H_\beta |i\rangle|^2 = \Sigma_{mf} \Sigma_{mi}^{av} |\Sigma_{h=1}^{A} \Sigma_{k=1}^{5} \int d\tau\, g_k (\Psi_f^* \Omega_k \tau_h^{(+)} \Psi_i)(\psi_e^* \Omega_k \psi_\nu)_h|^2 . \qquad (3.8)$$

Σ_{mf} indicates summation over the different orientations of the final nucleus specified by the magnetic quantum number m_f.

$\Sigma_{mi}^{av} = \frac{1}{2j_i+1} \Sigma_{mi}$ indicates that we must average over the different orientations of the initial nucleus if we take an initial state consisting of an ensemble of nuclei without preferential orientation in space (j_i = spin of initial nucleus; m_i its magnetic quantum number).

Ψ_i and Ψ_f in (3.8) represent the wave functions of the initial and final nucleus (A nucleons).

$\Sigma_{h=1}^{A} \tau_h^{(+)}$ indicates that β-emission is possible in principle for each nucleon ($\tau_h^{(+)}$: transition operator for the h^{th} nucleon). The subscript h with $(\psi_e^* \Omega_k \psi_i)_h$ indicates that these wave functions must be taken at the place where the nucleon is transformed. $\int d\tau$ indicates integration (or summation) over the variables occurring in the nuclear wave function.

For the calculation of allowed β-transitions we shall neglect the "small" terms of the $\mathcal{M}_k$ (cf (3.6)) (we shall only carry along the "small" term of $\mathcal{M}_P$ since g_P might be large in comparison with the other g_k's).

We want to calculate allowed transitions, neglecting the influence of the Coulomb field of the nucleus on the lepton wave functions. We can then simply take the lepton wave functions as constant over the nuclear volume and can thus place them in front of the integral sign in (3.8). (This is the second approximation made for the calculation of allowed transitions.) In this way the matrix element (3.8) for β^--emission takes the following form (we no longer write Σ_{mf} and Σ^{av}_{mi} explicitly for the sake of simplicity.)

$$|\langle f|H_\beta|i\rangle|^2 = |\Sigma_{k=1}^5 g_k (\psi_e^* \Omega_k \psi_\nu) \int \Omega_k|^2 , \tag{3.9}$$

where $\int \Omega_k$ is introduced as a concise notation for the nuclear matrix element, for β^--emission

$$\int \Omega_k = \Sigma_h \int d\tau \Psi_f^* \Omega_k \tau_h^{(+)} \Psi_i . \tag{3.10}$$

For β^+-emission we could write

$$\int \Omega_k = \Sigma_h \int d\tau \Psi_f^* \Omega_k \tau_h^{(-)} \Psi_i . \tag{3.11}$$

We note that for T- and A-interaction the operator Ω_k contains $\vec{\sigma}$ and the product of the lepton factor and the nuclear matrix element is then a 3-dimensional inner product.

The expression (3.9) can be worked out in the following way

$$|\langle f|H_\beta|i\rangle|^2 = \Sigma_{k,\ell=1}^5 g_k g_\ell^* (\psi_e^* \Omega_k \psi_\nu)(\psi_\nu^* \Omega_\ell \psi_e)(\int \Omega_k)(\int \Omega_\ell)^* . \tag{3.12}$$

Here we made use of the fact that the Ω_k are all hermitian. $\rho_2 = -i\beta\gamma_5$ is hermitian; we have used $\Omega_5 = \rho_2$; note that we used the non-hermitian matrix $\beta\gamma_5$ in (3.6)(e). For a further reduction of (3.12) we make use of the reduction (the Greek subscripts indicate the components of the Dirac spinors):

$$\begin{aligned}
&\Sigma_e \Sigma_\nu (\psi_e^* \Omega_k \psi_\nu)(\psi_\nu^* \Omega_\ell \psi_e) \\
&= \Sigma_e \Sigma_\nu [\Sigma_{\rho,\sigma} (\psi_e^*)_\rho (\Omega_k)_{\rho\sigma} (\psi_\nu)_\sigma][\Sigma_{\lambda,\mu} (\psi_\nu^*)_\lambda (\Omega_\ell)_{\lambda\mu} (\psi_e)_\mu] \\
&= \Sigma_e \Sigma_\nu \Sigma_{\lambda,\mu,\rho,\sigma} (\Omega_k)_{\rho\sigma} (\psi_\nu)_\sigma (\psi_\nu^*)_\lambda (\Omega_\ell)_{\lambda\mu} (\psi_e)_\mu (\psi_e^*)_\rho \\
&= \mathrm{Tr}\,(\Omega_k D_\nu \Omega_\ell D_e).
\end{aligned} \tag{3.13}$$

The 4×4 matrices D_ν and D_e are defined by

$$\begin{aligned}
(D_\nu)_{\sigma\lambda} &= \Sigma_\nu (\psi_\nu)_\sigma (\psi_\nu^*)_\lambda \\
(D_e)_{\mu\rho} &= \Sigma_e (\psi_e)_\mu (\psi_e^*)_\rho .
\end{aligned} \tag{3.14}$$

By Tr $(\Omega) = \Sigma_\rho \Omega_{\rho\rho}$ we indicate the trace of a Dirac matrix (the sum of the diagonal elements). Using (3.12) and (3.13) we see that we can write

$$\Sigma_e \Sigma_\nu |\langle f|H_\beta|i\rangle|^2 = \Sigma_{k,\ell=1}^{5} g_k g_\ell^* \left(\int \Omega_k\right) \left(\int \Omega_\ell\right)^* \mathrm{Tr}\,(\Omega_k D_\nu \Omega_\ell D_e)\,. \qquad (3.15)$$

The calculation can in this way be reduced to the calculation of traces of products of Dirac-matrices. It is easy to show that D_e and D_ν are the projection operators for "positive" or "negative" energies and thus to give their explicit form. For β^--emission we must write (cf. App. III)

$$D_e = \tfrac{1}{2}\left[1 - \frac{\vec{\alpha}\cdot\vec{p}+\beta}{E}\right] = \tfrac{1}{2}\left[1 - \frac{\rho_1 \vec{\sigma}\cdot\vec{p}+\rho_3}{E}\right] \qquad (3.16)$$

$$D_\nu = \tfrac{1}{2}\left[1 - \frac{\vec{\alpha}\cdot\vec{q}}{E_\nu}\right] = \tfrac{1}{2}\left[1 - \frac{\rho_1 \vec{\sigma}\cdot\vec{q}}{E_\nu}\right] \quad (q = E_\nu). \qquad (3.17)$$

For the calculation of the traces we use the following important property: The only matrix of the 16 Dirac matrices for which the trace is different from zero is the unit matrix

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for which Tr}\,(1) = 4. \qquad (3.18)$$

Because of the property (3.18) we must look in the terms of $\Omega_k D_\nu \Omega_\ell D_e$ only for those products which give the unit matrix, as other products give zero when calculating the trace. It is more convenient to use the representation of the Dirac matrices with the aid of the ρ- and σ-matrices than to use the general γ_μ-matrices because the introduction of a non-relativistic approximation for the nucleons can be represented conveniently with the ρ-matrices. (cf. App. I).

We note that the traces for the ρ- and σ-matrices may be calculated separately: the ρ- and σ-matrices commute mutually and can be considered as (2×2) matrices acting on two mutually independent variables (cf. Appendix I). We can write:

$\mathrm{Tr}\,[\ldots] = \mathrm{Tr}_{\rho,\sigma}[\ldots] = \mathrm{Tr}_\rho\ \mathrm{Tr}_\sigma\,[\ldots]$. We have then $\mathrm{Tr}_\sigma\,(1) = 2$, $\mathrm{Tr}_\sigma\,(\sigma_\kappa) = 0$ etc. One might say that taking the traces with respect to the ρ's only, means a reduction from the 4-component Dirac spinors to 2-component spinors.

In order to obtain the result for (3.15) we have to calculate the trace of the products resulting from all possible combinations in the following expression (each time provided with the corresponding factors g_k and $\int \Omega_k$)

$$\begin{matrix} g_S \\ g_V \\ g_T \\ \\ \\ g_A \\ \\ g_P \end{matrix} \left\{ \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \right. \begin{matrix} \frac{1}{4}\,\mathrm{Tr}\Big[\\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \left\{ \begin{matrix} \rho_3 \\ 1 \\ \rho_3\sigma_1 \\ \rho_3\sigma_2 \\ \rho_3\sigma_3 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ i\rho_2 \end{matrix} \right\} \left(1 - \frac{\rho_1 \vec{\sigma}\cdot\vec{q}}{E_\nu}\right) \left\{ \begin{matrix} \rho_3 \\ 1 \\ \rho_3\sigma_1 \\ \rho_3\sigma_2 \\ \rho_3\sigma_3 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ i\rho_2 \end{matrix} \right\} \left(1 - \frac{\rho_1\vec{\sigma}\cdot\vec{p} + \rho_3}{E}\right) \Big] . \qquad (3.19)$$

We shall give as examples the detailed calculation of some of the terms of the entire expression. We first take the term which has the factor $|g_S|^2\,|\int\beta|^2$ in front; for this term we have to evaluate the trace

$$\begin{aligned} &\tfrac{1}{4}\,\mathrm{Tr}\,[\rho_3(1 - \rho_1\frac{\vec{\sigma}\cdot\vec{q}}{E_\nu})\,\rho_3\,(1 - \frac{\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3}{E})] \\ &= \tfrac{1}{4}\,\mathrm{Tr}\,[\rho_3^2(1 + \rho_1\frac{\vec{\sigma}\cdot\vec{q}}{E_\nu})(1 - \frac{\rho_1\vec{\sigma}\cdot\vec{p} + \rho_3}{E})] \qquad (3.20) \\ &= \tfrac{1}{2}\,\mathrm{Tr}_\sigma[\,1 - \frac{(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{p})}{E_\nu E}\,], \end{aligned}$$

where we have used $\rho_3^2 = 1$ and have taken the trace for the ρ's. From the properties of the Pauli-matrices one sees immediately

$$\tfrac{1}{2}\,\mathrm{Tr}_\sigma[(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{p})] = \vec{p}\cdot\vec{q}\ . \qquad (3.21)$$

Hence we see that (3.20) reduces to

$$1 - (\vec{p}\cdot\vec{q})/EE_\nu. \qquad (3.22)$$

The calculation of the terms caused by the T- and A- interactions is slightly more complicated. If we abbreviate the matrix element $\int\vec{\sigma}$ as $\vec{\ell} = \int\vec{\sigma}$ the contribution from the A- interaction (with coefficient $|g_A|^2$) can be written as

$$\begin{aligned} &\tfrac{1}{4}\,\mathrm{Tr}\,[(\vec{\sigma}\cdot\vec{\ell})(1 - \rho_1(\vec{\sigma}\cdot\vec{q})/E_\nu)(\vec{\sigma}\cdot\vec{\ell}^*)(1 - (\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3)/E)] \\ &= \tfrac{1}{2}\,\mathrm{Tr}\,[(\vec{\sigma}\cdot\vec{\ell})(\vec{\sigma}\cdot\vec{\ell}^*) + (\vec{\sigma}\cdot\vec{\ell})(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{\ell}^*)(\vec{\sigma}\cdot\vec{p})/EE_\nu] \qquad (3.23) \end{aligned}$$

In order to calculate the trace of the fourfold product in (3.23) we decompose $\vec{\ell}$ in components parallel to $\vec{q}$ and perpendicular to $\vec{q}$:

$$\vec{\ell} = \vec{\ell}_\parallel + \vec{\ell}_\perp;\ \vec{\ell}_\parallel\cdot\vec{q} = \vec{\ell}\cdot\vec{q};\ \vec{\ell}_\perp\cdot\vec{q} = 0;\ \vec{\ell}_\parallel\cdot\vec{\ell}_\perp = \vec{\ell}_\parallel\cdot\vec{\ell}_\perp^* = 0. \qquad (3.24)$$

It results that $\vec{\sigma}\cdot\vec{q}$ commutes with $\vec{\sigma}\cdot\vec{\ell}_\parallel$ (and $\vec{\sigma}\cdot\vec{\ell}_\parallel^*$) and anticommutes with $\vec{\sigma}\cdot\vec{\ell}_\perp$ (and $\vec{\sigma}\cdot\vec{\ell}_\perp^*$). It should be noted that $\vec{q}$ is a real vector but that $\vec{\ell}$ is in

general a complex vector. We can now reduce the fourfold product in (3.23) in the following way

$$(\vec{\sigma}.\vec{\ell})(\vec{\sigma}.\vec{q})(\vec{\sigma}.\vec{\ell}^*)(\vec{\sigma}.\vec{p}) = (\vec{\sigma}.\vec{\ell})(\vec{\sigma}.\vec{\ell}_{\parallel}^*) - \vec{\sigma}.\vec{\ell}_{\perp}^*)(\vec{\sigma}.\vec{q})(\vec{\sigma}.\vec{p}) . \quad (3.25)$$

We make use of the following identity with Pauli matrices

$$(\vec{\sigma}.\vec{a})(\vec{\sigma}.\vec{b}) = \vec{a}.\vec{b} + i\vec{\sigma}.(\vec{a}\times\vec{b}) \quad (3.26)$$

so that

$$(\vec{\sigma}.\vec{\ell})(\vec{\sigma}.\vec{\ell}_{\parallel}^* - \vec{\sigma}.\vec{\ell}_{\perp}^*) = |\vec{\ell}_{\parallel}|^2 - |\vec{\ell}_{\perp}|^2 + i\vec{\sigma}.(\vec{\ell}\times\vec{\ell}_{\parallel}^* - \vec{\ell}\times\vec{\ell}_{\perp}^*) . \quad (3.27)$$

In the problem under consideration we have to average over the orientations of the initial and final nucleus; we thus obtain

$$\Sigma_{m_f}\Sigma_{m_i}^{av}|\vec{\ell}_{\parallel}|^2 = \tfrac{1}{3}\left|\int\vec{\sigma}\right|^2 \quad (3.28)$$

$$\Sigma_{m_f}\Sigma_{m_i}^{av}|\vec{\ell}_{\perp}|^2 = \tfrac{2}{3}\left|\int\vec{\sigma}\right|^2 \quad (3.29)$$

putting

$$\Sigma_{m_f}\left[\left|\int\sigma_x\right|^2 + \left|\int\sigma_y\right|^2 + \left|\int\sigma_z\right|^2\right] = \left|\int\vec{\sigma}\right|^2 . \quad (3.30)$$

It can be proved that (3.30) is independent of m_i: once the summation Σ_{m_f} for the final nucleus is carried out, the result is independent of the quantum number m_i characterizing the orientation of the initial nucleus. One must further take $\Sigma_{m_f}\Sigma_{m_i}^{av}$ for the vectors $\vec{\ell}\times\vec{\ell}_{\parallel}^*$ and $\vec{\ell}\times\vec{\ell}_{\perp}^*$. It is simply proved that both vectors average out to zero: if we take e.g. the direction of $\vec{q}$ as the z-axis, it is seen that the components of both vectors correspond to components of $\vec{\ell}\times\vec{\ell}^*$. It is clear that averaging must make this vector zero, for after averaging no preferential direction in space subsists which could determine a direction for $\vec{\ell}\times\vec{\ell}^*$.

We thus find for (3.23), using (3.21) and (3.25)...(3.30),

$$\left|\int\vec{\sigma}\right|^2[1 - (\vec{p}.\vec{q})/3EE_\nu] . \quad (3.31)$$

The calculation of the other terms is performed in entirely analogous ways. The only simplification which we can still make is by putting

$$\int\beta = -\int 1 \quad \text{and} \quad \int\beta\vec{\sigma} = -\int\vec{\sigma}. \quad (3.32)$$

This can be done in the non-relativistic approximation for the nucleons, because in our representation

$$\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

while the 3 and 4-components are the "large" ones in the non-relativistic approximation. The final formula, which is thus found for β^--emission, is

$$P_-(E, \vec{p}, \vec{q}) = pEq^2/(2\pi)^5\{(|g_S|^2 + |g_V|^2)\left|\int 1\right|^2 + (|g_T|^2 + |g_A|^2)1\left|\int \vec{\sigma}\right|^2 + |g_P|^2 \left|\int \beta\gamma_5\right|^2$$

$$- (\vec{p}.\vec{q}/qE)[(|g_S|^2 - |g_V|^2)\left|\int 1\right|^2 - \tfrac{1}{3}(|g_T|^2 - |g_A|^2)\left|\int \vec{\sigma}\right|^2 + |g_P|^2 \left|\int \beta\gamma_5\right|^2]$$

$$+ (2/E)[\mathrm{Re}(g_S g_V^*)\left|\int 1\right|^2 + \mathrm{Re}(g_T g_A^*)\left|\int \vec{\sigma}\right|^2]\} \quad . \tag{3.33}$$

For the calculation of β^+-emission we need the second part (herm. conj.) of the Hamiltonian (3.5). We thus obtain, analogous to (3.15),

$$\Sigma_e \Sigma_\nu |\langle f|H_\beta|i\rangle|^2 = \Sigma_{k,\ell=1}^5 g_k^* g_\ell \left(\int \Omega_k^*\right)\left(\int \Omega_\ell\right) \mathrm{Tr}\,(\Omega_k D_\nu \Omega_\ell D_e) \quad . \tag{3.34}$$

We must now take negative energy solutions for the neutrinos. The final result for the transition probability is

$$P_+(E, \vec{p}, \vec{q}) = pEq^2/(2\pi)^5\{(|g_S|^2 + |g_V|^2)\left|\int 1\right|^2 + (|g_T|^2 + |g_A|^2)\left|\int \vec{\sigma}\right|^2 + |g_P|^2\left|\int \beta\gamma_5\right|^2$$

$$- (\vec{p}.\vec{q}/qE)[(|g_S|^2 - |g_V|^2)\left|\int 1\right|^2 - \tfrac{1}{3}(|g_T|^2 - |g_A|^2)\left|\int \vec{\sigma}\right|^2 + |g_P|^2\left|\int \beta\gamma_5\right|^2]$$

$$- (2/E)[\mathrm{Re}(g_S g_V^*)\left|\int 1\right|^2 + \mathrm{Re}\,(g_T g_A^*)\left|\int \vec{\sigma}\right|^2]\} \quad . \tag{3.34a}$$

3.3. Directional distribution of β-rays from oriented nuclei

We now want to generalize the treatment of 3.2 to a more general Hamiltonian, which is not necessarily invariant for C, P or T. We shall use instead of (3.5) an interaction Hamiltonian specified by

$$\mathcal{H}_\beta = \tfrac{1}{\sqrt{2}} \Sigma_{k=1}^5 (g_k \mathcal{H}_k + g_k' \mathcal{H}_k') + \mathrm{h.c.} \tag{3.35}$$

with

$$\mathcal{H}_k = (\Psi_f^* \Omega_k \tau^{(+)} \Psi_i)(\psi_e^* \Omega_k \psi_\nu) \tag{3.36}$$

$$\mathcal{H}_k' = (\Psi_f^* \Omega_k \tau^{(+)} \Psi_i)(\psi_e^* \Omega_k \gamma_5 \psi_\nu). \tag{3.37}$$

We admit complex values for g_k and g_k'; this means that in general the Hamiltonian will not be invariant neither for C, P nor T transformations.

We have not specialized for the 2-component neutrino theory, but have formulated an interaction for general 4-component neutrinos.

Specialization for the two-component neutrino theory is obtained for $g_k' = \pm g_k$. For $g_k' = g_k$ the interaction may be written e.g. as

$$\mathcal{H}_\beta = \frac{1}{\sqrt{2}} \Sigma_{k=1}^{5} g_k (\psi_f^* \Omega_k \tau^{(+)} \psi_i)(\psi_e^* \Omega_k (1 + \gamma_5) \psi_\nu) + \text{h. c.} \qquad (3.38)$$

The factor $\frac{1}{\sqrt{2}}$ which we have added is purely conventional; it has the advantage that the numerical values of g_k obtained from (3.38) and experiments are the same as the "old" value obtained using the "parity-conserving" theory of 3.2. We shall not give detailed calculations for all possible "effects", which are a consequence of the interaction (3.35), but we shall confine ourselves to the effects which can be measured most easily: directional distribution of β-rays from oriented nuclei, β-γ circular polarization correlation and polarization of β-rays. The calculations can be made according to the lines of 3.2; we give the explicit calculations for a number of typical terms.

We first consider the directional distribution of β-rays from oriented nuclei for a mixture of V- and A-interactions. We should then not carry out the averaging $\Sigma_{m_i}^{av}$ indicated in (3.8), but should take the summation over the final states Σ_{m_f}. We want to average over the directions of emission of the neutrino. It is seen from (3.15) and (3.17) that this averaging is simply carried out by omitting the term linear in $\vec{q}$ in (3.17), hence by taking $D_\nu = \frac{1}{2}$. We now list the traces which have to be calculated for a mixture of V- and A-interactions (they are analogous to (3.20) and (3.23)). We add the factor containing the coupling constant, by which the trace has to be multiplied. We do not yet give the results for the traces, although we indicate that some are equal to zero.

$$\tfrac{1}{2} |g_V|^2 \; |\int 1|^2 \cdot \tfrac{1}{4} \operatorname{Tr} [\, 1 \cdot 1 \cdot 1 \cdot (1 - \frac{\rho_1 \vec{\sigma} \cdot \vec{p} + \rho_3}{E}) \,]. \qquad (3.39)$$

For the term with $|g_V'|^2$ we have to add $\gamma_5 = \rho_1$ to Ω_k in the lepton operator; this provides

$$\tfrac{1}{2} |g_V'|^2 \; |\int 1|^2 \cdot \tfrac{1}{4} \operatorname{Tr} [\, \rho_1 \cdot 1 \cdot \rho_1 \cdot (1 - \frac{\rho_1 \vec{\sigma} \cdot \vec{p} + \rho_3}{E}) \,]. \qquad (3.40)$$

Further we have to consider the cross terms with g_V and g_V'

$$\tfrac{1}{2} g_V g_V'^* \; |\int 1|^2 \cdot \tfrac{1}{4} \operatorname{Tr} [\, 1 \cdot 1 \cdot \rho_1 \cdot (1 - \frac{\rho_1 \vec{\sigma} \cdot \vec{p} + \rho_3}{E}) \,] = 0 \qquad (3.41)$$

$$\tfrac{1}{2} g_V' g_V^* \; |\int 1|^2 \cdot \tfrac{1}{4} \operatorname{Tr} [\, \rho_1 \cdot 1 \cdot 1 \cdot (1 - \frac{\rho_1 \vec{\sigma} \cdot \vec{p} + \rho_3}{E}) \,] = 0. \qquad (3.42)$$

If we again abbreviate $\vec{\ell} = \int \vec{\sigma}$, we find as terms with the A-interaction

$$\tfrac{1}{2} |g_A|^2 \; \tfrac{1}{4} \operatorname{Tr} [\, (\vec{\sigma} \cdot \vec{\ell}) \cdot 1 \cdot (\vec{\sigma} \cdot \vec{\ell}^*) (1 - \frac{\rho_1 \vec{\sigma} \cdot \vec{p} + \rho_3}{E}) \,] \qquad (3.43)$$

$$\tfrac{1}{2}\,|g_A'|^2 \quad \tfrac{1}{4}\,\mathrm{Tr}\,[\rho_1(\vec{\sigma}\cdot\vec{\ell})\cdot 1\cdot\rho_1(\vec{\sigma}\cdot\vec{\ell}^*)(1-\frac{\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3}{E})] \tag{3.44}$$

$$\tfrac{1}{2}\,g_A g_A'^* \quad \tfrac{1}{4}\,\mathrm{Tr}\,[(\vec{\sigma}\cdot\vec{\ell})\cdot 1\cdot\rho_1(\vec{\sigma}\cdot\vec{\ell}^*)(1-\frac{\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3}{E})] \tag{3.45}$$

$$\tfrac{1}{2}\,g_A' g_A^* \quad \tfrac{1}{4}\,\mathrm{Tr}\,[(\rho_1(\vec{\sigma}\cdot\vec{\ell})\cdot 1\cdot(\vec{\sigma}\cdot\vec{\ell}^*)(1-\frac{\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3}{E})]\,. \tag{3.46}$$

Finally we have to consider the (V, A) cross terms

$$\tfrac{1}{2}\,g_V g_A^* \quad (\int 1)\,\tfrac{1}{4}\,\mathrm{Tr}\,[1\cdot 1\cdot(\vec{\sigma}\cdot\vec{\ell}^*)\,(1-\frac{\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3}{E})] = 0. \tag{3.47}$$

Analogously the terms with the factors $g_A g_V^*$, $g_V' g_A'^*$, $g_A' g_V'^*$ are equal to zero. However, in general we have a result which is different from zero for the terms

$$\tfrac{1}{2}\,g_V g_A'^* \quad (\int 1)\,\tfrac{1}{4}\,\mathrm{Tr}\,[1\cdot 1\cdot\rho_1(\vec{\sigma}\cdot\vec{\ell}^*)\cdot(1-\frac{\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3}{E})] \tag{3.48}$$

$$\tfrac{1}{2}\,g_A' g_V^* \quad (\int 1)^*\,\tfrac{1}{4}\,\mathrm{Tr}\,[\rho_1(\vec{\sigma}\cdot\vec{\ell})\cdot 1\cdot 1\cdot(1-\frac{\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3}{E})] \tag{3.49}$$

$$\tfrac{1}{2}\,g_V' g_A^* \quad (\int 1)\,\tfrac{1}{4}\,\mathrm{Tr}\,[\rho_1\cdot 1\cdot(\vec{\sigma}\cdot\vec{\ell}^*)\cdot(1-\frac{\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3}{E})] \tag{3.50}$$

$$\tfrac{1}{2}\,g_A g_V'^* \quad (\int 1)^*\,\tfrac{1}{4}\,\mathrm{Tr}\,[(\vec{\sigma}\cdot\vec{\ell})\cdot 1\cdot\rho_1(1-\frac{\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3}{E})]\,. \tag{3.51}$$

The calculation of the traces is performed along the same lines as in 3.2. We find as the sum of the contributions from the V-interaction (3.39)..(3.42)

$$\tfrac{1}{2}\,(|g_V|^2 + |g_V'|^2)\,|\int 1\,|^2\,. \tag{3.52}$$

Analogously we have for the sum of the contributions (3.43) and (3.44)

$$\tfrac{1}{2}\,(|g_A|^2 + |g_A'|^2)\,|\int\vec{\sigma}|^2, \tag{3.53}$$

where we again introduced $|\int\vec{\sigma}|^2$ instead of $|\vec{\ell}|^2$. For the calculation of the traces in (3.45) and (3.46) we first take the trace with respect to the ρ's; this provides

$$-\tfrac{1}{2}\,\mathrm{Tr}_\sigma\,[(\vec{\sigma}\cdot\vec{\ell})(\vec{\sigma}\cdot\vec{\ell}^*)\cdot(\vec{\sigma}\cdot\vec{p}/E)\,. \tag{3.54}$$

Applying the identity (3.26) to $(\vec{\sigma}\cdot\vec{\ell})\,(\vec{\sigma}\cdot\vec{\ell}^*)$ we have

$$(\vec{\sigma}\cdot\vec{\ell})(\vec{\sigma}\cdot\vec{\ell}^{*}) = |\vec{\ell}|^2 + i\vec{\sigma}\cdot(\vec{\ell}\times\vec{\ell}^{*}) . \tag{3.55}$$

Substituting this into (3.54) and taking the trace provides

$$- i\,(\vec{\ell}\times\vec{\ell}^{*})\cdot(\vec{p}/E). \tag{3.56}$$

Analogously we obtain from (3.48) and (3.50)

$$- (\int 1)\,\vec{\ell}^{*}\cdot(\vec{p}/E) \tag{3.57}$$

while (3.49) and (3.51) give

$$- (\int 1)^{*}\,(\vec{\ell}\cdot\vec{p}/E). \tag{3.58}$$

Collecting all the terms we thus obtain

$$\tfrac{1}{2}\,(|g_V|^2 + |g_V'|^2)\,|\int 1|^2 + \tfrac{1}{2}\,(|g_A|^2 + |g_A'|^2)\,|\int\vec{\sigma}|^2 - \mathrm{Re}(g_A g_A'^{*})\,i(\vec{\ell}\times\vec{\ell}^{*})\cdot(\vec{p}/E)$$

$$- \mathrm{Re}\,(g_V g_A'^{*} + g_V' g_A^{*})\,|\int 1|\,(\vec{\ell}^{*}\cdot\vec{p}/E) \tag{3.59}$$

(we can take (3.57) and (3.58) together because $(\int 1)\,\vec{\ell}^{*}$ is real; cf. paragraph after formula (3.77)). We shall see that for nuclei, showing Gamow-Teller transitions, the asymmetric directional distribution of the electrons is caused by the terms (3.56)..(3.58). For this purpose the expressions $i(\vec{\ell}\times\vec{\ell}^{*})$ and $(\int 1)\,\vec{\ell}^{*}$ containing nuclear matrix elements must be expressed in terms of the degree of polarization of the nuclei. We need here the expressions which specify the dependence on m_i and m_f of the matrix elements of the different components of $\vec{\sigma}$. This dependence is provided by general considerations of angular momentum theory and given by Clebsch-Gordan coefficients. We call the quantum numbers for the total angular momentum and the magnetic quantum number of the initial state j_i, m_i and for the final state j_f, m_f. The dependence on m_i and m_f is specified most conveniently by using

$$\begin{aligned}\sigma_+ &= \tfrac{1}{2}\,(\sigma_x + i\sigma_y)\\ \sigma_- &= \tfrac{1}{2}\,(\sigma_x - i\sigma_y)\end{aligned} \tag{3.60}$$

The dependence on m_i and m_f can then be given by (a, b and c are constants; we do not start with a definite normalization, as this is carried out in the course of the calculation):

$$\begin{aligned} j_f = j_i - 1;\ &\langle m_f = m_i+1\,|\sigma_+|\,m_i\rangle = \tfrac{1}{2}\,a\sqrt{(j_i - m_i)(j_i - m_i - 1)}\\ &\langle m_f = m_i-1\,|\sigma_-|\,m_i\rangle = -\tfrac{1}{2}\,a\sqrt{(j_i + m_i)(j_i + m_i - 1)}\\ &\langle m_f = m_i\ \ |\sigma_z|\,m_i\rangle = \ a\sqrt{(j_i + m_i)(j_i - m_i)} \end{aligned} \tag{3.61}$$

$$j_f = j_i \qquad \left.\begin{aligned} \langle m_f = m_i+1 |\sigma_+| m_i\rangle &= \tfrac{1}{2}\, b\, \sqrt{(j_i + m_i + 1)(j_i - m_i)} \\ \langle m_f = m_i - 1 |\sigma_-| m_i\rangle &= \tfrac{1}{2}\, b\, \sqrt{(j_i + m_i)(j_i - m_i + 1)} \end{aligned}\right\} \quad (3.62)$$

$$\langle m_f = m_i \quad |\sigma_z| m_i\rangle = b\, m_i$$

$$j_f = j_i + 1 \qquad \left.\begin{aligned} \langle m_f = m_i+1 |\sigma_+| m_i\rangle &= -\tfrac{1}{2}\, c\, \sqrt{(j_i + m_i + 2)(j_i + m_i + 1)} \\ \langle m_f = m_i - 1 |\sigma_-| m_i\rangle &= \tfrac{1}{2}\, c\, \sqrt{(j_i - m_i + 2)(j_i - m_i + 1)} \end{aligned}\right\} (3.63)$$

$$\langle m_f = m_i \quad |\sigma_z| m_i\rangle = c\, \sqrt{(j_i + m_i + 1)(j_i - m_i + 1)}$$

All matrix elements which are not written are equal to zero. From the definitions of σ_+ and σ_- it follows immediately that:

$$4\left|\int \sigma_+\right|^2 = \left|\int \sigma_x\right|^2 + \left|\int \sigma_y\right|^2 + i\left(\int \sigma_y\right)\left(\int \sigma_x\right)^* - i\left(\int \sigma_x\right)\left(\int \sigma_y\right)^* \quad (3.64)$$

$$4\left|\int \sigma_-\right|^2 = \left|\int \sigma_x\right|^2 + \left|\int \sigma_y\right|^2 - i\left(\int \sigma_y\right)\left(\int \sigma_x\right)^* + i\left(\int \sigma_x\right)\left(\int \sigma_y\right)^* \quad (3.65)$$

By adding and subtracting one finds

$$\left|\int \vec{\sigma}\right|^2 = 2\left[\left|\int \sigma_+\right|^2 + \left|\int \sigma_-\right|^2\right] + \left|\int \sigma_z\right|^2 \quad (3.66)$$

$$i\left[\left(\int \vec{\sigma}\right) \times \left(\int \vec{\sigma}\right)^*\right]_z = i\left(\int \sigma_x\right)\left(\int \sigma_y\right)^* - i\left(\int \sigma_y\right)\left(\int \sigma_x\right)^* = 2\left[\left|\int \sigma_-\right|^2 - \left|\int \sigma_+\right|^2\right] . \quad (3.67)$$

We have already noted that Σ_{m_f} must be carried out for our case; if we specify for the case that $j_f = j_i$, we obtain by squaring and adding from (3.62), (3.66) and (3.67)

$$\Sigma_{m_f}\left|\int \vec{\sigma}\right|^2 = |b^2|\, j_i\,(j_i + 1) \quad (3.68)$$

$$\Sigma_{m_f} 2\left[\left|\int \sigma_-\right|^2 - \left|\int \sigma_+\right|^2\right] = |b^2| m_i . \quad (3.69)$$

We now think of polarized nuclei with the z-axis as the direction of polarization (axis of rotational symmetry). We indicate the unit vector in this direction by $\vec{j}$. From (3.68) and (3.69) it follows immediately that:

$$\sum_{m_f} i\left[\left(\int \vec{\sigma}\right) \times \left(\int \vec{\sigma}\right)^*\right]_z = \sum_{m_f} 2\left[\left|\int \sigma_-\right|^2 - \left|\int \sigma_+\right|^2\right] = (m_i / j_i(j_i + 1)) \sum_{m_f}\left|\int \vec{\sigma}\right|^2 . \quad (3.70)$$

As $\vec{j}$ is an axis of rotational symmetry it is clear that $\Sigma_{m_f}(\int \vec{\sigma}) \times (\int \vec{\sigma})^*$ can

only have the direction $\vec{j}$. This is also easily concluded formally from (3.62), for if one writes

$$i\,[(\int\vec{\sigma})\times(\int\vec{\sigma})^*]_x = i\,(\int\sigma_y)(\int\sigma_z)^* - i\,(\int\sigma_z)(\int\sigma_y)^* \tag{3.71}$$

it is seen from (3.62) that $\int\sigma_y$ and $\int\sigma_z$ can not simultaneously differ from zero if the same m_i and m_f are taken for both. Hence (3.71) is equal to zero for every m_f and $\Sigma_{m_f} i\,[(\int\vec{\sigma})\times(\int\vec{\sigma})^*]_x = 0$; similarly $\Sigma_{m_f} i\,[(\int\vec{\sigma})\times(\int\vec{\sigma})^*]_y = 0$.

If we now consider an ensemble of nuclei, for which the population of the different substates with magnetic quantum number m_i is given by a_{m_i}, (normalization according to $\Sigma_{m_i} a_{m_i} = 1$) we still have to take this additional average. This provides

$$\Sigma_{m_i} a_{m_i} \Sigma_{m_f} i[(\int\vec{\sigma})\times(\int\vec{\sigma})^*] = (j_i+1)^{-1}\,(\Sigma_{m_i}\frac{m_i}{j_i}\,a_{m_i})\,\Sigma_{m_f}|\int\vec{\sigma}|^2\,\vec{j} = (j_i+1)^{-1} f_1 \vec{j}\;|\,M_{GT}|^2. \tag{3.72}$$

We have introduced here the degree of polarization f_1 of the nuclei, defined as

$$f_1 = \Sigma_{m_i}(m_i/j_i)\,a_{m_i}. \tag{3.73}$$

We have further put

$$|M_{GT}|^2 = \Sigma_{m_f}|\int\vec{\sigma}|^2. \tag{3.74}$$

In (3.72) we have found the final form for the term (3.56). We have to make a similar reduction for the terms (3.57), (3.58) with $(\int 1)\,(\int\vec{\sigma})^*$. We have here that $\int 1 = \langle m_f\,|\,1\,|\,m_i\rangle$ differs from zero only for $m_f = m_i$. Thus it is seen that the combination $(\int 1)\times(\int\vec{\sigma})^*$ differs from zero only for $m_f = m_i$ and is then given by

$$(\int 1)(\int\sigma_z)^* = b^*\,m_i\,(\int 1). \tag{3.75}$$

We can now put (if $m_i = m_f$)

$$(\int 1)(\int\sigma_z)^* = [m_i/(j_i(j_i+1))^{\frac{1}{2}}]\;M_F M_{GT}^*. \tag{3.76}$$

where $M_F = (\int 1)$.

This agrees with the absolute value of $|M_{GT}|^2$ according to (3.68) and (3.74). As regards the phase (especially the sign) of $M_F M_{GT}^*$, (3.76) has to be considered a definition.

If we again take an average for an ensemble of nuclei, we obtain

$$\Sigma_{m_i} a_{m_i} \Sigma_{m_f}(\int 1)(\int\vec{\sigma})^* = [j_i/(j_i+1)]^{\frac{1}{2}} f_1 \vec{j}\,M_F M_{GT}^*. \tag{3.77}$$

The combination $(\int 1)^*\,(\int\vec{\sigma})$, which occurs in (3.58) is the complex conjugate of this. If the Hamiltonian determining the nuclear structure (strong and

electromagnetic interactions) is invariant for time reversal (which is generally assumed), it can be shown that $M_F M_{GT}^*$ is real. It follows that (3.48) and (3.49) can be written together as one term with $M_F M_{GT}^*$ and with $\mathrm{Re}(g_V g_A'^*)$ as a factor.

Substituting (3.72) and (3.77) into (3.59) we obtain the final result for the directional distribution of β-rays from oriented nuclei, in the case of V- and A-interactions; we write it in the form

$$W_-(\theta) = 1 + f_1 (v/c) A_- \cos\theta. \qquad (3.78)$$

We have put $\frac{\vec{j}\cdot\vec{p}}{E} = (v/c)\cos\theta$ (θ = angle between $\vec{j}$ and $\vec{p}$); we have used as abbreviations

$$\xi = (|g_V|^2 + |g_V'|^2)|M_F|^2 + (|g_A|^2 + |g_A'|^2)\,|M_{GT}|^2 \qquad (3.79)$$

$$A_-\xi = (-2\,\mathrm{Re}(g_A g_A'^*)/(j_i+1))\,|M_{GT}|^2 - (j_i/j_i+1)^{\frac{1}{2}}\,2\,\mathrm{Re}(g_V g_A'^* + g_V' g_A^*)\,M_F M_{GT}^*. \qquad (3.80)$$

This concerns the special case $j_f = j_i$ which we have treated in some detail as an example. At the end of this section we shall give the general formula for arbitrary interaction and arbitrary spin change of the nucleus.

3.4. The β, γ-circular polarization correlation

The calculation of this effect is to a great extent analogous to the calculation of the directional distribution of β-rays from oriented nuclei (cf. Fig. 1).

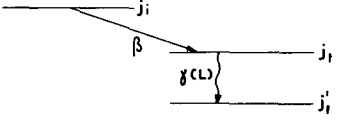


Fig. 1

Nuclear spins in the case considered for β, γ-circular polarization correlation

The difference between the calculations of the two effects is that for β,γ-circular polarization correlation the effect is determined by the degree of polarization of the nuclei after the β-emission in the state with nuclear spin j_f. In order to find this polarization one has to take the average $\Sigma_{m_i}^{av}$ for the orientations of the initial nucleus with spin j_i, but one should not take the sum Σ_{m_f}. For the directional distribution of the β-rays one must take the sum Σ_{m_f} for the different orientations of the final state, but the states with different m_i should be considered separately. We shall again give the calculation in some detail for mixed V- and A-interactions and for the case that $j_f = j_i$.

We again make use of the expressions (3.39)...(3.59) for the transition probability, but we now have to carry out the averaging $\Sigma_{m_i}^{av}$ in order to find the populations a_{m_f} of the final states of the nucleus after β-emission (with nuclear spin j_f). The natural choice of the coordinate system is to take the

direction of emission of the β-particle as the quantization-axis (z-axis), to which the magnetic quantum numbers are related. This is an axis of rotational symmetry for all further phenomena. If we denote the partial transition probability for a transition from a state with magnetic quantum number m_i before the β-emission (nuclear spin j_i) to a magnetic quantum number m_f after the β-emission (nuclear spin j_f) as $P_{m_i \to m_f}$, the populations a_{mf} can be written as

$$a_{m_f} = (\Sigma_{m_i}^{av} P_{m_i \to m_f}) / (\Sigma_{m_f} \Sigma_{m_i}^{av} P_{m_i \to m_f}). \tag{3.81}$$

It is evident that a common factor (independent of m_i and m_f) may be omitted from $P_{m_i \to m_f}$, if we merely wish to calculate the populations a_{mf}. If we first disregard normalization (determining the normalization at the end of the calculation), we may start to calculate non-normalized a'_{mf} according to

$$a'_{m_f} = \Sigma P_{m_i \to m_f}. \tag{3.82}$$

We thus find the following contributions from (3.52) and (3.53)

$$\tfrac{1}{2} (|g_V|^2 + |g'_V|^2) \Sigma_{m_i} \left| \int 1 \right|^2 \tag{3.83}$$

and

$$\tfrac{1}{2} (|g_A|^2 + |g'_A|^2) \Sigma_{m_i} \left| \int \vec{\sigma} \right|^2. \tag{3.84}$$

For the calculation of the contributions according to (3.45), (3.46) and (3.56) we have to carry out summations analogous to those for the calculation of (3.68) and (3.69), using (3.62) (it is convenient to express (3.62) in terms of j_f and m_f for this purpose).
One finds:

$$\Sigma_{m_i} i [(\int \vec{\sigma}) \times (\int \vec{\sigma})^*]_z = -|b|^2 m_f \tag{3.85}$$

and

$$\Sigma_{m_i} \left| \int \vec{\sigma} \right|^2 = -|b|^2 j_i (j_i + 1). \tag{3.86}$$

From (3.85) and (3.86) it is seen that the term with $\mathrm{Re}(g_A g'^*_A)$ can be written as

$$\mathrm{Re}\ (g_A g'^*_A) \frac{m_f \Sigma_{m_i} \left| \int \vec{\sigma} \right|^2}{j_i (j_i + 1)} \to (p/E). \tag{3.87}$$

Finally we have the contributions from (3.48) ... (3.51) which can be calculated as was done previously for (3.57) and (3.58) using (3.62). One thus finds as contributions from (3.48) and (3.49)

$$-\mathrm{Re}\ (g_V g'^*_A) (\int 1) b^* m_f \frac{p}{E}. \tag{3.88}$$

We may put, when $j_f = j_i$,

$$|M_F|^2 = \Sigma_{m_f}\left|\int 1\right|^2 = \Sigma_{m_i}\left|\int 1\right|^2 \quad , \tag{3.89}$$

and

$$|M_{GT}|^2 = \Sigma_{m_f}\left|\int \vec{\sigma}\right|^2 = \Sigma_{m_i}\left|\int \vec{\sigma}\right|^2 \quad . \tag{3.90}$$

Using (3.86) and (3.90) we can now give (3.88) the form

$$-\,\mathrm{Re}\,(g_V g_A'^{*})\; m_f\; (M_F M_{GT}^{*}/[j_i(j_i+1)]^{\frac{1}{2}})(p/E). \tag{3.91}$$

In an analogous way one obtains as the contribution from (3.50) and (3.51)

$$-\,\mathrm{Re}\,(g_V' g_A^{*})\; m_f\; (M_F M_{GT}^{*}/[j_i(j_i+1)]^{\frac{1}{2}})(p/E). \tag{3.92}$$

Collecting all the terms one obtains finally for a'_{m_f} according to (3.82)

$$a'_{m_f} = C\,(1 + A_1 m_f) \quad , \tag{3.93}$$

(C is a normalization constant) with,

$$A_1 = \frac{2\,\mathrm{Re}\,(g_A g_A'^{*})(|M_{GT}|^2/[j_i(j_i+1)]) - 2\,\mathrm{Re}\,(g_V g_A'^{*} + g_V' g_A^{*})(M_F M_{GT}^{*}/[j_i(j_i+1)]^{\frac{1}{2}})}{(|g_V|^2 + |g_V'|^2)|M_F|^2 + (|g_A|^2 + |g_A'|^2)|M_{GT}|^2}(p/E). \tag{3.94}$$

Calculating the degree of polarization f_1 from the populations a'_{m_f} according to (3.93) and (3.94) one obtains

$$f_1 = \frac{\sum_{m_f=-j_f}^{+j_f} \frac{m_f}{j_f}\, a'_{m_f}}{\sum_{m_f=-j_f}^{+j_f} a'_{m_f}} = \tfrac{1}{3}\,(j_f+1)\,A_1 \, . \tag{3.95}$$

Hence substituting (3.94) into (3.95) we find

$$f_1 = \tfrac{1}{3}\,\frac{(1/j_i)\,2\,\mathrm{Re}\,(g_A g_A'^{*})\,|M_{GT}|^2 - [(j_i+1)/j_i]^{\frac{1}{2}}\;2\,\mathrm{Re}\,(g_V g_A'^{*} + g_V' g_A^{*}) M_F M_{GT}^{*}}{(|g_V|^2 + |g_V'|^2)|M_F|^2 + (|g_A|^2 + |g_A'|^2)|M_{GT}|^2}(p/E). \tag{3.96}$$

Comparing (3.96) and (3.80) it is seen that $|M_{GT}|^2$ and $M_F M_{GT}^{*}$ are multiplied by factors for which the absolute value of their ratios is the same in both cases, although these ratios have a different sign. Once the value of f_1 is calculated the subsequent angular distribution of the γ-rays, taking the circular polarization into account, depends on the multipolarity L of the γ-radi-

ation and the nuclear spin values j_f and j_f'. We can give the β,γ-circular polarization correlation by the formula

$$W^{(\beta\gamma)}(\theta,\tau) = 1 + \tfrac{1}{3} A^{\gamma} (v/c)\tau Q_{\gamma}\cos\theta . \qquad (3.97)$$

θ is the angle between the directions of emission of the β and γ-ray; $\tau = +1$ if photons of helicity + 1 are observed; $\tau = -1$ for observation of the opposite polarization. Q_γ is a factor calculated in the theory of the angular distribution of radiations from oriented nuclei (we refer to [52]).

We give Q_γ for two simple but important cases

$$Q_{\gamma} = \begin{cases} \frac{3}{2} & \text{if} \quad j_f - j_f' = L = 1 \\ 1 & \text{if} \quad j_f - j_f' = L = 2 . \end{cases}$$

A^γ is for the special case $j_f = j_i$, given by

$$\xi A^{\gamma} = -[(j_f+1)/j_f]\{-2\,\mathrm{Re}[(g_A g_A'^{*})/(j_i+1)]\,|M_{GT}|^2 + [j/(j+1)]^{\frac{1}{2}} 2\,\mathrm{Re}\,(g_V g_A'^{*} + g_V' g_A^{*}) M_F M_{GT}^{*}\}. \qquad (3.98)$$

We have discussed the phenomena of β,γ-circular polarization here, considering γ-radiation emitted as a "detector" of the polarization of the nuclei after β-emission. Comparing (3.98) with (3.80) we see that, apart from a constant factor, the only difference is a reversal of the sign of the second term relative to the first term. In both cases we can get information about the same combinations of coupling constants.

3.5. The polarization of electrons emitted in β-emission

Before we deal with the polarization of β-radiation as it occurs for an interaction Hamiltonian which does not conserve parity, we want to consider how the polarization of relativistic fermions described by the Dirac equation can be characterized. For this purpose we write down the general plane wave solution of the Dirac equation (normalized to one particle per unit volume) in the following form:

$$\psi = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \phi \\ -\dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi \end{pmatrix} e^{i(\vec{p}\cdot\vec{x} - Et)} \qquad (3.99)$$

where

$$\phi = \begin{pmatrix} A \\ B \end{pmatrix} \qquad (|A|^2 + |B|^2 = 1) \qquad (3.100)$$

is a two-component spinor. For $\phi = \binom{1}{0}$ one has the spin-direction along the positive z-axis, for $\phi = \binom{0}{1}$ along the negative z-axis. A general state of polarization (3.99) is specified by $\binom{A}{B}$, where A and B are complex numbers, related by the normalization condition $|A|^2 + |B|^2 = 1$. Since a common phase-factor in A and B is not essential, it is seen that the state of polarization

is specified essentially by two physical parameters. An alternative characterization of the state of the fermion polarization is possible by means of a 3-dimensional unit vector $\vec{\zeta}$ related to A and B according to

$$\vec{\zeta} = \phi^* \vec{\sigma} \phi \,, \tag{3.101}$$

or written in components

$$\begin{aligned} \zeta_x &= (A^* B^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = AB^* + A^* B \,. \\ \zeta_y &= (A^* B^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = i(AB^* - BA^*) \\ \zeta_z &= (A^* B^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = |A|^2 - |B|^2 \,. \end{aligned} \tag{3.102}$$

In the non-relativistic Pauli-spin theory $\vec{\zeta}$ is simply the direction of the spin angular momentum or the spin magnetic moment. In the Dirac theory the meaning of $\vec{\zeta}$ is less direct since e.g. the operators for the spin angular momentum and the magnetic dipole moment are different. One also calculates that the expectation values of these operators for (3.99) are different and also differ from $\vec{\zeta}$. Still, $\vec{\zeta}$ is a correct characterization of the state of polarization. It can be shown that $\vec{\zeta}$ can be considered as the spin direction in that coordinate system in which the fermion is brought to rest by a special Lorentz transformation in the direction of $\vec{p}$. Transition probabilities, taking electron polarization into account, can be calculated using formula (3.15) so that the calculation is reduced again to the calculation of a trace of the products of Dirac-matrices. However, instead of D_e one should take a projection operator P_e defined by

$$(P_e)_{\mu\rho} = \psi_\mu \psi_\rho^* \tag{3.103}$$

without summation over the electron spin states. The resulting P_e is much more complicated than D_e given according to (3.16). P_e according to (3.103) can be considered as a projection operator for a one-dimensional subspace specified by a definite $\vec{p}$ and a certain polarization. We shall show how the complete expression for P_e is calculated, although this expression gives more than is needed for the limited purpose of calculating the polarization of the β-rays from non-oriented nuclei. We first note that a general 2×2 matrix $\underline{a}$ can be developed as a sum of Pauli matrices, writing

$$\underline{a} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = c_0 + c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3 \tag{3.104}$$

where

$$\begin{aligned} c_0 &= \tfrac{1}{2} \mathrm{Tr}_\sigma (\underline{a}) = \tfrac{1}{2} (a_{11} + a_{22}) \\ c_1 &= \tfrac{1}{2} \mathrm{Tr}_\sigma (\underline{a}\sigma_1) = \tfrac{1}{2} (a_{12} + a_{21}) \\ c_2 &= \tfrac{1}{2} \mathrm{Tr}_\sigma (\underline{a}\sigma_2) = \tfrac{1}{2} (a_{12} - a_{21}) \\ c_3 &= \tfrac{1}{2} \mathrm{Tr}_\sigma (\underline{a}\sigma_3) = \tfrac{1}{2} (a_{11} - a_{22}) \,. \end{aligned} \tag{3.105}$$

Of course one may do the same thing for an expansion as the sum of ρ-matrices. The way in which the Dirac-plane wave ψ_λ is written in (3.99) is useful when splitting the spinor index λ (=1, 2, 3, 4) into 2 indices r (= 1, 2) and s (= 1, 2), which concern ρ- and σ-matrices respectively; hence λ (=(r, s)) summarizes both indices; ρ-matrices are acting on r; σ-matrices are acting on s.

Analogous to P_e defined by (3.103), we may also introduce a spin-projection operator for the s-variables only, according to

$$(P_e)_{ss'} = \phi_s \phi^*_{s'} = \begin{pmatrix} AA^* & AB^* \\ BA^* & BB^* \end{pmatrix} = \tfrac{1}{2}\,[1 + \vec{\xi}\cdot(\vec{\sigma})_{ss'}] \qquad (3.106)$$

where we used (3.102), (3.104) and (3.105). If we do not write the indices s, s' explicitly, we can simply put

$$P_e = \phi\phi^* = \tfrac{1}{2}\,(1 + \vec{\xi}\cdot\vec{\sigma}). \qquad (3.107)$$

The Dirac spin projection operator (3.103) can now be obtained e.g. by using (3.99) and by writing

$$[P_e\ (\vec{\xi})]_{\lambda\lambda'} = \psi_\lambda \psi^*_{\lambda'} = \psi_{r,s}\ \psi^*_{r',s'}. \qquad (3.108)$$

We then write r, r' in a 2×2 matrix, but we do not write explicitly the indices s, s for ϕ and ϕ^*. Except for (3.99), we also use its complex conjugate:

$$\psi^* = \sqrt{E+m/2E}\begin{pmatrix} -\phi^*\ \vec{p}\cdot\vec{\sigma}/(E+m) \\ \phi^* \end{pmatrix} \exp\{-i(\vec{p}\vec{x} - Et)\}. \qquad (3.109)$$

We thus obtain from (3.103) by means of (3.99) and (3.109)

$$[P_e(\vec{\xi})]_{r,r'} \equiv (\psi_s \psi^*_{s'})_{r,r'} = ((E+m)/2E)\begin{pmatrix} (\vec{p}\cdot\vec{\sigma}/(E+m))\phi\phi^*(\vec{p}\cdot\vec{\sigma}/(E+m)) & -(\vec{p}\cdot\vec{\sigma}/(E+m))\phi\phi^* \\ -\phi\phi^*\vec{p}\cdot\vec{\sigma}/(E+m) & \phi\phi^* \end{pmatrix} \qquad (3.110)$$

Substitution of (3.107) into (3.110) gives

$$[P_e(\vec{\xi})]_{r,r'} = (1/4E)\begin{pmatrix} (\vec{p}\cdot\vec{\sigma})(1+\vec{\xi}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma})/(E+m) & -(\vec{p}\cdot\vec{\sigma})(1+\vec{\xi}\cdot\vec{\sigma}) \\ -(1+\vec{\xi}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma}) & (E+m)(1+\vec{\xi}\cdot\vec{\sigma}) \end{pmatrix} \qquad (3.111)$$

Again applying the identity (3.26) we have

$$(\vec{p}\cdot\vec{\sigma})(\vec{\xi}\cdot\vec{\sigma}) = (\vec{p}\cdot\vec{\xi}) + i\vec{\sigma}\cdot(\vec{p}\times\vec{\xi}) \tag{3.112}$$

$$(\vec{p}\cdot\vec{\sigma})[(1+(\vec{\xi}\cdot\vec{\sigma})](\vec{p}\cdot\vec{\sigma}) = p^2(1-\vec{\xi}\cdot\vec{\sigma})+2(\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{\sigma}) \ . \tag{3.113}$$

Making use of these identities and remembering that $p^2 = E^2 - m^2$, we can reduce (3.111) to

$$[P_e(\xi)]_{rr'} = (1/4E \begin{pmatrix} (E-m)(1-\vec{\xi}\cdot\vec{\sigma}) + 2((\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{\sigma})/(E+m)) - (\vec{p}\cdot\vec{\sigma}) - (\vec{p}\cdot\vec{\xi}) - i\vec{\sigma}\cdot(\vec{p}\times\vec{\xi}) \\ - (\vec{p}\cdot\vec{\sigma}) - (\vec{p}\cdot\vec{\xi}) + i\vec{\sigma}\cdot(\vec{p}\times\vec{\xi}) \qquad (E+m)(1+\vec{\xi}\cdot\vec{\sigma}) \end{pmatrix} \tag{3.114}$$

Developing this according to (3.104) into a sum of ρ-matrices, we have

$$P_e(\vec{\xi}) = c_0 + c_1\rho_1 + c_2\rho_2 + c_3\rho_3 \tag{3.115}$$

where according to (3.105)

$$\begin{aligned} 4E\,c_0 &= E + m(\vec{\xi}\cdot\vec{\sigma}) + ((\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{\sigma})/(E+m)) \\ 4E\,c_1 &= -(\vec{p}\cdot\vec{\xi}) - (\vec{p}\cdot\vec{\sigma}) \\ 4E\,c_2 &= \vec{\sigma}\cdot(\vec{p}\times\vec{\xi}) \\ 4E\,c_3 &= -m - E(\vec{\xi}\cdot\vec{\sigma}) + ((\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{\sigma})/(E+m)). \end{aligned} \tag{3.116}$$

According to (3.115) and (3.116) we may also write $P_e(\vec{\xi})$ in the following form

$$P_e(\vec{\xi}) = \tfrac{1}{4}[1 - (\vec{p}\cdot\vec{\xi}/E)\rho_1 - (m/E)\rho_3 - \rho_1(\vec{p}\cdot\vec{\sigma}/E) + (\vec{p}\times\vec{\xi}/E)\cdot\rho_2\vec{\sigma}$$

$$+\{-\vec{\xi} + (\vec{p}\cdot\vec{\xi})\vec{p}/E(E+m)\}\cdot\rho_3\vec{\sigma} + \{(m/E)\vec{\xi} + [(\vec{p}\cdot\vec{\xi})\vec{p}/E(E+m)]\}\cdot\vec{\sigma}] \ . \tag{3.117}$$

It is seen that the spin projection operator has a rather complicated form in this notation. This is related to the characterization of the spin polarization by the 3-dimensional vector $\vec{\xi}$. A characterization of the spin polarization in a more clearly relativistically covariant way by a 4-vector s_μ gives a more compact notation for $P_e(\vec{\xi})$ in terms of γ_μ-matrices. However, we do not want to go further into this matter here.

We now want to calculate the polarization of β-rays for allowed transitions for non-oriented nuclei when the interaction does not conserve parity. According to (3.15) we have to calculate the following trace:

$$\sum_\nu |\langle f|H_\beta|i\rangle|^2 = \tfrac{1}{2}\sum_{k,\ell} g_k g_\ell^* (\int\Omega_k)(\int\Omega_\ell)^* \mathrm{Tr}\,[\Omega_k D_\nu \Omega_\ell P_e(\vec{\xi})]. \tag{3.118}$$

We want to average over the directions of emission of the neutrino so that we simply substitute $\frac{1}{2}$ for D_ν (cf. (3.17)). Further we substitute (3.117) for $P_e(\vec{\xi})$. We shall make the calculation for pure V-interaction; we obtain the following terms

$$
\begin{aligned}
&|g_V^2|\,|\textstyle\int 1|^2\ \tfrac{1}{4}\ \mathrm{Tr}\,[1\cdot 1\cdot 1\cdot P_e(\vec{\xi})]\\
&+|g_V'|^2|\textstyle\int 1|^2\ \tfrac{1}{4}\ \mathrm{Tr}\,[\rho_1\cdot 1\cdot\rho_1\cdot P_e(\vec{\xi})]\\
&+ g_V g_V'^*|\textstyle\int 1|^2\,\tfrac{1}{4}\ \mathrm{Tr}\,[1\cdot 1\cdot\rho_1\cdot P_e(\vec{\xi})]\\
&+ g_V' g_V^*|\textstyle\int 1|^2\,\tfrac{1}{4}\ \mathrm{Tr}\,[\rho_1\cdot 1\cdot 1\cdot P_e(\vec{\xi})]\\
&\qquad = (|g_V|^2+|g_V'|^2)|\textstyle\int 1|^2\,\tfrac{1}{4}\,\mathrm{Tr}\,[(1+\mu\rho_1)P_e(\vec{\xi})]
\end{aligned}
$$

with (3.119)

$$\mu = 2\,\mathrm{Re}\,(g_V g_V'^*)/(|g_V|^2+|g_V'|^2). \tag{3.120}$$

From (3.117) one sees immediately that the result for (3.119) is given by

$$(|g_V|^2+|g_V'|^2)|\textstyle\int 1|^2\ \tfrac{1}{4}\ (1-\mu\,\vec{p}\cdot\vec{\xi}/E). \tag{3.121}$$

Hence the transition probability has a factor

$$1-\mu\,\vec{p}\cdot\vec{\xi}/E\,,$$

which means that the degree of polarization for pure V-interaction is given by

$$P = -\mu\,\frac{p}{E} = +2\,(v/c)\,[2\,\mathrm{Re}\,(g_V g_V'^*)/(|g_V|^2+|g_V'|^2). \tag{3.122}$$

The calculation according to (3.118) was very simple because we had already obtained the expression (3.117) for $P_e(\vec{\xi})$. One may also derive the β-polarization directly, without using the expression (3.117). We show this again for the case of V-interaction. We have to calculate the square of a matrix element according to

$$
\begin{aligned}
&\tfrac{1}{2}\sum_\nu |g_V|\textstyle\int 1|\ (\psi_e^*\psi_\nu) + g_V'|\int 1|\ (\psi_e^*\psi_5\,\psi_\nu)\,|^2\\
&=\tfrac{1}{2}\,|\textstyle\int 1|^2\sum_\nu\,[\,|g_V|^2\ (\psi_e^*\psi_\nu)(\psi_\nu^*\psi_e) + [g_V'|^2\ (\psi_e^*\gamma_5\psi_\nu)(\psi_\nu^*\gamma_5\psi_e)\\
&+ g_V g_V'^*\ (\psi_e^*\psi_\nu)(\psi_\nu^*\gamma_5\psi_e) + g_V' g_V^*\,(\psi_e^*\gamma_5\psi_\nu)(\psi_\nu^*\psi_e)\,]\,.
\end{aligned}
\tag{3.123}
$$

If we average $\sum_\nu \psi_\nu\psi_\nu^* = D_\nu$ over the direction of emission $\vec{q}$ of the neutrino we obtain simply a constant. We have further $(\gamma_5)^2 = 1$ and $\gamma_5 = \rho_1$. In this way (3.123) reduces to

$$\tfrac{1}{2}\,(|g_V|^2 + |g_V'|^2)\,|\textstyle\int 1|^2\,[\psi_e^*(1+\mu\rho_1)\psi_e\ ]\,. \tag{3.124}$$

μ is again the constant given by (3.120). The expression is further reduced by substituting (3.99) and (3.109), which provides

$$\begin{aligned}
&\psi_e^*(1+\mu\rho_1)\,\psi_e \\
&= \frac{E+m}{2E}\,[\,\phi^*\{(\vec{p}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma})/(E+m)^2\}\phi + \phi^*\phi - 2\mu\ \phi^*\ (\vec{p}\cdot\vec{\sigma}/(E+m))\phi] \\
&= \frac{1}{2E}\ [\,\phi^*\phi\ \ (p^2/(E+m)) + (E+m)\ \phi^*\phi - 2\mu\ \ \phi^*(\vec{p}\cdot\vec{\sigma})\,\phi\ ] \\
&= \frac{1}{2E}\ [\,2E - 2\mu\ (\vec{p}\cdot\vec{\xi})\,] \ = 1 - \mu\ (\vec{p}\cdot\vec{\xi}/E).
\end{aligned} \tag{3.125}$$

We have used here: $\phi^*\phi = |A|^2 + |B|^2 = 1$ and $\phi^*\vec{\sigma}\phi = \vec{\xi}$ (cf. (3.100) and (3.101)). We have thus derived the same factor in the transition probability in (3.125) as we had in (3.121), so that we have again derived the result (3.122). Taken separately, this second derivation of the degree of polarization of β-rays is shorter than the derivation of the expression (3.117) for the spin projection operator. However, it is useful to have the complete expression of the spin projection operator available in case one wants to calculate more complicated effects with electron polarization.

We shall now show that the β-polarization for S-interaction has an opposite sign to the V-interaction. We then have instead of (3.123)

$$\begin{aligned}
&\textstyle\sum_\nu \tfrac{1}{2}\ |g_S\,(\int 1)\,(\psi_e^*\rho_3\psi_\nu) + g_S'\,(\int 1)(\psi_e^*\rho_3\gamma_5\psi_\nu)\,|^2 \\
&= \tfrac{1}{2}\ |\textstyle\int 1|^2 \sum_\nu [\ |g_S|^2\ (\psi_e^*\rho_3\psi_\nu)(\psi_\nu^*\rho_3\psi_e) + |g_S'|^2\ \ (\psi_e^*\rho_3\rho_1\psi_\nu)(\psi_\nu^*\rho_1\rho_3\psi_e) \\
&+ g_S g_S'^*\,(\psi_e^*\rho_3\psi_\nu)(\psi_\nu^*\rho_1\rho_3\psi_e) + g_S g_S^*\,(\psi_e^*\rho_3\rho_1\,\psi_\nu)(\psi_\nu^*\rho_3\psi_e)\ ] \\
&= \tfrac{1}{2}\ (|g_S|^2 + |g_S'|^2)\,|\textstyle\int 1|^2\,[\psi_e^*\,(1 - \nu\,\rho_1)\psi_e] \\
&= \tfrac{1}{2}\ (|g_S|^2 + |g_S'|^2)\,|\textstyle\int 1|^2\,[1 + \nu\ (\vec{p}\cdot\vec{\xi}/E)]
\end{aligned} \tag{3.126}$$

where we have put

$$\nu = 2\ \mathrm{Re}\,(g_S g_S'^*)/\ |g_S|^2 + |g_S|^2\,. \tag{3.127}$$

In the reduction of (3.126) we have made use of $\rho_3\rho_1\rho_3 = -\rho_3\rho_3\rho_1 = -\rho_1$. The anticommutation of ρ_3 and ρ_1 provides a different sign for S than for V-interaction. For the same reason one finds opposite signs for the β-polarization for T- and A-interactions. However, we do not go further into the

calculations for T, A or P-interactions as they do not contain any new elements.

3.6. Summary of some formulae for β-radioactivity

For the purpose of further discussion we list below a number of formulae with some more details than we have derived hitherto. We give the transition probability, for when the emitted electron has momentum $\vec{p}$ and the neutrino has momentum $\vec{q}$. We assume that the nuclei may be polarized and that $\vec{j}$ is a unit vector, which is an axis of rotational symmetry for the orientation. The nuclear orientation may then be specified by the parameters f_1 and f_2:

$$f_1 = (1/j) \sum_m m a_m \tag{3.128}$$

$$f_2 = (1/j^2) \left[\sum_m m^2 a_m - \tfrac{1}{3} j (j+1) \right]. \tag{3.129}$$

The transition probability may now be as follows (units with $\hbar = c = 1$ are used; Coulomb-corrections are taken into account up to the first order; cf. JACKSON, TREIMAN and WYLD [17]):

$$\begin{aligned} W_\mp (\vec{p}, \vec{q}, j) = {} & 2^{-1}(2\pi)^{-5} F(\pm Z, E_e)\, p\, E_e (E_0 - E_e)^2 \xi^2 \\ & \{1 + a_\mp (\vec{p}\cdot\vec{q}/E_e E_\nu) + b_\mp (m/E_e) \\ & - (3 j_i / 2 j_i - 1) C_\mp f_2 \left[(\vec{p}\cdot\vec{q}/3 E_e E_\nu) - ((\vec{j}\cdot\vec{p})(\vec{j}\cdot\vec{q})/E_e E_\nu) \right] \\ & + f_1 \left[A_\mp (\vec{j}\cdot\vec{p}/E_e) + B_\mp (\vec{j}\cdot\vec{q}/E_\nu) + D_\mp (\vec{j}\cdot(\vec{p}\times\vec{q})/E_e E_\nu) \right] \}. \end{aligned} \tag{3.130}$$

$E_\nu = E_0 - E_e$ is the neutrino energy. $F(\pm Z, E_e)$ is the Fermi function, which specifies the influence of the Coulomb field of the nucleus on the β-spectrum. Averages are taken for the electron and neutrino polarization. The following effects are described by this formula:

(a) shape of the allowed spectrum (average over $\vec{p}$ and $\vec{q}$ and nuclear orientation $\vec{j}$);

(b) e-ν angular correlation; this is determined by the term $a_\mp (\vec{p}\cdot\vec{q})/E_e E_\nu$ (average over nuclear orientation);

(c) the angular distribution of β-rays from oriented nuclei is obtained by averaging over $\vec{q}$ (θ = angle between $\vec{j}$ and $\vec{p}$):

$$W_\mp (\theta) = 1 + b_\mp (m/E_e) + f_1 (v/c) A_\mp \cos\theta; \tag{3.131}$$

(d) for recoil experiments with polarized nuclei the important terms are $(\vec{j}\cdot\vec{q})/E_\nu$ and $\vec{j}\cdot(\vec{p}\times\vec{q})$.

We give below the expressions for $\xi, \ldots, D_\mp$ expressed in the coupling constants g_j and g_j' of the Hamiltonian.

The upper and lower signs refer to β^- and β^+ emission respectively:

$$\xi = (|g_S|^2 + |g_S'|^2 + |g_V|^2 + |g_V'|^2)|M_F|^2 + (|g_T|^2 + |g_T'|^2 + |g_A'|^2 + |g_A|^2)|M_{GT}|^2 \tag{3.132}$$

$$a_\mp \xi = \{[-|g_S|^2 - |g_S'|^2 + |g_V|^2 + |g_V'|^2] \mp \frac{\alpha Zm}{p} \cdot 2\,\mathrm{Im}\,(g_S g_V^* + g_S' g_V'^*)\}|M_F|^2$$

$$+ \{[|g_T|^2 + |g_T'|^2 - |g_A|^2 - |g_A'|^2] \pm \frac{\alpha Zm}{p} \cdot 2\,\mathrm{Im}\,(g_T g_A^* + g_T' g_A'^*)\}\tfrac{1}{3}|M_{GT}|^2 \tag{3.133}$$

$$b_\mp \xi = \pm 2\gamma\,\mathrm{Re}[(g_S g_V^* + g_S' g_V'^*)|M_F|^2 + (g_T g_A^* + g_T' g_A'^*)|M_{GT}|^2] \tag{3.134}$$

$$A_\mp \xi = \lambda_{jijf}[\pm 2\,\mathrm{Re}\,(g_T g_T'^* - g_A g_A'^*) + \frac{\alpha Zm}{p} 2\,\mathrm{Im}\,(g_T g_A'^* + g_T g_A^*)]|M_{GT}|^2$$

$$+ \delta_{jijf}\sqrt{\frac{j_i}{j_i+1}}[2\,\mathrm{Re}\,(g_S g_T'^* + g_S' g_T^* - g_V g_A'^* - g_V' g_A^*)$$

$$\pm \frac{\alpha Zm}{p} 2\,\mathrm{Im}\,(g_S g_A'^* + g_S' g_A^* - g_V g_T'^* - g_V' g_T^*)]\, M_F M_{GT}^*; \tag{3.135}$$

$$B_\mp \xi = 2\,\mathrm{Re}\{|M_{GT}|^2 \lambda_{jijf}[\frac{\gamma m}{E}(g_T g_A'^* + g_T' g_A^*) \pm (g_T g_T'^* + g_A g_A'^*)]$$

$$- \delta_{jijf} M_F M_{GT}^* \sqrt{\frac{j_i}{j_i+1}}[(g_S g_T'^* + g_S g_T^* + g_V g_A'^* + g_V g_A^*)$$

$$\pm \frac{\gamma m}{E_e}(g_S g_A'^* + g_S' g_A^* + g_V g_T'^* + g_V' g_T^*)\}; \tag{3.136}$$

$$D_\mp \xi = \delta_{jijf}\sqrt{\frac{j_i}{j_i+1}}[2\,\mathrm{Im}\,(g_S g_T^* + g_S' g_T'^* - g_V g_A^* - g_V g_A^*)$$

$$\mp \frac{\alpha Zm}{p} 2\,\mathrm{Re}\,(g_S g_A^* + g_S g_A^* - g_V g_T^* - g_V g_T^*)]\, M_F M_{GT}^*. \tag{3.137}$$

In these expressions we have used the following notations:

j_f is the nuclear spin after β-emission (3.138)

$$\gamma = \sqrt{1 - \alpha^2 Z^2} \tag{3.139}$$

$F(\pm Z, E_e)$ is the well-known Fermi function which gives the influence of the nuclear charge Z on the β-spectrum ($F(\pm Z, E_e) = 1$ for $Z = 0$).

$$\lambda_{jijf} = \begin{cases} 1 & \text{for} \quad j_i \to j_f = j_i - 1 \\ 1/(j_i+1) & \text{for} \quad j_i \to j_f = j_i \\ -j_i/(j_i+1) & \text{for} \quad j_i \to j_f = j_i + 1. \end{cases} \tag{3.140}$$

With respect to the interference term $M_F M_{GT}^*$ it should be noted that it can be proved to be real (assuming that the nuclear Hamiltonian is invariant for time reversal).

3.6.1. Polarization of β-rays

In formula (3.130) an average is taken for the electron and the neutrino polarization. If parity is not conserved, we have the important effect of the longitudinal polarization of β-rays for non-oriented nuclei. This effect is not yet described by (3.130) but we can describe it by the following formula

$$W_\mp(\vec{p}, \vec{\xi}) = (1/2(2\pi)^4) F(\pm Z, E_e) p E_e (E_0 - E_e)^2 \xi \times \{1 + b_\mp (m/E_e) + G_\mp (\vec{\xi} \cdot \vec{p}/E_e)\}. \tag{3.141}$$

$\vec{\xi}$ is the polarization vector for electrons.

$G_\mp$ is given by

$$G_\mp \xi = |M_F|^2 [\pm 2 \operatorname{Re}(g_S g_S'^* - g_V g_V'^*) + (\alpha Zm/p) 2 \operatorname{Im}(g_S g_V'^* + g_S' g_V^*)] + |M_{GT}|^2 [\pm 2 \operatorname{Re}(g_T g_T'^* - g_A g_A'^*) + (\alpha Zm/p) 2 \operatorname{Im}(g_T g_A'^* + g_T' g_A^*)] . \tag{3.142}$$

It follows from (3.141) that the degree of (longitudinal) polarization of electrons emitted in an allowed β-decay is given by

$$P_\mp = \frac{G_\mp (v/c)}{1 + b_\mp (m/E_e)} \tag{3.143}$$

3.6.2. β, γ-circular polarization correlation

This is another effect showing the non-conservation of parity as was discussed before (cf. Fig. 2).

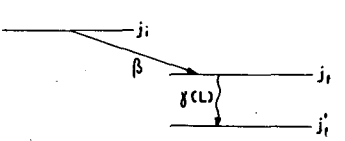


Fig. 2

Nuclear spins for β-emission followed by 2^L-pole γ-radiation for which the circular polarization is measured

The following result can be derived for the β, γ-circular polarization correlation:

$$W_{\mp}^{(\beta,\gamma)}(\theta,\tau) = 1 + b_{\mp}(m/E_e) + \tfrac{1}{3} A_{\mp}^{\gamma} \tau Q_\gamma \cos\theta \ . \qquad (3.144)$$

θ is the angle between the directions of emission of β and γ ; $\tau = +1$ for photons of helicity + 1 and $\tau = -1$ for the opposite polarization. The value of $b_{\mp}$ is given by (3.134). Q_γ is a function of j_f, j_f' and the multipolarity L of the γ-radiation; we specify Q_γ for two important cases

$$Q_\gamma = \begin{cases} \frac{3}{2} & \text{if} \quad j_f - j_f' = L = 1 \\ 1 & \text{if} \quad j_f - j_f' = L = 2 \ . \end{cases} \qquad (3.145)$$

The value of $A_{\mp}^{\gamma}\,\xi$ is given by

$$A_{\mp}^{\gamma}\,\xi = -((j_f+1)/j_f)\{\sigma_{jfji}\,[\pm 2\,\mathrm{Re}\,(g_T g_T'^{*} - g_A g_A'^{*}) + ((\alpha Zm)/p)\,2\,\mathrm{Im}\,(g_T g_A'^{*} + g_T' g_A^{*})]\,|M_{GT}|^2$$

$$- \sigma_{jfji}\sqrt{(j_i/(j_i+1))}\,[2\,\mathrm{Re}\,(g_S g_T'^{*} + g_S' g_T^{*} - g_V g_A'^{*} - g_V' g_A^{*})$$

$$\pm(\alpha Zm/p)\,2\,\mathrm{Im}\,(g_S g_A'^{*} + g_S' g_A^{*} - g_V g_T'^{*} - g_V' g_T^{*})]\}\,M_F M_{GT}^{*}. \qquad (3.146)$$

where

$$\sigma_{jfji} = \begin{cases} -j_f/(j_f+1) & \text{if} \quad j_f = j_i - 1 \\ 1/(j_f+1) & \text{if} \quad j_f = j_i \\ 1 & \text{if} \quad j_f = j_i + 1 \end{cases} \qquad (3.147)$$

It is useful to compare (3.135) and (3.146). These formulae are analogous although the factors λ_{jfji} and σ_{jfji} differ. If $j_f = j_i$, we may have the interference term $M_F M_{GT}^{*}$ in addition to the main term $|M_{GT}|^2$. In this respect it is remarkable that the main term and the interference term in (3.135) and (3.146) have opposite relative signs.

3.6.3. Some general remarks about the structure of the formulae

(1) It should be noted that $g_T g_T'^{*}$ has opposite sign to $g_A g_A'^{*}$ in the formulae for $A_{\mp}$ (3.135) determining the directional distribution of β-rays from oriented nuclei, as well as for $A_{\mp}^{\gamma}$ (3.146) for the β-γ circular polarization correlation, and the expression $G_{\mp}$ (3.142) for the β-ray longitudinal polarization. Similarly $g_S g_S'^{*}$ has opposite sign to $g_V g_V'^{*}$ in the expression $G_{\mp}$ for β-ray polarization.

From this structure of the formulae it is seen that the experiments providing $A_{\mp}$, $A_{\mp}^{\gamma}$ or $G_{\mp}$ can teach us about the helicity of the emitted neutrino related to the sign in $g_i' = \pm g_i$ (when assuming two-component neutrino theory) only when it is known whether the β-interaction has V or A character (as to its Gamow-Teller part) and S or V character (as to its Fermi part).

(2) The terms with $\vec{p}\cdot\vec{q}$, $\vec{j}\cdot\vec{p}$ and $\vec{j}\cdot\vec{q}$ in (3.130) are all even with respect to time reversal. In 2 (see in particular (2.18) - (2.21)) we discussed in general terms in what way time-reversal invariance can be tested. In formulae (3.133) - (3.137) we have examples of these general considerations. It is seen that the term with $D_{\pm}$ $(\vec{j}\cdot(\vec{p}\times\vec{q})/E_e E_\nu)$ is the only term of which time-reversal invariance can be tested, when we neglect final-state interactions (i.e. if we omit the terms with αZ); in the expression for D one still retains in this case the term with Im $(g_S g_T^* + g_S' g_T'^* - g_V g_A^* - g_V' g_A'^*)$ which should be equal to zero if invariance for time-reversal holds.

On the other hand, if final state interactions are considered (i.e. if the terms with αZ are taken into consideration), the expressions for $a_{\mp}$, $A_{\mp}$ and $B_{\mp}$ contain terms which could demonstrate non-invariance for time reversal (cf. the corresponding general formula (2.20)). However, such terms with αZ have the character of correction terms and are not easily measured.

3.7. The experimental situation in β-radioactivity

In this section we review concisely the experimental situation concerning β-radioactivity (cf. [10] for a more complete review).

(1) From the half-lives and end-point energies of (super-allowed) β-transitions one can learn the coupling constants for the Fermi and Gamow-Teller parts of the β-interactions (using the so-called ft-values). Nuclei are needed for which the matrix elements can be supposed to be known on theoretical grounds; in this respect the most important are:

(a) the neutron, spin $\frac{1}{2}$; $|M_F|^2 = 1$, $|M_{GT}|^2 = 3$.

(b) 0+ → 0+ - transitions such as occur for O^{14}, Al^{26}, Cl^{34}; because of the selection rules $|M_{GT}|^2 = 0$; further $|M_F|^2 = 2$.

(c) mirror nuclei, especially those with closed shells ± one nucleon, such as O^{15}, F^{17}, Ca^{39}.

The results of the ft-values measured for the nuclei (b), especially O^{14}, can be given as

$$g_F = (1.470 \pm 0.022) \times 10^{-49}\ \text{erg} \times \text{cm}^3, \tag{3.148}$$

where g_F is the "Fermi coupling constant", which may be expressed in terms of the coupling constants used hitherto as

$$g_F^2 = \tfrac{1}{2}\left[\,|g_S|^2 + |g_S'|^2 + |g_V|^2 + |g_V'|^2\,\right]. \tag{3.149}$$

Further one often introduces $R = \lambda^2$ according to

$$R = \lambda^2 = \frac{|g_T|^2 + |g_T'|^2 + |g_A|^2 + |g_A'|^2}{|g_S|^2 + |g_S'|^2 + |g_V|^2 + |g_V'|^2}. \tag{3.150}$$

The value obtained for λ on the basis of the ft-values for the n and O^{14} is

$$|\lambda| = 1.18 \pm 0.05. \tag{3.151}$$

The value obtained on the basis of O^{14} and the mirror nuclei is somewhat lower ($|\lambda| \approx 1.08$); it is probable that this is due to a number of cor-

rections which have to be applied in the case of mirror nuclei; of course, it is a most important problem that even the explanation of the ft-values of the simplest β-transitions require quite an amount of detail on the nuclear wave functions; for a survey of this problem we refer to [18].

(2) Recoil experiments for allowed β-transitions providing the constant $\alpha_{\mp}$ in the electron neutrino directional correlation are a most important source of information. For pure interactions one would have a $= -1, +1, +\frac{1}{3}, -\frac{1}{3}$ for S, V, T and A interactions respectively. Hence these experiments can distinguish between S or V interaction for the Fermi part and between T or A interaction for the Gamow-Teller part of the interaction, which is not possible on the basis of ft-values alone.

Recoil experiments have been performed for n, He^6, Ne^{19}, Ne^{23}, A^{35}. These nuclei have different ratios for $|M_F|^2$ and $|M_{GT}|^2$. The experiments show clearly that one must assume V and A interaction, while the upper limits for S and T interaction are found to be

$$\begin{aligned} (|g_T|^2 + |g_T'|^2/|g_A|^2 + |g_A'|^2) &\lesssim 0.07. \\ (|g_S|^2 + |g_S'|^2/|g_V|^2 + |g_V'|^2) &\lesssim 0.07. \end{aligned} \tag{3.152}$$

The situation as to recoil experiments and the T or A, and S or V character of the β-interaction was confused up to 1958, especially as a consequence of a He^6 recoil experiment which indicated T instead of A interaction due to experimental errors.

(3) The so-called Fierz terms, which can occur in the shape of β-spectra (terms with coefficients $b_{\mp}$ according to (3.134)) were shown to be absent experimentally within the accuracy of the experiments:

$$\begin{aligned} |b_{GT}| &\lesssim 0.02 \text{ for Gamow-Teller transition;} \\ |b_F| &\lesssim 0.1 \text{ for Fermi transitions.} \end{aligned} \tag{3.153}$$

For a Hamiltonian conserving parity ($g_i' = 0$) one could then conclude that S and V interaction cannot occur simultaneously (also T and A interaction cannot occur simultaneously). However, the conclusion for a more general Hamiltonian with $g_i' \neq 0$ is less simple, (cf. formula (3.134)).

(4) Since 1957 a large number of experiments demonstrating the non-conservation of parity in β-decay have been performed. Most of the experiments belong to the three types:

(a) asymmetry of β-rays from oriented nuclei;

(b) (β, γ) circular polarization correlation;

(c) longitudinal polarization of β-rays.

The experiments showed very soon that the effects have about the maximum possible size, and this has since been confirmed (e.g. the degree of longitudinal polarization was found to be $P = -v/c$ for β^--rays and $P = +v/c$ for β^+-rays within a few per cent). This indicates $g_i' = g_i$, hence two-component neutrino theory.

Once the V A character of the β-interaction was established (1958), one could then conclude immediately that $g_V = g_V'$ and $g_A = g_A'$, hence "left-handed neutrino", from the signs of these effects. The relative signs of the V and A coupling could be determined from the directional distribution of β-rays

from polarized neutrons, as one has here a V, A interference term (cf. (3.135)). Also the term with $B_{\mp}$ could be determined (measurement of the recoil protons). It was found that the relative sign of the V and A contributions was negative.

(5) A further experiment for detecting the term with $D_{\mp}$ (cf. (3.137)) for polarized neutrons gave no indication that time-reversal invariance would be violated.

From the TCP theorem it follows that T-invariance is equivalent to invariance for the "combined inversion" CP. From the values of the coupling constants obtained it is clear that both P-invariance and C-invariance are violated.

We may further summarize the situation concerning β-interaction by stating that all experiments are compatible with

(a) V - λ A coupling with $\lambda \approx 1.2$,

(b) 2-component neutrino theory, with left-handed neutrinos,

(c) lepton conservation.

The following basic experiments, which agree with this too, may also be mentioned:

(6) The experiment of Reines and Cowan on the inverse beta-process with neutrinos from a reactor

$$\bar{\nu} + p \rightarrow n + e^+$$

gave a cross-section $\sigma = (11 \pm 2.6) \times 10^{-44} \mathrm{cm}^2$ to be compared with a theoretical value of $14 \times 10^{-44} \mathrm{cm}^2$.

(7) Experiments showed that double β-decay without the emission of neutrinos is absent (within experimental errors), as is required by lepton conservation.

4. MUON DECAY

We shall review the situation concerning muon decay only very briefly. Muon decay is a three-body decay, which may be written (for μ^-) as

$$\mu^- \rightarrow e^- + \nu + \bar{\nu}. \tag{4.1}$$

In a three-body decay one has a continuous energy spectrum. The electron energy spectrum in (4.1) extends up to a maximum energy of about 52 MeV.

Besides the electron spectrum non-conservation of parity permits the observation of another effect: the muons resulting from pion decay

$$\pi^- \rightarrow \mu^- + \bar{\nu}$$

are completely polarized, as a consequence of angular momentum conservation, if we have two-component neutrinos with an intrinsic polarization.

The decaying polarized muons may give rise to an asymmetric angular distribution of the resulting electrons analogous to the angular distribution of electrons resulting in β-decay from polarized nuclei. Both electron spectrum and angular distribution are given by the formula (for the derivation see KINOSHITA and SIRLIN [19] and BOUCHIAT and MICHEL [20]

$$W_{\pm}(x,\theta) = \text{const}\ \{3(1-x) + 2\rho\,(\tfrac{4}{3}x - 1) \mp \xi\cos\theta\,[(1-x) + 2\delta\,(\tfrac{4}{3}x - 1)]\}x^2. \tag{4.2}$$

Radiative corrections and the electron mass are neglected $x = \frac{p}{p_0}$; p = electron momentum; p_0 = maximum electron momentum. The three constants ρ, ξ and δ are functions of the coupling constants: ρ is known as the Michel parameter (cf. MICHEL [21]); it determines the shape of the electron spectrum.

The value of ρ may vary between 0 and 1 for the decay scheme (4.1). In addition to the scheme (4.1) the following possibilities also exist

$$\mu^- \to e^- + \nu + \nu \tag{4.3}$$

$$\mu^- \to e^- + \bar{\nu} + \bar{\nu}. \tag{4.4}$$

For the schemes (4.3) and (4.4) ρ may vary between 0 and $\frac{3}{4}$. When a two-component theory for the neutrinos is assumed ($\gamma_5\psi_\nu = \psi_\nu$), ρ attains a unique value, namely $\rho = 0$ for (4.3) or (4.4) and $\rho = \frac{3}{4}$ for (4.1) [22], [23].

$$\left.\begin{aligned} dN &= 2x^2(3-2x)\,dx \quad \text{for} \quad \rho = \tfrac{3}{4} \quad \text{(a)} \\ dN &= 12x^2(1-x)\,dx \quad \text{for} \quad \rho = 0 \quad \text{(b)} \end{aligned}\right\}. \tag{4.5}$$

It is now well-established that the first case ($\rho = \frac{3}{4}$) is realized in nature. The interaction Hamiltonian for muon decay for the scheme (4.1) may be written as

$$\mathcal{H}^\mu = \Sigma^\mu_{i=V,A} f_i^\mu\,(\bar{\psi}_e \Gamma_i \psi_\mu)(\bar{\psi}_\nu \Gamma_i \psi_\nu) + \text{h.c.} \tag{4.6}$$

as well as

$$\mathcal{H}^\mu = \Sigma_i g_i^\mu\ (\bar{\psi}_e \Gamma_i\ \psi_\nu)(\bar{\psi}_\nu \Gamma_i\ \psi_\mu) + \text{h.c.} \tag{4.7}$$

In both expressions we assume that we have the two-component neutrino theory with $\gamma_5\,\psi_\nu = \psi_\nu$.

The two notations are equivalent provided the coupling constants are related as follows

$$\begin{aligned} g_S &= -g_P = f_V + f_A \\ g_V &= -g_A = \tfrac{1}{2}\,(f_A - f_V). \end{aligned} \tag{4.8}$$

For the two-component theory the normalized electron distribution can be written as

$$dN_{\pm} = 2x^2[(3-2x) \pm \xi\cos\theta\,(1-2x)], \tag{4.9}$$

where ξ is given by

$$\xi = (2 f_V f_A/(f_V)^2 + (f_A)^2). \tag{4.10}$$

Hence for $f_V = -f_A$ we have $\xi = -1$.

We may summarize this in the following way. For the two-component neutrino theory and V-A interaction the constants ρ, ξ and δ from (4.2) have the values

$$\rho = \tfrac{3}{4}, \quad \xi = -1, \quad \delta = \tfrac{3}{4}. \tag{4.11}$$

The expression for V-A interaction may be written in both the following ways (cf. (4.8) for $f_V = -f_A = \sqrt{2}\, g^\mu$)

$$\begin{aligned} \mathcal{M}^\mu &= -\frac{g^\mu}{\sqrt{2}} [\bar{\psi}_e \gamma_\lambda (1+\gamma_5)\psi_\mu][\bar{\psi}_\nu \gamma_\lambda (1+\gamma_5)\psi_\nu] \\ &= \frac{g^\mu}{\sqrt{2}} [\bar{\psi}_e \gamma_\lambda (1+\gamma_5)\psi_\nu][\bar{\psi}_\nu \gamma_\lambda (1+\gamma_5)\psi_\mu] \quad . \end{aligned} \tag{4.12}$$

Finally, the decay rate for the interaction (4.12) is given by

$$\Lambda\mu = (m_\mu^5 \, g^{\mu 2} / 3.2^6 \pi^3) \quad . \tag{4.13}$$

We now give a summary of experimental results (we make no attempt at completeness but give only the most precise and recent ones, cf. also [24] for a general review of the experiments).

$$\left.\begin{aligned} \rho = \; & 0.741 \pm 0.027 \quad [27] \\ & 0.785 \pm 0.020 \quad [28] \\ & 0.764 \pm 0.032 \quad [29] \end{aligned}\right\} \tag{4.14}$$

$$\xi = -0.94 \pm 0.07 \quad [28] \tag{4.15}$$

$$\delta = 0.78 \pm 0.05 \quad [28] \tag{4.16}$$

It should be noted that radiative corrections of about 5% have to be applied to the Michel parameter ρ (cf. [25] and [26] for their calculation).

The following precise values were obtained for the mean life-time

$$\tau_\mu = (2.210 \pm 0.003) \;\; \mu\text{sec} \;\; [30] \tag{4.17}$$

leading to

$$g^\mu = (1.428 \pm 0.001) \times 10^{-49} \text{ erg cm}^3 \tag{4.18a}$$

or

$$g^\mu = \frac{1.018 \times 10^{-5}}{m_p^2} \, . \tag{4.18b}$$

The situation concerning muon decay may be summarized as follows: the measured values of ρ, ξ and δ are in good agreement with the two-component neutrino theory and V-A coupling; muon decay proceeds via scheme (4.1) while (4.3) or (4.4) have to be rejected.

5. UNIVERSAL FOUR-FERMION INTERACTION; THE PROPOSAL FOR A UNIVERSAL V-A INTERACTION

In section 3 we have seen that all experiments on β-decay are in agreement with the V - λ A form for the interaction Hamiltonian with a two-component neutrino.

$$\mathcal{H}_\beta = (g_\beta/\sqrt{2})\,(\bar{\psi}_p \gamma_\mu (1 + \lambda\gamma_5)\psi_n)(\bar{\psi}_e \gamma_\mu (1+\gamma_5)\psi_\nu) + \text{h.c.}. \qquad (5.1)$$

From the ft-value of the β^+ decay of O^{14}, a pure Fermi transition, we get a very accurate value for g_β [31]

$$g_\beta = (1.4170 \pm 0.0022) \times 10^{-49} \text{ erg cm}^3. \qquad (5.2)$$

The ratio of the axial-vector and vector coupling constants, λ, is then determined from the life-time of the neutron [31]:

$$\lambda = \left| \frac{g_A}{g_V} \right| = 1.18 \pm 0.05. \qquad (5.3)$$

For several mirror-nuclei, in which case one can calculate the matrix elements, one finds values for λ which are significantly different from (5.3). BLIN STOYLE [31] has discussed possible causes of this discrepancy.

Several authors [32] [33] [34] independently remarked that by a simple prescription, one can get the form (5.1) with $\lambda = 1$ from the general form given by

$$2^{-\frac{1}{2}} \sum_{i=S,V,T,A,P} (\bar{\psi}_p \Gamma_i \psi_n)(\bar{\psi}_e \Gamma_i)(g_i + g_i' \gamma_5)\psi_\nu) + \text{h.c.}. \qquad (5.4)$$

We substitute in (5.4) for every ψ, $a\psi$ with $a = \frac{1}{2}(1 + \gamma_5)$. Then $\bar{\psi}$ becomes $\bar{\psi}\bar{a}$ ($\bar{a} = \frac{1}{2}(1 - \gamma_5)$) because of the commutation relations (A.36); we have:

$$a^2 = \tfrac{1}{4}(1 + \gamma_5)(1 + \gamma_5) = \tfrac{1}{2}(1 + \gamma_5) = a$$

$$\bar{a}a = \tfrac{1}{4}(1 - \gamma_5)(1 + \gamma_5) = \tfrac{1}{4}(1 - \gamma_5^2) = 0$$

$$\overline{a\psi} = \{\tfrac{1}{2}(1 + \gamma_5)\psi\}^* \gamma_4 = \psi^* \tfrac{1}{2}(1 + \gamma_5)\gamma_4 = \tfrac{1}{2}\psi^* \gamma_4 (1 - \gamma_5) = \bar{\psi}\bar{a}.$$

Thus

$$\bar{\psi}\,\Gamma_j \psi \to \bar{\psi}\,\bar{a}\,\Gamma_j\,a\,\psi\,,$$

with

$$\bar{a}\,\Gamma_j\,a = \Gamma_j \bar{a}a = 0 \qquad \text{if} \quad j = S,T,P$$

$$\bar{a}\Gamma_j\,a = \Gamma_j\,a^2 = \Gamma_j\,a \qquad \text{if} \quad j = V,A\,.$$

It is required that the interaction be invariant for the substitution

$$\psi \to a\psi\,.$$

Only the A and V terms in (5.4) remain. The result can be written in the exact V-A form as

$$\mathcal{M}_\beta^{(0)} = (g_\beta^{(0)}/\sqrt{2})\,(\bar{\psi}_p \gamma_\mu (1+\gamma_5)\psi_n)(\bar{\psi}_e \gamma_\mu (1+\gamma_5)\psi_\nu) \quad . \tag{5.5}$$

The addition of the superscript (0) to $\mathcal{M}_\beta$ and g_β in (5.5) will be explained later on.

Several authors have given theoretical speculations, which might be important for a deeper understanding of the prescription $\psi \to a\psi$. FEYNMAN and GELL-MANN [39] explain that one might use for quantum electrodynamics, instead of the first-order Dirac equation for the 4-component field

$$[\,\gamma_\mu\,(\frac{\partial}{\partial x_\mu} - ie A_\mu) + m\,]\,\psi = 0, \tag{5.6}$$

also a second-order equation for a 2-component field (A_μ in (5.6) specifies the 4-potential of the electromagnetic field, with which the particle is interacting). This equation can be obtained by putting

$$\psi = -\frac{1}{m}\,[\,\gamma_\mu\,(\frac{\partial}{\partial x_\mu} - ie A_\mu) - m]\,\omega, \tag{5.6a}$$

from which one obtains the iterated Dirac equation for ω,

$$[\,(\frac{\partial}{\partial x_\mu} - ie A_\mu)^2 - m^2 - \tfrac{1}{2}\,ie\,\gamma_\mu\gamma_\nu F_{\mu\nu}\,]\,\omega = 0, \tag{5.7}$$

where $F_{\mu\nu}$ is the electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

The equation is a second-order equation, however, for the 4-component field ω. One may pass to a second-order equation for a 2-component field by noticing that γ_5 commutes with $\gamma_\mu\gamma_\nu$ so that (5.7) can be split into two equations for 2-component fields satisfying

$$\text{either} \quad \gamma_5\,\omega = \omega\,, \tag{5.7a}$$

$$\text{or} \quad \gamma_5\,\omega = -\omega\,. \tag{5.7b}$$

Choosing (5.7a), one obtains by multiplying (5.6a) from the left by $(1+\gamma_5)$ (γ_5 anticommutes with γ_μ)

$$\omega = \tfrac{1}{2}\,(1+\gamma_5)\psi \quad . \tag{5.7c}$$

(5.6a) and (5.7c) establish a (1, 1)-correspondence between the solutions of the Dirac equation (5.6) and the second-order equation (5.7) with the additional condition (5.7a). The quantity ω satisfying (5.7a) has really only two independent components as is seen by writing $\psi = \binom{\chi}{\varphi}$ so that ω can be written as

$$\omega = \tfrac{1}{2}(1+\gamma_5)\psi = \begin{pmatrix} \tfrac{1}{2}(\varphi+\chi) \\ \tfrac{1}{2}(\varphi+\chi) \end{pmatrix} = \begin{pmatrix} \Phi \\ \Phi \end{pmatrix}.$$

It is then easily deduced that Φ satisfies the equation

$$[\,(\frac{\delta}{dx_\mu} - ieA_\mu)^2 - m^2 + e\vec{\sigma}\cdot(\vec{B} + i\vec{E})\,]\,\Phi = 0\,. \qquad (5.7d)$$

It can be shown that (5.6) and (5.7d) are equivalent as far as quantum electrodynamics is concerned. However, the two equations may lead to different theoretical proposals for 4-fermion interactions in the following way:

One may postulate theoretically that 4-fermion-interactions should be linear in the fields themselves but should not contain derivatives of the fields (think of the rejection on experimental grounds of the Konopinski-Uhlenbeck interaction, containing derivatives for β-radioactivity).

It then makes a difference whether this requirement is formulated for the field ψ or the 2-component field ϕ:

I. When formulated with ψ, the interaction may also be formulated with the aid of ω (or ϕ) by substituting (5.6a), but it then follows from this formula that the interaction formulated by means of ϕ contains derivatives.

II. On the other hand, if one requires the interaction to be linear in ϕ (or ω) it follows from (5.7c) that the interaction, when formulated with the aid of ψ should contain ψ only in the combination $a\psi$, which is equivalent to the rule we have just formulated.

It is this second proposal which was made by Feynman and Gell-Mann.

SAKURAI [33] requires that the interaction should be invariant for the so-called mass-reversal transformation:

$$\psi \rightarrow \gamma_5\psi, \quad \bar{\psi} = -\bar{\psi}\gamma_5, \quad m \rightarrow -m\,. \qquad (5.8)$$

This has to be applied for each of the four fermions separately. Note that the Dirac equation is invariant for (5.8). We now consider the combination $\bar{\psi}_2\Gamma_j(a+b\gamma_5)\psi_1$.

If we apply (5.8) to ψ_1 we obtain $a = b$.

Now we apply (5.8) to ψ_2;

$$\bar{\psi}_2\Gamma_j(1+\gamma_5)\psi_1 \rightarrow -\bar{\psi}_2\gamma_5\Gamma_j(1+\gamma_5)\psi_1.$$

From this we get $\gamma_5\Gamma_j = -\Gamma_j\gamma_5$ which means that only the V and A coupling remain.

Also the argument of MARSHAK and SUDARSHAN [34] implies the invariance of the interaction Hamiltonian for the transformation

$$\psi \rightarrow \gamma_5\psi\,. \qquad (5.9)$$

The eigenvalue of the operator γ_5 is called the chirality; $\psi^{(+)}$ is an eigenfunction of γ_5 with eigenvalue or chirality $+1$ (cf. (2.38)). The form (5.5) is such that it involves the conservation of chirality. Besides the β process

$$n \rightarrow p + e^- + \bar{\nu}\,,$$
$$p \rightarrow n + e^+ + \nu\,, \qquad (5.10)$$

two other four-fermion processes between nucleons and leptons have so far been studied experimentally as well as theoretically: the decay of the muon (cf. section 4)

$$\mu^- \rightarrow e^- + \nu + \bar{\nu}\,; \qquad (5.11)$$

and the capture of muons by nuclei

$$\mu^- + p \rightarrow \nu + n\,. \qquad (5.12)$$

The capture process will be considered in section 7; the errors involved both in theory and experiment are such that until now a precise determination of the coupling constants for this process has not been possible.

It was noticed already in 1949 that the magnitude of the coupling constants was about the same for the three processes (5.10), (5.11) and (5.12). Nowadays experiments and theory have improved to such an extent that we can investigate if there is really an exact Universal Fermi Interaction (U.F.I.). In 4 (cf. (4.12)) we have seen that the experiments on muon decay are in agreement with the V-A form for the interaction

$$\mathcal{H}_\mu = (g_\mu/\sqrt{2})\,(\bar{\psi}_e \gamma_\lambda (1+\gamma_5)\psi_\nu)(\bar{\psi}_\nu \gamma_\lambda (1+\gamma_5)\psi_\mu)\,. \qquad (5.13)$$

With respect to the "experimental" Hamiltonian for the β interaction (5.1), the only difference is that $\lambda = 1$ in (5.13). This should also follow from the prescription $\psi \rightarrow a\psi$ (cf. (5.5)).

From the life-time of the muon we obtained a precise value for g_μ (cf. (4.18a):

$$g_\mu = (1.428 \pm 0.001) \times 10^{-49} \text{ erg cm}^3. \qquad (5.14)$$

The discrepancy between (5.14) and (5.2) ($\approx 0.8\%$) is so small that one is inclined to assume that

$$g_\beta = g_\mu \qquad (5.15)$$

holds exactly.

This means that the vector coupling constants in β-decay and μ-decay are the same.

We should mention that the values (5.2) and (5.14) have to be corrected because of electromagnetic effects. DURAND et al. [35] find that the discrepancy then becomes less ($\approx 0.5\%$). However calculations by KINOSHITA and SIRLIN [36] make the agreement worse ($\approx 1.3\%$) but there are several uncertainties in these corrections. We shall assume (especially in the next section) that (5.15) would be exact, if one knew how to make all the necessary corrections. Now there is one important difference between the β-process and muon decay. In the first case one has to do with nucleons; these are particles which also have a strong interaction. One must imagine that around the nucleon there is a cloud of virtual pions, which are emitted and absorbed by the nucleon. A priori we expect that this will have an effect on the process

(5.10) (and also on muon capture). In muon decay we have only particles with a weak interaction (except for the well-known electromagnetic interaction). One now assumes that the hypothesis of the U.F.I. is valid for the interaction of the bare particles

$$g_\beta^{(0)} = g_\mu \quad (= g_\mu^{(0)}). \tag{5.16}$$

The deviation of λ from unity in the "experimental" Hamiltonian is then not surprising. It should follow from a difference in renormalization of the axial vector and vector part of the β-interaction. However the equality (5.16) or rather of g_β and $g_\beta^{(0)}$ (following from (5.16) and (5.15)) is then at first difficult to understand. We expect some renormalization effect. We explain this qualitatively by some Feynman diagrams (see Fig. 3).

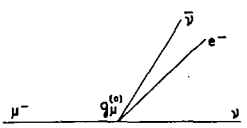


Fig. 3(a)

μ^- decay

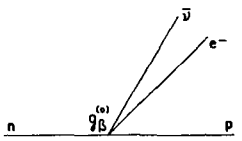


Fig. 3(b)

Decay of the bare neutron

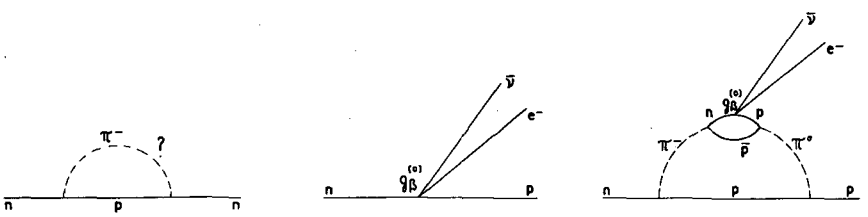


Fig. 3(c)

Decay of the physical neutron

The hypothesis of the U.F.I. states that the contribution of Fig. 3 (a) is the same as of Fig. 3 (b); $g_\beta^{(0)} = g_\mu^{(0)}$. However, it is not a priori clear that the sum of the processes Fig. 3 (c) gives the same result as Fig. 3 (b). It has not yet been possible to calculate the contributions of Fig. 3 (c) because one cannot apply perturbation calculations to strong interactions.

Nevertheless, in section 6 we shall discuss how one can, at least for the vector interaction, explain the equality $g_\beta^{(0)} = g_\beta$.

GOLDBERGER and TREIMAN [37] have given a general treatment of the weak interaction of nucleons with leptons.

Without doing explicit calculations of diagrams such as Fig. 3 (c) one can say something about the general structure of the effective matrix elements assuming some very general principles of invariance and space-time structure. If we assume that the "bare" Hamiltonian, i.e. the Hamiltonian without the influence of strong interactions, has a pure V, A form, we can write for the effective matrix element for the process (ℓ = e or μ)

$$n + \nu \leftrightarrow p + \ell \tag{5.17}$$

$$M = 2^{-\frac{1}{2}}\, \bar{u}_\nu\, i\gamma_5\, \gamma_\lambda\, (1+\gamma_5)\, u_\ell \langle n | g_A^0 J_\lambda^A | p \rangle$$

$$+ 2^{-\frac{1}{2}}\, \bar{u}_\nu \gamma_\lambda (1+\gamma_5)\, u_\ell \langle n | g_V^0 J_\lambda^V | p \rangle \tag{5.18}$$

$$J_\lambda^A = \bar{\psi}_N i\, \gamma_\lambda \gamma_5 \tau^{(-)} \psi_N \qquad (+ \text{ pion current ?})$$

$$J_\lambda^V = \bar{\psi}_N \gamma_\lambda \tau^{(-)}\, \psi_N \qquad (+ \text{ pion current ?}) \tag{5.19}$$

g_A^0 and g_V^0 are the "bare" coupling constants.

We are supposing that the U.F.I. holds for the bare couplings

$$g_V^0 = g_V^{\beta 0} = g_V^{\mu\, co} = g_V^\mu$$

$$g_A^0 = g_A^{\beta 0} = g_A^{\mu\, co} = g_A^\mu\ .$$

In the following ψ and $\bar{\psi}$ denote annihilation and creation operators respectively and not the ordinary wave functions (cf. App. II). u_ν and u_ℓ are the spinors (c numbers) for the neutrino and lepton, $|n\rangle$ and $|p\rangle$ are the physical states of the neutron and proton respectively. For the following the explicit expression for J_λ^A and J_λ^V is not relevant. The only thing that matters is the axial vector or vector behaviour of these expressions.

One imposes the following requirements on (5.18):

(1) Lorentz invariance;

(2) charge independence of the strong interactions;

(3) invariance for time reversal.

With conditions (1), (2), (3), one can show that the most general form for the "nucleon currents" is (cf. also 7.2)

$$\langle n | g_A^0 J_\lambda^A | p \rangle = A\, \bar{u}_p\, i\gamma_\lambda \gamma_5\, \bar{u}_n - B\, \bar{u}_n\, \gamma_\lambda \gamma_5 u_p + E\, \bar{u}_n \sigma_{\lambda\rho}\, q_\rho \gamma_5\, u_p, \tag{5.20}$$

$$\langle n | g_V^0 J_\lambda^V | p \rangle = C\, \bar{u}_n \gamma_\lambda u_p - iD\, \bar{u}_n\, \sigma_{\lambda\rho} q_\rho u_p + i\, F\, \bar{u}_n q_\lambda u_p\ . \tag{5.21}$$

The constants A ... F are functions of the invariant $q^2 = (p_\lambda - n_\lambda)^2$; $q_\lambda = p_\lambda - n_\lambda$; n_λ, p_λ are the four-momenta of the neutron and proton

$$\sigma_{\lambda\mu} = (1/2)\, (\gamma_\lambda \gamma_\mu - \gamma_\mu \gamma_\lambda).$$

In the limit $q \to 0$ (β decay), the terms with coefficients B, D, E, and F approach zero. This is the reason why in β experiments the terms with A and C are sufficient. In the following section we shall see that if one looks for more subtle effects the term with D has some measurable consequences. In the muon capture process with $q^2 \approx m_\mu^2$, we shall see that the effect of the terms with D and B becomes quite large (section 7). The terms with E and F were absent in the original paper by Goldberger and Treiman. How-

ever, one then has to assume that the nucleon currents transform "normally" for the transformation G.

(4)

$$G J_\lambda^A G^{-1} = -J_\lambda^A$$
$$G J_\lambda^A G^{-1} = +J_\lambda^A \qquad (5.22)$$

with $G = C\, e^{i\pi T^2}$.

C is the operator for charge conjugation. $e^{i\pi T^2}$ is a rotation over π on the 2-axis in the isospin space (cf. [8] and also 7.2). Requirement 4 has not yet been verified by experiment (see also next section). If we assume that invariance for time-reversal holds, it follows that the "form" factors A...F are real.

6. THE THEORY OF THE CONSERVED VECTOR CURRENT: THE PSEUDOSCALAR IN β-DECAY

6.1. The hypothesis of the conserved vector current

We shall confine ourselves in the first part of this section to the vector part of the interaction.

At the end of section 5 we discussed the fact that, although nucleons have strong interactions, one can nevertheless write down a general effective matrix element for the β-process. In β-decay it is almost sufficient, because of $q^2 \approx 0$, to consider only the first term of (5.18), which is the classical vector covariant with coupling constant: $C(q^2 \approx 0)$. This constant, which we also call $g_\beta = g_\beta^V$ is the result of a renormalization process and there is no direct connection with the unrenormalized coupling constant g_V^0. In muon decay there is no such renormalization effect (apart from small electromagnetic corrections) and the measured coupling constant g_μ is also the bare coupling constant (cf. Fig. 3).

Experiments on β-decay and μ-decay suggest that almost exactly (5.15)

$$C(0) = g_\beta = g_\mu. \qquad (6.1)$$

Now one can hardly believe that the bare coupling constant and the renormalization effects are such as to yield g_β equal g_μ. If one assumes a U.F.I. for the "bare" processes, the problem is to explain why here is no renormalization effect for the β-interaction:

$$g_\beta = g_\beta^{(0)}. \qquad (6.2)$$

FEYNMAN and GELL-MANN [39], and independently ZELDOVICH and GERSHTEIN [40] remarked that there is a striking analogy with the situation in electromagnetism.

The electromagnetic interaction of a proton is originally not influenced by the strong interaction: $e = e_0$; i.e. the total charge (this is the coupling with the photon field) of the physical proton is equal to the charge of the "bare" proton. We are supposing that the charge of the bare proton is equal to the charge of the electron.

In Fig. 4 we give the diagrams for the electromagnetic interaction of the proton.

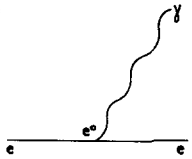


Fig. 4(a)

E. M. interaction of the electron

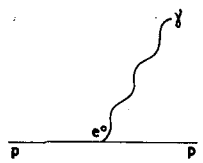
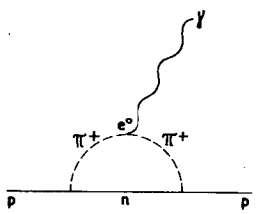
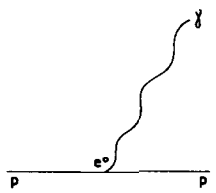


Fig. 4(b)

E. M. interaction of the bare proton





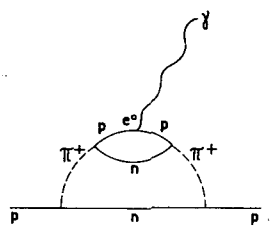


Fig. 4(c)

E. M. interaction of the physical proton

The reason why there is no renormalization effect is, of course, the fact that the charge is conserved in the strong interactions; the π^+ has the same charge as the proton.

We get an explanation of the equality $g_\beta = g_\beta^{(0)}$, by assuming that the pion has also a direct coupling with the lepton field, with the same coupling constant as the bare nucleon. Thus we replace the ? in Fig. 5(cf Fig. 3 (c)) by a diagram (Fig. 6) with a direct coupling of π with the lepton field.

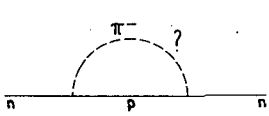


Fig. 5

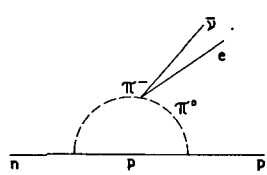


Fig. 6

Let us now consider the analogy more closely. The operator for the electromagnetic current can be divided into an isoscalar and an isovector

part. Both are conserved. If we consider only the contribution of the vector part to the electromagnetic interaction we can write:

$$\mathcal{H}^{V}_{e.m.} = -ie^{o}J^{3}_{\mu}A_{\mu}. \tag{6.3}$$

A_μ is the electromagnetic field. If we include the strong interaction, J^3_μ is the third component of the total isotopic vector current operator

$$\underset{\sim}{J_\mu} = \tfrac{1}{2}\,\bar{\psi}_\nu \gamma_\mu \underset{\sim}{\tau}\, \psi_\nu + \phi^*_\pi \underset{\sim}{T}\, \nabla_\mu \phi_\pi + \ldots \tag{6.4}$$

We use isotopic spin notation (cf. App. IV).

By a wavy line we indicate that we have a vector in the isospin space. If we also include the strong interaction of the nucleons with the strange particles (e.g. Σ) we have to add more terms to (6.4).

Conservation of isotopic spin implies that this current is so constructed that

$$\frac{\partial}{\partial x_\mu}\, \underset{\sim}{J_\mu} = 0\,. \tag{6.5}$$

Now the matrix element of the vector part of the electric current operator between two physical proton states can be written as

$$\langle p' | e^{o} J^{3}_{\mu} | p \rangle = \tfrac{1}{2}\bar{u}_{p'} e \left\{ F^{v}_{1}(q^2)\,\gamma_\mu - i\,\frac{\mu^{(a)}_p - \mu^{(a)}_n}{2M} F^{v}_{2}(q^2)\;\sigma_{\mu\lambda}\, q_\lambda \right\} u_p. \tag{6.6}$$

$\mu^{(a)}_p$ and $\mu^{(a)}_n$ are the anomalous magnetic moments of the proton and neutron

$$\mu^{(a)}_p = 1.8; \quad \mu^{(a)}_n = -1.9. \tag{6.7}$$

$F^{v}_{1}(q^2)$ and $F^{v}_{2}(q^2)$ are the well-known nucleon form factors which are measured in the Hofstadter experiments. They are normalized to 1 at $q^2 = 0$. One can now prove that because of the conservation law (6.5) for the third component $e^o = e$, which means that there is no renormalization due to the strong interactions. In analogy with (6.5) the vector part of interaction $n + \nu \rightarrow p + e$ can be written generally as (cf. (5.14))

$$\mathcal{H}^{v}_{\beta} = (1/2^{\frac{1}{2}})\, g^{o}_{v}\, J^{v}_{\mu}\, j_\mu \tag{6.8}$$

with

$$j_\mu = \bar{u}_\nu \gamma_\mu (1+\gamma_5)\, u_e. \tag{6.9}$$

Now we suppose that this current operator for the vector part J^v_μ (or μ capture; $e \rightarrow \mu$) is described by the - isovector component of the current (6.4). This means that we take for the operator J^v_μ (cf. (5.17)) the definite form

$$J^{-}_{\lambda} = J^{V}_{\lambda} = \bar{\psi}_N \gamma_\lambda \underset{\sim}{\tau^{(-)}}\, \psi_N + \phi^*_\pi\, \underset{\sim}{T^{(-)}}\, \nabla_\mu \phi_\pi + \ldots \tag{6.10}$$

Comparing (6.10) with (6.4) we see that the factor $\frac{1}{2}$ in the first term of (6.4) has disappeared in (6.10). The reason is the difference between the definition of $\tau^{(-)}$ and $T^{(-)}$ (cf. App. IV).

In analogy to the electromagnetic case (6. 6), the effective matrix element then becomes for the - component

$$\langle n | g_V^0 J_\lambda^V | p \rangle = \bar{u}_n g_V \left\{ F_V(q^2) \gamma_\lambda - i F_M(q^2) \sigma_{\lambda\mu} q_\mu \right\} u_p \qquad (6.11)$$

with $g_V = g_V^0$ and $F(c) = 1$, because of the conservation (6. 5).

Furthermore, we have a definite prediction for $F_V(q^2)$ and $F_M(q^2)$ (cf. FUJII and PRIMAKOFF [41]):

$$F_V(q^2) = F_1^V(q^2), \qquad (6.12)$$

$$F_n(q^2) = ((\mu_p^{(a)} - \mu_n^{(a)})/2M) F_2^V(q^2).$$

Comparing (6. 11) with the more general (5. 18) we see that the non-G-invariant term with F in (5. 18) is absent in (6. 11). This is easy to prove. Consider

$$\langle n | (\partial/\partial x_\lambda) J_\lambda^V | p \rangle = - i q_\lambda \langle n | J_\lambda^V | p \rangle. \qquad (6.13)$$

If we now substitute (5. 18) we find with the help of the Dirac equation that the first two terms (with C and D) become zero. We then have, using (6. 5), for the - component

$$0 = i F \bar{u}_n q_\lambda q_\lambda u_p \rightarrow F = 0.$$

6.2. Experimental evidence for the conserved vector current theory

In this section we consider several experimental consequences of the theory of the conserved vector current.

(a) As we have already remarked, the second term in (6. 9) put into (6. 7) will give a direct coupling of the pion to the lepton field. The process

$$\pi^+ \rightarrow \pi^0 + e^+ + \nu \qquad (6.14)$$

is not only predicted to occur but one can also calculate its lifetime (partial with respect to the main process $\pi^+ \rightarrow \mu^+ + \nu$). One then obtains 2. 4 sec. This process will also occur in other theories but it is not possible then to predict a definite value for the lifetime, because of the strong vertices (Fig. 7).

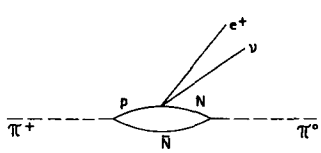


Fig. 7

Process (6. 14) has been measured by several groups but the uncertainty is as yet too large for any conclusion to be drawn.

The other tests of the conserved vector theory all rest on the detection of the second term in (6. 11). Here we should stress the point that this term

will also occur generally in other theories. However, the C. V. C. theory gives a definite prediction about its magnitude. It should be mentioned that (6. 11) with (6. 12) is derived for the case of free nucleons whereas in β-decay mostly the nucleons are bound by the nuclear potential. Luckily, it is sometimes possible, again from the analogy between electromagnetic vector interaction and β-vector interaction (in section 7 we will consider the effect on the muon capture interaction), to make certain definite predictions which can be tested experimentally. Another difficulty is that in β-decay the second term of (6. 11) generally makes a rather small contribution to the allowed transitions. Moreover, there is an uncertainty in the magnitude of often small terms coming from the first term in (6. 11) and from the G. T. coupling.

(b) The spectrum; GELL-MANN [42] drew attention to a very nice example where one can use the analogy to predict some consequences of the C. V. C. theory.

This is the case when A = 12.

In Fig. 8 we have a β^--transition from $B^{12} \to C^{12}$ a β^+-transition from $N^{12} \to C^{12}$ and a parallel γ-transition from $C^{12*} \to C^{12}$. The ground states

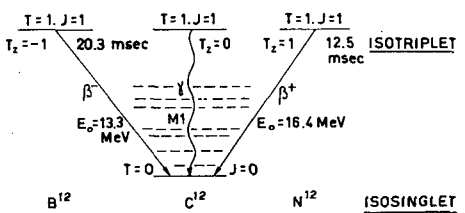


Fig. 8

of B^{12} and N^{12} and the excited state of C^{12} are members of an isotriplet with $T = 1$ (T_z resp. -1, 1 and 0). The spin J is 1 and the parity + 1. The ground state of C^{12} has I = 0, J = 1 and parity + 1. The β-transitions are allowed G. T. transitions, the γ-transition M 1. One now finds the following correction factor for the allowed spectrum (cf. SCHOPPER [43]):

$$K_1 = 1 \pm \frac{2}{3} a (2E - E_0 - 1/E) - \frac{2}{3} b (E_0 - 1/E) \qquad (6.15)$$

with

$$a = \left| \frac{g_V}{g_A} \right| \cdot \frac{\int (\vec{\alpha} \times \vec{x})}{\int \vec{\sigma}}; \quad b = \frac{i \int \gamma_5 \vec{x}}{\int \sigma} . \qquad (6.15a)$$

The minus sign is for a β^+, the plus sign for a β^--transition. This comes from the fact that the term with a is an interference term between $\int \vec{\alpha} \times \vec{x}$ coming mainly from the second term in (6. 11) and $\int \vec{\sigma}$ the main matrix element from the G. T. coupling. If one goes from β^- to β^+ the only thing that differs is the sign of g_A and thus of the term with a, this being in contrast to the term with coefficient b which is an interference between two G. T. matrix elements. This is very helpful because if we now measure the ratio of the β^+ and β^--spectra we get rid of the unknown matrix elements from the G. T. coupling. The coefficient a can now be predicted from the analogy. If one assumes that the decaying nucleon is a free particle one easily gets by means of (6. 11) and (6. 12)

$$a = ((1 + \mu_p^{(a)} - \mu_n^{(a)})/\lambda M) \approx 4/M. \qquad (6.16)$$

If one uses the analogy of the matrix element $\int \vec{\alpha} \times \vec{x}$ with the corresponding μ matrix element from the γ-transition in C^{12} one has

$$a = (2^{\frac{1}{2}}/\lambda M)(\mu / \int \vec{\sigma}). \tag{6.17}$$

With

$$\Gamma_\gamma = (\mu^2/3.137)(E_\gamma^3/M^2) = (53 \pm 11) \text{ e V [44]} \tag{6.18}$$

one has

$$a \approx (4.6 / M) \tag{6.19}$$

which is in agreement with the rough estimation (6.16). A more detailed calculation by GELL-MANN and BERMAN [45] (cf. also WEIDENMÜLLER [46]), who take into account also electromagnetic effects, gives for the ratio of the β^+ and β^--spectrum

$$(K_1(E, B^{12})/K_1(E, N^{12})) = \text{const } \{1 + (\tfrac{8}{3}a + \delta)E\} f(E); \tag{6.20}$$

δ and f(E) are electromagnetic corrections.

One now predicts using (6.18),

$$\begin{aligned} \tfrac{8}{3}a + \delta &= (1.33 \pm 0.15) + (-0.25 \pm 0.15)\ \% \\ &= (1.08 \pm 0.30)\%. \end{aligned} \tag{6.21}$$

Recently the spectra from B^{12} and N^{12} were measured by MAYER - KUCKUK and MICHEL [47]. They found:

$$\tfrac{8}{3}a + \delta = (1.13 \pm 0.15)\%. \tag{6.22}$$

This agrees very well with (6.21) and the conserved vector current theory seems nicely confirmed. However, the experiment is very difficult and an independent determination by a different group remains very desirable.

(c) $\beta - \gamma$ and $\beta - \alpha$ angular correlation. The matrix elements from the 2nd forbidden category to which the second term of (6.11) belongs, give anisotropic $\beta - \gamma$ and $\beta - \alpha$ angular correlations.

One has

$$W(\theta) = 1 + A_2(\cos^2\theta - \tfrac{1}{3}); \tag{6.23}$$

θ is the angle between the β and γ or β and α-particle

$$A_2 \sim (E - \tfrac{1}{E})(\pm a + 2b); \tag{6.24}$$

a and b are defined in (6.15).

NORDBERG et al. [48] have measured the $\beta - \alpha$ angular correlation for the case when A = 8. The situation here is analogous to the case when A = 12. However, there is no experimental value of the analogous γ-transition in

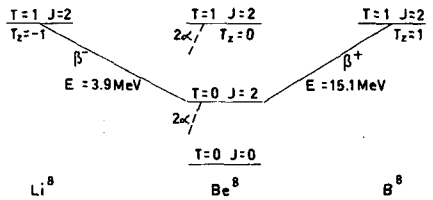


Fig. 9

Be^8 since a precise measurement of the β spectra of Li^8 and Be^8 has not yet been made.

The result obtained by NORDBERG et al. [48] is

$$\delta = A_2^{Li} - A_2^{Be} = (0.0069 \pm 0.008)\,E. \tag{6.25}$$

A calculation by WEIDENMÜLLER [49] gives with the C.V.C. theory

$$0.005\,E < \delta < 0.009\,E, \tag{6.26}$$

which means that there is at least no contradiction between theory and experiment.

BOEHM et al. [50] have measured a $\beta - \gamma$ angular correlation for the sequence $F^{20}(1.1^+) \xrightarrow{\beta} Ne^{20}\ (0.2^+) \xrightarrow{\gamma} Ne^{20}\ (0.0^+)$. They found $A_2 = (0.94 \pm \pm 0.28)\%$ whereas the C.V.C. theory predicts a value of about 0.45%. Here the influence of the b term is not cancelled because a ratio is not measured and therefore the predicted value is very uncertain.

(d) BOUCHIAT [51] considered another consequence of the C.V.C. theory. In this theory one expects that the Fermi matrix element $M_F\ (= \int 1)$ is strictly zero in the case of a $\Delta J = 0, \Delta T \neq 0$ transition.

This is so because now the Fermi operator (cf. (6.10)) contains the total isospin (if one takes e.g. (5.16) without pion terms the virtual pions in the physical nucleon states can cause a Fermi transition with $\Delta T \neq 0$).

Information on the ratio of M_F to M_{GT} is now obtained by measuring the $\beta - \gamma$ circular polarization correlation.

This angular correlation is given by (cf. section 3; (3.97) (3.98))

$$\omega^{\beta\gamma}(\theta, \tau) = 1 \pm \frac{1}{3} A_{\pm}^{\gamma} \frac{v}{c} \tau Q_{\gamma} \cos\theta. \tag{6.27}$$

The coefficient A^{γ} contains essentially the ratio of M_F to M_{GT} (cf. (3.98)). The + sign is for the β^-, the - sign for the analogous β^+ transition. A complication now arises. Because of the Coulomb interaction there can be an admixture of wave functions with different T value in the initial and final states. Bouchiat shows that under charge conjugation (i.e. $\beta^+ \leftrightarrow \beta^-$) M_F is even if its contribution comes from this admixture, odd if there are mesic effects (i.e. when the C.V.C. theory is not correct). Thus the C.V.C. theory predicts that exactly

$$A_+^{\gamma} + A_-^{\gamma} = 0. \tag{6.28}$$

There is an interesting case when A = 24:

$$Al^{24}\ (4^+, 1) \xrightarrow{\beta^-} Mg^{24}\ (4^+, 0) \xrightarrow{\gamma} Mg^{24}\ (0^+, 0)$$

$$Na^{24}\ (4^+, 1) \xrightarrow{\beta^+} Mg^{24}\ (4^+, 0) \xrightarrow{\gamma} Mg^{24}\ (0^+, 0).$$

For Na^{24} the experiment has been done

$$A_+^\gamma = 0.07 \pm 0.03. \tag{6.29}$$

As yet Al^{24} has not been measured (it is a very short living nucleus). From (6.28) and (6.29) one predicts $A^\gamma \approx -0.07$.

Summarizing, we can say that there are several experiments which indicate that the C.V.C. theory is right; at least they are not in contradiction with this theory. However, bearing in mind that one should have quantitative agreement, more experiments (and a theory to compute certain corrections) are desirable.

6.3. The axial vector current

If it is assumed that $g_V^0 = -g_A^0$ and if the C.V.C. theory is accepted then it seems that there is a renormalization effect in the axial vector coupling constant because $\lambda = |g_A/g_V| \neq 1$. Attempts have been made to compute the magnitude of this renormalization effect but then the perturbation theory has to be applied to the strong interaction (cf. e.g. [40]). Therefore, the results are very doubtful. Moreover, nothing is known about possible pion terms in the bare coupling (5.16). That there is no conservation law analogous to (6.5) for the current J_μ^A can be proved in two different ways.

(1) The first argument is given by GOLDBERGER and TREIMAN [53]. Suppose

$$\frac{\partial}{\partial x_\mu} J_\mu^A = 0. \tag{6.30}$$

Now we can compute

$$\langle p \left| \frac{\partial}{\partial x_\mu} J_\mu^A \right| n \rangle = i q_\mu \langle p | J_\mu^A | n \rangle. \tag{6.31}$$

If we now take the general expression for $\langle p | J_\mu^A | n \rangle$ from (5.17) and use the Dirac equation, which states that the term with E vanishes, we get a relation between the coefficient $A = g_A$ and B (with $m_e B = g_p^{eff}$; see 6.4.).

$$B = 2MA/(p - n)^2 \tag{6.32}$$

or

$$g_p^{eff}/g_A = 2m_e/q^2 \gtrsim 10^3. \tag{6.33}$$

However, such a large (induced) pseudoscalar is certainly not measured in β-decay (see 6.4).

(2) TAYLOR [54] remarks that the conservation law (6.30) is in contradiction to the occurrence of the leptonic decay of the pion

$$\pi^- \longrightarrow \mu^- + \nu. \tag{6.34}$$

If we assume a primary V, A interaction Hamiltonian the effective matrix element for (6.34) can be written as

$$\langle 0 \mid J_\mu^A \mid \pi \rangle \bar{u}_\mu \gamma_\mu (1 + \gamma_5) u_\nu . \tag{6.35}$$

The vector current does not make a contribution because of the conservation of parity in the strong interaction (the pion has an intrinsic negative parity).

Now the momentum vector k_μ of the pion is the only vector that occurs in this process. If follows that

$$\langle 0 \mid J_\mu^A \mid \pi \rangle = C(k^2) k_\mu . \tag{6.36}$$

From (6.30) we obtain

$$-i k_\mu \langle 0 \mid J_\mu^A \mid \pi \rangle = \langle 0 \mid \frac{\partial}{\partial x_\mu} J_\mu^A \mid \pi \rangle = 0. \tag{6.37}$$

With (6.36) we then have

$$k_\mu \, k_\mu \, C(k^2) = 0 .$$

Therefore ($k_\mu \, k_\mu = -m^2$) $C(k^2) = 0$ and process (6.34) should not occur.

6.4. The pseudoscalar in β-decay

Pseudoscalar coupling can appear in the effective matrix element from two sources.

(1) The pseudoscalar is already present in the bare interaction, as are the V and A coupling terms:

(2) The pseudoscalar can be induced by mesonic effects from the original axial vector coupling term; if we put the second term (with coefficient B) of (5.17) into (5.14) and apply the Dirac equation, using $p_\lambda - n_\lambda = \ell_\lambda - \nu_\lambda$, for the leptons we get

$$\frac{1}{\sqrt{2}} m_\ell B \bar{u}_\nu (1 + \gamma_5) u_\ell \bar{u}_n \gamma_5 u_p . \tag{6.38}$$

This shows that the induced pseudoscalar appears with the effective coupling constant $m_\ell B$. GOLDBERGER and TREIMAN [55] (cf. also WOLFENSTEIN [56]) have computed the coefficient B using methods of dispersion theory. They assume that the main contribution comes from Fig. 10.

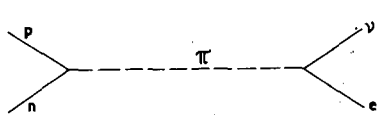


Fig. 10

The first vertex describes strong interaction whereas the second describes the process

$$\pi \longrightarrow \nu + \ell , \tag{6.39}$$

It is found that

$$g_p^{\text{eff}} \approx (1/2\pi)(G^2/4\pi)(m_p m_\ell/(m_\ell^2 + m_\pi^2))|g_A|. \tag{6.40}$$

G is the effective coupling for the strong vertex ($n \rightarrow p + \pi$). If $\ell = \mu$ (in muon capture) we have

$$g_p^{\text{eff}} \approx 8|g_A|, \tag{6.41}$$

whereas in β-decay, $m_\ell = m_e$, we get

$$g_p^{\text{eff}} \approx (1/20)\ |g_A|. \tag{6.42}$$

This, as we shall see, is certainly not detectable in β-decay experiments. If one assumes that the π-meson decay (6.39) is caused by the primary axial vector coupling one finds that the sign of g_p is the same as the sign of g_A.

The reason why it is so difficult to detect the pseudoscalar is that the main contribution belongs to the category of the first forbidden transitions because of the selection rules $\Delta J = 0$ and change of parity. The nuclear matrix element is $\int \beta \gamma_5$. Now there are other terms coming from the axial vector interaction with nuclear matrix elements $\int \gamma_5$ and $\int \vec{\sigma} \cdot \vec{x}$. The matrix elements $\int \gamma_5$ and $\int \beta \gamma_5$ are relativistic: γ_5 connects the non-relativistic part of e.g. the initial nucleon wave function with the relativistic part of the final wave function. As yet there is almost no information regarding the relativistic part of the nucleon wave function in a nucleus. The only thing which can be done is to approximate these matrix elements. This is done with the Foldy-Wouthuysen transformation, which reduces odd operators such as γ_5 (odd means that the operator connects relativistic with non-relativistic parts) to even operators with a certain power of $1/M$ (M is the nucleon mass).

The first step (i.e. to an order of $1/M$, which is sufficient in our case because of the relatively low energy of the nucleons) can be made by expressing simply the small components in terms of the large ones, using the Dirac equation in an approximate form.

The Dirac equation for a nucleon in an external field U(r) (the potential of the nucleus) is as follows

$$(\vec{\alpha}\vec{p} - \beta M + U(r))\psi = E\psi \tag{6.43}$$

We put $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, ψ_1 and ψ_2 are two component spinors.

(6.43) one can then be written as two coupled equations for ψ_1 and ψ_2.

$$\begin{aligned} \vec{\sigma} \cdot \vec{p}\psi_2 + (M + U(r) - E)\psi_1 &= 0 \\ \vec{\sigma} \cdot \vec{p}\psi_1 + (-M - U(r) - E)\psi_2 &= 0 \end{aligned} \tag{6.44}$$

From (6.44) we get

$$\psi_2 = \frac{\vec{\sigma} \cdot \vec{p}}{M + E + U(r)} \psi_1 = f(r) \cdot \vec{\sigma} \cdot \vec{p}\psi_1 \tag{6.45}$$

with

$$f(r) = \frac{1}{M + E + U(r)} \approx \frac{1}{2M} \tag{6.46}$$

Thus ψ_2 is the relativistic part of ψ. If one applies this result to the matrix elements $\int \beta\gamma_5$ and $\int \gamma_5$ one gets the following effective matrix-elements for $o^+ \to o^-$ transition ($r = |\vec{x}|$)

$$M = \frac{1}{\sqrt{2}} g_A \left[-i \int (\vec{\sigma} \cdot \vec{x}) \quad \vec{\sigma} \cdot \vec{k} + \int \left(\frac{\vec{\sigma} \cdot \vec{p}}{M} - i \vec{\sigma} \cdot \vec{x} \cdot \xi(r) \right) \gamma_5 \right]$$

$$+ \frac{1}{\sqrt{2}} g_P \left[-i \int \frac{(\vec{\sigma} \cdot \vec{x})}{2M} \quad k^2 + i \int \vec{\sigma} \cdot \vec{x}\, \xi(r) \quad \beta \gamma_5 \right] \qquad (6.47)$$

$$\xi(r) = -\frac{1}{4M^2} \quad \frac{1}{r} \quad \frac{dU}{dr}$$

$$\vec{k} = \vec{p}_e + \vec{p}_r \quad . \qquad (6.48)$$

With (6.47) one can now calculate e.g. the spectrum and the polarization of the electrons. However, one has three parameters, $\int (\vec{\sigma} \cdot \vec{p})$, $\int (\vec{\sigma} \cdot \vec{x})$ and $\int (\vec{\sigma} \cdot \vec{x} \xi(r))$.

ROSE and BHALLA [57] find on the basis of an accurate analysis of the spectrum and the β-polarization, especially in the case of Pr_{144}, as an upper limit for g :

$$\left| \frac{g_P}{g_A} \right| < 90. \qquad (6.49)$$

The experiments are in agreement with $g_P = 0$.

TADIĆ [58] has argued that a strong increase of the nuclear matrix elements occurs as a consequence of the Coulomb field of the nucleus, from which one would conclude a lower value than (6.43). However, BLOKHINTSEV and DOLINSKY have shown [59] that requirements of gauge invariance invalidate Tadić's argument.

Comparing (6.43) with (6.42) we conclude that nothing can be said about the induced pseudoscalar in β-decay.

7. MUON CAPTURE

7.1. Introduction

When muons are stopped in solid matter the process of slowing down is so fast that they generally come to rest before disintegrating. When positive muons are stopped they will be repulsed by the atomic nuclei and will also decay in matter according to $\mu^+ \longrightarrow e^+ + \nu + \bar{\nu}$. However, the negative muons are attracted by the atomic nuclei and are captured at the end of their trajectory in a Bohr orbit around a nucleus, with which they then form a "muonic atom". The Bohr orbit will mostly be highly excited and subsequently de-excitation will occur by emission of muonic X-rays or by the Auger-effect. It turns out that de-excitation is so fast that the muon has only a negligible chance to disintegrate during this process. After the de-excitation the muon is found in a "K-orbit" of the muonic atom for which the radius of the Bohr orbit is given by

$$a_{\mu_o} = \frac{\hbar^2}{m_\mu e^2 Z} = \frac{m_e}{m_\mu} \cdot \frac{1}{Z} a_o, \qquad (7.1)$$

where Z is the nuclear charge and

$$a_o = \frac{\hbar^2}{m_e e^2} = 0.529 \times 10^{-8} \text{cm} \qquad (7.2)$$

is the radius of the first Bohr orbit in a hydrogen atom. We see from (7.2) that the radius of the muon orbit is at least 200 × smaller than a_o because $m_\mu = 206.7\, m_e$. This means that the muon in the muonic atom is so close to the nucleus in the centre of the electron cloud, that there is practically no screening of the nuclear charge by the electrons. $a_{\mu o}$ becomes so small for high Z that the muon wave function (for the K-orbit) is situated for a considerable part within the nucleus (for example for Pb). We should then take into account that the nucleus has a field deviating from the Coulomb field of

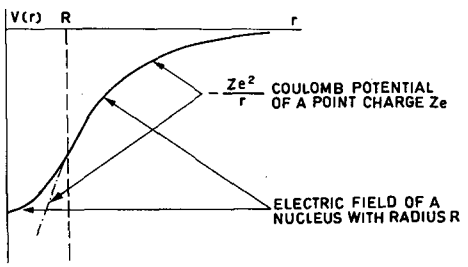


Fig. 11

Electric potential of a nucleus

a point charge (cf. Fig. 11). Once the muon has arrived in a K-orbit we have the possibility of two (competing) processes

$$\mu^- \rightarrow e^- + \nu + \bar{\nu} \qquad (7.3)$$

$$\mu^- + p \rightarrow n + \nu. \qquad (7.4)$$

If (7.4) is considered as a process occurring for free particles initially at rest the energy balance is

$$E_\mu = m_\mu c^2 = 206.7\, m_e c^2 = 105.6 \text{ MeV} = p_\nu c + (p_\nu{}^2/2m_n). \qquad (7.5)$$

p_ν is the magnitude of the momentum, which the neutrino and the neutron obtain ($\vec{p}_\nu$ and $-\vec{p}_\nu$ respectively). Solving (7.5) for p_ν, one obtains: $p_\nu = 196\, m_e c$, from which one calculates for the velocity and energy of the emitted neutron and neutrino

$$v_n = 0.11\, c, \quad E_n = 5.3 \text{ MeV}, \quad E_\nu = 100.3 \text{ MeV}. \qquad (7.6)$$

This shows that most of the energy corresponding to the rest mass of the muon is carried away by the neutrino.

When the process (7.4) occurs in a complex nucleus the protons in the initial state may have a certain motion. However, (7.6) shows qualitatively what must then be expected: the major part of the energy $m_\mu c^2 = 105.6$ MeV will be carried away by the neutrino; the remaining nucleus will be relatively little excited (of the order of 5 MeV); sometimes one or even two neutrons may be emitted from the nucleus. Although the average excitation of the

nucleus is not too high (it may be some 20 MeV) there will also exist smaller probabilities for high excitations in which a large fraction of $m_\mu c^2$ is transmitted to the nucleus as excitation energy.

The total rate for muon capture by a complex nucleus consists of many partial transition rates. The final nucleus will be de-excited by γ-emission for the lower excitations and often by n-emission for the higher excitations.

In Fig. 12 we represent the muon capture by a complex nucleus in a simple diagram. A characteristic difference between muon-capture and

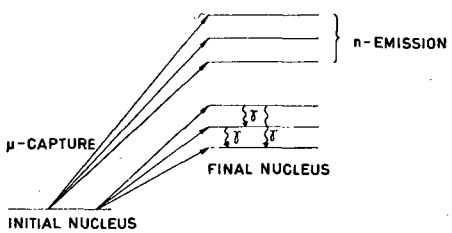


Fig. 12

electron capture is that in muon capture many partial transitions occur, while only one or a few partial transitions are found in electron capture. Of course this difference is due to the fact that $m_\mu >> m_e$.

For negative muons the processes (7.3) and (7.4) compete once the muon has arrived in a K-orbit. It now appears that muon capture (7.4) increases strongly with the nuclear charge Z, while the decay (7.3) in a K-orbit depends only slightly on Z. Experimentally it turns out that for $Z > 11$ muon capture is the preponderant process, while for $Z < 11$ the decay (7.3) is most probable.

The strong dependence of the muon capture probability $\Lambda_{\mu c}$ on Z is mainly caused by the fact that $\Lambda_{\mu c}$ contains as a factor $|\phi_\mu(0)|^2$, the square of the muon wave function evaluated in the origin. It is easily seen that

$$|\phi_\mu(0)|^2 \sim \frac{1}{(a_{\mu o})^3} = \left(\frac{m_\mu}{m_e}\right)^3 \frac{Z^3}{a_0^3}. \qquad (7.7)$$

Hence this factor provides a Z^3 dependence in $\Lambda_{\mu c}$. If the sum of nuclear matrix elements were proportional to Z, one would have a total dependence of $\Lambda_{\mu c}$ on Z proportional to Z^4. Often an approximate law proportional to Z_{eff}^4 is used, where a certain difference between Z_{eff} and Z takes account of the finite extent of the nuclear charge.

From the strong dependence on Z, roughly $\Lambda_{\mu c} \sim Z_{eff}^4$ and the approximate equality of the rates for (7.3) and (7.4) at $Z = 11$, it follows that muon capture occurs only in a small fraction of the cases for hydrogen or other light nuclei, when it is very difficult to detect. Muon capture is most easily observed for complex nuclei, which are not too light. For such nuclei muon capture rates were already reasonably known about 1950, while muon capture in H_2 was observed for the first time only in 1961. The investigation of muon-capture interaction involves the difficulty, in comparison with the investigation of beta-interaction (to which it is analogous to a great extent), that for obvious energy reasons only the process (7.4) analogous to electron capture can be

observed, but not the processes analogous to β^+ or β^--decay (neither for free nucleons nor for nucleons contained in nuclei):

$$\begin{aligned} n &\rightarrow p + e^- + \bar{\nu} \\ p &\rightarrow n + e^+ + \nu . \end{aligned} \tag{7.8}$$

Hence one is forced to exploit all the possibilities for observations of processes fundamentally given by (7.4) occurring either for free nucleons or nucleons contained in nuclei. We mention here the following effects which may be observed:

(1) Partial and total capture rates for muon capture. Up till now mostly total capture rates have been observed. Partial capture rates can be observed in principle by measuring the radioactivity of the final nucleus, by measuring the emitted gamma rays, etc.

(2) The hyperfine structure effect: dependence of total capture rate on relative orientation of muon spin and nuclear spin.

(3) Neutron emission after muon capture; both absolute numbers and energy spectra.

(4) Non-conservation of parity in muon capture will have as a consequence:

(a) After capture of polarized muons the emitted (primary) neutrons will have an asymmetric angular distribution (i. e. the latter contains a $\cos\theta$-term);

(b) the emitted neutrons will be polarized.

(5) Analogous to inner bremsstrahlung in electron capture the possibility of the emission of an additional photon exists in muon capture: in radiative muon capture,

$$\mu^- + p \rightarrow n + \nu + \gamma , \tag{7.9}$$

the energy $m_\mu c^2$ is mainly distributed over the neutrino and photon. Although (7.9) is a rather rare process its observation seems quite important.

Because muon capture in hydrogen is such a rare process, one will probably be forced for quite a number of years to study the above-mentioned effects (except for total capture rates) for complex nuclei and not for free nucleons. This means that the analysis of the phenomena requires a combination of elementary particle physics and a knowledge of nuclear structure.

After these general remarks we shall now consider some points in more detail.

7.2. The coupling constants for muon capture: an effective Hamiltonian

When a universal V-A interaction (with the two-component neutrino theory) is assumed, the Hamiltonian for the muon capture interaction should read (cf. (3.35))

$$\mathcal{H}_{\mu c} = (g^{\mu oc}\, 2^{-\frac{1}{2}})(\overline{\Psi}_f\, \gamma_\mu (1 + \gamma_5)(\tau^{(+)}\, \Psi_i)(\overline{\psi}_\nu \gamma_\mu (1 + \gamma_5)\psi_\mu) + \text{h.c.} \tag{7.10}$$

(Ψ_i and Ψ_f initial and final nucleon wave functions; ψ_μ muon wave function, ψ_ν neutrino wave function).

When calculating the matrix element ME of (7.10) for a process (7.4) where the nucleons are in plane wave states, we obtain from (7.10)

$$\sqrt{2}\cdot ME = g^{0\mu c}[(\bar{u}_\nu(1-\gamma_5)\,i\gamma_\lambda\gamma_5 u_\mu)(\bar{u}_n\, i\,\gamma_\lambda\gamma_5 u_p)$$

$$+ (\bar{u}_\nu(1-\gamma_5)\,\gamma_\lambda u_\mu)(\bar{u}_n\gamma_\lambda u_p)]\,; \qquad (7.11)$$

u_ν, u_μ, u_n and u_p are the 4-component Dirac spinors for neutrino, muon, neutron and proton respectively. However, as in the case of β-radioactivity, the nucleons involved also have strong interactions. It follows that the expression for the "currents" related to nucleons may deviate from the expressions given in (7.11). The more general expression for the matrix element, which is in accordance with general invariance requirements is (cf. [55], [38])

$$\sqrt{2}\,ME = (\bar{u}_\nu(1-\gamma_5)\,i\gamma_\lambda\gamma_5 u_\mu)[A(\bar{u}_n\, i\gamma_\lambda\gamma_5 u_p) - B(\bar{u}_n q_\lambda\gamma_5 u_p) + E(\bar{u}_n\sigma_{\lambda\rho}q_\rho\gamma_5 u_p)]$$

$$+ (\bar{u}_\nu(1-\gamma_5)\gamma_\lambda u_\mu)[C(\bar{u}_n\gamma_\lambda u_p) - i\,D(\bar{u}_n\sigma_{\lambda\ell}\,q_\ell u_p) + i\,F(\bar{u}_n q_\lambda u_p)]. \qquad (7.12)$$

We have put here

$q_\lambda = p_\lambda - n_\lambda$ 4-momentum transfer in the process

p_λ 4-momentum of the proton

n_λ 4-momentum of the neutron

$$\sigma_{\lambda\mu} = \tfrac{1}{2}(\gamma_\lambda\gamma_\mu - \gamma_\mu\gamma_\lambda).$$

For B=E=D=F=0 and C= -A =g^0, (7.12) would reduce to (7.11), i.e. simple V-A interaction. However, we assume that only the "bare" interaction is V-A, but that the "effective" interaction which is observed and in which the strong interactions also have an influence may deviate from this.

The coefficients A, B, C, D, E, F need not be constants but are "form factors", which may depend on the magnitude q^2 of the 4-momentum transfer q_λ.

They are real if T-invariance holds. When q^2 does not vary too much they may be "constants" for practical purposes and we can consider them as a kind of "apparent coupling constants". We introduce the following notation so that all "coupling constants" have the usual dimension of a 4-fermion coupling constant:

$C = g_V$	vector apparent	coupling constant	
$A = g_A$	axial vector apparent	" "	
$m_\mu B = g_P$	(induced) pseudoscalar	" "	
$2MD = g_M$	weak magnetism	" "	(7.12a)
$m_\mu F = g_S$	(induced) scalar	" "	
$2ME = g_T$	(induced) tensor	" "	

Hence we assume that the strong interactions may cause g_V and g_A to deviate from g_V^0 and g_A^0 (while $g_A^0 = -g_V^0$) and may "induce" effective couplings characterized by "coupling constants" g_P, g_M, g_S and g_T. Some further hypotheses may restrict the values of these apparent coupling constants (see below).

We shall give here a treatment of muon capture in which we introduce an effective Hamiltonian for two-component spinors (cf. [65] for such a treatment). Hence we proceed here somewhat differently from our calculations for β-radioactivity; this is not a matter of principle but is merely to demon-

strate somewhat different procedures. In order to derive the effective Hamiltonian we make use of the Dirac equation for free particles

$$(\gamma_\lambda p_\lambda - im)\, u = 0. \tag{7.13}$$

We write the 4-component Dirac fields as

$$\psi = ((E+m)/2E)^{\frac{1}{2}} \begin{matrix} \chi \\ \varphi \end{matrix}, \tag{7.14}$$

where χ and Φ are 2-component spinors related by

$$\chi = -((\vec{\sigma}\cdot\vec{p})/E+m)\varphi \quad (\sigma_k = i\gamma_k\gamma_4\gamma_5) \tag{7.15}$$

as follows from (7.13) where we are using a representation such that

$$\gamma_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.16}$$

Of course the use of (7.13) and (7.15) involves a certain approximation, which seems, however, rather reasonable. Substitution of (7.15) into (7.12) provides the following "effective Hamiltonian" for a system of A nucleons (to be applied to the 2-component spinors φ, which give the "large" components of ψ):

$$H^{\mu}_{\text{eff}} = \tfrac{1}{2}\tau^{(+)}(1 - \vec{\sigma}\cdot\hat{\vec{\nu}})\Sigma_{i=1}^{A}\tau_i^{(-)}[G_V 1\cdot 1_i + G_A\vec{\sigma}\cdot\vec{\sigma}_i - G_p(\nu/2M)(\vec{\sigma}\cdot\hat{\vec{\nu}})(\vec{\sigma}_i\cdot\hat{\vec{\nu}})$$
$$- g_V(\vec{\sigma}\cdot\hat{\vec{\nu}})(\vec{\sigma}\cdot\vec{p}_i/M) - g_V(\vec{\sigma}\cdot\hat{\vec{\nu}})(\vec{\sigma}_i\cdot\vec{p}_i/M)]\,\delta(\vec{r} - \vec{r}_i); \tag{7.17}$$

$\tau_i^{(-)}$, 1_i, $\vec{\sigma}_i$, $\vec{p}_i$	are operators for the i^{th} nucleon,
$\tau^{(+)}$, $\vec{\sigma}$, 1	are lepton operators,
$\vec{\nu}$	is the neutrino momentum,
$\hat{\vec{\nu}}$	is the unit vector in the direction of $\vec{\nu}$,
$\tau^{(-)}$ and $\tau^{(+)}$	are the operators decreasing and increasing, the charge of the nucleons (or leptons) by one unit.

G_V, G_A and G_P are combinations of the quantities (7.12a) and might be called "effective coupling constants"; they are given by

$$\begin{aligned} G_V &= C(1 + (\nu/2M)) + m_\mu F = g_V(1 + (\nu/2M)) + g_s \\ G_A &= A - ((C/2M) + D)\nu = g_A - (g_V + g_M)(\nu/2M) \\ G_P &= (m_\mu B - A)(\nu/2M) - ((C/2M) + D - E)\nu \\ &= [(g_P - g_A) - (g_V + g_M - g_T)](\nu/2M)\,. \end{aligned} \tag{7.18}$$

In (7.17) the first order terms in $\nu/2M$ and p_i/M are retained, but second and higher order terms in these quantities are neglected. We do not give the calculation for the transition from (7.12) to (7.17) in full, but indicate the reduction for a typical term. For the reduction from (7.12) to (7.17) it is convenient to note an alternative form in which (7.12) can be written as

$$\sqrt{2}\,ME = A[\bar{u}_\nu(1-\gamma_5)i\gamma_\lambda\gamma_5 u_\mu](\bar{u}_n i\gamma_\lambda\gamma_5 u_p)$$
$$+ m_\mu B[\bar{u}_\nu(1-\gamma_5)\gamma_5 u_\mu](\bar{u}_n\gamma_5 u_p)$$
$$- (iC/M)[\bar{u}_\nu(1-\gamma_5)\gamma_\lambda u_\mu](\bar{u}_n p_\lambda u_p)$$
$$-(iC/M)[\bar{u}_\nu(1-\gamma_5)\gamma_\lambda(\mu_\lambda-\nu_\lambda)u_\mu](\bar{u}_n u_p)$$
$$+ i((C/2M)+D)[\bar{u}_\nu(1-\gamma_5)\gamma_\lambda(\mu_\rho-\nu_\rho)u_\mu](\bar{u}_n\sigma_{\lambda\rho}u_p)$$
$$+ E[\bar{u}_\nu(1-\gamma_5)i\gamma_\lambda\gamma_5 u_\mu](\bar{u}_n\sigma_{\lambda\rho}(p_\rho-n_\rho)\gamma_5 u_p)$$
$$+ m_\mu F[\bar{u}_\nu(1-\gamma_5)u_\mu](\bar{u}_n u_p). \qquad (7.19)$$

As an example of a reduction from (7.12) to (7.19) we consider the term with B as coefficient; in the reduction we make use of (7.13) and of the equation of the conservation of 4-momentum in the process

$$p_\lambda - n_\lambda = \nu_\lambda - \mu_\lambda \qquad (7.20)$$

(μ_λ and ν_λ give the 4-momenta of muon and neutrino respectively). We can thus reduce

$$[\bar{u}_\nu(1-\gamma_5)i\gamma_\lambda\gamma_5 u_\mu](\bar{u}_n(p_\lambda-n_\lambda)\gamma_5 u_p)$$
$$= -[\bar{u}_\nu(1-\gamma_5)i\gamma_\lambda\gamma_5(\mu_\lambda-\nu_\lambda)u_\mu](\bar{u}_n\gamma_5 u_p)$$
$$= [\bar{u}_\nu(1-\gamma_5)\gamma_5(i\gamma_\lambda\mu_\lambda u_\mu)+(\bar{u}_\nu\gamma_\lambda\nu_\lambda)(1+\gamma_5)i\gamma_5 u_\mu](\bar{u}_n\gamma_5 u_p)$$
$$= -m_\mu[\bar{u}_\nu(1-\gamma_5)\gamma_5 u_\mu](\bar{u}_n\gamma_5 u_p) \qquad (7.21)$$

(as $\gamma_\lambda\mu_\lambda u_\mu = im_\mu u_\mu$ and $\bar{u}_\nu\gamma_\lambda\nu_\lambda = 0$).

This shows the reduction of the term with B from (7.12) to (7.19); from this term one then obtains a term in the interaction density given by (apart from the factor $m_\mu B$)

$$[T_B] = [\psi_\nu^*(\vec{x})\gamma_4(1-\gamma_5)\gamma_5\psi_\mu(\vec{x})](\psi_n^*\gamma_4\gamma_5\psi_p). \qquad (7.22)$$

Introducing 2-component spinors according to (7.14), we obtain, using (7.16) and taking a non-relativistic approximation for the muon,

$$\psi_\nu^*\gamma_4(\gamma_5-1)\psi_\mu = (1/\sqrt{2})\begin{pmatrix}-\vec{\sigma}\cdot\hat{\nu}\,\phi_\nu\\ \phi_\nu\end{pmatrix}^*\gamma_4(\gamma_5-1)\begin{pmatrix}0\\ \phi_\mu\end{pmatrix}$$
$$= (1/\sqrt{2})\begin{pmatrix}(\vec{\sigma}\cdot\hat{\nu})\,\phi_\nu\\ \phi_\nu\end{pmatrix}^*\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]\begin{pmatrix}0\\ \phi_\mu\end{pmatrix} = (1/\sqrt{2})\begin{pmatrix}(\vec{\sigma}\cdot\hat{\nu})\,\phi_\nu\\ \phi_\nu\end{pmatrix}^*\left[\begin{pmatrix}\phi_\mu\\0\end{pmatrix}-\begin{pmatrix}0\\ \phi_\mu\end{pmatrix}\right]$$
$$= -(1/\sqrt{2})\,\phi_\nu^*(1-\vec{\sigma}\cdot\hat{\nu})\,\phi_\mu. \qquad (7.23)$$

We have further (the nucleon normalization factors are equal to unity in the approximation considered).

$$\psi_n^* \gamma_4 \gamma_5 \psi_p = \begin{pmatrix} -(\vec{\sigma}\cdot\vec{n}/2M)\phi_n \\ \phi_n \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -(\vec{\sigma}\cdot\vec{p}/2M)\phi_p \\ \phi_p \end{pmatrix}$$

$$= \phi_n^* (\vec{\sigma}\cdot\vec{n}/2M)\phi_p - \phi_n^* (\vec{\sigma}\cdot\vec{p}/2M)\phi_p = -\phi_n^* (\vec{\sigma}\cdot\vec{\nu}/2M)\phi_p. \quad (7.24)$$

We have used here $\vec{p} = \vec{n} + \vec{\nu}$, which is a consequence of (7.20) when taking $\vec{\mu} = 0$. From (7.23) and (7.24) we obtain

$$[T_B] = (1/\sqrt{2})[\phi_\nu^*(1 - \vec{\sigma}\cdot\hat{\vec{\nu}})\phi_\mu][\phi_n^*(\vec{\sigma}\cdot\vec{\nu}/2M)\phi_p]. \quad (7.25)$$

This term then provides the contribution to H_{eff} arising from the term with B as coefficient

$$H^{\mu}_{eff,B} = -\tfrac{1}{2} m_\mu B \tau^{(+)}(1 - \vec{\sigma}\cdot\hat{\vec{\nu}}) \sum_{i=1}^{A} \tau_i^{(-)} (\nu/2M)\ (\vec{\sigma}\cdot\hat{\vec{\nu}})\ (\vec{\sigma}_i \cdot \hat{\vec{\nu}}) \quad (7.26)$$

we have used here $(1 - \vec{\sigma}\cdot\hat{\vec{\nu}})(\vec{\sigma}\cdot\hat{\vec{\nu}}) = -(1 - \vec{\sigma}\cdot\hat{\vec{\nu}})$.

By means of analogous reductions the other terms of (7.17) can be found.

As regards the magnitude of G_V, G_A and G_P, it seems probable that G_V and g_V as well as G_A and g_A do not differ very much; at least it seems probable that g_S is substantially smaller than g_V and that g_M is not much larger than g_A. With $\nu/2M \approx 0.04 \ll 1$ it then follows that the values of G_V and g_V will be quite close as well as those of G_A and g_A.

The last two terms in (7.17) have the character of correction terms; hence we see that no appreciable difference should be expected if we substitute G_V for g_V and G_A for g_A. We thus see from (7.17) that one can scarcely hope to obtain more than the three combinations G_V, G_A and G_P of the six quantities g_V, g_A, g_P, g_M, g_S and g_T from experiments on effects, for which (7.17) applies. Until now we have considered the "coupling constants" in this section as purely phenomenological coefficients. However, a number of theoretical proposals have been formulated which restrict the values of these parameters.

(I) Only interactions of the first class exist (in Weinberg's terminology, cf. [38])

$$g_S = 0,\ g_T = 0. \quad (7.27)$$

One may say that interactions are restricted to those with the property of "G-invariance" or "normal G-symmetry". G is the transformation given by

$$G = C e^{i\pi T^{(2)}} \quad (7.28)$$

where C is the charge conjugation operator and $e^{i\pi T^{(2)}}$ is the operator for a rotation of π about the 2-axis in isospace.

Currents defined according to

$$J_i = \bar{\psi}_p \Gamma_i \psi_n$$

have as a transformation property for the G-transformation

$$G\, J_i G^{-1} = -\xi_i J_i, \quad (7.29)$$

with $\xi_i = +1$ for S, A, P, and $\xi_i = -1$ for V, T.

We refer to the property that possible additional terms to J_i should have the transformation property for G expressed by (7.29) as "G-invariance". This has (7.27) as a result.

(II) The validity of the conserved vector current hypothesis means that the weak magnetism coupling constant has the special value

$$g_M = (\mu_p^{(a)} - \mu_n^{(a)})\, g_V \approx 3.7\, g_V \tag{7.30}$$

($\mu_p^{(a)}$ and $\mu_n^{(a)}$ are the proton and neutron anomalous magnetic moments in nuclear magnetons).

(III) Several considerations (cf. [55, 56]) provide a plausible value for the induced pseudoscalar coupling constant

$$g_P \approx 8\, g_A . \tag{7.31}$$

(IV) The hypothesis of a Universal Fermi Interaction, supplemented by considerations of the conserved vector current hypothesis and dispersion relations, suggests that the coupling constants for the V and A parts of the interaction should be equal for muon capture and beta radioactivity

$$g_V^{\mu c} = g_V^{\beta}, \quad g_A^{\mu c} = g_A^{\beta}, \tag{7.31a}$$

at least within 3%.

Proposals II and III have already been discussed in 6. We emphasize that we do not yet consider (7.27) (7.30) and (7.31) as firmly established theoretical values but more as hypotheses to be tested experimentally.

Using H_{eff}^{μ} according to (7.17) we obtain for the muon capture rate for a transition from an initial state $|a\rangle$ to a final nuclear state $|b\rangle$

$$\Lambda_{\mu c}(a \to b) = \frac{\nu_{ab}^2}{2\pi} \int \frac{d\hat{\nu}_{ab}}{4\pi} \left\{ G_V^2 \left| \int 1 \right|^2 + G_A^2 \left| \int \vec{\sigma} \right|^2 + (G_p^{\circ} - 2G_p G_A) \left| \hat{\nu}_{ab} \cdot \int \vec{\sigma} \right|^2 \right.$$

$$\left. - \frac{G_V g_V}{M} \left[\left(\int 1 \right)^* \left(\int \vec{p} \right) \cdot \hat{\nu}_{ab} + \text{c.c.} \right] - \frac{G_A g_A - G_p g_A}{M} \left[(\vec{\nu}_{ab} \cdot \int \vec{\sigma})^* \int \vec{p} \cdot \vec{\sigma} + \text{c.c.} \right] \right\} . \tag{7.32}$$

We have used here the abbreviations

$$\begin{aligned}
\int 1 &= \langle b \,|\, \sum_i \tau_i^{(-)} \exp(-i\vec{\nu} \cdot \vec{r}_i)\, \phi_\mu(\vec{r}_i) \,|\, a \rangle \\
\int \vec{\sigma} &= \langle b \,|\, \sum_i \tau_i^{(-)} \exp(-i\vec{\nu} \cdot \vec{r}_i)\, \phi_\mu(\vec{r}_i)\, \vec{\sigma}_i \,|\, a \rangle \\
\int \vec{p} &= \langle b \,|\, \sum_i \tau_i^{(-)} \exp(-i\vec{\nu} \cdot \vec{r}_i)\, \phi_\mu(\vec{r}_i)\, \vec{p}_i \,|\, a \rangle \\
\int \vec{p} \cdot \vec{\sigma} &= \langle b \,|\, \sum_i \tau_i^{(-)} \exp(-i\vec{\nu} \cdot \vec{r}_i)\, \phi_\mu(\vec{r}_i)\, \vec{p}_i \cdot \vec{\sigma}_i \,|\, a \rangle .
\end{aligned} \tag{7.33}$$

$\phi_\mu(\vec{r})$ is here the radial muon wave function. In (7.32) averages are carried out for the initial muon polarization and initial nuclear orientation and summations for final neutrino polarization and final nuclear orientation. It should be noted that we have used the notation (7.33) for the sake of conciseness,

although it differs from that usual for beta matrix elements as the neutrino and muon wave functions still occur under the integral sign. Therefore the matrix elements (7.33) still contain "forbidden" contributions.

The expression (7.32) can be easily derived from (7.17) in a manner which is quite analogous to that used in section 3 for β-decay; however, it is somewhat simpler as we already made a reduction to 2-component spinors; we can give the following reduction of the square of a lepton factor with 2-component spinors (Ω is an operator acting on the spinors, $v^{(\nu)}$ and $v^{(\mu)}$ are the 2-component spinors for neutrino and muon respectively)

$$\sum_{\mu\nu} \left| v_\alpha^{(\nu)*} \Omega_{\alpha\beta} v_\beta^{(\mu)} \right|^2 = \sum_{\mu\nu} (v_\alpha^{(\nu)*} \Omega_{\alpha\beta} v_\beta^{(\mu)}) (v_\gamma^{(\nu)*} \Omega_{\gamma\delta} v_\delta^{(\mu)})^*$$

$$= \sum_{\mu\nu} v_\alpha^{(\nu)*} \Omega_{\alpha\beta} v_\beta^{(\mu)} v_\delta^{(\mu)*} (\Omega^+)_{\delta\gamma} v_\gamma^{(\nu)} = \mathrm{Tr}_\sigma (D^{(\nu)} \Omega D^{(\mu)} \Omega^+) = \mathrm{Tr}_\sigma (\Omega \Omega^+) \qquad (7.34)$$

(summation over repeated spinor indices α, β, γ and δ is not written explicitly; Tr_σ indicates that we have only 2-component spinors).

We have introduced here 2-dimensional "projection operators" analogous to (3.14) in section 3

$$D_{\alpha\beta}^{(\nu)} = \sum_\nu v_\alpha^{(\nu)} v_\beta^{(\nu)*} = 1$$

$$D_{\alpha\beta}^{(\mu)} = \sum_\mu v_\alpha^{(\mu)} v_\beta^{(\mu)*} = 1. \qquad (7.35)$$

However in the 2-component case they are simply equal to unity when performing the sums $\sum_\nu$, $\sum_\mu$ over the polarizations of neutrino and muon.

As an example of the calculation of (7.32) from (7.17) we deduce the term with G_A^2 (we leave the derivation of the factors in front to the reader). We put $\vec{\ell} = \int \vec{\sigma}$ and have to evaluate

$$\sum_\nu \sum_\mu \left| [v^{(\nu)*} (1 - \vec{\sigma}.\hat{\vec{\nu}})\, \vec{\sigma}\, v^{(\mu)}] \cdot \vec{\ell} \right|^2$$

$$= \mathrm{Tr}_\sigma [(1 - \vec{\sigma}.\hat{\vec{\nu}})(\vec{\sigma}.\vec{\ell})(\vec{\sigma}.\vec{\ell}^*)(1 - \vec{\sigma}.\hat{\vec{\nu}})]$$

$$= \mathrm{Tr}_\sigma [(1 - \vec{\sigma}.\hat{\vec{\nu}})(1 - \vec{\sigma}.\hat{\vec{\nu}})(\vec{\sigma}.\vec{\ell})(\vec{\sigma}.\vec{\ell}^*)]$$

$$= 2\mathrm{Tr}_\sigma [(1 - \vec{\sigma}.\hat{\vec{\nu}})\{(\vec{\ell}.\vec{\ell}^*) + i\vec{\sigma}.(\vec{\ell} \times \vec{\ell}^*)\}]$$

$$= 4(\vec{\ell}.\vec{\ell}^*) - 4i\hat{\vec{\nu}}.(\vec{\ell} \times \vec{\ell}^*). \qquad (7.36)$$

When averaged over neutrino direction and nuclear orientation the second term gives zero and we are left with the first term, which gives the term with $G_A^2 |\vec{\sigma}|^2$ in (7.32). Similar reductions provide the other terms in (7.32).

The total capture rate to all final states is obtained by an additional summation

$$\Lambda_{\mu c} = \sum_b \Lambda_{\mu c}(a \to b). \qquad (7.37)$$

In a first approximation (7.32) can be simplified in the following ways:

(1) The terms in the second line of (7.32) have the character of correction terms and can be neglected in a first approximation.

(2) The muon wave function in (7.35) can be considered with good approximation as constant within the nucleus. A value $|\phi_\mu|^2_{av}$ will be assumed

which presents an average of $|\phi_\mu|^2$ over the nuclear volume. Under certain conditions one has for partial transitions

$$(4\pi)^{-1}\int d\hat{\vec{\nu}}_{ab}\ |\hat{\vec{\nu}}_{ab}\cdot\int\vec{\sigma}\,|^2 = \frac{1}{3}\cdot\left|\int\vec{\sigma}\right|^2. \tag{7.38}$$

Hence, for such cases

$$\Lambda_{\mu c}(a\to b) = (\nu_{ab}^2/2\pi)\int(d\hat{\vec{\nu}}_{ab}/4\pi)\,[G_F^2\left|\int 1\right|^2 + G_{GT}^2\left|\int\vec{\sigma}\right|^2] \tag{7.39}$$

with

$$\begin{aligned} G_F &= G_V \\ G_{GT}^2 &= G_A^2 + \frac{1}{3}(G_P^2 - 2G_PG_A). \end{aligned} \tag{7.40}$$

For the total capture rates for certain nuclei (e.g. O^{16} and Ca^{40}) one can further put

$$\sum_b\int(d\hat{\vec{\nu}}_{ab}/4\pi)\,|\,\hat{\vec{\nu}}_{ab}\cdot\int\vec{\sigma}\,|^2 = \frac{1}{3}\sum_b\int(d\hat{\vec{\nu}}_{ab}/4\pi)\left|\int\vec{\sigma}\right|^2 \tag{7.41}$$

and

$$\sum_b\int(d\hat{\vec{\nu}}_{ab}/4\pi)\left|\int\vec{\sigma}\right|^2 = 3\sum_b\int(d\hat{\vec{\nu}}_{ab}/4\pi)\,\left|\int 1\right|^2. \tag{7.42}$$

Hence for such nuclei one can finally write in first approximation

$$\Lambda_{\mu c} = (\nu_\mu^2/2\pi)\,|\phi_\mu|_{av}^2\,[G_F^2 + 3G_{GT}^2]M^2, \tag{7.43}$$

with

$$M^2 = \sum_b(\nu_{ab}/\nu_\mu)^2\int(d\hat{\vec{\nu}}_{ab}/4\pi)\left|\langle b\,|\sum_i\tau_i^{(-)}\exp(-i\vec{\nu}_{ab}\cdot\vec{r}_i)\,|\,a\rangle\right|^2 \tag{7.44}$$

where ν_μ is the maximum possible neutrino momentum given in our units by

$$\nu_\mu = m_\mu\,(105.6\,\mathrm{MeV}).$$

From (7.43) it is seen that the combination of coupling constants, which is determined in the first instance from total capture rates, is

$$G_F^2 + 3G_{GT}^2.$$

7.3. The application of the closure method to a statistical model

Total rates for muon capture are generally composed of many partial transition rates. The separate and precise calculation of all partial transition rates involves much work and generally more knowledge of the nuclear wave functions than is available at present. However, various approximations may be used for calculating total capture rates. The most drastic approximation is the closure approximation. This consists in using the completeness relation for the final states

$$\sum_b|b\rangle\langle b| = 1. \tag{7.45}$$

We can apply (7.45) to (7.44) only when substituting one average value ν for all neutrino energies ν_{ab}. This average value is related to ν_μ according to

$$\nu = \nu_\mu - \langle E_b - E_a \rangle_{av}, \tag{7.46}$$

where $\langle E_b - E_a \rangle_{av}$ is the average energy difference between the final state $|b\rangle$ and the initial state $|a\rangle$. One often takes $\nu \approx 85$ MeV, $\langle E_b - E_a \rangle_{av} \sim 20$ MeV ($\nu_\mu = 105.6$ MeV). It is seen that this approximation is reasonable in the first approximation, but cannot be expected to be very precise.

When using this closure approximation (7.43) and (7.44) can be rewritten as

$$\Lambda_{\mu c} = (\nu^2/2\pi)\,|\phi_\mu|^2_{av}[G_F^2 + 3G_{GT}^2]\mathcal{M}^2 \tag{7.47}$$

with

$$\mathcal{M}^2 = \sum_b \int (d\hat{\vec{\nu}}/4\pi)\,|\langle b|\sum_i \tau_i^{(-)} \exp(-i\vec{\nu}\cdot\vec{r}_i)|a\rangle|^2$$

$$= \sum_b \int (d\hat{\vec{\nu}}/4\pi)\langle a|\sum_i \tau_i^{(+)} \exp(+i\vec{\nu}\cdot\vec{r}_i)|b\rangle\langle b|\sum_j \tau_j^{(-)} \exp(-i\vec{\nu}\cdot\vec{r}_j)|a\rangle$$

$$= \int (d\hat{\vec{\nu}}/4\pi)\langle a|\sum_{ij} \tau_i^{(+)}\tau_j^{(-)} \exp(i\vec{\nu}\cdot\vec{r}_{ij})|a\rangle. \tag{7.48}$$

Use was made of (7.45); $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$.

The expression (7.48) is remarkable as it refers only to the wave function of the initial state; it can still be applied to nuclear wave functions according to any model.

In this subsection we shall use a statistical model for evaluating (7.48) (cf. [60], [61] and [63]). For this purpose we consider the evaluation of the average of an operator

$$\langle \Omega \rangle_{av} = \langle \Psi | \Omega | \Psi \rangle = \mathrm{Tr}\,[\Omega\rho] \tag{7.49}$$

where ρ is a density matrix given by

$$\rho = |\Psi\rangle\langle\Psi|, \tag{7.50}$$

using Dirac notation. We assume that Ω is a sum of one and two-particle operators:

$$\Omega = \sum_i \Omega_i + \sum'_{ij} \Omega_{ij}, \tag{7.51}$$

where Σ'_{ij} means Σ_{ij} with $i \neq j$. If $x_1, \ldots, x_A$ are the coordinates occurring in the wave function Ψ, it is easily seen that (7.49) can be reduced to

$$\langle \Omega \rangle_{av} = \int dx_1 \int d\bar{x}_1 \langle x_1|\Omega_1|\bar{x}_1\rangle\langle \bar{x}_1|\rho^{(1)}|x_1\rangle$$

$$+ \int dx_1 \int dx_2 \int d\bar{x}_1 \int d\bar{x}_2 \langle x_1 x_2|\Omega_{12}|\bar{x}_1\bar{x}_2\rangle\langle \bar{x}_1\bar{x}_2|\rho^{(2)}|x_1 x_2\rangle. \tag{7.52}$$

The k-particle density $\rho^{(k)}$ is introduced here (it is a kind of "partial density matrix") as follows

$$\langle x_1 \ldots x_k \,|\, \rho^{(k)} \,|\, x'_1 \ldots x'_k \rangle$$

$$= \frac{A!}{(A-k)!} \int \Psi(x_1 \ldots x_A) \Psi^*(x'_1 \ldots x'_k, x_{k+1} \ldots x_A) dx'_{1 \ldots k} \tag{7.53}$$

$\int dx'_1 \ldots_k$ indicates integration over all coordinates $x_{k+1}, \ldots x_A$.

Applying (7.52) to (7.48) we can write

$$\mathcal{M}^2 = \sum_{\zeta_1 \zeta'_1} \int d\vec{r}_1 \langle \vec{r}_1 \zeta_1 | D | \vec{r}_1 \zeta'_1 \rangle \langle \vec{r}_1 \zeta'_1 | \Omega_1 | \vec{r}_1 \zeta_1 \rangle$$

$$+ \sum_{\zeta_1 \zeta'_1 \zeta_2 \zeta'_2} \int d\vec{r}_1 \int d\vec{r}_2 \langle \vec{r}_1 \vec{r}_2 \zeta_1 \zeta_2 | g | \vec{r}_1 \vec{r}_2 \zeta'_1 \zeta'_2 \rangle \langle \vec{r}_1 \vec{r}_2 \zeta'_1 \zeta'_2 | \Omega_{12} | \vec{r}_1 \vec{r}_2 \zeta_1 \zeta_2 \rangle . \tag{7.54}$$

We have designated the one- and two-particle densities $\rho^{(1)}$ and $\rho^{(2)}$ as D (density distribution) and g (pair correlation); the coordinates x_k are written more explicitly as $\vec{r}_k$, ζ_k, where $\vec{r}_k$ are the space coordinates and ζ_k give spin- and isospin-coordinates. The expressions for Ω_i and Ω_{ij} are diagonal with respect to space coordinates; they are given by

$$\Omega_i = \tfrac{1}{2} (1 + \tau_i^{(3)}) \tag{7.55}$$

$$\Omega_{ij} = \tau_i^{(+)} \tau_j^{(-)} \cdot \exp [i \vec{\nu} \cdot (\vec{r}_i - \vec{r}_j)] . \tag{7.56}$$

When averaging over the neutrino directions ($\int (d\hat{\vec{\nu}}/4\pi)$) this becomes

$$\Omega_{ij} = \tau_i^{(+)} \tau_j^{(-)} (\sin \nu r / \nu r) (r = |\vec{r}_i - \vec{r}_j|) \tag{7.57}$$

It follows that we need only the matrix elements of D and g diagonal with respect to the space coordinates; however, the non-diagonal elements with respect to spin and isospin are needed. In the case of a system which can be specified by a Slater determinant (κ indicating the individual nucleon states ψ_κ)

$$\Psi(x_1 \ldots x_A) = (A!)^{-\frac{1}{2}} \sum_P \delta_P \prod_{\kappa=1}^{A} \psi_\kappa(x_{P\kappa}) \tag{7.58}$$

($\sum_P$ denotes the sum over all permutations) D and g are given by

$$\langle \vec{r}_1 \zeta_1 | D | \vec{r}_1 \zeta'_1 \rangle = \sum_\kappa \psi_\kappa(\vec{r}_1 \zeta_1) \psi_\kappa^*(\vec{r}_1 \zeta'_1) \tag{7.59}$$

$$\langle \vec{r}_1 \vec{r}_2 \zeta_1 \zeta_2 | g | \vec{r}_1 \vec{r}_2 \zeta'_1 \zeta'_2 \rangle = \sum_{\kappa,\lambda} [\psi_\kappa(\vec{r}_1 \zeta_1) \psi_\lambda(\vec{r}_2 \zeta_2) \psi_\kappa^*(\vec{r}_1 \zeta_1) \psi_\lambda^*(\vec{r}_2 \zeta'_2)$$

$$- \psi_\kappa(\vec{r}_1 \zeta_1) \psi_\lambda(\vec{r}_2 \zeta_2) \psi_\lambda^*(\vec{r}_1 \zeta'_1) \psi_\kappa^*(\vec{r}_2 \zeta'_2)] . \tag{7.60}$$

Up to here the treatment has been fairly general. If we now want to specify for a statistical model for the nucleus we must have definite expressions for D and g; we may make the following assumptions in this respect:

(a) The pair correlation function is calculated as if the wave functions were given by a Slater determinant for plane waves contained in a large cubic box (with periodic boundary conditions).

(b) For the radial density distribution one may assume, e. g.

$$D(r) = D_0 [1 + \exp (r - R_D / a_D)]^{-1} \tag{7.61}$$

with parameters a_D and R_D determined from the Stanford experiments on the scattering of electrons by nuclei.

(c) Every space state occupied by a nucleon with a certain spin also contains a nucleon of the same type with opposite spin.

We shall confine ourselves here to nuclei with $N = Z = \frac{1}{2}A$. From the assumptions (a), (b), (c) we obtain for such nuclei

$$\langle \vec{r}_1 \zeta_1 | D | \vec{r}_1' \zeta_1' \rangle = \frac{1}{4} A \, \delta_{\zeta_1 \zeta_1'} \, D(r_1) \tag{7.62}$$

$$\langle \vec{r}_1 \vec{r}_2 \zeta_1 \zeta_2 | g | \vec{r}_1' \vec{r}_2' \zeta_1' \zeta_2' \rangle = (1/16 A^2) D(r_1) \cdot D(r_2) \cdot [\delta_{\zeta_1 \zeta_1'} \delta_{\zeta_2 \zeta_2'} - f(r) \delta_{\zeta_1 \zeta_2'} \delta_{\zeta_2 \zeta_1'}] \tag{7.63}$$

with

$$(r = |\vec{r}_1 - \vec{r}_2|)$$

$$f(r) = [S(k_F r)]^2 \tag{7.64}$$

$$k_F = (3\pi^2)^{1/3} (A/2V)^{1/3}. \tag{7.65}$$

where

$$S(x) = 3x^{-3} (\sin x - x \cos x)$$
$$= 3(\tfrac{\pi}{2})^{\frac{1}{2}} x^{-\frac{3}{2}} J_{\frac{3}{2}}(x).$$

k_F is the Fermi-momentum; V is the nuclear volume. It should be emphasized that (7.62) and (7.63) are not rigorous deductions from (a), (b) and (c); they are made plausible on this basis. However, the assumption is introduced that the dependence of g on $r = |\vec{r}_1 - \vec{r}_2|$ is the same as for an infinite Fermi-gas, while the dependence on $|\vec{r}_1|$ and $|\vec{r}_2|$ is given by $D(r_1)$ and $D(r_2)$

The difficulty is that a simple dependence of g on r can only be given for infinite matter while the problem we have here is one where the finite extent of the nucleus is essential. Hence it is clear that (7.63) can only be an approximate expression. It is also of importance to check whether the expressions (7.62) and (7.63) are at least consistent with their definition (7.53). From this definition it follows simply that D and g have to satisfy the following conditions

$$\int d\vec{r}_1 \sum_{\zeta_1} \langle \vec{r}_1 \zeta_1 | D | \vec{r}_1 \zeta_1 \rangle = A \tag{7.66}$$

$$\int d\vec{r}_2 \sum_{\zeta_2} \langle \vec{r}_1 \vec{r}_2 \zeta_1 \zeta_2 | g | \vec{r}_1 \vec{r}_2 \zeta_1' \zeta_2 \rangle = (A - 1) \langle \vec{r}_1 \zeta_1 | D | \vec{r}_1 \zeta_1' \rangle \tag{6.67}$$

Hence g has also to satisfy the normalization condition

$$\int d\vec{r}_1 \int d\vec{r}_2 \sum_{\zeta_1} \sum_{\zeta_2} \langle \vec{r}_1 \vec{r}_2 \zeta_1 \zeta_2 | g | \vec{r}_1 \vec{r}_2 \zeta_1 \zeta_2 \rangle = A (A - 1). \tag{7.68}$$

However, a number of additional "detailed normalization conditions" may be obtained in the following way: consider an operator acting on spin and isospin variables only

$$U = \sum_{ij}{}' U_{ij}. \tag{7.69}$$

It is then easily shown that the expectation value of U for the state of the nucleus $|a\rangle$ can be expressed with the aid of g

$$\langle a \,|\, U \,|\, a \rangle = \sum_{\zeta_1 \zeta_2 \zeta_1' \zeta_2'} \int d\vec{r}_1 \int d\vec{r}_2 \, \langle \vec{r}_1 \vec{r}_2 \zeta_1 \zeta_2 \,|\, g \,|\, \vec{r}_1 \vec{r}_2 \zeta_1' \zeta_2' \rangle \times \langle \zeta_1' \zeta_2' \,|\, U_{12} \,|\, \zeta_1 \zeta_2 \rangle . \tag{7.70}$$

When $\langle a \,|\, U \,|\, a \rangle$ can be easily evaluated without detailed assumptions, we obtain what we call a "detailed normalization condition". For the simplest choice $U_{ij} = 1$ we come back to the normalization condition (7.68). Other choices which may be made for U_{ij} are

$$\Pi_{ij}^{(+)} \; ; \; \Pi_{ij}^{(-)} \; ; \quad \tau_i^{(+)} \tau_j^{(-)} \Pi_{ij}^{(+)} \; ; \quad \tau_i^{(+)} \tau_j^{(-)} \Pi_{ij}^{(-)}$$

where $\Pi_{ij}^{(+)}$ and $\Pi_{ij}^{(-)}$ are the projection operators for symmetric and antisymmetric pairs. It is found that (7.63) does not rigorously satisfy the detailed normalization conditions for these operators. However, the following slightly generalized expression for (7.63)

$$\Omega_g = (A^2/16)\, D(r_1) \cdot D(r_2) \left\{ [1 + f(r)]\, N_+ \Pi_{12}^{(+)} + [1 - f(r)]\, N_- \Pi_{12}^{(-)} \right\}, \tag{7.71}$$

(Ω_g operator corresponding to g) improves the situation. It is calculated that if N_+ and N_- have the values

$$N_+ = 1 + \epsilon \; ; \quad N_- = 1 - \epsilon \quad \text{with} \quad \epsilon = (4/A) - (J'/V) \tag{7.72}$$

the detailed normalization conditions are satisfied up to an order $1/A$ of the main terms (but neglecting quantities of order $1/A^2$). J'/V is defined as

$$J'/V = \iint D(r_1) \cdot D(r_2) \cdot F(r)\, d\vec{r}_1\, d\vec{r}_2 . \tag{7.73}$$

From (7.64), (7.65), (7.73) it can then be seen that J'/V if of the order $1/A$: hence ϵ is also of the order $1/A$. (7.71) can be expressed alternatively as

$$\langle \vec{r}_1 \vec{r}_2 \zeta_1 \zeta_2 | g | \vec{r}_1 \vec{r}_2 \zeta_1' \zeta_2' \rangle = (A^2/16)\, D(r_1) \cdot D(r_2) \cdot \left\{ [1 + \epsilon\, f(r)]\, \delta_{\zeta_1 \zeta_1'} \delta_{\zeta_2 \zeta_2'} - [f(r) + \epsilon]\, \delta_{\zeta_1 \zeta_2'} \delta_{\zeta_2 \zeta_1'} \right\} . \tag{7.74}$$

From (7.54), (7.55), (7.56), (7.62), (7.74) and (7.72) one now obtains

$$\mathcal{M}^2 = \tfrac{1}{2} A [1 - \tfrac{1}{4} A I_f - I_1 (1 - (AJ'/4V))] \tag{7.75}$$

with

$$I_1 = \iint (\sin \nu r / \nu r)\, D(r_1) \cdot D(r_2)\, d\vec{r}_1\, d\vec{r}_2 \tag{7.76}$$

$$I_f = \iint (\sin \nu r / \nu r)\, D(r_1) \cdot D(r_2)\, f(r)\, d\vec{r}_1\, d\vec{r}_2 . \tag{7.77}$$

It is not so easy to tell how reliable the result (7.47) with (7.75) may be. Errors may arise through

(a) inadequacy of the physical model, expressed here by (7.62) and (7.71),
(b) rough mathematical approximations.

It seems that the mathematical approximations made in deriving the result (7.75) are not important. We have tried [63] to obtain some idea of the sensitivity of the result to the physical assumptions made by comparing different models (statistical model and shell model) for O^{16} and Ca^{40}, (cf. 7.4 for a table in which this comparison is made).

PRIMAKOFF [66] has given an estimate of $\mathcal{M}^2$ for nuclei with $N \neq Z$. His result can be written as

$$\mathcal{M}^2 = Z[1 - \delta((A - Z)/2A)]. \tag{7.78}$$

The value of the constant δ is estimated by him at about $\delta \approx 3$. The experimental results of SENS [62] on muon capture up to the heaviest elements are fitted quite reasonably by Primakoff's formula (7.47) with (7.78), when adjusting somewhat the "parameters" ν and δ (with $\delta = 3.13$), taking the coupling constants in accordance with a UFI. However, the derivation of Primakoff's formula contains appreciable mathematical approximations and it is difficult to tell what reliability the coupling constant value, obtained in this way, may have. It seems questionable to us that the coupling constant thus obtained could be relied upon to have an error better than, say, 40% (cf. also section 7.4 in this respect). Of course (7.47) with (7.78) may be used as an empirical formula fitting two parameters (ν and δ) but it should be noted that the result for $\mathcal{M}^2$ is very sensitive to the value of δ (a 10% change of δ may change $\mathcal{M}^2$ by 30%).

7.4. Shell model calculations for total capture rates

Shell model calculations for total muon capture rates can be made with or without the use of the closure approximation, using either (7.47) with (7.48) or (7.43) with (7.44) as a starting point. The use of the closure approximation means a substantial simplification; however, it requires that a value for the average neutrino momentum ν is assumed, on which the result depends rather strongly, and this value is not found from the closure method itself. Hence the calculation of the transition rate, performing summation over partial transitions according (7.44), could in principle give more reliable results.

LUYTEN, ROOD and TOLHOEK [63] made calculations for shell model wave functions with and without closure approximation for O^{16} and Ca^{40}, also comparing them with the results of the statistical model.

For O^{16} and Ca^{40} the wave function may be written as a Slater determinant

$$\Psi(x_1 \ldots x_A) = \frac{1}{\sqrt{A!}} \Sigma_p \delta_p \prod_{\kappa=1}^{A} \psi_\kappa (x_{p_\kappa}) \tag{7.79}$$

x_κ specifies the coordinates for the κ^{th} nucleon; the $\psi_\kappa(x)$ give the individual particle states

$$\Psi_\kappa(x) = R^t_{n\ell}(r)\, \Psi_{\ell m}(\theta, \Phi)\, \alpha_s(\xi)\, \nu_t(\eta) = \omega^t_\lambda(\vec{r})\, \alpha_s(\xi)\, \nu_t(\eta). \tag{7.80}$$

κ summarizes all other subscripts $\kappa = (n, l, m, s, t)$, while
λ summarizes three subscripts, $\lambda = (n, l, m)$;
$\alpha_s(\xi)$ is a spin function ($s = +$ or $-$) and
$\nu_t(\eta)$ gives the i-spin dependence ($t = +$ or $-$).

Substituting (7.79) with (7.80) into (7.48) one finds for $\mathcal{M}^2$, separating the terms with $i = j$ and $i \neq j$ in Σ_{ij}

$$\mathcal{M}^2 = Z - Q \tag{7.81}$$

with

$$Q = 2\int \left(\frac{d\hat{\vec{\nu}}}{4\pi}\right) \Sigma_{\lambda_1\lambda_2} [\omega^{+*}_{\lambda_1}(\vec{r}_1) \exp(i\vec{\nu}\cdot\vec{r}_1)\omega_\lambda(\vec{r}_1)d\vec{r}_1] \times [\int \omega^{-*}_{\lambda_2}(\vec{r}_2) \exp(-i\vec{\nu}\cdot\vec{r}_2)\omega^{+}_{\lambda_1}(\vec{r}_2)d\vec{r}_2]. \tag{7.82}$$

Using standard operations of angular momentum theory this reduces to

$$Q = 2\Sigma_\ell \Sigma_{n_1\ell_1 n_2\ell_2} X_{\ell_1\ell_2\ell} I^2_{n_1\ell_1 n_2\ell_2\ell} \tag{7.83}$$

with

$$X_{\ell_1\ell_2\ell} = (2\ell+1)(2\ell_1+1)(2\ell_2+1)\begin{pmatrix} \ell_1 & \ell_2 & \ell \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{7.84}$$

$$I_{n_1\ell_1 n_2\ell_2} = \int R^{+}_{n_1\ell_1}(r) R^{-}_{n_2\ell_2}(r) j_\ell(\nu r) r^2 dr . \tag{7.85}$$

In order to see how sensitive the results are for details of the wave functions the calculations were made for shell model wave functions belonging to

(A) a harmonic oscillator well;

(B) an infinitely deep potential well;

(C) a finite nuclear potential well with rounded-off edges.

As the transition rates are rather sensitive to the nuclear radius, the radius parameters of the potential wells were so fixed that the corresponding radial proton distribution had an r.m.s. radius equal to the r.m.s. radius found from the Stanford e-scattering experiments.

Without using the closure approximation one has to evaluate M^2 according to (7.44) with the same wave functions, which provides:

$$M^2 = \Sigma_{n_a\ell_a n_b\ell_b\ell} 2 I^2_{n_a\ell_a n_b\ell_b\ell} X_{\ell_a\ell_b\ell} (\nu_{ab}/\nu_\mu)^2 . \tag{7.86}$$

This requires more extensive summations than for (7.83) especially for heavier nuclei. A summary of the various results is given in Table I.

The value for the effective coupling constant obtained from the value of K (IIC) in Table I and the experimental capture rate [64] is

$$\Lambda_{\mu c} = (0.97 \pm 0.03) \times 10^5 \text{ sec}^{-1} \tag{7.87}$$

$$G^{\mu c^2}_F + 3\, G^{\mu c^2}_{GT} = 0.62 \times 10^{-97} \text{erg}^2 \text{cm}^6 . \tag{7.88}$$

For Ca^{40} we get about the same result for the coupling constant. This should be compared with the value obtained for this coupling constant combination from the hypothesis of a UFI and the coupling constants for β-radioactivity

TABLE I

A COMPARISON OF THE VALUES OF $\mathcal{M}^2$ OR M^2 AND K CALCULATED ACCORDING TO VARIOUS MODELS FOR O^{16}

$$\Lambda_{\mu c} = K(G_F^{\mu c^2} + G_{GT}^{\mu c^2})$$

Method of calculation	$K \times 10^{-102}$ (erg^{-2}, cm^{-6}. sec^{-1})	
I Closure approximation $\hbar c\nu = 85$ MeV	$\mathcal{M}^2$	
shell model: A harm. osc. well	1.95	1.34
shell model: B inf. well	1.75	1.20
shell model: C finite well	1.85	1.26
shell model: D statist. model [+]	2.22	1.51
II Summation over partial transitions	M^2	
shell model: A harm. osc. well	1.46	1.54
shell model: B inf. wel	1.26	1.33
shell model: C finite well	1.48	1.56

[+](The values $k_F = 1.33$ fm^{-1} and $a_D = 0.51$ fm were chosen)

(supplemented by assumptions concerning the conserved vector current and the induced pseudoscalar as specified in [65] cf. 7.2, I-IV),

$$G_F^{\mu c^2} + 3\, G_{GT}^{\mu c^2} = 1.10 \times 10^{-97} \text{erg}^2 \text{cm}^6 . \qquad (7.89)$$

We conjecture that this discrepancy with the U.F.I. is due to wave functions in our model, which may be simplified too much. The contribution from velocity dependent (relativistic) terms are not yet contained in the table. It can be calculated in the closure approximation and can be expressed as

$$(\Delta\mathcal{M}^2)_{vel} = R_G(\nu/M)\,Q , \qquad (7.90)$$

where R_G is the following coupling constant ratio

$$R_G = \frac{G_V g_V + (G_A - G_P)\, g_A}{G_F^2 + 3\, G_{GT}^2} \approx 0.35$$

(numerical result for assumption about coupling constants according to [65], or 7.2).

The result for the contribution of the velocity dependent terms is found to be

$$(\Delta\mathcal{M}^2)_{vel} / \mathcal{M}^2 = 10\%.$$

This holds within 2% for O^{16} as well as Ca^{40} for all shell model wave functions used.

7.5. Muon capture in hydrogen

The ideal process of muon capture from the point of view of elementary particle physics is muon capture in hydrogen

$$\mu^- + p \rightarrow n + \nu \,. \tag{7.91}$$

This avoids the complications of nuclear physics which are no longer very simple even for D or He^3. However, atomic and molecular complications exist for capture in hydrogen, but great progress has been made during the last few years in analysing these problems of atomic physics. The first preliminary experimental value for the capture rate in hydrogen was published in 1961; at the moment 3 independent determinations are available. One of the experimental problems is that exceedingly pure hydrogen is required; heavier atoms show a far stronger muon capture and even deuterium must be removed to a great extent.

We give a very brief account (cf.[66],[67],[67a]) of the atomic complications which exist here. If a beam of negative muons is stopped in (liquid) hydrogen both hyperfine structure states of the muonic atom $(p\mu^-)$ are formed rapidly (about 10^{-10} sec) with a ratio 3 : 1 for $(p\mu^-)_{\uparrow\uparrow}$ (triplet) to $(p\mu^-)_{\uparrow\downarrow}$ (singlet). By means of exchange collisions

$$(p\mu^-)_{\uparrow\uparrow} + (p)_{\downarrow} \rightarrow (p\mu^-)_{\downarrow\uparrow} + (p)_{\uparrow} \tag{7.92}$$

a conversion to essentially pure $(p\mu^-)_{\uparrow\downarrow}$ takes place in a time of the order of 10^{-9} sec. By means of the reaction

$$(p\mu^-)_{\downarrow\uparrow} + (p)_{\downarrow} \rightarrow (p\mu^-p)_{\downarrow\uparrow\downarrow} \tag{7.93}$$

one has then the formation of a molecule with two protons bound by means of a muon, mostly in an ortho-state: ($S_{pp} = 1$, $J_{p\mu p} = \frac{1}{2}$) at a rate of $(1.4 \pm 0.6) \times 10^6$. sec^{-1} (in liquid hydrogen). For a gaseous target the process would play only a minor role. One has now to consider the capture rates in the muonic atoms and the muonic molecule

$$\Lambda\,(p\mu^-)_{\uparrow\downarrow}\,, \quad \Lambda\,(p\mu^-)_{\uparrow\uparrow} \quad \text{and} \quad \Lambda(p\mu^-p)_{\uparrow\downarrow\uparrow}.$$

The capture rate in the molecule can be expressed in the capture rates in the atoms:

$$\Lambda\,(p\mu^-p)_{\uparrow\downarrow\uparrow} = 2\gamma\,[\tfrac{3}{4}\,\Lambda\,(p\mu^-)_{\downarrow\uparrow} + \tfrac{1}{4}\Lambda(p\mu^-)_{\uparrow\uparrow}] \tag{7.94}$$

where

$$\gamma = \frac{\text{muon prob. density at either proton in } (p\mu^-p)_{\uparrow\downarrow\uparrow}}{\text{muon prob. density at proton in } (p\mu^-)_{\downarrow\uparrow}} = 0.585. \tag{7.95}$$

We now mention the result for the ratio of the capture rates in the singlet and the triplet states for a simple V-λ Λ theory; this is

$$R = \frac{\Lambda\,(p\mu^-)\downarrow\uparrow}{\Lambda\,(p\mu^-)\uparrow\uparrow} = \frac{(G_V - 3G_A)^2}{(G_V + G_A)^2}\,. \qquad (7.96)$$

Hence we see that $R = \infty$ for $G_V = -G_A$ and $R = 1$ for $G_V = G_A$. Thus R is seen to be strongly dependent on the relative sign of the V and A interaction. This also holds for $\Lambda\,(p\mu^-p)\uparrow\downarrow\uparrow$ and is still true after introducing the terms for weak magnetism and induced pseudoscalar.

Taking these refinements into account and assuming UFI one obtains the theoretical capture rates (using the β-decay coupling constants) mentioned in Table II, which compares theoretical and experimental results.

TABLE II

THEORETICAL AND EXPERIMENTAL CAPTURE RATES IN HYDROGEN AND He^3 (SEC^{-1})

	Theory	Experiment
$(\mu^- p)$ $(F=0)$	636	
$(\mu^- p)$ $(F=1)$	13	
$(p\mu^- p)$	560	515 ± 85 (Columbia)[1]
$0.7(p\mu^- p) + 0.3(\mu^- p)$ $(F=0)$	583	426 ± 60 (CERN+Chicago)[2]*
$\mu^- + {}^3He \rightarrow {}^3H + \nu$	1400 ± 150	1410 ± 140 (Dubna)

[1] Counter experiment
[2] Bubble chamber experiments
*) Proc. CERN Conf. High Energy Phys. 1962.

From a comparison of the experimental values with the theory one may draw the following conclusions:

(1) Within the limits of the rather large errors of the present experiments, the experimental values are in agreement with the theoretical value according to (7.95).
(2) The present experimental values are not sufficiently accurate for a test of the existence of the weak magnetism or induced pseudoscalar terms.
(3) The experimental values give a clear indication that the relative sign of the V and A-interactions is negative. The experimental accuracy only allows the following limits for G_A/G_V,

$$-2.0 < (G_A/G_V) < -1.1. \qquad (7.97)$$

7.6. Comparison of pion decay into electrons and muons

The decay of pions into electrons or muons is supposed to proceed primarily via an intermediate state with one nucleon and one anti-nucleon with subsequent decay via either the β-decay interaction or the muon-decay interaction. Supposing a UFI, or more particularly e, μ universality, one can calculate the ratio for the decay of the pion into electrons or muons. As-

suming the same V - λ A interaction for electrons and muons such that the currents j (e ν) and j (μ ν) coupled to the nucleons are identical in form, one finds for the ratio

$$R_{\text{theor}} = \frac{\Lambda(\pi \to e + \nu)}{\Lambda(\pi \to \mu + \nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.36 \times 10^{-4} .$$

This value is slightly changed when applying an estimate for the electromagnetic corrections

$$R_{\text{theor.}} = 1.23 \times 10^{-4} . \qquad (7.98)$$

This value is in very good agreement with the experimental value, first measured at CERN in 1958 and recently determined very accurately by H. L. ANDERSON et al. [71] as

$$R_{\text{exp.}} = (1.21 \pm 0.07) \times 10^{-4} . \qquad (7.99)$$

The good agreement between (7.99) and (7.98) is at the moment the best evidence available for the equality of $g_A^{\mu c}$ and g_A^{β} (to within a few per cent).

7.7. The hyperfine structure effect and isotope effects in muon capture

It has been mentioned already that the capture rates for $(p\mu^-)$ are quite different in the singlet and triplet states. For a V - λ A interaction they are proportional to

$$\Lambda_0 \equiv \Lambda(p\mu^-)_{\uparrow\downarrow} = C\,(G_V - 3\,G_A)^2 \qquad (7.100)$$

$$\Lambda_1 \equiv \Lambda(p\mu^-)_{\uparrow\uparrow} = C\,(G_V + G_A)^2 . \qquad (7.101)$$

Hence the statistical average is

$$\bar{\Lambda} = \tfrac{1}{4}\,\Lambda_0 + \tfrac{3}{4}\,\Lambda_1 = C\,(G_V^2 + 3\,G_A^2). \qquad (7.102)$$

From the capture rates in complex (even-even) nuclei it is mostly this combination $G_V^2 + 3\,G_A^2$ which determines the overall capture rates.

However, it is a matter of much interest as to what ratio exists in muon capture for the V- and A-interactions:

(a) Total capture rates (of even-even nuclei) are proportional to the combination $G_V^2 |\int 1|^2 + G_A^2 |\int \vec{\sigma}|^2$ (taking the leading terms only). If $|\int \vec{\sigma}|^2$ differs substantially from $3\,|\int 1|^2$ this permits G_A^2/G_V^2 to be determined at least in principle. This requires an appropriate choice of nuclei (often separated isotopes).

(b) Odd A-nuclei have a nuclear spin I, which has the interesting possibility of a hyperfine structure effect in the muon capture rate. The muon in a K-orbit may couple with the nucleus to a resulting angular momentum $F = I \pm \frac{1}{2}$ (cf. Fig. 13). The capture rates Λ_+^{cap} and Λ_-^{cap} in both hyperfine structure states may be different corresponding to different probabilities

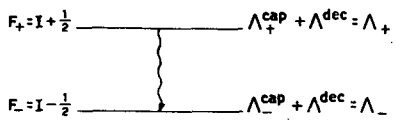


Fig. 13

for singlet or triplet states of $(p\mu^-)$; this difference will depend on G_A/C_V, not only on the magnitude of this ratio but also on its sign.

The quantitative analysis of both effects requires a certain knowledge of the nuclear wave functions. It turns out that the effect is most sensitive to the ratio G_A/G_V, and thus the most suitable for obtaining information on this aspect of the muon capture interaction. Once the muon capture interaction is sufficiently known the muon capture rates present a tool for obtaining information on nuclear structure.

The suggestion that a hyperfine structure effect might be observable in muon capture was made by BERNSTEIN, LEE, YANG and PRIMAKOFF [72]. We cannot go into detail here, but indicate simply that it is an effect for complex nuclei, which goes back to the different capture rates in the singlet and triplet state of $(p\mu^-)$. Hence we may write, comparing the hyperfine structure effect in complex nuclei with that in hydrogen,

$$\frac{\Delta\Lambda(Z)}{\bar{\Lambda}^{cap}(Z)} = (1/Z\xi)(\Delta\Lambda(p)/\bar{\Lambda}^{cap}(p)). \qquad (7.103)$$

$\Delta\Lambda = \Lambda_- - \Lambda_+$: Z indicates a nucleus with nuclear charge Z: p indicates a proton; ξ (≤ 1) is a factor which should be calculated on the basis of a nuclear model. If all protons contributed equally to muon capture, and only the last odd proton to the hyperfine structure effect, one should expect $\xi \approx 1$. However, in general, the last odd proton will contribute more to the muon capture than the average proton and one expects $\xi < 1$; Überall obtained $\xi = 0.57$ from shell model calculations for F^{19}. We note the following about the way in which the hyperfine structure effect might be observed.

The total disappearance rates in both hyperfine structure states are composed of the decay rate and the capture rate (different for both values of F). The way in which $\Delta\Lambda = \Lambda_- - \Lambda_+$ can be observed depends strongly on the rate R, with which conversion from the $F_+ = I + \frac{1}{2}$ to the $F_- = I - \frac{1}{2}$ state takes place. The main contribution to R consists of a magnetic Auger conversion process; the transition rate was calculated by TELEGDI [73]. Depending on the value of R one must distinguish the following cases:

(1) If $R \ll \Lambda_+$ one observes (e.g. by measuring the decay electrons) a superposition of two exponentials (corresponding to Λ_+ and Λ_- for the decay).

(2) If $R \gg \Lambda_+$ a rapid conversion from F_+ to F_- takes place and the total capture rate which is measured is essentially Λ_-. This rate Λ_- may show a marked deviation from the value interpolated from the capture rates of neighbouring even-even nuclei thus indicating the hyperfine structure effect.

The difference $\Delta\Lambda = \Lambda_+ - \Lambda_+$ can be measured directly when measuring the neutrals (neutrons and gammas) emitted after the capture process, as was indicated by WINSTON [74]; it requires a good measurement of the time dependence of the neutrals between $0.1\,\mu$sec and $1\,\mu$sec after a muon is stopped. This experiment was performed by the Chicago group [75] for F^{19}. It was found that

$$(\Lambda_-^{cap}/\Lambda^{cap}) \approx 1.45 \pm 0.05 \ .$$

This excludes V + A interaction and is in agreement with estimates for V-A interaction (a shell model calculation was given by ÜBERALL [76]). Hence the relative sign of the V and A contributions seems now well established. It seems that a further discussion of the results could also provide a reasonable value for G_A/G_V.

The fact that total muon capture rates for suitably chosen isotopes will depend appreciably on the relative contributions of Fermi and Gamow-Teller interaction to the muon capture interaction was first pointed out by TOLHOEK and LUYTEN [77]. Such effects depend essentially on the nuclear model. We indicate some typical features on the basis of the shell model (cf. Fig. 14).

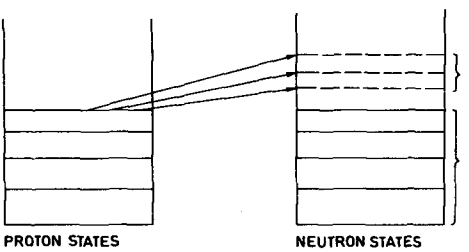


Fig. 14

The arrows indicate some transitions from occupied proton states to non-occupied neutron states. (Only a small part of all the possible arrows is drawn)

When considering muon capture in a simple shell model picture, the total transition rate is composed of all possible partial transitions from occupied proton states to non-occupied neutron states (all states are considered as single particle states). The Pauli principle forbids protons going to already occupied neutron states; hence most protons have to go to "different" non-occupied neutron states. For some outer proton states analogous empty neutron states may be available. Transitions could also occur from $j = \ell + \frac{1}{2}$ states to $j = \ell - \frac{1}{2}$ states (e.g. $p_{\frac{3}{2}} \rightarrow p_{\frac{1}{2}}$ or $f_{\frac{7}{2}} \rightarrow f_{\frac{5}{2}}$); such transitions involve spin-flip and can result only from the Gamow-Teller part of the interaction and not from the Fermi part.

Estimates by TOLHOEK and LUYTEN [77] and by G. GOULARD and B. GOULARD [78] by means of simple shell model calculations showed indeed that capture ratios

$$\Lambda(V)/\Lambda(Ca),\ \Lambda(Mn)/\Lambda(Ca),\ \Lambda(Co)/\Lambda(Ca),\ \Lambda(Ni)/\Lambda(Ca)$$

are 20% to 40% higher for pure A- than for pure V-interaction (the difference resulting for a considerable part from the $f_{\frac{7}{2}} \rightarrow f_{\frac{5}{2}}$ transitions). The difficulties for a quantitative theory of such capture rates is that even the ratios depend appreciably on details of the nuclear model. Further, the statistical weight of the Gamow-Teller contribution in $G_V^2 \left| \int 1 \right|^2 + G_A^2 \left| \int \vec{\sigma} \right|^2$ is roughly three times larger for the Gamow-Teller contribution than for the Fermi contribution and although the difference between pure Fermi and pure Gamow Teller may be considerable, the difference between mixed interactions with $\lambda = G_A/G_V$ varying between -1 and -2 are far less striking.

Hence it seems most appropriate to determine λ from the hyperfine structure effect and to consider the discussion of the variation of muon capture probabilities mainly as a problem concerned with nuclear structure, using muon capture as a tool. In this respect it should be stressed that present experimental data on muon capture rates are generally only available for the natural isotope mixtures, while it would be interesting to study the variation with Z and N separately in certain regions, which would require measurement for separated isotopes. Only recently the capture rates for some separated isotopes were measured; Cl^{35} and C^{37} [70a], Ca^{40} and Ca^{44} [70b], U^{235} and U^{230} [70c], and Li^6 and Li^7, B^{10} and B^{11} [64].

BURKHARDT and CAINE [79] have considered total capture rates for the nuclei N^{14}, O^{16} and F^{19} which appeared to be not very dependent on the ratio G_A/G_V. The advantage of these nuclei is that quite some knowledge has been accumulated on their wave functions. BELTRAMETTI and RADICATI [80] and DUCK [81] also made shell model calculations on muon capture, especially on partial transitions to bound states in particular for O^{16}.

7.8. Partial muon capture rate in C^{12}

As early as 1954 an experiment was performed by Godfrey which aimed at a direct comparison of the coupling constants of the Gamow-Teller parts in muon capture and β-decay.

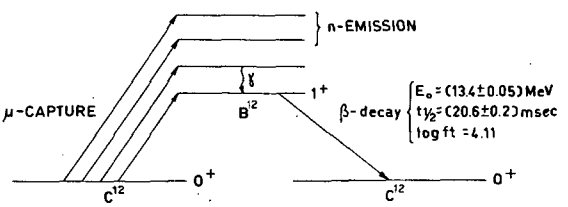


Fig. 15

A comparison is to be made between the reactions (cf. Fig. 15)

$$\mu^- + C^{12} \rightarrow B^{12} + \nu \qquad (7.104)$$

and

$$B^{12} \rightarrow C^{12} + e^- + \bar{\nu}\,. \qquad (7.105)$$

The β-decay in this $1^+ \rightarrow 0^+$ transition is determined by the $\int \vec{\sigma}$ nuclear matrix element; if one could compare the partial muon capture rate between the same nuclear states one might hope to obtain the ratio $|G_A^{\mu c}/G_A^{\beta}|$ by dividing out the matrix element $\int \vec{\sigma}$ occurring for both processes. Experimentally, one measures the β-radioactivity which occurs after the muon capture. This provides the muon capture probability to those states of B^{12} which show no particle emission (mostly neutrons), i.e. to the ground state and the lowest excited states decaying back to the ground state by γ-emission. It can be shown that the muon capture leads only in quite a small fraction of the cases to these excited states, in comparison with the transitions to the ground state of B^{12}. This is one of the corrections which has to be applied. Another correction is that the muon capture does not depend only on

$\int \vec{\sigma}$ but also on $\int \vec{\sigma}\, r^2$ (using the terminology of β-radioactivity), because of the shorter wavelength of the neutrino in muon capture.

It follows that the analysis of the experiment requires a certain knowledge of the structure of the nuclei. Hence it is seen that the theoretical analysis of the experiment is not without complications.

Unfortunately the experimental situation is worse as appreciably different values were found by different groups, differing more than the quoted errors

$$\Lambda(\mu^- + C^{12} \to B^{12} + \nu) \quad \begin{cases} 5.9 \pm 1.5 & [82] \\ 9.05 \pm 0.95 & [83] \\ 9.18 \pm 0.5 & [84] \\ 6.8 \pm 1.1 & [85] \\ 5.8 \pm 1.3 & [86] \\ 6.31 \pm 0.24 & [87] \\ 6.75 \,{}^{+\,0.30}_{-\,0.75} & [87a] \end{cases} \times 10^3 \mathrm{sec}^{-1} \qquad (7.106)$$

It would be desirable for the experimental discrepancies to be cleared up. At the moment one can only conclude that $|G_A^{\mu c}/G_A^{\beta}|$ equals unity within about 30% (using the "most plausible" assumptions mentioned in section 7.2 about weak magnetism and the pseudoscalar).

7.9. Partial transitions in O^{16} and the pseudoscalar interaction

In 7.2 we considered the effective Hamiltonian for muon capture and showed that it seems to be determined by the 3 constants G_V, G_A and G_P even if we start with the general expression (7.12) containing 6 parameters.

For the sake of simplicity we discussed the hyperfine structure effect and the capture in hydrogen as if there were mainly two parameters to be determined, namely G_V and G_A.

However, it is seen from (7.17) and (7.32) that G_P also plays an independent role and that it could be determined independently from a phenomenological analysis of suitable experiments. *In this respect it should be noted* that

$$\int d\hat{\vec{\nu}}_{ab}/4\pi) \; | \; \hat{\vec{\nu}}_{ab} \cdot \int \vec{\sigma} \, |^2 = \tfrac{1}{3} \left| \int \vec{\sigma} \, \right|^2$$

does not generally hold and for the transitions, for which it does not hold, G_V, G_A and G_P occur independently and not only in the combinations $G_F = G_V$ and G_{GT}.

An interesting proposal for such a determination of G_P was made by SHAPIRO and BLOKHINTSEV [88]. They pointed out that in the partial transitions from $O^{16} \to N^{16}$ the ratio of the transition rates to the 0^- and 1^- excited levels of N^{16} (both characterized by the configuration $(1p_{\frac{1}{2}})_p^{-1} (2J_{\frac{1}{2}})^1_n$) is quite sensitive to the value of G_P (cf. Fig. 16). It can be shown that the ratio of the partial capture probabilities to the 0^- and 1^- levels is given in a first approximation by

$$\Lambda(0^+ \to 0^-)/\Lambda(0^+ \to 1^-) = (G_P - G_A)^2/G_V^2 + 2G_A^2. \qquad (7.107)$$

Hence it follows that this ratio is sensitive to the value of G_P. (7.107) is not yet suitable in this form for an analysis of a possible experiment as it

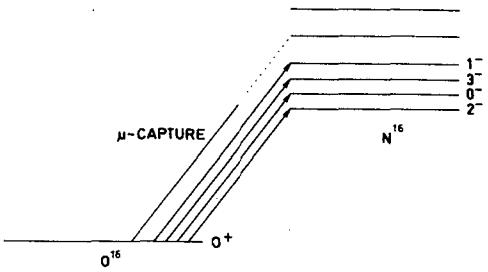


Fig. 16

can be shown that the velocity-dependent terms of the Hamiltonian and configuration mixing in the nuclear wave functions cause appreciable deviations [88, 89]. If the calculation of these corrections were sufficiently reliable, measurement of the above-mentioned ratio would indeed be a good experiment for determining G_P .

7.10. The angular distribution of recoil nuclei and of emitted neutrons

When (polarized) negative muons are captured in a K-orbit of a spin zero nucleus they have a residual polarization which amounts to 15 to 20%. This residual polarization can be measured by means of the anisotropy of the decay electrons from the polarized muons. As a consequence of the non-conservation of parity the following effects can be expected if the muon captured in the K-orbit is polarized.

(a) An anisotropic angular distribution may occur for the recoil nuclei

$$W_{rec}\ (\theta_{rec}) = 1 + P_\mu\, \alpha_{rec} \cos\theta_{rec} . \qquad (7.108)$$

(b) An anisotropic angular distribution may occur for the emitted (primary) neutrons

$$W_n\,(\theta_n) = 1 + P_\mu\, \alpha_n\, \cos\,\theta_n ; \qquad (7.109)$$

P_μ = degree of polarization of the muon captured in the K-orbit;
$\vec{s}_\mu$ = spin direction of this muon,
θ_{rec} = angle between $\vec{s}_\mu$ and recoil nucleus momentum $\vec{p}_{rec}$,
θ_n = angle between $\vec{s}_\mu$ and momentum of the emitted neutrons $\vec{p}_n$.

α_{rec} and α_n depend on the coupling constants G_V, G_A, G_P; they are seen to be rather similar; especially, if most of the final states are unbound, it seems plausible to suppose that most of the recoil momentum is carried off by a neutron.

As an example of the dependence of the asymmetry coefficient on the coupling constants we give an estimate of α_{rec} made by PRIMAKOFF [90] for the heavier nuclei:

$$\alpha_{rec} = \frac{(G_V)^2 - (G_A)^2 + (G_P)^2 - 2\,G_A\,G_P}{(G_V)^2 + 3\,(G_A)^2 + (G_P)^2 - 2\,G_A\,G_P} \quad (= -\,0.39) . \qquad (7.110)$$

This shows

$$\alpha_{rec} = 0 \text{ for } |G_V| = |G_A| \text{ and } G_p = 0 .$$

However, it is seen that for the values of the coupling constants according to (7.18) and (7.27) - (7.31) an appreciable asymmetry should occur.

In the first instance the value of α_n is something like (7.110). ÜBERALL [91] has considered on the basis of a Fermi gas model how α_n is changed for complex nuclei. DOLINSKY, BLOKHINTSEV and AKIMOVA [92] did this on the basis of a shell model. Various experimental results are now available for α_n for S^{32} from the groups in Liverpool [94] and Chicago [93] which average to

$$\alpha_n = -0.24 \pm 0.06. \qquad (7.112)$$

A value for Ca was recently obtained in Dubna [95]. Although the accuracy is not yet very high, the existence of an asymmetry seems to be now well established. This seems to be a rather clear indication of the existence of an induced pseudoscalar coupling. The data were recently extensively analysed by BLOKHINTSEV and DOLINSKY [96] .

A further consequence of the non-conservation of parity, which we want to just mention here, is that one must expect that recoil nuclei and emitted neutrons will in general be polarized.

7.11. Radiative muon capture: spectrum; directional distribution of the photons; circular polarization of the photons [60] [97] [98] [99]

Radiative muon capture is a second-order process, with one weak interaction vertex and one electromagnetic interaction vertex; the main contribution is represented by the Feynman diagram, (see Fig. 17 (a)). However also the following contributions must be considered (interaction with charge of proton) (cf. Fig. 17(b)) and the interactions via the anomalous electromagnetic moments (cf. Fig. 17(c) and (d)).

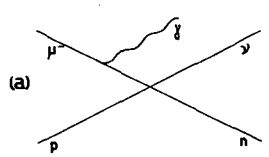


Fig. 17 (a)

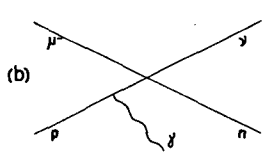


Fig. 17 (b)

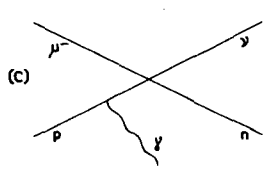


Fig. 17 (c)

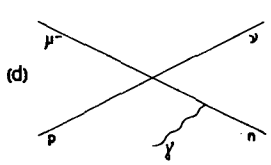


Fig. 17 (d)

The energy $m_\mu c^2 = 105$ MeV is mostly divided between neutrino and photon so that the I. B. photons (internal bremsstrahlung photons) may have

quite a high energy here; they have a continuous spectrum extending to energies which are considerably higher than the gammas from the de-excitation of the final nucleus. However, radiative capture is estimated to occur only in about 1 in 10^4 of all captures so that one can expect that it will be studied only for complex nuclei (and not for hydrogen) in the near future. An experiment at CERN (CONVERSI et al., [99a]) has established experimentally the occurrence of radiative μ-capture at about the expected rate (cf. Fig. 18).

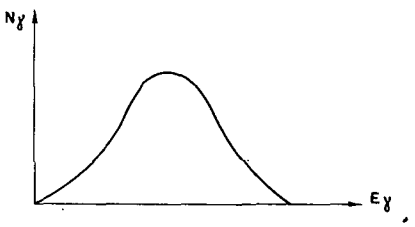


Fig. 18

The probability of the radiative muon capture

Radiative muon capture can provide independent information on the muon capture coupling constants; the experiments will, however, be difficult. The interesting feature in radiative muon capture is that for high E_γ the muon in the intermediate state is not "at rest" with respect to the nucleon as is always (approximately) the case for normal muon capture. This can enable us in principle to obtain different combinations of the coupling constants g_V, g_A, g_M, g_P, g_S, g_T than are given by G_V, G_A, G_P.

We shall not discuss now specific theoretical results but we mention concisely what can be observed about radiative muon capture:

(a) Form and intensity of the photon spectrum.

(b) If the muons, which are captured, have an initial polarization P_μ, the I.B. photons can be expected to have an anisotropic angular distribution

$$W(\theta) = 1 + P_\mu \beta \cos\theta_\gamma \quad . \qquad (7.113)$$

θ_γ is the angle between the direction of emission of the photon and the direction of the muon spin $\vec{s}_\mu$. The anisotropy coefficient would be equal to one: $\beta = 1$ for $G_V = g_V$, $G_A = g_A$, $G_P = 0$. However the induced pseudoscalar interaction will change this value, which provides a means of determining G_P.

(c) Even when the initial captured muons are not polarized it must be expected that the emitted I.B. photons have a circular polarization β'. It can be shown that this parameter $\beta' = \beta$, so that it provides the same information on the coupling constants as the I.B. photon angular distribution.

The experimental study of radiative muon capture has only just started. Much work has also still to be performed on the theory, both on the influence of the binding of the nucleons in nuclei and on the influence of diagrams (b), (c) and (d) of Fig. 17.

8. NEUTRINO PROCESSES; THE HYPOTHESIS OF AN INTERMEDIATE VECTOR BOSON; ARE THERE TWO KINDS OF NEUTRINO?

In this section we want to consider the theory of weak interactions in a somewhat wider context. If we consider all kinds of weak interaction processes, the four following types may be distinguished:

I Processes with leptons only; example:

$$\mu^{\pm} \rightarrow e^{\pm} + \nu + \bar{\nu}, \tag{8.1}$$

II Processes in which baryons or mesons occur in addition to leptons, but in which the strangeness does not change; examples:

$$n \rightarrow p + e^- + \bar{\nu} \tag{8.2}$$

$$e^- + p \rightarrow n + \nu \tag{8.3}$$

$$\mu^- + p \rightarrow n + \nu \tag{8.4}$$

$$\pi^- \rightarrow \mu^- + \bar{\nu}, \tag{8.5}$$

III Processes, in which baryons or mesons occur in addition to leptons and in which the strangeness changes; examples:

$$K\mu 2 \qquad K^{\pm} \rightarrow \mu^{\pm} + \nu \tag{8.6}$$

$$K\mu 3 \qquad K^{\pm} \rightarrow \mu^{\pm} + \nu + \pi_0 \tag{8.7}$$

$$Ke3 \qquad K^{\pm} \rightarrow e^{\pm} + \nu + \pi_0 \tag{8.8}$$

IV Processes, in which no leptons occur and in which the strangeness changes; examples:

$$K\pi 2 \qquad K^+ \rightarrow \pi^+ + \pi^o \tag{8.9}$$

$$K\pi 3 \qquad K^+ \rightarrow \pi^+ + \pi^- + \pi^+ \tag{8.10}$$

$$\Sigma^+ \rightarrow \begin{matrix} n + \pi^+ \\ p + \pi^o \end{matrix} \Big\} \tag{8.11}$$

$$\Lambda^o \rightarrow \begin{matrix} p + \pi^- \\ n + \pi^o \end{matrix} \Big\} \tag{8.12}$$

Up to now we have confined our attention to processes of types I and II, in which the strangeness does not change. The four types of processes have characteristic differences, although they are all weak interaction processes. We shall mention only a few points about processes III and IV, as the time available does not allow us to go into any detail in this field in which much work has been done during the last few years.

It is possible to try whether the processes of all four types I..IV can be characterized by 4-fermion interactions. We now indicate which formal expressions should then be chosen for the types I...IV. We use the following expressions for lepton currents and nucleon currents.

Lepton currents: we write for electrons

$$j^{(-)}_{e,\lambda} = \bar{\psi}_e \gamma_\lambda (1 + \gamma_5) \psi_\nu \tag{8.13}$$

$$j^{(+)}_{e,\lambda} = \bar{\psi}_\nu \gamma_\lambda (1 + \gamma_5) \psi_e . \tag{8.14}$$

Analogously we write for muons

$$j^{(-)}_{\mu,\lambda} = \bar{\psi}_\mu \gamma_\lambda (1 + \gamma_5) \psi_\nu \tag{8.15}$$

$$j^{(+)}_{\mu,\lambda} = \bar{\psi}_\nu \gamma_\lambda (1 + \gamma_5) \psi_\mu . \tag{8.16}$$

The superscripts (+) and (-) indicate whether the electric charge is increased or decreased. We can also write a lepton current for electrons and muons together

$$j^{(-)}_\lambda = j^{(-)}_{e,\lambda} + j^{(-)}_{\mu,\lambda} \tag{8.17}$$

$$j^{(+)}_\lambda = j^{(+)}_{e,\lambda} + j^{(+)}_{\mu,\lambda} \tag{8.18}$$

Nucleon currents: we write

$$J^{(-)}_\lambda = \bar{\psi}_n \gamma_\lambda (1 + \lambda\gamma_5) \psi_p \tag{8.19}$$

$$J^{(+)}_\lambda = \bar{\psi}_p \gamma_\lambda (1 + \lambda\gamma_5) \psi_n . \tag{8.20}$$

The constants $\lambda \approx 1.2$ provides the ratio of A and V interaction in β-decay. The usual Hamiltonian for β-radioactivity can be written in the following form, using the preceding notation

$$\mathcal{H}^\beta_i = g_\beta/\sqrt{2} \; [\, J^{(+)}_\lambda \, j^{(-)}_{e,\lambda} + J^{(-)}_\lambda \, j^{(+)}_{e,\lambda} \,] . \tag{8.21}$$

The two terms in (8.21) are each others Hermitian conjugate. The factor $1/\sqrt{2}$ is just added to conform to the historical convention for the coupling constant. If the hypothesis of the conserved vector current is accepted, (8.19) and (8.20) have to be supplemented with terms for the pion-field as was explained in 6. The interaction for muon capture can be written in a form analogous to (8.21) as

$$\mathcal{H}^{\mu c}_i = \frac{g_{\mu c}}{\sqrt{2}} \; [\, J^{(+)}_\lambda \, j^{(-)}_{\mu,\lambda} + J^{(-)}_\lambda \, j^{(+)}_{\mu,\lambda} \,] . \tag{8.22}$$

$\mathcal{H}^\beta_i$ as well as $\mathcal{H}^{\mu c}_i$ are concerned with processes of type II. Muon decay is a process of type I and can be characterized by

$$\mathcal{H}^\mu_i = \frac{g_\mu}{\sqrt{2}} \; [\, j^{(+)}_{e,\lambda} \, j^{(-)}_{\mu,\lambda} + j^{(-)}_{e,\lambda} \, j^{(+)}_{\mu,\lambda} \,] . \tag{8.23}$$

Experimentally it was found that $g_{\mu i} \approx g_{\mu c} \approx g_\beta$ and it is tempting to suppose that all weak interactions amongst nucleons and leptons can be summarized in the expression

$$\mathcal{H}_i = \frac{g}{\sqrt{2}} \left[J_\lambda^{(+)} + j_\lambda^{(+)} \right] \left[J_\lambda^{(-)} + j_\lambda^{(-)} \right] . \tag{8.24}$$

This expression has an attractive simplicity, being simply of the form

$$\mathcal{H}_i = g \times \text{current} \times \text{current} . \tag{8.25}$$

(8.24) can be considered to be the expression of the hypothesis of a Universal Fermi Interaction (UFI) amongst nucleons and leptons. The expression (8.24) contains the interactions (8.21), (8.22) and (8.23) as cross terms; however, it contains in addition terms such as

$$\mathcal{H}_i^{e,\nu} = \frac{g}{\sqrt{2}} j_{e,\lambda}^{(+)} \; j_{e,\lambda}^{(-)} , \tag{8.26}$$

which provide the possibility of e-ν scattering by a UFI. Hence (8.24) implies that electron neutrino scattering

$$e + \nu \rightarrow e + \nu \tag{8.27}$$

should occur with a small cross-section, which is easily calculated. Of course, it would be of the utmost importance if it could be tested experimentally whether (8.27) occurs in nature.

8.1. Generalization to processes of types III and IV with a change of strangeness S

It seems a plausible generalization to introduce for such processes a strangeness-non-conserving current S_λ. One can form such currents with the Λ-hyperon in the following way

$$S_\lambda^{(+)} = \bar{\psi}_p \gamma_\lambda (1 + \gamma_5) \psi_\Lambda \tag{8.28}$$

$$S_\lambda^{(-)} = \bar{\psi}_\Lambda \gamma_\lambda (1 + \gamma_5) \psi_p . \tag{8.29}$$

Of course, analogous expressions can be formed with the Σ- and Ξ hyperons. In contrast to S_λ, J_λ can be called a strangeness-conserving current. For processes of type III (leptonic decays with change of strangeness) one may now write down an interaction

$$\mathcal{H}_i = \frac{g'}{\sqrt{2}} \left[j_\lambda^{(+)} \; S_\lambda^{(-)} + j_\lambda^{(-)} \; S_\lambda^{(+)} \right] . \tag{8.30}$$

For processes of type IV (non-leptonic decays with change of strangeness) one may write finally

$$\mathcal{H}_i = \frac{g''}{\sqrt{2}} \left[J_\lambda^{(+)} S_\lambda^{(-)} + J_\lambda^{(-)} S_\lambda^{(+)} \right] . \tag{8.31}$$

When considering (8.30) and (8.31) as the fundamental weak interactions, the decays of K-mesons and hyperons have to be considered as second- or higher-order processes. One can easily draw Feynman diagrams with one weak interaction vertex and one or more strong interaction vertices, which explain these decays qualitatively. One might attempt to generalize the UFI for nucleons and leptons, such as formulated in (8.24), also for strange particles, writing for example

$$\mu_i = \frac{g}{\sqrt{2}} (J_\lambda^{(+)} + j_\lambda^{(+)} + S_\lambda^{(+)})(J_\lambda^{(-)} + j_\lambda^{(-)} + S_\lambda^{(-)}). \tag{8.32}$$

Such an expression would be an elegant unification of all weak interactions into one expression of the type (8.25). However, much research will be required before it can really be established whether a form more or less of the type (8.32) might really be possible. We cannot go into a discussion here of the different rules which have been proposed and are under investigation for processes II and IV, we only mention the rules concisely:

(1) no $|\Delta S| = 2$ rule,
(2) $\Delta Q = \Delta S$ rule (leptonic decays),
(3) $|\Delta \vec{T}| = \frac{1}{2}$ rule for non-leptonic decays ($\vec{T}$ is isospin),
(4) $|\Delta \vec{T}| = \frac{1}{2}$ rule for leptonic decays.

Rule No. (3) now seems to have been quite well established by experiments; recent experiments seem to be in contradiction with (2) and (4); rule No. (1) has not yet been extensively tested.

A rule, which seems well established by experiments for processes both conserving and not conserving strangeness is: no four-fermion interactions occur with neutral lepton currents. This means that processes which need a four-fermion interaction, in which a lepton pair with zero total charge necessarily occurs, are strictly forbidden. Examples of such forbidden processes are

$$\mu^- \to e^- + e^+ + e^- , \tag{8.33}$$

$$\mu^- + p \to p + e^- . \tag{8.34}$$

The non-occurrence of these processes has been verified experimentally with great precision. This is an important experimental fact for weak interactions between nucleons and leptons.

8.2. The hypothesis of an intermediate vector boson

It is possible to replace a weak 4-fermion interaction by two "semi-weak" interactions of fermions with an infermediate boson B. In many respects this gives identical results. We specify by means of Feynman diagrams which couplings have to be assumed for the boson B (cf. Figs. 19 and 20).

We note here the following points:

(1) The diagram drawn for β-decay or muon capture is the same as is assumed for the main contribution to the induced pseudoscalar in β-radioactivity or muon capture. However in this case the boson B is the pseudoscalar pion from which there results an effective pseudoscalar coupling.
(2) In order to obtain an effective V-A coupling one has to take a (charged) vector boson for B. If we indicate by B_λ the field oper-

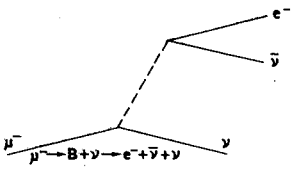


Fig. 19

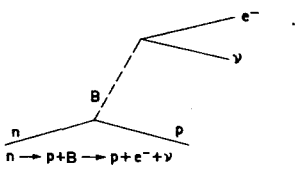


Fig. 20(a)

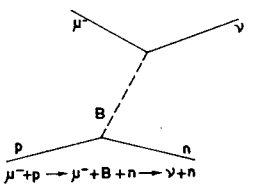


Fig. 20(b)

ator for this particle, the couplings which have to be assumed for B can be written as

$$\left.\begin{aligned} \mathcal{H}_i^{e} &= f_e B_\lambda j_{e,\lambda}^{(+)} + \text{h.c.} \qquad \text{(a)} \\ \mathcal{H}_i^{\mu} &= f_\mu B_\lambda j_{\mu,\lambda}^{(+)} + \text{h.c.} \qquad \text{(b)} \\ \mathcal{H}_i^{N} &= f_N B_\lambda J_\lambda^{(+)} + \text{h.c.} \qquad \text{(c)} \\ \mathcal{H}_i^{S} &= f_S B_\lambda S_\lambda^{(+)} + \text{h.c.} \qquad \text{(d)} \end{aligned}\right\} . \tag{8.35}$$

The effective coupling constants for processes I and II which result from the interactions (8.35) are (m_B = mass of the boson)

$$\left.\begin{aligned} g_B/\sqrt{2} &= f_e f_N / m_B^2 \\ g_{\mu c}/\sqrt{2} &= f_\mu f_N / m_B^2 \\ g_\mu/\sqrt{2} &= f_e f_\mu / m_B^2 \end{aligned}\right\} . \tag{8.36}$$

Because $g_B = g_{\mu c} = g_\mu$, it must be assumed that

$$f_e = f_\mu = f_N .$$

(3) As regards the mass of the boson, it must be assumed that

$$m_B \gtrsim m_K \tag{8.37}$$

in order to explain the absence of the process

$$K^+ \to B^+ + \gamma . \tag{8.38}$$

(4) Possible decay modes of the boson are

$$\begin{array}{ll} B^+ \rightarrow \ell^+ + \nu & (\ell : \text{e or } \mu) \\ \searrow 2\pi & \\ \searrow K^+ + \pi & (\text{if } m_B > m_K + m_\pi). \end{array} \qquad (8.39)$$

One calculates for the decay rate

$$\Lambda(B^+ \rightarrow \mu^+ + \nu) = \Lambda(B^+ \rightarrow e^+ + \nu) = \frac{g_\beta m_B^3}{6\pi\sqrt{2}} > 8 \times 10^{16} \text{ sec}^{-1}. \qquad (8.40)$$

Hence we see that the intermediate boson should be a particle with a very short lifetime ($\lesssim 10^{-17}$ s) as was to be expected from the fact that it decays via a "semi-weak" interaction and not via a "weak" interaction.

(5) When only charged intermediate bosons are assumed, this has the attractive feature that a kind of "explanation" is given for the rule that no neutral lepton currents are found.

(6) The theory with charged intermediate vector boson, is not renormalizable. However, this can scarely be considered as an argument against the existence of such a particle.

Up to now we have mentioned the coupling scheme with an intermediate boson only to the extent that it is equivalent in many ways to a 4-fermion interaction, which is considered as a fundamental interaction. But it is not clear whether the scheme would in any way be more attractive than an elementary 4-fermion interaction. However, the intermediate boson hypothesis was worked out by Lee and Yang and by d'Espagnat in such a way that certain advantages are indeed obtained. They try to formulate such a coupling scheme for the intermediate bosons that rule No.(3): $|\Delta\vec{T}| = \frac{1}{2}$ for non-leptonic decays, results from it. It turns out that this is still possible in various ways and one can use the remaining arbitrariness to incorporate one or more of the other rules (1) to (4) in the scheme (for a detailed description cf.[101, 32]). Lee and Yang propose a scheme, in which the bosons (which they indicate by W and call "schizons") are coupled in the W-J_λ and the W-S_λ interactions in different ways as regards their isospin properties. They postulate 4 bosons W^+, W^-, W_a^0, W_b^0 which can be grouped either in a triplet ($W^- W_a^0 W^+$) and a singlet W_b^0, or in two doublets (W^0, W^+) and (W^-, $\overline{W}^0$) (where $W_a^0 = -\frac{1}{\sqrt{2}}(W^0 + \overline{W}^0)$; $W_b^0 = \frac{1}{\sqrt{2}}(W^0 - \overline{W}^0)$).

One may say that in the proposed coupling scheme the isospin is conserved both in the W-J_λ and in the W-S_λ interaction, but that the boson behaves as an isospin-1-particle in its coupling to J_λ and as an isospin-$\frac{1}{2}$-particle in its coupling to S_λ. This two-fold "schizoid" behaviour has then the $|\Delta\vec{T}| = \frac{1}{2}$ - rule (3) as a result and also the leptonic $|\Delta\vec{T}| = \frac{1}{2}$-rule can be incorporated in the scheme. Also rule (1) (no $|\Delta S| = 2$) can easily be incorporated in this scheme. The rule that no neutral lepton currents occur has to be added in a rather artificial way.

D'Espagnat has postulated in his "veton"-coupling scheme only charged bosons (one or two kinds). This results in neutral lepton currents being forbidden in a natural way, hence the name "vetons". Rules (3) and (4) are incorporated in the scheme either rigorously or in a certain approximation. In one of his proposals the rule that no $|\Delta S| = 2$ processes take place is not valid for leptonic decays, so that for example

$$\Xi^- \to n + e^- + \bar{\nu}$$

should occur with a measurable probability (which cannot, however, be excluded on the basis of the present experimental data). Although many results are the same with or without an intermediate boson, one can of course concentrate on cases in which it makes a difference whether intermediate bosons do or do not exist in order to make a decision on the basis of experiments. Possibilities in this respect exist in the form of reactions in which real bosons B should be produced, for example, by means of pions or neutrinos of high energy. Because of the short lifetime of B ($\lesssim 19^{-17}$ sec) such production processes (occurring with small cross-sections) will not be easily observed: one cannot observe the bosons by means of real "tracks", but one has to make an analysis of the decay products. However, such investigations seem possible with existing techniques.

8.3. γ-decay of the muon

The possible occurrence of γ-decay of the muon is a very fundamental problem. If an intermediate boson B is assumed, this decay is possible up to first order in the weak interaction according to the diagram given

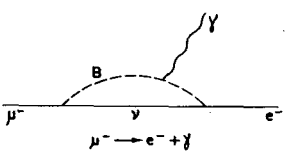


Fig. 21

Fig. 21 diverges; a cut-off $\approx m_B$ gives as an estimate for the branching ratio

$$\frac{\Lambda(\mu \to e + \gamma)}{\Lambda(\mu \to e + \nu + \bar{\nu})} \approx 3 \times 10^{-4} \quad \text{(theoretical estimate with intermediate boson B).} \tag{8.42}$$

Experimentally a very low upper limit was determined for this process (cf. [106] and [107]).

$$\frac{\Lambda(\mu \to e + \gamma)}{\Lambda(\mu \to e + \nu + \bar{\nu})} < 0.6 \times 10^{-7} \quad \text{(experiment).} \tag{8.43}$$

The experimental value (8.43) obviously contradicts the theoretical value (8.42). The theoretical result (8.42) depends strongly on the chosen cut-off, but it seems impossible to obtain agreement with experiment with a reasonable value of the cut-off (cf. MEYER and SALZMAN [108]). In order to maintain the intermediate boson theory the following hypothesis has now been introduced:

Two-neutrino hypothesis: it is assumed that two kinds of neutrinos exist, of which one kind ν_e is connected to the electron and the other kind ν_μ to the muon. One should then write

$$\left.\begin{array}{ll} n \rightarrow p + e^- + \bar{\nu}_e & \text{(a)} \\ \mu^- + p \rightarrow n + \nu_\mu & \text{(b)} \\ \mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu & \text{(c)} \\ \pi^- \rightarrow \mu^- + \bar{\nu}_\mu & \text{(d)} \end{array}\right\} . \qquad (8.44)$$

The intermediate boson should then be coupled both to the pair (e, ν_e) and to the pair (μ, ν_μ). Introducing two kinds of neutrinos in this way means that we have two lepton conservation laws; we may introduce two kinds of lepton numbers

$$\begin{aligned} &\ell_e = +1 \text{ for } e^- \text{ and } \nu_e ; \ \ell_e = -1 \text{ for } e^+ \text{ and } \bar{\nu}_e \\ &\ell_\mu = +1 \text{ for } \mu^- \text{ and } \nu_\mu ; \ \ell_e = -1 \text{ for } \mu^+ \text{ and } \bar{\nu}_\mu . \end{aligned} \qquad (8.45)$$

ℓ_e and ℓ_μ should then be conserved separately. This provides an additional selection rule so that (8.41) is rigorously forbidden; we see that in the diagram (Fig. 21) first emission of ν_μ takes place, but for reabsorption by the electron a ν_e would be required, which is not available.

With regard to their helicity, both neutrinos ν_e and ν_μ are supposed to be identical. The hypothesis that two kinds of neutrinos exist does not seem very attractive at first sight; one would probably prefer one kind of neutrino rejecting the intermediate boson. However, the process $\mu \rightarrow e + \gamma$, also rejecting the intermediate boson, is not necessarily excluded. For example, one obtains a contribution to this process from the following diagram, of second order in the weak interaction.

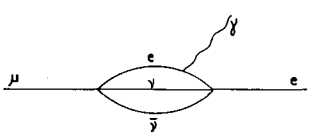


Fig. 22

It is assumed here that an expression of the type (8.24) is valid (cf. JOFFE [109] for quantitative results; if no $e + \nu \rightarrow e + \nu$ existed this diagram would not contribute). The diagram is strongly divergent and the result greatly depends on the cut-off which is assumed. If a cut-off $\Lambda \approx 30$ GeV is assumed the contribution of the last diagram reduces sufficiently to explain (8.43). However, introducing a cut-off resembles introducing a certain "non-locality" of the interaction and this again threatens to cause the process $\mu \rightarrow e + \gamma$ (up to the first order in the weak interaction, as for the intermediate boson). Hence it turns out that it is not easy to forbid effectively the process $\mu \rightarrow e + \gamma$ even without supposing an intermediate boson. Thus it may be attractive to introduce the two-neutrino hypothesis, causing an additional selection rule, even when not assuming an intermediate boson.

The electron and muon are two fermions, which seem very analogous in their behaviour in weak interactions as well as in their electromagnetic behaviour (e.g. magnetic moment); only their masses are different. It is tempting to suppose that the existence of two charged leptons, which are

thus analogous, also corresponds to the existence of two strongly analogous neutral leptons ν_e and ν_μ.

Of course these arguments are purely theoretical speculation. It is of the utmost importance to try to find experimental checks of the two-neutrino hypothesis. This seems possible by means of an experiment with neutrinos of a high energy in the following way (we write the processes as if two kinds of neutrinos existed).

Neutrinos of a high energy are obtained from the decay of very energetic pions (which are produced in proton-nucleon collisions)

$$\pi^+ \rightarrow \mu^+ + \nu_\mu \quad . \tag{8.46}$$

One then looks for processes of the following kind when the neutrinos of very high energy fall on nuclei

$$\nu_\mu + n \rightarrow p + \mu^- \tag{8.47}$$

$$\nu_\mu + n \not\rightarrow p + e^- \quad . \tag{8.48}$$

When two kinds of neutrino exist, one should find only process (8.47) (of course the energy of the neutrinos should be sufficient for muon production). In case only one kind of neutrino exists one should also find process (8.48).

The practical realization of this very fundamental neutrino-experiment is under way at CERN and in Brookhaven. The pions are first produced by means of protons in a 30 GeV proton synchrotron. The difficulty is the very small cross-section. This means that only very few events can be expected, of the order of one event per ton of matter per day. For the experiment one has, e.g., a propane bubble chamber with about a ton of matter, while also a spark chamber with 10 tons of iron plates could be a very useful instrument. Although the experiment is very exacting, it can already provide a decision on the very fundamental issue of whether the two-neutrino hypothesis is correct or not without reaching a high accuracy. Recently (July 1962), the existence of two types of neutrino has been established in the Brookhaven National Laboratory (see Phys. Rev. Lett. 9 (1962) 36 and Proc. 1962 High Energy Phys. Conf. at CERN).

8.4. Experiment Brookhaven

In this experiment 10 tons of material were used in the detector, consisting of a combination of 10 spark chambers of 1 ton each. Extensive shielding consisting of concrete and iron was essential. Anticoincidence counters were used to suppress the cosmic-ray background. It was essential in this respect to have a gating system which makes the detector sensitive only when the protons arrive at the target; the protons are circulating in 12 bunches per pulse, each 20 μsec long separated by 220 μsec; the machine produces 2×10^{11} protons per second; in the experiment 3.5×10^7 protons came to the target.

The spark chamber photographs were examined for muon tracks of high energy ($p_\mu > 300$ MeV/c) produced according to (8.47), and for showers arising from electrons, in case reaction (8.48) occurred.

After correcting for the cosmic-ray background, 29 events were obtained for muons produced according to (8.47) while no events due to the process (8.48) could be observed. The observed cross-section for (8.47) was in agreement with the value predicted by theory.

No evidence was found for production of intermediate bosons according to

$$\nu + Z \rightarrow \mu^- + B^+ + Z, \tag{8.49}$$

and subsequent decay of B^+ according to (8.39). Of course, this does not yet prove the non-existence of intermediate bosons either.

Except for their importance with regard to the two neutrino hypothesis, high energy neutrino experiments are very important in the following respects:

(1) As was discussed in connection with the matrix elements (7.12) the "coupling constants" are as a matter of fact "form factors" which can depend on the 4-momentum transfer.

In the normal processes of β-radioactivity, muon capture or muon decay the 4-momentum transfer is nearly the same for all processes which are studied. However, the neutrino reactions provide the possibility of studying the behaviour at different values of the 4-momentum transfer.

(2) For sufficiently high neutrino energies one should expect the production of real bosons B if the intermediate boson hypothesis were correct. Although we have already mentioned that their lifetime is too short to give visible tracks in bubble chambers, for example, one would still expect observable consequences, e.g. in cross-sections of such boson production.

It is clear that high-energy neutrino physics will become a field of great importance. Because of the small cross-sections the experimental difficulties are great and precise values for cross-sections will become available probably only very slowly. It is relevant in this connection to stress that the processes (8.47) and (8.48) will have to be studied in complex nuclei (pure neutron samples not being available for this purpose). This means that nuclear complications again appear in the analysis of these processes. For example, the influence of the Pauli principle should be studied carefully for the reactions (8.47) and (8.48) occurring in nuclei. Berman has made some calculations on this point. (cf. [102]).

APPENDIX I

NOTATIONS FOR THE DIRAC EQUATION AND THE DIRAC MATRICES

As various notations are used for the Dirac equation by different authors, we specify here the notation which we use. We also give explicit representations for the Dirac matrices which are used in some cases.

We may use the Dirac equation in the form

$$[\, E/c + \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \alpha_4 mc \,]\, \psi(x, y, z, t) = 0 \tag{A.1}$$

or

$$[\, E/c + \vec{\alpha} \cdot \vec{p} + \alpha_4 mc \,]\, \psi = 0 \tag{A.2}$$

E and p are the operators

$$E = -(\hbar/i)\, \partial/\partial t; \quad p_x = (\hbar/i)(\partial/\partial x); \tag{A.3}$$

$\alpha_1, \ldots, \alpha_4$ are (4×4) matrices satisfying

$$\alpha_i^2 = 1 \tag{A.4}$$

$$\alpha_i \alpha_j = -\alpha_j \alpha_i \qquad (i \neq j). \tag{A.5}$$

The α's may be chosen as follows

$$\alpha_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} ; \quad \alpha_2 = \begin{pmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{pmatrix} ; \quad \alpha_3 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} ; \quad \alpha_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{A.6}$$

(elements which are not written are 0). We shall often write β instead of α_4. We note that these α's are Hermitian. The wave function has 4 components and thus summarizes 4 functions of x, y, z, t:

$$\psi = \begin{pmatrix} \psi_1(x, y, z, t) \\ \psi_2(x, y, z, t) \\ \psi_3(x, y, z, t) \\ \psi_4(x, y, z, t) \end{pmatrix} \tag{A.7}$$

We may also write the Dirac equation in components $\psi_1 \ldots \psi_4$. Using the representation (A. 6) for the α's we obtain

$$\left.\begin{aligned} (E/c + mc)\,\psi_1 + (p_x - ip_y)\,\psi_4 + p_z\,\psi_3 &= 0 \\ (E/c + mc)\,\psi_2 + (p_x + ip_y)\,\psi_3 - p_z\,\psi_4 &= 0 \\ (E/c - mc)\,\psi_3 + (p_x - ip_y)\,\psi_2 + p_z\,\psi_1 &= 0 \\ (E/c - mc)\,\psi_4 + (p_x + ip_y)\,\psi_1 - p_z\,\psi_2 &= 0. \end{aligned}\right\} \tag{A.8}$$

An alternative form for the Dirac equation is

$$(E/c + \rho_1\, \vec{\sigma} \cdot \vec{p} + \rho_3 mc)\,\psi = 0. \tag{A.9}$$

The ρ's and σ's are related in the following way to the α's

$$\left.\begin{aligned} \alpha_k &= \rho_1 \sigma_k \qquad (k = 1, 2, 3) \\ \alpha_4 &= \rho_3. \end{aligned}\right\} \tag{A.10}$$

The ρ's and σ's are represented by the following matrices

$$\rho_1 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} ; \quad \rho_2 = \begin{pmatrix} & & -i & \\ & & & -i \\ i & & & \\ & i & & \end{pmatrix} ; \quad \rho_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{A.11}$$

$$\sigma_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} ; \quad \sigma_2 = \begin{pmatrix} & -i & & \\ i & & & \\ & & & -i \\ & & i & \end{pmatrix} ; \quad \sigma_3 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \tag{A.12}$$

The matrices ρ and σ satisfy the following relations

$$\rho_1^2 = 1;\ \rho_i \rho_j + \rho_j \rho_i = 0 \ (i \neq j);\ \rho_1 \rho_2 = i\rho_3 \quad \text{(and cyclic)}, \tag{A.13}$$

$$\sigma_1^2 = 1;\ \sigma_i \sigma_j + \sigma_j \sigma_i = 0 \ (i \neq j);\ \sigma_1 \sigma_2 = i\sigma_3 \quad \text{(and cyclic)}; \tag{A.14}$$

every matrix ρ commutes with every matrix σ. (A.15)

The form (A. 1) is not the most convenient one, if one wants to consider the relativistic transformation properties. With this in view we write

$$p_\mu = (\hbar/i)(\partial/\partial x_\mu) \quad (x_4 = ict)\,. \tag{A.16}$$

We multiply (A. 1) to the left by $i\alpha_4$ and obtain

$$(i\alpha_4\alpha_1 p_x + i\alpha_4\alpha_2 p_y + i\alpha_4\alpha_3 p_z + \alpha_4 \frac{i}{c} E + imc)\psi = 0 \tag{A.17}$$

or putting (in the following, summation over repeated Greek indices from 1 to 4 is understood)

$$(\gamma_\mu p_\mu - imc)\psi = 0\,, \tag{A.18}$$

putting

$$\begin{aligned} &\gamma_k = -i\alpha_4\alpha_k \quad (k=1,2,3); && \alpha_k = -i\gamma_4\gamma_k \\ &\gamma_4 = -\alpha_4 && \alpha_4 = -\gamma_4. \end{aligned} \tag{A.19}$$

The γ matrices which are so defined are hermitian as well as the α's. We shall use the Dirac equation mostly in the form (A. 18), or writing p_μ explicitly according to (A. 16)

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar})\,\psi = 0. \tag{A.20}$$

The γ_μ's satisfy the commutation relations

$$\tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \delta_{\mu\nu}\,. \tag{A.21}$$

Taking the complex conjugate of (A. 18) and changing the order of the ψ^*'s and the γ's one obtains

$$\sum_{k=1,2,3} \frac{\partial\psi^*}{\partial x_k}\gamma_k - \frac{\partial\psi^*}{\partial x_4}\gamma_4 + \frac{mc}{\hbar}\psi^* = 0. \tag{A.22}$$

Putting

$$\bar{\psi} = \psi^*\gamma_4 \tag{A.23}$$

(A. 20) takes the following form after multiplication to the right by γ_4 (adjoint Dirac equation)

$$\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu - \frac{mc}{\hbar}\bar{\psi} = 0\,. \tag{A.24}$$

A Lorentz transformation is specified by

$$\left.\begin{aligned} x'_\mu &= a_{\mu\nu}x_\nu \\ x_\nu &= a_{\mu\nu}x'_\mu \end{aligned}\right\} \tag{A.25}$$

where

$$\left.\begin{aligned} a_{\mu\ell}\,a_{\mu\sigma} &= \delta_{\ell\sigma} \\ a_{\ell\mu}\,a_{\sigma\mu} &= \delta_{\ell\sigma} \end{aligned}\right\}. \tag{A.26}$$

It follows from (A. 25) that

$$\frac{\partial}{\partial x_\mu} = a_{\mu\nu} \frac{\partial}{\partial x_\nu} . \tag{A. 27}$$

We shall write (A. 25) in an abbreviated form as

$$x' = L\, x$$
$$x = L^{-1} x' . \tag{A. 28}$$

The Dirac equation and its adjoint are invariant for Lorentz transformations; this means that the equation has the same form in x and x'-coordinates:

$$(\gamma_\mu \cdot (\partial/\partial x_\mu) + (mc/\hbar))\psi(x) = 0 \tag{A. 29}$$

$$(\gamma_\mu \cdot (\partial/\partial x_\mu) + (mc/\hbar))\psi'(x') = 0. \tag{A. 30}$$

ψ' and ψ are related for coordinates related according to (A. 28) by

$$\psi'(x') = S\psi\,(L^{-1} x') = S\psi\,(x) . \tag{A. 31}$$

S is a (4 x 4) matrix satisfying

$$S^{-1} \gamma_\mu S = a_{\mu\nu} \gamma_\nu . \tag{A. 32}$$

$\bar{\psi}$ transforms according to

$$\bar{\psi}'(x') = \bar{\psi}(L^{-1}x') . S^{-1} = \bar{\psi}(x)\, S^{-1} \tag{A. 33}$$

S has also to satisfy

$$S^{-1} = \gamma_4 S^{+} \gamma_4 . \tag{A. 34}$$

It is often useful to introduce γ_5 according to

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 . \tag{A. 35}$$

We note that for γ_5

$$(\gamma_5)^2 = 1, \qquad \gamma_5 \gamma_\mu + \gamma_\mu \gamma_5 = 0. \tag{A. 36}$$

Hence we see that (A. 21) remains valid for $\mu, \nu = 1, \ldots, 5$ when introducing the matrix γ_5.

The covariants of the Dirac theory

If ψ_1 and ψ_2 are Dirac wave functions transforming according to (A. 31), (A. 33) it is easily proved that the following quantities have the indicated transformation properties ((A. 32) is used for this purpose) :

S scalar	$\bar{\psi}_1 \psi_2$	(A. 37)
V vector	$\bar{\psi}_1 \gamma_\mu \psi_2$	(A. 38)
T tensor (antisymmetric)	$\bar{\psi}_1 \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \psi_2$	(A. 39)
A axial vector (pseudovector)	$\bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2$	(A. 40)
P pseudoscalar	$\bar{\psi}_1 \gamma_5 \psi_2$	(A. 41)

The relations between our different notations of Dirac matrices are given by

$$\alpha_k = \rho_1\sigma_k\,; \quad \alpha_4 = \rho_3 (= \beta) \tag{A.42}$$

$$\alpha_k = -i\gamma_4\gamma_k\,; \quad \alpha_4 = -\gamma_4 \tag{A.43}$$

$$\gamma_k = -i\alpha_4\alpha_k\,; \quad \gamma_4 = -\alpha_4 \tag{A.44}$$

$$\left.\begin{aligned}\gamma_k &= \rho_2\sigma_k \\ \gamma_4 &= -\rho_3\end{aligned}\right\} \tag{A.45}$$

We specify the 16 independent Dirac matrices, used in the representation, in the following table (these matrices are all Hermitian).

Table of the Dirac matrices

$$\beta = \rho_3 = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0 \\ & & 0 & -1 \end{pmatrix} \qquad 1 = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$$

$$\alpha_1 = \rho_1\sigma_1 = \begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ 0 & 1 & & \\ 1 & 0 & & \end{pmatrix}; \quad \alpha_2 = \rho_1\sigma_2 = \begin{pmatrix} & & 0 & -i \\ & & i & 0 \\ 0 & -i & & \\ i & 0 & & \end{pmatrix}; \quad \alpha_3 = \rho_1\sigma_3 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & -1 & & \end{pmatrix}$$

$$\beta\sigma_1 = \rho_3\sigma_1 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix}; \quad \beta\sigma_2 = \rho_3\sigma_2 = \begin{pmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & i \\ & & -i & 0 \end{pmatrix}; \quad \beta\sigma_3 = \rho_3\sigma_3 = \begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & -1 & 0 \\ & & 0 & 1 \end{pmatrix}$$

$$i\beta\alpha_1 = -\rho_2\sigma_1 = \begin{pmatrix} & & 0 & i \\ & & i & 0 \\ 0 & -i & & \\ -i & 0 & & \end{pmatrix}; \quad i\beta\alpha_2 = -\rho_2\sigma_2 = \begin{pmatrix} & & 0 & 1 \\ & & -1 & 0 \\ 0 & 1 & & \\ 1 & 0 & & \end{pmatrix}; \quad i\beta\alpha_3 = -\rho_2\sigma_3 = \begin{pmatrix} & & i & 0 \\ & & 0 & -i \\ -i & 0 & & \\ 0 & i & & \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix}$$

$$\gamma_5 = \rho_1 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix} \qquad i\beta\gamma_5 = -\rho_2 = \begin{pmatrix} & & i & 0 \\ & & 0 & i \\ -i & 0 & & \\ 0 & -i & & \end{pmatrix}$$

APPENDIX II

THE USE OF FIELD OPERATORS OR WAVE FUNCTIONS IN THE FORMULATION OF FOUR-FERMION INTERACTIONS

It is often not stated clearly whether field operators or wave functions are meant, when formulating a four-fermion interaction. We want to state here precisely how both formulations are related in order to avoid confusion.

In this appendix $\psi(\vec{x}, t)$ will mean the operator for the quantized fermion field. We can write down the following Fourier development for ψ and ψ^* (an asterisk with the field operator indicates the hermetian conjugate operator)

$$\psi(\vec{x}, t) = \frac{1}{\sqrt{V}} \cdot \sum_r \sum_{i=1,2} \left[a_i(\vec{p_r}) \cdot u_i(\vec{p_r}) \exp\{i(\vec{p_r}\cdot\vec{x} - E_r t)\} + b_i^*(\vec{p_r}) \cdot v_i(\vec{p_r}) \exp\{-i(\vec{p_r}\cdot\vec{x} - E_r t)\} \right] \tag{A.46}$$

$$\psi^*(\vec{x}, t) = \frac{1}{\sqrt{V}} \cdot \sum_r \sum_{i=1,2} \left[a_i^*(\vec{p_r}) \cdot u_i^*(\vec{p_r}) \exp\{-i(\vec{p_r}\cdot\vec{x} - E_r t)\} + b_i(\vec{p_r}) \cdot v_i^*(-\vec{p_r}) \exp\{i(\vec{p_r}\cdot\vec{x} - E_r t)\} \right] \tag{A.47}$$

$a_i(\vec{p_r})$, $a^*_i(\vec{p_r})$, $b_i(\vec{p_r})$, $b^*_i(\vec{p_r})$ are operators,

$a^*_i(\vec{p_r})$ is the creation operator for a negaton in the state (i, p_r),

$a_i(\vec{p_r})$ is the annihilation operator for a negaton in the state (i, p_r),

$b^*_i(\vec{p_r})$ is the creation operator for a positon in the state (i, p_r),

$b_i(\vec{p_r})$ is the annihilation operator for a positon in the state (i, p_r),

$u_i(\vec{p_r})$ and $v_i(-\vec{p_r})$ are spinors (not operators but c-numbers),

i indicates the state of polarization; p specifies the momentum. We have considered a cube with volume $V = L^3$ as normalization volume, taking periodic boundary conditions. The possible momentum eigenstates can then be specified as $\vec{p_r}$, where r is a discrete index. We shall often indicate a fermion state by one index s, which summarizes $(i, \vec{p_r})$.

In the expression for a four-fermion interaction, e.g. the β-interaction, the ψ's can be considered as field operators

$$\mathcal{H}_\beta(x) = g_\beta [(\psi^*_p \Omega \psi_n)(\psi^*_e \Omega \psi_\nu) + (\psi^*_n \Omega^+ \psi_p)(\psi^*_\nu \Omega^+ \psi_e)] . \qquad (A.48)$$

The initial and final states may be characterized by the numbers n_s^+ = number of fermions in the state s and n_s^- = number of antifermions in the state s. The field operators ψ and ψ^* have the following matrix elements for transitions between states with zero and one particle

$$\langle n_s^+ = 0 | \psi(\vec{x}) | n_s^+ = 1 \rangle = \frac{1}{\sqrt{V}} u_i(\vec{p_r}) \exp\{i(\vec{p_r}\cdot\vec{x} - E_r t)\} \qquad (a)$$

$$\langle n_s^- = 1 | \psi(\vec{x}) | n_s^- = 0 \rangle = \frac{1}{\sqrt{V}} v_i(-\vec{p_r}) \exp\{-i(\vec{p_r}\cdot\vec{x} - E_r t)\} \qquad (b)$$

$$\langle n_s^- = 0 | \psi^*(\vec{x}) | n_s^- = 1 \rangle = \frac{1}{\sqrt{V}} v^*_i(-\vec{p_r}) \exp\{i(\vec{p_r}\cdot\vec{x} - E_r t)\} \qquad (c) \qquad (A.49)$$

$$\langle n_s^+ = 1 | \psi^*(\vec{x}) | n_s^+ = 0 \rangle = \frac{1}{\sqrt{V}} u^*_i(\vec{p_r}) \exp\{-i(\vec{p_r}\cdot\vec{x} - E_r t)\} \qquad (d)$$

(all other matrix elements between states with zero and one particle are zero).

In the right-hand terms of (A.49) we find:

(a) the one-particle wave function of the absorbed negaton,

(d) the complex conjugate of the one-particle wave function of the emitted negaton,

(b) the 'negative energy' one-particle wave function corresponding to the emitted positon and

(c) the complex conjugate of this positon wave function.

Using (A.49) one obtains immediately the matrix element of $H_\beta = \int d^3 \vec{x} \mathcal{H}_\beta(\vec{x})$ (where $\mathcal{H}_\beta(\vec{x})$ is specified by (A.48)) for a process of β^--emission $n \to p + e^- + \bar{\nu}$ (all particles in plane wave states). The matrix element obtained corresponds exactly to the first term of (A.48), where the ψ's have now to be taken as one-particle wave functions. Analogously one obtains the matrix element for β^+-emission: $p \to n + e^+ + \nu$. This corresponds to the second term of (A.48), again taking the ψ's as one-particle wave functions. In this way we have specified in what sense the ψs in (A.48) can be taken to be either field operators or one-particle wave functions. Generalizations to particles with more complicated wave functions are possible.

APPENDIX III

THE PROJECTION OPERATORS D
(cf. (3.14), (3.16) and (3.17)).

A general plane wave solution with momentum p may be written as

$$\psi = C_1\psi^{(1)} + C_2\psi^{(2)} + C_3\psi^{(3)} + C_4\psi^{(4)} \qquad (A.50)$$

positive energy solution — negative energy solution

The indices (1) and (2) refer to two opposite spin states; also (3) and (4) define opposite spin states. It is easily shown that the four states $\psi^{(k)}$ are mutually orthogonal

$$\psi^{(k)*}\,\psi^{(\ell)} = \delta_{k\ell}. \tag{A.51}$$

It now follows that D^+ according to (3.14) is the projection operator into the (2-dimensional) space of positive energy solutions; according to (3.14) and (A.51) one has

$$(D^+\psi)_\mu = \Sigma_\rho D^+_{\mu\rho}\ \psi_\rho = \Sigma_\rho \Sigma_{k=1,2}(\psi^{(k)}_\mu \psi^{(k)*}_\rho)(\Sigma_{i=1}^{4} C_i \psi^{(i)}_\rho) = \Sigma_{k=1,2} C_k \psi^{(k)}_\mu. \tag{A.52}$$

This expresses that D^+ projects out the positive energy part of ψ. Now it is easily seen that this (uniquely determined) projection operator can be written as

$$D^+ = \frac{H + |E|}{2|E|}. \tag{A.53}$$

When acting on a positive energy solution $\psi^{(+)}$ ($H\psi^{(+)} = E\psi^{(+)}$) one has $D^+\psi^{(+)} = \psi^{(+)}$, while for a negative energy solution $\psi^{(-)}$ ($H\psi^{(-)} = -E\psi^{(-)}$) one has $D^+\psi^{(-)} = 0$. It follows from (A.1) and (A.53) that D^+ can be written as

$$D^+ = \tfrac{1}{2}\left[1 - \frac{\vec{\alpha}\cdot\vec{p} + \beta m}{E}\right], \tag{A.54}$$

where $\vec{p}$ and E now have to be considered as numbers (not operators).

In an entirely analogous way one sees that the projection operator for negative energies has to be written as

$$D^- = \frac{-H + |E|}{2|E|}. \tag{A.55}$$

When we consider the projection operator for negative energy -E and momentum $-\vec{p}$ (absorption of a particle in this state corresponds to creation of the anti-particle with energy E and momentum $\vec{p}$) we obtain

$$D^- = \tfrac{1}{2}\left[1 - \frac{\vec{\alpha}\cdot\vec{p} - \beta m}{E}\right] \tag{A.56}$$

($\vec{p}$ and E are again numbers in this formula).

APPENDIX IV

THE ISOSPIN OF NUCLEON AND PION

One of the reasons that the concept of the isospin is so useful is the fact that the nuclear forces (i.e. strong interaction, neglecting electromagnetic and weak interaction) are charge independent. There is an analogy with the ordinary spin in the magnetic field. As long as the field is not present, the two spin states cannot be distinguished. We shall not go into detail. We just want to specify our notation for systems with nucleons and pions.

We introduce a specific "isotopic spin" or charge wave function; this is a two-component spinor in the case when we have a nucleon

$$\eta_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \eta_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.57}$$

The nucleon wave function (or operator) is written as

$$\psi_N = \psi_p\,\eta_p + \psi_n\,\eta_n = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

In analogy with the ordinary spin we introduce the isopsin operators;

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.58}$$

and

$$\tau^{(+)} = \tfrac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\tau^{(-)} = \tfrac{1}{2}(\tau_1 - i\tau_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{A.59}$$

with the properties

$$\begin{aligned} \tau_3\,\eta_p &= \eta_p & \tau_3\,\eta_n &= -\eta_n \\ \tau^{(+)}\eta_n &= \eta_p & \tau^{(-)}\eta_p &= \eta_n \end{aligned} \tag{A.60}$$

The isopsin is defined by $\frac{1}{2}\,\tau$, from which it follows that the isospin of a nucleon is $\frac{1}{2}$. The pions occur in a triplet of charges $\pi^{\pm}$, π^0. We attribute to them an isospin $T = 1$. The T spin operators are defined by

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{A.61}$$

$$\eta_{\pi^+} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \qquad \eta_{\pi^0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \qquad \eta_{\pi^-} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T^{(\pm)} = T_1 \pm iT_2 .$$

Generally we have

$$\varphi_\pi = \varphi_\pi\,\eta_\pi + \varphi_{\pi^0}\,\eta_{\pi^0} + \varphi_{\pi^-}\,\eta_{\pi^-} = \begin{pmatrix} \varphi_{\pi^+} \\ \varphi_{\pi^0} \\ \varphi_{\pi^-} \end{pmatrix}$$

The isospin of a system of nucleons and pions is denoted by T. We have a relation between the total charge Q and the third component of the total isospin of a system of pions and nucleons

$$Q = T_3 + \tfrac{1}{2}N \tag{A.62}$$

where N is the number of nucleons.

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