Maxwell-Klein-Gordon system with nontrivial coupling on four dimensional Minkowski spacetime

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Abstract. In this paper, we study about Maxwell-Klein-Gordon system with non-trivial coupling in four dimensional Minkowski spacetime with potential turned on. We start from Lagrangian of non-trivially coupled Maxwell-Klein-Gordon, then we derive the equations of motion and energy of the system. The coupling and potential function is chosen such that the Lagrangian is gauge invariant. Using Coulomb gauge and conservation of energy, we prove some inequality for energy which will be important to proving the existence of solution.

1. Maxwell-Klein-Gordon System with Nontrivial Coupling

In this section, we will write the Lagrangian for four dimensional Maxwell-Klein-Gordon system with nontrivial coupling.

Our spacetime is a four dimensional Lorentzian manifold, M^4 with standard coordinate, x^μ, $\mu = 0, 1, 2, 3$ and equipped by Minkowski metric denoted by $\eta_{\mu\nu} = \text{diag}\{-1, 1, 1, 1\}$. As usual, we split the spacetime coordinates x^μ into time coordinate $x^0 = t$ and space coordinates $x = (x^i)_{i=1,2,3}$.

The Maxwell-Klein-Gordon system in four dimensions consist of multiplet (ϕ, A_μ) where ϕ is the complex valued scalar field and A_μ is the real vector valued Abelian gauge field. The Lagrangian of Maxwell-Klein-Gordon is given by,

$$\mathcal{L} = -\frac{1}{2} D_\mu \phi D^\mu \bar{\phi} - \frac{1}{4} h(\phi, \bar{\phi}) F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} k(\phi, \bar{\phi}) F_{\mu\nu} \tilde{F}^{\mu\nu} - V(\phi, \bar{\phi}) \,, \tag{1}$$

where $D_\mu \phi$ is a gauge covariant derivative, $F_{\mu\nu}$ is the Abelian field strength tensor, $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ denotes the Hodge dual of field strength with $\epsilon_{\mu\nu\rho\sigma}$ is the standard Levi-Civita tensor. Relative to A_μ, we have

$$
\begin{aligned}
D_\mu \phi &= \partial_\mu + \mathrm{i} A_\mu \phi \,, \tag{2}\\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{3}
\end{aligned}
$$

We also have the Bianchi identity,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \,, \tag{4}$$

which equivalent with Maxwell homogeneous equations.

We split the field strength tensor $F_{\mu\nu}$ into electric and magnetic fields,

$$E_i = F_{0i}, \qquad B_i = \tilde{F}_{0i} = \frac{1}{2}\epsilon_{ijk}F^{jk} \ . \tag{5}$$

We also decompose the gauge fields A_μ into temporal component A_0 and spatial components A_i. For given function $\phi(t,x)$, we denote the spatial gradient by $\nabla\phi = (\partial_i\phi)_{i=1,2,3}$ and the spacetime gradient by $\partial\phi = (\partial_0, \nabla\phi)$. We also denote the wave operator (or D'Alembertian operator) by $\Box = -\partial_t^2 + \triangle = -\partial_t^2 + \partial_i\partial^i$.

2. Gauge Invariant Conditions

The coupling between scalar and Maxwell fields are determined by three scalar dependent real functions (h, k, V). The non trivial coupling is introduced by a real functions $h(\phi, \bar{\phi})$ and $k(\phi, \bar{\phi})$. This form of nontrivial coupling function is motivated by the coupling in $N = 1$ global supesymmetric gauge theory in four dimensions [1, 2, 3]. The real function $V(\phi, \bar{\phi})$ denotes the scalar potential which we assume to be non-negative, smooth and satisfy the following conditions,

(i) $V(0) = 0$,

(ii) $\partial_\phi V(0) = 0$.

The conditions above tell that the lowest order must be at least a quadratic term which correspond to mass term. The conditions are satisfied by several known scalar potential, for examples, the ϕ^4 theory and the sine-Gordon potential. We also impose that Lagrangian (1) invariant with respect to local $U(1)$ transformation. Hence we have,

$$\begin{aligned} h(U\phi, \bar{U}\bar{\phi}) &= h(\phi, \bar{\phi}) \\ k(U\phi, \bar{U}\bar{\phi}) &= k(\phi, \bar{\phi}) \\ V(U\phi, \bar{U}\bar{\phi}) &= V(\phi, \bar{\phi}) \end{aligned} \tag{6}$$

where $U(x) = e^{-\mathrm{i}\lambda(x)} \in U(1)$. Based on discussion above, for the sake of simplicity, we take the simple form which the functions are only depend on $|\phi| = \sqrt{\phi\bar{\phi}}$,

$$\begin{aligned} h(\phi, \bar{\phi}) &= h(|\phi|) \\ k(U\phi, \bar{\phi}) &= k(|\phi|) \\ V(U\phi, \bar{\phi}) &= V(|\phi|^2) \end{aligned} \tag{7}$$

The corresponding equations of motion are

$$\begin{aligned} D_\mu D^\mu \phi &= \frac{1}{2}F^{\mu\nu}\partial_\phi \mathcal{G}_{\mu\nu} + 2\partial_\phi V \end{aligned} \tag{8}$$

$$\begin{aligned} \partial_\mu \mathcal{G}^{\mu\nu} &= -\mathrm{Im}\left(\phi D^\nu \bar{\phi}\right) \ , \end{aligned} \tag{9}$$

where $\mathcal{G}_{\mu\nu} = hF_{\mu\nu} - k\tilde{F}_{\mu\nu}$.

Since we impose the gauge invariant condition to Lagrangian (1), then we have a freedom to choice a particular gauge condition. In this paper, we construct solutions of Maxwell-Klein-Gordon with additional condition, namely Coulomb gauge,

$$\partial_i A^i = 0 \ . \tag{10}$$

The Coulomb gauge condition implies that the gauge field are uniquely determined from the field strength tensor $F_{\mu\nu}$ by solving elliptic equations [4],

$$\begin{aligned} \triangle A_0 &= \partial^i F_{i0} \ , \end{aligned} \tag{11}$$

$$\begin{aligned} \triangle A_j &= \partial^i F_{ij} \ . \end{aligned} \tag{12}$$

In particular we have the following,

Preposition 1. *Let $F_{\mu\nu}$ is a field strength tensor defined by (3). Given $F_{\mu\nu} \in L^2(\mathbb{R}^3)$, there exists a unique A_μ belongs to homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$ such that (10) holds. In particular, we have*

$$\sum_{i,\mu} \|\partial_i A_\mu(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq \|E(t,\cdot)\|_{L^2(\mathbb{R}^3)} + \|B(t,\cdot)\|_{L^2(\mathbb{R}^3)} . \tag{13}$$

Proof. In Coulomb gauge, the field A_i is pure transversal. Then, from (3), we can express A_i as,

$$A_i = \int_{\mathbb{R}^3} \epsilon_{ijk} K^j(x-y)\, B^k(y)\, dy , \tag{14}$$

where $K^j(x-y) = -\frac{1}{4\pi} \frac{x^j - y^j}{|x-y|^3}$. Taking the Fourier transform of $\partial_j A_i$ and using Fubini theorem, we get

$$\widehat{(\partial_j A_i)} = \frac{k_j k_l}{|k|^2} \epsilon_{ilk} \hat{B}^k . \tag{15}$$

Thus, from Plancherel's theorem, we have

$$\|\partial_j A_i\|_{L^2(\mathbb{R}^3)} \leq \|B\|_{L^2(\mathbb{R}^3)} \tag{16}$$

Since curl grad = 0 and from (3) and (10), we conclude that $\partial_i A_0$ is a longitudinal components of electric fields E^i. Thus, we have

$$\partial_i A_0 = \int_{\mathbb{R}^3} \delta_{jk} \partial_i K^j(x-y)\, E^k(y)\, dy , \tag{17}$$

Using similar step as before, we have

$$\|\partial_i A_0\|_{L^2(\mathbb{R}^3)} \leq \|E\|_{L^2(\mathbb{R}^3)} \tag{18}$$

and complete the proof.

$\square$

3. Energy Inequality

The energy-momentum tensor for Lagrangian (1) is given by,

$$T_{\mu\nu} = -\frac{1}{2}\eta_{\mu\nu} D_\rho \phi D^\rho \bar{\phi} + \mathrm{Re}\left(D_\mu \phi D_\nu \bar{\phi}\right) + \frac{1}{2}\left(\mathcal{G}_{\mu\rho} F_\nu^\beta + \tilde{\mathcal{G}}_{\mu\rho} \tilde{F}_\nu^\beta\right) - \eta_{\mu\nu} V , \tag{19}$$

where $\tilde{\mathcal{G}}_{\mu\rho}$ is the Hodge dual of $\mathcal{G}_{\mu\nu}$. The conservation laws are encoded in equation,

$$\partial_\mu T^{\mu\nu} = 0 . \tag{20}$$

In particular, we have conservation of total energy,

$$\mathcal{H}(t) = \mathcal{H}(0) , \tag{21}$$

where the total energy at time t is given by,

$$\begin{aligned}
\mathcal{H}(t) &= \int_{\mathbb{R}^3} T^{00}(t,x)\, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^3} \left\{ |D_0\phi|^2 + \sum_{i=1}^3 |D_i\phi|^2 + h\left(|E|^2 + |B|^2\right) \right\} dx
\end{aligned} \tag{22}$$

In this paper, we construct the global solution of non-trivial coupled Maxwell-Klein-Gordon system in Coulomb gauge by assuming that the initial data, $\phi(0,\cdot)$, $\partial_t\phi(0,\cdot)$, $E(0,\cdot)$ and $B(0,\cdot)$ satisfy finite energy condition,

$$\mathcal{H}(0) < \infty \,, \tag{23}$$

Based on (13), we define an approximate energy norm,

$$\mathcal{J}(A_0, A, \phi)(t) = \sum_{\mu,\nu} \|h^{\frac{1}{2}}(\phi(t,\cdot))\partial_\mu A_\nu(t,\cdot)\|_{L^2(\mathbb{R}^3)} + \|\phi(t,\cdot)\|_{L^2(\mathbb{R}^3)} + \sum_{\mu} \|\partial_\mu\phi(t,\cdot)\|_{L^2(\mathbb{R}^3)} \,. \tag{24}$$

Then we have the following,

Preposition 2. *Let ϕ, A_0 and A be a smooth solution of Maxwell-Klein-Gordon system given by (8)-(9) in the Coulomb gauge with*

$$\mathcal{J}_0 = \mathcal{J}(0) < \infty \,, \tag{25}$$

Then, for all $t \geq 0$ and some positive constant C depends only on $\mathcal{J}_0$, we have

$$\mathcal{J}(t) \leq C(1 + t) \,. \tag{26}$$

Proof. Clearly, $\mathcal{H}(0) = \mathcal{H}(t) \leq \mathcal{J}_0^2$. From the proof of preposition 1, we get

$$\sum_{i,\mu} \|h^{\frac{1}{2}}(\phi(t,\cdot))\partial_i A_\mu(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq \|h^{\frac{1}{2}}(\phi(t,\cdot))E(t,\cdot)\|_{L^2(\mathbb{R}^3)} + \|h^{\frac{1}{2}}(\phi(t,\cdot))B(t,\cdot)\|_{L^2(\mathbb{R}^3)}$$

$$\leq C\mathcal{H}^{\frac{1}{2}}(t) \leq C\mathcal{J}_0 \tag{27}$$

Consider,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} |\phi|^2 \, dx = 2\,\mathrm{Re} \int_{\mathbb{R}^3} \phi D_0 \bar{\phi} \, dx$$

$$\leq 2\|\phi\|_{L^2(\mathbb{R}^3)}\|D_0\phi\|_{L^2(\mathbb{R}^3)} \leq C\mathcal{J}_0\|\phi\|_{L^2(\mathbb{R}^3)} \,.$$

Integrating the inequality, we get

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq C\mathcal{J}_0(1 + t) \,. \tag{28}$$

From Sobolev inequality, we have $\|\phi\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla\phi\|_{L^2(\mathbb{R}^3)}$ as well as $\|A\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla A\|_{L^2(\mathbb{R}^3)}$. Since $h(\phi,\bar{\phi}) \geq 1$, then $|\nabla A|^2 \leq h|\nabla A|^2$ which implies

$$\|A\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla A\|_{L^2(\mathbb{R}^3)} \leq C\|h^{1/2}\nabla A\|_{L^2(\mathbb{R}^3)} \leq C\mathcal{J}_0 \,.$$

Making use of Hölder inequality, we derive

$$\|\phi\|_{L^3(\mathbb{R}^3)} \leq \|\phi\|_{L^2(\mathbb{R}^3)}^{1/2}\|\phi\|_{L^6(\mathbb{R}^3)}^{1/2}$$

$$\leq C\mathcal{J}_0^{1/2}(1 + t)^{1/2}\left(\sum_{i} \|\nabla_i\phi\|_{L^2(\mathbb{R}^3)}\right)^{1/2}$$

$$\leq C\mathcal{J}_0^{1/2}(1 + t)^{1/2}\left(\sum_{i} \|D_i\phi\|_{L^2(\mathbb{R}^3)} + \|A_i\phi\|_{L^2(\mathbb{R}^3)}\right)^{1/2}$$

$$\leq C\mathcal{J}_0^{1/2}(1 + t)^{1/2}\left(C\mathcal{J}_0 + \|A_i\|_{L^6(\mathbb{R}^3)}\|\phi\|_{L^3(\mathbb{R}^3)}\right)^{1/2}$$

$$\leq C\mathcal{J}_0(1 + t)\left(1 + \|\phi\|_{L^3(\mathbb{R}^3)}\right)^{1/2} \,,$$

which proves that

$$\|\phi\|_{L^3(\mathbb{R}^3)} \le C\mathcal{J}_0(1+t)\,,$$

for some constant C. Thus, we get

$$
\begin{aligned}
\|\nabla_i \phi\|_{L^2(\mathbb{R}^3)} &\le& \sum_i \|D_i\phi\|_{L^2(\mathbb{R}^3)} + \|A_i\phi\|_{L^2(\mathbb{R}^3)} \\
&\le& C\mathcal{J}_0 + \|A_i\|_{L^6(\mathbb{R}^3)}\|\phi\|_{L^3(\mathbb{R}^3)} \\
&\le& C\mathcal{J}_0(1+t)\,.
\end{aligned}
\tag{29}
$$

Similarly, we have $\|A_0\|_{L^6(\mathbb{R}^3)} \le C\|\nabla A_0\|_{L^2(\mathbb{R}^3)} \le C\|h^{1/2}\nabla A_0\|_{L^2(\mathbb{R}^3)} \le C\mathcal{J}_0$, then by definition of $D_0\phi$, we get

$$
\begin{aligned}
\|\partial_t \phi\|_{L^2(\mathbb{R}^3)} &\le& \|D_0\phi\|_{L^2(\mathbb{R}^3)} + \|A_0\phi\|_{L^2(\mathbb{R}^3)} \\
&\le& C\mathcal{J}_0 + \|A_0\|_{L^6(\mathbb{R}^3)}\|\phi\|_{L^3(\mathbb{R}^3)} \\
&\le& C\mathcal{J}_0(1+t)\,.
\end{aligned}
\tag{30}
$$

Hence, the proof of the preposition is complete. $\qquad\square$

The Maxwell-Klein-Gordon system in Coulomb gauge can be formulated as follows,

$$
\begin{aligned}
h\triangle A^0 &=& -\mathrm{Im}\left(\phi D^0\bar{\phi}\right) - \epsilon^{ijk}\partial_i k\,\partial_j A_k - \partial_i h\left(\partial^i A^0 - \partial^0 A^i\right) \tag{31} \\
h\square A^i &=& h\partial^i \partial_0 A^0 + \epsilon^{ijk}\left(\partial_0 k\,\partial_j A_k - \partial_j k\,\partial_k A_0 + \partial_j k\,\partial_0 A_k\right) \\
&& -\partial_0 h\left(\partial^0 A^i - \partial^i A^0\right) - \partial_j h\left(\partial^j A^i - \partial^i A^j\right) - \mathrm{Im}\left(\phi D^i\bar{\phi}\right) \tag{32} \\
D_\mu D^\mu \phi &=& \frac{1}{2}F^{\mu\nu}\partial_\phi \mathcal{G}_{\mu\nu} + 2\partial_\phi V \tag{33} \\
\partial_i A^i &=& 0\,. \tag{34}
\end{aligned}
$$

We have the following preposition:

Preposition 3. *If the initial data for gauge field A satisfy divergence-free condition,*

$$\partial_i A^i(0,\cdot) = 0\,, \qquad \partial_t\,\partial_i A^i(0,\cdot) = 0\,, \tag{35}$$

then for all $t > 0$, we have

$$\partial_i A^i(t,\cdot) = 0\,. \tag{36}$$

Proof. Let $\psi(t,x) = \partial_i A^i(t,x)$. Taking divergence to the equation (32), we get

$$h\square\psi = \partial_i J^i + \partial_i\left(\partial_\mu k\,\tilde{F}^{\mu i} - \partial_\mu h\,F^{\mu i}\right) - \frac{\partial_i h}{h}\left(J^i + \partial_\mu k\,\tilde{F}^{\mu i} - \partial_\mu h\,F^{\mu i}\right) + h\partial_0\triangle A^0\,, \tag{37}$$

where $J^\mu = -\mathrm{Im}\left(\phi\,D^\mu\bar{\phi}\right)$. From equation (31), we have

$$h\partial_0\triangle A^0 = \partial_0 J^0 + \partial_0\left(\partial_\mu k\,\tilde{F}^{\mu 0} - \partial_\mu h\,F^{\mu 0}\right) - \frac{\partial_0 h}{h}\left(J^0 + \partial_\mu k\,\tilde{F}^{\mu 0} - \partial_\mu h\,F^{\mu 0}\right)\,. \tag{38}$$

Thus, from equations (37) and (38), we get

$$h\square\psi = \partial_\mu J^\mu + \partial_\nu\left(\partial_\mu k\,\tilde{F}^{\mu\nu} - \partial_\mu h\,F^{\mu\nu}\right) - \frac{\partial_\nu h}{h}\left(J^\nu + \partial_\mu k\,\tilde{F}^{\mu\nu} - \partial_\mu h\,F^{\mu\nu}\right)\,, \tag{39}$$

Note,

$$
\begin{aligned}
\partial_\mu \partial_\nu h \, F^{\mu\nu} &= \frac{1}{2} \left(\partial_\mu \partial_\nu h \, F^{\mu\nu} + \partial_\mu \partial_\nu h \, F^{\mu\nu} \right) = \frac{1}{2} \left(\partial_\mu \partial_\nu h \, F^{\mu\nu} + \partial_\nu \partial_\mu h \, F^{\nu\mu} \right) \\
&= \frac{1}{2} \left(\partial_\mu \partial_\nu h \, F^{\mu\nu} - \partial_\mu \partial_\nu h \, F^{\mu\nu} \right) = 0 \,,
\end{aligned}
$$

then using Bianchi identity and equation (9), we get

$$
h \Box \psi = \partial_\mu J^\mu \,, \tag{40}
$$

By definition of gauge covariant derivative and equation (8), we have

$$
\partial_\mu D^\mu \bar\phi = \frac{1}{2} F^{\mu\nu} \partial_\phi \mathcal{G}_{\mu\nu} - 2\partial_\phi V + \mathrm{i} A_\mu D^\mu \bar\phi \,. \tag{41}
$$

Thus,

$$
\partial_\mu \left(\phi \, D^\mu \bar\phi \right) = D_\mu \phi \, D^\mu \bar\phi + \phi \left(\frac{1}{2} F^{\mu\nu} \partial_\phi \mathcal{G}_{\mu\nu} - 2\partial_\phi V \right) \,. \tag{42}
$$

Using equation (7), we find that the right hand side is a real function. Hence,

$$
h \Box \psi = \partial_\mu J^\mu = -\mathrm{Im} \, \partial_\mu \phi \, D^\mu \bar\phi = 0 \,,
$$

and from Kirchoff's formula for solution of linear wave equation, the proof is complete. $\Box$

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