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Doctoral Thesis in Theoretical Physics

ON MARGINALLY DEFORMED AdS/CFT

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Chapter 1

Introduction and motivations

The aim of this work is investigating marginal deformations of a specific class of supersymmetric field theories. These theories appear in the framework of the AdS/CFT correspondence. Here we will review the general scenario of this correspondence and make a link to the specific research presented in the thesis.

It was already since t'Hooft's [9] work anticipated that string theories are linked to gauge theories. The basic idea of t'Hooft was based on two simple facts. The first fact is that pure gauge theories consist of fields in the adjoint representation of the gauge group which appears in the product of fundamental and anti-fundamental representations (for $SU(N)$), and thus every adjoint index can be described by two indices - one fundamental and one anti-fundamental. In the Feynman diagrams we can describe each index by a line, so the propagators in pure gauge theories can be represented by two lines. The other simple fact is that in the double line notation we can attach for any Feynman diagram to every index loop a surface. Thus, we can view a diagram as a decomposition of some closed surface (for vacuum or gauge invariant diagrams). The main result here is that if we take the large N limit with fixed $g_{YM}^2 N$ then the N dependence of any diagram will now be determined by the topology of the surface which it decomposes: the power of N we get is $(2-2g)$, where g is the genus of the surface. For a sphere $g=0$, for a torus $g=1$. In particular, g simply counts the handles of the surface (In other words: when we say that a diagram has a topology of surface $\mathcal{G}$, we mean that it can be drawn on surface $\mathcal{G}$ in double line notation without any line crossings).

When the number of colors, N, is taken to infinity[1], one can expand the path integral in a power series in $\frac{1}{N}$, such that the leading contribution is of the planar diagrams. So we get a power series with powers being linear functions of the genera of oriented closed surfaces, exactly like in oriented closed string theory (If we add matter fields in (anti)fundamental representation we get open surfaces, leading to an open string theory-like expansion, and if we look at gauge group $SO(N)$ for example we get an unoriented string theories-like expansion, because here the adjoint is a product of two fundamentals). We stress that we don't see from here any well defined string theory appearing, but only that the perturbation series is very similar to the perturbation series of string theory, with the string coupling constant being $\frac{1}{N}$.

It was Maldacena's work [52] that for the first time translated t'Hooft's idea of similarity be-

[1] N is taken to infinity while keeping $g_{YM}^2 N$ fixed.

tween large N gauge theories and string theories to a definite, although still conjectured, relation between a subclass of conformal field theories and a class of well defined string theories. Maldacena's conjecture was based on the following observation. In superstring theories appear solitonic, non-perturbative, objects called Dq-branes. These objects have at least two descriptions:

- In string-perturbative language they are defined as manifolds (extended in q directions) on which an open string can end.
- In the supergravity language, which is supposed to describe the low energy limit of string theories, they are defined as extended (in q directions) black hole solutions.

We now look at a system of N coincident D3-branes. In the string-perturbation theory language, in the low energy limit, the physics of the system is described by $\mathcal{N} = 4$ SYM with $U(N)$ gauge group[2] on the brane and by supergravity in the bulk, with these two systems decoupled. It is well known that in order for field theory perturbation theory to work $g_{YM}^2 N$ should be much smaller than one. In the supergravity language we will have some black hole solution, which in the near horizon limit is described by $AdS_5 \times S^5$ geometry. Here again we can describe our physics by two decoupled systems: supergravity in the bulk and the type IIB string theory on $AdS_5 \times S^5$. The supergravity solution is valid only if the radius of curvature is much larger than the string scale, which leads us to demand large N, since the radius in string units of AdS and the radius of the sphere are both proportional to $(g_{string} N)^{\frac{1}{4}}$. Thus, we see that the same object is described on one hand by field theory and supergravity and on the other hand by string theory and supergravity. This led Maldacena to conjecture that :

$\mathcal{N} = 4$ $d = 4$ $SU(N)$ SYM is equivalent to type IIB string theory on $AdS_5 \times S^5$ in the large N limit.

There is also a stronger conjecture that these theories describe the same physics for every value of N.

There are many different indirect checks of this conjecture. One such check is the striking property of S-duality. S-duality relates two theories, one with small and the other with large coupling. Both $\mathcal{N} = 4$ SYM and type IIB string theory are believed to be self dual under the S-duality.

The AdS/CFT correspondence relates expectation values in string theory to coupling constants in the field theory. For instance we get from the correspondence that $g_{YM}^2 \propto g_{string}$, and from string theory we know that g_{string} is related to the vacuum expectation value of the dilaton field. Thus we conclude that changing the gauge coupling on the field theory side, which is done by adding some marginal operator, is equivalent to changing the expectation value of the dilaton field on the string theory side. The marginal operators of the field theory are related to some moduli of the string theory[3]. In general, scalar supergravity fields ϕ which live in AdS couple to operators $\mathcal{O}$ which live on the boundary of AdS via $\int_{\mathbb{R}^{3,1}} \phi_0 \mathcal{O}$, where ϕ_0 is a restriction of ϕ to the boundary (up to some power of the radial coordinate). The dimension of $\mathcal{O}$, Δ, is related to the mass m^2 of the scalar field by:

[2]The $U(1)$ part is free so we will discuss essentially only the $SU(N)$ part.

[3]We need operators to be marginal in order not to spoil the conformal properties of the field theory.

$$m^2 = \Delta(\Delta - d). \tag{1.0.1}$$

Here d is the dimensionality of the space-time which the field theory lives in. We see that the massless, massive and tachyonic fields on the supergravity side correspond to marginal, irrelevant and relevant operators, respectively, on the field theory side.

The classification of operators to marginal, relevant and irrelevant in this way is meaningful before we deform our theory with them. After we deform our theory with these oprators the conformal dimensions of operators can receive corrections (via the anomalous dimensions). The marginality of an operator, as defined in the previous paragraph, can not assure that it will remain marginal after deforming the theory: the operators can be exactly marginal, marginally relevant or marginally irrelevant.

On the field theory side adding an irrelevant operator strongly affects the UV limit of the theory. Thus, because usually we define field theories in the UV and then flow to the IR, it does not make sense to discuss theories with irrelevant deformations. On the other hand, relevant deformations affect weakly the UV limit but break the conformal invariance. Finally, the exactly marginal operators keep the conformal properties of the theory.

On the string theory side giving a VEV to a massive field will change significantly the behavior on the boundary of AdS, which is equivalent to demanding a new UV description on the field theory side. The tachyonic fields will go to zero on the boundary, thus this deformation will affect only the interior and asymptotically we will still have an AdS background. Giving a VEV to a massless field (if it corresponds to an *exactly* marginal operator) will always leave us with an AdS factor.

Thus we see that by finding *exactly* marginal operators on the field theory side we can learn about the moduli of string theory.

Adding additional operators to the theory will in general change the supergravity background. The AdS_5 space has as its symmetry group $SO(2,4)$, which is exactly the conformal group in four dimensions (the boundary of AdS_5 is four dimensional). Thus, if we demand conformality, this factor will remain even after deforming the original theory. The second factor (the five-sphere) is related to the $SU(4)$ global symmetry of the SYM in some sense, and thus can be and will be deformed after deforming the original theory, if we break some supersymmetry[4]. Thus, we can say that if we deform the original $\mathcal{N} = 4$, $d = 4$ SYM by some marginal operators, then on the string theory side we have to deform the supergravity solution: $AdS_5 \times S^5 \to AdS_5 \times M$, where M is some five dimensional compact manifold and sometimes we will also have to turn on some fields.

1.1 Marginally deformed backgrounds

Marginal deformations of $\mathcal{N} = 4$ super Yang–Mills theory have recently drawn much attention in the context of conformal generalizations of AdS/CFT correspondence. The so–called

[4]The breaking of supersymmetry here is inevitable, because essentially there is only one renormalizable, consistent $\mathcal{N} = 4$ theory in $d = 4$ which is the SYM theory. Thus, by adding additional operators, other than the change of the gauge coupling, we always break the $\mathcal{N} = 4$ SUSY.

β–deformation is an interesting example of this class of theories thanks to the work of Lunin and Maldacena [66] where its gravity dual description has been found. From the field theory point of view this deformation is realized by enlarging the space of parameters of the original $\mathcal{N} = 4$ theory with the following modification of the superpotential:

$$i\, g_{YM}\, \mathrm{Tr}\, (\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \;\longrightarrow\; i\, h\, \mathrm{Tr}\, \Big(e^{i\pi\beta}\, \Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\, \Phi_1\Phi_3\Phi_2\Big) \tag{1.1.1}$$

where h and β are two new complex coupling constants in addition to the gauge coupling g_{YM}, which is chosen to be real. The resulting theory preserves $\mathcal{N} = 1$ supersymmetry and a $U(1)\times U(1)$ non-R-symmetry[5]. It is expected that this theory becomes conformally invariant only if a precise relation among the coupling constants exists [57].

This deformation can be viewed as arising from a new definition of the product of fields in the Lagrangian, namely

$$\Phi_i\, \Phi_j \;\longrightarrow\; \Phi_i * \Phi_j \equiv e^{i\pi\beta(Q^1_i Q^2_j - Q^2_i Q^1_j)}\, \Phi_i\, \Phi_j \tag{1.1.2}$$

where $\Phi_i\, \Phi_j$ is an ordinary product and (Q^1, Q^2) are the $U(1)\times U(1)$ charges of the fields. Though this prescription is similar in spirit to the one used to define non-commutative field theories [1, 2], the resulting theory is an ordinary field theory. All that happens is that (1.1.2) introduces some phases in the Lagrangian, see (1.1.1).

Suppose that we know the gravity dual of the original theory and that this geometry has two isometries associated to the two $U(1)$ global symmetries. Thus the geometry contains a two torus. The gravity description of the deformation (1.1.2) is surprisingly simple. We just need make the following replacement

$$\tau \equiv B + i\sqrt{g} \longrightarrow \tau_\beta = \frac{\tau}{1+\beta\tau} \tag{1.1.3}$$

in the original solution, where $\sqrt{g}$ is the volume of the two torus. We can view (1.1.3) as a solution generating transformation. Namely, we reduce the ten dimensional theory to eight dimensions on the two torus. The eight dimensional gravity theory is invariant under $SL(2,R)$ transformations acting on τ. The deformation (1.1.3) is one particular element of $SL(2,R)$. This particular element has the interesting property that it produces a non-singular metric if the original metric was non-singular. The $SL(2,R)$ transformation could only produce singularities when $\tau \to 0$. But we see from (1.1.3) that $\tau_\beta = \tau + o(\tau^2)$ for small τ. Therefore, near the possible singularities the ten dimensional metric is actually same as the original metric, which was non-singular by assumption.

As a first example, let us consider a string theory background with two $U(1)$ symmetries that are realized geometrically. Namely there are two coordinates φ_1, φ_2 and the two $U(1)$ symmetries act on these two coordinates as shifts of φ_i. Then we will have a two torus parametrized by φ_i, which, in general, will be fibered over an eight dimensional manifold. A simple example is the metric of R^4

$$ds^2 = d\rho_1^2 + d\rho_2^2 + \rho_1^2 d\varphi_1^2 + \rho_2^2 d\varphi_2^2 \tag{1.1.4}$$

As this example shows, the two torus could contract to zero size at some points but nevertheless the whole manifold is non-singular.

[5] By a non-R-symmetry we mean a symmetry that leaves the $\mathcal{N} = 1$ supercharges invariant. In addition, $\mathcal{N} = 1$ superconformal theories have a $U(1)_R$ symmetry.

When we compactify a closed string theory on a two torus the resulting eight dimensional theory has an exact $SL(2,Z) \times SL(2,Z)$ symmetry which acts on the complex structure of the torus and on the parameter[6]

$$\tau = B_{12} + i\sqrt{g} \tag{1.1.5}$$

where $\sqrt{g}$ is the volume of the two torus in string metric. The $SL(2,Z)$ that acts on the complex structure will not play an important role and we forget about it for the moment. At the level of supergravity we have an $SL(2,R) \times SL(2,R)$ symmetry. This is not a symmetry of the full string theory. The $SL(2,R)$ symmetries of supergravity can be used as solution generating transformations. The $SL(2,R)$ symmetry that plays a central role in this paper is the one acting as

$$\tau \to \tau' = \frac{\tau}{1+\gamma\tau} \tag{1.1.6}$$

where τ is given by (1.1.5). Of course, we can also think of (1.1.6) as the result of doing a T-duality on one circle, a change of coordinates, followed by another T-duality. When γ is an integer (1.1.6) is an $SL(2,Z)$ transformation, but for general γ it is not. This transformation generates a new solution. For example, applying this to (1.1.4) we get

$$\tau = i\rho_1\rho_2 \to \tau' = \frac{\gamma\rho_1^2\rho_2^2}{1+\gamma^2\rho_1^2\rho_2^2} + i\frac{\rho_1\rho_2}{1+\gamma^2\rho_1^2\rho_2^2} \tag{1.1.7}$$

The metric after the transformation (1.1.7) is

$$\begin{aligned} ds^2 &= d\rho_1^2 + d\rho_2^2 + \frac{1}{1+\gamma^2\rho_1^2\rho_2^2}(\rho_1^2 d\varphi_1^2 + \rho_2^2 d\varphi_2^2) \\ B_{12} &= \frac{\gamma\rho_1^2\rho_2^2}{1+\gamma^2\rho_1^2\rho_2^2} \\ e^{2\phi} &= e^{2\phi_0}\frac{1}{1+\gamma^2\rho_1^2\rho_2^2} \end{aligned} \tag{1.1.8}$$

where the change in the dilaton is due to the fact that the $SL(2,R)$ transformation leaves the eight dimensional dilaton invariant, not the ten dimensional one. (ϕ_0 is the original ten dimensional dilaton).

Moreover, if we start with a non-singular ten dimensional geometry after applying (1.1.6), the resulting geometry turns out to be non-singular[66]. The reason is that the only points where we could possibly introduce a singularity by performing an $SL(2,R)$ transformation is where the two torus shrinks to zero size. In this case τ is small and τ' becomes equal to τ. The region near the possible singularity becomes equal to what it was before the transformation. Thus the metric remains non-singular.

Let us consider a D-brane on the original background that is invariant under both $U(1)$ symmetries. Such a brane will be left invariant under the action of (1.1.6). In other words, there is a corresponding brane on the new background. Now, what is the theory on this brane in the new background? Lunin and Maldacena conjecture give us the following answer. Suppose that the

[6] Throughout this section we are setting $\alpha' = 1$ and we are using a normalization for the B field such that its period is $B_{12} \sim B_{12} + 1$.

original brane, on the original background, gave rise to a certain open string field theory. Then the open string field theory on the brane living on the new background is given by changing the standard product as in (1.1.2)

$$\Phi_i \, \Phi_j \;\longrightarrow\; \Phi_i * \Phi_j \equiv e^{i\pi\beta(Q_i^1 Q_j^2 - Q_i^2 Q_j^1)} \, \Phi_i \, \Phi_j \tag{1.1.9}$$

The basic idea leading to this conjecture is the following. In [2] it was pointed out that in the presence of a B field the open string field theory is defined in terms of an open string metric and non-commutativity parameter

$$G^{ij}_{open} + \Theta^{ij} = \left(\frac{1}{g+B} \right)^{ij} \sim \frac{1}{\tau} \tag{1.1.10}$$

where the last expression is schematic. Note that under the transformation (1.1.6) $1/\tau \to 1/\tau' = 1/\tau + \gamma$. All that happens in (1.1.10) is that we introduce a non-commutativity parameter $\Theta^{12} = \gamma$. The open string metric remains the same. The reason we called this a "conjecture" rather than a derivation is that [2] considered a constant metric and B field while here we are applying their formulas in a case where these fields vary in spacetime.

Let us now consider branes sitting at the origin. In general, the "origin" is the point where both circles shrink to zero size. Notice that for this brane the transformation (1.1.9) does *not* lead to a non-commutative field theory at low energies, since the $U(1)$ directions are not along its worldvolume but they are global symmetries of the field theory. The net effect of (1.1.9) for the field theory living on a brane is to introduce certain phases in the Lagrangian. In other words, starting with the low energy conventional field theory living on the brane, we obtain another conventional field theory with some phases in the Lagrangian. Viewing the deformation as a $*$–product allows us to show that all planar diagrams in the new theory are the same as in the old theory [3]. Then, for example, if the original theory was conformal, then this is a marginal deformation to leading order in N.

An interesting question is whether the deformation that we are doing preserves supersymmetry. In principle we can perform this transformation independently of whether we break or preserve supersymmetry, but sometimes we are interested in the ones that preserve it. If the original ten dimensional background is supersymmetric under a supersymmetry that is invariant under $U(1) \times U(1)$, then the deformed background will also be invariant under this supersymmetry. In particular, a D3-brane at the origin leads to an $\mathcal{N} = 1$ theory.

1.1.1 The Lunin–Maldacena Supergravity solution

It follows from the analysis of the Kaluza Klein spectrum on $AdS_5 \times S^5$ that there is a massless field in AdS_5 that corresponds to the deformation in question. In fact, there are more massless fields in AdS_5 than there are exactly marginal deformations. In [55] the super-gravity equations were analyzed in a perturbative fashion and a constraint was found. There are as many solutions to this constraint as there are exactly marginal deformations of $\mathcal{N} = 4$.

Now we will review the exact solution found in [66] for deformations which preserve $U(1) \times U(1)$ global symmetry. All we need to do is to apply the method described in the previous section. After we apply the transformation (1.1.6) to a particular two torus inside the S^5 geometry in the

$AdS_5 \times S^5$ background, we can find the solution corresponding to the gravity dual of the deformed theory

$$
\begin{aligned}
ds^2 &= R^2\left[ds^2_{AdS_5} + \sum_i (d\rho_i^2 + G\rho_i^2 d\phi_i^2) + \hat{\gamma}^2 G\rho_1^2\rho_2^2\rho_3^2(\sum_i d\phi_i)^2\right] \\
G^{-1} &= 1 + \hat{\gamma}^2(\rho_1^2\rho_2^2 + \rho_2^2\rho_3^2 + \rho_1^2\rho_3^2)\ , \qquad \hat{\gamma} = R^2\gamma\ , \qquad R^4 \equiv 4\pi e^{\phi_0} N = \lambda \\
e^{2\phi} &= e^{2\phi_0} G \\
B &= \hat{\gamma} R^2 G(\rho_1^2\rho_2^2 d\phi_1 \wedge d\phi_2 + \rho_2^2\rho_3^2 d\phi_2 \wedge d\phi_3 + \rho_3^2\rho_1^2 d\phi_3 \wedge d\phi_1) \\
C_2 &= -4\hat{\gamma} R^2 e^{-\phi_0}\omega_1 \wedge \sum_{i=1}^3 d\phi_i\ , \qquad d\omega_1 = \frac{r^3}{R^4}\sin\theta\cos\theta\, dr \wedge d\theta \\
C_4 &= 4R^4 e^{-\phi_0}\left(\omega_4 + G\,\omega_1 \wedge \sum_{i=1}^3 d\phi_i\right)\ , \qquad \omega_{AdS_5} = d\omega_4
\end{aligned}
\tag{1.1.11}
$$

where we find convenient to parametrize ρ_i coordinates via $\rho_1^2 = 1 - \frac{r^2}{R^2}\ ,\ \rho_2^2 = \frac{r^2}{R^2}\cos^2\theta\ ,\ \rho_3^2 = \frac{r^2}{R^2}\sin^2\theta$. Note that $\sum_{i=1}^3 \rho_i^2 = 1$ and we have $0 \leq r \leq R$. We consider only the case of real deformation parameters $\hat{\gamma}$ where the axion field C_0 is a constant and can be set to zero. Moreover, ϕ_0 is the dilaton of the undeformed background. The metric is written in string frame.

The corresponding R-R field strengths are given by the general prescription $\tilde{F}_q = dC_{q-1} - dB \wedge C_{q-3}$. In particular, the five form field strength of the background is

$$
F_5 = dC_4 - C_2 \wedge dB \qquad \star F_5 = F_5 \tag{1.1.12}
$$

The missing forms of higher degrees can be found by applying the ten–dimensional Hodge duality operator

$$
\tilde{F}_7 = -\star \tilde{F}_3, \qquad \tilde{F}_9 = \star\tilde{F}_1 \tag{1.1.13}
$$

From the first identity and using the equation of motion for C_2

$$
d(\star\tilde{F}_3) = dC_4 \wedge dB, \tag{1.1.14}
$$

it is easy to see that $d(C_6 - B \wedge C_4) = 0$, i.e. $C_6 - B \wedge C_4 = dX$ for an arbitrary 5–form X. In what follows

$$
C_6 = C_4 \wedge B \tag{1.1.15}
$$

Finally, from the second identity in (1.1.13), by using (1.1.15) and taking into account that $B \wedge B = 0$ and $C_0 = 0$ we find $\tilde{F}_9 = dC_8 = 0$. Therefore, we also set $C_8 = 0$.

Let us now examine the regime of validity of this solution. The solution which is presented here has small curvature as long as

$$
R\gamma \ll 1\ , \qquad R \gg 1 \tag{1.1.16}
$$

The first inequality can be understood as the condition that (at a generic point) the two torus does not become smaller than the string scale after the transformation. We also computed the square of the Riemann tensor on the deformed fivesphere and looked at the region where it is a maximum as a check of the first condition (1.1.16). The topology of this solution (1.1.11) is always $AdS_5 \times S^5$, since our transformation (1.1.6) does not change the topology.

In the case of real deformation parameter $\beta \equiv \gamma$ the new $AdS_5 \times \tilde{S}^5$ background can be also obtained from the original $AdS_5 \times S^5$ solution by applying a TsT (T–duality, shift, T–duality) transformation in S^5. A natural non–supersymmetric generalization of the Lunin–Maldacena background has been obtained in [67] by performing a series of TsT transformations on each of the three tori of S^5 but with different shift parameters $\hat{\gamma}_i$. This background is believed to be dual to a non–supersymmetric but still conformal gauge theory obtained by a related three–parameter deformation of the $\mathcal{N} = 4$ SYM. If all the $\hat{\gamma}_i$ are equal, the deformation reduces to the Lunin–Maldacena one. We will review these constructions in the following sections [7].

1.1.2 T–duality transformation

First we present the T-duality transformation useful to obtain these deformed backgrounds [67]. We start with the following string theory action

$$S = -\frac{\sqrt{\lambda}}{2} \int d\tau \frac{d\sigma}{2\pi} \left[\gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN}(X^i) - \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N B_{MN}(X^i) \right] . \tag{1.1.17}$$

Here $\epsilon^{01} = 1$, $M = 1, 2, 3 \ldots$, $i = 2, 3 \ldots$, and the background fields G_{MN} and B_{MN} do not depend on X^1. The action can be represented in the following equivalent form

$$\begin{aligned} S = -\sqrt{\lambda} \int d\tau \frac{d\sigma}{2\pi} \Bigg[& p^\alpha \left(\partial_\alpha X^M \frac{G_{1M}}{G_{11}} - \gamma_{\alpha\beta} \epsilon^{\beta\rho} \partial_\rho X^M \frac{B_{1M}}{G_{11}} \right) - \frac{1}{2\, G_{11}} \gamma_{\alpha\beta}\, p^\alpha p^\beta \\ & + \frac{1}{2} \gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j \left(G_{ij} - \frac{G_{1i} G_{1j} - B_{1i} B_{1j}}{G_{11}} \right) - \frac{1}{2} \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j \left(B_{ij} - \frac{G_{1i} B_{1j} - B_{1i} G_{1j}}{G_{11}} \right) \Bigg] . \end{aligned} \tag{1.1.18}$$

Indeed, variating with respect to p^α, one gets the following equation of motion for p^α

$$p^\alpha = \gamma^{\alpha\beta} \partial_\beta X^M G_{1M} \epsilon^{\alpha\beta} \partial_\beta X^M B_{1M} . \tag{1.1.19}$$

Substituting (1.1.19) into (1.1.18) and using the identity $\epsilon^{\alpha\gamma} \gamma_{\gamma\rho} \epsilon^{\rho\beta} = \gamma^{\alpha\beta}$, we reproduce the action (1.1.17). On the other hand, variating (1.1.18) with respect to X^1 gives

$$\partial_\alpha\, p^\alpha = 0 . \tag{1.1.20}$$

The general solution to this equation can be written in the form

$$p^\alpha = \epsilon^{\alpha\beta} \partial_\beta \tilde{X}^1 , \tag{1.1.21}$$

where $\tilde{X}^1$ is the scalar T-dual to X^1. Substituting (1.1.21) into the action (1.1.18), we obtain the following T-dual action

$$\tilde{S} = -\frac{\sqrt{\lambda}}{2} \int d\tau \frac{d\sigma}{2\pi} \left[\gamma^{\alpha\beta} \partial_\alpha \tilde{X}^M \partial_\beta \tilde{X}^N \tilde{G}_{MN} - \epsilon^{\alpha\beta} \partial_\alpha \tilde{X}^M \partial_\beta \tilde{X}^N \tilde{B}_{MN} \right] , \tag{1.1.22}$$

[7]Other interesting generalizations can be found in [68, 69]

where

$$\tilde{G}_{11} = \frac{1}{G_{11}}\,, \quad \tilde{G}_{ij} = G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}}\,, \quad \tilde{G}_{1i} = \frac{B_{1i}}{G_{11}}\,, \tag{1.1.23}$$
$$\tilde{B}_{ij} = B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}\,, \quad \tilde{B}_{1i} = \frac{G_{1i}}{G_{11}}\,,$$
$$\epsilon^{\alpha\beta}\partial_\beta \tilde{X}^1 = \gamma^{\alpha\beta}\partial_\beta X^M G_{1M}\epsilon^{\alpha\beta}\partial_\beta X^M B_{1M}\,, \quad \tilde{X}^i = X^i\,.$$

We can also apply the rules of T-duality for RR fields [4] to find C_2 and C_4.

1.1.3 The Frolov argument

We start from the string theory sigma model action on $AdS_5 \times S^5$, and derive the metric and the two-form field part of the LM–background in the case of real $\beta \equiv \gamma$ by using a T-duality on one circle of S^5, a shift of a second angle variable, followed by another T-duality.

Since the TsT-transformation involves only variables of S^5 it is sufficient to consider the S^5 part of the string action that can be written in the form

$$\tilde{\tilde{S}} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\left(\partial_\alpha\rho_i\partial_\beta\rho_i + \rho_i^2\partial_\alpha\tilde{\tilde{\phi}}_i\partial_\beta\tilde{\tilde{\phi}}_i\right) + \Lambda(\rho_i^2 - 1)\right]\,. \tag{1.1.24}$$

Here $\sqrt{\lambda} = R^2/\alpha'$, R is the radius of S^5, Λ is a Lagrange multiplier, $i = 1,2,3$, and $\gamma^{\alpha\beta} \equiv \sqrt{-h}\,h^{\alpha\beta}$, where $h^{\alpha\beta}$ is a world-sheet metric with Minkowski signature. In the conformal gauge $\gamma^{\alpha\beta} = \mathrm{diag}(-1,1)$ but we do not fix the world-sheet metric in this section. The action is invariant under the $SO(6)$ group, and the three $U(1)$ isometry transformations are realized as shifts of the angle variables $\tilde{\tilde{\phi}}_i$.

To derive the γ–deformed background it is convenient to make the following change of variables [66]

$$\tilde{\tilde{\phi}}_1 = \tilde{\tilde{\varphi}}_3 - \tilde{\tilde{\varphi}}_2\,, \qquad \tilde{\tilde{\phi}}_2 = \tilde{\tilde{\varphi}}_3 + \tilde{\tilde{\varphi}}_1 + \tilde{\tilde{\varphi}}_2\,, \qquad \tilde{\tilde{\phi}}_3 = \tilde{\tilde{\varphi}}_3 - \tilde{\tilde{\varphi}}_1\,, \qquad \tilde{\tilde{\varphi}}_3 \equiv \tilde{\tilde{\psi}}\,. \tag{1.1.25}$$

In terms of these new angle variables the action (1.1.24) takes the form

$$\tilde{\tilde{S}} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\left(\partial_\alpha\rho_i\partial_\beta\rho_i + g_{ij}\,\partial_\alpha\tilde{\tilde{\varphi}}_i\partial_\beta\tilde{\tilde{\varphi}}_j\right) + \Lambda(\rho_i^2 - 1)\right]\,, \tag{1.1.26}$$

where the metric components g_{ij} are

$$g_{11} = \rho_2^2 + \rho_3^2\,, \qquad g_{22} = \rho_1^2 + \rho_2^2\,, \qquad g_{33} = 1\,, \tag{1.1.27}$$
$$g_{12} = \rho_2^2\,, \qquad g_{13} = \rho_2^2 - \rho_3^2\,, \qquad g_{23} = \rho_2^2 - \rho_1^2\,.$$

Then we make the T-duality transformation on the circle parameterized by $\tilde{\tilde{\varphi}}_1$. By using the formulas collected in the previous Section, we get the action for the T-dual theory

$$\tilde{S} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\left(\partial_\alpha\rho_i\partial_\beta\rho_i + \tilde{g}_{ij}\partial_\alpha\tilde{\varphi}_i\partial_\beta\tilde{\varphi}_j\right) - \epsilon^{\alpha\beta}\tilde{b}_{ij}\partial_\alpha\tilde{\varphi}_i\partial_\beta\tilde{\varphi}_j + \Lambda(\rho_i^2 - 1)\right]\,. \tag{1.1.28}$$

Here $\epsilon^{01} = 1$, the T-transformed metric $\tilde{g}$ and the skew-symmetric B-field $\tilde{b}_{ij}$ are given by

$$\tilde{g}_{11} = \frac{1}{\rho_2^2 + \rho_3^2} \ , \quad \tilde{g}_{22} = \frac{\rho_1^2\rho_2^2 + \rho_1^2\rho_3^2 + \rho_2^2\rho_3^2}{\rho_2^2 + \rho_3^2} \ , \quad \tilde{g}_{33} = 1 - \frac{(\rho_2^2 - \rho_3^2)^2}{\rho_2^2 + \rho_3^2} \ , \quad \tilde{g}_{12} = \tilde{g}_{13} = 0 \,,$$
$$\tilde{g}_{23} = \frac{2\rho_2^2\rho_3^2 - \rho_1^2\rho_2^2 - \rho_1^2\rho_3^2}{\rho_2^2 + \rho_3^2} \ , \quad \tilde{b}_{12} = \frac{\rho_2^2}{\rho_2^2 + \rho_3^2} \ , \quad \tilde{b}_{13} = \frac{\rho_2^2 - \rho_3^2}{\rho_2^2 + \rho_3^2} \ , \quad \tilde{b}_{23} = 0 \,.$$

The T-dual variables $\tilde{\varphi}_i$ are related to $\tilde{\tilde{\varphi}}_i$ as follows

$$\epsilon^{\alpha\beta}\partial_\beta\tilde{\varphi}_1 = \gamma^{\alpha\beta}\partial_\beta\tilde{\tilde{\varphi}}_i\, g_{1i} \quad \Leftrightarrow \quad \partial_\alpha\tilde{\tilde{\varphi}}_1 = \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{\varphi}_1\,\tilde{g}_{11} - \partial_\alpha\tilde{\varphi}_i\,\tilde{b}_{1i} \,, \qquad \tilde{\tilde{\varphi}}_2 = \tilde{\varphi}_2 \,, \qquad \tilde{\tilde{\varphi}}_3 = \tilde{\varphi}_3 \ . \tag{1.1.29}$$

The relations (1.1.29) are satisfied only on-shell, that means that their consistency conditions lead to the equations of motion for $\tilde{\tilde{\varphi}}_i$ and $\tilde{\varphi}_i$.

Next, we make the following shift of the angle variable $\tilde{\varphi}_2$

$$\tilde{\varphi}_2 \to \tilde{\varphi}_2 + \hat{\gamma}\,\tilde{\varphi}_1 \,, \tag{1.1.30}$$

where $\hat{\gamma}$ is any constant. After the shift the T-transformed metric $\tilde{g}$ acquires the following form, $\tilde{g}_{ij} \to \tilde{G}_{ij}$:

$$\tilde{G}_{11} = \tilde{g}_{11} + \hat{\gamma}^2\,\tilde{g}_{22} = \frac{G^{-1}}{\rho_2^2 + \rho_3^2} \,, \quad G^{-1} = 1 + \hat{\gamma}^2\,(\rho_1^2\rho_2^2 + \rho_1^2\rho_3^2 + \rho_2^2\rho_3^2) \,,$$
$$\tilde{G}_{22} = \tilde{g}_{22} \ , \quad \tilde{G}_{33} = \tilde{g}_{33} \ , \quad \tilde{G}_{12} = \hat{\gamma}\,\tilde{g}_{22} \ , \quad \tilde{G}_{13} = \hat{\gamma}\,\tilde{g}_{23} \ , \quad \tilde{G}_{23} = \tilde{g}_{23} \,,$$

and the eq.(1.1.29) transforms into

$$\partial_\alpha\tilde{\tilde{\varphi}}_1 = \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{\varphi}_1\,\tilde{g}_{11} - \partial_\alpha\tilde{\varphi}_i\,\tilde{b}_{1i} - \hat{\gamma}\,\partial_\alpha\tilde{\varphi}_1\,\tilde{b}_{12} \,. \tag{1.1.31}$$

The final step is to make again the T-duality transformation on the circle parameterized by $\tilde{\varphi}_1$. This leads to the string action on the γ–deformed background

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\left(\partial_\alpha\rho_i\partial_\beta\rho_i + G_{ij}\partial_\alpha\varphi_i\partial_\beta\varphi_j\right) - \epsilon^{\alpha\beta}B_{ij}\,\partial_\alpha\varphi_i\partial_\beta\varphi_j + \Lambda(\rho_i^2 - 1)\right] . \tag{1.1.32}$$

The variables $\tilde{\varphi}_i$ are related to the T-dual variables φ_i as follows

$$\epsilon^{\alpha\beta}\partial_\beta\tilde{\varphi}_1 = \gamma^{\alpha\beta}\partial_\beta\tilde{\varphi}_i\,\tilde{G}_{1i} - \epsilon^{\alpha\beta}\partial_\beta\tilde{\varphi}_i\,\tilde{b}_{1i} \quad \Leftrightarrow \quad \partial_\alpha\tilde{\varphi}_1 = \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\varphi_i\,G_{1i} - \partial_\alpha\varphi_i\,B_{1i} \,, \qquad \tilde{\varphi}_2 = \varphi_2 \ , \qquad \tilde{\varphi}_3 = \varphi_3 \,. \tag{1.1.33}$$

The eqs.(1.1.29), (1.1.31) and (1.1.33) allow us to determine the following relations between the angle variables $\tilde{\tilde{\varphi}}_i$ and the TsT-transformed variables φ_i:

$$\partial_\alpha\tilde{\tilde{\varphi}}_1 = \left(\tilde{g}_{11}G_{1i} + \hat{\gamma}\,\tilde{b}_{12}B_{1i} - \tilde{b}_{1i}\right)\partial_\alpha\varphi_i - \left(\hat{\gamma}\,\tilde{b}_{12}G_{1i} + \tilde{g}_{11}B_{1i}\right)\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\varphi_i \,, \tag{1.1.34}$$
$$\partial_\alpha\tilde{\tilde{\varphi}}_2 = \partial_\alpha\varphi_2 - \hat{\gamma}\,B_{1i}\partial_\alpha\varphi_i \ + \hat{\gamma}\,G_{1i}\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\varphi_i \ ,$$
$$\partial_\alpha\tilde{\tilde{\varphi}}_3 = \partial_\alpha\varphi_3 \,.$$

The γ–deformed metric in (1.1.32) is given by

$$G_{ij} = G\, g_{ij}\ ,\ \text{if both } i,j \neq 3\ ; \qquad G_{33} = G\, g_{33} + 9\,\hat\gamma^2\, G\,\rho_1^2\rho_2^2\rho_3^2\ .$$

It is easy to see from this form of the metric that in terms of the angle variables ϕ_i, eq.(1.1.25), the action takes the following simple form

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\Big[\gamma^{\alpha\beta}\left(\partial_\alpha\rho_i\partial_\beta\rho_i + G\,\rho_i^2\partial_\alpha\phi_i\partial_\beta\phi_i + \hat\gamma^2\,G\,\rho_1^2\rho_2^2\rho_3^2(\sum_i\partial_\alpha\phi_i)(\sum_j\partial_\beta\phi_j)\right)$$
$$-2\,\hat\gamma\,G\,\epsilon^{\alpha\beta}\left(\rho_1^2\rho_2^2\partial_\alpha\phi_1\partial_\beta\phi_2 + \rho_2^2\rho_3^2\partial_\alpha\phi_2\partial_\beta\phi_3 + \rho_3^2\rho_1^2\partial_\alpha\phi_3\partial_\beta\phi_1\right) + \Lambda(\rho_i^2-1)\Big]\,. \qquad (1.1.35)$$

It is in this form the gravity background was written in (1.1.11).

It is possible to use a chain of TsT transformations to generate a three-parameter deformation of the $AdS_5\times S^5$ supergravity background. In the case when all the parameters are equal to each other the deformed background reduces to the one-parameter Lunin-Maldacena background we just discussed.

We saw that to obtain the γ–deformed supergravity background from $AdS_5\times S^5$ by using a TsT transformation we had to choose a very special torus in S^5. This choice is related to supersymmetry of the Lunin–Maldacena background but in general one may be interested in studying non–supersymmetric deformations too. In that case, the choice of the torus looks rather superficial. On the other hand, in the parametrization of S^5 we use in (1.1.24) there are three natural tori: (ϕ_1,ϕ_2), (ϕ_2,ϕ_3) and (ϕ_3,ϕ_1). One may ask how one could get the γ–deformed background by using TsT transformations on the three tori. The answer appears to be very simple. One should just perform a chain of three consecutive TsT transformations on each of the three tori with the same shift parameter $\hat\gamma$. If we allow the TsT transformations to have different shift parameters $\hat\gamma_i$ we get a non–supersymmetric deformation of $AdS_5\times S^5$. The three-parameter supergravity background should be dual to a non–supersymmetric but marginal deformation of $\mathcal{N}=4$ SYM.

Since the details of the derivation are very similar to the case of the γ–deformed background we present here only the final results. We apply the first TsT transformation with T-duality acting on the first angle ϕ_1 and the shift parameter equal to $\hat\gamma_3$ to the torus (ϕ_1,ϕ_2), then the second TsT transformation with the shift parameter equal to $\hat\gamma_1$ to the torus (ϕ_2,ϕ_3), and finally the third TsT transformation with the shift parameter equal to $\hat\gamma_2$ to the torus (ϕ_3,ϕ_1).

The resulting type IIB supergravity background written in string frame takes the form

$$\begin{aligned}
ds^2 &= R^2\left[ds^2_{AdS_5} + \sum_i(d\rho_i^2 + G\rho_i^2 d\phi_i^2) + G\rho_1^2\rho_2^2\rho_3^2(\sum_i\hat\gamma_i d\phi_i)^2\right]\\
G^{-1} &= 1 + \hat\gamma_3^2\rho_1^2\rho_2^2 + \hat\gamma_1^2\rho_2^2\rho_3^2 + \hat\gamma_2^2\rho_1^2\rho_3^2\ , \qquad \hat\gamma_i = R^2\gamma_i\ , \qquad R^4 \equiv 4\pi e^{\phi_0}N = \lambda\\
e^{2\phi} &= e^{2\phi_0}G\\
B &= R^2 G(\hat\gamma_3\rho_1^2\rho_2^2 d\phi_1\wedge d\phi_2 + \hat\gamma_1\rho_2^2\rho_3^2 d\phi_2\wedge d\phi_3 + \hat\gamma_2\rho_3^2\rho_1^2 d\phi_3\wedge d\phi_1)\\
C_2 &= -4R^2 e^{-\phi_0}\omega_1\wedge\sum_{i=1}^3\hat\gamma_i d\phi_i\ , \qquad d\omega_1 = \frac{r^3}{R^4}\sin\theta\cos\theta\, dr\wedge d\theta
\end{aligned}$$

$$C_4 \;=\; 4R^4 e^{-\phi_0}\left(\omega_4 + G\,\omega_1 \wedge \sum_{i=1}^{3} d\phi_i\right)\,, \qquad\qquad \omega_{AdS_5} = d\omega_4 \tag{1.1.36}$$

where again $(0 \leq r \leq R)$ $\rho_1^2 = 1 - \frac{r^2}{R^2}\,,\ \rho_2^2 = \frac{r^2}{R^2}\cos^2\theta\,,\ \rho_3^2 = \frac{r^2}{R^2}\sin^2\theta$ so that $\sum_{i=1}^3 \rho_i^2 = 1$. Moreover,

$$C_0 = C_8 = 0 \qquad\qquad \text{and} \qquad\qquad C_6 = C_4 \wedge B \tag{1.1.37}$$

as in the supersymmetric case.

Chapter 2

Conformal invariance and chiral ring for non–planar Leigh–Strassler theories

In this chapter we discuss deformations of the conformal field theory in order to generalize the standard AdS/CFT correspondence towards more realistic theories.

Conformal field theories have many applications in their own right, but since our main interest (at least in the context of four dimensional field theories) is in studying non-conformal field theories like QCD, it is interesting to ask how we can learn about non-conformal field theories from conformal field theories. One way to break conformal invariance while preserving Lorentz invariance, is to deform the action by local operators,

$$S \to S + h \int d^4x \mathcal{O}(x) \tag{2.0.1}$$

for some Lorentz scalar operator $\mathcal{O}$ and some coefficient h.

The analysis of such a deformation depends on the scaling dimension Δ of the operator $\mathcal{O}$ [1]. If $\Delta < 4$, the effect of the deformation is strong in the IR and weak in the UV, and the deformation is called *relevant*. If $\Delta > 4$, the deformation is called *irrelevant*, and its effect becomes stronger as the energy increases. Since we generally describe field theories by starting with some UV fixed point and flowing to the IR, it does not really make sense to start with a CFT and perform an irrelevant deformation, since this would really require a new UV description of the theory. Thus, we will not discuss irrelevant deformations here. The last case is $\Delta = 4$, which is called a *marginal deformation*, and which does not break conformal invariance to leading order in the deformation. Generally, even if the dimension of an operator equals 4 in some CFT, this will no longer be true after deforming by the operator, and conformal invariance will be broken. Such deformations can be either *marginally relevant* or *marginally irrelevant*, depending on the dimension of the operator $\mathcal{O}$ for finite small values of h. In special cases the dimension of the operator will remain $\Delta = 4$ for any value of h, and conformal invariance will be present for any value of h. In such a case

[1] If the operator does not have a fixed scaling dimension we can write it as a sum of operators which are eigenfunctions of the scaling operator, and treat the deformation as a sum of the appropriate deformations.

the deformation is called *exactly marginal*, and the conformal field theories for all values of h are called a *fixed line* (generalizing the concept of a conformal field theory as a fixed point of the renormalization group flow). The *exactly marginal* deformations of the $\mathcal{N}=4$ SYM have been first classified in [57] and extensively studied in a field theory approach [57, 58, 59, 60] and in the context of the AdS/CFT correspondence [61, 62, 63, 64, 65].

The interest in marginal deformed SYM theories has recently received a considerable boost thanks to the work of Lunin–Maldacena [66] where the gravity dual of the so called β–deformed theory has been proposed[2]. It corresponds to the low energy limit of a string theory on a deformed background $\mathrm{AdS}_5 \times \mathrm{S}^5_\beta$ obtained by SL(2,R) transforming the τ modulus of a two–torus inside S^5. Alternatively, it can be obtained from the original $\mathrm{AdS}_5 \times \mathrm{S}^5$ solution by applying a TsT transformation in S^5 [66, 67, 68, 69].

A considerable effort has been devoted so far to provide tests of the correspondence in its marginal deformed version. As for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ original correspondence, perturbative properties of the field theory have been investigated: For the $SU(N)$ case the condition which constrains the couplings of the theory in order to have $\mathcal{N}=1$ superconformal invariance has been determined perturbatively up to three loops [70, 71, 72]. In the large N limit the exact superconformal condition has been found in [74]. Nonrenormalization properties of operators in the chiral ring have been established perturbatively [70, 71, 72] and multiloop amplitudes have been computed [71, 72, 76]. The exact anomalous dimensions for spin–2 operators of the form $\mathrm{Tr}(\Phi_1^J \Phi_2)$ have been determined [74] for N, J unrelated and large[3]. Finally, the gauge one–loop effective action has been computed [78] for a particular background configuration. Nonperturbative instantonic effects have been also considered [79].

Integrability properties of the original $\mathcal{N}=4$ SYM theory (see [80] for a review and list of references) survive the β–deformation [81, 82, 67, 140, 83] and Bethe ansatz techniques can be used also in this case to compute the spectrum of anomalous dimensions of composite operators.

On the string theory side BPS states have been investigated in [84] for orbifold configurations. Integrability properties have been exploited on the two sides of the correspondence in order to match the energies of semiclassical fast rotating strings with one–loop anomalous dimensions of scalar operators [85, 86, 87, 88, 89]. The spectrum of states has been also investigated in the BMN limit [90, 91].

Non–supersymmetric generalizations of the Lunin–Maldacena β–deformation have been proposed [67] and further investigations have been carried on [92, 68, 93, 94, 95]. Very recently, deformations obtained by acting with TsT transformations in AdS_5 have been also proposed [96].

Finally, the Lunin–Maldacena deformation has been applied in the context of dipole theories with the purpose of disentangling the KK modes (whose dynamics gets affected by the deformation) from the gauge modes [97, 98, 99, 100, 101].

In a previous paper [71] it has been initiated the study of the chiral ring of the $SU(N)$ β–deformed SYM theory by exploiting perturbative techniques in $\mathcal{N}=1$ superspace [102, 103, 106, 107, 108]. There, the single–trace sector of the chiral ring has been considered: For the lowest dimensional scalar operators it has been proved the vanishing of their anomalous dimensions up

[2] First steps in the direction of studying the correspondence with a lower number of supersymmetries were undertaken in [55].

[3] The same kind of limit has been recently considered in [77] for studying magnons in the $\mathcal{N}=4$ SYM theory.

to two loops and the appearance of finite corrections to their correlation functions, in contradistinction to the $\mathcal{N} = 4$ case. In particular, the two–loop results confirmed the protection [70] of the operator $\text{Tr}(\Phi_i \Phi_j)$, $i \neq j$ which was missing in the list of CPO's of the theory [61, 62, 66].

In this chapter we intend to pursue our investigation and extend it to higher dimensional sectors of the chiral ring for scalar chiral superfields. We work at finite N and take into account mixing among sectors with different trace structures. Exploiting the definition of quantum chiral ring we reduce the determination of protected operators up to order n in perturbation theory to the evaluation of the effective superpotential up to order $(n-1)$. Precisely, from the knowledge of the effective superpotential we determine perturbatively all the quantum descendant operators of naive scale dimension Δ_0, and find the CPO's as the operators which are orthogonal order by order to the descendants.

For the β–deformed theory we investigate the spin–2 sector[4] and applying our procedure to simple cases ($\Delta_0 = 4, 5$) we determine the protected operators up to three loops. In the sectors we have studied we can always define descendant operators which do not receive quantum corrections. This seems to be a general property of the spin–2 operators: Despite the nontrivial appearance of finite perturbative corrections to the effective action, the quantum descendant operators defined in terms of the effective superpotential coincide with their expressions given in terms of the classical superpotential (up to possible mixing among them).

We then investigate the spin–3 sector where, due to the appearance of Konishi–like anomalies, we need restrict our analysis at two loops in order to avoid dealing with mixed gauge/scalar operators. Up to this order the descendant operators we consider are the classical ones. However, in this sector we expect higher order corrections to the descendants to appear together with a nontrivial dependence on the anomaly term. Therefore, the non–renormalization properties of the descendant operators that we experiment for the spin–2 sector are not a general feature of the theory.

We generalize our procedure to the study of protected operators for the $\mathcal{N} = 1$ superconformal theory associated to the full Leigh–Strassler deformation. Even if the gravity dual of this theory is not known yet, it is anyway interesting to figure out the general structure of its chiral ring. Still at finite N, we study explicitly the weight–2 and weight–3 sectors up to two loops and perform a preliminary analysis of the general sectors at least at lowest order in the couplings. An interesting result we find is that, because of the discrete Z_3 symmetries of the theory, the sectors corresponding to conformal weights which are multiple of 3 have a different operator structure from the other ones.

Then, we study the conformal invariance of non–planar β–deformed $\mathcal{N} = 4$ SYM theory for complex values of the deformation parameter. Recall that from the field theory side this can be realized by enlarging the space of parameters of the original $\mathcal{N} = 4$ theory (1.1.1):

$$\{g\} \longrightarrow \{g,\, h,\, \beta\} \qquad g \in \mathbb{R} \qquad h,\, \beta \in \mathbb{C} \tag{2.0.2}$$

The resulting theory preserves $\mathcal{N} = 1$ supersymmetry and it is expected to become conformally invariant only if a precise relation among the coupling constants exists [57]. Several papers have

[4]We use the notation of [85] and call "spin–n" the sector containing operators made by products of n different flavors.

been devoted to the study of an explicit realization of this condition in the planar case ([74]-[45]). Keeping β real, the Leigh–Strassler constraint turns out to be satisfied at all order in perturbation theory by the exact solution $h\bar{h} = g^2$ [74]. The case of complex β requires a more careful investigation since the conformal condition gets perturbatively corrected. In order to properly describe the fixed point surface in the space of couplings, the coupling constant reduction (CCR) program has shown to be a powerful tool ([46]-[51]). Using this approach, in [44] it is claimed that conformal invariance and scheme independence of the theory can not be achieved at the same time for the complex β deformed case in the planar limit[5].
Our aim is to achieve a better understanding of the problem by looking at the finite N case (see also [70]-[143]). Working perturbatively we will ask for the chiral and gauge beta functions to vanish in order to define the theory at its conformal point.

The plan of the chapter is as follows: After introductory sections on the full Leigh–Strassler superconformal theory, in Section 2.2 we introduce the definition of perturbative chiral ring and discuss the general procedure we adopt to determine the CPO's of the theory. In Section 2.2.2 we compute the perturbative effective superpotential up to two loops as required to determine protected operators up to three loops. These are then the subject of Sections 2.2.3 and 2.2.4 for the spin–2 and spin–3 sectors, respectively. In Section 2.3 we study the more general $\mathcal{N} = 1$ superconformal theory described by the full Leigh–Strassler superpotential.

In Section 2.4.1 we analyze the properties of the two–point chiral correlator for complex β. We make use of the CCR procedure to obtain the vanishing of the anomalous dimension. This amounts to express the chiral couplings as functions of the gauge coupling g. As a consequence the perturbation theory is naturally defined in terms of powers of g instead of powers of loops. At order g^6 we meet the first non–trivial situation because at this stage different loop diagrams start contributing at the same order in g. We will see that up to order g^{10}, differently from the planar case, there is enough freedom to remove the scheme dependence without reducing to the real β case.

Then we turn to consider the gauge beta function in Section 2.4.2. As CCR approach allows different loop orders to mix, it is not obvious that standard finiteness theorems [144, 145] should hold. So, having canceled the chiral beta function up to $\mathcal{O}(g^7)$ does not automatically imply the vanishing of the gauge beta function at $\mathcal{O}(g^9)$. The fact that this is still the case is a highly non–trivial check that we will cover in details in Section 3. The same problem was studied in [44] in the planar case where it was shown by an explicit computation that the condition for the vanishing of the anomalous dimension γ at $\mathcal{O}(g^8)$ actually ensures the vanishing of the gauge beta function at $\mathcal{O}(g^{11})$. This result was obtained making use of background field method combined with covariant ∇–algebra. However it is worth noting that the procedure followed in [44] is not the standard one (extensively explained in [146]), which turned out to be too involved. Here, working at a lower order in g but keeping N finite, we will be able to get through the calculation adopting both of the methods and checking explicitly the equivalence of the two.

[5]The possible scheme dependence of the vanishing γ condition has been first noted by the authors of [73]. In [44] we explicitly considered this feature and studied its implications.

2.1 The Leigh–Strassler deformations

In this section we discuss the marginal deformations of field theories coming from a system of N coincident D3-branes. The whole discussion is done from the field theory perspective. The theories we discuss have $\mathcal{N} = 4, 1$ supersymmetry.

An operator is *exactly* marginal if upon adding it to the original conformal theory all the β-functions still vanish. Generally if we have p couplings in the theory we also have p β-functions. The conditions for the theory to be conformal are:

$$\begin{aligned} 0 &= \beta_{g_1}(g_i, h_j) \\ &\vdots \\ 0 &= \beta_{g_n}(g_i, h_j) \\ 0 &= \beta_{h_1}(g_i, h_j) \\ &\vdots \\ 0 &= \beta_{h_k}(g_i, h_j) \end{aligned} \qquad (2.1.1)$$

(Here h_i are the couplings and g_i are the gauge couplings of the system.) We have $n + k$ equations in $n + k$ variables. Thus, in general, we expect to have isolated, if at all we will have any, solutions of this system of equations.

However, in supersymmetric field theories we have several simplifications. The first one is that from nonrenormalization of the superpotential in supersymmetric theories we get a relation between the anomalous dimensions of the fields and the coupling associated with the superpotential term(see [8]). For a superpotential $W = \frac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k$ we get:

$$\beta_Y^{ijk} = Y^{p(ij}\gamma_p^{k)} = Y^{ijp}\gamma_p^k + (k \leftrightarrow i) + (k \leftrightarrow j). \qquad (2.1.2)$$

Here γ_p^k is the anomalous dimension related to the $\langle\bar{\Phi}^k\Phi_p\rangle$ correlator. The second simplification is the relation between the gauge coupling and the anomalous dimensions - the NSVZ β-function [114],

$$\beta_g = \frac{g^3}{16\pi^2}\left[\frac{Q - 2r^{-1}Tr\left[\gamma C(R)\right]}{1 - 2C(G)g^2(16\pi^2)^{-1}}\right]. \qquad (2.1.3)$$

The symbols appearing here will be defined later. We conclude that in a supersymmetric field theory, in order to find *exactly* marginal deformations we have to solve a set of linear equations in the anomalous dimensions. These equations can be linearly dependent, giving a manifold of solutions [57]. The equations (2.1.1) become:

$$\begin{aligned} 0 &= \beta_{g_1}(\gamma_l, g_i, h_j) \\ &\vdots \\ 0 &= \beta_{g_n}(\gamma_l, g_i, h_j) \end{aligned}$$

$$
\begin{aligned}
0 &= \beta_{h_1}(\gamma_l, g_i, h_j) \\
&\vdots \\
0 &= \beta_{h_k}(\gamma_l, g_i, h_j)
\end{aligned}
\tag{2.1.4}
$$

Here usually we will get that the righthand sides of these equations depend only on γs which will greatly simplify our job.

In order to find *exactly* marginal directions we have to solve a set of linear equations, to find the possible values for the anomalous dimensions (γs) such that all βs vanish. Then, by loop calculations we calculate the dependence of the γs on the couplings and other parameters of the theory, and finally we impose the conditions from the first step on the γs and see if they can be satisfied. This will be the strategy in our search for *exactly* marginal deformations throughout this work.

When solving the set of linear equations (2.1.4) we can get possible solutions which will be ruled out from the loop calculations[6]. We will see examples of this below.

2.1.1 $\mathcal{N} = 4$ theory

First we review some basic properties of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. In $\mathcal{N} = 0$ language the theory contains six scalar fields, four Weyl fermions and a real vector field. All fields are in the adjoint representation of $SU(N)$. In $\mathcal{N} = 1$ language the six scalars can be coupled to form three complex scalars which together with three Weyl fermions form three chiral superfields Φ_i, while the vector and the remaining Weyl spinor can be joined to form a vector superfield V. The Lagrangian in $\mathcal{N} = 1$ language is then:

$$
\begin{aligned}
S &= \int d^8z \; \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \; \mathrm{Tr}W^{\alpha}W_{\alpha} \\
&\quad + \frac{ig}{3!}\int d^6z \; \epsilon_{ijk}\mathrm{Tr}(\Phi^i[\Phi^j, \Phi^k]) \; + \; h.c.
\end{aligned}
\tag{2.1.5}
$$

This is a pure Yang Mills theory with sixteen supercharges, in particular for $U(1)$ gauge group this theory becomes free. The β-function of the gauge coupling vanishes identically (at one loop it is a trivial consequence of having three chiral superfields in the adjoint representation), thus it is a conformal theory. It is believed to be exactly self S-dual. This symmetry of the theory exchanges the strong coupling regime with a weak coupling regime, and the perturbative, electric, degrees of freedom with non-perturbative, magnetic degrees of freedom.

In string theory we get $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group by putting N D3-branes in type IIB string theory together. In this picture we have six "vibrational" modes of the branes (which are related to the six transverse directions to the brane) which become six scalars, which in turn when joined in pairs comprise the $\mathcal{N} = 1$ scalar part of three complex chiral supermultiplets. The possibility of the fundamental string to end on one of the N branes gives an $SU(N)$ gauge group and puts the scalars (as well as the other fields) in the adjoint representation. To all these

[6] We can count on the loop calculations only in the weak coupling regime. Thus, we can not rule out these solutions from appearing in the strong coupling regime.

integer spin fields we have fermionic counterparts, and all in all we get an $SU(N)$ gauge group in 4d with three chiral and one vector multiplets. In type IIB superstrings we have 32 supercharges, D-branes are BPS states and thus they break half of the supersymmetry. Finally we have 16 supercharges in d=4 which give us $\mathcal{N}=4$ supersymmetry. Three $\mathcal{N}=1$ chiral multiplets and the vector multiplet in $\mathcal{N}=4$ language give an $\mathcal{N}=4$ vector multiplet. Thus to summarize we get on the D-branes $\mathcal{N}=4$ pure Yang Mills with $SU(N)$ gauge group.

Another, related, way to obtain $\mathcal{N}=4$ SYM in d=4 is [7] to look at pure $\mathcal{N}=1$ SYM in d=10 and then do the dimensional reduction procedure to d=4. In d=10 we had only the vector multiplet, six scalar components of which lose their vector nature after the reduction. They can be coupled in pairs to form complex scalars which will be the scalars of three chiral $\mathcal{N}=1$, d=4 multiplets. Of course in this procedure we have a global $SO(6)\sim SU(4)$ symmetry, which becomes the $\mathcal{R}$ symmetry of $\mathcal{N}=4$.

There is extensive literature on this field theory, in particular regarding its finiteness and the exact S-duality of this theory. There is also research concerning the relevant deformations of $\mathcal{N}=4$ ([5, 108, 6] for example). Relevant deformations break the conformal invariance by introducing a scale to the theory. We will be interested only in marginal deformations throughout this work.

2.1.2 Marginal deformations

In this section we investigate some *exactly* marginal deformations of $\mathcal{N}=4$ SYM. There are essentially only three types of marginal deformations which one can add to the lagrangian above[7]. The obvious deformation is just changing the gauge coupling constant, the two other types are superpotentials of the form:

$$\begin{aligned}&\frac{ih_1}{3!}\epsilon_{ijk}Tr(\Phi^i[\Phi^j,\Phi^k])\\&\frac{h_{ijk}}{3!}Tr(\Phi^i\{\Phi^j,\Phi^k\}),\end{aligned} \tag{2.1.6}$$

where h_{ijk} is totally symmetric. These operators are marginal (by power counting), obey gauge invariance and preserve $\mathcal{N}=1$ supersymmetry. So by adding these operators we get $\mathcal{N}=1$ SQCD. What has to be determined is under what conditions these marginal deformations are *exactly* marginal, i.e. the β-functions vanish to all orders in perturbation theory.

$\mathcal{N}=1$ SQCD was analyzed for general superpotentials and general simple gauge group G ([104] and references therein). We will briefly summarize the general results:

We write the superpotential as:

$$W=\frac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k. \tag{2.1.7}$$

We assume that the gauge group is simple and that there are no gauge singlets. The $\beta-function$ of Y can be written in terms of the anomalous dimensions:

[7] There are also relevant deformations, inserting mass terms for the fields, and they were discussed in [5, 108].

$$\beta_Y^{ijk} = Y^{p(ij}\gamma_p^{k)} = Y^{ijp}\gamma_p^k + (k \leftrightarrow i) + (k \leftrightarrow j). \tag{2.1.8}$$

The one loop gauge β-function and the anomalous dimensions are given by:

$$16\pi^2 \beta_g^{(1)} = g^3 Q, \quad \text{and} \quad 16\pi^2 \gamma_j^{(1)i} = P_j^i, \tag{2.1.9}$$

where we have defined:

$$Q = T(R) - 3C(G), \quad \text{and} \quad P_j^i = \frac{1}{2} Y^{ikl} Y_{jkl} - 2g^2 C(R)_j^i, \tag{2.1.10}$$

and:

$$T(R)\delta_{AB} = Tr(R_A R_B), \quad C(G)\delta_{AB} = f_{ACD} f_{BCD} \quad \text{and} \quad C(R)_j^i = (R_A R_A)_j^i. \tag{2.1.11}$$

R_A is the representation of the chiral superfields. A,B,C,D are indices in the adjoint representation. For the gauge coupling we use the NSVZ $\beta - function$ [114]:

$$\beta_g = \frac{g^3}{16\pi^2}\left[\frac{Q - 2r^{-1} Tr\left[\gamma C(R)\right]}{1 - 2C(G)g^2 (16\pi^2)^{-1}}\right], \tag{2.1.12}$$

(here $r = \delta_{AA}$) which at one loop gives (as in (2.1.9)):

$$16\pi^2 \beta_g^{(1)} = g^3 Q. \tag{2.1.13}$$

Now we have set the general stage and return to the specific marginally deformed $\mathcal{N} = 4$ theory. The superpotential can be rewritten in the form:

$$W = (-h_g f_{abc}\epsilon_{ijk} + d_{abc} h_{ijk})\frac{1}{6}\Phi_i^a \Phi_j^b \Phi_k^c \tag{2.1.14}$$

$$Y_{abc}^{ijk} = (-h_g f_{abc}\epsilon_{ijk} + d_{abc} h_{ijk}) \tag{2.1.15}$$

where $f_{abc} = -iTr\left(T_a[T_b, T_c]\right)$ and $d_{abc} = Tr\left(T_a\left\{T_b, T_c\right\}\right)$. Here T_a are the generators of the Lie algebra of G in the fundamental representation of the group and $h_g = g + h_1$. The groups for which d_{abc} is not vanishing are only $SU(N \geq 3)$ or E_8. Here we discuss only the $SU(N)$ case.

Φ_i are in the adjoint of $SU(N)$:

$$R_A = \begin{pmatrix} T_A^{adj} & 0 & 0 \\ 0 & T_A^{adj} & 0 \\ 0 & 0 & T_A^{adj} \end{pmatrix} \tag{2.1.16}$$

Here T_A^{adj} are the adjoint representation matrices. From here we can calculate the parameters appearing in the general setup above:

$$
\begin{aligned}
C(G)\delta_{AB} &= f_{ACD}f_{BCD} \equiv C_1\delta_{AB} \\
T(R)\delta^A_B &= Tr(R_A R_B) = 3Tr(T^a_A T^a_B) = 3C_1\delta^A_B(= -3f_{ACD}f_{BDC}) \\
C(R)^{Ai}_{Bj} &= (R_D R_D)^{Ai}_{Bj} = C_1\delta^A_B\delta^i_j \\
r &= N^2 - 1.
\end{aligned}
\tag{2.1.17}
$$

C_1 depends on the normalization of the Lie algebra generators, for the moment we will keep it arbitrary which will not affect our results. The one loop gauge $\beta - function$ is proportional to Q:

$$
Q = T(R) - 3C(G) = 3C_1 - 3C_1 = 0. \tag{2.1.18}
$$

So the gauge $\beta - function$ vanishes at one-loop. The gauge $\beta - function$ vanishes at one loop in general gauge theories with three chiral superfields in the adjoint representation.

First we do the general Leigh-Strassler analysis [57]. We have here:

$$
\begin{aligned}
\beta_g &\propto Tr(\gamma) \\
\beta_{h_1} &\propto Tr(\gamma)
\end{aligned}
\tag{2.1.19}
$$

So in general we have 10 h_{ijk}, h_1 and the gauge coupling g, total of 12 couplings, we have to demand that $Tr(\gamma) = 0$ and $\beta_{h_{ijk}} = 0$ giving a total of 11 conditions. So we expect a one dimensional manifold of the fixed points which we have already in $\mathcal{N} = 4$ and it is parameterized by the gauge coupling, with $h_1 = h_{ijk} = 0$. But we can do a more complicated thing. If we assume also that γ is proportional to identity matrix[8] we get $\beta_{h_{ijk}} \propto Tr(\gamma)$. So we will have 12 couplings, one condition $Tr(\gamma) = 0$ and 8 conditions for $\gamma^j_i \propto \delta^j_i$, giving a total of 12-8-1=3 free parameters. So we expect to have a three dimensional manifold of exactly marginal deformations. We will see below that we essentially get only these three marginal directions.

Now we continue with the perturbation theory analysis. For $SU(N)$: $d_{acd}d_{bcd} = 2\frac{N^2-4}{N}C_2^3\delta_{ab}$, where $C_2\delta_{ab} \equiv Tr(T_a T_b)$. So we can write:

$$
P^{ai}_{bj} = (2C_1(|h_g|^2 - g^2)\delta_{ij} + \frac{N^2-4}{N}C_2^3 h^{(2)}_{ij})\delta_{ab}. \tag{2.1.20}
$$

Here $h^{(2)}_{ij} \equiv h_{ilm}\bar{h}_{jlm}$. And finally we get the one-loop anomalous dimensions and $\beta - functions$:

$$
\gamma^{(1)ai}_{bj} = \frac{1}{16\pi^2}(2C_1(|h_g|^2 - g^2)\delta_{ij} + \frac{N^2-4}{N}C_2^3 h^{(2)}_{ij})\delta_{ab} \tag{2.1.21}
$$

$$
\beta^{(1)ijk}_{abc} = \frac{1}{16\pi^2}\left\{6C_1(|h_g| - g^2)Y^{ijk}_{abc} + \frac{N^2-4}{N}C_2^3\left(-h_g f_{abc}\tilde{h}_{ijk} + d_{abc}h^{(3)}_{ijk}\right)\right\} \tag{2.1.22}
$$

[8]There are also other restrictions we can make on the γs and get the same dimensionality of the manifold of fixed points, but they all are related by the global SU(3) symmetry we have here.

$$h^{(3)}_{ijk} \equiv \bar{h}_{plm}(h_{ijp}h_{klm} + h_{kjp}h_{ilm} + h_{ikp}h_{jlm}) \tag{2.1.23}$$
$$\tilde{h}_{ijk} \equiv \epsilon_{ijp}h^{(2)}_{kp} + \epsilon_{pjk}h^{(2)}_{ip} + \epsilon_{ipk}h^{(2)}_{jp} \tag{2.1.24}$$

$\tilde{h}_{ijk}$ is totally antisymmetric, thus because (i j k) run over (1 2 3), $\tilde{h}_{ijk}$ has only one independent component:

$$\tilde{h}_{123} = \epsilon_{123}h^{(2)}_{33} + \epsilon_{123}h^{(2)}_{22} + \epsilon_{123}h^{(2)}_{11} = Tr(h^{(2)}) \tag{2.1.25}$$
$$\tilde{h}_{ijk} = Tr(h^{(2)})\epsilon_{ijk}. \tag{2.1.26}$$

Now we can look separately on the part of the $\beta - function$ proportional to f_{abc} and on the part proportional to d_{abc}:

$$\beta^{(1)}_{h_g} = \frac{h_g}{16\pi^2}\left\{6C_1(|h_g|^2 - g^2) + \frac{N^2-4}{N}C_2^3 Tr(h^{(2)})\right\} \tag{2.1.27}$$
$$\beta^{(1)}_{ijk} = \frac{1}{16\pi^2}\left\{6C_1(|h_g|^2 - g^2)h_{ijk} + \frac{N^2-4}{N}C_2^3 h^{(3)}_{ijk}\right\}. \tag{2.1.28}$$

When we constrain ourselves only to the case of h_{123}, $h_{111} = h_{222} = h_{333}$, h_g non zero (which is the only case where we will get *exactly* marginal deformations as we will see later), we get:

$$h^{(3)}_{ijk} = h_{ijk}Tr(h^{(2)}). \tag{2.1.29}$$

And in this case:

$$\frac{\beta_{h_g}}{h_g} = \frac{\beta_{ijk}}{h_{ijk}}. \tag{2.1.30}$$

So if we are looking for fixed points we have only one condition on four couplings, and thus we have a three dimensional manifold of fixed points in the coupling constants space [57].

2.1.3 RG–flow analysis

Here we will analyze the $\beta - functions$ obtained in the previous section. The equations will simplify if we rescale the coupling constants:

$$g \to \frac{\sqrt{C_1}}{4\pi}g \quad h_g \to \frac{\sqrt{C_1}}{4\pi}h_g \quad \text{and} \quad h_{ijk} \to \frac{\sqrt{C_2^3}}{4\pi}\sqrt{\frac{N^2-4}{N}}h_{ijk}. \tag{2.1.31}$$

The $\beta - functions$ become:

$$\beta_g = -\frac{2g^3}{1-2g^2}Tr(\gamma), \quad \beta_{h_g} = h_g Tr(\gamma). \tag{2.1.32}$$

Here the trace is taken only over the $SU(3)$ indices and not over gauge indices. From these $\beta - functions$ we can obtain a differential equation:

$$-\frac{1}{2g^3}dg + \frac{1}{g}dg = \frac{dh_g}{h_g}. \tag{2.1.33}$$

This can be easily solved to give:

$$\frac{h_g}{h_g^*} = \frac{g e^{\frac{1}{4g^2}}}{g^* e^{\frac{1}{4(g^*)^2}}}. \tag{2.1.34}$$

This result means that the RG flow lines in the $h_g - g$ plane are exactly known (to the extent that we can count on the NSVZ $\beta - function$). It is easy to convince oneself that there is no line with the couplings going to zero in the UV, except the trivial case when one of the couplings is constantly zero. This implies that there is no choice of coupling constants for which this theory is asymptotically free.

Another interesting question is the existence of fixed points. In order to have a fixed point we have to satisfy $Tr(\gamma) = 0$ which implies at one loop that:

$$Tr(h^{(2)}) = -6(|h_g|^2 - g^2) \tag{2.1.35}$$

And we can substitute this into β_{ijk} to get another condition:

$$Tr(h^{(2)})h_{ijk} = h^{(3)}_{ijk}. \tag{2.1.36}$$

We will argue that these conditions can be satisfied (in the limit $g \to 0$) only if the anomalous dimensions matrix is proportional to identity matrix.

First we don't assume any special property of γ. By multiplying (2.1.36) on both sides by $\bar{h}_{ijk}$ we get:

$$3Tr((h^{(2)})^2) = (Tr(h^{(2)}))^2. \tag{2.1.37}$$

We denote:

$$h^{(2)} \equiv \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \tag{2.1.38}$$

$$\tag{2.1.39}$$

And then:

$$(Tr(h^{(2)}))^2 = (a+e+k)^2 \tag{2.1.40}$$

$$Tr((h^{(2)})^2) = (a^2+e^2+k^2) + 2(bd+cg+hf). \tag{2.1.41}$$

So (2.1.37) implies:

$$(a-e)^2+(a-k)^2+(k-e)^2+6(cg+bd+fh)=0 \tag{2.1.42}$$

But remembering that $h^{(2)}$ is hermitian:

$$c=\bar{g} \quad f=\bar{h} \quad b=\bar{d}, \tag{2.1.43}$$

we get that the only possibility for (2.1.42) to hold is if:

$$a=e=k, \quad h=f=b=g=c=d=0, \tag{2.1.44}$$

which implies: $h^{(2)}_{ij}=\alpha^2\delta_{ij}$ and γ which is proportional to identity matrix. So the theory at weak coupling only has fixed points when the anomalous dimensions matrix is proportional to identity matrix.

If the anomalous dimensions matrix is proportional to the identity then:

$$\gamma^i_j \equiv \rho\, \delta^i_j. \tag{2.1.45}$$

From here and from (2.1.8) we obtain:

$$\beta_{ijk}=Tr(\gamma)\cdot h_{ijk}. \tag{2.1.46}$$

The one loop γ implies further that:

$$h^{(2)}_{ij}=\alpha^2\delta_{ij}, \quad \alpha^2 \equiv \frac{1}{3}\sum_{i,j,k}|h_{ijk}|^2. \tag{2.1.47}$$

So the condition (2.1.36) is automatically satisfied and from (2.1.35) we get:

$$\alpha^2=-2(|h_g|^2-g^2). \tag{2.1.48}$$

The fixed points we found are essentially IR stable fixed points, we have:

$$Tr(\gamma)=3(2(|h_g|^2-g^2)+\alpha^2), \tag{2.1.49}$$

and the condition for a fixed point is $Tr(\gamma)=0$. From the $\beta-functions$ we calculated we see that if we increase one of the couplings h_g, h_{ijk}, $Tr(\gamma)$ becomes positive thus decreasing these couplings and increasing the gauge coupling in IR, till we get again zero. And the same if we decrease the couplings. Thus we can conclude that in the weak coupling limit all fixed points that exist imply diagonal γ and are IR stable. Consequently nothing is known of the UV behavior of the theory, and we can unambiguously define it only at the conformal fixed points. Now if we restrict ourselves only to the case of $h_{111}=h_{222}=h_{333}\equiv h'\neq 0$ and $h_{123}\neq 0$, with all other couplings vanishing, then from the symmetry of the interactions we see that γ^i_j is proportional to the identity ($\gamma^i_j \equiv \gamma\delta^i_j$).

Finally, turning back to the original couplings (before the rescaling (2.1.31)), working with our conventions in which $C_1=N$, $C_2=1$ (see Appendix A) and making the last coupling redefinition

$h_{123} + ih_g \equiv he^{i\pi\beta}$ (and $h_{123} - ih_g \equiv he^{-i\pi\beta}$ so that the new parameter β is real), the conformal condition becomes

$$\left[|h|^2 \left(1 - \frac{1}{N^2}\left|e^{i\pi\beta} - e^{-i\pi\beta}\right|^2\right) + |h'|^2 \frac{N^2-4}{2N^2}\right] = g^2 \tag{2.1.50}$$

All the calculations in this section were done based on the one loop anomalous dimensions. An interesting question is how the results are altered by higher loop calculations. As just discussed the request for the anomalous dimensions of the elementary chiral superfields to vanish guarantees the theory to be superconformal invariant. In general we do not know the superconformal condition exactly. However it is possible to perform a perturbative analysis and define the superconformal theory order by order in the couplings.

To summarize, the full Leigh-Strassler $\mathcal{N} = 1$ deformation of the $\mathcal{N} = 4$ SYM theory can be rewritten using the following action [57]

$$\begin{aligned} S &= \int d^8 z \mathrm{Tr}(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i) + \frac{1}{2g^2}\int d^6 z \mathrm{Tr}(W^\alpha W_\alpha) \\ &\quad + \left\{ ih \int d^6 z \mathrm{Tr}(q\,\Phi_1\Phi_2\Phi_3 - \bar{q}\,\Phi_1\Phi_3\Phi_2) + \frac{ih'}{3}\int d^6 z \mathrm{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) \ + \ \text{h.c.}\right\} \end{aligned} \tag{2.1.51}$$

where we have set $q \equiv e^{i\pi\beta}$, $\bar{q} \equiv e^{-i\pi\beta}$, β real. The gauge coupling g has been chosen to be real in order to avoid dealing with instantonic effects, whereas h is generically complex.

Recall that the superfield strength $W_\alpha = i\overline{D}^2(e^{-gV}D_\alpha e^{gV})$ is given in terms of a real prepotential V and $\Phi_{1,2,3}$ contain the six scalars of the original $\mathcal{N} = 4$ SYM theory organized into the $\mathbf{3} \times \bar{\mathbf{3}}$ of $SU(3)$ subgroup of the R–symmetry group $SU(4)$. We write $V = V^a T_a$, $\Phi_i = \Phi_i^a T_a$ where T_a are $SU(N)$ matrices in the fundamental representation[9].

First set $h' = 0$. The so called β–deformation breaks $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 1$ and the original $SU(4)$ R–symmetry to $U(1)_R$. However, two extra non–R–symmetry global $U(1)$'s survive. Applying the a–maximization procedure [109] and the conditions of vanishing ABJ anomalies it turns out that $U(1)_R$ is the one which assigns the same R–charge ω to the three elementary superfields, whereas the charges with respect to the two non–R–symmetries $U(1)_1 \times U(1)_2$ can be chosen to be $(\Phi_1, \Phi_2, \Phi_3) \to (0, 1, -1)$ and $(-1, 1, 0)$, respectively.

The action (2.1.51) possesses two extra discrete symmetries. One is the Z_3 associated to cyclic permutations of (Φ_1, Φ_2, Φ_3) which is a remnant of the original $SU(3) \subset SU(4)$ symmetry of the undeformed theory, whereas the other one corresponds to exchanges

$$\Phi_i \leftrightarrow \Phi_j \ , \quad i \neq j \qquad \text{and} \qquad q \to -\bar{q} \quad (\beta \to 1-\beta) \tag{2.1.52}$$

The equations of motion for the chiral superfields are

$$\bar{D}^2(e^{-gV}\bar{\Phi}_1^a e^{gV}) = -ih\Phi_2^b\Phi_3^c\ [q(abc) - \bar{q}(acb)] \tag{2.1.53}$$

and cyclic, where $(abc) \equiv \mathrm{Tr}(T^a T^b T^c)$.

[9]For more details on our conventions we refer to Appendix A and paper [106, 107, 108, 71].

At the quantum level the theory is superconformal invariant (and then finite) up to two loops if the coupling constants satisfy the following condition (vanishing of the beta functions) [70, 71]

$$|h|^2 \left[1 - \frac{1}{N^2} \, |q - \bar{q}|^2 \right] = g^2 \tag{2.1.54}$$

Superconformal invariance at three loops has been discussed in [72] for any N. In the large N limit this condition reduces simply to $|h|^2 = g^2$, independently of the value of q. In [74] it has been proven that this is the *exact* superconformal invariance condition for the large N theory dual to the Lunin–Maldacena supergravity background [66].

Now consider the full superpotential with $h' \neq 0$. The original $SU(4)$ R–symmetry is broken to $U(1)_R$ and no extra $U(1)$'s are left. However, the action is still invariant under the cyclic permutation of (Φ_1, Φ_2, Φ_3) and the symmetry (2.1.52). Moreover, a second Z_3 is left corresponding to

$$(\Phi_1, \Phi_2, \Phi_3) \quad \rightarrow \quad (\Phi_1, z\Phi_2, z^2\Phi_3) \tag{2.1.55}$$

where z is a cubic root of unity.

The equations of motion derived from (2.1.51) are

$$\begin{aligned}
\bar{D}^2(e^{-gV}\bar{\Phi}^a_1 e^{gV}) &= -ih\Phi^b_2\Phi^c_3\,[q(abc) - \bar{q}(acb)] - ih'\Phi^b_1\Phi^c_1(abc) \\
\bar{D}^2(e^{-gV}\bar{\Phi}^b_2 e^{gV}) &= -ih\Phi^a_1\Phi^c_3\,[q(abc) - \bar{q}(acb)] - ih'\Phi^a_2\Phi^c_2(abc) \\
\bar{D}^2(e^{-gV}\bar{\Phi}^c_3 e^{gV}) &= -ih\Phi^a_1\Phi^b_2\,[q(abc) - \bar{q}(acb)] - ih'\Phi^a_3\Phi^b_3(abc)
\end{aligned} \tag{2.1.56}$$

Since the three chirals have the same anomalous dimension due to the cyclic Z_3 symmetry, superconformal invariance requires a single condition $\gamma(g, h, h', \beta) = 0$ and we find a three–dimensional complex manifold of fixed points.

To this purpose we evaluate the anomalous dimension of the chiral superfield Φ_i up to two loops. The calculation can be carried on exactly as in the case of $h' = 0$ by taking into account that compared to the previous case the present action contains three extra chiral vertices of the form $\frac{ih'}{6} d_{abc}\Phi^a_i\Phi^b_i\Phi^c_i$, $i = 1, 2, 3$.

As long as we deal with diagrams which do not contain the new h' vertices we have exactly the same contributions as in the $h' = 0$ theory [70, 71]. We only need evaluate all the diagrams which contain these extra vertices.

At one loop, besides the h–chiral and the mixed gauge–chiral self–energy diagrams [71] we have a h'–chiral self–energy graph whose contribution is proportional to $|h'|^2$. This new diagram modifies the one–loop superconformal condition (2.1.54) as

$$\left[|h|^2 \left(1 - \frac{1}{N^2} \, |q - \bar{q}|^2 \right) + |h'|^2 \frac{N^2 - 4}{2N^2} \right] = g^2 \tag{2.1.57}$$

in agreement with [115, 58, 65]. As for the $h' = 0$ case it is easy to verify that the one–loop condition is sufficient to guarantee the vanishing of the beta functions (i.e. superconformal invariance) up to two loops.

2.2 The chiral ring of the β-deformed theory

We are now interested in studying perturbatively the structure of the chiral ring for the β–deformed theory (2.1.51) with $h' = 0$. As discussed in [110], for a generic $\mathcal{N} = 1$ SYM theory scalar operators in the chiral ring can be constructed as products of scalar chiral superfields Φ_i and/or times $(W^\alpha W_\alpha)$, where W_α is the chiral field strength. In this paper we will focus only on the Φ–sector, neglecting operators with a dependence on W_α.

In [61, 62, 66] the single–trace sector of the chiral ring has been identified as given by chiral operators of the form $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3})$ with weight $\Delta_0 = J_1 + J_2 + J_3$ and $(J_1, J_2, J_3) = (J, 0, 0), (0, J, 0), (0, 0, J), (J, J, J)$. In [70, 71] it has been shown perturbatively that also the assignements $(J_1, J_2, J_3) = (1, 1, 0), (1, 0, 1), (0, 1, 1)$ give protected operators.

This classification identifies the CPO's according to their dimension and their charges with respect to the two $U(1)$ global invariances of the theory. However, it does not give any information on the precise form of the protected operator corresponding to a given set (J_1, J_2, J_3), which turns out to be in general a linear combination of single–trace operators with different order of the fields inside the trace. Moreover, if we work at finite N, mixing with multi–trace operators is also allowed.

A first example has been studied in [70] for the weight–3 sector. There, it has been shown that the correct expression for the protected operator correponding to $(J_1, J_2, J_3) = (1, 1, 1)$ is a linear combination

$$\mathrm{Tr}(\Phi_1\Phi_2\Phi_3) + \alpha \mathrm{Tr}(\Phi_1\Phi_3\Phi_2) \tag{2.2.1}$$

where at one–loop

$$\alpha = \frac{(N^2 - 2)\bar{q}^2 + 2}{N^2 - 2 + 2\bar{q}^2} \tag{2.2.2}$$

showing an explicit dependence on the coupling β.

We are interested in the generalization of this result to higher loops in order to investigate whether and how the linear combination gets modified order by order. Moreover, we extend this analysis to other sectors of the chiral ring in order to discuss mixing at finite N.

In general, given a set of primary operators $\mathcal{O}_i$ with the same dimension Δ_0 and the same global charges, we can read their anomalous dimensions perturbatively from the matrix of the two–point correlation functions. Precisely, this matrix has the form

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{1}{x^{2\Delta_0}} \left(A_{ij} - \rho_{ij} \log \mu^2 x^2 \ + \ \cdots \ \right) \tag{2.2.3}$$

where dots stay for higher powers in $\log \mu^2 x^2$. Here A is the mixing matrix, whereas ρ signals the appearance of anomalous dimensions. Both matrices are given as power series in the couplings.

In order to determine the anomalous dimensions we need diagonalize the two matrices by performing the linear transformation $\mathcal{O}' = L\mathcal{O}$ which maps the operators into an orthogonal basis of quasi–primaries. In a perturbative approach it is easy to see [111, 112] that the diagonalization of the ρ matrix at order n fixes the correct orthogonalization (resolution of the mixing) at order $(n-1)$ uniquely, up to a residual rotation among operators with the same anomalous dimension. This means that in general an order n calculation is required to determine the anomalous dimensions at this order and the correct linear combinations of operators $\mathcal{O}_i$ at order $(n-1)$ which correspond to quasi–primaries with well–defined anomalous dimensions up to order n.

In our case, since we are interested into **chiral primary operators**, the procedure to determine perturbatively the correct linear combination which corresponds to a protected operator is made simpler if we also use the definition of chiral ring.

In our conventions the chiral ring is the set of chiral operators which cannot be written, by using the equations of motion, as $\bar{D}^2 X$, being X any primary operator.

In general, given a set of linearly independent chiral operators $\mathcal{C}_i$, $i = 1, \cdots, s$ characterized by the same classical scale dimension Δ_0 and the same charges under the two $U(1)$ flavor groups they will mix and we need solve the mixing in order to compute their anomalous dimensions. Since we are working with chiral operators, we know a priori that once we have orthogonalized as $\mathcal{C}'_i = L_{ij}\mathcal{C}_j$ in order to have well–defined quasi–primary operators, some of them will turn out to be descendant, i.e. they can be written as $\bar{D}^2 X$ for some primary X. The remaining operators will be necessarily primary chirals with vanishing anomalous dimensions.

Exploiting this simple observation, in order to find the correct expression for the protected operators, we then proceed as follows: In a given (J_1, J_2, J_3) sector, we first select all the descendants, that is all the linear combinations

$$\mathcal{D}_i = \sum_j d_j^{(i)} \mathcal{C}_j \tag{2.2.4}$$

which satisfy the condition

$$\mathcal{D}_i = \bar{D}^2 X_i \tag{2.2.5}$$

Let us suppose that there are $i = 1, \cdots, r \leq s$ independent linear combinations of this type. Then, for a generic operator $\mathcal{P} = \sum_j c_j \mathcal{C}_j$ we impose the orthogonality condition

$$\langle \mathcal{P}\bar{\mathcal{D}}_i \rangle = 0 \qquad i = 1, \cdots, r \tag{2.2.6}$$

where $\bar{\mathcal{D}}$ indicates the hermitian conjugate of $\mathcal{D}$. These constraints provide r equations for the s unknowns c_j. In this way we select a $(s-r)$-dimensional subspace of operators orthogonal to the descendant ones. We can choose an appropriate (orthogonal) basis in this subset, obtaining $(s-r)$ independent operators which are protected. This procedure has been already applied in the undeformed $\mathcal{N} = 4$ case [113].

The problem of determining the CPO's of the theory is then traslated into the problem of finding *all* the linear combinations of operators which satisfy the condition (2.2.5). In particular, since we are interested into a perturbative determination of the chiral ring we need find descendants which solve eq. (2.2.5) order by order in perturbation theory. This can be done by introducing a perturbative definition of quantum chiral ring, as we are now going to explain in detail.

2.2.1 The perturbative quantum chiral ring

As previously discussed, the chiral ring is defined as the set of chiral operators orthogonal to null operators, i.e. linear combinations of chirals which can be written in the form $\bar{D}^2 X$, X primary. At the classical level a linear combination (2.2.4) gives rise to a null operator every time the coefficients $d_j^{(i)}$ are such that the operator $\mathcal{D}_i$ can be rewritten as a product of chiral superfields times $\frac{\delta W}{\delta \Phi_k}$, where W is the classical superpotential[10]

$$W = ih \left[q \ \mathrm{Tr}(\Phi_1 \Phi_2 \Phi_3) - \bar{q} \ \mathrm{Tr}(\Phi_1 \Phi_3 \Phi_2) \right] \tag{2.2.7}$$

Indeed, if this is the case, we can use the classical equations of motion $\bar{D}^2 \bar{\Phi}_k = -\frac{\delta W}{\delta \Phi_k}$ to express the operator as in (2.2.5). It follows that we can alternatively define the chiral ring as

$$\mathcal{C} = \{ \text{chiral op.}'\text{s } \mathcal{P} \mid \langle \mathcal{P} \bar{\mathcal{D}} \rangle = 0, \text{ for any } \mathcal{D} \sim (...\Phi..\Phi..\frac{\delta W}{\delta \Phi}) \} \tag{2.2.8}$$

where in $\mathcal{D}$ we do not indicate trace structures and flavor charges explicitly. In the undeformed $\mathcal{N}=4$ theory, an immediate consequence of the definition (2.2.8) is that all the CPO's correspond to completly symmetric representations of the $SU(3) \subset SU(4)$ R–symmetry group [54].

This definition for the chiral ring allows for a straightforward generalization at the quantum level. Since the quantum dynamics of the elementary superfields is driven by the effective superpotential rather than the classical W, it appears natural to define the quantum chiral ring as

$$\mathcal{C}_Q = \{ \text{chiral op.}'\text{s } \mathcal{P} \mid \langle \mathcal{P} \bar{\mathcal{D}}_Q \rangle = 0, \text{ for any } \mathcal{D}_Q \sim (..\Phi...\Phi...\frac{\delta W_{eff}}{\delta \Phi}) \} \tag{2.2.9}$$

where now $\mathcal{D}_Q$ is a *quantum* null operator. Using the quantum equations of motion $\bar{D}^2 \frac{\delta K}{\delta \Phi_i} = -\frac{\delta W_{eff}}{\delta \Phi_i}$ where K is the effective Kähler potential which takes into account possible perturbative D–term corrections, it is easy to see that $\mathcal{D}_Q$ is a null operator at the quantum level. In the undeformed $\mathcal{N}=4$ case the symmetries of the theory constrain $\mathcal{D}_Q$ to be proportional to $\mathcal{D}$ and the quantum chiral ring coincides with the classical one (2.2.8).

When W_{eff} is determined perturbatively, eq. (2.2.9) gives a perturbative definition of chiral ring. Precisely, given W_{eff} at a fixed perturbative order[11]

$$W_{eff} = W + \lambda W_{eff}^{(1)} + \lambda^2 W_{eff}^{(2)} + \cdots + \lambda^L W_{eff}^{(L)} \tag{2.2.10}$$

we can construct independent descendants [12] at that order as

$$\mathcal{D} = \mathcal{D}_0 + \lambda \mathcal{D}_1 + \lambda^2 \mathcal{D}_2 + \cdots + \lambda^L \mathcal{D}_L \qquad , \qquad \mathcal{D}_i = \Phi \ldots \frac{\delta W_{eff}^{(i)}}{\delta \Phi} \tag{2.2.11}$$

and determine the protected operators $\mathcal{P}$ by imposing the orthogonality condition $\langle \mathcal{P} \bar{\mathcal{D}} \rangle = 0$ order by order. Since $\mathcal{P}$ will be in general a linear combination of single/multitrace operators, these conditions allow to determine the coefficients of the linear combination order by order in the couplings. If we set

$$\mathcal{P} = \mathcal{P}_0 + \lambda \mathcal{P}_1 + \lambda^2 \mathcal{P}_2 + \cdots + \lambda^L \mathcal{P}_L \tag{2.2.12}$$

[10]This is true only for operators which are not affected by Konishi–like anomalies or as long as these anomalies do not enter the actual calculation (see the discussion at the beginning of Section 2.2.4).

[11]In principle, perturbative corrections to W_{eff} would depend on both g and h couplings. Here we mean to use the superconformal invariance condition to express $|h|^2$ as a function of g^2 and write the perturbative expansion in powers of the 't Hooft coupling $\lambda = \frac{g^2 N}{4\pi^2}$.

[12]As long as we are interested in orthogonalizing with respect to the whole space generated by the descendants, we do not need the precise form of pure descendants, but just a suitable set of linear independent states. From now on we will refer to this definition of quantum descendants.

the perturbative corrections $\mathcal{P}_j$ will be determined by

$$\begin{array}{lll} O(\lambda^0) & : & \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_0 = 0 \\ O(\lambda^1) & : & \langle \mathcal{P}_0 \bar{\mathcal{D}}_1 \rangle_0 + \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_1 + \langle \mathcal{P}_1 \bar{\mathcal{D}}_0 \rangle_0 = 0 \\ \vdots & & \vdots \\ O(\lambda^L) & : & \langle \mathcal{P}_0 \bar{\mathcal{D}}_L \rangle_0 + \langle \mathcal{P}_0 \bar{\mathcal{D}}_{L-1} \rangle_1 + \cdots + \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_L + \langle \mathcal{P}_1 \bar{\mathcal{D}}_{L-1} \rangle_0 + \cdots + \langle \mathcal{P}_L \bar{\mathcal{D}}_0 \rangle_0 = 0 \end{array} \tag{2.2.13}$$

where $\langle\ \rangle_j$ stands for the two–point function at order λ^j.

Conditions (2.2.13) together with the general statement that orthogonalization at order $(n-1)$ is sufficient for having well–defined quasi–primary operators at order n, brings us to formulate the following prescription: In order to determine perturbatively the correct form of chiral operators with vanishing anomalous dimension at order n it is sufficient to determine the effective superpotential at order $(n-1)$, select all the descendant operators at that order by (2.2.11) and impose the conditions (2.2.13) up to order $(n-1)$. In so doing, we gain a perturbative order at each step. Moreover, in order to have all the descendants at a given order it is sufficient to compute the effective superpotential once for all.

As follows from its definition, the structure of the chiral ring is directly related to the structure of the effective superpotential. Therefore, the perturbative corrections to the CPO's depend on the perturbative corrections to the effective superpotential. In particular, this explains universality properties of the protected operators we will discuss in Section 2.2.3, as for example the fact that in any case the orthogonalization at tree level is sufficient for the protection up to two loops.

2.2.2 The effective superpotential at two–loops

Since we are dealing with a superconformal (finite) theory any correction to the effective action must be finite. By definition, the effective superpotential corresponds to perturbative, finite F–terms evaluated at zero momenta. It is given by *local* contributions which are constrained by dimensions, $U(1)\times U(1)$ flavor symmetry charges, reality and symmetry (2.1.52) to have necessarily the form

$$W_{eff} = ih\left[b\ \mathrm{Tr}(\Phi_1\Phi_2\Phi_3) - \bar{b}\ \mathrm{Tr}(\Phi_1\Phi_3\Phi_2)\right] \ + \ \mathrm{h.c.} \tag{2.2.14}$$

The constant b is given as an expansion in the couplings, $b = q(1 + b_1\lambda + b_2\lambda^2 + \cdots)$, with coefficients b_j which are functions of q and N, whereas $\bar{b}$ is the hermitian conjugate. We note that in principle the symmetries of the theory would only constrain the form of the superpotential to $W_{eff} = \{ih\left[b(q)\ \mathrm{Tr}(\Phi_1\Phi_2\Phi_3) + b(-\bar{q})\ \mathrm{Tr}(\Phi_1\Phi_3\Phi_2)\right] \ + \ \mathrm{h.c.}\}$. However, it is easy to show that $b(-\bar{q}) = -\overline{b(q)}$ since the b_j coefficients are rational functions of q^2 with real coefficients (loop diagrams always give real contributions and they always contain an even number of extra chiral vertices compared to the tree–level vertex).

At a given order L we can have two kinds of corrections to W_{eff}: Corrections which do not mix the two terms in the superpotential and are then of the form

$$W_{eff}^{(L)} \sim \lambda^L\ W \tag{2.2.15}$$

where W is the classical superpotential. These contributions do not affect the structure of the descendant operators at order L since $\frac{\delta W_{eff}^{(L)}}{\delta\Phi} \sim \frac{\delta W}{\delta\Phi}$ and $\mathcal{D}_L \sim \mathcal{D}_0$. As a consequence at order L

the correlation function $\langle \mathcal{P}_0 \bar{\mathcal{D}}_L \rangle_0$ in (2.2.13) vanishes and the protected operator is determined only by loop corrections to its two–point function with descendants of lower orders.

The second kind of corrections to W_{eff} mixes the two terms in W and gives rise to a linear combination $W_{eff}^{(L)}$ of the form (2.2.14) which is not proportional to the classical superpotential anymore. For these corrections the request for the protected operator to be orthogonal to a descendant proportional to $\frac{\delta W_{eff}^{(L)}}{\delta \Phi}$ modifies in general its structure by contributions of order λ^L proportional to $\langle \mathcal{P}_0 \bar{\mathcal{D}}_L \rangle_0$.

In this section we evaluate explicitly the effective superpotential up to two loops. Our result is useful for determining the correct CPO's up to three loops.

The diagrams contributing to the effective superpotential up to this order are given in Fig. 2.1 where the grey bullets indicate the one–loop corrections to the chiral and gauge–chiral vertices, respectively. These corrections are exactly the ones of the undeformed $\mathcal{N} = 4$ theory once we use the one–loop superconformal invariance condition (2.1.54).

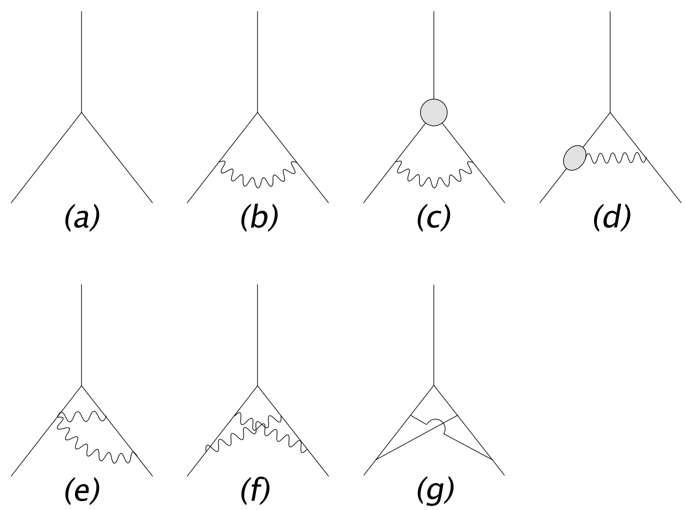

Figure 2.1: Diagrams contributing to the effective superpotential up to two loops.

The one–loop diagram 2.1b), compared with the tree level diagram 2.1a), does not contain any extra q–deformed vertex. Moreover, using standard color identities it is easy to see that its contribution is proportional to λW, where W is the classical superpotential.

The same happens at two loops for the diagrams 2.1c), 2.1d) and 2.1e) which do not contain any extra q–deformed vertex and have a color structure which does not mix the two traces, so reproducing W.

Diagram 2.1f) vanishes for color reasons.

Diagram 2.1g) contains four extra q–deformed vertices. Moreover, by direct inspection one can easily see that the nonplanar chiral structure which corrects the tree level diagram mixes nontrivially the two terms of W. As a result at two loops the superpotential undergoes a nontrivial modification of the form

$$W_{eff}^{(2)} \sim ih\left[q\,P\;\mathrm{Tr}(\Phi_1\Phi_2\Phi_3) - \bar q\,\bar P\;\mathrm{Tr}(\Phi_1\Phi_3\Phi_2)\right] \;+\; \mathrm{h.c.} \tag{2.2.16}$$

with

$$P = \frac{(q^2-1)^3[N^2+3+q^2(3N^2-10+7q^2)]}{q^2[q^4+1+(N^2-2)q^2]^2} \tag{2.2.17}$$

Here we have used $\bar q = 1/q$. We note that the nontrivial q–dependence of this diagram is a direct consequence of its nonplanarity. In fact, as discussed in [74] planar diagrams depend on the particular combination $q\bar q = 1$, while the nonplanar ones have generically nontrivial phases. Moreover, a q–dependence has also been introduced by using the superconformal condition (2.1.54) to express the coefficient $|h|^4$ from the four chiral vertices in terms of λ^2.

To evaluate the various contributions from Fig. 2.1 we first perfom D–algebra to reduce superdiagrams to ordinary loop diagrams and compute the corresponding integrals in momentum space (for the description of the procedure and our conventions we refer to [106, 107, 108, 71]). As reported in Appendix A the one and two–loop integrals are all finite and they give a well–defined, local value for external momenta set to zero. Therefore, collecting all the contributions, at two loops the superpotential has the structure (2.2.14) with

$$b = q\left[(1+\lambda c_1+\lambda^2 c_2) + \lambda^2\frac{3}{8}\zeta(3)P\right] \tag{2.2.18}$$

where the coefficients c_1, c_2 are numbers, independent of q and N, determined by the loop integrals 2.1b) and 2.1c)–2.1e), respectively (we do not need their explicit values).

It follows that in general a descendant at this order will have the form

$$\mathcal{D}_Q = (1+\lambda c_1+\lambda^2 c_2)\mathcal{D}_0 + \lambda^2\mathcal{D}_2 \tag{2.2.19}$$

with $\mathcal{D}_2 \neq \mathcal{D}_0$.

2.2.3 Chiral Primary Operators in the spin–2 sector

The $(J,1,0)$ flavor

We start considering operators of the form $\mathrm{Tr}(\Phi_1^J\Phi_2)$. In this case, due to the ciclicity of the trace, there is no ambiguity in the ordering of the operators inside the trace. In the large N limit these operators do not belong to the chiral ring, they are descendants and their anomalous dimensions have been computed exactly [74] for J large. However, for finite N they can mix with multitraces and give rise to linear combinations of single and multi–trace operators which are protected. We are going to construct them perturbatively up to three loops. For simplicity we consider first the particular cases of $J = 3, 4$ and postpone the discussion for generic J at the end of this section.

The $(3,1,0)$ case: The first nontrivial example where mixing conspires to give rise to protected operators is for $J = 3$. This sector contains the two operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^3\Phi_2) \qquad , \qquad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2) \tag{2.2.20}$$

Using the classical equations of motion (2.1.53), it is easy to see that

$$\bar{D}^2 \mathrm{Tr}(\Phi_1^2 e^{-gV} \bar{\Phi}_3 e^{gV}) = \mathrm{Tr}\left(\Phi_1^2 \frac{\delta W}{\delta \Phi_3}\right) = -ih\,(q-\bar{q})\,[\mathrm{Tr}(\Phi_1^3\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)] \quad (2.2.21)$$

and a descendant can be constructed as (we always forget about the normalization of the operators)

$$\mathcal{D}_0 = \mathcal{O}_1 - \frac{1}{N}\mathcal{O}_2 \quad (2.2.22)$$

The knowledge of $\mathcal{D}_0$ allows us to determine the one–loop protected operator. We consider the linear combination

$$\mathcal{P}_0 = \mathcal{O}_1 + \alpha_0\,\mathcal{O}_2 \quad (2.2.23)$$

which, for any $\alpha_0 \neq -\frac{1}{N}$, gives an operator in the chiral ring. We then impose the orthogonality condition $\langle \mathcal{P}_0\,\bar{\mathcal{D}}_0\rangle_0 = 0$ and find

$$\alpha_0 = -\frac{N^2-6}{2N} \quad (2.2.24)$$

This result coincides with the one found in [72] where the one–loop CPO has been determined by diagonalizing directly the one–loop anomalous dimension matrix.

In order to extend our analysis to higher loops we need establish the correct form of the descendant operator order by order, as described in Section 2.2. If we look at its perturbative definition (2.2.11) and the way the equations of motion work in this case, we easily realize that as long as the effective superpotential has the structure (2.2.14) we obtain

$$\mathrm{Tr}\left(\Phi_1^2 \frac{\delta W_{eff}}{\delta \Phi_3}\right) = -ih\left(b-\bar{b}\right)[\mathrm{Tr}(\Phi_1^3\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)] \quad (2.2.25)$$

whatever b might be (determined perturbatively at a given order). It follows that the linear combination on the r.h.s. of this equation, which is nothing but the operator (2.2.22), is always a descendant operator independently of the order we have computed the coefficient b. Therefore we conclude that (2.2.22) is the *exact* quantum descendant up to an overall coupling–dependent normalization factor, that is $\mathcal{D}_Q \sim \mathcal{D}_0$.

An alternative way [113] to establish the relation $\mathcal{D}_Q \sim \mathcal{D}_0$ is to consider the combination

$$\bar{D}^2 \mathrm{Tr}(\Phi_1^2 e^{-gV} \bar{\Phi}_3 e^{gV}) + ih\,(q-\bar{q})\,[\mathrm{Tr}(\Phi_1^3\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)] \quad (2.2.26)$$

which is zero at tree level and check that it is order by order orthogonal to the three monomials $\bar{D}^2\mathrm{Tr}(\Phi_1^2 e^{-gV}\bar{\Phi}_3 e^{gV})$, $\mathrm{Tr}(\Phi_1^3\Phi_2)$ and $\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)$, separately. In fact, if this is the case, there is no extra mixing of the linear combination (2.2.26) with the three operators at the quantum level and (2.2.26) must be necessarily zero at any order in perturbation theory. We have checked the absence of mixing perturbatively up to two loops confirming our conclusion.

In order to determine the protected operator we consider the linear combination

$$\mathcal{P} = \mathcal{O}_1 + \alpha\,\mathcal{O}_2 \quad (2.2.27)$$

with α given as an expansion in λ

$$\alpha = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + O(\lambda^3) \tag{2.2.28}$$

In the notation of Section 2.2 we have $\mathcal{P}_0 = \mathcal{O}_1 + \alpha_0 \mathcal{O}_2$ with α_0 already determined in (2.2.24) and $\mathcal{P}_j = \alpha_j \mathcal{O}_2$.

As a consequence of the relation $\mathcal{D}_Q \sim \mathcal{D}_0$ the orthogonality conditions (2.2.13) become

$$O(\lambda) \ : \qquad \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_1 + \langle \mathcal{P}_1 \bar{\mathcal{D}}_0 \rangle_0 = 0 \tag{2.2.29}$$

$$O(\lambda^2) : \qquad \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_2 + \langle \mathcal{P}_1 \bar{\mathcal{D}}_0 \rangle_1 + \langle \mathcal{P}_2 \bar{\mathcal{D}}_0 \rangle_0 = 0 \tag{2.2.30}$$

The first condition (2.2.29) gives

$$\alpha_1 = - \frac{\langle (\mathcal{O}_1 + \alpha_0 \mathcal{O}_2) \bar{\mathcal{D}}_0 \rangle_1}{\langle \mathcal{O}_2 \bar{\mathcal{D}}_0 \rangle_0} \tag{2.2.31}$$

In order to select the diagrams which contribute to the two–point function at the numerator we note that the tree level correlation function at the denominator, when computed in momentum space and in dimensional regularization ($n = 4 - 2\epsilon$), is $1/\epsilon$ divergent. This divergence signals the well–known short distance singularity of any two–point function of a conformal field theory.

If the denominator of (2.2.31) goes as $1/\epsilon$, in the numerator we can consider only divergent diagrams (finite diagrams would not contribute in the $\epsilon \to 0$ limit). It is easy to show that at this order the only diagram which we need take into account is the one in Fig. 2.2 where on the left hand side we have an insertion of the operator $(\mathcal{O}_1 + \alpha_0 \mathcal{O}_2)$ while on the right hand side we have $\bar{\mathcal{D}}_0$.

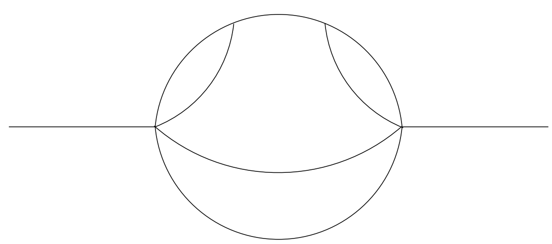

Figure 2.2: One–loop diagram contributing to the evaluation of α_1.

By a direct calculation one realizes that if α_0 is chosen as in (2.2.24) this diagram vanishes. The reason is very simple to understand: If we cut the diagram vertically at the very right end, close to the $\bar{\mathcal{D}}_0$ vertex, from the calculation it comes out that the left part would be nothing but a one–loop divergent contribution to the operator $(\mathcal{O}_1 + \alpha_0 \mathcal{O}_2)$ which vanishes since α_0 has been determined just to give a protected (not renormalized) operator at one–loop.

From the one–loop constraint we then read $\alpha_1 = 0$ and the expression (2.2.23) with α_0 as in (2.2.24) corresponds to the protected chiral operator up to two loops.

Next we analyze the constraint (2.2.30). Setting $\mathcal{P}_1 = 0$ there, we obtain

$$\alpha_2 = - \frac{\langle (\mathcal{O}_1 + \alpha_0 \mathcal{O}_2) \, \bar{\mathcal{D}}_0 \rangle_2}{\langle \mathcal{O}_2 \, \bar{\mathcal{D}}_0 \rangle_0} \tag{2.2.32}$$

and consequently the exact expression for the CPO up to three loops.

Again we select only divergent diagrams contributing to the numerator. They are given in Fig. 2.3. We have not drawn diagrams associated to the two–loop anomalous dimension of the operator $(\mathcal{O}_1 + \alpha_0 \mathcal{O}_2)$ which vanish when α_0 is chosen as in (2.2.24).

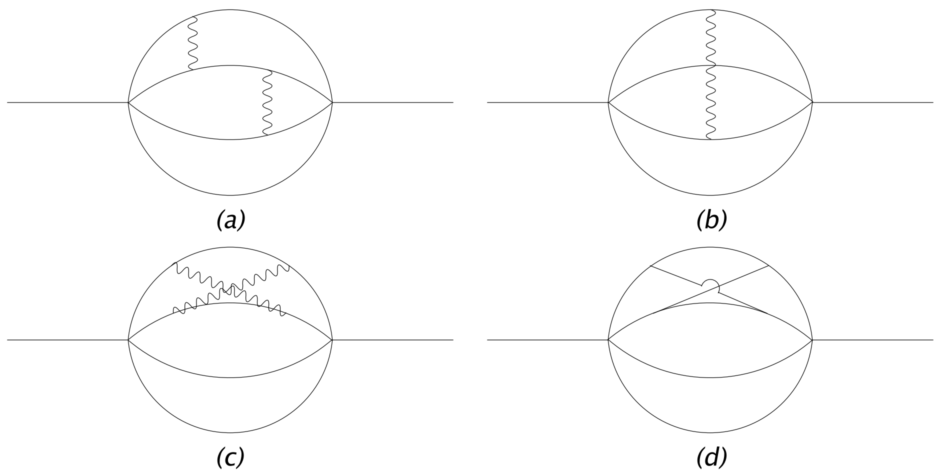

Figure 2.3: Two-loop diagrams contributing to the evaluation of α_2.

These diagrams contribute nontrivially to α_2 since, cutting the graphs at the very right hand side, their left parts cannot be recognized as corrections to the tree–level operator (nontrivial mixing between $\mathcal{O}_1$ and $\mathcal{O}_2$ occurs). Evaluating the diagrams by using the results in Appendix A we obtain

$$\alpha_2 = \frac{9(N^2-9)(q^2-1)^2[(N^4-8N^2-8)(q^4+1)+2(N^4+8)q^2]}{80N[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \qquad (2.2.33)$$

where we have used the one–loop superconformal condition (2.1.54) to express all the contributions of Fig. 2.3 in terms of λ^2 and set $\bar{q} = 1/q$.

Therefore the protected operator $\mathcal{P}$ up to three–loops can be written as

$$\mathcal{P} = \mathcal{O}_1 - \frac{N^2-6}{2N}(1 + r\,\lambda^2)\,\mathcal{O}_2 \qquad (2.2.34)$$

with

$$r = \frac{\alpha_2}{\alpha_0} = -\frac{9(N^2-9)(q^2-1)^2[(N^4-8N^2-8)(q^4+1)+2(N^4+8)q^2]}{40(N^2-6)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \qquad (2.2.35)$$

We note that in the 't Hooft limit, $N \to \infty$ and λ fixed, $\mathcal{O}_2$ dominates and gives the protected operator up to three loops. This is consistent with the fact that, in the absence of mixing, the only primary operators in a given Δ_0 sector are necessarily products of single–trace primaries $\mathrm{Tr}(\Phi_1^m)$ and $\mathrm{Tr}(\Phi_1\Phi_2)$.

The $(4,1,0)$ case: It is interesting to analyze this case in detail since it is the first case where more than one descendant appears.

This sector contains three independent operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^4\Phi_2) \quad , \quad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1^3)\mathrm{Tr}(\Phi_1\Phi_2) \quad , \quad \mathcal{O}_3 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1^2\Phi_2) \tag{2.2.36}$$

Using the classical equations of motion (2.1.53), we can write

$$\bar{D}^2\mathrm{Tr}(\Phi_1^3 e^{-gV}\bar{\Phi}_3 e^{gV}) = \mathrm{Tr}\left(\Phi_1^3\frac{\delta W}{\delta\Phi_3}\right) = -ih\,(q-\bar{q})\,[\mathrm{Tr}(\Phi_1^4\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^3)\mathrm{Tr}(\Phi_1\Phi_2)] \tag{2.2.37}$$

$$\bar{D}^2\left[\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1 e^{-gV}\bar{\Phi}_3 e^{gV})\right] = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}\left(\Phi_1\frac{\delta W}{\delta\Phi_3}\right) = -ih\,(q-\bar{q})\,\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1^2\Phi_2) \tag{2.2.38}$$

Therefore, in this case we can consider the two descendants

$$\mathcal{D}_0^{(1)} = \mathcal{O}_1 - \frac{1}{N}\mathcal{O}_2 \qquad , \qquad \mathcal{D}_0^{(2)} = \mathcal{O}_3 \tag{2.2.39}$$

or any linear combination which realizes an orthogonal basis in the subspace of weight–5 descendants.

As in the previous example it is easy to prove that, given the particular structure (2.2.14) of the effective superpotential and the way the equations of motion enter the calculation, the linear combinations $\mathcal{D}_0^{(1)}$ and $\mathcal{D}_0^{(2)}$ provide two independent descendants even at the quantum level.

Proceeding as before we consider the linear combination

$$\mathcal{P} = \mathcal{O}_1 + \alpha\,\mathcal{O}_2 + \beta\,\mathcal{O}_3 \tag{2.2.40}$$

and choose the constants α and β (expanded in powers of λ) by requiring $\mathcal{P}$ to be orthogonal to the two descendants up to two loops.

Solving the constraints $\langle\mathcal{P}_0\bar{\mathcal{D}}_0^{(i)}\rangle_0$ at tree level we determine the correct expression for the operator characterized by a vanishing one–loop anomalous dimension

$$\mathcal{P}_0 = \mathcal{O}_1 - \frac{N^2-12}{3N}\,\mathcal{O}_2 - \frac{2}{N}\,\mathcal{O}_3 \tag{2.2.41}$$

As in the previous case, this operator is automatically orthogonal to $\mathcal{D}_0^{(1)}$ and $\mathcal{D}_0^{(2)}$ also at one loop and so we expect it to be protected up to two loops.

The orthogonality at two loops can be imposed exactly as in the previous case and allows to determine the corrections α_2 and β_2. The diagrams contributing are still the ones in Fig. 2.3 with one extra free chiral line running between the two vertices. Performing the calculation we find the final expression for the operator protected up to three loops

$$\mathcal{P} = \mathcal{O}_1 - \frac{N^2-12}{3N}\,(1+s_1\,\lambda^2)\,\mathcal{O}_2 - \frac{2}{N}\,(1+s_2\,\lambda^2)\,\mathcal{O}_3 \tag{2.2.42}$$

where

$$\begin{aligned} s_1 = \frac{\alpha_2}{\alpha_0} &= \frac{(N^2-16)(q^2-1)^2\,[(11N^2+21)(q^4+1)+2(N^2-21)q^2]}{4(N^2-12)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \\ s_2 = \frac{\beta_2}{\beta_0} &= -\frac{(N^2-16)(q^2-1)^2[(N^2+5)(q^4+1)+2(N^2-5)q^2]}{8[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \end{aligned} \tag{2.2.43}$$

Again, the coefficients depend on N in such a way that in the large N limit only the $\mathcal{O}_2$ operator in (2.2.36) survives in agreement with the chiral ring content of the theory in the planar limit.

We note that these coefficients, as well as r in (2.2.35) are real. This is a consequence of the fact that in the sectors studied so far the descendant operators are q–independent and the two–point correlation functions are real.

The previous analysis can be applied to the generic operators of the form $(\Phi_1^J\Phi_2)$. The peculiar pattern $\mathcal{D}_Q \sim \mathcal{D}_0$ for the descendants occurs in any $(J,1,0)$ sector since it only depends on the particular structure of the superpotential and the particular way the equations of motion work for this class of operators. Therefore, the determination of CPO's proceeds as before. In particular, we expect the tree level orthogonality condition to be still sufficient for protection up to two loops since the only one–loop diagram relevant for the calculation would be the vanishing one–loop anomalous dimension diagram in Fig. 2.2. At two loops diagrams of the kind drawn in Fig. 2.3 should be still the only relevant ones.

Without entering the details of the calculations which would be quite involved and not very illuminating, we can determine the dimension of the corresponding chiral ring subspace, i.e. the number of independent protected operators corresponding to $U(1)$ flavors $(J,1,0)$.

To be definite we consider J even $(J=2p)$. In this case the list of chirals we can construct is

$$\begin{array}{ll}
\text{single}-\text{trace} & \text{Tr}(\Phi_1^{2p}\Phi_2) \\
\text{double}-\text{trace} & \text{Tr}(\Phi_1^{m_1})\ \text{Tr}(\Phi_1^{2p-m_1}\Phi_2) \qquad\qquad m_1=2,\cdots,2p-1 \\
\text{triple}-\text{trace} & \text{Tr}(\Phi_1^{m_1})\ \text{Tr}(\Phi_1^{m_2})\ \text{Tr}(\Phi_1^{2p-m_1-m_2}\Phi_2) \\
 & \qquad m_1=2,\cdots,p-1, \quad m_2=m_1,\cdots,2p-1-m_1 \\
\vdots & \\
p-\text{trace} & \text{Tr}(\Phi_1^2)\cdots\text{Tr}(\Phi_1^2)\ \text{Tr}(\Phi_1^2\Phi_2)\ ,\ \text{Tr}(\Phi_1^3)\ \text{Tr}(\Phi_1^2)\cdots\text{Tr}(\Phi_1^2)\ \text{Tr}(\Phi_1\Phi_2)
\end{array} \tag{2.2.44}$$

In order to find how many independent primaries we can construct out of (2.2.44) we need first count how many descendants of the form (2.2.5) we have. As explained in the previous simple examples, given the generic n–trace, $\Delta_0 = J$ sector, null conditions come from considering the operators

$$\text{Tr}(\Phi_1^{m_1})\cdots\text{Tr}(\Phi_1^{m_{n-1}})\bar{D}^2\text{Tr}(\Phi_1^{2p-1-m_1-\ldots-m_{n-1}}e^{-gV}\bar{\Phi}_3 e^{gV}) \tag{2.2.45}$$

as long as $2p-1-m_1-\ldots-m_{n-1}\geq 1$. In fact, once we act with $\bar{D}^2$ on $\bar{\Phi}_3$ and use the equations of motion (2.1.53) we generate the linear combination

$$\begin{aligned}
&\text{Tr}(\Phi_1^{m_1})\cdots\text{Tr}(\Phi_1^{m_{n-1}})\text{Tr}(\Phi_1^{2p-m_1-\ldots-m_{n-1}}\Phi_2) \\
&\qquad -\frac{1}{N}\text{Tr}(\Phi_1^{m_1})\cdots\text{Tr}(\Phi_1^{m_{n-1}})\text{Tr}(\Phi_1^{2p-1-m_1-\ldots-m_{n-1}})\text{Tr}(\Phi_1\Phi_2)
\end{aligned} \tag{2.2.46}$$

which is then a descendant. Therefore, the complete list of descendants is

$$\text{single}-\text{trace} \qquad \bar{D}^2\ \text{Tr}(\Phi_1^{2p-1}e^{-gV}\bar{\Phi}_3 e^{gV})$$

$$
\begin{array}{lll}
\text{double}-\text{trace} & \bar{D}^2\left[\mathrm{Tr}(\Phi_1^{m_1})\ \mathrm{Tr}(\Phi_1^{2p-1-m_1}e^{-gV}\bar{\Phi}_3e^{gV})\right] & m_1=2,\cdots,2p-2\\
\text{triple}-\text{trace} & \bar{D}^2\left[\mathrm{Tr}(\Phi_1^{m_1})\ \mathrm{Tr}(\Phi_1^{m_2})\ \mathrm{Tr}(\Phi_1^{2p-1-m_1-m_2}e^{-gV}\bar{\Phi}_3e^{gV})\right] & \\
 & \qquad m_1=2,\cdots,p-1,\quad m_2=m_1,\cdots,2p-2-m_1 & \\
\vdots & & \\
p-\text{trace} & \bar{D}^2\left[\mathrm{Tr}(\Phi_1^2)\cdots\ \mathrm{Tr}(\Phi_1^2)\ \mathrm{Tr}(\Phi_1e^{-gV}\bar{\Phi}_3e^{gV})\right] &
\end{array}
\tag{2.2.47}
$$

Counting how many operators we have in (2.2.44) and subtracting the number of descendants in (2.2.47) we find that the number of protected chiral operators is $\sum_{n=2}^{p} X_n$ where X_n is the number of partitions of $(2p-1)$ objects into $(n-1)$ boxes with at least 2 objects per box. Analogously, the number of chiral primary operators for J odd is $\sum_{n=2}^{p+1} X_n$.

This result is consistent with the number of primary operators which survive in the large N limit where mixing effects disappear and the chiral ring reduces to products of single–trace operators $\mathrm{Tr}(\Phi_1^k)$, $\mathrm{Tr}(\Phi_1\Phi_2)$.

The $(2,2,0)$ flavor

In the class of more general operators with weights $(J_1,J_2,0)$ we consider the particular case $J_1=J_2=2$. This sector contains four operators, two single– and two double–traces

$$
\begin{array}{lcl}
\mathcal{O}_1=\mathrm{Tr}(\Phi_1^2\Phi_2^2) & , & \mathcal{O}_2=\mathrm{Tr}(\Phi_1\Phi_2\Phi_1\Phi_2)\\
\mathcal{O}_3=\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_2^2) & , & \mathcal{O}_4=\mathrm{Tr}(\Phi_1\Phi_2)\mathrm{Tr}(\Phi_1\Phi_2)
\end{array}
\tag{2.2.48}
$$

Using the classical equations of motion (2.1.53), we can write

$$
\begin{aligned}
\bar{D}^2\left[\mathrm{Tr}(\Phi_1\Phi_2e^{-gV}\bar{\Phi}_3e^{gV})-\mathrm{Tr}(\Phi_2\Phi_1e^{-gV}\bar{\Phi}_3e^{gV})\right] &= -ih(q+\bar{q})[\mathcal{O}_2-\mathcal{O}_1]\\
\bar{D}^2\left[\mathrm{Tr}(\Phi_1\Phi_2e^{-gV}\bar{\Phi}_3e^{gV})+\mathrm{Tr}(\Phi_2\Phi_1e^{-gV}\bar{\Phi}_3e^{gV})\right] &= -ih(q-\bar{q})[\mathcal{O}_1+\mathcal{O}_2-\frac{2}{N}\mathcal{O}_4]
\end{aligned}
\tag{2.2.49}
$$

We note that on the right hand side of these equations the q–dependence is still factored out as it happened in the previous cases (see eqs. (2.2.21, 2.2.38)). Therefore, tree level descendants can be defined as linear combinations

$$
\begin{aligned}
\mathcal{D}_0^{(1)} &= \mathcal{O}_2-\mathcal{O}_1\\
\mathcal{D}_0^{(2)} &= \mathcal{O}_1+\mathcal{O}_2-\frac{2}{N}\mathcal{O}_4
\end{aligned}
\tag{2.2.50}
$$

Because of their q–independence these operators correspond indeed to a suitable choice of quantum descendants.

The general structure of a chiral primary operator in this sector is

$$
\mathcal{P}=\alpha\mathcal{O}_1+\beta\,\mathcal{O}_2+\gamma\,\mathcal{O}_3+\delta\,\mathcal{O}_4 \tag{2.2.51}
$$

where the coefficients are determined order by order by the orthogonality conditions $\langle\mathcal{P}\bar{\mathcal{D}}_0^{(1)}\rangle$ and $\langle\mathcal{P}\bar{\mathcal{D}}_0^{(2)}\rangle$. Having two conditions for four unknowns we expect to single out two protected operators.

At tree level, for the particular choice $\alpha_0 = 2, \beta_0 = 1$ and $\alpha_0 = 1, \beta_0 = -1$, we find

$$\begin{aligned}
\mathcal{P}^{(1)} &= 2\mathcal{O}_1 + \mathcal{O}_2 - \frac{N^2-6}{2N}(\mathcal{O}_3 + 2\mathcal{O}_4) \\
\mathcal{P}^{(2)} &= \mathcal{O}_1 - \mathcal{O}_2 - \frac{N}{4}\mathcal{O}_3 + N\mathcal{O}_4
\end{aligned} \tag{2.2.52}$$

These are one–loop protected operators and coincide with the ones found in [72]. They are not orthogonal but a basis can be easily constructed by considering linear combinations.

According to the general pattern already discussed for the previous cases we expect the operators (2.2.52) to be protected up to two loops. The condition for these operators to be protected up to three loops requires instead nontrivial λ^2–corrections to (2.2.52) which can be determined by solving the orthogonality constraints at this order. The diagrams contributing nontrivially to the 2–point functions are still the ones in Fig. 2.3. Since the final expressions are quite unreadable, we find convenient to fix $\alpha_2 = \beta_2 = 0$ for both the CPO's and we obtain

$$\begin{aligned}
\mathcal{P}^{(1)} &= 2\mathcal{O}_1 + \mathcal{O}_2 - \frac{N^2-6}{2N}(1 + t_1\,\lambda^2)\mathcal{O}_3 - \frac{N^2-6}{N}(1 + t_2\,\lambda^2)\mathcal{O}_4 \\
\mathcal{P}^{(2)} &= \mathcal{O}_1 - \mathcal{O}_2 - \frac{N}{4}(1 + u_1\,\lambda^2)\mathcal{O}_3 + N(1 + u_2\,\lambda^2)\mathcal{O}_4
\end{aligned} \tag{2.2.53}$$

where

$$\begin{aligned}
t_1 &= -\frac{9(N^2-9)(q^2-1)^2[(N^4-6N^2-4)(q^4+1) + 2(N^4-2N^2+4)q^2]}{20(N^2-6)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \\
t_2 &= \frac{9(N^2-9)(N^2+2)(q^2-1)^4}{10(N^2-6)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3)
\end{aligned} \tag{2.2.54}$$

and

$$\begin{aligned}
u_1 &= -\frac{9(q^2-1)^2[(N^6-9N^4-16N^2+18)(q^4+1) + 2(N^6-14N^4+34N^2-18)q^2]}{20N^2[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \\
u_2 &= \frac{9(q^2-1)^2[(N^4-31N^2-18)(q^4+1) - 2(7N^4-13N^2-18)q^2]}{40N^2[q^4+1+(N^2-2)q^2]^2}\,\zeta(3)
\end{aligned} \tag{2.2.55}$$

2.2.4 Chiral Primary Operators in the spin–3 sector

This sector contains operators of the form $(\Phi_1^k \Phi_2^l \Phi_3^m)$ with all possible trace structures.

The simplest case is for $k = l = m = 1$ and involves the two weight–3 operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1\Phi_2\Phi_3) \qquad , \qquad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1\Phi_3\Phi_2) \tag{2.2.56}$$

As already mentioned, the correct one–loop expression for the protected operator has been determined in [70] by computing directly the anomalous dimension at that order. It turns out that the protected operator is a linear combination of the two operators (2.2.56) with coefficient α as in (2.2.2). The result has been confirmed in [72] by using a simplified approach based on the evaluation of the difference between the one–loop two–point function of the deformed theory and the one for the $\mathcal{N} = 4$ case. This approach is very convenient since it avoids computing many graphs containing gauge vertices but, as recognized by the authors, in this case it cannot be pushed beyond one loop.

Using our procedure, we can easily re-derive the Freedman–Gursoy result by working at tree level and extend it to two–loops by performing a one–loop calculation. The correct application of our procedure beyond this order would require a substantial modification in the definition of quantum chiral ring (2.2.9) since in this sector descendants of Konishi–like operators are present and the equations of motion need be supplemented by the Konishi anomaly term. As a consequence the corresponding chiral ring sector necessarily contains operators depending on $W^\alpha W_\alpha$.

In fact, from the anomalous conservation equation for the Konishi current we can write

$$\bar{D}^2 \mathrm{Tr}(e^{-gV}\bar{\Phi}_i e^{gV}\Phi_i) = -3ih[q\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3) - \bar{q}\mathrm{Tr}(\Phi_1\Phi_3\Phi_2)] + \frac{1}{32\pi^2}\mathrm{Tr}(W^\alpha W_\alpha) \tag{2.2.57}$$

We remind that in our conventions $W_\alpha = i\bar{D}^2(e^{-gV}D_\alpha e^{gV})$ and it is at least of order g. From the previous identity it follows that a descendant operator has to be constructed out of the two operators (2.2.56) plus the anomaly term

$$\mathcal{D}_0 = q\mathcal{O}_1 - \bar{q}\mathcal{O}_2 + \frac{i}{96\pi^2\,h}\mathrm{Tr}(W^\alpha W_\alpha) \tag{2.2.58}$$

However, since the operator $\mathrm{Tr}(W^\alpha W_\alpha)$ is of order g^2 and has vanishing tree level two–point function with $\mathcal{O}_1$ and $\mathcal{O}_2$ it does not enter the orthogonality conditions at tree level and one–loop. Therefore we can safely use our procedure to find CPO's up to two loops forgetting about the anomaly.

Thus we consider the linear combination

$$\mathcal{P}_0 = \mathcal{O}_1 + \alpha_0\,\mathcal{O}_2 \tag{2.2.59}$$

for any value of $\alpha_0 \neq -\bar{q}^2$. In order to determine the exact expression for the CPO at one–loop we need impose the operator to be orthogonal to the descendant (2.2.58) at tree level. A simple calculation proves that $\langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_0 = 0$ iff α_0 is given in (2.2.2), in agreement with the result of [70].

At one loop first we need determine the correct expression for the descendant at this order. As it follows from the calculations of Section 2.2 at one loop the effective superpotential is proportional to the tree level W and the corresponding descendant operator is still proportional to $\mathcal{D}_0$ in eq. (2.2.58). Given the generic linear combination $\mathcal{P} = \mathcal{O}_1 + (\alpha_0 + \alpha_1\lambda)\,\mathcal{O}_2$ we then impose the orthogonality condition up to order λ to uniquely determine α_1 as in (2.2.31). As in the previous examples, if α_0 is given in (2.2.2) the α_1 coefficient is identically zero being this a consequence of the one–loop protection of $\mathcal{P}_0$. Therefore the expression (2.2.59) with α_0 given in (2.2.2) corresponds to the protected chiral operator up to two loops.

The next case we investigate is for $k = 2$, $l = m = 1$. There are five operators

$$\begin{aligned} &\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^2\Phi_2\Phi_3) \quad , \quad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1^2\Phi_3\Phi_2) \quad , \quad \mathcal{O}_3 = \mathrm{Tr}(\Phi_1\Phi_2\Phi_1\Phi_3) \\ &\mathcal{O}_4 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_2\Phi_3) \quad , \quad \mathcal{O}_5 = \mathrm{Tr}(\Phi_1\Phi_2)\mathrm{Tr}(\Phi_1\Phi_3) \end{aligned} \tag{2.2.60}$$

Using the classical equations of motion (2.1.53) we can write three descendants

$$\begin{aligned} \mathcal{D}_0^{(1)} &= q\mathcal{O}_3 - \bar{q}\mathcal{O}_2 - \frac{1}{N}(q-\bar{q})\mathcal{O}_5 \\ \mathcal{D}_0^{(2)} &= q\mathcal{O}_1 - \bar{q}\mathcal{O}_3 - \frac{1}{N}(q-\bar{q})\mathcal{O}_5 \\ \mathcal{D}_0^{(3)} &= q\mathcal{O}_1 - \bar{q}\mathcal{O}_2 - \frac{1}{N}(q-\bar{q})\mathcal{O}_4 \end{aligned} \tag{2.2.61}$$

We expect to find out two protected operators of the form

$$\mathcal{P} = \alpha\,\mathcal{O}_1 + \beta\,\mathcal{O}_2 + \gamma\,\mathcal{O}_3 + \delta\,\mathcal{O}_4 + \epsilon\,\mathcal{O}_5 \tag{2.2.62}$$

By imposing the tree-level orthogonality condition with respect to the three $\mathcal{D}_0^{(i)}$ we can fix for instance γ, δ and ϵ in terms of α and β. The calculation proceeds exactly as in the previous case and we find

$$\begin{aligned} \gamma &= \frac{\alpha[q^4 - 2q^2 + 1 - N^2] - \beta[(1-N^2)q^4 - 2q^2 + 1]}{N^2(q^4-1)} \\ \delta &= \frac{\alpha[(N^2+2)q^4 + 2(N^2-2)q^2 + N^4 - 5N^2 + 2]}{2N^3(q^4-1)} \\ &\quad - \frac{\beta[(N^4 - 5N^2 + 2)q^4 + 2(N^2-2)q^2 + N^2 + 2]}{2N^3(q^4-1)} \\ \epsilon &= \frac{\alpha[2(N^2+1)q^4 + (N^4-4)q^2 + N^4 - 4N^2 + 2]}{N^3(q^4-1)} \\ &\quad - \frac{\beta[(N^4 - 4N^2 + 2)q^4 + (N^4-4)q^2 + 2(N^2+1)]}{N^3(q^4-1)} \end{aligned} \tag{2.2.63}$$

We expect these operators to have a vanishing anomalous dimension at one loop. If we set $\alpha = \beta = 1$ and $\alpha = -\beta = 1$, we recover the two protected operators found in [72].

As in the previous cases, the operators $\mathcal{D}_0^{(1)}$, $\mathcal{D}_0^{(2)}$ and $\mathcal{D}_0^{(3)}$ keep being good descendants at one loop. Moreover, the one–loop orthogonality conditions do not modify the CPO's (2.2.62, 2.2.63) and we expect these operators to have a vanishing two–loop anomalous dimension.

If we were to push our calculation beyond this order we should first determine the descendant operators at two loops. It is easy to realize that in this case the relation $\mathcal{D}_Q \sim \mathcal{D}_0$ does not hold anymore, for two simple reasons:

1) At higher orders the Konishi anomaly cannot be ignored anymore. In particular, the correct expression for the descendant operators from two loops on will have a nontrivial dependence on $(W^\alpha W_\alpha)$.

2) Differently from the spin–2 case, the nontrivial corrections to the effective superpotential which appear at two loops determine nontrivial corrections to the descendants since in this case they depend on q not only through an overall coefficient (see eq. (2.2.61)).

2.3 The full Leigh–Strassler deformation

Now we consider the chiral ring of the full Leigh–Strassler theory given in (2.1.51). Once the theory is made finite with (2.1.57), we are interested in the perturbative evaluation of *finite* corrections to the superpotential, as observed in the previous sections. Taking $h' \neq 0$, the symmetries of the theory force the effective superpotential to have the form

$$W_{eff} = ih \int d^6 z \mathrm{Tr}[b(q)\,\Phi_1\Phi_2\Phi_3 + b(-\bar{q})\,\Phi_1\Phi_3\Phi_2] + \frac{ih'}{3} d \int d^6 z \mathrm{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) \; + \; \mathrm{h.c.} \quad (2.3.1)$$

where the coefficients b and d are determined as double power expansions in the couplings h and h' [13]. In particular, the invariance under cyclic permutations of the superfields requires the d correction to be the same for the three Φ_i^3 terms, whereas the other global symmetries force the particular q dependence of the corrections to $(\Phi_1\Phi_2\Phi_3)$ and $(\Phi_1\Phi_3\Phi_2)$. We note that in this case we cannot apply the previous arguments (see the discussion after eq. (2.2.14)) to state that $b(-\bar{q}) = -\overline{b(q)}$ since the perturbative corrections to $(\Phi_1\Phi_2\Phi_3)$ and $(\Phi_1\Phi_3\Phi_2)$ are not always proportional to q times functions of q^2. In fact, it is still true that diagrams contributing to the effective potential contain an even number of extra chiral vertices compared to the tree level diagrams, but part of these vertices could be h'–vertices not carrying any q–dependence.

The topologies of diagrams contributing to the superpotential up to two loops are still the ones in Fig. 2.1 where now chiral vertices may be either h or h' vertices. Performing the explicit calculation as in Section 2.2.2 we discover that at one loop the various terms in the superpotential do not mix and receive separate corrections still proportional to the classical terms. Precisely, we find that $W_{eff}^{(1)}$ coincides with W, up to an overall constant coefficient. This is also true at two loops for the diagrams 2.1c), 2.1d) and 2.1e), whereas the diagram 2.1g) with all possible configurations of h and h' vertices mixes nontrivially the various terms of the superpotential. Similarly to what happens for the β–deformed theory, this leads to a nontrivial correction $W_{eff}^{(2)}$ which has the form (2.3.1) but with the b and d coefficients nontrivially corrected by functions of q and N. We then expect descendant operators to get modified at this order as in the previous case (see discussion around eq. (2.2.19)).

The exact supergravity dual of the theory (2.1.51) is still unknown even if few steps towards it have been undertaken in [55]. However, it is interesting to investigate the nature of composite operators of the superconformal field theory waiting for the discovery of the exact correspondence of these operators to superstring states.

The chiral ring for the h'-deformed theory is not known in general (however, see [62]). Compared to the chiral ring of the β–deformed theory ($h' = 0$) which contains operators of the form $\mathrm{Tr}(\Phi_i^J)$, $\mathrm{Tr}(\Phi_1^J\Phi_2^J\Phi_3^J)$ plus the particular operators $\mathrm{Tr}(\Phi_i\Phi_j)$, $i \neq j$, we expect the chiral ring of the present theory to be more complicated because of the lower number of global symmetries present.

Here we exploit the general procedure described in Section 2.2 to move the first steps towards the determination of chiral primary operators. In particular, we concentrate on the first simple cases of matter chiral operators with dimensions $\Delta_0 = 2, 3$ and study how turning on the h'-interaction may affect their quantum properties. We then take advantage of these results to make a preliminary discussion of the CPO content for generic scale dimensions.

[13] Here we use the superconformal condition (2.1.57) to express g^2 as a function of h and h'. Any other choice would be equally acceptable.

2.3.1 Chiral ring: The $\Delta_0 = 2$ sector

Weight–2 chiral operators are $\mathrm{Tr}(\Phi_i^2)$ and $\mathrm{Tr}(\Phi_i\Phi_j)$, $i \neq j$. These operators can be classified as in Table 1 according to their charge $\mathcal{Q}$ with respect to the Z_3 symmetry (2.1.55).

$\mathcal{Q}=0$	$\mathcal{Q}=1$	$\mathcal{Q}=2$
$\mathcal{O}_{11} = \mathrm{Tr}(\Phi_1^2)$	$\mathcal{O}_{33} = \mathrm{Tr}(\Phi_3^2)$	$\mathcal{O}_{22} = \mathrm{Tr}(\Phi_2^2)$
$\mathcal{O}_{23} = \mathrm{Tr}(\Phi_2\Phi_3)$	$\mathcal{O}_{12} = \mathrm{Tr}(\Phi_1\Phi_2)$	$\mathcal{O}_{13} = \mathrm{Tr}(\Phi_1\Phi_3)$

Table 2.1: Operators with $\Delta_0 = 2$.

The charged sectors can be obtained from the $\mathcal{Q} = 0$ one by successive applications of cyclic Z_3–permutations $\Phi_i \to \Phi_{i+1}$. This is the reason why the three sectors contain the same number of operators. In the $h' = 0$ theory their anomalous dimensions have been computed up to two loops and found to be vanishing [70, 71]. According to our discussion in Section 2.2 this was an expected result since for these operators there is no way to use the equations of motion (2.1.53) to write them as $\bar{D}^2 X$. Therefore they must be necessarily primaries and belong to the classical chiral ring. Since this sector does not contain descendants this property is mantained at the quantum level. In the $h' = 0$ case these operators have different $U(1)$ flavor charges and do not mix. The matrix of their two–point functions is then diagonal and receives finite corrections at two loops [71].

The same analysis can be applied in the present case. Again, there is no way to write these operators as descendants by using the classical equations of motion (2.1.56). Therefore, we expect them to belong to the chiral ring.

In order to check that these operators do not get renormalized but their correlators might receive finite corrections we compute directly their two–point functions.

The smaller number of global symmetries surviving the h'–deformation do not prevent the operators to mix. For instance the operator $\mathrm{Tr}(\Phi_1^2)$ can mix with $\mathrm{Tr}(\Phi_2\Phi_3)$ since they have the same charge under the Z_3 symmetry (2.1.55). Therefore, we need compute the non–diagonal matrix of their two–point functions.

To this purpose we concentrate on the operators $\mathcal{O}_{11}$ and $\mathcal{O}_{23}$ and evaluate all the correlators up to two loops. The calculation goes exactly as in the $h' = 0$ theory with the understanding of adding contributions from diagrams containing the new h'–vertices.

At one–loop, as in the undeformed [106, 107] and the β–deformed cases [71] we do not find any divergent nor finite contributions to the two–point functions as long as the superconformal condition (2.1.57) holds.

At two loops the topologies of diagrams which contribute to $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ and $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$ are the ones in Fig. 2.4.

Here the grey bullets indicate two–loop corrections to the chiral propagator and one–loop corrections to the mixed gauge–chiral vertex. Using the superconformal condition (2.1.57) their q, h, h' dependence disappears and these corrections coincide with the ones of the $\mathcal{N} = 4$ theory [102, 106, 107]. Therefore the first three diagrams give the same kind of contribution to both correlators.

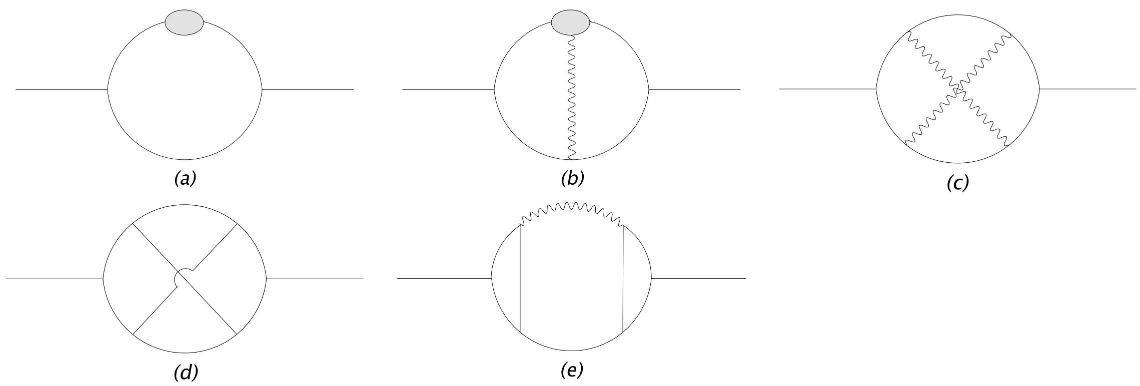

Figure 2.4: Two–loop diagrams for $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ and $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$.

The last two diagrams contain chiral vertices and they instead differ in the two cases for the number of h vs. h' insertions: Diagram 2.4d) gives contributions proportional to $|h|^4$ and $|h'|^4$ to $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$, and contributions proportional to $|h|^4$ and $|h|^2|h'|^2$ to $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$. Analogously, diagram 2.4e) contributes to $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ with a term proportional to $g^2|h'|^2$ and to $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$ with $g^2|h|^2$.

Diagrams contributing to the mixed two–point function $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{23}\rangle$ at two loops are of the type 2.4d) with two h and two h' vertices (contributions proportional to $\bar{h}^2 h'^2$), with three h and one h' (contributions proportional to $|h|^2\bar{h}'h$) and 2.4e) with one h and one h' vertices (contributions proportional to $g^2 h\bar{h}'$).

Performing the D–algebra and computing the corresponding loop integrals in momentum space and dimensional regularization, it is easy to verify that the diagrams 2.4a)–d) have at most $1/\epsilon$ poles which correspond to finite corrections to the two–point functions when transformed back to the configuration space.

The only potential source of anomalous dimension terms would be the graph 2.4e) since, after D–algebra, the corresponding integral has a $1/\epsilon^2$ pole, that is a $\log(\mu^2 x^2)$ divergence in configuration space. However, when computing the correlators $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ and $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{23}\rangle$ this diagram gives a vanishing color factor, whereas for the third correlator there is a complete cancellation between the contribution corresponding to a particular configuration of the $\bar{\Phi}_2$, $\bar{\Phi}_3$ lines coming out from the $\bar{\mathcal{O}}_{23}$ vertex and the one with the two lines interchanged (the same happens in the $h' = 0$ theory [71]).

Therefore, all the correlators in configuration space are two–loop finite, the anomalous dimension matrix vanishes and the two operators are protected up to this order.

It is interesting to give the explicit result for the two–loop corrections to the correlators. We find

$$
\begin{aligned}
\langle \mathrm{Tr}(\Phi_1^2)(z_1)\mathrm{Tr}(\bar{\Phi}_1^2)(z_2)\rangle_{2-\mathrm{loops}} &\sim \frac{\delta^{(4)}(\theta_1-\theta_2)}{[(x_1-x_2)^2]^2}\,\mathcal{F}_1 \\
\langle \mathrm{Tr}(\Phi_2\Phi_3)(z_1)\mathrm{Tr}(\bar{\Phi}_2\bar{\Phi}_3)(z_2)\rangle_{2-\mathrm{loops}} &\sim \frac{\delta^{(4)}(\theta_1-\theta_2)}{[(x_1-x_2)^2]^2}\,\mathcal{F}_2
\end{aligned} \tag{2.3.2}
$$

where

$$\mathcal{F}_1 = \left[|h|^4 \frac{N^2-4}{N^2} |q-\bar{q}|^2 \left(\frac{N^2-1}{4N^2} |q-\bar{q}|^2 - 1\right) \right.$$
$$\left. + \; |h'|^4 \frac{(N^2-20)(N^2-4)}{8N^4} - |h|^2 |h'|^2 \frac{N^2-4}{2N^2} \left(1 - \frac{1}{N^2} |q-\bar{q}|^2\right)\right] \quad (2.3.3)$$

and

$$\mathcal{F}_2 = \left[|h|^4 \frac{N^2-4}{4N^4} |q-\bar{q}|^4 + |h'|^4 \frac{(N^2-4)^2}{8N^4} \right.$$
$$\left. + \; |h|^2 |h'|^2 \frac{N^2-4}{2N^2} \left(3 - \frac{N^2-5}{N^2} |q-\bar{q}|^2\right)\right] \quad (2.3.4)$$

We note that all the g^4 contributions cancel and we are left with expressions which vanish in the $\mathcal{N}=4$ limit ($\beta = h' = 0$, $|h|^2 = g^2$). Moreover, both the contributions survive in the large N limit in contradistinction to the $h'=0$ case where $\mathcal{F}_2$ is subleading [71].

2.3.2 Chiral ring: The $\Delta_0 = 3$ sector

The next sector we investigate contains operators with naive scale dimension $\Delta_0 = 3$. We classify them according to their Z_3–charge as in Table 2.

$\mathcal{Q}=0$	$\mathcal{Q}=1$	$\mathcal{Q}=2$
$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^3)$	$\mathcal{O}_6 = \mathrm{Tr}(\Phi_1^2\Phi_2)$	$\mathcal{O}_9 = \mathrm{Tr}(\Phi_1^2\Phi_3)$
$\mathcal{O}_2 = \mathrm{Tr}(\Phi_2^3)$	$\mathcal{O}_7 = \mathrm{Tr}(\Phi_2^2\Phi_3)$	$\mathcal{O}_{10} = \mathrm{Tr}(\Phi_3^2\Phi_2)$
$\mathcal{O}_3 = \mathrm{Tr}(\Phi_3^3)$	$\mathcal{O}_8 = \mathrm{Tr}(\Phi_3^2\Phi_1)$	$\mathcal{O}_{11} = \mathrm{Tr}(\Phi_2^2\Phi_1)$
$\mathcal{O}_4 = \mathrm{Tr}(\Phi_1\Phi_2\Phi_3)$		
$\mathcal{O}_5 = \mathrm{Tr}(\Phi_1\Phi_3\Phi_2)$		

Table 2.2: Operators with $\Delta_0 = 3$.

We note that the neutral sector does not contain the same number of operators as the charged ones. This is due to the fact that, in contradistinction to the previous case, the $\mathcal{Q}=0$ sector is closed under the application of cyclic permutations $\Phi_i \to \Phi_{i+1}$ and tranformations (2.1.52). Therefore, we cannot generate the charged sectors from the neutral one by using these mappings.

The charged sectors are also closed under permutations but they get exchanged under transformations (2.1.52). This is the reason why they still have the same number of operators.

We first focus on the set of operators with $\mathcal{Q}=0$. As for the $h'=0$ theory, in this sector the Konishi anomaly enters the game when we try to use the equations of motion to write descendants which involve $\mathcal{O}_4$ and $\mathcal{O}_5$. However, as discussed in Section 2.2.4, the Konishi anomaly can be neglected as long as we are interested in the construction of CPO's up to two loops. We will then restrict our analysis at this order.

Using the equations of motion (2.1.56) we can write three descendant operators

$$\begin{aligned}\mathcal{D}^{(1)} &= h\,(q\,\mathcal{O}_4 - \bar{q}\,\mathcal{O}_5) + h'\,\mathcal{O}_1\\ \mathcal{D}^{(2)} &= h\,(q\,\mathcal{O}_4 - \bar{q}\,\mathcal{O}_5) + h'\,\mathcal{O}_2\\ \mathcal{D}^{(3)} &= h\,(q\,\mathcal{O}_4 - \bar{q}\,\mathcal{O}_5) + h'\,\mathcal{O}_3\end{aligned} \tag{2.3.5}$$

According to the discussion of Section 2.2 we expect to single out two protected operators. We consider the most general linear combination

$$\mathcal{P} = \alpha\,\mathcal{O}_1 + \beta\,\mathcal{O}_2 + \gamma\,\mathcal{O}_3 + \delta\,\mathcal{O}_4 + \epsilon\,\mathcal{O}_5 \tag{2.3.6}$$

and require tree-level orthogonality to the three descendants. These constraints provide the condition $\alpha = \beta = \gamma \equiv a$ (as expected because of the Z_3 symmetries of this sector) and the extra relation

$$3a\bar{h}'(N^2-4)q + \bar{h}\left[\delta\,\left(N^2-2+2q^2\right) - \epsilon\,\left((N^2-2)q^2+2\right)\right] = 0 \tag{2.3.7}$$

which can be used to express a in terms of two arbitrary constants.

Any CPO in this sector has then the following form

$$\mathcal{P} = a\,(\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3) + \delta\,\mathcal{O}_4 + \epsilon\,\mathcal{O}_5 \tag{2.3.8}$$

An explicit check on its two–point function at one loop leads to $\langle \mathcal{P}\,\bar{\mathcal{P}}\rangle_1$ finite, independently of the choice of δ and ϵ. One can choose the two constants in order to select two mutually orthogonal operators.

As it happened in the previous cases, these operators are guaranteed to be protected up to two loops as a consequence of their one–loop protection plus the result $W^{(1)}_{eff} \sim W$ which insures that the classical descendants (2.3.5) keep being good descendants also at one loop.

The sectors characterized by Z_3 charges $\mathcal{Q} = 1, 2$ do not contain protected operators. In fact, one can see that any charged operator in Table 2 can be written as $\mathcal{O}_i = \bar{D}^2 X_i$ by using the classical equations of motion. We expect this result to be valid at any order of perturbation theory since the structure of the effective superpotential for what concerns its superfield dependence cannot change.

To summarize, in the $\Delta_0 = 3$ sector we have found two protected operators which are linear combinations of $\mathrm{Tr}(\Phi_i^3)$, $i = 1, 2, 3$, $\mathrm{Tr}(\Phi_1\Phi_2\Phi_3)$ and $\mathrm{Tr}(\Phi_1\Phi_3\Phi_2)$. We note that among all possible weight–3 operators these are the only ones which belong to the chiral ring of the β–deformed theory. The rest of weight–3 operators which were descendants for $h' = 0$ keep being descendants.

The protected operators we have found are neutral under the Z_3 symmetry (2.1.55). As discussed in [62], the neutral sector of the chiral ring (the untwisted sector) coincides with the center of the quantum algebra generated by the F–terms constraints. In particular, for the h'–deformation one element of the center has been constructed explicitly (eq. (4.83) in [62]). This element coincides with one of the CPO's (2.3.8) we have found, once we set $\mathcal{D}^{(i)} = 0$ in the chiral ring (see eq. (2.3.5)), use these identities to express the operator $\mathcal{O}_5$ in terms of the other ones and make a suitable choice for the coefficients δ and ϵ.

2.3.3 Comments on the general structure of the chiral ring

The $\Delta_0 = 2, 3$ sectors studied in the previous section are very peculiar and do not provide enough informations to guess the structure of the sectors for generic scale dimension. In fact, for $\Delta_0 = 2$ no descendants are present and we cannot even apply the orthogonality procedure to construct CPO's. The $\Delta_0 = 3$ sector contains only protected operators which are Z_3 neutral and are linear combinations of "old" CPO's, that is operators which were protected for $h' = 0$.

A naive generalization of our results to higher dimensional sectors would lead to the conjecture that the chiral ring for the h'–deformed theory, at least for what concerns its neutral sector with $\Delta_0 = 3J$, would be given by linear combinations of $\mathrm{Tr}(\Phi_i^{3J})$ and $\mathrm{Tr}(\Phi_1^J \Phi_2^J \Phi_3^J)$. However, we expect more general operators of the form $\mathrm{Tr}(\Phi_1^{3J-m-n} \Phi_2^m \Phi_3^n)$, $m + 2n = \mathrm{mod}(3)$ to appear. Moreover, nontrivial Z_3–charged sectors should appear for $\Delta_0 = 3J$ even if they are absent in the particular case $\Delta_0 = 3$.

To investigate these issues we should extend our analysis to higher dimensional sectors and this would require quite a bit of technical effort. However, without entering any calculative detail, but simply performing dimensional and Z_3–charge balances we can figure out few general properties of the $\mathcal{Q}$–sectors of the chiral ring.

We consider the generic chiral operator $\mathcal{O}_1 = (\Phi_1^a \Phi_2^b \Phi_3^c)$ for any trace structure with scale dimension $\Delta_0 = a + b + c$ and Z_3–charge $\mathcal{Q}_1 \equiv b + 2c$ with respect to the symmetry (2.1.55).

We now perform $\Phi_i \leftrightarrow \Phi_j$ exchanges according to the symmetry (2.1.52) and Z_3 permutations. In this way of doing we generate all the operators with the same trace structure in a given Δ_0 sector. Let us consider for example the operators $\mathcal{O}_2 = (\Phi_2^a \Phi_1^b \Phi_3^c)$ and $\mathcal{O}_3 = (\Phi_3^a \Phi_1^b \Phi_2^c)$ obtained by a $\Phi_1 \leftrightarrow \Phi_2$ exchange and a cyclic permutation, respectively. They have charges $\mathcal{Q}_2 = a + 2c$ and $\mathcal{Q}_3 = 2a + c$. It is easy to see that if $\Delta_0 = 3J$ then $\mathcal{Q}_2 = \mathcal{Q}_3 = 0\ (\mathrm{mod}(3))$ *iff* $\mathcal{Q}_1 = 0\ (\mathrm{mod}(3))$. This property holds for any operator that we can construct from $\mathcal{O}_1$ by the application of the two discrete symmetries. On the other hand, if $\mathcal{Q}_1 = 1, 2(\mathrm{mod}(3))$ operators obtained from it by cyclic permutations still mantain the same charge, but the application of field exchanges (2.1.52) map charge–1 operators into charge–2 operators and viceversa.

Therefore, for $\Delta_0 = 3J$ the $\mathcal{Q} = 0$ class is closed under the action of Z_3–permutations and (2.1.52) symmetry, and being independent, may contain a different number of operators compared to the charged sectors which instead are related by (2.1.52) mappings. In particular, as it happens for $\Delta_0 = 3$ charged classes of the chiral ring might be empty while the corresponding neutral one is not.

If $\Delta_0 \neq 3J$ a simple calculation leads to the conclusion that starting from operators with zero Z_3–charge we generate operators with $\mathcal{Q} = 1$ by applying $\Phi_1 \leftrightarrow \Phi_2$ if $\Delta_0 = 3J + 1$ and a cyclic permutation if $\Delta_0 = 3J + 2$. Correspondingly, we obtain operators with $\mathcal{Q} = 2$ by applying a cyclic permutation in the first case and a $\Phi_1 \leftrightarrow \Phi_2$ exchange in the second case. Therefore, in any sector with $\Delta_0 \neq 3J$ the number of operators with $\mathcal{Q} = 1$ is the same as the ones with $\mathcal{Q} = 2$ and coincides with the number of neutral operators.

If we apply the same reasoning to the descendant operators of each sector (to simplify the analysis we work at large N to avoid mixing among different trace structures) we discover that every time $\Delta_0 \neq 3J$ the descendants of the charged classes can be obtained from the neutral ones by field exchanges. As a consequence, the three classes contain the same number of descendants and then the *same* number of protected operators.

To summarize, the sectors of the chiral ring behave differently according to their scale dimension: If $\Delta_0 \neq 3J$ the corresponding operators are equally split into the three $\mathcal{Q}$ classes. On the contrary, if $\Delta_0 = 3J$ the neutral class is independent and may contain a different number of CPO's.

As a further example we have studied the $\Delta_0 = 4$ operators. In the large N limit and at the lowest order in perturbation theory we have found that the neutral single–trace sector contains one independent CPO (we have eight single–trace chirals and seven descendants). Therefore, we conclude that also the charged sectors contain one single protected operator and we know how to construct it once we have found the $\mathcal{Q} = 0$ operator explicitly. In the single–trace sector the protected operator turns out to be a linear combination of

$$\begin{array}{l} \mathrm{Tr}(\Phi_1^4) \\ \mathrm{Tr}(\Phi_1\Phi_2^3) \quad , \quad \mathrm{Tr}(\Phi_1\Phi_3^3) \quad , \quad \mathrm{Tr}(\Phi_2^2\Phi_3^2) \quad , \quad \mathrm{Tr}(\Phi_2\Phi_3\Phi_2\Phi_3) \\ \mathrm{Tr}(\Phi_1^2\Phi_2\Phi_3) \quad , \quad \mathrm{Tr}(\Phi_1^2\Phi_3\Phi_2) \quad , \quad \mathrm{Tr}(\Phi_1\Phi_2\Phi_1\Phi_3) \end{array} \tag{2.3.9}$$

It remains the open question whether for $\Delta_0 = 3J$, $J > 1$, the charged sectors are trivial as in the weight–3 case. A systematic analysis of the charged protected operators is a difficult task in general. However, working at large N it is easy to realize that for J *even* and $J > 1$, there are nontrivial protected operators for $\mathcal{Q} = 1$ and $\mathcal{Q} = 2$. These are operators with the $3J$ chiral superfields split into the maximal number of traces allowed by $SU(N)$, i.e. $3J/2$. In fact, for these operators it is impossible to exploit the equations of motion and write them as descendants. For J *odd* the same arguments do not lead to any definite conclusion. However, we expect to generate nontrivial charged protected operators by multiplying the neutral CPO's of weight 3 previously constructed by $3(J-1)/2$ traces containing two operators each and carrying the right Z_3 charge.

2.4 Conformal invariance and finiteness theorems for complex β–deformation

2.4.1 Chiral Beta Function and Conformal Condition

Let us consider the $\mathcal{N} = 1$ β–deformed action for complex values of β. We rewrite the action as follows

$$\begin{aligned} S \;=\; & \int d^8z \; \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \; \mathrm{Tr}(W^\alpha W_\alpha) \\ & + ih\int d^6z \; \mathrm{Tr}(\; q\; \Phi_1\Phi_2\Phi_3 - \frac{1}{q}\; \Phi_1\Phi_3\Phi_2 \;) \\ & + i\bar{h}\int d^6\bar{z} \; \mathrm{Tr}(\; \frac{1}{\bar{q}}\; \overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - \bar{q}\; \overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2 \;) \qquad q \equiv e^{i\pi\beta} \end{aligned} \tag{2.4.1}$$

Here h and β are complex couplings and g is the real gauge coupling constant. In the undeformed $\mathcal{N} = 4$ SYM theory one has $h = g$ and $q = 1$. From now on we will be considering 't Hooft rescaled quantities

$$h \to \frac{h}{\sqrt{N}} \qquad g \to \frac{g}{\sqrt{N}} \tag{2.4.2}$$

in order to easily make contact with the planar limit. Moreover we notice that the phase of h can always be reabsorbed by a field redefinition, so that the effective number of independent real parameters in the superpotential is actually three. For later convenience we choose them to be $|h_1|^2$, $|h_2|^2$ and $|h_3|^2$, where

$$h_1 \equiv h\,q \qquad\qquad h_2 \equiv \frac{h}{q} \qquad\qquad h_3 \equiv h\,q - \frac{h}{q} \tag{2.4.3}$$

In the spirit of [57] the idea is to find a surface of renormalization group fixed points in the space of the coupling constants. To this end one can consider the coupling constant reduction program ([46]-[51]) and express the renormalized Yukawa couplings in terms of the gauge one:

$$\begin{aligned} |h_1|^2 &= a_1 g^2 + a_2 g^4 + a_3 g^6 + \dots \\ |h_2|^2 &= b_1 g^2 + b_2 g^4 + b_3 g^6 + \dots \\ |h_3|^2 &= c_1 g^2 + c_2 g^4 + c_3 g^6 + \dots \end{aligned} \tag{2.4.4}$$

This operation has an immediate consequence: we are forced to work perturbatively in powers of g instead of powers of loops. To single out a conformal theory we will ask for the chiral and gauge beta functions to vanish. In this section we will concentrate on β_h and adopt dimensional regularization within minimal subtraction scheme. The chiral beta function is proportional to the anomalous dimension γ of the elementary fields and the condition $\beta_h = 0$ can be conveniently traded with $\gamma = 0$. Even working in a generic scheme, one can easily convince oneself that at a given order in g^2 the proportionality relation between β_h and γ gets affected only by terms proportional to lower order contributions to γ. Therefore, if we set $\gamma = 0$ order by order in the coupling, we are guaranteed that β_h vanishes as well [104]. So the object we will be mainly interested in is the two–point chiral correlator.

In [44] this issue has been analyzed by considering the planar limit where only two independent real constants enter the color factors, namely $|h_1|^2$ and $|h_2|^2$. As a result the definition of the conformal theory was found to be scheme dependent as long as β was complex. In the non–planar case all of the three parameters enter the calculation of the two–point chiral correlator. We will see that this difference will be important in the definition of the fixed point surface.

The idea is to proceed perturbatively in superspace. Supergraphs will be evaluated performing the D–algebra inside the loops and the corresponding divergent integrals will be computed using dimensional regularization in $n = 4-2\epsilon$. In this framework one could allow the coefficients a_i, b_i, c_i in (2.4.4) to be expanded in power series of ϵ [45]. Doing this, evanescent terms are introduced *ad hoc* in order to deal with the $1/\epsilon$ poles and ensure the complete finiteness of the theory. However, after sending $\epsilon \to 0$, they do not enter the relation between renormalized coupling constants so we will neglect their possible presence hereafter.

Let us start at order g^2. As first proposed in [72] it is convenient to consider the difference between divergent diagrams in the β–deformed and in the $\mathcal{N} = 4$ theory. This amounts to the evaluation of the chiral bubbles in Fig. 2.5 which give the following divergent contribution to the chiral propagator

$$\frac{1}{(4\pi)^2}\left[|h_1|^2+|h_2|^2-\frac{2}{N^2}|h_3|^2-2g^2\right]\frac{1}{\epsilon}\left(\frac{\mu^2}{p^2}\right)^{\epsilon} \tag{2.4.5}$$

where we have explicitly indicated the factors coming from dimensionally regulated integral (here p is the external momentum and μ is the standard renormalization mass).

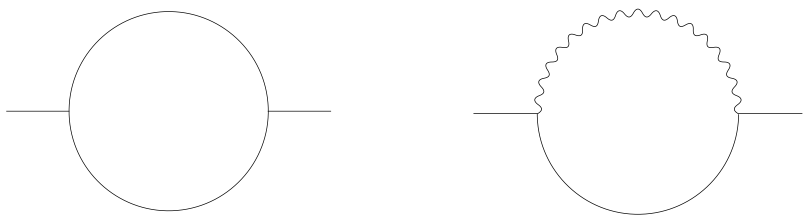

Figure 2.5: One–loop diagrams.

At this stage, in order to obtain a vanishing chiral beta function, the following condition has to be imposed

$$\mathcal{O}(g^2): \qquad a_1+b_1-\frac{2}{N^2}c_1=2 \tag{2.4.6}$$

Moreover, it is well known that

$$|h_1|^2+|h_2|^2-\frac{2}{N^2}|h_3|^2=2g^2 \tag{2.4.7}$$

ensures $\gamma=0$ up to two loops [71]. So, looking at the chiral two–point contribution (2.4.5) at order g^4, we have the following additional requirement

$$\mathcal{O}(g^4): \qquad a_2+b_2-\frac{2}{N^2}c_2=0 \tag{2.4.8}$$

It is easy to see that equations (2.4.6) and (2.4.8) reduce to the ones found in [44] in the large N limit. When we move up to the next order the situation becomes more involved with respect to the planar case. In fact, working with finite N we need to consider the non–planar graph in Fig. 2.6, whose contribution is:

$$\frac{1}{(4\pi)^6}\;2\zeta(3)\;\mathcal{F}\;\frac{1}{\epsilon}\;\left(\frac{\mu^2}{p^2}\right)^{3\epsilon} \tag{2.4.9}$$

where $\mathcal{F} \equiv \mathcal{F}\left(|h_1|^2, |h_2|^2, |h_3|^2, N^2\right)$ reads [72, 143]

$$\mathcal{F} = \frac{N^2-4}{N^4}\ |h_3|^2\ \left[\frac{N^2+5}{N^2}\ |h_3|^4 - 3|h_3|^2(|h_1|^2+|h_2|^2) + 3(|h_1|^2-|h_2|^2)^2\right] \tag{2.4.10}$$

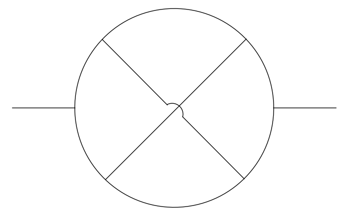

Figure 2.6: Three loop non–planar diagram.

Notice that the color factor in (2.4.10) is suppressed as $1/N^2$ for large N. Due to the expansion in (2.4.4) both the one loop (2.4.5) and three loops (2.4.9) structures contribute to the evaluation of γ at $\mathcal{O}(g^6)$. The final result can be recast as

$$\frac{1}{\epsilon}\left[A\left(\frac{\mu^2}{p^2}\right)^{\epsilon} + \frac{B}{N^2}\left(\frac{\mu^2}{p^2}\right)^{3\epsilon}\right] \tag{2.4.11}$$

where we have defined for concision

$$A \equiv \frac{1}{(4\pi)^2}\ (a_3 + b_3 - \frac{2}{N^2}c_3) \tag{2.4.12}$$

$$B \equiv \frac{2\zeta(3)}{(4\pi)^6}\ \frac{N^2-4}{N^2}\ c_1\ \left[\frac{N^2+5}{N^2}\ c_1^2 - 3c_1(a_1+b_1) + 3(a_1-b_1)^2\right] \tag{2.4.13}$$

The vanishing condition of the anomalous dimension at order g^6 can be read directly from the finite *log* term in (2.4.11):

$$\mathcal{O}(g^6): \qquad\qquad A + \frac{3B}{N^2} = 0 \tag{2.4.14}$$

We emphasize that at this order the condition for the vanishing of γ and β_h is completely scheme independent. However, from now on we will have to care about the scheme dependence in the definition of the fixed points. To see this, let us consider the counterterm needed at this stage to properly renormalize the propagator in an arbitrary scheme:

$$g^6\ (A + \frac{B}{N^2})\ (\frac{1}{\epsilon} + \rho) \tag{2.4.15}$$

where ρ is a constant related to the choice of finite renormalization. In fact, if we were to push the conformal invariance condition one order higher we should compute the chiral beta function at order g^9. We expect to have several sources of nontrivial contributions to γ at this order: one coming from the one–loop bubble proportional to $(a_4+b_4-\frac{2}{N^2}c_4)$, then from two–loop, three–loop and four–loop diagrams. All of the diagrams containing subdivergences, namely the two and four loop contributions, will be subtracted making use of the appropriate counterterms. To be specific, a term like

$$g^8\ (A+\frac{B}{N^2})\ (\frac{1}{\epsilon}+\rho)\ \frac{1}{\epsilon}\ \left(\frac{\mu^2}{p^2}\right)^{\epsilon} \tag{2.4.16}$$

will appear in the calculation of γ. Therefore the request for vanishing anomalous dimension depends unavoidably on the arbitrary constant ρ which appears in the form

$$(A+\frac{B}{N^2})\ \rho \tag{2.4.17}$$

If we wanted to kill the scheme dependence of the result we would also need to impose the vanishing of the combination $A+B/N^2$ which together with (2.4.14) would lead immediately to $A=B=0$. The crucial observation is that in the non–planar case we deal with three parameters and at this stage we have enough freedom to eliminate the scheme dependence from the conformal condition without reducing to the real β case. In fact, the constraint $A=0$ gives

$$a_3+b_3-\frac{2}{N^2}c_3=0 \tag{2.4.18}$$

while the condition $B=0$ combined with equation (2.4.6) yields

$$\left\{\begin{array}{lcl} a_1 & + & b_1=2 \\ c_1 & = & 0 \end{array}\right. \tag{2.4.19}$$

or, if $c_1\neq 0$

$$\left\{\begin{array}{lcl} a_1+b_1 & = & 2\left(1+\frac{c_1}{N^2}\right) \\ \\ a_1-b_1 & = & \pm\sqrt{2\,c_1\,\left(1-\frac{N^2-1}{6\,N^2}\,c_1\right)} \end{array}\right. \tag{2.4.20}$$

These solutions allow for a non vanishing imaginary part of β (which is proportional to the combination $|h_1|^2-|h_2|^2$). At the same time, they define the surface of renormalization fixed points without any ambiguity related to the choice of regularization scheme. It is clear that in the planar limit only the condition coming from $A=0$ survives as the $B=0$ condition is subleading. So we are left with $a_3+b_3=0$, in complete agreement with the result found in [44].

If we move to the next order, a new scenario will show up. Having imposed (2.4.19) or (2.4.20) only three graphs will contribute to the anomalous dimension at order g^8 (Fig. 2.7).

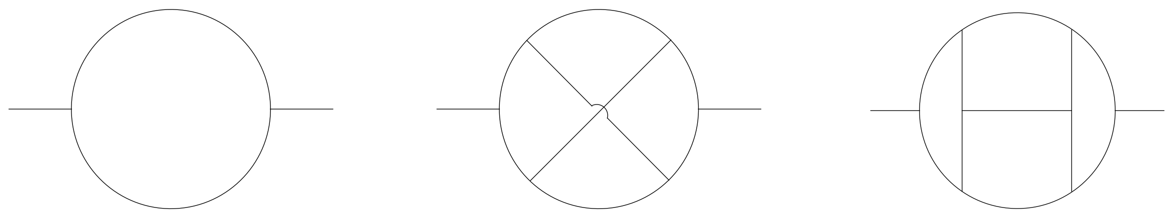

Figure 2.7: Diagrams contributing to γ at order g^8.

Since these diagrams are primitively divergent (no subdivergences are present) the condition for $\gamma = 0$ at this order turns out to be completely scheme independent.In fact we have to consider the following expression:

$$\frac{1}{\epsilon}\left[A'\left(\frac{\mu^2}{p^2}\right)^{\epsilon} + \frac{B'}{N^2}\left(\frac{\mu^2}{p^2}\right)^{3\epsilon} + H\left(\frac{\mu^2}{p^2}\right)^{4\epsilon}\right] \tag{2.4.21}$$

where we have denoted

$$A' \equiv \frac{1}{(4\pi)^2}\left(a_4 + b_4 - \frac{2}{N^2}\,c_4\right) \tag{2.4.22}$$

$$B' \equiv \frac{6\,\zeta(3)}{(4\pi)^6}\frac{N^2-4}{N^2}\left[(a_1-b_1)^2 c_2 + c_1\Big(\frac{N^2-1}{N^2}\,c_1 c_2 + 4\,(a_1-b_1)(a_2 - \frac{c_2}{N^2}) - 4c_2\Big)\right] \tag{2.4.23}$$

$$H \equiv -\frac{5\,\zeta(5)}{2(4\pi)^8}\left[(a_1-b_1)^4 + (a_1+b_1)^4 + \frac{1}{N^2}\,f\Big(a_1, b_1, c_1, \frac{1}{N^2}\Big) - \frac{16(N^2+12)}{N^2}\right] \tag{2.4.24}$$

where we have used the relations (2.4.6) and (2.4.8). The form of the function f can be read in Appendix B.

The vanishing of γ reads

$$\mathcal{O}(g^8): \qquad\qquad A' + \frac{3B'}{N^2} + 4H = 0 \tag{2.4.25}$$

Again, in order to remove scheme dependence from the $\mathcal{O}(g^{10})$ conformal condition we have to impose:

$$A' + \frac{B'}{N^2} + H = 0 \tag{2.4.26}$$

At this stage, independently of the choice (2.4.19) or (2.4.20), we have enough parameters to solve both equations without restricting to the real β case as in the planar theory. If one sends $N \to \infty$, equations (2.4.25) and (2.4.26) reduce to the ones found in [44]. This large N limit turns out to be smooth and does not present any sort of singularity, so there is no contradiction between our results and those found in [44]. We observe that a scheme–independent definition of the complex β conformal theory can be achieved only thanks to subleading coefficients which are projected out by the planar limit.

2.4.2 Gauge Beta Function and Finiteness Theorems

Now we turn to consider the gauge beta function. Standard finiteness theorems [144, 145] ensure the vanishing of β_g at L+1–loops once β_h has been set to zero at L–loops. Here, as a consequence of coupling constant reduction, we are forced to work order by order in g^2 instead of loop by loop and it is not obvious that such theorems still hold. Nevertheless in [44] it was shown that in the planar β–deformed theory the vanishing condition for β_h at $\mathcal{O}(g^9)$ was sufficient to have vanishing β_g at $\mathcal{O}(g^{11})$. This result was a strong indication that finiteness theorems could be generalized as follows: if the matter chiral beta function vanishes up to order g^{2n+1} then the gauge beta function vanishes as well up to order g^{2n+3}. Here we are going to check this result at finite N and for $n = 3$. In order to do this, we take advantage of covariant supergraph techniques combined with background field method [146].

The standard procedure consists in looking at vacuum diagrams at a given perturbative order and performing covariant ∇–algebra. Then by expanding propagators one extracts tadpole type contributions with vector connections as external legs. Moreover one only selects diagrams containing at least a $1/\epsilon^2$ pole (see [145] for details). In the present case, contributions to the gauge beta function at $\mathcal{O}(g^9)$ come from two and four loop vacuum diagrams (Fig. 2.8).

Figure 2.8: Two and four loop vacuum diagrams.

The analysis of the two loop diagram is straightforward and completely analogous to the one in [145]. Expanding the covariant propagators one obtains three times the diagram in Fig. 2.9 which corresponds to the term

$$\frac{1}{2}\ \mathrm{Tr}\ (\Gamma^a \Gamma_a) \int \frac{d^n k\ d^n q}{(2\pi)^{2n}} \frac{1}{q^2 (q+k)^2 k^4} \tag{2.4.27}$$

where Γ_a is the vector connection.

This integral contains a one–loop ultraviolet subdivergence and it is infrared divergent. It is convenient to remove the IR divergence using the R^* subtraction procedure of [105]. After UV and IR subtractions one isolates the $1/\epsilon^2$ term and rewrites the result in a covariant form, obtaining the following contribution to the two loop effective action:

$$\frac{1}{(4\pi)^2} \frac{3\,(N^2-1)}{4\,N}\, A\, \frac{1}{\epsilon}\, \mathrm{Tr} \int d^4x\, d^2\theta\, W^\alpha W_\alpha \tag{2.4.28}$$

where we have inserted the A factor defined in (2.4.12).

Now we turn to consider the four loop contributions. In this case the computation is much more

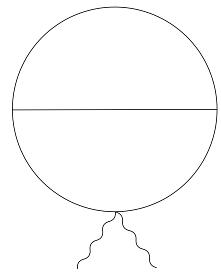

Figure 2.9: Two loops tadpole diagram.

involved because we need to perform very non trivial ∇–algebra operations. In [44] an analogous problem was solved by using an alternative procedure, though different from the one just described which turned out to be too hard to deal with. Here we want to consider both methods and show that they indeed give the same result. Let us start with the standard procedure. A detailed explanation of ∇–algebra operations can be found in Appendix B. Finally, the only surviving terms sum up to give nine times the same diagram, shown in Fig. 2.10.

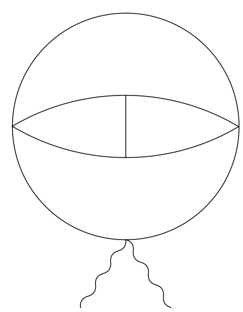

Figure 2.10: Four loop total contribution to the gauge beta function.

The corresponding bosonic integral is:

$$\frac{1}{2}\,\mathrm{Tr}(\Gamma^a\Gamma_a)\int \frac{d^nk\,d^nq\,d^nr\,d^nt}{(2\pi)^{4n}}\,\frac{1}{k^4\,q^2\,t^2\,(r-q)^2\,(r+t)^2\,(t+q)^2\,(r+k)^2} \tag{2.4.29}$$

So the total four–loop contribution to the effective action, after inserting color and combinatorial factors and subtracting IR and UV subdivergences is given by:

$$\frac{1}{(4\pi)^2}\,\frac{9\,(N^2-1)}{8\,N^3}\,B\,\frac{1}{\epsilon}\,\mathrm{Tr}\int d^4x\,d^2\theta\,W^\alpha W_\alpha \tag{2.4.30}$$

with B defined as in (2.4.13). This completes the computation of the four loops contribution with the standard method.

Had we followed the alternative procedure developed in [44] we would have first expanded each of the nine propagators of the four–loop vacuum diagram in Fig. 2.8 and then performed ∇–algebra. In this case, the only possible contributions would come from two types of diagrams:
I. the ones with flat D^2 and $\bar{D}^2$ factors at the vertices, flat propagators and one tadpole insertion, for which now standard D–algebra can be performed
and
II. the vacuum diagrams with flat propagators but ∇^2 and $\bar{\nabla}^2$ at the chiral vertices in which the tadpole insertion will have to appear after completion of the ∇–algebra.
Analogously to [44], it is easy to see that only type I diagrams contribute. The computation is now straightforward. As the vacuum diagram is completely symmetric we have nine equivalent choices for the propagator to expand. Once a choice has been made the standard D–algebra gives rise to a unique contribution, producing precisely the result depicted in Fig. 2.10. We have therefore checked that as expected the two methods actually give the same answer.
Now we come back to the computation of the gauge beta function and combine (2.4.28) and (2.4.30). We can easily read the vanishing condition at order g^9:

$$A + \frac{3\,B}{N^2} = 0 \tag{2.4.31}$$

which is exactly the one obtained by requiring the vanishing of β_h at order g^7. Thus we provide one more confirmation that finiteness theorems for the gauge beta functions hold even in the CCR context.

2.5 Summary

In this chapter we have considered $\mathcal{N} = 1$ $SU(N)$ SYM theories obtained as marginal deformations of the $\mathcal{N} = 4$ theory. In particular, we have focused on the perturbative structure of the matter (not gauge) quantum chiral ring defined as in (2.2.9) in terms of the effective superpotential. According to our general prescription, CPO's can be determined by imposing order by order the orthogonality condition (2.2.6) to all the descendants of a given sector. This requires constructing first the descendants as a power expansion in the couplings. According to the definition (2.2.9), this can be easily accomplished once the effective superpotential is known at a given order.

For the Lunin–Maldacena β–deformed theory (2.1.51) we have studied quite extensively the spin–2 sector of the theory. For the particular examples of weights $(J, 1, 0)$ and $(2, 2, 0)$ we have considered, a special pattern arises which allows for a drastic simplification in the study of the orthogonality condition: In any of these sectors descendants can be always constructed at tree level which turn out to be good independent descendants even at the quantum level. This is due to the particular form (2.2.14) of the superpotential and the peculiar way the equations of motion work which allow for constructing q–independent descendants, insensible to the quantum corrections of the theory. This property persists even for other examples of the form $(J_1, J_2, 0)$. Therefore, we conjecture that it might be a property of the entire spin–2 sector: For any weight $(J_1, J_2, 0)$ quantum descendant operators can be constructed which coincide with the descendants determined classically.

We have then studied the spin–3 sector. In this case the determination of quantum descendants of weights (J_1, J_2, J_3) cannot ignore the Konishi anomaly term. Being its effect of order λ it only enters nontrivially the orthogonality condition from two loops on, that is it will affect the form of the protected operators at least at three loops. For weights $(1,1,1)$ and $(2,1,1)$ we have determined the CPO's up to two loops. In particular, for the first case we have proved that up to this order the correct CPO is the one found in [70]. Higher order calculations would require computing two–point correlation functions between matter chiral operators and $\mathrm{Tr}(W^\alpha W_\alpha)$. It would be interesting to pursue this direction since it represents the first case where the descendant operators, apart from acquiring an explicit dependence on the Konishi anomaly term, get modified nontrivially at the quantum level due to the nontrivial corrections to the superpotential which start appearing at order λ^2.

We have extended our procedure to the study of protected operators for the full Leigh–Strassler deformation. We can think of this theory as a marginal perturbation of the β–deformed theory induced by the h'–terms in (2.1.51). In this case the determination of the complete chiral ring is a difficult task and only few insights have been discussed in [62]. We have moved few steps in this direction by studying perturbatively the simple $\Delta_0 = 2, 3$ sectors. For operators of scale dimension two we have found that the h'–deformed theory has still the same CPO's as the $h' = 0$ one, i.e. $\mathrm{Tr}(\Phi_i^2)$ and $\mathrm{Tr}(\Phi_i \Phi_j)$, $i \neq j$.

For the $\Delta_0 = 3$ sector we have found a two–dimensional plane of CPO's given as linear combinations of the CPO's of the corresponding $h' = 0$ theory, i.e. $\mathrm{Tr}(\Phi_i^3)$ and $\mathrm{Tr}(\Phi_1\Phi_2\Phi_3)$. In fact, in this case the lower number of global symmetries surviving the deformation allows for mixing among the operators who were protected in the previous case and belonged to different $U(1) \times U(1)$ sectors. The class of protected operators we have found contains the central element of the quantum algebra proposed in [62].

What turns out is that in the $\Delta_0 = 2$ sector the chiral ring is made by operators which are both charged and neutral with respect to the Z_3–symmetry (2.1.55) that the theory inherits from the parent $h' = 0$ theory. On the other hand, in the $\Delta_0 = 3$ sector *all* CPO's we can construct are neutral under (2.1.55). The generalization of our results to higher dimensional sectors leads to the result that the chiral ring for the h'–deformed theory can be divided into two subsets: Sectors with scale dimension $\Delta_0 = 3J$ have an independent $\mathcal{Q} = 0$ class which may contain in general a different number of CPO's. Instead, whenever $\Delta_0 \neq 3J$ we can generate the chiral primary operators of the charged classes from neutral CPO's by the use of the other discrete symmetries, i.e. cyclic permutations of the three superfields and the symmetry (2.1.52). It then follows that the three classes contain the same number of protected operators. In particular, for any non–empty neutral sector (for instance $\Delta_0 = 2, 4$) the corresponding charged ones are nontrivial. Neutral CPO's will be in general linear combinations of operators of the form $\mathrm{Tr}(\Phi_1^{J-m-n}\Phi_2^m\Phi_3^n)$ with $m + 2n = 3p$.

The Z_3 periodicity we have found in the chiral ring structure should have a counterpart in the spectrum of BPS states of the dual supergravity theory. Therefore, it might be of some help in the construction of the dual spectrum.

For *all* the cases we have investigated the CPO's do not get corrected at one–loop, whereas they start being modified at order λ^2. This one–loop non–renormalization found for a large class of chiral operators is probably universal for all the CPO's and might be traced back to the one–loop non–renormalization properties of the theories. Precisely, the conditions (2.1.54, 2.1.57) which insure superconformal invariance at one–loop are maintained at two loops, i.e. the superconformal theories at one and two loops are the same. It is then natural to speculate that the corresponding chiral rings should be the same. The theory instead changes at three loops where the superconformal condition gets modified by terms of order λ^2 [72]. Therefore we expect that at this order the chiral ring will be modified by effects of the same order.

Then, we have focused on the superconformal condition working perturbatively with a complex deformation parameter β at finite N.

We have addressed the issue of finding a surface of renormalization fixed points by requiring the theory to have vanishing beta functions and using the coupling constant reduction (CCR) procedure. In the CCR prescription the renormalized chiral couplings are expressed in terms of a power expansion in the real gauge coupling constant g and this amounts to face loop mixing at a given order of g.

First, we have concentrated on the chiral beta function (β_h) up to $\mathcal{O}(g^7)$. To this end we have fixed the arbitrary coefficients which appear in the power expansions of the chiral couplings (2.4.4) by requiring $\gamma = 0$ order by order. If we want to work with a well–defined and a physically meaningful quantum field theory, we believe that the condition $\beta_h = 0$ should not be affected by scheme dependence. Scheme independence of the conformal definition of the theory introduces a further constraint on the couplings. Here comes the novelty with respect to the planar case studied in [44]. The planar limit involves only two of the three independent constants in (2.4.3) and scheme independence of the theory forces β to be *real*. On the other hand, keeping N finite, all of the three parameters $|h_1|^2, |h_2|^2, |h_3|^2$ enter the superconformal condition allowing for a *complex* deformed theory which is scheme–independent at least at $\mathcal{O}(g^{10})$. We expect this pattern should hold even for higher orders.

Then we have considered the gauge beta function β_g. Working in the CCR context we are not guaranteed that standard finiteness theorems [144, 145] are valid. In [44] a generalization of these theorems was proposed: if $\beta_h = 0$ up to $\mathcal{O}(g^{2n+1})$ then $\beta_g = 0$ up to $\mathcal{O}(g^{2n+3})$. This statement was checked in the planar limit for $n = 4$ using an alternative procedure for covariant ∇–algebra. Here we have provided another highly non–trivial confirmation of this proposal in the non–planar theory for $n = 3$. Moreover, we have explicitly checked that the simplified ∇–algebra technique used in [44] is equivalent to the standard one.

Chapter 3

Giants on deformed backgrounds

Dualities in string theory have proved to be powerful tools in our understanding of its physics. As we have seen, the most studied of these is the AdS/CFT correspondence which relates the $\mathcal{N} = 4$ SYM conformal field theory to the type IIB string theory on $AdS_5 \times S^5$. This being a strong-weak duality to test, one usually relies on some non-renormalisation theorems. In this context the 1/2-BPS operators of the CFT, corresponding to the 1/2-BPS states on $S^3 \times R$ via the state-operator correspondence, played a very important role as their conformal dimensions are protected from quantum corrections. Under the AdS/CFT correspondence these states are dual to 1/2-BPS states in the type IIB string theory on $AdS_5 \times S^5$. However it is well known that the chiral primary operators have many possible dual descriptions on the string theory side when the effect of angular momentum along S^5 (the R–charge in the holographic dictionary) is taken into account. For small values of R-charge they are dual to multiparticle supergravity/closed string states. As the R-charge increases to $J \sim N$ the point like states are no longer good descriptions and they are better described by large D3-branes[1].

In fact, inspired by the work of Myers [120], the authors of [121] found a stable expanded brane configuration in the $AdS_5 \times S^5$ background with exactly the same quantum numbers of a point particle: The giant graviton. It was described as a D3–brane sitting at the center of AdS_5, wrapping an S^3 onto the S^5 part of the geometry and traveling around an equator of the internal space. Although spherical branes are unstable against shrinking due to their own tensions in the trivial vacuum, there is an additional repulsive force due to the coupling to the background Ramond-Ramond field in its presence. The main feature of the giant graviton is that it provides another example of IR/UV non-decoupling that often occurs in AdS/CFT theories. In fact, in conventional (20th century) physics, high energy or high momentum came to be associated with small distances. The physics of the 21st century is likely to be dominated by a very different perspective. According to the Infrared/Ultraviolet connection [28] which underlies much of our new understanding of string theory and its connection to gravity, physics of increasing energy or momentum is governed by increasingly large distances. Examples include the growth of particle size with momentum [29, 30] and the IR/UV connection in AdS spaces. Another important manifestation is the spacetime uncertainty principle of string theory [32, 33, 34]

$$\Delta x \Delta t \sim \alpha. \tag{3.0.1}$$

[1]For $J \sim N^2$ new geometries arise [11].

Similar uncertainty principles occur in non-commutative geometry where the coordinates of space do not commute. An important consequence of the non-commutativity is the fact that the particles described by non-commutative field theories have a spatial extension which is proportional to their momentum [35, 36]. The angular momentum of a single quantum field in non–commutative theory turns out to be bounded and it is a large distant effect. Probe branes seem to share the same feature. In fact, the motion of the giant is characterized by the angular momentum J and as this increases the probe brane blows up in size very much like the quanta of non–commutative field theories. When the size reaches the radius of the S^5, the growth can no longer continue and the tower of Kaluza–Klein states terminates. This is the origin of the stringy exclusion principle [40, 41, 42].

In [122, 123] it was shown that also stable configurations blown up into the AdS part of the geometry exist: The dual giant gravitons. In this case, they have a completely different behavior due to the fact that the AdS space–time is non–compact and then there are no constraints on their size. However, it is also possible to explain how stringy exclusion principle manifests itself in terms of the dual–giants. Just as there is an upper bound on the angular momentum of a single giant, it turns out that there is an upper bound, namely N, on the number of dual-giants [160].

A remarkable fact is that both the configurations saturate a BPS bound for their energy, which turns out to be equal to their angular momentum in units of the radius of the background. The BPS bound follows from their embedding in a supersymmetric theory because they preserve half of the supersymmetries involved [122, 123]. This makes (dual) giant graviton a natural object to study in the framework of AdS/CFT correspondence. A lot is known from the field theory side [124, 125, 126] and the elegant description of these states in terms of free fermions [127] has led to a complete classification of all the half–BPS solutions of Type IIB supergravity [128]. Other results on giant gravitons can be found in [129].

Our aim is to study giant graviton probes in the framework of theories with less (or no) supersymmetries which preserve their conformal feature. This is the case of the three–parameter deformation of the $AdS_5 \times S^5$ background described discussed in this dissertation. In [130] giant graviton configurations were analyzed on the non–supersymmetric three–parameter deformation of the $AdS_5 \times S^5$ background. They did not find energetically favorable solutions making the giants unstable states. On the other hand, they showed a striking quantitative agreement between the open string sigma model and the open spin chain arising from the Yang–Mills theory. Moreover, as noted in the recent paper [131] it seems strange that giant gravitons have been not found in the supersymmetric $\hat{\gamma}_i = \hat{\gamma}$ Lunin–Maldacena background yet and it would be also interesting to study giants which expand in AdS directions. In this chapter we try to shed light on these problems, revisiting the construction of (dual) giant gravitons in the three–parameter deformed background. Our results can be easily translated to the superconformal Lunin–Maldacena deformation by setting $\hat{\gamma}_i = \hat{\gamma}$.

The plan of this chapter is as follows. After introductory sections on the (dual) giant graviton story of the standard $AdS_5 \times S^5$ background we turn to its three–parameter LM–Frolov deformations. In Section 3.2.1 we propose an analysis from a point–particle point of view to understand how (and if) the deformation manifests itself in the study of geodesics of the deformed background. In Section 3.2.2 we give an ansatz for extended brane solutions blown up in the deformed $\tilde{S}^5$ part of the geometry (giant gravitons) and also in the AdS_5 space–time (dual giant gravitons). We find potentially stable states in both cases and an identical scenario to the undeformed one where (dual) giant gravitons behave as point–like gravitons. We note that the symmetric $\hat{\gamma}_i = \hat{\gamma}$ case is not special as long as the procedure seems to be independent of the specific value of the deformation parameters. In Sections 3.2.3 we prove that our giants are effectively solutions which minimize the action. Moreover, we examine the bosonic spectrum of small fluctuations around the classical solutions where the deformation of the background plays a crucial role and we show that all fluctuation modes have real frequencies. This signals that (dual) giant gravitons are stable over perturbation even in the presence of non–vanishing $\hat{\gamma}_i$ parameters. In Section 3.2.4 we compare our Dirac–Born–Infeld results with qualitative and, where possible, quantitative expectations from the dual CFT pictures. The main focus of this section is on possible directions along which our work can be extended. Then we summarize and conclude.

3.1 Giant gravitons and BPS bounds

Let us start with the analysis of dynamical probe brane in the string theory background of the form of $AdS_5 \times S^5$. To be more specific let us recall the coordinate system for AdS_5 in global coordinates:

$$ds^2_{AdS_5} = -(1+\frac{l^2}{R^2})dt^2 + \frac{dl^2}{1+\frac{l^2}{R^2}} + l^2\left[d\alpha_1^2 + \sin^2\alpha_1\left(d\alpha_2^2 + \sin^2\alpha_2 d\alpha_3^2\right)\right] \tag{3.1.1}$$

The scale of the AdS space–time is R, which is also the radius of S^5 and there is a constant 5-form flux on S^5 with N quanta of flux. Let us analyze the sphere S^5 embedded in R^6 with coordinates $X^1 \cdots X^6$

$$(X^1)^2 + \cdots (X^6)^2 = R^2 \tag{3.1.2}$$

and we choose the following parametrization

$$\begin{aligned} X^1 &= \sqrt{R^2 - r^2}\ \cos\phi_1 \\ X^2 &= \sqrt{R^2 - r^2}\ \sin\phi_1 \end{aligned} \tag{3.1.3}$$

where $0 \le r \le R$. The remaining $X^3 \cdots X^6$ are chosen to satisfy

$$(X^3)^2 + \cdots (X^6)^2 = r^2 \tag{3.1.4}$$

These may be written in terms of three angles, we take (θ, ϕ_2, ϕ_3) for future convenience, and r in the form of standard spherical polar coordinates in four dimensions. Then the metric on S^5 becomes

$$ds^2 = \frac{R^2}{R^2 - r^2}dr^2 + (R^2 - r^2)d\phi_1^2 + r^2 d\Omega_3^2 \tag{3.1.5}$$

where $d\Omega_3^2 = d\theta^2 + \cos\theta^2 d\phi_2^2 + \sin\theta^2 d\phi_3^2$ is the volume element on a unit 3-sphere.

Giant gravitons [121] are D3-branes wrapping an S^3 inside the S^5 and rotating along one of the transverse directions within the S^5. We now consider giants sitting at the center of AdS. Since they do not wrap any homological cycle they do not carry any net D3-brane charge, but they do have a D3 dipole moment. They preserve 16 of the 32 supersymmetries[2] of $AdS_5 \times S^5$

[2]There are also giant gravitons that carry more than one R-charge which are 1/4 or 1/8 supersymmetric [12]. We will not consider these configurations here.

[122, 123]. The time coordinate in AdS is denoted by t. In the 4 dimensional world volume of the brane with coordinates $(\tau, \sigma_1, \sigma_2, \sigma_3)$ we choose a static gauge

$$\tau = t \qquad \sigma_i = (\theta, \phi_2, \phi_3) \tag{3.1.6}$$

The dynamical coordinates are now $r(\tau, \sigma_i)$ and $\phi_1(\tau, \sigma_i)$. We will look at motions of the brane where there are no oscillations, i.e. r, ϕ_1 are independent of the angles σ_i, so that our ansatz is

$$r = r(\tau, \sigma_i) \qquad \phi_1 = \phi_1(\tau, \sigma_i) \qquad l = \alpha_1 = \alpha_2 = \alpha_3 = 0 \tag{3.1.7}$$

In general, the dynamics of a D3–brane in a given background is described by the action (see Appendix C)

$$S = S_{DBI} + S_{WZ} \tag{3.1.8}$$

where the Dirac–Born–Infeld term is

$$S_{DBI} = -T_3 \int_{\Sigma_4} d\tau d^3\sigma \, e^{-\phi} \sqrt{-det(g_{ab} + \mathcal{F}_{ab})} \tag{3.1.9}$$

With $g_{ab} = G_{MN} \partial_a X^M \partial_b X^N$ we mean the pull–back of the ten–dimensional space–time metric G_{MN} on the worldvolume Σ_4 of the brane. T_3 is the D3–brane tension[3]. The gauge potential A_a enters the action through a $U(1)$ worldvolume gauge field strength F_{ab} in the modified field strength $\mathcal{F}_{ab} = 2\pi F_{ab} - b_{ab}$, where b_{ab} is the pull-back to the worldvolume of the target NS–NS two-form potential, $b_{ab} = B_{MN} \partial_a X^M \partial_b X^N$. We are now setting $\alpha' = 1$. D–branes are charged under R–R potentials and this feature determines that their action should contain a term (the Wess–Zumino term) coupling the brane to these fields,

$$S_{WZ} = T_3 \int_{\Sigma_4} P\left[\sum_q C_q \, e^{-B}\right] e^{2\pi F} \tag{3.1.10}$$

where $P[...]$ denotes again the pull–back and the wedge–product is implicit.

Our analysis focuses on purely scalar solutions so we drop all the fermions and we also set the gauge potential A_a on the brane to be vanishing.

Taking the configuration ansatz (3.1.7), the effective brane lagrangian coming from (3.1.8) is given by

$$L = -\xi \left[r^3 \sqrt{1 - G_{rr}(r)\dot{r}^2 - G_{\phi_1\phi_1}(r)\dot{\phi_1}^2} - r^4 \dot{\phi_1} \right] \tag{3.1.11}$$

where

$$\begin{aligned} \xi &= \frac{N}{R^4} \\ G_{rr}(r) &= \frac{R^2}{R^2 - r^2} \\ G_{\phi_1\phi_1}(r) &= R^2 - r^2 \end{aligned} \tag{3.1.12}$$

[3] In our conventions $T_3 = \frac{1}{(2\pi)^3}$, see [132] for example.

The first term is the Dirac-Born-Infeld (DBI) term. The coefficient ξ is a rewriting of the tension of the brane in terms of N and R. This follows from the corresponding classical supergravity solution. It is crucial in what follows that we have exactly the same coefficient in the second term, the Wess–Zumino term. This is the coupling of the brane with the 5–form field strength and the precise coefficient follows from standard flux quantization.

The canonical momenta for r and ϕ_1 are p_r and p_{ϕ_1} respectively and are given by

$$\begin{aligned} p_r \equiv \xi P &= \frac{\xi r^3 G_{rr}(r)\dot{r}}{\sqrt{1 - G_{rr}(r)\dot{r}^2 - G_{\phi_1\phi_1}(r)\dot{\phi_1}^2}} \\ p_{\phi_1} \equiv \xi j &= \frac{\xi r^3 G_{\phi_1\phi_1}\dot{\phi_1}}{\sqrt{1 - G_{rr}(r)\dot{r}^2 - G_{\phi_1\phi_1}(r)\dot{\phi_1}^2}} + \xi r^4 \end{aligned} \tag{3.1.13}$$

The lagrangian (3.1.11) is independent of ϕ_1 so the momentum p_{ϕ_1} is an angular momentum and is conserved. On the other hand p_r is not conserved. From (3.1.13) one gets

$$\left[1 - G_{rr}(r)\dot{r}^2 - G_{\phi_1\phi_1}(r)\dot{\phi_1}^2\right]^{1/2} = r^3\left[r^6 + \frac{P^2}{G_{rr}(r)} + \frac{(j - r^4)^2}{G_{\phi_1\phi_1}(r)}\right]^{-1/2} \tag{3.1.14}$$

The canonical hamiltonian can be now derived in a standard fashion and becomes

$$H = p_{\phi_1}\dot{\phi_1} + p_r\dot{r} - L = \xi\sqrt{r^6 + \frac{P^2}{G_{rr}(r)} + \frac{(j - r^4)^2}{G_{\phi_1\phi_1}(r)}} \tag{3.1.15}$$

3.1.1 BPS bounds

Motion can be labelled by the quantum number j. It is easy to show that for some given j there is a lower bound on the energy, a BPS bound. This is not immediately obvious from the form of the hamiltonian (3.1.15). However a straightforward algebra allows us to rewrite H in the following form

$$H = \xi\sqrt{\frac{j^2}{R^2} + \frac{P^2}{G_{rr}(r)} + \frac{r^2(j - R^2r^2)^2}{R^2G_{\phi_1\phi_1}(r)}} \tag{3.1.16}$$

Since $G_{rr}(r) = (R^2 - r^2)^{-1}$ and $G_{\phi_1\phi_1}(r) = R^2 - r^2$ are positive it is clear that

$$H \geq \frac{\xi j}{R} \tag{3.1.17}$$

This is the BPS bound.

In deriving the form of the hamiltonian given in (3.1.16) it is absolutely crucial that the relative coefficient between the DBI term and the Wess–Zumino term is what it is. This happens because the 5-form flux is quantized in the standard way. Furthermore the exact form of the metric on the sphere is also crucial. All the details of working in a consistent supergravity background has entered in the calculation.

3.1.2 BPS saturated states and angular momentum bounds

The bound is saturated when

$$p_r = 0 \tag{3.1.18}$$

and

$$r(j - R^2 r^2) = 0 \tag{3.1.19}$$

The latter has two solutions

$$r = 0 \qquad r \equiv r_0 = \frac{\sqrt{j}}{R} \tag{3.1.20}$$

Thus BPS motions have constant r, which is the size of the brane.

The potential energy for such motion is

$$V(r) = \frac{r^2(j - R^2 r^2)^2}{R^2(R^2 - r^2)} \tag{3.1.21}$$

This potential has two minima with a maximum inbetween. These minima are precisely $r = 0$ and r_0 given in (3.1.20). Thus there are two kinds of BPS states : the one which correspond to zero size branes and the other with branes with sizes scaling as $\sqrt{j}/R$. Since the range of r is between 0 and R this immediately implies that there is an upper bound for j

$$j \leq R^4 \tag{3.1.22}$$

This implies that the physical angular momentum has a maximum value given by

$$p_{\phi_1} = N \tag{3.1.23}$$

When the brane has maximal size R, the angular momentum is the maximum value N and this turns out to be interbreted as the manifestation of the stringy exclusion principle. In a quantum theory one expects that p_{ϕ_1} is quantised. A general configuration of giants is then given by an N-vector $\vec{b}_1 = (r_1, r_2, \cdots, r_N)$ where the integers $r_k \in [0, \infty)$ denote the number of giant gravitons with angular momentum $p_{\phi_1} = k$. The total energy (and therefore the angular momentum) of this configuration is $\sum_{k=1}^{N} k\, r_k$.

For a BPS state, $H = \xi j = p_{\phi_1}/R$. This is the *same* dispersion relation as that of a massless graviton which is moving purely on the sphere. What is surprising is that states of branes, which are by themselves heavy objects, can lead to a light state. The reason this behind this is of course the coupling to the 5-form field strength. The effect of this cancelled the effect of brane tension.

From the point of view of 5-dimensional supergravity in the AdS space, the stable brane configurations correspond to massive states with $M = p_{\phi_1}/R$. The motion on the S^5 means that these states are also charged under a $U(1)$ subgroup of the $SO(6)$ gauge symmetry in the reduced supergravity theory. With the appropriate normalizations, the charge is $Q = p_{\phi_1}/R$, and hence one finds that these configurations satisfy the appropriate BPS bound $Q = M$. In the above discussion, we have used the phrase "BPS configuration" in its original sense. In a supersymmetric theory one would expect that these configurations also preserve some of the supersymmetries and this fact has been proved in [122, 123].

It is also interesting to consider in details the motion of these stable configurations. First, considering the fixed size brane $\dot{r} = 0$, we can invert the second of (3.1.13) to write

$$\dot{\phi}_1 = \frac{p_{\phi_1} - \xi r^4}{G_{\phi_1\phi_1}(r)\sqrt{\xi^2 r^6 + \frac{(p_{\phi_1} - \xi r^4)^2}{G_{\phi_1\phi_1}(r)}}} \tag{3.1.24}$$

Evaluating this expression for any of the BPS solutions, remarkably one finds the same result: $\dot{\phi}_1 = 1/R$, independent of p_{ϕ_1}! Note then that the center of mass motion for any of the configurations in the full 10-dimensional background is along a null trajectory, since

$$ds^2 = \left[G_{tt} + G_{\phi_1\phi_1}(r)\dot{\phi}_1^{\,2}\right] dt^2 = \left[-1 + (R^2 - r^2)\dot{\phi}_1^{\,2}\right] dt^2 = 0 \tag{3.1.25}$$

when evaluated for $r = 0$ and $\dot{\phi}_1 = 1/R$. This is, of course, the expected result for a massless 'point-like' graviton, but it applies equally well for the expanded brane configurations. However, note that in the expanded configurations, the motion of each element of the sphere is along a timelike trajectory, with $ds^2 = -(r^2/R^2)dt^2$.

It is important to specify that we have chosen global coordinates in AdS and we have put the brane probe at the center of AdS ($l = 0$). Then, the giant does not move in the radial direction and the energy in global coordinates is equal (in units of the radius) to the angular momentum.

3.1.3 Dual giant gravitons

In the previous section, we have seen that a spherical D3–brane configuration has the same quantum numbers as the point-like graviton. Motivated by the analysis in [120], one might also consider the possibility of a brane expanding into the AdS part of the spacetime. In this section, we will show that there is in fact a stable expanded D3–brane configuration in the AdS space, which again carries the same quantum numbers as the point-like graviton.

In this case, we again begin with the same world–volume action (3.1.8). Now, however, we wish to find stable solutions where a D3–brane has expanded into the AdS_5 space to a sphere of constant radius l while it orbits in the ϕ_1 direction on the S^5. Choosing again a static gauge, we identify

$$\tau = t\,, \;\; \sigma_i = \alpha_i \tag{3.1.26}$$

Our trial solution will be

$$r = 0\ , \qquad \phi_1 = \phi_1(\tau)\ , \qquad l = \text{constant}\ . \tag{3.1.27}$$

Now one can calculate the pull-backs of the metric and the 4-form potential, substitute the trial solution and integrate over the angular directions. The resulting Lagrangian is

$$L = -\xi\left[l^3\sqrt{1 + \frac{l^2}{R^2} - R^2\dot{\phi}_1^{\,2}} - \frac{l^4}{R}\right], \tag{3.1.28}$$

where again $\xi = N/R^4$.

The conjugate momentum for ϕ_1 now becomes

$$p_{\phi_1} = \frac{\xi l^3 R^2 \dot{\phi}_1}{\sqrt{1 + \frac{l^2}{R^2} - R^2 \dot{\phi_1}^2}} \tag{3.1.29}$$

Then, one can calculate the Hamiltonian to be

$$H = p_{\phi_1}\dot{\phi}_1 - L = \sqrt{\left(1 + \frac{l^2}{R^2}\right)\left(\xi^2 l^6 + \frac{p_{\phi_1}^2}{R^2}\right)} - \xi \frac{l^4}{R} \tag{3.1.30}$$

Examining $\partial H/\partial l = 0$, one finds minima located at

$$l = 0 \qquad \text{and} \qquad (l/R)^2 = p_{\phi_1}/N \ . \tag{3.1.31}$$

The energy at each of the minima is $H = p_{\phi_1}/R$, matching the BPS mass found in the previous section. The physical reasons for this structure are the same as in the case for the branes expanding on the 5-sphere. An essential difference from that case, however, is that the minima corresponding to expanded branes persist for arbitrarily large values of p_{ϕ_1}.

As we did above, we consider the center of mass motion of these brane configurations. One again finds for any of the stable minima that $\dot{\phi}_1 = 1/R$, independent of p_{ϕ_1}. The center of mass motion then follows a null trajectory in the full 10-dimensional background spacetime for either the point-like state or the branes that have expanded into the AdS space. In the latter case, the motion of each element of the sphere is along a time-like trajectory with $ds^2 = -(l/R)^2\, dt^2$.

Several comments are in order regarding the spherical brane configuration in AdS_5.

- The spherical brane in AdS_5 couples electrically to the background Ramond-Ramond field and should be thought of as a dielectric brane. The spherical brane in S^5 couples magnetically and should be thought of as a dimagnetic brane.

- There are two solutions, one at $l = R\sqrt{p_{\phi_1}/N}$ and the other at $l = 0$, just as in the previous section. All of these brane configuration preserve the same 16 of the 32 supersymmetries of type IIB theory on $AdS_5 \times S^5$. At first sight this is natural for they saturate the BPS bound. Nonetheless, this is a very non-trivial statement since different patches of the brane world volume are oriented in different directions.

- All of the solutions $l = R\sqrt{p_{\phi_1}/N}$, $l = 0$, $r = R\sqrt{p_{\phi_1}/N}$, and $r = 0$ have the same energy and angular momentum quantum numbers.

Since l is not bounded (it ranges from 0 to ∞), the dual-giants can have arbitrary (integer valued) angular momenta p_{ϕ_1}.

This raises the question of how the stringy exclusion principle manifests itself for the dual-giants. To answer this let us note a subtle effect which restricts the total number of dual-giants that one can place in $AdS_5 \times S^5$ (see also [13]). Since a dual-giant occupies three of the four spacelike coordinates in AdS_5, it acts like a domain-wall and the flux of $F^{(5)}$ measured on either side of this domain-wall differs by one unit with the lesser value on the inside of S^3 that the

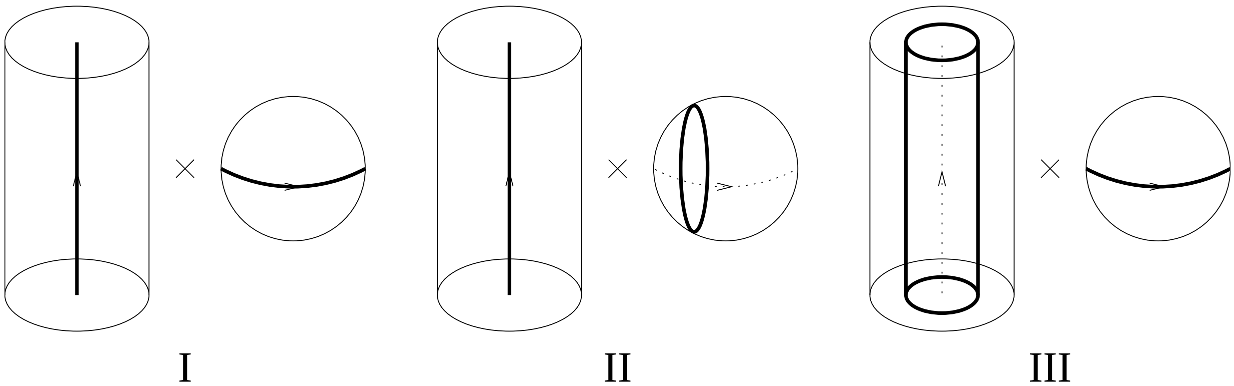

Figure 3.1: **I** collapsed spherical D3-brane of zero size, **II** spherical D3-brane embedded in S^5, and **III** spherical D3-brane embedded in AdS_5. These states are degenerate in energy and angular momentum quantum numbers.

dual-giant wraps [123]. So if we have m dual-giants in AdS_5 the $F^{(5)}$ flux measured inside the inner most dual-giant will be $N - m$ units. For $m = N$ the five form flux inside the innermost dual-giant vanishes. Since it is crucial to have non-zero flux to stabilise a dual-giant at a non-zero radius and to produce a geometry where there are closed orbits, it follows that we can not have any more dual-giants in the system. This is the manifestation of 'stringy exclusion principle' for the dual-giants.

Taking this into account a general configuration of dual-giants is also given by an N-vector $\vec{b}_2$ $= (s_1, s_2, \cdots, s_N)$. Here the integers s_k are such that $0 \leq s_N \leq \cdots \leq s_1 < \infty$ and s_k denotes the angular momentum of the kth dual-giant away from the boundary of AdS_5. The total energy of this configuration is given by $H_{\vec{b}_2} = \sum_{k=1}^{N} s_k$.

3.1.4 Gravitons vs Expanded Branes

Summarizing, in the $AdS_5 \times S^5$ background there are three different configurations characterized by the same quantum numbers. The first one is a point–like graviton spinning around an S^1 direction contained in S^5, then there is a giant graviton corresponding to a D3–brane wrapping an $S^3 \subset S^5$ and the third one is the so called dual giant graviton with the topology of an $S^3 \subset AdS_5$. These configurations are illustrated in figure 3.1.

Extended objects possess a set of low energy excitations arising from small vibrations about their equilibrium configuration. Such modes are very important to provide a way of checking whether the spherical brane ansatze used in the previous sections are stable over the whole spectrum of harmonic fluctuations. Since we will cover this issue for the brane probes expanding in the deformed geometry and since there exists a smooth limit to the undeformed cases, we refer to Section 3.2.3 for a detailed analysis of the problem. The solution defines the ground state for a subsector of the D3–brane field theory, based on normal coordinates expansion, which turns out to be stable since all fluctuation modes have real and positive frequencies.

In general one would expect that the descriptions in terms of a graviton and an expanded brane state are vaild in different regions of parameter space[4].

In analyzing the graviton states one can think of doing a Kaluza Klein reduction on the S^5. The graviton then turns into a massive state with mass

$$M \sim \frac{p_{\phi_1}}{R} \tag{3.1.32}$$

where again p_{ϕ_1} refers to the angular momentum. In order to demand that the resulting mass is smaller than the string scale and treat the graviton in a controlled manner we need

$$p_{\phi_1} \ll \frac{R}{l_s} \tag{3.1.33}$$

The alternative description in this case involves an expanded 3-brane. This description in under control when the corrections to the space–time geometry and the Born–Infeld action can be ignored. In string theory the former requires $g_s \ll 1$ and $g_s N \gg 1$, so that the dilaton is small ($e^{\phi} \ll 1$). On the other hand, the corrections to the Born–Infeld action are suppressed if the induced curvature scale on the world–volume is much larger than the string scale which gives

$$\sqrt{\frac{p_{\phi_1}}{N}} \gg \frac{l_s}{R} \tag{3.1.34}$$

Now, if (3.1.33) and (3.1.34) are simultaneously valid,

$$\frac{R}{l_s} \gg N \left(\frac{l_s}{R}\right)^2 \tag{3.1.35}$$

But then it follows, from the standard $AdS_5 \times S^5$ relation $R^4 \sim g_s N l_s^4$, that

$$g_s^3 \gg N \tag{3.1.36}$$

which is in contrast with the previous requirement $e^{\phi} \ll 1$.

In summary then, we have seen above that the massless particle description and the expanded brane description are valid for different values of the angular momentum. As the rotational energy for the graviton increases and becomes larger than the string scale the gravitons turn into an expanded brane configuration. This is made all the more plausible by the fact that in several cases even without supersymmetry, as we will see later, the expanded brane solutions has the same energy, for fixed angular momentum, as the massless particle.

Even though there are dualities between different descriptions of chiral primaries on the string theory side, one has to keep in mind that for most of the situations only one of the three candidates, namely the point like KK modes, the giant gravitons or the dual-giant gravitons, is a good description but not all. In some cases none of them alone describes the true physics in which case one has to work with the full supergravity solution [11].

[4]The use of the word graviton should not be taken literally. We simply mean a fluctuation about the $AdS_5 \times S^5$ supergravity background which is massless in 10 dimensions.

3.1.5 The CFT picture

The type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ and the dual $\mathcal{N} = 4$ super Yang-Mills in four dimensions have different realizations of the same superconformal group. In $\mathrm{AdS}_5 \times S^5$ we have the isometry group $SO(2,4) \times SO(6)$ or better its covering group $SU(2,2) \times SU(4)$ (since spinors are also involved on this background) and there are thirty-two real supercharges that enhance the invariance group to the supergroup $SU(2,2|4)$. On the field theory side, the $SU(2,2)$ part is realized as the conformal group of flat four-dimensional Minkowski space-time, while the $SU(4)$ part corresponds to the R-symmetry group. Although, at first sight, there are only sixteen real supercharges Q, the extension to the superconformal group provides the necessary sixteen extra real supercharges S, to reach the grand total of thirty-two real supercharges.

As we have seen, the checks of this conjecture are mostly restricted to the strong coupling limit of the 't Hooft coupling constant, in the large N approximation of the SYM theory, corresponding to the supergravity regime of superstring theory. In this limit, the analysis of Kaluza-Klein excitations due to compactification on S^5 leads to several families of field modes with well-defined transformation properties under the $SU(2,2|4)$ group. At this point, a study of superconformal representations is needed, since the conjecture translates into a series of predictions concerning the spectrum of SYM operators. In particular, *short* representations are specially useful due to the fact that some of their properties are protected from quantum corrections. In fact, chiral fields (fields belonging to these representations) in SYM theory correspond to Kaluza-Klein harmonics on the gravity side.

Primary fields are defined as fields annihilated by all supercharge operators S and all generators of special conformal transformations K at the origin. Chiral primary fields are additionally annihilated by some of the Q. For example, we construct the half BPS family by considering symmetric traceless combination of the scalar fields X^I of $\mathcal{N} = 4$ SYM of the form $\mathcal{O}^{I_1 \dots I_n} = tr\left(X^{(I_1} \cdots X^{I_n)}\right)$ [5]. These operators have protected scaling dimension Δ, coinciding with their R-symmetry charge J

$$\Delta = J \tag{3.1.37}$$

Supersymmetry protects the conformal dimensions of chiral primaries from receiving quantum corrections. X^I is in the **6** representation of the R-symmetry group $SU(4)$ and therefore $\mathcal{O}^{I_1 \dots I_n}$ has weight $(0, n, 0)$, which matches precisely one of the unitarity bounds for short representations of the superconformal group[6]. The full chiral multiplet is generated by the repeated action of the operators Q and P on the chiral primary. All the multiplet is annihilated by some of the Q and, due to the structure of the superconformal algebra $[Q, K] \sim S$, half of the S also give zero on all the states of the multiplet, recovering in this way the notion of sixteen conserved real supersymmetries, i.e. eight generated by the Q and eight by the S.

Normally, single trace operators in the CFT side are related to single particle states in the gravity side since, in the large N limit, single trace operators form an orthogonal set. Nevertheless, this is only correct if the R-symmetry charge of the single trace operators is not comparable with N. If this is not the case, the orthogonality property is lost, and we have to use a different type

[5] The $U(N)$ and $SU(N)$ gauge group indices are here denoted $i, j, \dots = 1, \dots, N$ and for the R-charges we use $m, n, \dots = 1, \dots, 6$ indices when described in terms of the fundamental of $SO(6)$ and capital $I, J, \dots = 1, 2, 3$ indices when described in terms of the fundamental of $SU(4)$.

[6] See [14] for a short review.

of operators to describe the corresponding dual single particle states. Giant gravitons and duals are among this type of particles with very high R-charge. Therefore, they are not expected to be described by single trace operators.

Giant gravitons have been identified with a particular class of half-BPS operators made out of the real scalars X^m of $\mathcal{N}=4$ SYM theory. We now turn to the $\mathcal{N}=1$ decomposition, where the scalars are usually written as three complex scalars $\Phi^I = \frac{1}{\sqrt{2}}(X^I + iX^{I+3})$, with $I = 1,2,3$, and all the fields transform in the adjoint representation of $U(N)$. Giant gravitons are a combination of single-trace and multi-trace operators in Φ^I, labeled by their R-charge n. These operators are then identified with Schur polynomial in Φ^I, written either in the totally symmetric representation U of the associated symmetric group S_n (corresponding to a dual giant graviton in AdS$_5$) or the totally antisymmetric representation U' of the symmetric group S_n (corresponding to a giant graviton in S^5) [154]. To be more precise, Schur polynomial operators are defined as

$$\chi_{(n,R)}(\Phi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\, \Phi^{i_1}_{\sigma(i_1)} \cdots \Phi^{i_n}_{\sigma(i_n)} \tag{3.1.38}$$

where, without loss of generality, we have set the SU(4) indices I to 1 and will neglect it for the rest of this section. We also have written explicitly the $U(N)$ indices 'i', taking values from 1 to N. The sum is over all the group elements σ of the symmetric group S_n and $\chi(\sigma)$ is the character of the element σ in the chosen representation R. The result of the permutation σ acting on the natural number 'i' is written as $\sigma(i)$.

For example,

$$\chi_{(2,U)} = \frac{1}{2}\left[(tr\ \Phi)^2 + tr\,(\Phi^2)\right] \qquad \text{and} \qquad \chi_{(2,U')} = \frac{1}{2}\left[(tr\ \Phi)^2 - tr\,(\Phi^2)\right] \tag{3.1.39}$$

are respectively the Schur polynomials of degree $n = 2$ in the U and U' representations[7].

The operator corresponding to an individual giant graviton with R-charge n can be rewritten also as a subdeterminant operator [155]:

$$\chi_{(n,U')} \sim \frac{1}{n!}\epsilon_{i_1\cdots i_n i_{n+1}\cdots i_N}\,\epsilon^{j_1\cdots j_n i_{n+1}\cdots i_n}\,\Phi^{i_1}_{j_1}\cdots\Phi^{i_n}_{j_n}. \tag{3.1.40}$$

These operators have the correct orthogonality property when n is comparable to N and therefore are good candidates to describe single particle states. They belong to a short representation preserving half of the total supersymmetry, more precisely to a chiral family of $SU(4)$ with $(0,n,0)$ weight. Note that these operators reproduce the correct bound for the R-charge, saturated by giant gravitons with $n = N$. In fact they do not exist for $n > N$ which is the manifestation of the stringy exclusion principle.

On the other hand, the explict local gauge theory operator representing an AdS giant graviton with n units of angular momentum is proposed to be

$$\chi_{(n,U)} \sim S^{j_1\cdots j_n}_{i_1\cdots i_n}\,\Phi^{i_1}_{j_1}\cdots\Phi^{i_n}_{j_n}, \tag{3.1.41}$$

[7]This is just an example to understand the structure of the Schur polynomial, and it must be remembered that we always work in the case where n is comparable to N.

where $S^{j_1\cdots j_n}_{i_1\cdots i_n}$ is a tensor totally symmetric in all its indices. One very concrete and interesting observation is that the spherical branes in AdS_5 (as opposed to the spherical branes in S^5 and the point-like brane) turns out to have a concrete interpretation as a *classical* solution from the field theory point of view. Since the deformed case is related to the undeformed one performing a smooth limit in the space of deformation parameters, we will discuss in details this issue in the deformed case (see Section 3.2.4).

3.2 Giants on deformed backgrounds

In this section we study giant graviton probes in the framework of the three–parameter deformation of the $AdS_5 \times S^5$ background. We examine both the case when the brane expands in the deformed $\tilde{S}^5$ part of the geometry and the case when it blows up into AdS_5. Then we perform a detailed analysis of small fluctuations around the giants to understand the stability properties of these configurations.

The Type IIB supergravity background we will study is related by T–dualities and shift transformations to the usual $AdS_5 \times S^5$ and is the generalization of the background first proposed in [66] to the case of three unequal $\hat{\gamma}_i$ parameters [67]. The corresponding background is a non–supersymmetric deformation of $AdS_5 \times S^5$ and should be dual to a non–supersymmetric but marginal deformation of $\mathcal{N} = 4$ SYM. Since the deformation is exactly marginal, the AdS factor remains unchanged. The metric of the so called $AdS_5 \times \tilde{S}^5$ solution (written in string frame and with $\alpha' = 1$) can be read from

$$ds^2 = ds^2_{AdS_5} + ds^2_{\tilde{S}^5} \tag{3.2.1}$$

where again

$$ds^2_{AdS_5} = -(1 + \frac{l^2}{R^2})dt^2 + \frac{dl^2}{1 + \frac{l^2}{R^2}} + l^2 \left[d\alpha_1^2 + \sin^2\alpha_1 \left(d\alpha_2^2 + \sin^2\alpha_2 d\alpha_3^2\right)\right] \tag{3.2.2}$$

represents the usual AdS_5 space–time and

$$ds^2_{\tilde{S}^5} = R^2 \left(\frac{dr^2}{R^2 - r^2} + \frac{r^2}{R^2}d\theta^2 + G\sum_{i=1}^{3} \rho_i^2 d\phi_i^2\right) + R^2 G \rho_1^2\rho_2^2\rho_3^2 \left(\sum_{i=1}^{3} \hat{\gamma}_i d\phi_i\right)^2 \tag{3.2.3}$$

is the deformed five–sphere. Here

$$G^{-1} = 1 + \hat{\gamma}_1^2\rho_2^2\rho_3^2 + \hat{\gamma}_2^2\rho_1^2\rho_3^2 + \hat{\gamma}_3^2\rho_1^2\rho_2^2\,, \qquad\qquad \hat{\gamma}_i = R^2\gamma_i \tag{3.2.4}$$

and it is convenient to parametrize ρ_i coordinates via $\rho_1^2 = 1 - \frac{r^2}{R^2}\,,\ \ \rho_2^2 = \frac{r^2}{R^2}\cos^2\theta\,,\ \ \rho_3^2 = \frac{r^2}{R^2}\sin^2\theta$. Note that $\sum_{i=1}^{3}\rho_i^2 = 1$ and we have $0 \le r \le R$. We consider only the case of real deformation parameters $\hat{\gamma}_i$, when the axion field χ is a constant and is set to zero. With respect to the dilaton ϕ_0 of the undeformed background, the dilaton ϕ of the solution is

$$e^{2\phi} = e^{2\phi_0}G \tag{3.2.5}$$

and we have the usual AdS/CFT relation $R^4 = 4\pi e^{\phi_0} N = \lambda$, relating the radius of the background and the 't Hooft coupling constant. Note that the dilaton field ϕ is not simply a constant, but it depends on the coordinates of the deformed sphere $\tilde{S}^5$.

There is a non–zero NS–NS two form

$$B = R^2 G \left(\hat{\gamma}_3 \rho_1^2 \rho_2^2 d\phi_1 \wedge d\phi_2 + \hat{\gamma}_1 \rho_2^2 \rho_3^2 d\phi_2 \wedge d\phi_3 + \hat{\gamma}_2 \rho_3^2 \rho_1^2 d\phi_3 \wedge d\phi_1 \right) \,, \tag{3.2.6}$$

while the R–R forms are

$$C_2 = -4R^2 e^{-\phi_0} \omega_1 \wedge \sum_{i=1}^{3} \hat{\gamma}_i d\phi_i \,, \qquad d\omega_1 = \frac{r^3}{R^4} \sin\theta \cos\theta \, dr \wedge d\theta \tag{3.2.7}$$

and

$$\begin{aligned} C_4 &= e^{-\phi_0} \frac{l^4}{R} \sin^2 \alpha_1 \sin \alpha_2 dt \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 + \\ &+ 4R^4 e^{-\phi_0} G \, \omega_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \end{aligned} \tag{3.2.8}$$

The five form field strength of the background is

$$F_5 = dC_4 - C_2 \wedge dB \qquad * F_5 = F_5 \tag{3.2.9}$$

When all the three deformation parameters are equal, $\hat{\gamma}_i = \hat{\gamma}$, we recover the Lunin–Maldacena supersymmetric background [66].

3.2.1 A rotating point particle probe

As a warm up for what follows, we focus on the motion of a massless point–like particle in the deformed $AdS_5 \times \tilde{S}^5$ background which rotates on the $\tilde{S}^5$ and minimizes its energy in this internal space. For convenience we start from the action for a massive particle in ten dimensions and later take the mass M to zero,

$$S = -M \int dt \sqrt{-(g - b)} \tag{3.2.10}$$

where g and b are, respectively, the pull–backs of the space–time metric and of the NS–NS two form onto the particle's worldline and are given by

$$g = G_{MN} \dot{X}^M \dot{X}^N \qquad b = B_{MN} \dot{X}^M \dot{X}^N \tag{3.2.11}$$

Here X^M are coordinates on the ten–dimensional space–time with $X^0 = t$ and $\dot{X}^M$ denotes the derivative of X^M with respect to t. The metric G_{MN} and the NS–NS two form B_{MN} can be read in (3.2.1) and in (3.2.6), respectively. The rotating point particle we want to analyze sits at the center of AdS_5 and spins in the ϕ_1 direction. For this configuration we have $g = G_{tt} + G_{\phi_1 \phi_1} \dot{\phi}_1^2$, $b = 0$ and the action becomes

$$S = -M \int dt \sqrt{1 - R^2 G \rho_1^2 (1 + \hat{\gamma}_1^2 \rho_2^2 \rho_3^2) \dot{\phi}_1^2} \tag{3.2.12}$$

From now on, to save space we introduce the positive quantity $Q^2 = R^2 G\rho_1^2(1+\hat{\gamma}_1^2\rho_2^2\rho_3^2)$. Since the action we have written down presents no explicit dependence on the cyclic coordinate ϕ_1, we can replace $\dot{\phi}_1$ with its conjugate momentum

$$J = \frac{\partial L}{\partial \dot{\phi}_1} = \frac{Q^2 M \dot{\phi}_1}{\sqrt{1 - Q^2\dot{\phi}_1^2}} \tag{3.2.13}$$

which is conserved in time. So we can define the Hamiltonian in the standard way

$$H = \dot{\phi}_1 J - L = \frac{J}{Q} \tag{3.2.14}$$

where we have already taken the limit $M \to 0$. We need to find the minimum of the Hamiltonian and it is easy to convince that this occurs when Q is maximum, namely when $r = 0$ and so $Q = R$. Substituting this value in equation (3.2.14) we obtain the energy of the rotating point particle

$$E = \frac{J}{R} \tag{3.2.15}$$

Finally, we find a geodesic which represents a BPS state[8] with energy E equal to the angular momentum J (in units of $1/R$) and does not depend on the deformation parameters, i.e. is the same as in the undeformed theory. This is one of the cases already analyzed in [92] (see also [69]).

3.2.2 The equilibrium configurations

Our main purpose is to probe the deformed and non–supersymmetric background with giant gravitons. We want to understand if it is possible to find minimum energy configurations, study their stability and eventually their dependence on the deformation parameters. Recall that in the standard $AdS_5 \times S^5$ background there are three different configurations characterized by the same quantum numbers: the point–like graviton, the giant graviton and the dual giant graviton. What about the deformed case?

Branes expanding in the deformed $\tilde{S}^5$ space–time: Giant gravitons

The first solutions we want to study are D3–branes wrapped on the deformed sphere part of the geometry, moving entirely in the $\tilde{S}^5$ and sitting at the center of AdS_5. The time coordinate in AdS_5 is denoted by t. In what follows it is convenient to choose a static gauge such that the worldvolume coordinates of the brane (τ, σ_i) are identified with the appropriate space–time coordinates. In particular the brane wraps the (θ, ϕ_2, ϕ_3) directions,

$$\tau = t\,, \qquad \sigma_1 = \theta \in \left[0, \frac{\pi}{2}\right]\,, \qquad \sigma_2 = \phi_2 \in [0, 2\pi]\,, \qquad \sigma_3 = \phi_3 \in [0, 2\pi] \tag{3.2.16}$$

The D3–brane action (3.1.8) can be rewritten as

[8] In all our discussions we use the term BPS in its original sense. We do not refer to supersymmetry.

$$S = -T_3 \int_{\Sigma_4} dt d\theta d\phi_2 d\phi_3 \, e^{-\phi} \sqrt{-det(g_{ab} - b_{ab})} + T_3 \int_{\Sigma_4} P\left[C_4 - C_2 \wedge B\right] \tag{3.2.17}$$

Our giant graviton has constant radius (r_0), it orbits the $\tilde{S}^5$ in the ϕ_1 direction with a constant angular velocity (ω_0) and all the worldvolume modes are frozen. While it is not a priori obvious that this is a consistent way of embedding the brane, we will see that it gives in fact a minimal energy configuration. So we propose an ansatz of the form

$$r = r_0 \qquad \phi_1 = \omega_0 t \qquad l = \alpha_1 = \alpha_2 = \alpha_3 = 0 \tag{3.2.18}$$

which, after integration on the spatial coordinates of the worldvolume, leads to the effective Lagrangian

$$L = -h\sqrt{1 - a^2 \dot{\phi}_1^{\,2}} + m\dot{\phi}_1 \tag{3.2.19}$$

with

$$h = N\frac{r_0^3}{R^4}\,, \qquad a^2 = R^2 - r_0^2\,, \qquad m = N\frac{r_0^4}{R^4} \tag{3.2.20}$$

We have the constraint $r_0 \leq R$ because the size of the brane cannot exceed the radius of $\tilde{S}^5$ and so $a^2 \geq 0$. We have also used $A_3 \, T_3 \, e^{-\phi_0} = \frac{N}{R^4}$, where A_3 is the area of a unit 3–sphere. Note that the effective Lagrangian is exactly the same found in the undeformed case [121, 122, 123] and this appears to be strange at first sight because the giant has blown up in the deformed $\tilde{S}^5$. We will comment later on this particular behavior which is in contrast with the results obtained in [130].

The conjugate momentum to ϕ_1 is

$$J = \frac{\partial L}{\partial \dot{\phi}_1} = \frac{h a^2 \dot{\phi}_1}{\sqrt{1 - a^2 \dot{\phi}_1^{\,2}}} + m \tag{3.2.21}$$

This relation can be easily inverted to obtain

$$\dot{\phi}_1 = \frac{J - m}{a^2\sqrt{h^2 + \frac{(J-m)^2}{a^2}}} \tag{3.2.22}$$

The corresponding Hamiltonian of the giant graviton becomes

$$H = \dot{\phi}_1 J - L = \sqrt{h^2 + \frac{(J-m)^2}{a^2}} \tag{3.2.23}$$

and it is independent of ϕ_1, so that the equations of motion can be solved with constant momentum. For fixed J, we have two extrema of (3.2.23) now regarded as the potential that determines the equilibrium radius. In particular, there are two degenerate minima at $r_0 = 0$ and at $r_0 = R\sqrt{\frac{J}{N}}$, where the energy is $E = \frac{J}{R}$, as for the point graviton, and $\omega_0 = \dot{\phi}_1 = \frac{1}{R}$. This analysis obviously gives the same results already found in the undeformed case and the stringy exclusion principle manifests itself in the relation between the radius of the giant and its angular momentum.

Branes expanding in AdS_5 space–time: Dual giant gravitons

In the previous section we have seen that there is a D3–brane configuration with the same quantum numbers as the point–like graviton, even in the deformed $AdS_5 \times \tilde{S}^5$ background. Now we also consider the possibility of dual giant graviton solutions where the D3–branes are wrapped in the 3–sphere $(\alpha_1, \alpha_2, \alpha_3)$ contained in the AdS_5 part of the geometry. In contradistinction to the previous case we expect a priori the effective Lagrangian not to depend on the deformation parameters because they do not enter the AdS space–time [130, 131]. Again the dynamics is described by the action (3.1.8) and we use the static gauge for the worldvolume coordinates of the brane (τ, σ_i),

$$\tau = t\,, \qquad \sigma_1 = \alpha_1 \in [0,\pi]\ , \qquad \sigma_2 = \alpha_2 \in [0,\pi]\ , \qquad \sigma_3 = \alpha_3 \in [0,2\pi] \tag{3.2.24}$$

The giant graviton has constant radius (l_0) and again orbits rigidly in the ϕ_1 direction on the $\tilde{S}^5$. Our ansatz is

$$l = l_0 \qquad \phi_1 = \omega_0 t \qquad r = \phi_2 = \phi_3 = 0 \qquad \theta = \frac{\pi}{4} \tag{3.2.25}$$

We will see that with the parametrization of the deformed 5–sphere as in (3.2.3), the choice $\theta = \pi/4$ is the most natural one in the study of fluctuations around the giant. The dependence on the deformation parameters of the vibrations turns out to depend on the position of the giant into the internal space. This ansatz yields the effective Lagrangian

$$L = -\tilde{h}\sqrt{\tilde{b}^2 - R^2\dot{\phi}_1^{\,2}} + \tilde{m} \tag{3.2.26}$$

with

$$\tilde{h} = N\frac{l_0^3}{R^4}\,, \qquad \tilde{b}^2 = 1 + \frac{l_0^2}{R^2}\,, \qquad \tilde{m} = N\frac{l_0^4}{R^5} \tag{3.2.27}$$

as in the undeformed case [122, 123]. Again we have used $A_3\, T_3\, e^{-\phi_0} = \frac{N}{R^4}$. The conjugate momentum to ϕ_1 now becomes

$$J = \frac{\partial L}{\partial \dot{\phi}_1} = \frac{\tilde{h}R^2\dot{\phi}_1}{\sqrt{\tilde{b}^2 - R^2\dot{\phi}_1^{\,2}}} \tag{3.2.28}$$

and from this relation we obtain

$$\dot{\phi}_1 = \frac{J\tilde{b}}{R^2\sqrt{\tilde{h}^2 + \frac{J^2}{R^2}}} \tag{3.2.29}$$

We can calculate the corresponding Hamiltonian of the dual giant graviton and obtain

$$H = \dot{\phi}_1 J - L = \tilde{b}\sqrt{\tilde{h}^2 + \frac{J^2}{R^2}} - \tilde{m} \tag{3.2.30}$$

Again H, as a function of l_0, has two minima located at $l_0 = 0$ and $l_0 = R\sqrt{\frac{J}{N}}$. The energy at each minima is

$$E = \frac{J}{R} \tag{3.2.31}$$

and $\omega_0 = \dot{\phi}_1 = \frac{1}{R}$, matching the results of the previous sections. Of course now there is no upper bound on the angular momentum J because AdS space–time is non–compact and the radius l_0 of the giant can be greater than R [122, 123].

So far we have seen that even for the deformed background $AdS_5 \times \tilde{S}^5$, there are three potential configurations to describe a graviton carrying angular momentum J: The point–like graviton, the giant graviton of section 3.2.2 consisting of a 3–brane expanded into the deformed 5–sphere, and a dual giant graviton consisting of a spherical 3–brane which expands into the AdS space. This is exactly the same situation known from the standard undeformed $AdS_5 \times S^5$ background. Moreover, if we consider the collective motion of both brane configurations, we see that their center of mass travels along a null trajectory in the ten–dimensional space–time once evaluated in $\dot{\phi}_1 = 1/R$. We stress that this is the expected result for a massless point–like graviton, but it is also true for the expanded (dual) giant gravitons. So we have really found that giant graviton states which are degenerated with massless particle states exist classically even in a background which in general preserves no supersymmetries. This result is not so strange because it is a feature of a large class of non–supersymmetric backgrounds [133] and of particular configurations in theories with non zero NS–NS B field [134].

3.2.3 Stability analysis and vibration modes

One of the main issues related to giant gravitons is their stability under the perturbation around the equilibrium configurations. In the last two sections we found expanded branes with the same energy of a point graviton and so they should be stable. In order to verify this expectation we will consider the spectrum of small fluctuations around the giants, as first studied in [135]. A vibration of the brane can be described by expanding our previous ansatz as follows

$$X = X_0 + \varepsilon \delta X(t, \sigma_i) \tag{3.2.32}$$

where X is a generic space–time coordinate, X_0 denotes the solution of the unperturbed equilibrium configuration, the fluctuation $\delta X(t, \sigma_i)$ is a function of the worldvolume coordinates (t, σ_i) and ε is a small perturbation parameter. We work in a Lagrangian setup [135] and we expand the action of the probe brane in powers of ε as

$$S = \int dt d^3\sigma \{\mathcal{L}_0 + \varepsilon \mathcal{L}_1 + \varepsilon^2 \mathcal{L}_2 + \cdots\} \tag{3.2.33}$$

Obviously $\mathcal{L}_0$ gives a zeroth order Lagrangian density related to that we have found in the previous sections. To state that those solutions really minimize the action we have to focus on the $\mathcal{L}_1$ term. The second order term $\mathcal{L}_2$ is useful to study the stability of the configurations we have found and the bosonic fluctuation spectrum, which we expect to depend on the deformation parameters, as in the analysis of vibrations around other BPS states of this background [92]. Perturbative instability will manifest in the spectrum as a tachyonic mode. We closely follow [135]. A slightly different method has been proposed in [136].

Giant graviton fluctuations

To study the fluctuations around the configurations found in section 3.2.2 it is useful to rewrite the AdS_5 part of the metric as suggested in [135]

$$ds^2_{AdS_5} = -\left(1 + \sum_{k=1}^{4} v_k^2\right) dt^2 + R^2 \left(\delta_{ij} + \frac{v_i v_j}{1 + \sum_{k=1}^{4} v_k^2}\right) dv_i dv_j \tag{3.2.34}$$

Then we change our previous ansatz as

$$r = r_0 + \varepsilon\, \delta r(t, \sigma_i) \qquad \phi_1 = \omega_0 t + \varepsilon\, \delta\phi_1(t, \sigma_i) \qquad v_k = \varepsilon\, \delta v_k(t, \sigma_i) \tag{3.2.35}$$

with $\sigma_i = (\theta, \phi_2, \phi_3)$. Expanding the action to the linear order we get

$$\begin{aligned} \mathcal{L}_1 &= -T_3\, e^{-\phi_0}\, \sin\theta \cos\theta \\ r_0^2 & \left\{ \left[\frac{4r_0^2\omega_0^2 + 3(1 - R^2\omega_0^2)}{\sqrt{1 - (R^2 - r_0^2)\omega_0^2}} - 4r_0\omega_0 \right] \delta r + \right. \\ - & \left. \left[\frac{(R^2 - r_0^2) r_0 \omega_0}{\sqrt{1 - (R^2 - r_0^2)\omega_0^2}} + r_0^2 \right] \frac{\partial \delta\phi_1}{\partial t} \right\} \end{aligned} \tag{3.2.36}$$

The first order Lagrangian density (3.2.36) does not contain the deformation parameters and is exactly the same found in the undeformed analysis [135]. The term in front of $\frac{\partial \delta\phi_1}{\partial t}$ is a constant and so it brings no contribution to the variation of the action with fixed boundary values. The coefficient of the term δr vanishes if we take

$$\omega_0 = \frac{1}{R} \tag{3.2.37}$$

This confirms that the giant graviton described in the previous section (the zeroth order solution) is the right solution which really minimizes the action. Now we consider the second order term in ε. With the choice (3.2.37) we get

$$\begin{aligned} \mathcal{L}_2 &= T_3\, e^{-\phi_0}\, r_0^2\, \sin\theta\cos\theta \\ \Bigg\{ \Bigg[&- \frac{R^3}{2(R^2 - r_0^2)} \frac{\partial^2 \delta r}{\partial t^2} + \frac{R}{2(R^2 - r_0^2)} \Delta_{S^3}\, \delta r + \\ &+ \frac{1}{2R} \left(\hat\gamma_3^2 \frac{\partial^2 \delta r}{\partial \phi_2^2} + \hat\gamma_2^2 \frac{\partial^2 \delta r}{\partial \phi_3^2} - 2\hat\gamma_2\hat\gamma_3 \frac{\partial^2 \delta r}{\partial \phi_2 \partial \phi_3} \right) \Bigg] \delta r + \\ \Bigg[&- \frac{R^3 (R^2 - r_0^2)}{2r_0^2} \frac{\partial^2 \delta\phi_1}{\partial t^2} + \frac{R(R^2 - r_0^2)}{2r_0^2} \Delta_{S^3}\, \delta\phi_1 \Bigg] \delta\phi_1 + \\ &+ \frac{2R^2}{r_0} \frac{\partial \delta\phi_1}{\partial t} \delta r + \end{aligned}$$

$$\Big[\quad - \quad \frac{R^3}{2}\frac{\partial^2 \delta v_k}{\partial t^2} + \frac{R}{2}\,\Delta_{S^3}\,\delta v_k - \frac{R}{2}\,\delta v_k$$

$$+ \quad \frac{R^2-r_0^2}{2R}\left(\hat{\gamma}_3^2\frac{\partial^2\delta v_k}{\partial\phi_2^2} + \hat{\gamma}_2^2\frac{\partial^2\delta v_k}{\partial\phi_3^2} - 2\hat{\gamma}_2\hat{\gamma}_3\frac{\partial^2\delta v_k}{\partial\phi_2\partial\phi_3}\right)\Big]\delta v_k\Big\} \tag{3.2.38}$$

where the sum over k is understood and Δ_{S^3} is the Laplacian on the unit 3–sphere. In writing $\mathcal{L}_2$ some terms are integrated by parts; there are no surface contributions because the worldvolume of the brane is a closed surface and the variations are assumed to vanish at $t = \pm\infty$.

Because of the $U(1) \times U(1)$ worldvolume symmetry, corresponding to translations of ϕ_2 and ϕ_3, it is convenient to introduce spherical harmonics $\mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3)$ with definite $U(1)\times U(1)$ quantum numbers (m_2, m_3) [130, 131]. In particular we have

$$\begin{aligned}
\Delta_{S^3}\,\mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3) &= -Q_s^2\mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3) \\
\frac{\partial}{\partial\phi_{2,3}}\,\mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3) &= im_{2,3}\mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3)
\end{aligned} \tag{3.2.39}$$

For spherical harmonics on S^3, $Q_s^2 = s(s+2)$. We expand the perturbations as

$$\begin{aligned}
\delta r(t,\theta,\phi_2,\phi_3) &= A_r\, e^{-i\omega t}\,\mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3) \\
\delta\phi_1(t,\theta,\phi_2,\phi_3) &= A_{\phi_1}\, e^{-i\omega t}\,\mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3) \\
\delta v_k(t,\theta,\phi_2,\phi_3) &= A_{v_k}\, e^{-i\omega t}\,\mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3)
\end{aligned} \tag{3.2.40}$$

The form of $\mathcal{L}_2$ tells us that the δv_k perturbations decouple from δr, $\delta\phi_1$ and have frequencies given by

$$\omega_k^2 = \frac{1}{R^2}\left(1 + Q_s^2 + \hat{\Gamma}^2\right) \tag{3.2.41}$$

where we have defined the positive quantity

$$\hat{\Gamma}^2 = \left(1 - \frac{r_0^2}{R^2}\right)(\hat{\gamma}_3 m_2 - \hat{\gamma}_2 m_3)^2 \tag{3.2.42}$$

which contains the whole dependence on the deformation parameters and on the radius $r_0 = R\sqrt{J/N}$ of the giant. The fluctuations δr, $\delta\phi_1$ are coupled and the resulting frequencies are obtained solving the following matrix equation

$$\begin{bmatrix} \frac{R}{R^2-r_0^2}\left(\omega^2R^2 - Q_s^2 - \hat{\Gamma}^2\right) & -2i\omega\frac{R^2}{r_0} \\ 2i\omega\frac{R^2}{r_0} & \frac{R(R^2-r_0^2)}{r_0^2}\left(\omega^2R^2 - Q_s^2\right) \end{bmatrix}\begin{bmatrix} A_r \\ A_{\phi_1}\end{bmatrix} = 0 \tag{3.2.43}$$

The determinant brings us to a quadratic equation for ω^2 from which we obtain

$$\omega_\pm^2 = \frac{1}{R^2}\left[2 + Q_s^2 + \frac{\hat{\Gamma}^2}{2} \pm 2\sqrt{1 + Q_s^2 + \frac{\hat{\Gamma}^2}{2}\left(1 + \frac{\hat{\Gamma}^2}{8}\right)}\,\right] \tag{3.2.44}$$

The condition for a giant graviton to be stable over the perturbations is that all the frequencies are real, i.e. $\omega^2 \geq 0$. The existence of imaginary part in ω means that the $e^{-i\omega t}$ term can grow exponentially, which gives instability to the configuration, a tachyonic mode. We have the constraint $r_0 \leq R$ and so it is easy to conclude that there are not unstable modes in the system at this quadratic order, as all the ω^2 we found are real and nonnegative. Note that because of the deformation parameters these frequencies depend on the radius r_0 of the giant (3.2.42). In the undeformed background all of the frequencies are independent of r_0 [135] and this is the main difference with respect to the deformed theory.

Dual giant graviton fluctuations

Now we want to study the fluctuations around the configurations found in section 3.2.2. The AdS space–time is now better described by the global coordinate metric (3.2.2). Hence the ansatz becomes

$$l = l_0 + \varepsilon\, \delta l(t,\sigma_i) \qquad \phi_1 = \omega_0 t + \varepsilon\, \delta\phi_1(t,\sigma_i) \tag{3.2.45}$$

and

$$r = \varepsilon\, \delta r(t,\sigma_i) \qquad \theta = \frac{\pi}{4} + \varepsilon\, \delta\theta(t,\sigma_i) \qquad \phi_2 = \varepsilon\, \delta\phi_2(t,\sigma_i) \qquad \phi_3 = \varepsilon\, \delta\phi_3(t,\sigma_i) \tag{3.2.46}$$

with $\sigma_i = (\alpha_1, \alpha_2, \alpha_3)$. Expanding the action to the linear order we get the same contribution as in the undeformed background [135]

$$\begin{aligned}
\mathcal{L}_1 &= -\frac{T_3}{R}\, e^{-\phi_0}\, \sin^2\alpha_1 \sin\alpha_2 \\
& l_0^2 \left\{ \left[\frac{4l_0^2 + 3R^2(1 - R^2\omega_0^2)}{\sqrt{l_0^2 + R^2(1 - R^2\omega_0^2)}} - 4l_0 \right] \delta l + \right. \\
& - \left. \frac{l_0 \omega_0 R^4}{\sqrt{l_0^2 + R^2(1 - R^2\omega_0^2)}} \frac{\partial \delta\phi_1}{\partial t} \right\}
\end{aligned}$$

Again, the coefficient of the term $\frac{\partial\delta\phi_1}{\partial t}$ is a constant and so it brings no contribution to the variation of the action with fixed boundary values. The coefficient of the term δl vanishes if we take

$$\omega_0 = \frac{1}{R} \tag{3.2.47}$$

This fact confirms that the giant graviton written in the previous section is a solution to the equation of motion following from the D3–brane action. With this choice the term linear in ε vanishes, while the second order term is

$$\mathcal{L}_2 = T_3\, e^{-\phi_0}\, l_0^2 \sin^2\alpha_1 \sin\alpha_2$$

$$\left\{\left[-\frac{R^3}{2(l_0^2+R^2)}\frac{\partial^2\delta l}{\partial t^2}+\frac{R}{2(l_0^2+R^2)}\Delta_{S^3}\,\delta l\right]\delta l+\right.$$
$$\left[-\frac{R^3(l_0^2+R^2)}{2l_0^2}\frac{\partial^2\delta\phi_1}{\partial t^2}+\frac{R(l_0^2+R^2)}{2l_0^2}\Delta_{S^3}\,\delta\phi_1\right]\delta\phi_1+$$
$$+\frac{2R^2}{l_0}\frac{\partial\delta\phi_1}{\partial t}\,\delta l+$$
$$\left.\left[-\frac{R}{2}\frac{\partial^2\delta r}{\partial t^2}+\frac{1}{2R}\Delta_{S^3}\,\delta r-\frac{1}{2R}\left(1+\frac{\tilde{b}^2}{2}\left(\hat{\gamma}_2^2+\hat{\gamma}_3^2\right)\right)\delta r\right]\delta r\right\} \tag{3.2.48}$$

Of course Δ_{S^3} is the Laplacian on a 3–sphere and $\tilde{b}^2 = 1+\frac{l_0^2}{R^2}$, as in (3.2.27). Let $\tilde{\mathcal{Y}}_s(\alpha_1,\alpha_2,\alpha_3)$ be spherical harmonics so that the usual relation holds

$$\Delta_{S^3}\,\tilde{\mathcal{Y}}_s(\alpha_1,\alpha_2,\alpha_3) = -Q_s^2\tilde{\mathcal{Y}}_s(\alpha_1,\alpha_2,\alpha_3) \tag{3.2.49}$$

We expand the perturbations as

$$\begin{aligned}
\delta l(t,\alpha_1,\alpha_2,\alpha_3) &= \tilde{A}_l\,e^{-i\tilde{\omega}t}\,\tilde{\mathcal{Y}}_s(\alpha_1,\alpha_2,\alpha_3)\\
\delta\phi_1(t,\alpha_1,\alpha_2,\alpha_3) &= \tilde{A}_{\phi_1}\,e^{-i\tilde{\omega}t}\,\tilde{\mathcal{Y}}_s(\alpha_1,\alpha_2,\alpha_3)\\
\delta r(t,\alpha_1,\alpha_2,\alpha_3) &= \tilde{A}_r\,e^{-i\tilde{\omega}t}\,\tilde{\mathcal{Y}}_s(\alpha_1,\alpha_2,\alpha_3)
\end{aligned} \tag{3.2.50}$$

The δr perturbation decouples from δl, $\delta\phi_1$ and it has a frequency given by

$$\tilde{\omega}_r^2 = \frac{1}{R^2}\left[1+Q_s^2+\frac{\tilde{b}^2}{2}\left(\hat{\gamma}_2^2+\hat{\gamma}_3^2\right)\right] \tag{3.2.51}$$

The δl, $\delta\phi_1$ fluctuations are coupled and the resulting normal frequencies are obtained solving

$$\left[\begin{array}{cc} \frac{R}{l_0^2+R^2}\left(\tilde{\omega}^2R^2-Q_s^2\right) & -2i\omega\frac{R^2}{l_0} \\ 2i\omega\frac{R^2}{l_0} & \frac{R(l_0^2+R^2)}{l_0^2}\left(\tilde{\omega}^2R^2-Q_s^2\right)\end{array}\right]\left[\begin{array}{c}\tilde{A}_l\\ \tilde{A}_{\phi_1}\end{array}\right]=0 \tag{3.2.52}$$

which yields

$$\tilde{\omega}_\pm^2 \;=\; \frac{1}{R^2}\left(2+Q_s^2\pm 2\sqrt{1+Q_s^2}\right) \tag{3.2.53}$$

Again there are not unstable modes in the system at this quadratic order, as all the frequencies are real. The deformation parameters $\hat{\gamma}_{2,3}$ enter the frequency $\tilde{\omega}_r^2$ which brings a dependence on the radius $l_0 = R\sqrt{J/N}$. The frequencies $\tilde{\omega}_\pm^2$ are the same as in the undeformed case and do not depend on l_0 [135].

Summary of the excitation spectrum and role of deformation

In this section we discuss how the deformation enters the vibration modes. First of all, we stress that turning off $\hat{\gamma}_i$ manifestly reduces all the frequencies to those of the undeformed case. This is a good test of our results.

- When the giant graviton expands into the deformed sphere, it has six transverse scalar fluctuations, of which four correspond to fluctuations into AdS_5 (ω_k^2) and two are fluctuations within $\tilde{S}^5$ ($\omega_\pm^2$). In particular from (3.2.41) and (3.2.44)

$$\begin{aligned} \delta v_k \quad &\rightarrow \quad \omega_k^2 = \frac{1}{R^2}\left(1 + Q_s^2 + \hat{\Gamma}^2\right) \\ (\delta r, \delta\phi_1) \quad &\rightarrow \quad \omega_\pm^2 = \frac{1}{R^2}\left[2 + Q_s^2 + \frac{\hat{\Gamma}^2}{2} \pm 2\sqrt{1 + Q_s^2 + \frac{\hat{\Gamma}^2}{2}\left(1 + \frac{\hat{\Gamma}^2}{8}\right)}\right] \end{aligned} \tag{3.2.54}$$

  All the vibrations involve the deformation parameters $\hat{\gamma}_{2,3}$ (3.2.42) because the perturbations $\delta X(t, \theta, \phi_2, \phi_3)$ are functions of the worldvolume coordinates of the brane and in particular they depend on $\phi_2\,, \phi_3$. So, once we perturb the giant around the equilibrium configuration in X_0 the fluctuations feel the effect of the deformed background. Note that a similar $\hat{\gamma}_{2,3}$ dependence appears also in [92] in the calculation of quadratic fluctuations near a $(J, 0, 0)$ geodesic. The frequencies just discussed are very similar to the ones obtained in [131]; the main difference is our dependence on the radius of the giant.

- Similarly, the vibration mode frequencies corresponding to the giant graviton expanded in the AdS part, are (3.2.51) and (3.2.53)

$$\begin{aligned} \delta r \quad &\rightarrow \quad \tilde{\omega}_r^2 = \frac{1}{R^2}\left[1 + Q_s^2 + \frac{\tilde{b}^2}{2}(\hat{\gamma}_2^2 + \hat{\gamma}_3^2)\right] \\ (\delta l, \delta\phi_1) \quad &\rightarrow \quad \tilde{\omega}_\pm^2 = \frac{1}{R^2}\left(2 + Q_s^2 \pm 2\sqrt{1 + Q_s^2}\right) \end{aligned} \tag{3.2.55}$$

  An accurate analysis of the quadratic expansion tells us that $G_{\phi_1\phi_1}$ brings the whole dependence on the deformation, once one is calculating the pull–back. In section 3.2.2, we have mentioned that the choice of the parametrization of the ρ_i in (3.2.3) is important in the study of the dual giant vibrations. Physically, their dependence on the deformation is expected due to the location of the giant into the deformed sphere. The coordinates ρ_i are functions of the angle θ and we are now expanding around $\pi/4$. So, up to ε^2 we obtain $G_{\phi_1\phi_1} \sim R^2 - \varepsilon^2(2 + \hat{\gamma}_2^2 + \hat{\gamma}_3^2)\delta r^2/2$ and the $\hat{\gamma}_{2,3}$ dependence manifests itself only when we study perturbations in $\tilde{S}^5$, as for $\tilde{\omega}_r^2$. The original ansatz $\theta = \pi/4$ does not select a particular deformation parameter. The frequency $\tilde{\omega}_r^2$ is symmetric in the exchange $\hat{\gamma}_2 \leftrightarrow \hat{\gamma}_3$ and depends on the radius l_0 of the dual giant. On the other hand, we expect independence from the deformation when studying perturbations in AdS directions.

From (3.2.54) and (3.2.55) we see that $\omega_-^2 = 0$ and $\tilde{\omega}_-^2 = 0$ are solutions when $Q_s^2 = 0$. These zero modes correspond to the fact that we have no constraints on r_0 and l_0, namely they can be taken to have any value allowed by the geometry.

3.2.4 Undeformed giants in a deformed background

At a classical level we have found that the effective Lagrangian and hence the energy of a giant in the $\hat{\gamma}_i$–deformed background are independent of the deformation parameters. This is an expected result for the dual giant (brane expanded in the AdS part of the geometry), but seems quite strange if the brane expands into the deformed 5–sphere. Analytically, this is due to the particular form of the D3–brane action. The kinetic part (3.1.9) is independent of the deformation because of the presence of the modified dilaton (the same behavior found in [130]). The Wess–Zumino part of the action is

$$S_{WZ} = T_3 \int_{\Sigma_4} P\left[C_4 - C_2 \wedge B\right] \tag{3.2.56}$$

It is important to note that, even before taking the pull–back on the worldvolume of the brane, the combination

$$\begin{aligned} C_4 - C_2 \wedge B &= e^{-\phi_0} \frac{l^4}{R} \sin^2\alpha_1 \sin\alpha_2 dt \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 + \\ &+ 4R^4 e^{-\phi_0}\, \omega_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \end{aligned} \tag{3.2.57}$$

is exactly the same as the R–R 4–form in the undeformed $AdS_5 \times S^5$ space–time (recovered after setting the deformation parameters $\hat{\gamma}_i$ to zero). So, the independence of the deformation seems to be a feature of the Wess–Zumino term for a D3–brane configuration with vanishing worldvolume gauge field strength F in this particular background[9].

Can we speculate more on our $\hat{\gamma}_i$–independent results? Remember that we have pointed out that the existence of degenerate point–like and giant graviton states is not a new feature even in non–supersymmetric backgrounds [133] and in theories characterized by $B \neq 0$ [134]. Moreover, our giant graviton solutions are classically BPS states in the deformed model, i.e. states that have the minimal energy for the given charge. The authors of [92] discuss geodesics on $\hat{\gamma}_i$–deformed $\tilde{S}^5$ labeled by three conserved angular momenta (J_1, J_2, J_3). These geodesics depend in general on the deformation parameters. In the standard $AdS_5 \times S^5$ background all geodesics represent BPS states with energy E equal to the total angular momentum $J = J_1 + J_2 + J_3$, while in the deformed case only few of them are characterized by this property. In particular, in the $\hat{\gamma}_i$–deformed model special solutions with energies that do not depend on the deformation parameters exist, i.e. they are the same as in the undeformed theory. This is the case for states labeled by $(J, 0, 0)$. We want to stress that our giant gravitons are $(J, 0, 0)$ BPS states and follow a geodesics of $\tilde{S}^5$, so that their classical independence on the deformation parameters is not a new feature. Moreover, studying giant gravitons on a deformed $(J, 0, 0)$ PP–wave, the authors of [131] also found a classical

[9] The authors of [130] get a dependence on the deformation parameters but their conventions do not coincide with ours and with those of [137, 66, 67].

configuration independent of the deformation and with a spectrum of small fluctuations almost identical to the one obtained in section 3.2.3. This similar behavior could be an interesting point to study in detail.

The background we have studied breaks all the supersymmetries of $AdS_5 \times S^5$ and so it should be dual to a non–supersymmetric but marginal deformation of the $\mathcal{N} = 4$ $SU(N)$ SYM [67]. More precisely, the gauge theory is conformal in the large N limit [92, 138, 139], which we assume from now on. The bosonic part of the deformed YM has the following form[10]

$$\mathcal{W} = \mathrm{Tr}\left(-\frac{1}{2}\left[\Phi_i\,,\,\Phi_j\right]_{C_{ij}}\left[\bar{\Phi}^i\,,\,\bar{\Phi}^j\right]_{C_{ij}} + \frac{1}{4}\left[\Phi_i\,,\,\bar{\Phi}^i\right]\left[\Phi_j\,,\,\bar{\Phi}^j\right]\right) \tag{3.2.58}$$

where Φ_i are the three holomorphic scalars of $\mathcal{N} = 4$ SYM. The deformation manifests itself in the modified commutators

$$\left[\Phi_i\,,\,\Phi_j\right]_{C_{ij}} \equiv e^{iC_{ij}}\,\Phi_i\,\Phi_j - e^{-iC_{ij}}\,\Phi_j\,\Phi_i\,, \qquad\qquad i,j = 1,2,3 \tag{3.2.59}$$

and similarly for the conjugate fields $\bar{\Phi}_i$. The matrix C reads [140]

$$C = \pi\begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix} \tag{3.2.60}$$

The real deformation parameters $\hat{\gamma}_i$ appearing in (3.2.3) are related to the γ_i deformations on the gauge theory side (3.2.60) via the simple rescaling $\hat{\gamma}_i = R^2\gamma_i$. The potential can be also obtained from the undeformed one by replacing the usual product $\Phi_i\Phi_j$ by the associative $\star$-product of [66, 92].

The fact that the energy is independent of the deformation parameters is general and persists both in the case of unequal $\hat{\gamma}_i$ and in the $\mathcal{N} = 1$ supersymmetric $\hat{\gamma}_i = \hat{\gamma}$ theory. In order to simplify our analysis of the dual CFT picture of the giant gravitons, we restrict to the more studied $\mathcal{N} = 1$ case where we are protected by supersymmetry. We have not checked that in the supersymmetric case our giant gravitons preserve some of the supersymmetries but the fact that they saturate a BPS bound is an indication of this feature. It would be interesting to prove this expectation. From now on we set $\gamma_i = \gamma$.

Via AdS/CFT, states in supergravity are expected to map onto states of Yang–Mills theory on $\mathbb{R} \times S^3$ and the energy in space–time maps to energy in the field theory. Using the state–operator correspondence, the energy of states on $\mathbb{R} \times S^3$ maps to the dimension $\Delta = RE$ of operators on $\mathbb{R}^4$. In the undeformed case, the operators corresponding to (dual) giant gravitons have been first introduced in [124, 125]. Our giant graviton solutions correspond to the case where we have only one non–vanishing angular momentum (a $(J,0,0)$ BPS state in the language of [92]) and we should construct the dual operators on the CFT side with only one holomorphic scalar field. Let $Z \equiv \Phi_1 = \phi^5 + i\phi^6$ be a complex combination of two of the six adjoint scalars in the YM theory, then in the undeformed case giant gravitons are dual to states created by a family of subdeterminants [124]

[10] We use the notations of [139].

$$O_J = \frac{1}{J!}\epsilon_{i_1\,i_2\cdots i_J\,a_1\,a_2\cdots a_{N-J}}\,\epsilon^{l_1\,l_2\cdots l_J\,a_1\,a_2\cdots a_{N-J}}\,Z^{i_1}_{l_1}\,Z^{i_2}_{l_2}\cdots Z^{i_J}_{l_J} \tag{3.2.61}$$

Moreover,

$$\tilde{O}_J = \frac{1}{J!}\sum_{\sigma\in\mathcal{S}_J} Z^{i_1}_{i_{\sigma(1)}}\,Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_J}_{i_{\sigma(J)}} \tag{3.2.62}$$

with $\mathcal{S}_J$ the permutation group of length J, is supposed to describe a dual giant graviton in the undeformed theory [125]. Once the deformation is turned on we are instructed to use the $\star$-product among the fields, so introducing a set of relative phases [141]. However, the field content of the operators (3.2.61) and (3.2.62) implies a vanishing phase factor, and so we guess that the same operators could describe giant graviton states even in the γ–deformed theory [131]. All these operators form a good basis in the large $J \sim N$ limit and have classical scaling dimension $\Delta = J$, matching the results of sections 3.2.2 and 3.2.2. This is an agreement between a strong and a weak coupling limits and so the operators (3.2.61) and (3.2.62) seem to be protected even in this less–supersymmetric case. Remember that single trace operators of the form $(J, 0, 0)$ are BPS states of the γ–deformed gauge theory which have zero anomalous dimension [66, 65, 142, 143] but we expect this property to hold also for the more complicated operators (3.2.61) and (3.2.62) because they can always be written as (Schur) polynomials in Z [125, 147][11].

Comments on the dual gauge theory picture of giant gravitons

We have seen that the deformation seems to manifest itself in the vibration modes around the stable configurations. It would be very interesting to find the CFT dual of these scalar fluctuations, as in [148, 149]. In general, most fluctuations of giant gravitons are not BPS and so from the field theory side we expect anomalous dimensions to develop quantum mechanically: The calculation would involve the full potential (3.2.58) and of course the deformation parameters. From the brane side we read $\Delta = RE_\omega$, where E_ω is the excited energy of the giant graviton, i.e. if we switch to the quantum–mechanical system $E_\omega \sim E+\omega$ (with $\hbar = 1$), and ω is a general fluctuation frequency. To be more explicit[12] let us focus on the spectrum of small AdS fluctuations when the giant graviton expands into the deformed 5–sphere $\tilde{S}^5$. The frequencies of the four modes are given by (3.2.54)

$$\omega_k = \frac{\sqrt{(s+1)^2+\hat{\Gamma}^2}}{R} \tag{3.2.63}$$

with $Q_s^2 = s(s+2)$. The radius r_0 of the spherical D3–brane enters in the definition of $\hat{\Gamma}^2$ (3.2.42) and the energy now reads

$$E_{\omega_k} = \frac{J}{R} + \frac{\sqrt{(s+1)^2+\lambda\,\Gamma^2}}{R} \tag{3.2.64}$$

[11] The authors of [92] have shown that also in the non–supersymmetric case of three unequal γ_i, operators of the class $(J, 0, 0)$ are protected in the limit of large N. It is possible that the operators (3.2.61) and (3.2.62) could represent giant graviton states even in the non–supersymmetric case.

[12] The following analysis can be extended in the same way to the other giant fluctuations.

We have used $R^4 = 4\pi e^{\phi_0} N = g_{YM}^2 N = \lambda$ and $\hat{\gamma} = R^2\gamma$, so that from (3.2.42) and $r_0^2 = JR^2/N$, the relation

$$\hat{\Gamma}^2 = \lambda\,\Gamma^2 = \lambda\left(1 - \frac{J}{N}\right)\gamma^2(m_2 - m_3)^2 \tag{3.2.65}$$

naturally follows. Note that for a maximal giant graviton $J = N$, $\Gamma^2 = 0$ and we recover the frequency obtained in the standard $AdS_5 \times S^5$ case [135]. If we want to find the dual description of these fluctuations, we can introduce suitable impurities in (3.2.61) as first proposed in [148] (the $\star$-product is implicit)

$$\mathcal{O}_k^s \sim \epsilon_{i_1\,i_2\cdots i_J\,a_1\,a_2\cdots a_{N-J}}\,\epsilon^{l_1\,l_2\cdots l_J\,a_1\,a_2\cdots a_{N-J}}\,Z_{l_1}^{i_1}\,Z_{l_2}^{i_2}\cdots Z_{l_{J-1}}^{i_{J-1}}(W_k^s)_{l_J}^{i_J} \tag{3.2.66}$$

Here W_k^s is a word built out of the sth symmetric traceless product of the other four scalars ϕ_i of the YM theory ($i = 1, \cdots, 4$) to match the scalar spherical harmonics of S^3 on the brane side. In order to consider fluctuations along the AdS directions we have to include a covariant derivative D_k in the word, so the index $k = 1\cdots 4$ refers to the four Cartesian directions of $\mathbb{R}^4$ in radial quantization of $\mathbb{R}\times S^3$. We stress that the deformation parameters introduce a dependence on the 't Hooft coupling λ and, if the AdS/CFT correspondence holds, the energy E_{ω_k} gives the scaling dimension of $\mathcal{O}_k^s$ in the limit of large 't Hooft coupling

$$\Delta = J + \sqrt{(s+1)^2 + \lambda\,\Gamma^2} \tag{3.2.67}$$

We do not exclude the possibility that the interactions of the Yang–Mills theory do produce a perturbative (weak coupling constant $\lambda \ll 1$) anomalous dimension for the operators just introduced, related to that predicted by the other side of the correspondence. This is a heuristic discussion, since the precise form of a general operator of the type (3.2.66) is still unknown. Moreover, we are now talking about non–protected quantities and a direct comparison is a very difficult task because we are facing a strong/weak coupling duality. If we want to match the results, it is simpler to study the correspondence in novel limits, for example where quantum numbers become large with N [150].

Dual giants and semi–classical solutions of CFT

The fluctuations around dual giants can be similarly described using operators on the field theory side (see the recent [151]). However, a more efficient approach is to identify a classical field theory configuration which encodes the same properties of the spherical brane in AdS [123] and then try to study fluctuations around this solution similarly to [149].

Configuration of spherical branes in AdS_5 is such that the flux of RR 5-form in the interior of the spherical D3-brane is less by one unit compared to the exterior. In light of the UV/IR relation of the AdS/CFT correspondence, this suggests that the gauge symmetry is broken from $SU(N)$ to $SU(N-1)\times U(1)$ at low energies. Therefore we should look for a classical configuration involving Higgs expectation values.

Since the D3-branes do not act as a source for the dilaton and the axion, the supergravity back reaction of the spherical D3-branes is trivial in the dilaton/axion sector. Trivial dilaton/axion background corresponds to trivial F^2 and $F\tilde{F}$ expectation values. The field theory counterpart of the spherical brane is therefore not likely to involve the gauge fields. Furthermore, the fact that the energy (3.2.31) of the solution we are after does not depend on the coupling constant suggests that the commutator term in the action of the SYM should not play any role. We are therefore left with the bosonic part of the dual CFT which lives on the boundary of AdS_5, namely on $\mathbb{R} \times S^3$ with metric $ds^2 = h_{\mu\nu} dx^\mu dx^\nu$

$$S = -\frac{1}{g_{YM}^2} \int d^4x \sqrt{-h} \left[\mathrm{Tr} \left(\partial^\mu \bar{\Phi}_i \partial_\mu \Phi^i + \frac{1}{R^2} \bar{\Phi}_i \Phi^i \right) + \mathcal{W} \right] \tag{3.2.68}$$

where $\mathcal{W}$ is defined in (3.2.58) with $\gamma_i = \gamma$. Since the background is of the form $\mathbb{R} \times S^3$, the conformal invariance of the theory imposes a mass term for Φ_i and R is the radius of AdS_5. By rescaling the Φ_i fields

$$\Phi_i(t, \Omega) \rightarrow \sqrt{\frac{g_{YM}^2 N}{4\pi^2 R^2}}\, \Phi_i(t, \Omega) \tag{3.2.69}$$

the action can be rewritten as

$$S = \frac{N}{4\pi^2 R^2} \int d^4x \sqrt{-h} \left[\mathrm{Tr} \left(-\partial^\mu \bar{\Phi}_i \partial_\mu \Phi^i - \frac{1}{R^2} \bar{\Phi}_i \Phi^i \right) + \mathcal{W}_\lambda \right] \tag{3.2.70}$$

The rescaled potential $\mathcal{W}_\lambda$ is

$$\mathcal{W}_\lambda = \mathrm{Tr} \left(\frac{\lambda}{8\pi^2 R^2} \left[\Phi_i \,, \Phi_j \right]_{C_{ij}} \left[\bar{\Phi}^i \,, \bar{\Phi}^j \right]_{C_{ij}} - \frac{\lambda}{16\pi^2 R^2} \left[\Phi_i \,, \bar{\Phi}^i \right] \left[\Phi_j \,, \bar{\Phi}^j \right] \right) \tag{3.2.71}$$

The matrix C_{ij} is defined in (3.2.60) with now $\gamma_i = \gamma$. Next, we consider the ansatz

$$\Phi_1(t, \Omega) = \mathrm{diag}(\eta, 0, 0, \cdots, 0) e^{i\theta(t)} \quad \text{with} \quad \eta = \text{const.}\,, \qquad \Phi_{2,3}(t, \Omega) = 0 \tag{3.2.72}$$

To properly account for the $SU(N)$ field content of the SYM, simply parameterize $\Phi_1(t, \Omega)$ according to

$$\Phi_1(t, \Omega) = \hat{\eta}\, e^{i\theta(t)} \tag{3.2.73}$$

where $\hat{\eta}$ is a traceless diagonal $N \times N$ matrix

$$\hat{\eta} = \sqrt{\frac{N-1}{N}} \begin{pmatrix} \eta & & & \\ & -\frac{\eta}{N-1} & & \\ & & \ddots & \\ & & & -\frac{\eta}{N-1} \end{pmatrix} . \tag{3.2.74}$$

To leading order in $1/N$, all but the first diagonal element can be ignored and the analysis reduces to treating Φ_1 as an ordinary scalar field. The subleading $1/N$ correction can be thought of as the back reaction of the spherical brane to the background geometry.

The Lagrangian turns out to be

$$L = \frac{NR}{2} \left(\eta^2 \dot{\theta}^2 - \frac{\eta^2}{R^2} \right) \tag{3.2.75}$$

We see that the angular momentum $J = \partial L/\partial\dot{\theta}$ is conserved and the energy

$$E = J\dot{\theta} - L = \frac{J^2}{2NR\eta^2} + \frac{N\eta^2}{2R} \tag{3.2.76}$$

is minimized at $\eta^2 = \eta_0^2 = J/N$ where its value is $E = J/R$. Let us make some comments regarding this solution

- The energy of the classical solution $E = J/R$ is precisely the energy of the spherical brane in AdS_5.
- The classical solution is invariant under half of the supersymmetries. This can be verified easily by acting on the solution with the supersymmetry transformation rules given in [156]. (Strictly speaking, one can check that the solution is invariant with respect to 8 out of 16 Poincare supersymmetries [123].)

The fact that the classical solution of the SYM shares many properties in common with the spherical brane configuration in AdS_5 is a good indication that the former is the field theory realization of the latter. There are some subtle differences, however. The potential (3.2.76) is the field theory counterpart of (3.2.30). To be more precise, (3.2.30) is the effective action for the spontaneously broken $U(1)$ at large λ after integrating out the massive W-bosons. Equation (3.2.76) can simply be thought of as the small λ limit of the same quantity. To facilitate the comparison, let us re-express (3.2.30) in terms of $\eta = l_0/R$

$$H = \frac{N}{R}\left(\sqrt{1+\eta^2}\sqrt{\frac{J^2}{N^2}+\eta^6} - \eta^4\right) . \tag{3.2.77}$$

Potentials (3.2.76) and (3.2.77) differ from each other in one very important sense. The potential at strong coupling (3.2.77) has two minima, one at $\eta = 0$ and the other at $\eta = \sqrt{J/N}$. At small coupling, (3.2.76) has only one minima, at $\eta = \sqrt{J/N}$.

Thus, we have found an argument based on duality that the minima at $\eta = 0$ is lifted by $1/\lambda$ corrections. When $\lambda \ll 1$, semi-classical description of the SYM becomes reliable, but the configuration at $\eta = 0$ simply does not exist as a solution of the classical equation of motion. It would be very interesting to understand the status of $\eta = 0$ solution when the quantum effects on the SYM side is taken into account. Studying the quantum correction to (3.2.76) perturbatively should teach us a lot about this issue.

Unlike the solution at $\eta = 0$, the solution at $\eta = \sqrt{J/N}$ is a robust result. This can be seen in the following way. For large values of J/N, the spherical brane will grow to have size much greater than the radius of AdS_5. In [152], Seiberg and Witten showed that the DBI+WZ action of the n-brane in AdS_{n+1} has the following form for $n > 2$ near the boundary of the AdS (see eq.(3.17) of that paper)

$$S \sim \int \sqrt{g}\left((\partial\phi)^2 + \frac{n-2}{4(n-1)}\phi^2\tilde{R} + \mathcal{O}(\phi^{\frac{2(n-4)}{n-2}})\right) \tag{3.2.78}$$

The form of this action is dictated by the fact that the extension of the metric on the boundary of AdS to the bulk is unique in the neighborhood of the boundary. The leading term in large ϕ of (3.2.78) exactly matches the field theory action (3.2.70).

This effect can be also seen by studying the transverse fluctuations of dual giants which in the gauge theory are represented by modes of the scalars ϕ_i for $i = 1, \cdots, 4$ as explained in the previous section. The coordinates (ρ_i, ϕ_i) which parametrize the deformed sphere $\tilde{S}^5$ (3.2.3) correspond to the three complex scalars Φ_i of Yang–Mills theory and in particular the dictionary tells us that $\Phi_i = \rho_i e^{i\phi_i}$. On the supergravity side the modified ansatz (3.2.46) yields to $\rho_{2,3} \sim \varepsilon \delta r/\sqrt{2}$ and so if we want to translate this vibrations in the dual CFT it seems natural to consider diagonal fluctuations of the form

$$\Phi_2(t,\Omega) = \Phi_3(t,\Omega) = \varepsilon \, \mathrm{diag}(\frac{\delta\rho(t,\Omega)}{\sqrt{2}}, 0, 0, \cdots, 0) \tag{3.2.79}$$

Since in this CFT analysis η covers the role of the radius of the giant, while $\dot{\theta}$ is the angular velocity (at the minimum of the energy its value is $\dot{\theta} = \theta_0 = 1/R$ as in (3.2.47)), we guess that the study of small fluctuations in radius and in the orientation of angular momentum could be performed thanks to the modified ansatz

$$\Phi_1(t,\Omega) = \mathrm{diag}(\eta + \varepsilon\, \delta\eta(t,\Omega), 0, 0, \cdots, 0) e^{i(\theta(t) + \varepsilon\, \delta\theta(t,\Omega))} \tag{3.2.80}$$

Exactly as in section 3.2.3, we study the action up to second order in ε and we expand the generic perturbation $\delta x(t,\Omega)$ in spherical harmonics

$$\delta x(t,\Omega) = A_x e^{-i\tilde{\omega}_x t} \mathcal{Y}_s(\Omega) \tag{3.2.81}$$

The calculation runs parallel to that of section 3.2.3 so we are free to omit the details; we only stress that the linear term in ε vanishes when evaluated in the classical vacuum and the first commutator in (3.2.71) covers a crucial role in what follows. The $\delta\eta$, $\delta\theta$ perturbations are coupled and the resulting frequencies are

$$\tilde{\omega}_\pm^2 \; = \; \frac{1}{R^2}\left(2 + Q_s^2 \pm 2\sqrt{1+Q_s^2}\right) \tag{3.2.82}$$

in perfect agreement with (3.2.53). The $\delta\rho$ perturbation decouples from $\delta\eta$, $\delta\theta$ and has a frequency

$$\tilde{\omega}_\rho^2 = \frac{1}{R^2}\left(1 + Q_s^2 + \frac{\lambda}{4\pi^2}|q-\bar{q}|^2 \frac{J}{N}\right) \tag{3.2.83}$$

where we have defined $q = e^{i\pi\gamma}$. Note that the frequency (3.2.83) is very similar to the exact anomalous dimension obtained in [74]. When the deformation parameter is set to zero ($q = \bar{q} = 1$), we recover the frequencies obtained in the undeformed theory [149]. Because of the λ–dependence of (3.2.83) we have to be careful in comparing it to the result of section 3.2.3. Quantum mechanically, the energy (in units of $1/R$ and with $\hbar = 1$) has the form

$$E_{CFT} = J + \sqrt{1 + Q_s^2 + \frac{\lambda}{4\pi^2}|q-\bar{q}|^2 \frac{J}{N}} \tag{3.2.84}$$

On the other hand, from the value of the small fluctuation frequency given in (3.2.51) and with $\hat{\gamma}_2^2 = \hat{\gamma}_3^2 = \hat{\gamma}^2 = \lambda\gamma^2$, the energy of the brane is

$$E_{BRANE} = J + \sqrt{1 + Q_s^2 + \lambda\left(1 + \frac{J}{N}\right)\gamma^2} \qquad (3.2.85)$$

What happened? The two energies are remarkably similar but again we have to check the regime of validity of our analysis of the small vibrations, both in the gauge theory and in the supergravity side. To be more precise, the energy E_{BRANE} (3.2.85) is a well defined quantity at large λ and in the small γ limit, with $\hat{\gamma}^2 = \lambda\gamma^2$ fixed [66]. The CFT energy (3.2.84) was computed for small λ, where the semi–classical description of the Yang–Mills theory becomes reliable, and at arbitrary q. So we expect a function to exist which smoothly interpolates between the weak coupling result (3.2.84) and the strong coupling one (3.2.85). Note that if we expand the $|q - \bar{q}|^2$ term into the square root of (3.2.84) for a particularly small value of γ, we obtain

$$E_{CFT} \sim J + \sqrt{1 + Q_s^2 + \lambda\,\gamma^2\,\frac{J}{N}} \qquad (3.2.86)$$

On the other hand, if $J/N \gg 1$ we can safely ignore the 1 appearing in (3.2.85) and up to their regime of validity, the two energies are identical. This is the same limit analyzed before in order to show that for large values of J/N the dual giant becomes a large brane and the leading term of its Dirac–Born–Infeld and Wess–Zumino action in AdS exactly matches the CFT action. We leave the complete understanding of these features for future works. Another useful strategy to interpret these results could be the one used in [153].

Our CFT analysis applies equally well to the case of unequal γ_i and reproduces the $\gamma_{2,3}$ behavior obtained on the brane side. So, let us conclude noting that in particular the authors of [92] and [139] have found non–trivial examples where implications of the AdS/CFT duality are observed even in the non–supersymmetric case and where the non–renormalization theorem seems not to be dictated by supersymmetry. We do not exclude a possible extension of this AdS/CFT comparison to the more general case of unequal γ_i deformation parameters.

3.3 Summary

The main subject of this chapter is the analysis of giant graviton configurations on the Type IIB supergravity background which can be obtained by a non–supersymmetric but marginal three–parameter deformation of the original $AdS_5 \times S^5$ solution. In particular, we have shown the existence of giants which are energetically indistinguishable from the point graviton, even in absence of supersymmetry. This feature holds for both the two sets of giant graviton solutions, namely when the D3–brane expands into the deformed 5–sphere part of the geometry and when it blows up into AdS_5. The (dual) giant dynamics turns out to be independent of the deformation parameters with a behavior which is exactly the same found in the undeformed theory. The deformation of the background affects both the NS–NS and the R–R sectors. The D3–brane couples to the two and four forms but with a precise mechanism which exactly compensates the changes induced by the deformation. More striking, this complete cancellation of the deformation parameters does not depend on their values and remains valid in the presence of unequal $\hat{\gamma}_i$ (the non–supersymmetric case) and in the special case $\hat{\gamma}_i = \hat{\gamma}$, corresponding to the supersymmetric Lunin–Maldacena deformation. In order to understand the stability of the configurations we have found, we have also performed a systematic study of the spectrum of small fluctuations around the giant graviton solutions. This is where the deformation manifests itself providing the first important difference with respect to the undeformed case. In fact, the deformed spectrum turns out to depend on the radius of the (dual) giant which is always coupled to the deformation parameters. Despite this fact, the deformation enters into the spectrum as a positive contribution and the frequencies do not allow tachyonic modes. The (dual) giant gravitons are perturbatively stable and this characteristic works in favor of the perfect quantitative agreement between the gauge theory and the string theory found in [130]. Finally, restricting to the supersymmetric case of equal $\hat{\gamma}_i$, we have proposed qualitative and quantitative comparisons obtained from the dual gauge theory picture, generalizing what is known in the original undeformed correspondence. In the case of dual giant gravitons, a semi–classical CFT picture seems to capture a lot of the physics of the brane configuration, giving the correct energy, angular momentum and a remarkable similar spectrum of small fluctuations.

The study of giant graviton dynamics is certain a fascinating subject. One of their most striking features is their ability to relate UV and IR regimes by enlarging their size with the increasing of the energy. Another interesting feature of giant graviton solutions is their stability even in a non–supersymmetric background. Further investigations of this property could give new insight in the understanding of the role played by supersymmetry in the gauge/gravity dualities.

Chapter 4

Mesons in marginally deformed AdS/CFT

In the large N limit the Feynman diagrams of $SU(N)$ Yang-Mills theory reorganize themselves into a genus expansion of closed string theory [9]. This closed string is believed to propagate in a five dimensional background [10]. At each point on the worldvolume of the string, we have to specify its position in 4d Minkowski space and its thickness, which is represented by its position in the 5th dimension. The metric structure of the 5th dimension describes the internal structure of the string. This expectation has been realized in many supersymmetric examples starting with [52], where in addition to the 5th dimension, the closed string background includes an internal compact space, representing the additional fields in the theory. Thus, pure Yang-Mills, with only glue as degrees of freedom, is expected to map to an entirely 5d non-critical string theory background [10].

It is hoped that methods based on gauge-gravity duality will eventually be applicable to QCD. A difficulty with describing QCD in this way arises due to the asymptotic freedom of QCD. The vanishing of the 't Hooft coupling in the UV requires the dual geometry to be infinitely curved in the region corresponding to the UV. In this case classical supergravity is insufficient and one needs to use full string theory. Formulating string theory in the relevant backgrounds has thus far proven difficult. The existing glueball calculations involve geometries with small curvature that return asymptotically to AdS (the field theory returns to the strongly coupled $\mathcal{N} = 4$ theory in the UV), and are in the same coupling regime as strong coupling lattice calculations far from the continuum limit. There is nevertheless optimism that the glueball calculations are fairly accurate, based on comparisons with lattice data [161, 162, 163].

The standard AdS/CFT duality conjectures the equivalence of a particular string (or supergravity) theory and a pure Yang-Mills theory with matter in the adjoint representation of the gauge group. For a more realistic gauge-gravity duality the inclusion of matter in the *fundamental* representation ("quarks") is a mandatory requirement. The introduction of quarks into the AdS/CFT correspondence is a prerequisite for studying a number of non-perturbative phenomena in QCD in terms of a weakly coupled string theory. Examples are the formation of hadrons, spontaneous chiral symmetry breaking, pion scattering and decay, quark confinement, etc., to mention only the most prominent among the strong coupling phenomena.

Adding fundamental flavors effectively introduces boundaries in the 't Hooft expansion, that is one adds an open string sector. Since the open strings should be allowed to have a thickness as well, one is led to believe that adding fundamental flavors in the gauge theory maps to adding spacetime filling D–branes in the 5d bulk theory. In the limit where N_f, the number of flavors, is much smaller than N, the backreaction of the D-branes on the bulk geometry can be ignored: this corresponds to the quenched approximation of lattice gauge theory. The spacetime filling D-branes should be stable. Moreover, making the quarks very heavy should decouple them from the IR theory, so D–branes dual to massive quarks should be spacetime filling in the UV region but than end at a finite distance in the 5th dimension, and be absent in the IR. The gauge fields living on the D–branes map to global flavor currents in the gauge theory.

In particular, for the $AdS_5 \times S^5$ geometry the appropriate flavor branes are D7-branes which fill the spacetime directions of the gauge theory and are extended along the holographic direction [164, 166].

So, the dual description of a 4D supersymmetric Yang–Mills theory with fundamental matter can be obtained by considering a system of D3–D7 branes which intersect along three common spatial directions. If the number of D3-branes is large we can take the decoupling limit and substitute them by the $AdS_5 \times S^5$ geometry. Moreover, when the number of D7–branes is small compared to the number of D3–branes, we can assume that the D7-branes do not backreact on the geometry and treat them as probes. The fluctuations of the probes correspond to degrees of freedom of open strings connecting the brane probe and those that generated the background [168, 169].

Finally, the near horizon geometry of a system of N D3–branes in the presence of N_f spacetime–filling D7–branes, in the large N limit and N_f fixed, gives the dual description of a $\mathcal{N} = 4$ $SU(N)$ SYM theory living on the D3–branes with supersymmetry broken to $\mathcal{N} = 2$ by N_f hypermultiplets of dynamical quarks in the fundamental representation of $SU(N)$. The field content of the hypermultiplets is given by excitations of fundamental strings stretching between D3 and D7–branes.

When the D3 and the D7–branes are separated along the mutual orthogonal directions the hypermultiplets acquire a mass which is proportional to the distance between the branes. For coincident branes (vanishing masses) the $\mathcal{N} = 2$ theory is superconformal invariant.

As proposed in [166] (see also [165]), excitations of fundamental strings with both ends on the D7–branes represent mesonic states of the corresponding SYM field theory. Studying these fluctuations allows for determining the mass spectrum of the mesonic excitations. The spectrum turns out to be discrete with a mass gap [167].

Since the original proposal of inserting D7–branes in the standard $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry, a lot of work has been done in the direction of finding generalizations to less supersymmetric and/or non–conformal backgrounds. In particular, flavors and meson spectra on the conifold and in the Klebanov–Strassler model have been studied in [173]. The Maldacena–Nunez background has been considered in [174], the class of metrics of the form $\mathrm{AdS}_5 \times Y^{p,q}$ and $\mathrm{AdS}_5 \times L^{a,b,c}$ in [175], while for the Polchinski–Strassler set–up see [176]. Supersymmetric embeddings of D–branes and their fluctuations in non–commutative theories have been investigated in [177]. Further generalizations concern other stable brane systems [178, 179]. Chiral symmetry breaking and theories at finite temperature have been first studied in [180, 181]. Moreover, several attempts have been devoted to going beyond the probe approximation and studying full back–reacted (super)gravity solutions

[182]. Further interesting results can be found in [183, 184, 185, 186, 187].

Among the formulations of the AdS/CFT correspondence with less supersymmetry, the one–parameter Lunin–Maldacena (LM) background [66] corresponding to $\mathcal{N}=1$ β–deformed SYM theories plays an interesting role, being the field theory and the dual string geometry explicitly known. The gravitational background is $\mathrm{AdS}_5\times\tilde{\mathrm{S}}^5$ where $\tilde{\mathrm{S}}^5$ is the β–deformed five sphere obtained by performing a TsT transformation on a 2–torus inside the S^5 of the original background. This operation breaks the $SO(6)$ symmetry group of the five sphere down to $U(1)\times U(1)\times U(1)$. On the field theory side, this deformation corresponds to promoting the ordinary products among the fields in the $\mathcal{N}=4$ action to a $*$–product which depends on the charges of the fields under two $U(1)$'s and allowing for the chiral coupling constant to be different from the gauge coupling. Consistently with what happens on the string side, these operations break $\mathcal{N}=4$ to $\mathcal{N}=1$ supersymmetry, as the third $U(1)$ (the one not involved in the $*$–product) corresponds to the R–symmetry. Further generalizations [188] lead to a dual correspondence between a non–supersymmetric Yang–Mills theory and a deformed LM background depending on three different real parameters γ_1, γ_2 and γ_3 [1].

All these models are (super)conformal invariant since the string geometry still has an AdS factor. As such they cannot be used to give a realistic description of the RG flow of a gauge theory towards a confining phase. However, it is interesting to investigate what happens if we insert D7–branes in these deformed backgrounds [2]. In particular, we expect to find a parametric dependence of the mesonic spectrum on γ_i's which could then be used to fine–tune the results.

In what follows we accomplish this project by studying the effects of inserting D7–branes in the more general non–supersymmetric LM–Frolov background. In the probe approximation ($N_f \ll N$), we first study the stability of the D3–D7 configuration. We find that, independently of the value of the deformation parameters, an embedding can be found which is stable, BPS and in the $\gamma_1=\gamma_2=\gamma_3$ case it is also supersymmetric.

We then study fluctuations of a D7–brane around the static embedding which correspond to scalar and vector mesons of the dual field theory. We consider the equations of motion for the tower of Kaluza–Klein modes arising from the compactification of the D7–brane on a deformed three–sphere. The background deformation induces a non–trivial coupling between scalar and vector modes. However, with a suitable field redefinition, we manage to simplify the equations and solve them analytically, so determining the mass spectrum exactly.

The effects of the deformation on the mesonic mass spectrum and on the corresponding KK modes are the following: i) As in the undeformed case the mass spectrum is discrete and with a mass gap, but it acquires a non–trivial dependence on the deformation parameters. Precisely, it depends on the parameters γ_2, γ_3 which are associated to TsT transformations along the tori with a direction orthogonal to the probe branes, whereas the parameter γ_1 associated to the deformation along the torus inside the D7 worldvolume never enters the equations of motion for quadratic fluctuations and does not affect the mass spectrum. ii) Since the deformation breaks $SO(4)$ (the isomorphisms of the three–sphere) to $U(1)\times U(1)$ a Zeeman–like effect occurs and the masses exhibit a non–trivial dependence on the (m_2, m_3) quantum numbers associated to the two $U(1)$'s.

[1] We use the standard convention to name *real* deformation parameters with γ.

[2] Several works in the literature are devoted to the study of D–branes in this context [189, 190, 191, 192, 193, 194, 195].

The dependence is through the linear combination $(\gamma_2 m_3 - \gamma_3 m_2)^2$ so that the mass eigenvalues are smoothly related to the ones of the undeformed case by sending $\gamma_i \to 0$. iii) The corresponding eigenstates are classified according to their $SO(4)$ and $U(1) \times U(1)$ quantum numbers. Expanding in vector and scalar harmonics on the three–sphere, we find Type I elementary fluctuations [3] in the $(\frac{l\mp 1}{2}, \frac{l\pm 1}{2})_{(m_2,m_3)}$ representations and Type II, Type III and scalar modes in the $(\frac{l}{2}, \frac{l}{2})_{(m_2,m_3)}$. For a given l the total number of degrees of freedom is $8(l+1)^2$ as in the undeformed theory but, given the degeneracy breaking, they split among different eigenvalues. For any given triplet (l, m_2, m_3) we compute the degeneracy of the corresponding mass eigenvalue. We find that the splitting is different according to the choice $\gamma_2 \neq \gamma_3$ or $\gamma_2 = \gamma_3$ (which includes the $\mathcal{N} = 1$ supersymmetric deformation). In the last case the spectrum exhibits a mass degeneracy between scalars and vectors which is remnant of the $\mathcal{N} = 2$ supersymmetric, undeformed case.

The chapter is organized as follows.

First, we give the foundations for a holographic study of Yang-Mills theories with flavour and to show that some of the non-perturbative phenomena can be understood in a string theoretical framework at least in a qualitative way.

Then, we study the embedding of spacetime filling D7–branes in β–deformed backgrounds which, according to the AdS/CFT dictionary, corresponds to flavoring β–deformed $\mathcal{N} = 4$ super Yang–Mills. We consider supersymmetric and more general non–supersymmetric three parameter deformations. In Section 4.2 we review the three–parameter deformation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ by using a set of coordinates suitable for the introduction of D7–branes. In Section 4.2.1 we study the static embedding of a D7–brane and discuss its stability. In the $\gamma_1 = \gamma_2 = \gamma_3$ case, using the results of [194] we argue that our configuration is supersymmetric. We then find the equations of motion for the bosonic fluctuations of a D7–brane in Section 4.3.1 and solve them analytically in Section 4.3.2 determining the exact mass spectrum. In Section 4.3.3 we discuss the properties of the spectrum and analyze in detail the splitting of the mass levels and the corresponding degeneracy. Finally, in Section 4.4 we formulate the field theory dual to our configuration, whereas our conclusions, comments and perspectives are collected in Section 4.5.

4.1 The undeformed case

Let us consider an orthogonal intersection of a p_1-brane and a p_2-brane along d common spatial directions ($p_2 \geq p_1$). We denote this intersection as $(d|p_1 \perp p_2)$. We shall treat the lower dimensional p_1-brane as a background, whereas the p_2-brane will be considered as a probe. The background metric will be taken as:

$$ds^2 = \left[\frac{u^2}{R^2}\right]^{\alpha_1} \left(-dt^2 + (dx_1)^2 + \cdots + (dx_{p_1})^2\right) + \left[\frac{R^2}{u^2}\right]^{\alpha_2} d\vec{X} \cdot d\vec{X} \,, \tag{4.1.1}$$

where R, α_1 and α_2 are constants that depend on the case considered, $\vec{X} = (X_1, \cdots, X_{9-p_1})$ and $u^2 = \vec{X} \cdot \vec{X}$. The supergravity solution also contains a dilaton ϕ, which we will parametrize as:

$$e^{-\phi(r)} = \left[\frac{R^2}{u^2}\right]^{\alpha_3} \,, \tag{4.1.2}$$

[3] We use the classification of [167].

with α_3 being constant.

Let us now place a p_2-brane in this background extended along the directions:

$$(\, t, x_1, \cdots, x_d, X_1, \cdots, X_{p_2-d}\,) \ . \tag{4.1.3}$$

We shall denote by $\vec{z}$ the set of X coordinates transverse to the probe:

$$\vec{z} \,=\, (z_1, \cdots, z_{9-p_1-p_2+d}) \ , \tag{4.1.4}$$

with $z_m = X_{p_2-d+m}$ for $m = 1, \cdots, 9 - p_1 - p_2 + d$. Notice that the $\vec{z}$ coordinates are transverse to both background and probe branes. Moreover, we shall choose spherical coordinates on the p_2-brane worldvolume which is transverse to the p_1-brane. If we define:

$$\rho^2 \,=\, (X_1)^2 \,+\, \cdots + \,(X_{p_2-d})^2 \ , \tag{4.1.5}$$

clearly, one has:

$$(dX_1)^2 \,+ \cdots (dX_{p_2-d})^2 \,=\, d\rho^2 \,+\, \rho^2 d\Omega^2_{p_2-d-1} \ , \tag{4.1.6}$$

where $d\Omega^2_{p_2-d-1}$ is the line element of a unit (p_2-d-1)-sphere. Obviously we are assuming that $p_2 - d \geq 2$.

Let us consider first a configuration in which the probe is located at a constant value of $|\vec{z}|$, *i.e.* at $|\vec{z}| = L$. If ξ^a are a set of worldvolume coordinates, the induced metric on the probe worldvolume for such a static configuration will be denoted by:

$$ds^2_I \,=\, \mathcal{G}_{ab}\, d\xi^a d\xi^b \ . \tag{4.1.7}$$

In what follows we will use as worldvolume coordinates the cartesian ones $x_0 \cdots x_d$ and the radial and angular variables introduced in eqs. (4.1.5) and (4.1.6). Taking into account that, for an embedding with $|\vec{z}| = L$, one has $u^2 = \rho^2 + \vec{z}^{\,2} \,=\, \rho^2 + L^2$, the induced metric can be written as:

$$ds^2_I \,=\, \left[\frac{\rho^2 + L^2}{R^2}\right]^{\alpha_1} (\,-dt^2 \,+\, (dx_1)^2 \,+\, \cdots + (dx_d)^2\,) \,+\, \left[\frac{R^2}{\rho^2 + L^2}\right]^{\alpha_2} (\,d\rho^2 \,+\, \rho^2 d\Omega^2_{p_2-d-1}\,) \ . \tag{4.1.8}$$

The action of the probe is given by the Dirac-Born-Infeld action. In the configurations we study in this section the worldvolume gauge field vanishes and it is easy to verify that the lagrangian density reduces to:

$$\mathcal{L} \,=\, -e^{-\phi}\sqrt{-\det \mathcal{G}} \ . \tag{4.1.9}$$

For a static configuration such as the one with $|\vec{z}| = L$, the energy density $\mathcal{H}$ is just $\mathcal{H} = -\mathcal{L}$. By using the explicit form of $\mathcal{G}$ in (4.1.8), one can verify that, for the $|\vec{z}| = L$ embedding, $\mathcal{H}$ is given by:

$$\mathcal{H} \,=\, \left[\frac{\rho^2 + L^2}{R^2}\right]^{\frac{\alpha_1}{2}(d+1) \,-\, \frac{\alpha_2}{2}(p_2-d) \,-\alpha_3} \rho^{p_2-d-1}\,\sqrt{\det g} \ , \tag{4.1.10}$$

where g is the metric of the unit (p_2-d-1)-sphere. In a BPS configuration the no-force condition of a supersymmetric intersection requires that $\mathcal{H}$ be independent of the distance L between the branes. Clearly, this can be achieved if the α_i-coefficients are related as:

$$\alpha_3 \,=\, \frac{\alpha_1}{2}\,(d+1) \,-\, \frac{\alpha_2}{2}\,(p_2 - d) \ . \tag{4.1.11}$$

Let us rewrite this last equation as:

$$d \;=\; \frac{\alpha_2}{\alpha_1+\alpha_2}\,p_2 \;+\; \frac{2\alpha_3-\alpha_1}{\alpha_1+\alpha_2}\;, \tag{4.1.12}$$

which gives the number d of common dimensions of the intersection in terms of the parameters α_i of the background and of the dimension p_2 of the probe brane.

In the string frame, the supergravity solution corresponding to a Dp-brane with $p<7$ has the form displayed in eqs. (4.1.1) and (4.1.2) with $p_1=p$, R given by

$$R^{7-p} \;=\; 2^{5-p}\,\pi^{\frac{5-p}{2}}\,\Gamma\Big(\frac{7-p}{2}\Big)\,g_s\,N\,(\alpha')^{\frac{7-p}{2}}\;, \tag{4.1.13}$$

and with the following values for the exponents α_i:

$$\alpha_1 \;=\; \alpha_2 \;=\; \frac{7-p}{4}\;, \qquad\qquad \alpha_3 \;=\; \frac{(7-p)(p-3)}{8}\;. \tag{4.1.14}$$

Moreover, the Dp-brane solution is endowed with a Ramond-Ramond $(p+1)$-form potential, whose component along the Minkowski coordinates $x_0\cdots x_p$ can be taken as:

$$\Big[\,C^{(p+1)}\,\Big]_{x_0\cdots x_p} \;=\; \Bigg[\,\frac{u^2}{R^2}\,\Bigg]^{\frac{7-p}{2}}\;. \tag{4.1.15}$$

Applying eq. (4.1.12) to this background, we get the following relation between d and p_2:

$$d=\frac{p_2+p-4}{2}\;. \tag{4.1.16}$$

Let us now consider the case in which the probe brane is another D-brane. As the brane of the background and the probe should live in the same Type II theory, p_2-p should be even. Since $d\le p$, we are left with the following three possibilities:

$$(p|Dp\perp D(p+4))\;,\qquad (p-1|Dp\perp D(p+2))\;,\qquad (p-2|Dp\perp Dp)\;. \tag{4.1.17}$$

In the standard AdS/CFT correspondence, a system of N coincident D3-branes is considered within Type IIB string theory. In the Maldacena limit, the metric of the D3-branes reduces to its near-horizon (throat) region which is $AdS_5\times S^5$. This is a product space of a five-dimensional Anti-de-Sitter space and a five-sphere. Since we want to generalize the AdS/CFT correspondence in this direction, we set $p=3$ and we study the embedding of an additional Dk–probe brane directly into the $AdS_5\times S^5$ background, according to (4.1.17). Depending on the dimension of the probe brane, the dual field theory of this supergravity set-up is then a conformal field theory with a space-time defect. These defect conformal field theories (dCFT) involve fields which are confined to a lower-dimensional subspace of the original four-dimensional space-time. For these dCFT the four-dimensional conformal symmetry is broken to the lower-dimensional conformal group of the defect.

There are various strings in the set-up: As usual, open string modes with both endpoints on the D3-branes generate the $\mathcal{N}=4$ super Yang-Mills theory, while closed string modes give rise to type IIB supergravity on $AdS_5 \times S^5$. However, we have additional strings due to the embedding of a probe brane. First, there are strings stretching between the Dk–brane and the D3–branes. They give rise to *fundamental* hypermultiplets ("quarks") in the low-energy theory. Due to the decoupling of open strings on the Dk-brane in the infrared, the $U(1)$ gauge group on the Dk-brane, or $U(N_f)$ in case of N_f Dk-branes, turns into the flavour group of the fundamental matter. Second, there are open strings ending on the Dk-brane. In the probe approximation, one neglects the back-reaction of the Dk-brane on the near-horizon background of the D3-branes. Classically, the fluctuation modes of the probe are then described by the Dirac-Born-Infeld action of the Dk-brane (plus Wess-Zumino term).

As a special case we have a "defect" of codimension zero corresponding to flavour in four spacetime dimensions ($k = 7$): this is the D3–D7 brane configuration, first studied by Karch and Katz [164, 166], where a spacetime filling D7-brane was added to the AdS_5/CFT$_4$ correspondence. The D7-brane completely fills the AdS_5 space and wraps a maximal S^3 inside S^5. This supergravity configuration is dual to a four-dimensional $\mathcal{N} = 2$ Yang-Mills theory describing open strings in the presence of one D7 and N D3-branes sharing 3+1 dimensions. The degrees of freedom are those of the $\mathcal{N} = 4$ super Yang-Mills theory, coupled to an $\mathcal{N} = 2$ hypermultiplet with fields in the fundamental representation of $SU(N)$. The latter arise from strings stretched between the D7 and D3-branes.

The D3-D7 configuration is special since fundamental fields are allowed to propagate in all four space-time dimensions. This opens up the possibility for studying flavour in supersymmetric extensions of QCD. It is possible to introduce mass for the fundamental matter by separating the D7-brane from the D3-branes. The dual description involves a probe D7 on which the induced metric is only asymptotically $AdS_5 \times S^3$. In this case there is a discrete spectrum of mesons. This spectrum has been computed (exactly!) at large 't Hooft coupling [167] using an approach analogous to the glueball calculations in deformed AdS backgrounds. The novel feature here is that the "quark" bound states are described by the scalar fields in the Dirac-Born-Infeld action of the D7-brane probe.

4.1.1 The probe approximation

The massless open string degrees of freedom of the D3–D7 intersection correspond to a $\mathcal{N} = 4$ super-Yang-Mills multiplet (generated by 3-3 strings) coupled to a fundamental hypermultiplet (3-7 and 7-3 strings). The decoupling of closed strings is achieved by scaling $N \to \infty$ while keeping the 't Hooft coupling $\lambda \equiv g_{YM}^2 N = 4\pi g_s N$ fixed. This is the usual 't Hooft limit for the gauge theory describing the N D3-branes. The 't Hooft coupling for the N_f orthogonal D7-branes is (see for example [132])

$$\lambda_f = \lambda (2\pi l_s)^4 N_f / N \qquad (4.1.18)$$

which vanishes in the above limit if N_f is kept fixed. This implies that the $SU(N_f)$ gauge theory on the D7-branes (generated by 7-7 strings) decouples and the group $SU(N_f)$ becomes the flavour symmetry of N_f flavours. For $\lambda \ll 1$ the appropriate description of this system is given by a four-dimensional $\mathcal{N} = 4$ $SU(N)$ gauge theory coupled to N_f hypermultiplets.

For $\lambda \gg 1$ one may replace the N D3-branes by the geometry $AdS_5 \times S^5$, according to the usual AdS/CFT correspondence. The D7-branes may be treated as a probe of the $AdS_5 \times S^5$

geometry. Comparing the tension of both stacks of branes (see Appendix C),

$$T_7 = \frac{\nu}{(2\pi l_s)^4} T_3 \qquad (\nu = N_f/N)\,, \tag{4.1.19}$$

we see that the tension T_7 and thus the backreaction of the D7-branes can be neglected in the probe limit $\nu \to 0$ keeping $\nu/l_s^4 \ll 1$. As we will see shortly, the D7-branes act as $AdS_5 \times S^3$ probe branes. Consequently, for large 't Hooft coupling, the generating function for correlation functions of the CFT should be given by the classical action of IIB supergravity on $AdS_5 \times S^5$ coupled to a Dirac-Born-Infeld theory on $AdS_5 \times S^3$.

4.2 Flavoring the marginally deformed case

Now we skip to the main topic of this chapter, namely the introduction of flavor in marginal deformations of AdS/CFT.

Following [66, 188] we consider a type IIB supergravity background obtained as a three–parameter deformation of $\mathrm{AdS}_5 \times \mathrm{S}^5$. It is realized by three TsT transformations (T duality – angle shift – T duality) along three tori inside S^5 and driven by three different real parameters γ_i. The corresponding metric is usually written in terms of radial/toroidal coordinates (ρ_i, ϕ_i), $i = 1, 2, 3$, $\sum_i \rho_i^2 = 1$ on the deformed sphere, and in string frame it reads (we set $\alpha' = 1$)

$$ds^2 = \frac{u^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{u^2}du^2 + R^2\left[\sum_i (d\rho_i^2 + G\rho_i^2 d\phi_i^2) + G\rho_1^2\rho_2^2\rho_3^2\left(\sum_i \hat{\gamma}_i d\phi_i\right)^2\right]$$

$$G^{-1} = 1 + \hat{\gamma}_3^2\rho_1^2\rho_2^2 + \hat{\gamma}_2^2\rho_3^2\rho_1^2 + \hat{\gamma}_1^2\rho_2^2\rho_3^2 \qquad\qquad \hat{\gamma}_i \equiv R^2\gamma_i \tag{4.2.1}$$

where R is the AdS_5 and S^5 radius. A further change of coordinates may be useful (we use the notation $c_\xi \equiv \cos\xi, s_\xi \equiv \sin\xi$ for any angle ξ)

$$\rho_1 = c_\alpha \quad , \quad \rho_2 = s_\alpha c_\theta \quad , \quad \rho_3 = s_\alpha s_\theta \tag{4.2.2}$$

leading to the description of this background in terms of Minkowski coordinates x^μ plus the AdS_5 coordinate u and five angular coordinates $(\alpha, \theta, \phi_1, \phi_2, \phi_3)$. The deformations correspond to TsT transformations along the three tori (ϕ_1, ϕ_2), (ϕ_1, ϕ_3), (ϕ_2, ϕ_3) and are parametrized by constants $\hat{\gamma}_3$, $\hat{\gamma}_2$ and $\hat{\gamma}_1$ respectively.

This background is non–supersymmetric and it is dual to a non–supersymmetric but marginal deformation of $\mathcal{N} = 4$ SYM (the deformation has to be exactly marginal since the AdS factor is not affected by TsT's). The $\mathcal{N} = 1$ supersymmetric background of [66] can be recovered by setting $\hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}_3$.

With the aim of embedding D7–branes in this background we find more convenient to express the metric in terms of a slightly different set of coordinates. We describe the six dimensional internal space in terms of $X^m \equiv \{\rho, \theta, \phi_2, \phi_3, X_5, X_6\}$ which are mapped into the previous set of coordinates by the change of variables

$$\rho = u\, s_\alpha \quad , \quad X_5 = u\, c_\alpha\, c_{\phi_1} \quad , \quad X_6 = u\, c_\alpha\, s_{\phi_1} \tag{4.2.3}$$

In string frame and still setting $\alpha' = 1$, we then have

$$ds^2 = \frac{u^2}{R^2}\,\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{u^2}\,G_{mn}dX^m dX^n \tag{4.2.4}$$

where the non–vanishing components of the metric G_{mn} are

$$
\begin{aligned}
G_{\rho\rho} &= 1 \qquad\qquad G_{\theta\theta} = \rho^2 \\
G_{\phi_2\phi_2} &= G\left(1+\hat\gamma_2^2\rho_1^2\rho_3^2\right)\rho_2^2\,u^2 \qquad\qquad G_{\phi_3\phi_3} = G\left(1+\hat\gamma_3^2\rho_1^2\rho_2^2\right)\rho_3^2\,u^2 \\
& G_{\phi_2\phi_3} = G\,\hat\gamma_2\hat\gamma_3\,\rho_1^2\rho_2^2\rho_3^2\,u^2 \\
G_{\phi_2 X_5} &= -G\,\hat\gamma_1\hat\gamma_2\,\rho_2^2\rho_3^2\,X_6 \qquad\qquad G_{\phi_2 X_6} = G\,\hat\gamma_1\hat\gamma_2\,\rho_2^2\rho_3^2\,X_5 \\
G_{\phi_3 X_5} &= -G\,\hat\gamma_1\hat\gamma_3\,\rho_2^2\rho_3^2\,X_6 \qquad\qquad G_{\phi_3 X_6} = G\,\hat\gamma_1\hat\gamma_3\,\rho_2^2\rho_3^2\,X_5 \\
G_{X_5X_5} &= 1 - \frac{X_6^2}{u^2\rho_1^2}\left[1 - G\left(1+\hat\gamma_1^2\rho_2^2\rho_3^2\right)\right] \qquad G_{X_6X_6} = 1 - \frac{X_5^2}{u^2\rho_1^2}\left[1 - G\left(1+\hat\gamma_1^2\rho_2^2\rho_3^2\right)\right] \\
& G_{X_5X_6} = \frac{X_5X_6}{u^2\rho_1^2}\left[1 - G\left(1+\hat\gamma_1^2\rho_2^2\rho_3^2\right)\right]
\end{aligned}
\tag{4.2.5}
$$

where G is given in (4.2.1) and now

$$\rho_1^2 = \frac{X_5^2 + X_6^2}{u^2} \quad , \quad \rho_2^2 = \frac{\rho^2 c_\theta^2}{u^2} \quad , \quad \rho_3^2 = \frac{\rho^2 s_\theta^2}{u^2} \tag{4.2.6}$$

The constraint $\sum_{i=1}^3 \rho_i^2 = 1$ is traded with the condition $u^2 = \rho^2 + X_5^2 + X_6^2$.

The LM–Frolov supergravity solution is characterized by a non–constant dilaton

$$e^{2\phi} = e^{2\phi_0} G \tag{4.2.7}$$

where ϕ_0 is the constant dilaton of the undeformed background related to the AdS radius by $R^4 = 4\pi e^{\phi_0} N \equiv \lambda$. For real deformation parameters $\hat\gamma_i$ the axion field C_0 is a constant and can be set to zero.

This background carries also a non–vanishing NS-NS two–form and R-R forms as well. In our set of coordinates they read

$$
\begin{aligned}
B &= \frac{R^2 G}{u^2}\Bigg((X_5 dX_6 - X_6 dX_5)\wedge(\hat\gamma_3\rho_2^2 d\phi_2 - \hat\gamma_2\rho_3^2 d\phi_3) + \hat\gamma_1\rho_2^2\rho_3^2\,u^2 d\phi_2\wedge d\phi_3\Bigg) \\
C_2 &= 4R^2 e^{-\phi_0}\omega_1\wedge\left(\hat\gamma_1\frac{X_5 dX_6 - X_6 dX_5}{u^2\rho_1^2} + \hat\gamma_2 d\phi_2 + \hat\gamma_3 d\phi_3\right) \quad , \qquad \omega_1 = \frac{\rho^4}{4u^4}c_\theta s_\theta d\theta \\
C_4 &= 4R^4 e^{-\phi_0}\left(\frac{u^4}{4R^8}dt\wedge dx_1\wedge dx_2\wedge dx_3 \; - G\,\omega_1\wedge\frac{X_5 dX_6 - X_6 dX_5}{u^2\rho_1^2}\wedge d\phi_2\wedge d\phi_3\right)
\end{aligned}
\tag{4.2.8}
$$

Moreover, we have

$$C_6 = C_4 \wedge B \tag{4.2.9}$$

and in what follows we also set $C_8 = 0$ (see Section 1.1.1).

The deformed background written in terms of the original internal coordinates $(\rho, \alpha, \theta, \phi_1, \phi_2, \phi_3)$ has a manifest invariance under constant shifts of the toroidal coordinates (ϕ_1, ϕ_2, ϕ_3) which correspond to three $U(1)$ symmetries. With our choice of coordinates the invariance under $\phi_{2,3} \to \phi_{2,3} + \text{const.}$ is still manifest, whereas the third $U(1)$ associated to shifts of ϕ_1 is realized as a rotation in the (X_5, X_6) plane.

4.2.1 The embedding of D7–branes

We now study the embedding of $N_f \ll N$ D7–branes in the deformed background described in the previous Section. For simplicity we consider the case of a single spacetime filling D7–brane ($N_f = 1$) which extends in the internal directions $(\rho, \theta, \phi_2, \phi_3)$ (we work in the static gauge where the worldvolume coordinates σ^a of the brane are identified with the appropriate ten dimensional coordinates). The X_5, X_6 coordinates parametrize the mutual orthogonal directions of the intersecting system of N sources D3–branes and one flavor D7–brane.

The dynamics of bosonic degrees of freedom of the D7–brane is described by the action (see Appendix C)

$$S = S_{DBI} + S_{WZ} \tag{4.2.10}$$

where recall that S_{DBI} is the abelian Dirac–Born–Infeld term (in what follows latin labels $a, b, ...$ stand for worldvolume components)

$$S_{DBI} = -T_7 \int_{\Sigma_8} d^8\sigma \, e^{-\phi} \sqrt{-det(g_{ab} + \mathcal{F}_{ab})} \tag{4.2.11}$$

whereas S_{WZ} is the Wess–Zumino term describing the coupling of the brane to the R-R potentials

$$S_{WZ} \;=\; T_7 \int_{\Sigma_8} \left\{ \frac{(2\pi\alpha')^3}{6} P[C_2] \wedge F \wedge F \wedge F + \frac{(2\pi\alpha')^2}{2} P[C_4 - C_2 \wedge B] \wedge F \wedge F \right\} \tag{4.2.12}$$

Here $g_{ab} \equiv G_{MN} \partial_a X^M \partial_b X^N$ is the pull–back of the ten–dimensional spacetime metric (4.2.4, 4.2.5) on the worldvolume Σ_8 and T_7 is the D7–brane tension. The $U(1)$ worldvolume gauge field strength F_{ab} enters the action through the modified field strength $\mathcal{F}_{ab} = 2\pi\alpha' F_{ab} - b_{ab}$, where b_{ab} is the pull–back of the target NS-NS two–form potential in (4.2.8), $b_{ab} = B_{MN} \partial_a X^M \partial_b X^N$. Moreover, in (4.2.12) $P[...]$ denotes the pull–back of the R-R forms on Σ_8.

We look for ground state configurations of the D7–brane. These are static solutions of the equations of motion for X_5, X_6 and εF ($\varepsilon \equiv 2\pi\alpha'$) derived from (4.2.10).

In the ordinary $AdS_5 \times S^5$ background static embeddings (see for example [180]) can be found by setting $X_6 = 0$, $F = 0$ and $X_5 = X_5(\rho)$ satisfying

$$\frac{d}{d\rho} \left(\frac{\rho^3}{\sqrt{1 + (\partial_\rho X_5)^2}} \frac{dX_5}{d\rho} \right) = 0 \tag{4.2.13}$$

with asymptotic behavior $X_5(\rho) = L + \frac{c}{\rho^2}$ for $\rho \gg 1$. The mass solution $X_5 = L$ is the only well–behaved solution and corresponds to fixing the location of the D7–brane in the 56–plane at $X_5^2 + X_6^2 = L^2$. This is a BPS configuration since the energy density turns out to be independent of L [196, 179].

In the deformed background we consider an embedding of the form

$$X^M = (x_\mu, \rho, \theta, \phi_2, \phi_3, X_5(\rho), X_6(\rho)) \quad , \quad F = F(X^M) \tag{4.2.14}$$

where, as in the ordinary case, we allow for a non–trivial dependence of the orthogonal directions on the non–compact internal coordinate ρ. Solving the equations of motion for X_5, X_6 and F in the present case requires a bit of care since the non–vanishing NS-NS 2–form in (4.2.8) can act as a source for the field strength εF.

We expand the action (4.2.10) up to second order in εF. The WZ action is simply

$$S_{WZ} = \frac{T_7}{2} \int_{\Sigma_8} P[C_4 - C_2 \wedge B] \wedge \varepsilon F \wedge \varepsilon F \tag{4.2.15}$$

whereas the expansion of S_{DBI} gives

$$\begin{aligned} \mathcal{L}_{DBI} &= -T_7 \frac{\sqrt{-det(g - b + \varepsilon F)}}{\sqrt{G}} \\ &= -T_7 \frac{\sqrt{-det(g-b)}}{\sqrt{G}} \sqrt{det(1+Y)} \\ &= -T_7\, \rho^3 s_\theta c_\theta \sqrt{\Omega_2} \left\{ 1 + \frac{1}{2}\mathrm{Tr}(Y) - \frac{1}{4}\mathrm{Tr}(Y^2) + \frac{1}{8}\left[\mathrm{Tr}(Y)\right]^2 + \cdots \right\} \end{aligned} \tag{4.2.16}$$

where we have defined

$$\begin{aligned} Y &\equiv (g-b)^{-1}\, \varepsilon F \\ \Omega_2 &\equiv 1 + (\partial_\rho X_5)^2 + (\partial_\rho X_6)^2 \end{aligned} \tag{4.2.17}$$

and set $e^{\phi_0} \equiv 1$.

The source for εF comes from the term

$$\frac{1}{2}\mathrm{Tr}(Y) = \frac{\varepsilon}{R^2 \Omega_2} \left[(X_5\, \partial_\rho X_6 - X_6\, \partial_\rho X_5)(\hat{\gamma}_2\, F_{\rho\phi_3} - \hat{\gamma}_3\, F_{\rho\phi_2}) - \hat{\gamma}_1 \Omega_2\, F_{\phi_2\phi_3} \right] \tag{4.2.18}$$

In the abelian case the last term is a total derivative and, once integrated on the worldvolume of the brane, it cancels. We are left with the first term which gives a non–trivial coupling between the scalars and the vectors. We note that these couplings are proportional to the deformation parameters and disappear for $\hat{\gamma}_i = 0$, consistently with the undeformed case.

Since all the F components except $F_{\rho\phi_2}$ and $F_{\rho\phi_3}$ satisfy homogeneous equations we can set them to zero and concentrate on the system of coupled equations of motion for $X_5, X_6, F_{\rho\phi_2}$ and $F_{\rho\phi_3}$. It is easy to realize that a solution is still given by $X_6 = 0$, $F_{\rho\phi_2} = F_{\rho\phi_3} = 0$, whereas $X_5(\rho)$ satisfies eq. (4.2.13) and can be chosen as $X_5 = L$.

Therefore, even in the deformed case, the ground state of the probe brane is given by a static location at $X_5^2 + X_6^2 = L^2$ with no F flux and absence of non–trivial quark condensate. The choice $X_5 = L$ and $X_6 = 0$ breaks the rotational invariance in the (X_5, X_6) plane.

This configuration is stable (BPS). In fact, the corresponding action

$$S = -T_7 \int_{\Sigma_8} d^8\sigma \rho^3 s_\theta c_\theta \tag{4.2.19}$$

coincides with the one of the undeformed case and satisfies the no–force condition [196, 179].

Setting $X_5^2 + X_6^2 = L^2$, the induced metric on the D7–brane reads

$$\begin{aligned} ds_I^2 &\equiv g_{ab}\, dX^a dX^b \\ &= \frac{L^2+\rho^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{L^2+\rho^2}(d\rho^2 + \rho^2 d\theta^2) \\ &+ \frac{R^2 G \rho^2}{(L^2+\rho^2)}\left[c_\theta^2 d\phi_2^2 + s_\theta^2 d\phi_3^2 + \frac{\rho^2 L^2 c_\theta^2 s_\theta^2 (\hat\gamma_2 d\phi_2 + \hat\gamma_3 d\phi_3)^2}{(L^2+\rho^2)^2}\right] \end{aligned} \tag{4.2.20}$$

where G in (4.2.1) takes the explicit form

$$G = \frac{(L^2+\rho^2)^2}{(L^2+\rho^2)^2 + \hat\gamma_1^2 \rho^4 s_\theta^2 c_\theta^2 + \hat\gamma_2^2 L^2 \rho^2 s_\theta^2 + \hat\gamma_3^2 L^2 \rho^2 c_\theta^2} \tag{4.2.21}$$

We note that, due to the particular embedding we have realized, the parameter $\hat\gamma_1$ associated to the TsT transformation on the (ϕ_2, ϕ_3) torus inside the D7 worldvolume enters the metric differently from $\hat\gamma_{2,3}$ which are instead associated to deformations on tori with one parallel and one orthogonal direction to the probe.

The different role played by $\hat\gamma_1$ respect to $(\hat\gamma_2, \hat\gamma_3)$ can be also understood by looking at the conformal case ($L = 0$) or the UV limit ($\rho \to \infty$) of the theory. In both cases the dependence on $(\hat\gamma_2, \hat\gamma_3)$ disappears and the worldvolume metric reduces to the one for $\mathrm{AdS}_5 \times \tilde{\mathrm{S}}^3$ where $\tilde{\mathrm{S}}^3$ is the deformed three–sphere with metric

$$\frac{ds^2_{\tilde{\mathrm{S}}^3}}{R^2} = d\theta^2 + G(c_\theta^2 d\phi_2^2 + s_\theta^2 d\phi_3^2) \quad , \quad G = \frac{1}{1 + \hat\gamma_1^2 c_\theta^2 s_\theta^2} \tag{4.2.22}$$

Instead, for ρ finite and $L \neq 0$ the AdS_5 factor is lost, the theory is no longer conformal and a non–trivial dependence on all the deformation parameters appears.

The particular probe brane configuration we have chosen is smoothly related to the one of the undeformed case. In fact, sending $\hat\gamma_i \to 0$ we recover the usual Karch–Katz [164] picture of flavor branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$. As we have just proved, the stability of the D3–D7 system survives the deformation.

We have embedded flavor D7–branes in a deformed background. When the D7–brane is spacetime filling and wraps the (ϕ_2, ϕ_3) torus the configuration is stable and no worldvolume flux is turned on. Alternatively, we could have started with a configuration of D7–branes in the undeformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background and perform the three TsT transformations as a second step. If the D7–branes were to be placed along the same directions as before, we would obtain exactly the same configuration of stable D7–branes in the deformed background with no flux turned on. In fact, along the directions (ϕ_1, ϕ_2, ϕ_3) affected by TsT transformations the probe branes have Dirichlet–Neumann–Neumann (DNN) boundary conditions. Considering the proposal in [191] and according to the analysis of [193] a DNN configuration with no flux is mapped into the same configuration, whatever is the TsT transformation we perform. Therefore, for the particular embedding we are analyzing the two operations i) Adding a probe to the deformed background and ii) Performing a TsT transformation on the undeformed brane scenario are equivalent processes. The stability of our brane configuration for any value of the deformation parameters then follows from the fact that TsT transformations do not affect the BPS nature of the original brane system [66] (see also [192]).

It is worth stressing that the possibility of applying equivalently prescriptions i) or ii) is peculiar of the particular brane configuration we have chosen. Had we considered different embeddings, the two procedures wouldn't had led necessarily to equivalent settings [191, 193]. Furthermore, the stability of the configuration would have become questionable.

When the deformation parameters $\hat{\gamma}_i$ are all equal the $\mathrm{AdS}_5 \times \tilde{\mathrm{S}}^5$ background has $\mathcal{N} = 1$ supersymmetry. The question is whether our D7–brane embedding preserves supersymmetry. The standard way of finding supersymmetric configurations is to look at the κ–symmetry condition of the probes. However, since the β–deformed background can be described by an $SU(2)$ structure manifold, it is more convenient to work using the formalism of G–structures [197] and Generalized Complex Geometry (GCG) [198]. In this framework the supersymmetry conditions for D–branes probing $SU(2)$ structure manifolds have been established in [194]. For spacetime filling D7–branes a class of supersymmetric embeddings is given by $z_1 \equiv X_5 + iX_6 = L$, with $z_2 \equiv X_1 + iX_2$ and $z_3 \equiv X_3 + iX_4$ arbitrarily fixed and no worldvolume flux turned on. This embeddings break one of the $U(1)$ global symmetries. Since our configuration belongs to this class we conclude that our embedding is supersymmetric.

4.3 The mesonic scenario

As proposed in [166, 165] D7–brane fluctuations around its ground state are dual to color singlets which may be interpreted as describing mesonic states of the four dimensional gauge theory. The mass spectrum of the mesons is given by the Kaluza–Klein spectrum of states which originate from the compactification of the D7–brane on the internal submanifold. In the ordinary undeformed scenario the spectrum is discrete and with a mass gap [167].

4.3.1 Probe fluctuations

Our main purpose is to investigate probe fluctuations in the deformed background. A generic vibration of the brane around its ground state can be described by

$$X_5 = L + \varepsilon\,\chi(\sigma^a)\,, \qquad\qquad X_6 = \varepsilon\,\varphi(\sigma^a) \tag{4.3.1}$$

together with a non–trivial flux $\varepsilon F_{ab} = \varepsilon(\partial_a A_b - \partial_b A_a)$. The fluctuations are functions of the worldvolume coordinates σ^a and ε is a small perturbation parameter.

We expand the action of the probe brane in powers of the small parameter

$$S = S_{DBI} + S_{WZ} = \int_{\Sigma_8} d^8\sigma \{\mathcal{L}_0 + \varepsilon \mathcal{L}_1 + \varepsilon^2 \mathcal{L}_2 + \cdots\} \tag{4.3.2}$$

and consider terms up to the quadratic order in ε.

We first concentrate on the DBI term

$$\mathcal{L}_{DBI} = -T_7 \frac{1}{\sqrt{G}} \sqrt{-det(g - b + \varepsilon F)} \tag{4.3.3}$$

where we have written the dilaton field as in (4.2.7) with $e^{\phi_0} \equiv 1$.

We expand the various terms by writing

$$\begin{aligned} g &= g^{(0)} + \varepsilon g^{(1)} + \varepsilon^2 g^{(2)}\,, \qquad b = b^{(0)} + \varepsilon b^{(1)} + \varepsilon^2 b^{(2)} \\ \frac{1}{\sqrt{G}} &= G^{(0)} + \varepsilon G^{(1)} + \varepsilon^2 G^{(2)} \end{aligned} \tag{4.3.4}$$

Therefore, the determinant can be written as

$$\begin{aligned} \sqrt{-det(g - b + \varepsilon F)} &= \sqrt{-det\left(g^{(0)} - b^{(0)}\right)} \sqrt{det(1 + Y)} \\ &= \sqrt{-det\left(g^{(0)} - b^{(0)}\right)} \left[1 + \frac{1}{2}\mathrm{Tr}(Y) - \frac{1}{4}\mathrm{Tr}(Y^2) + \frac{1}{8}\left[\mathrm{Tr}(Y)\right]^2 + \cdots\right] \end{aligned} \tag{4.3.5}$$

where the matrix Y is given by

$$Y = \left(g^{(0)} - b^{(0)}\right)^{-1} \left[\varepsilon\left(g^{(1)} - b^{(1)} + F\right) + \varepsilon^2\left(g^{(2)} - b^{(2)}\right) + \cdots\right] \tag{4.3.6}$$

At the lowest order the contribution $g^{(0)}$ is easily read from (4.2.20), whereas for the pull–back of B from eq. (4.2.8) we find that the only non–vanishing component is $b^{(0)}_{\phi_2\phi_3} = \hat{\gamma}_1 R^2 G \rho_2^2 \rho_3^2$.

It is convenient to introduce the undeformed induced metric

$$\mathcal{G} = diag\left(-\frac{L^2+\rho^2}{R^2}, \frac{L^2+\rho^2}{R^2}, \frac{L^2+\rho^2}{R^2}, \frac{L^2+\rho^2}{R^2}, \frac{R^2}{L^2+\rho^2}, \frac{R^2\rho^2}{L^2+\rho^2}, \frac{R^2\rho^2 c_\theta^2}{L^2+\rho^2}, \frac{R^2\rho^2 s_\theta^2}{L^2+\rho^2}\right) \tag{4.3.7}$$

the auxiliary metric $\mathcal{C}$ defined by

$$\begin{aligned} d\hat{s}^2 &\equiv \mathcal{C}_{ab} d\sigma^a d\sigma^b \\ &= \frac{L^2+\rho^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{L^2+\rho^2}(d\rho^2 + \rho^2 d\theta^2) \\ &+ \frac{R^2 \hat{G} \rho^2}{L^2+\rho^2}\left[c_\theta^2 d\phi_2^2 + s_\theta^2 d\phi_3^2 + \frac{\rho^2 L^2 c_\theta^2 s_\theta^2 (\hat{\gamma}_2 d\phi_2 + \hat{\gamma}_3 d\phi_3)^2}{(L^2+\rho^2)^2}\right] \end{aligned} \tag{4.3.8}$$

with

$$\hat{G} = \frac{(L^2+\rho^2)^2}{(L^2+\rho^2)^2 + \hat{\gamma}_2^2 L^2 \rho^2 s_\theta^2 + \hat{\gamma}_3^2 L^2 \rho^2 c_\theta^2} \tag{4.3.9}$$

and two deformation matrices $\mathcal{T}$ and $\mathcal{J}$ given by

$$\begin{array}{lll} \mathcal{T}^{\phi_2\phi_2} = \hat{\gamma}_3^2 & \mathcal{T}^{\phi_3\phi_3} = \hat{\gamma}_2^2 & \mathcal{T}^{\phi_2\phi_3} = \mathcal{T}^{\phi_3\phi_2} = -\hat{\gamma}_2\hat{\gamma}_3 \\ \mathcal{J}^{\phi_2\phi_2} = 0 & \mathcal{J}^{\phi_3\phi_3} = 0 & \mathcal{J}^{\phi_2\phi_3} = -\mathcal{J}^{\phi_3\phi_2} = \gamma_1 \end{array} \tag{4.3.10}$$

The metric $\mathcal{C}$ is nothing but the induced metric (4.2.20) evaluated at $\hat{\gamma}_1 = 0$. Its inverse can be expressed as

$$\mathcal{C}^{-1} = \mathcal{G}^{-1} + \frac{L^2}{R^2(L^2+\rho^2)}\mathcal{T} \tag{4.3.11}$$

It turns out that the matrix $\left(g^{(0)} - b^{(0)}\right)^{-1}$ in (4.3.6) can be written as

$$\left(g^{(0)} - b^{(0)}\right)^{-1} = \mathcal{C}^{-1} + \mathcal{J} = \mathcal{G}^{-1} + \frac{L^2}{R^2(L^2+\rho^2)}\mathcal{T} + \mathcal{J} \tag{4.3.12}$$

Since the whole dependence on the deformation parameters is encoded in $\mathcal{T}$ and $\mathcal{J}$, the $\hat{\gamma}_i \to 0$ limit is easily understood.

Now a long but straightforward calculation allows to determine the first order corrections $g^{(1)}, b^{(1)}, G^{(1)}$ as well as the second order ones $g^{(2)}, b^{(2)}, G^{(2)}$. Inserting in $\mathcal{L}_{DBI}$ we eventually find

$$\begin{aligned} \mathcal{L}^{(0)}_{DBI} &= -T_7\rho^3 c_\theta s_\theta \\ \mathcal{L}^{(1)}_{DBI} &= T_7\rho^3 c_\theta s_\theta \hat{\gamma}_1 F_{\phi_2\phi_3}/R^2 \\ \mathcal{L}^{(2)}_{DBI} &= -T_7\rho^3 c_\theta s_\theta \left[\frac{R^2}{2(L^2+\rho^2)}\,\mathcal{C}^{ab}\partial_a\chi\partial_b\chi + \frac{R^2}{2(L^2+\rho^2)}\,\mathcal{G}^{ab}\partial_a\varphi\partial_b\varphi \right. \\ &\quad \left. + \frac{1}{4}F_{ab}F^{ab} + \frac{L}{(L^2+\rho^2)}(\hat{\gamma}_2 F_{a\phi_3} - \hat{\gamma}_3 F_{a\phi_2})\,\mathcal{G}^{ab}\partial_b\varphi\right] \end{aligned} \tag{4.3.13}$$

where $F^{ab} \equiv \mathcal{C}^{ac}\mathcal{C}^{bd}F_{cd}$ and $\mathcal{C}^{ac}$ is given in (4.3.11). The first order Lagrangian is a total derivative since our embedding $X_5 = L$, $X_6 = 0$ is an exact solution of the equations of motion.

The Wess–Zumino Lagrangian starts with a second order term in ε given by

$$\mathcal{L}_{WZ} = T_7\,\frac{1}{2}P[C_4 - C_2\wedge B]\wedge F\wedge F = T_7\frac{(L^2+\rho^2)^2}{R^4}\,\epsilon^{ijk}\partial_\rho A_i\partial_j A_k \tag{4.3.14}$$

where we use latin indices to indicate coordinates on the three–sphere parametrized by (θ, ϕ_2, ϕ_3), A_i is the flux potential on it and ϵ^{ijk} is the Levi–Civita tensor density ($\epsilon^{\theta 23} = 1$). This term turns out to be independent of the deformation parameters since the combination $(C_4 - C_2\wedge B)$ at lowest order gives exactly the 4–form of the $AdS_5 \times S^5$ undeformed geometry.

Determining the equations of motion from the previous Lagrangian is now an easy task. Introducing the fixed vector

$$v^a = \hat{\gamma}_2\delta^a_3 - \hat{\gamma}_3\delta^a_2 \tag{4.3.15}$$

for the χ and φ scalars we find

$$\partial_a\left[\sqrt{-det(\mathcal{G})}\,\left(\frac{R^2}{(L^2+\rho^2)}\,\mathcal{G}^{ab} + \frac{L^2}{(L^2+\rho^2)^2}v^a v^b\right)\partial_b\chi\right] = 0 \tag{4.3.16}$$

$$\partial_a \left[\sqrt{-det(\mathcal{G})}\, \frac{R^2}{(L^2+\rho^2)}\, \mathcal{G}^{ab} \, \left(\partial_b \varphi + \frac{L}{R^2} v^c F_{bc} \right) \right] = 0 \tag{4.3.17}$$

whereas, using (4.3.17) the equations of motion for the gauge fields take the form

$$\begin{aligned} &\partial_a \left[\sqrt{-det(\mathcal{G})}\, \mathcal{G}^{ac}\mathcal{G}^{bd}\, F_{cd} \right] - \frac{4\rho(L^2+\rho^2)}{R^4} \epsilon^{bjk} \partial_j A_k \\ &\quad - \sqrt{-det(\mathcal{G})}\, \frac{L}{(L^2+\rho^2)}\, v^d \partial_d \left[\mathcal{G}^{bc} \left(\partial_c \varphi + \frac{L}{R^2} v^f F_{cf} \right) \right] = 0 \end{aligned} \tag{4.3.18}$$

It is interesting to note that the equations of motion depend only on the deformation parameters $\hat{\gamma}_2$ and $\hat{\gamma}_3$ hidden in the vector v. In fact, at this order the dependence on the parameter $\hat{\gamma}_1$ associated to the torus inside the D7 worldvolume completely cancels between the factors $\sqrt{-det(g-b+\varepsilon F)}$ and $1/\sqrt{G}$.

The scalar fluctuation χ along the direction where the branes are located at distance L decouples from the rest. The scalar φ, instead, interacts non–trivially with the worldvolume gauge fields through terms proportional to the deformation parameters.

The vector v has non–vanishing components only on the three–sphere and selects there a fixed direction. As a consequence, the equations of motion (4.3.16 – 4.3.18) loose $SO(4)$ invariance.

As a first application we consider the $L = 0$ conformal case. The vibration of the brane is given by $X_5 = \varepsilon\, \chi(\sigma^a)$ and $X_6 = \varepsilon\, \varphi(\sigma^a)$. The equations of motion reduce to

$$\begin{aligned} &\partial_a \left[\sqrt{-det(\mathcal{G})}\, \frac{R^2}{\rho^2}\, \mathcal{G}^{ab}\, \partial_b \Psi \right] = 0 \\ &\partial_a \left[\sqrt{-det(\mathcal{G})}\, \mathcal{G}^{ac}\mathcal{G}^{bd}\, F_{cd} \right] - \frac{4\rho^3}{R^4} \epsilon^{bjk} \partial_j A_k = 0\,. \end{aligned} \tag{4.3.19}$$

where $\Psi \equiv (\varphi, \chi)$ and $\mathcal{G}^{ab}$ is the inverse of the matrix (4.3.7) evaluated at $L = 0$. We see that the dependence on the deformation parameters disappears completely and the equations of motion reduce to the ones of the undeformed case [167]. In particular, the scalar and gauge fluctuations decouple. Written explicitly, the scalar equations read

$$\frac{R^4}{\rho^4} \partial^\mu \partial_\mu \Psi + \frac{1}{\rho^3} \partial_\rho (\rho^3 \partial_\rho \Psi) + \frac{1}{\rho^2} \Delta_{\mathrm{S}^3} \Psi = 0 \tag{4.3.20}$$

where

$$\Delta_{\mathrm{S}^3} \Psi \equiv \frac{1}{c_\theta s_\theta} \partial_\theta (c_\theta s_\theta \partial_\theta \Psi) + \frac{1}{c_\theta^2} \partial_2^2 \Psi + \frac{1}{s_\theta^2} \partial_3^2 \Psi \tag{4.3.21}$$

is the Laplacian on the unit 3–sphere ($\partial_2 \equiv \partial_{\phi_2}$, $\partial_3 \equiv \partial_{\phi_3}$).

According to the results in [164, 167] the corresponding AdS_5 masses are above the Breitenlohner–Freedman bound [199]. This is a further check of the stability of our brane configuration.

4.3.2 Mesonic spectrum

We now concentrate on the more general situation $X_5 = L + \varepsilon\, \chi(\sigma^a)$, $X_6 = \varepsilon\, \varphi(\sigma^a)$ and solve the equations of motion (4.3.16 – 4.3.18) for scalar and vector modes. We write the abelian flux in terms of its potential one–form, $F_{ab} = \partial_a A_b - \partial_b A_a$, and choose the Lorentz gauge $\partial_\mu A^\mu = 0$ on the spacetime components.

We find convenient to introduce covariant derivatives on the unit three–sphere (θ, ϕ_2, ϕ_3). Given its metric $g = diag(1, c_\theta^2, s_\theta^2)$, we have $\nabla_i V^j = \partial_i V^j + \Gamma^j_{ik} V^k$ with the only non–vanishing components being $\Gamma^\theta_{22} = -\Gamma^\theta_{33} = c_\theta s_\theta$, $\Gamma^2_{2\theta} = -\frac{s_\theta}{c_\theta}$ and $\Gamma^3_{3\theta} = \frac{c_\theta}{s_\theta}$.

In order to simplify the equations we introduce the special operators

$$\begin{aligned}\mathcal{O}_{\hat\gamma} &\equiv \frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu + \frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho) + \frac{1}{\rho^2}\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i) + \frac{L^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)^2 \\ \tilde{\mathcal{O}}_{\hat\gamma} &\equiv \frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu + \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left[\rho(L^2+\rho^2)^2\partial_\rho\right] + \frac{1}{\rho^2}\nabla_l\nabla^l \\ &\quad + \frac{L^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)^2 \end{aligned} \tag{4.3.22}$$

along with their undeformed versions $\mathcal{O}_0 \equiv \mathcal{O}_{\hat\gamma}|_{\hat\gamma_2=\hat\gamma_3=0}$, $\tilde{\mathcal{O}}_0 \equiv \tilde{\mathcal{O}}_{\hat\gamma}|_{\hat\gamma_2=\hat\gamma_3=0}$.

Equation (4.3.16) for the χ mode then takes the compact form

$$\mathcal{O}_{\hat\gamma}\,\chi = 0 \tag{4.3.23}$$

whereas equation (4.3.17) can be rewritten as

$$\mathcal{O}_0\,\Phi - \frac{L}{R^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)\left[\frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) + \frac{1}{\rho^2}\nabla_l A^l\right] = 0 \tag{4.3.24}$$

where we have defined

$$\Phi \equiv \varphi + \frac{L}{R^2}v^a A_a = \varphi + \frac{L}{R^2}(\hat\gamma_2 A_3 - \hat\gamma_3 A_2) \tag{4.3.25}$$

Equations (4.3.18) for the vector modes come into three classes, according to b being in Minkowski, or $b = \rho$ or $b = i \equiv \{\theta,\, \phi_2,\, \phi_3\}$. We list the three cases.

- b in Minkowski: For $b = \mu$ and expressing the F flux in terms of its one–form potential, equation (4.3.18) becomes

$$\mathcal{O}_{\hat\gamma}\,A_\mu - \partial_\mu\left[\frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) + \frac{1}{\rho^2}\nabla_l A^l + \frac{LR^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)\,\Phi\right] = 0 \tag{4.3.26}$$

  with Φ defined in (4.3.25).

  We apply ∂^μ to this equation and sum over μ. Using $[\partial^\mu, \mathcal{O}_{\hat\gamma}] = 0$ and Lorentz gauge, solutions corresponding to non–trivial dispersion relations ($k^2 \neq 0$) satisfy

$$\left[\frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) + \frac{1}{\rho^2}\nabla_l A^l + \frac{LR^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)\Phi\right] = 0 \quad , \quad \mathcal{O}_{\hat\gamma}\,A_\mu = 0 \tag{4.3.27}$$

- $b = \rho$: Again, expressing the flux in terms of the vector potential we obtain

$$\mathcal{O}_{\hat\gamma}\,A_\rho - \left[\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho A_\rho) + \frac{1}{\rho^2}\partial_\rho\nabla_l A^l + \frac{LR^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)\partial_\rho\Phi\right] = 0 \tag{4.3.28}$$

- $b=i$: On the internal $\tilde{\rm S}^3$ sphere we have

$$\tilde{\mathcal{O}}_{\hat\gamma}\, A_j - \frac{1}{\rho^2}\left(\nabla_l\nabla_j A^l + \frac{4\rho^2}{L^2+\rho^2}\frac{1}{c_\theta s_\theta}\epsilon_{jlm}\nabla^l A^m\right) - \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left[\rho(L^2+\rho^2)^2\partial_j A_\rho\right] - \frac{LR^2}{(L^2+\rho^2)^2}\left(\hat\gamma_2\partial_3-\hat\gamma_3\partial_2\right)\partial_j\Phi = 0 \tag{4.3.29}$$

where we have used $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}F^{ij}) = \nabla_i F^{ij} = \nabla_i\nabla^i A^j - \nabla_i\nabla^j A^i$.

Now, collecting all the equations and using the first of (4.3.27) in (4.3.24) the system of coupled equations we need solve is

$$\begin{aligned}
&(0) \qquad \mathcal{O}_{\hat\gamma}\chi = 0 \qquad ; \qquad \mathcal{O}_{\hat\gamma}\, A_\mu = 0 \\
&(1) \qquad \mathcal{O}_{\hat\gamma}\,\Phi = 0 \\
&(2) \qquad \left[\frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) + \frac{1}{\rho^2}\nabla^l A_l + \frac{LR^2}{(L^2+\rho^2)^2}\left(\hat\gamma_2\partial_3-\hat\gamma_3\partial_2\right)\Phi\right] = 0 \\
&(3) \qquad \mathcal{O}_{\hat\gamma}\, A_\rho - \left[\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho A_\rho) + \frac{1}{\rho^2}\partial_\rho\nabla^l A_l + \frac{LR^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3-\hat\gamma_3\partial_2)\partial_\rho\Phi\right] = 0 \\
&(4) \qquad \tilde{\mathcal{O}}_{\hat\gamma}\, A_j - \frac{1}{\rho^2}\left(\nabla_l\nabla_j A^l + \frac{4\rho^2}{L^2+\rho^2}\frac{1}{c_\theta s_\theta}\epsilon_{jlm}\nabla^l A^m\right) \\
&\qquad\qquad - \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left[\rho(L^2+\rho^2)^2\partial_j A_\rho\right] - \frac{LR^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3-\hat\gamma_3\partial_2)\partial_j\Phi = 0
\end{aligned} \tag{4.3.30}$$

Equations (1)–(4) exhibit a non–trivial interaction between the scalar Φ and the components of the vector potential along the internal directions. The modes χ and A_μ instead decouple.

It is convenient to search for solutions expanded in the particular spherical harmonics on S^3, described in Appendix D. Scalar spherical harmonics are a complete set of functions $\mathcal{Y}_l^{m_2,m_3}$ in the $\left(\frac{l}{2},\frac{l}{2}\right)$ representation of $SO(4)$ and with definite $U(1)\times U(1)$ quantum numbers (m_2,m_3) satisfying $|m_2+m_3| = |m_2-m_3| = l-2k$, $l,k=0,1,\ldots$. For fixed l the degeneracy is $(l+1)^2$. Their defining equations are [4]

$$\begin{aligned}
\Delta_{\mathrm{S}^3}\,\mathcal{Y}_l^{m_2,m_3} &= -l(l+2)\,\mathcal{Y}_l^{m_2,m_3} \\
\frac{\partial}{\partial\phi_{2,3}}\,\mathcal{Y}_l^{m_2,m_3} &= i m_{2,3}\,\mathcal{Y}_l^{m_2,m_3}
\end{aligned} \tag{4.3.31}$$

Vector spherical harmonics come into three classes. Choosing them to be also eigenfunctions of $\frac{\partial}{\partial\phi_{2,3}}$ we have longitudinal harmonics $\mathcal{H}_i = \nabla_i\mathcal{Y}_l^{m_2,m_3}$, $l\geq 1$ which are in the $(\frac{l}{2},\frac{l}{2})$ representation of $SO(4)$ with (m_2,m_3) ranging as before. Transverse harmonics are $\mathcal{M}_i^+ \equiv \mathcal{Y}_i^{(l,m_2,m_3);+}$ with $l\geq 1$ in the $\left(\frac{l-1}{2},\frac{l+1}{2}\right)$ and $\mathcal{M}_i^- \equiv \mathcal{Y}_i^{(l,m_2,m_3);-}$ with $l\geq 1$ in the $\left(\frac{l+1}{2},\frac{l-1}{2}\right)$. Their degeneracy is

[4] For their explicit realization see for instance [157, 190].

$l(l+2)$ and it is counted by $|m_2+m_3| = l \pm 1 - 2k, |m_2 - m_3| = l \mp 1 - 2k$. These harmonics satisfy

$$\begin{aligned}
\nabla_i \nabla^i \mathcal{M}^{\pm}_j - R^k_j \mathcal{M}^{\pm}_k &= -(l+1)^2\, \mathcal{M}^{\pm}_j \\
\epsilon_{ijk} \nabla^j \mathcal{M}^{\pm;k} &= \pm \sqrt{g}\,(l+1)\, \mathcal{M}^{\pm}_i \\
\nabla^i \mathcal{M}^{\pm}_i &= 0 \\
\frac{\partial}{\partial \phi_{2,3}} \mathcal{M}^{\pm}_i &= i m_{2,3}\, \mathcal{M}^{\pm}_i
\end{aligned} \tag{4.3.32}$$

where $\sqrt{g} = c_\theta s_\theta$ is the square root of the determinant of the metric on S^3, whereas $R^i_j = 2\delta^i_j$ is the Ricci tensor.

As in the undeformed case [167] we require the solutions to be regular at the origin ($\rho = 0$), normalizable and small enough to justify the quadratic approximation. All these conditions are used to select the actual mass spectrum of the mesonic excitations.

The scalar mode χ

We start solving the equation for the decoupled scalar χ. Using the general identity $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i s) = \nabla_i \nabla^i s$ valid for any scalar s, the equation $\mathcal{O}_{\hat{\gamma}}\chi = 0$ reads explicitly

$$\frac{R^4}{(L^2+\rho^2)^2}\partial^\nu \partial_\nu \chi + \frac{1}{\rho^3}\partial_\rho(\rho^3 \partial_\rho \chi) + \frac{1}{\rho^2}\nabla_l \nabla^l \chi + \frac{L^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 \partial_3 - \hat{\gamma}_3 \partial_2)^2 \chi = 0 \tag{4.3.33}$$

We look for single–mode solutions of the form

$$\chi(\sigma^a) = r(\rho)\, e^{ikx}\, \mathcal{Y}_l^{m_2,m_3}(\theta, \phi_2, \phi_3) \tag{4.3.34}$$

Inserting in (4.3.33) we obtain an equation for $r(\rho)$ that, after the redefinitions

$$\varrho = \frac{\rho}{L}\,, \qquad \hat{\Gamma}^2 = -\frac{k^2 R^4}{L^2} - (\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2 = \bar{M}^2 - (\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2\,, \tag{4.3.35}$$

becomes

$$\partial^2_\varrho r + \frac{3}{\varrho}\partial_\varrho r + \left[\frac{\hat{\Gamma}^2}{(1+\varrho^2)^2} - \frac{l(l+2)}{\varrho^2}\right] r = 0 \tag{4.3.36}$$

This has exactly the same structure of the equation found in the undeformed case [167]. The only difference is the presence of the deformation parameters in $\hat{\Gamma}^2$ which in the undeformed case reduces simply to $\bar{M}^2$. Following what has been done in that case [167] we find that the general solution is

$$r(\rho) = \rho^l (L^2+\rho^2)^{-\alpha} F(-\alpha, -\alpha+l+1; l+2; -\rho^2/L^2) \tag{4.3.37}$$

where F is the hypergeometric function and $\alpha = \frac{-1+\sqrt{1+\hat{\Gamma}^2}}{2}$. This solution satisfies the conditions of regularity and normalizability if the quantization condition

$$\hat{\Gamma}^2 = 4(n+l+1)(n+l+2) \qquad n \in N\,, \quad n, l \geq 0 \tag{4.3.38}$$

is imposed. Using (4.3.35) and $M^2 = -k^2$, the mass spectrum of scalar mesons then follows

$$M_\chi(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat\gamma_2 m_3 - \hat\gamma_3 m_2}{2}\right)^2} \tag{4.3.39}$$

with $n, l \geq 0$ and $|m_2 + m_3| = |m_2 - m_3| = l - 2k$, k a non–negative integer.

We see that the deformation parameters induce a non–trivial dependence of the mass spectrum on the two $U(1)$ quantum numbers (m_2, m_3), so breaking the degeneracy of the undeformed case.

The mass spectrum is smoothly related to the one of the undeformed case for $\hat\gamma_i \to 0$.

The Type II modes

We look for excitations of the form

$$A_\mu(\sigma^a) = \zeta_\mu\, Z_{II}(\rho)\, e^{ikx}\, \mathcal{Y}_l^{m_2,m_3}(\theta,\phi_2,\phi_3) \quad , \quad k\cdot\zeta = 0 \tag{4.3.40}$$

Following the classification introduced in [167] for the undeformed case we call them Type II modes. The equation $\mathcal{O}_{\hat\gamma} A_\mu = 0$ in (4.3.30) yields to

$$\frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu A_\mu + \frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho A_\mu) + \frac{1}{\rho^2}\nabla_l\nabla^l A_\mu + \frac{L^2}{(L^2+\rho^2)^2}(\hat\gamma_2\partial_3 - \hat\gamma_3\partial_2)^2 A_\mu = 0 \tag{4.3.41}$$

This is exactly the same equation as the one for the scalar mode χ. Therefore, for each component A_μ we follow the same strategy of subsection 5.1.1 and find the mass spectrum

$$M_{II}(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat\gamma_2 m_3 - \hat\gamma_3 m_2}{2}\right)^2} \tag{4.3.42}$$

with $n, l \geq 0$ and $|m_2 + m_3| = |m_2 - m_3| = l - 2k$.

Even for this type of vector fluctuations the spectrum is smoothly related to the undeformed one for $\hat\gamma_i \to 0$.

The Type I modes

Having performed the field redefinition (4.3.25) we solve the coupled equations (1)–(4) by considering elementary fluctuations of Φ, A_ρ and A_i.

Being in a different representation the harmonics $\mathcal{M}_i^\pm$ do not mix with the others. Therefore we can make the ansatz [5]

$$\Phi = 0, \qquad A_\rho = 0, \qquad A_i(\sigma^a) = Z_I^\pm(\rho)\, e^{ikx}\, \mathcal{M}_i^\pm(\theta,\phi_2,\phi_3) \tag{4.3.43}$$

[5] We note that if we were to follow closely the classification of [167] we would call Type I modes the elementary modes with $\varphi = 0$, i.e. with no fluctuations along the X^6 coordinate. However, given the structure of the equations of motion, in our case we find the definition (4.3.43) more convenient. In any case, the two definitions coincide for $\hat\gamma_i = 0$.

By using the identity $\nabla_i A^i = 0$ as follows from (4.3.32), equations (1), (2) and (3) in (4.3.30) are identically satisfied whereas eq. (4) reads

$$\tilde{\mathcal{O}}_{\hat{\gamma}}\, A_j - \frac{1}{\rho^2}\left(\nabla_l \nabla_j A^l + \frac{4\rho^2}{L^2+\rho^2}\frac{1}{c_\theta s_\theta}\epsilon_{jlm}\nabla^l A^m\right) = 0 \tag{4.3.44}$$

Considering the explicit expression for the operator $\tilde{\mathcal{O}}_{\hat{\gamma}}$ in (4.3.22) and using properties (4.3.32) we find that $Z_I^{\pm}(\rho)$ is a solution of the equation

$$\frac{1}{\varrho}\partial_\varrho\left[\varrho(\varrho^2+1)^2\partial_\varrho Z_I^{\pm}\right] + \left[\hat{\Gamma}^2 - \frac{(\varrho^2+1)^2}{\varrho^2}(l+1)^2 \mp 4(\varrho^2+1)(l+1)\right] Z_I^{\pm} = 0 \tag{4.3.45}$$

where we have used the definitions (4.3.35). This is formally the same equation as the one of the undeformed case, except for the different definition of $\hat{\Gamma}^2$. Therefore, following the same steps [167] we find that the solutions are still hypergeometric functions

$$\begin{aligned} Z_I^{+}(\rho) &= \rho^{l+1}(\rho^2+L^2)^{-\alpha-1}F(l+2-\alpha,-1-\alpha;l+2;-\rho^2/L^2) \\ Z_I^{-}(\rho) &= \rho^{l+1}(\rho^2+L^2)^{-\alpha-1}F(l-\alpha,1-\alpha;l+2;-\rho^2/L^2) \end{aligned} \tag{4.3.46}$$

where $\alpha = \frac{-1+\sqrt{1+\hat{\Gamma}^2}}{2}$. Requiring them to be regular at infinity we obtain the following quantization conditions

$$\begin{aligned} \hat{\Gamma}_+^2 &= 4(n+l+2)(n+l+3) \\ \hat{\Gamma}_-^2 &= 4(n+l)(n+l+1) \qquad\qquad n \geq 0 \end{aligned} \tag{4.3.47}$$

As a consequence the mass spectrum reads

$$\begin{aligned} M_{I,+} &= \frac{2L}{R^2}\sqrt{(n+l+2)(n+l+3) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2} \qquad \begin{cases} |m_2+m_3| = l-1-2k \\ |m_2-m_3| = l+1-2k \end{cases} \\ M_{I,-} &= \frac{2L}{R^2}\sqrt{(n+l)(n+l+1) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2} \qquad \begin{cases} |m_2+m_3| = l+1-2k \\ |m_2-m_3| = l-1-2k \end{cases} \end{aligned} \tag{4.3.48}$$

where $l \geq 1$ and k is a non–negative integer.

The Type III modes

Finally, we consider the following fluctuations

$$\begin{aligned} \Phi(\sigma^a) &= X_{III}(\rho)\, e^{ikx}\, \mathcal{Y}_l^{m_2,m_3}(\theta,\phi_2,\phi_3) \\ A_\rho(\sigma^a) &= Y_{III}(\rho)\, e^{ikx}\, \mathcal{Y}_l^{m_2,m_3}(\theta,\phi_2,\phi_3) \\ A_i(\sigma^a) &= Z_{III}(\rho)\, e^{ikx}\, \nabla_i\mathcal{Y}_l^{m_2,m_3}(\theta,\phi_2,\phi_3) \equiv \nabla_i A(\sigma^a) \end{aligned} \tag{4.3.49}$$

with $l \geq 1$. We note that $l = 0$ corresponds to having $A_i = 0$. We will comment on this particular case at the end of this Section.

Inserting in (4.3.30) and using the identities (4.3.31) for the scalar harmonics, after a bit of algebra the equations (1)–(4) can be rewritten as

$$(1) \qquad \left[\frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu + \frac{1}{\rho^3}\partial_\rho\left(\rho^3\partial_\rho\right) - \frac{l(l+2)}{\rho^2} - \frac{L^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2\right]\Phi = 0$$

$$(2) \qquad \frac{1}{\rho^3}\partial_\rho(\rho^3 A_\rho) - \frac{l(l+2)}{\rho^2}A + i\frac{LR^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)\,\Phi = 0$$

$$(3) \qquad \frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu A_\rho + \frac{1}{\rho^2}\partial_\rho\left(\frac{1}{\rho}\partial_\rho(\rho^3 A_\rho)\right) - \left[\frac{l(l+2)}{\rho^2} + \frac{L^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2\right]A_\rho + 2iLR^2\frac{(L^2-\rho^2)}{\rho(L^2+\rho^2)^3}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)\,\Phi = 0$$

$$(4) \qquad \frac{R^4}{(L^2+\rho^2)^2}\partial^\nu\partial_\nu A + \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left(\rho(L^2+\rho^2)^2\partial_\rho A\right) - \frac{L^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)^2 A - \frac{1}{\rho(L^2+\rho^2)^2}\partial_\rho\left[\rho(L^2+\rho^2)^2 A_\rho\right] - i\frac{LR^2}{(L^2+\rho^2)^2}(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)\,\Phi = 0 \qquad (4.3.50)$$

It is worth mentioning that eq. (1) in (4.3.30) contains the operator $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i)$ which acts differently on scalars and spherical vectors. Therefore, when this operator is applied on $\Phi = \varphi + \frac{L}{R^2}(\hat{\gamma}_2 A_3 - \hat{\gamma}_3 A_2)$, in principle one should split it as acting on φ and A_i separately. However, since in the present case $A_i = \nabla_i A$, exploiting the algebra of covariant derivatives and the properties of scalar harmonics in (4.3.49), it is easy to show that

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i\nabla_j A) = \nabla_i\nabla^i\nabla_j A - 2\nabla_j A = -l(l+2)\nabla_j A \qquad (4.3.51)$$

This is exactly the same relation satisfied by the scalar φ, so we are led to $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial^i\Phi) = -l(l+2)\Phi$. This confirms that considering Φ as an elementary scalar fluctuation is a consistent procedure.

Equations (4.3.50) are four equations for three unknowns $X_{III}, Y_{III}, Z_{III}$ and lead to non–trivial solutions only if they are compatible. Indeed it turns out that equation (4) is identically satisfied once the others are. We then concentrate on the first three equations.

We first solve equation (1). By observing that it is identical to the equation for the scalar χ (see eq. (4.3.33)) we immediately obtain

$$X_{III}(\rho) = \rho^l(L^2+\rho^2)^{-n-l-1}F(-(n+l+1), -n; l+2; -\rho^2/L^2) \qquad (4.3.52)$$

where the quantization condition (4.3.38) has been used. As a consequence, the mass spectrum is

$$M_\Phi(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat\gamma_2 m_3 - \hat\gamma_3 m_2}{2}\right)^2} \tag{4.3.53}$$

where $n \geq 0$, $l \geq 1$ and $|m_2+m_3| = |m_2-m_3| = l-2k$.

Equation (2) can be used to express the mode A in terms of Φ and A_ρ. Inserting the expressions (4.3.49) we obtain

$$Z_{III} = \frac{1}{l(l+2)}\left[\frac{1}{\rho}\partial_\rho(\rho^3 Y_{III}) + i\frac{LR^2\rho^2}{(L^2+\rho^2)^2}(\hat\gamma_2 m_3 - \hat\gamma_3 m_2)X_{III}\right] \tag{4.3.54}$$

We then consider equation (3) which exhibits an actual coupling between X_{III} and Y_{III}. In order to solve for Y_{III} given the solution (4.3.52) for X_{III} we set

$$Y_{III}(\varrho) = \varrho^{l-1}(1+\varrho^2)^{-\alpha}\,P(\varrho) \tag{4.3.55}$$

Using the definitions (4.3.35) together with the quantization condition (4.3.38) and defining $y \equiv -\varrho^2$, after some algebra the equation for P reads

$$\begin{aligned} y(1-y)P''(y) + [(l+2)+(2n+l)\,y]\,P'(y) - n(n+l+1)P(y) \\ = \eta\,\frac{(1+y)}{(1-y)^2}F(-(n+l+1),-n;l+2;y) \end{aligned} \tag{4.3.56}$$

where we have defined $\eta \equiv i\frac{R^2}{2L^2}(\hat\gamma_2 m_3 - \hat\gamma_3 m_2)$. This is an inhomogeneous hypergeometric equation whose source is a polynomial of degree n, solution of the corresponding homogeneous equation. The most general solution is then of the form

$$P(y) = c\,F(-(n+l+1),-n;l+2;y) + \bar P(y) \tag{4.3.57}$$

for arbitrary constant c, where $\bar P$ is a particular solution of (4.3.56). Exploiting the general identity

$$\begin{aligned} (1-y)\,F'(-(n+l+2),-n;l+1;y) \;+\; (n+l+2)\,F(-(n+l+2),-n;l+1;y) \\ = \frac{(n+l+1)(n+l+2)}{(l+1)}\,F(-(n+l+1),-n;l+2;y) \end{aligned} \tag{4.3.58}$$

valid for hypergeometric functions with integer coefficients, it is easy to show that a solution is given by

$$\bar P(y) = \eta\,\frac{(l+1)}{(n+l+1)(n+l+2)}\,\frac{F(-(n+l+2),-n;l+1;y)}{1-y} \tag{4.3.59}$$

The general solution of equation (3) is then

$$\begin{aligned} Y_{III}(\rho) &= \rho^{l-1}(L^2+\rho^2)^{-n-l-2}\Big[c\,(L^2+\rho^2)\,F(-(n+l+1),-n;l+2;-\rho^2/L^2) \\ &+ \eta\,\frac{(l+1)}{(n+l+1)(n+l+2)}F(-(n+l+2),-n;l+1;-\rho^2/L^2)\Big] \end{aligned} \tag{4.3.60}$$

This solution is regular at the origin and not divergent for $\rho \to \infty$. Due to the quantization condition (4.3.38) the corresponding mass spectrum is still given by

$$M_{III}(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2} \qquad (4.3.61)$$

with $n \geq 0$, $l \geq 1$ and $|m_2 + m_3| = |m_2 - m_3| = l - 2k$.

Before closing this Section we comment on the particular $l = m_2 = m_3 = 0$ mode. In (4.3.49) this corresponds to turn off $A_i = \nabla_i A$ since $A(\sigma^a)$ is independent of the three–sphere coordinates. Equation (2) reduces to $\partial_\rho(\rho^3 A_\rho) = 0$ which, together with the condition of regularity at $\rho = 0$, sets $A_\rho = 0$. Equations (3) and (4) in (4.3.50) are then automatically satisfied, whereas eq. (1) provides a non–trivial solution for Φ as given in (4.3.52) with mass (4.3.53) where we set $l = m_2 = m_3 = 0$.

As a slightly different attitude we can consider the configuration with all the vector modes turned off ($Y_{III} = Z_{III} = 0$) and study only scalar Φ fluctuations of the form (4.3.49). In this case Φ is still solution of equation (1) but, as follows from the rest of equations, it is constrained by the further condition

$$(\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2)\Phi = 0 \qquad (4.3.62)$$

In general, for non–vanishing and distinct deformation parameters, non–trivial solutions can be found only for $m_2 = m_3 = 0$, i.e. only the $U(1) \times U(1)$ zero–mode sector is selected and the fluctuations are independent of (ϕ_2, ϕ_3). A greater number of solutions, corresponding to the modes $m_2 = m_3$, is instead allowed when $\hat{\gamma}_2 = \hat{\gamma}_3$, therefore in particular for the supersymmetric deformation. In any case, the mass spectrum is given by

$$M_\Phi(n,l) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2)} \qquad n \geq 0 \quad l \text{ (even)} \geq 0 \qquad (4.3.63)$$

and coincides with the undeformed mass.

4.3.3 Analysis of the spectrum

From the previous discussion it follows that the bosonic modes arising from the compactification of the D7–brane on the deformed $\tilde{S}^3$ give rise to a mesonic spectrum which is given by

- 2 scalars and 1 vector in the $(\frac{l}{2}, \frac{l}{2})$ with $l \geq 0$, $|m_2 \pm m_3| = l - 2k$ and mass

$$M_{\chi,\Phi,II}(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2}$$

- 1 scalar in the $(\frac{l}{2}, \frac{l}{2})$ with $l \geq 1$, $|m_2 \pm m_3| = l - 2k$ and mass

$$M_{III}(n,l,m_2,m_3) = \frac{2L}{R^2}\sqrt{(n+l+1)(n+l+2) + \left(\frac{\hat{\gamma}_2 m_3 - \hat{\gamma}_3 m_2}{2}\right)^2}$$

- 1 scalar in the $(\frac{l-1}{2}, \frac{l+1}{2})$ with $l \geq 1$, $|m_2 \pm m_3| = l \mp 1 - 2k$ and mass

$$M_{I,+}(n,l,m_2,m_3) \;=\; \frac{2L}{R^2}\sqrt{(n+l+2)(n+l+3)+\left(\frac{\hat{\gamma}_2 m_3-\hat{\gamma}_3 m_2}{2}\right)^2}$$

- 1 scalar in the $(\frac{l+1}{2}, \frac{l-1}{2})$ with $l \geq 1$, $|m_2 \pm m_3| = l \pm 1 - 2k$ and mass

$$M_{I,-}(n,l,m_2,m_3) \;=\; \frac{2L}{R^2}\sqrt{(n+l)(n+l+1)+\left(\frac{\hat{\gamma}_2 m_3-\hat{\gamma}_3 m_2}{2}\right)^2}$$

for any $n \geq 0$. This matches exactly the bosonic content found in the undeformed case [167]. However, in this case the γ–deformation breaks $SO(4) \to U(1) \times U(1)$ and induces an explicit dependence of the mass spectrum on the the quantum numbers (m_2, m_3) with a pattern similar to the Zeeman effect for atomic electrons where the constant magnetic field which breaks $SU(2)$ rotational invariance down to $U(1)$ induces a dependence of the energy levels on the azimuthal quantum number m [6].

The dependence on the deformation parameters disappears completely in the $m_2 = m_3 = 0$ sector (or for $\hat{\gamma}_2 = \hat{\gamma}_3$ and $m_2 = m_3$) and the mass eigenvalues coincide with the ones of the undeformed theory. When $\hat{\gamma}_2 = \hat{\gamma}_3$ the mass spectrum acquires an extra symmetry under the exchange of the two $U(1)$'s and an extra degeneracy corresponding to $m_2 \to m_2+m$, $m_3 \to m_3+m$, m integer.

For any value of $\hat{\gamma}_i$ there are no tachyonic modes, so confirming the stability of our configuration. Moreover, massless states are absent and the spectrum has a mass gap given by

$$M_{gap} = 2\sqrt{2}\frac{L}{R^2} \tag{4.3.64}$$

This is exactly the mass gap present in the undeformed theory [167].

In order to analyze in detail the mass splitting induced by the deformation and study how the modes organize themselves among the different eigenvalues it is convenient to rewrite the mass of a generic eigenstate X as

$$M_X(n,l,m_2,m_3) = \sqrt{\left(M_X^{(0)}(n,l)\right)^2 + \frac{4L^2}{R^4}\left(\Delta M(m_2,m_3)\right)^2} \tag{4.3.65}$$

where $M_X^{(0)}$ is the undeformed mass, whereas

$$\Delta M(m_2,m_3) \equiv \left(\frac{\hat{\gamma}_2 m_3-\hat{\gamma}_3 m_2}{2}\right) \tag{4.3.66}$$

is the Zeeman–splitting term.

Since for any $l \geq 2$ the following mass degeneracy occurs

$$M^{(0)}_{\chi,\Phi,II}(n,l) = M^{(0)}_{III}(n,l) = M^{(0)}_{I,+}(n,l-1) = M^{(0)}_{I,-}(n,l+1) \tag{4.3.67}$$

[6] A similar effect has been observed in the case of backgrounds with B fields turned on in Minkowski [185, 200].

for $\hat{\gamma}_i = 0$ we have $8(l+1)^2$ bosonic degrees of freedom corresponding to the same mass eigenvalue. For the particular values $l = 0, 1$ the number of states is reduced since for $l = 0$ modes $A_{(I,+)}$ and A_{III} are both absent, whereas for $l = 1$ $A_{(I,+)}$ is still absent. For any value of l they match the bosonic content of massive $\mathcal{N} = 2$ supermultiplets [167].

In the present case mass degeneracy occurs among states which satisfy the above condition and have the same value of $\Delta M(m_2, m_3)$. Therefore, having performed the l–shift for the $(I, \pm)$ modes as in (4.3.67), we concentrate on the degeneracy in $\Delta M(m_2, m_3)$ for fixed values of (n, l). It is convenient to discuss the $\hat{\gamma}_2 = \hat{\gamma}_3$ and $\hat{\gamma}_2 \neq \hat{\gamma}_3$ cases, separately.

$\underline{\hat{\gamma}_2 = \hat{\gamma}_3 \equiv \hat{\gamma}}$: This case includes the supersymmetric LM–theory. The deformation enters the mass spectrum only through the difference $(m_2 - m_3)$ and the splitting term ΔM depends only on a single integer j

$$\begin{array}{llll} l & \text{even} & 2j \equiv |m_2 - m_3| = 0, 2, \cdots, l & \Delta M(j) = \hat{\gamma}\, j \\ l & \text{odd} & 2j+1 \equiv |m_2 - m_3| = 1, 3, \cdots, l & \Delta M(j) = \hat{\gamma}\left(j + \frac{1}{2}\right) \end{array} \qquad (4.3.68)$$

Excluding for the moment the $l = 0, 1$ cases, for any given value of $2j$ and $2j + 1$ the degeneracies of the corresponding mass levels are listed in Table 4.1 and Table 4.2, respectively.

State	$\|m_2 - m_3\| = 2j$	Degeneracy
χ, Φ, A_{III}	0	$l+1$
	$2, 4, \cdots, l$	$2(l+1)$
A_μ	0	$l+1$
	$2, 4, \cdots, l$	$2(l+1)$
$A_{I,+}$	0	$l-1$
	$2, 4, \cdots, l$	$2(l-1)$
$A_{I,-}$	0	$l+3$
	$2, 4, \cdots, l$	$2(l+3)$

Table 4.1: Degeneracy of states in the case $\hat{\gamma}_2 = \hat{\gamma}_3$ and $l \geq 2$ even. The degeneracy in the third column refers to every single value of j.

State	$\|m_2 - m_3\| = 2j+1$	Degeneracy
χ, Φ, A_{III}	$1, 3, \cdots, l$	$2(l+1)$
A_μ	$1, 3, \cdots, l$	$2(l+1)$
$A_{I,+}$	$1, 3, \cdots, l$	$2(l-1)$
$A_{I,-}$	$1, 3, \cdots, l$	$2(l+3)$

Table 4.2: Degeneracy of states in the case $\hat{\gamma}_2 = \hat{\gamma}_3$ and $l \geq 3$ odd.

For any value of $l \geq 2$ we observe Zeeman–like splitting as shown in Fig. 4.1. Precisely, the splitting occurs in the following way: For l even there are $8(l+1)$ d.o.f. corresponding to $j = 0$

and $16(l+1)$ for each $j \neq 0$. Since we have $l/2$ possible values of $j \neq 0$, the total number of states sum up correctly to $8(l+1)^2$. Analogously, for odd values of l the number of levels is $(l+1)/2$, each of them corresponds to $16(l+1)$ d.o.f., so we still have $8(l+1)^2$ modes.

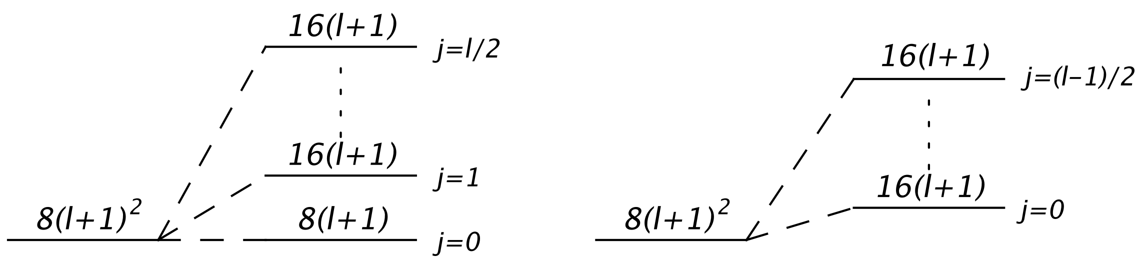

Figure 4.1: The Zeeman–splitting of the undeformed $8(l+1)^2$ d.o.f. for $\hat{\gamma}_2 = \hat{\gamma}_3$ and l even (left) or odd (right).

The $l = 0$ case corresponds to $m_2 = m_3 = 0$ $(j = 0)$. The deformation is then harmless and we are back to the bosonic content of the undeformed theory, that is three scalars χ, Φ, $A_{(I,-)}$ and one vector with $M^{(0)}(n, 0)$. Similarly, for $l = 1$ $(j = 0)$, excluding $A_{(I,+)}$ we have three scalars and one vector in the $(1/2, 1/2)$ of $SO(4)$ and one scalar in the $(3/2, 1/2)$, all corresponding to $M^2 = (M^{(0)}(n, 1))^2 + \hat{\gamma}^2 L^2/R^4$. These cases can be included in Tables 4.1 and 4.2 with the agreement to discharge modes which are not switched on.

We note that there is an accidental mass degeneracy which is remnant of the undeformed $\mathcal{N} = 2$ theory. In particular, in the supersymmetric LM case this allows to organize the bosonic states in $\mathcal{N} = 1$ supermultiplets.

In principle, this unexpected degeneracy could be related to the particular theories we are considering which are smooth deformations of their undeformed counterpart. In order to better understand $\mathcal{N} = 2$ vs. $\mathcal{N} = 1$ supersymmetry at the level of mesonic spectrum, the study of the fermionic sector is a mandatory requirement.

$\hat{\gamma}_2 \neq \hat{\gamma}_3$: The splitting term ΔM now depends on both $m_{2,3}$ and no longer on their difference. In order to make the comparison with the $\hat{\gamma}_2 = \hat{\gamma}_3$ case easier, for fixed l it is convenient to label ΔM by two numbers j and s

$$
\begin{aligned}
&l \quad \text{even} \qquad \Delta M(j,s) = \frac{(j+s)\,\hat{\gamma}_2 + (j-s)\,\hat{\gamma}_3}{2} \\
&l \quad \text{odd} \qquad \Delta M(j,s) = \frac{(j+\frac{1}{2}+s)\,\hat{\gamma}_2 + (j+\frac{1}{2}-s)\,\hat{\gamma}_3}{2}
\end{aligned}
\tag{4.3.69}
$$

where j is still defined as before, whereas s is integer if l is even and half–integer if l is odd. Its range can be read in Tables 4.3 and 4.4.

As appears in the Tables the degeneracy is almost completely broken. In fact, except for the $m_2 = m_3 = 0$ case, only a residual degeneracy 2 survives due to the fact that the mass (4.3.65) is invariant under the exchange $(m_2, m_3) \rightarrow (-m_2, -m_3)$.

State	$\|m_2 - m_3\| = 2j$	s	Degeneracy
χ, Φ, A_{III}	0	0	1
		$1, 2, \cdots, \frac{l}{2}$	2
	$2, 4, \cdots, l$	$-\frac{l}{2}, \cdots, 0, \cdots, \frac{l}{2}$	2
A_μ	0	0	1
		$1, 2, \cdots, \frac{l}{2}$	2
	$2, 4, \cdots, l$	$-\frac{l}{2}, \cdots, 0, \cdots, \frac{l}{2}$	2
$A_{I,+}$	0	0	1
		$1, 2, \cdots, \frac{l-2}{2}$	2
	$2, 4, \cdots, l$	$-\frac{l-2}{2}, \cdots, 0, \cdots, \frac{l-2}{2}$	2
$A_{I,-}$	0	0	1
		$1, 2, \cdots, \frac{l+2}{2}$	2
	$2, 4, \cdots, l$	$-\frac{l+2}{2}, \cdots, 0, \cdots, \frac{l+2}{2}$	2

Table 4.3: Degeneracy of states in the case $\hat{\gamma}_2 \neq \hat{\gamma}_3$ and $l \geq 2$ even. The degeneracy in the fourth column refers to every single pair (j, s).

State	$\|m_2 - m_3\| = 2j + 1$	s	Degeneracy
χ, Φ, A_{III}	$1, 3, \cdots, l$	$-\frac{l}{2}, \cdots, \frac{l}{2}$	2
A_μ	$1, 3, \cdots, l$	$-\frac{l}{2}, \cdots, \frac{l}{2}$	2
$A_{I,+}$	$1, 3, \cdots, l$	$-\frac{l-2}{2}, \cdots, \frac{l-2}{2}$	2
$A_{I,-}$	$1, 3, \cdots, l$	$-\frac{l+2}{2}, \cdots, \frac{l+2}{2}$	2

Table 4.4: Degeneracy of states in the case $\hat{\gamma}_2 \neq \hat{\gamma}_3$ and $l \geq 3$ odd.

To better understand the level splitting it is convenient to compare the present situation with the previous one. In fact, fixing j, the degenerate degrees of freedom of the $\hat{\gamma}_2 = \hat{\gamma}_3$ case further split according to the different values of s. If l is even and $j = 0$, the previous $8(l+1)$ degenerate levels split in $(l/2+2)$ new mass levels, while for $j \neq 0$ the $16(l+1)$ levels open up in $(l+3)$ levels (see Fig. 4.2). If l is odd we find $(l+3)$ different mass levels as drawn in Fig. 4.3.

The particular cases $l = 0, 1$ can be read from Tables 4.3 and 4.4 by discharging $(A_{(I,+)}, A_{III})$ and $A_{(I,+)}$, respectively. For $l = 0$ three modes χ, Φ and A_μ correspond to $\Delta M = 0$ $(j = s = 0)$, whereas the three degrees of freedom of $A_{(I,-)}$ split into one d.o.f. with $\Delta M = 0$ $(j = s = 0)$ and two with $\Delta M = \frac{\hat{\gamma}_2 - \hat{\gamma}_3}{2}$ $(j = s = 1)$. Already in the simplest $l = 0$ case the $SO(4)$ breaking is manifest. For $l = 1$ $(j = 0)$ the four degrees of freedom of each mode χ, Φ, A_{III} and A_μ now split into two states with $\Delta M = \hat{\gamma}_2/2$ and two states with $\Delta M = \hat{\gamma}_3/2$. On the other hand, the 8 d.o.f. corresponding to $A_{(I,-)}$ split into two states with $\Delta M = \hat{\gamma}_2/2$, two states with $\Delta M = \hat{\gamma}_3/2$, two states with $\Delta M = (2\hat{\gamma}_2 - \hat{\gamma}_3)/2$ and two with $\Delta M = (2\hat{\gamma}_3 - \hat{\gamma}_2)/2$.

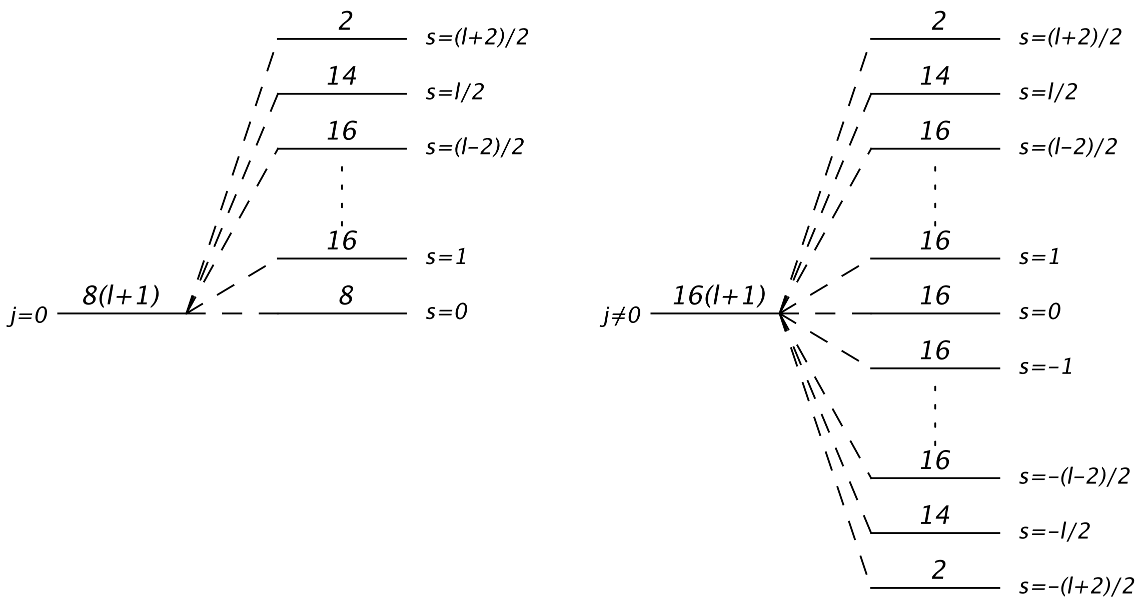

Figure 4.2: The Zeeman–splitting of the $\hat{\gamma}_2 = \hat{\gamma}_3 = \hat{\gamma}$ d.o.f. for $\hat{\gamma}_2 \neq \hat{\gamma}_3$ and l even. The value of ΔM here appearing is pictured considering the case $\hat{\gamma}_3 < \hat{\gamma} < \hat{\gamma}_2$.

As discussed in [167] the undeformed spectrum exhibits a huge degeneracy in $\nu \equiv n + l$ which can be traced back to a (non–exact) SO(5) symmetry. This originates from the fact that the induced metric on the D7–brane is conformally equivalent to $E^{(1,3)} \times \mathrm{S}^4$. If in the quadratic action for the fluctuations the conformal factor can be re–absorbed by a field redefinition the corresponding equations of motion are invariant under S^4 diffeomorphisms. Therefore, solutions can be found by expanding in spherical harmonics of S^4 and the mass spectrum of the elementary modes depends only on the $SO(5)$ quantum number ν. This happens for instance for scalar modes and vectors which, for a given ν, organize themselves into reducible representations $(0,0) \oplus (1/2,1/2) \cdots \oplus (\nu/2,\nu/2)$ of $SO(4)$. This is indeed the decomposition of the highest weight representation $[\nu, 0]$ of $SO(5)$ in $SO(4)$ representations.

In principle, the same analysis can be applied also to our case. Here the induced metric (4.2.20) is conformally equivalent to $E^{(1,3)} \times \tilde{\mathrm{S}}^4$ where $\tilde{\mathrm{S}}^4$ is the deformed four–sphere (set $\varrho = \rho/L$)

$$ds^2_{\tilde{\mathrm{S}}^4} = \frac{R^4}{4L^2}\frac{4}{(1+\varrho^2)^2}(d\varrho^2 + \varrho^2 d\tilde{\Omega}_3^2) \tag{4.3.70}$$

and

$$d\tilde{\Omega}_3^2 = d\theta^2 + G\left[c_\theta^2 d\phi_2^2 + s_\theta^2 d\phi_3^2 + \frac{\varrho^2 c_\theta^2 s_\theta^2 (\hat{\gamma}_2 d\phi_2 + \hat{\gamma}_3 d\phi_3)^2}{(1+\varrho^2)^2}\right] \tag{4.3.71}$$

is the deformed three–sphere.

It follows that a dependence on the $SO(5)$ quantum number $\nu = n + l$ still appears if the conformal factor $(1+\varrho^2)L^2/R^2$ can be compensated by a field redefinition and the action can be

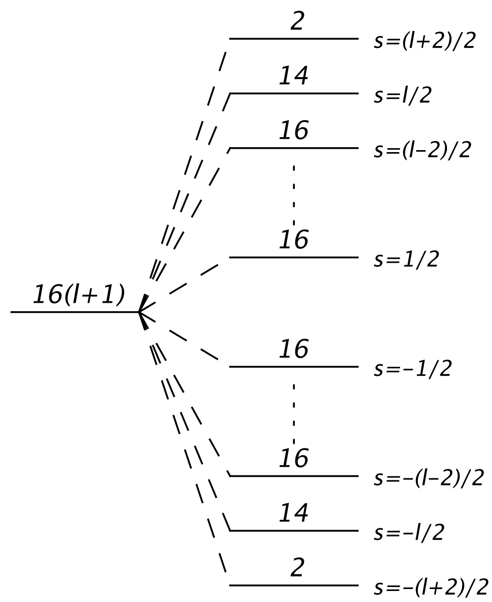

Figure 4.3: The Zeeman–splitting of the $\hat{\gamma}_2 = \hat{\gamma}_3$ d.o.f. for $\hat{\gamma}_2 \neq \hat{\gamma}_3$ and l odd. Once again $\hat{\gamma}_3 < \hat{\gamma} < \hat{\gamma}_2$.

entirely expressed in terms of the metric of $E^{(1,3)} \times \mathrm{S}^4$ plus deformations. A close look at the action (4.3.13) reveals that this is always the case for the decoupled modes χ, A_μ and also for Φ. Despite of the presence of the deformation terms which break explicitly the $SO(5)$ invariance, we can still search for solutions expanded in spherical harmonics on S^4 and, consequently, the mass spectrum exhibits a dependence on n and l only in the combination $n+l$. In particular, in the zero–mode sector $m_2 = m_3 = 0$ a degeneracy appears which is remnant of the $SO(5)$ invariance. Of course, the eigenstates corresponding to degenerate eigenvalues never reconstruct the complete $[\nu, 0]$ representation of $SO(5)$, being organized into a direct product of $SO(4)$ representations with integer spins only $(0,0) \oplus (1,1) \cdots ([\nu/2], [\nu/2])$, since $m_2 = m_3 = 0$ only occurs for even values of l.

4.4 The dual field theory

In this Section we construct the 4D conformal field theory whose composite operators are dual to the mesonic states just found.

As already discussed in Section 3, in the supergravity description the operations of TsT deforming the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background and adding D7–branes commute. Since on the field theory side TsT deformations correspond to promoting all the products among the fields to be $*$–products [66], whereas the addition of D7–branes corresponds to adding interacting fundamental matter [164] we expect that in determining the action for the dual field theory the operations of $*$–product deformation and addition of fundamental matter commute. Therefore, in order to obtain the dual action we proceed by promoting to $*$–products all the products in the $\mathcal{N}=2$ SYM action with fundamental matter corresponding to the undeformed Karch–Katz model.

Given N_f probe D7–branes embedded in the ordinary $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with N units of flux, $N \gg N_f$, in the large N limit the dual field theory on the D3–branes consists of $\mathcal{N}=4$ $SU(N)$ SYM coupled in a $\mathcal{N}=2$ fashion to N_f $\mathcal{N}=2$ hypermultiplets which contain new dynamical fields arising from open strings stretching between D3 and D7–branes. In $\mathcal{N}=1$ superspace language the $\mathcal{N}=4$ gauge multiplet is given in terms of one $\mathcal{N}=1$ gauge superfield W_α and three chirals Φ_1, Φ_2, Φ_3 all in the adjoint representation of $SU(N)$. The $\mathcal{N}=2$ hypermultiplets are described by N_f chiral superfields Q^r transforming in the $(N, \bar{N}_f)$ of $SU(N) \times SU(N_f)$ plus N_f chirals $\tilde{Q}_r$ transforming in the $(\bar{N}, N_f)$.

According to the AdS/CFT duality the lowest components of the three chirals Φ_i are in one–to–one correspondence with the three complex coordinates of the internal 6D space as (we use notations consistent with Section 2)

$$\begin{aligned} X^1 + iX^2 &\equiv u\rho_3 e^{i\phi_3} \to \Phi_3|_{\theta=\bar{\theta}=0} \\ X^3 + iX^4 &\equiv u\rho_2 e^{i\phi_2} \to \Phi_2|_{\theta=\bar{\theta}=0} \\ X^5 + iX^6 &\equiv u\rho_1 e^{i\phi_1} \to \Phi_1|_{\theta=\bar{\theta}=0} \end{aligned} \tag{4.4.1}$$

For a configuration of D7–branes placed at distance $X^5 + iX^6 = L$ from the D3–branes the Lagrangian of the corresponding gauge theory is [164]

$$\begin{aligned} \mathcal{L} &= \int d^4\theta \left[\mathrm{Tr}\left(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i\right) + \mathrm{tr}\left(\bar{Q}e^{gV}Q + \tilde{Q}e^{-gV}\bar{\tilde{Q}}\right)\right] + \frac{1}{2g^2}\int d^2\theta \, \mathrm{Tr}\left(W^\alpha W_\alpha\right) \\ &+ i\int d^2\theta \left[g\,\mathrm{Tr}\left(\Phi^1\left[\Phi^2, \Phi^3\right]\right) + g\,\mathrm{tr}\left(\tilde{Q}\Phi^1 Q\right) + m\,\mathrm{tr}\left(\tilde{Q}Q\right)\right] + h.c. \end{aligned} \tag{4.4.2}$$

where the trace Tr is over color indices and tr is over the flavor ones. This action is $\mathcal{N}=2$ supersymmetric with (W_α, Φ_1) realizing a $\mathcal{N}=2$ vector multiplet and (Φ_2, Φ_3) an adjoint matter hypermultiplet. The coupling of Φ_1 with massive matter fields leads to a non–trivial vev $\langle\Phi_1\rangle = -m/g$ which gives the displacement between the D3 and the D7–branes according to the identification $L \equiv -m/g$.

The theory has a $SU(2)_\Phi \times SU(2)_R$ invariance corresponding to a symmetry which exchanges (Φ_2, Φ_3) and to the $\mathcal{N}=2$ R–symmetry, respectively. In addition, for $m=0$, there is a $U(1)$ R–symmetry under which $(Q^r, \tilde{Q}_r)$ and (Φ_2, Φ_3) are neutral, whereas Φ_1 has charge 2 and W_α has charge 1 [201, 183]. In the dual supergravity description these symmetries originate from the $SO(4) \times SO(2)$ invariance which survives after the insertion of the D7–branes [164] and which are related to rotations in the (X^1, X^2, X^3, X^4) and (X^5, X^6) planes, respectively. Fixing $X^5 + iX^6 = L \neq 0$ breaks rotational invariance in the (X^5, X^6) plane and, correspondingly, the mass term breaks the $U(1)$ R–symmetry in the dual gauge theory. Finally, the theory also possesses a $U(1)$ baryonic symmetry under which only $(Q^r, \tilde{Q}_r)$ are charged $(1, -1)$. This is a residual of the original $U(N_f)$ invariance.

For $m=0$ and in the large N limit with N_f fixed the theory is superconformal invariant. In fact, the beta–function for the 't Hooft coupling $\lambda = g^2 N$ is proportional to $\lambda^2 N_f/N$ and vanishes for $N_f/N \to 0$.

Since we are interested in non–supersymmetric deformations of this theory we need the Lagrangian (4.4.2) expanded in components. Given the physical components of the multiplets being

$$\begin{aligned} \Phi^i &= \left(a^i, \psi^i_\alpha\right) & Q^r &= (q^r, \chi^r_\alpha) \\ W_\alpha &= \left(\lambda_\alpha, f_{\alpha\beta}\right) & \tilde{Q}_r &= (\tilde{q}_r, \tilde{\chi}_{r\alpha}) \end{aligned} \tag{4.4.3}$$

after eliminating the auxiliary fields through their algebraic equations of motion, the Lagrangian (4.4.2) takes the form

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{int} \tag{4.4.4}$$

where [7]

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=4} &= \mathrm{Tr}\left(-\frac{1}{2}f^{\alpha\beta}f_{\alpha\beta} + i\lambda\left[\nabla,\bar{\lambda}\right] + \bar{a}_i\Box a^i + i\psi^i\left[\nabla,\bar{\psi}_i\right]\right) \\ &\quad + g^2\,\mathrm{Tr}\left(-\frac{1}{4}\left[a^i,\bar{a}_i\right]\left[a^j,\bar{a}_j\right] + \frac{1}{2}\left[a^i,a^j\right]\left[\bar{a}_i,\bar{a}_j\right]\right) \\ &\quad + \left\{ ig\,\mathrm{Tr}\left(\left[\bar{\psi}_i,\bar{\lambda}\right]a^i + \frac{1}{2}\epsilon_{ijk}\left[\psi^i,\psi^j\right]a^k\right) \quad + h.c.\right\} \end{aligned} \tag{4.4.5}$$

is the ordinary $\mathcal{N}=4$ Lagrangian,

$$\begin{aligned} \mathcal{L}_b &= \mathrm{tr}\Big(\bar{q}\left(\Box - |m|^2\right)q + \tilde{q}\left(\Box - |m|^2\right)\bar{\tilde{q}}\Big) \\ &\quad - \frac{g^2}{4}\,\mathrm{tr}\left(\bar{q}\,q\,\bar{q}\,q + \tilde{q}\,\bar{\tilde{q}}\,\tilde{q}\,\bar{\tilde{q}} - 2\bar{q}\,\bar{\tilde{q}}\,\tilde{q}\,q + 4\tilde{q}\,\bar{\tilde{q}}\,\bar{q}\,q\right) + \frac{g^2}{2}\,\mathrm{tr}\left(\tilde{q}\left[a^i,\bar{a}_i\right]\bar{\tilde{q}} - \bar{q}\left[a^i,\bar{a}_i\right]q\right) \\ &\quad - \left\{\mathrm{tr}\left(g\bar{m}(\bar{q}a_1q + \tilde{q}a_1\bar{\tilde{q}}) + \frac{g^2}{2}\left(\bar{q}\bar{a}_1a^1q + \tilde{q}a^1\bar{a}_1\bar{\tilde{q}} + 2\tilde{q}\left[\bar{a}_2,\bar{a}_3\right]q\right)\right) \quad + h.c.\right\} \end{aligned} \tag{4.4.6}$$

describes the bosonic fundamental sector and its interactions with bosonic matter in the adjoint,

$$\mathcal{L}_f = i\,\mathrm{tr}\left(\bar{\chi}\overrightarrow{\nabla}\chi - \tilde{\chi}\overleftarrow{\nabla}\bar{\tilde{\chi}}\right) + \left\{im\,\mathrm{tr}\left(\tilde{\chi}\chi\right) \quad + h.c.\right\} \tag{4.4.7}$$

describes the free fermionic fundamental sector and

$$\mathcal{L}_{int} = ig\,\mathrm{tr}\left(\bar{\chi}\bar{\lambda}q - \tilde{q}\bar{\lambda}\bar{\tilde{\chi}} + \tilde{q}\psi^1\chi + \tilde{\chi}\psi^1q + \tilde{\chi}a^1\chi\right) \quad + h.c. \tag{4.4.8}$$

contains the interaction terms between bosons and fermions.

The most general non–supersymmetric marginal deformation of this theory can be obtained by promoting all the products among the fields in the Lagrangian to be $*$–products according to the following prescription [140, 92]

$$f\,g \;\longrightarrow\; f * g = e^{i\pi Q^f_i Q^g_j \epsilon_{ijk}\gamma_k}\,f\,g \tag{4.4.9}$$

[7] We use superspace conventions of [103]. When $\psi\lambda$ indicates the product of two chiral fermions it has to be understood as $\psi^\alpha\lambda_\alpha$. The same convention is used for antichiral fermions.

where γ_k are the deformation parameters, whereas (Q_1, Q_2, Q_3) are the charges of the fields under the three $U(1)$ global symmetries of the original $\mathcal{N}=4$ theory associated to the Cartan generators of $SU(4)$. On the dual supergravity side they correspond to angular shifts in (4.4.1). Accordingly, the charges of the chiral Φ_i superfields are chosen as in Table 4.5 [140, 92] with the additional requirement for the charges of the spinorial superspace coordinates to be $(1/2, 1/2, 1/2)$. This insures invariance of the superpotential term $\int d^2\theta \mathrm{Tr}(\Phi^1[\Phi^2, \Phi^3])$ under the three $U(1)$'s. The charges for the matter chiral superfields are determined by requiring the superpotential term $\int d^2\theta \mathrm{tr}(\tilde{Q}\Phi^1 Q)$ to respect the three global symmetries in addition to the condition for Q and $\tilde{Q}$ to have the same charges.

	Φ^1	Φ^2	Φ^3	Q	$\tilde{Q}$
Q_1	1	0	0	0	0
Q_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$
Q_3	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$

Table 4.5: $U(1)$ charges of the chiral superfields. The corresponding antichirals have opposite charges.

The gauge superfield W_α and the gaugino have charges $(1/2, 1/2, 1/2)$, whereas the gauge field strength $f_{\alpha\beta}$ is neutral under the three $U(1)$'s.

In the absence of mass term in (4.4.2) the corresponding currents $(J_{\phi_1}, J_{\phi_2}, J_{\phi_3})$ are conserved, whereas J_{ϕ_1} fails to be conserved when $m \neq 0$. Moreover, (J_{ϕ_2}, J_{ϕ_3}) are ABJ–anomaly free also in the presence of fundamental matter, whereas J_{ϕ_1} is non–anomalous only in the quenching limit $N_f/N \to 0$.

As is well–known, the ordinary Lunin–Maldacena $U(1) \times U(1)$ charges [66] are associated to (φ_1, φ_2) angular shifts after performing the change of variables (in our notations)

$$\varphi_1 = \frac{1}{3}(\phi_1 + \phi_2 - 2\phi_3), \quad \varphi_2 = \frac{1}{3}(\phi_2 + \phi_3 - 2\phi_1), \quad \varphi_3 = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3), \tag{4.4.10}$$

Expressing the $(J_{\varphi_1}, J_{\varphi_2})$ generators in terms of $(J_{\phi_1}, J_{\phi_2}, J_{\phi_3})$ we easily find that the Lunin–Maldacena charges are given by

$$Q_1^{(LM)} = Q_2 - Q_3 \qquad , \qquad Q_2^{(LM)} = Q_2 - Q_1 \tag{4.4.11}$$

In the case of supersymmetric deformations the third linear combination $Q_R \sim (Q_1 + Q_2 + Q_3)$ provides the R–symmetry charge.

We are now ready to derive the deformed action by using the prescription (4.4.9) in the original undeformed one.

We begin with the one–parameter deformation, $\gamma_1 = \gamma_2 = \gamma_3$. In this case $\mathcal{N}=1$ supersymmetry survives and we can work directly with the superspace action (4.4.2). Since only for $m = 0$ the $*$–product is well–defined being the three U(1) charges conserved, the correct way to proceed is to deform the massless theory and then add the mass operator as a perturbation. Following this prescription and taking into account the superfields charges given in Table 4.5, the Lagrangian of the deformed theory is

$$\mathcal{L} = \int d^4\theta \left[\mathrm{Tr}\left(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i\right) + \mathrm{tr}\left(\bar{Q}e^{gV}Q + \tilde{Q}e^{-gV}\bar{\tilde{Q}}\right)\right] + \frac{1}{2g^2}\int d^2\theta\,\mathrm{Tr}\,(W^\alpha W_\alpha)$$
$$+ \; ig\int d^2\theta\,\left[\mathrm{Tr}\left(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2\right) + \,\mathrm{tr}\left(\tilde{Q}\Phi_1 Q\right) + m\,\mathrm{tr}\left(\tilde{Q}Q\right)\right] \tag{4.4.12}$$

We note that a non–trivial deformation appears in the superpotential only in the pure adjoint sector. The interaction and the mass terms involving flavor matter do not change, so that the vev for Φ_1 which is related to the D7–brane location through the dictionary (4.4.1) is the same as in the undeformed theory, $\langle\Phi_1\rangle = -m/g \equiv L$. Since in the supergravity description we have chosen L to be real ($X^5 = L$, $X^6 = 0$) here and in what follows we restrict to real values of m.

As already stressed, for $m \neq 0$ the Q_1 charge is not conserved, neither is $Q_2^{(LM)}$. Therefore, this deformed theory possesses only one U(1) non–R–symmetry corresponding to $Q_1^{(LM)}$.

The action (4.4.12) has been obtained by $*$–product deforming the $\mathcal{N} = 2$ SYM action (4.4.2). However, it could have been equivalently obtained by adding fundamental chiral matter to the $\mathcal{N} = 1$ β–deformed SYM theory of [66]. In particular, the appearance of the gauge coupling constant in front of the adjoint chiral superpotential insures that for $m = 0$ and in the probe approximation the theory is superconformal invariant [74].

We now consider the more general non–supersymmetric case. We implement the $*$–product (4.4.9) in the action (4.4.4). Using the deformed commutator [140, 92]

$$[X_i, X_j]_{M_{ij}} \equiv e^{i\pi M_{ij}} X_i X_j - e^{-i\pi M_{ij}} X_j X_i \tag{4.4.13}$$

where for X_i fermions

$$M_{\text{fermions}} \equiv B = \begin{pmatrix} 0 & \frac{1}{2}(\gamma_1+\gamma_2) & -\frac{1}{2}(\gamma_1+\gamma_3) & -\frac{1}{2}(\gamma_2-\gamma_3) \\ -\frac{1}{2}(\gamma_1+\gamma_2) & 0 & \frac{1}{2}(\gamma_2+\gamma_3) & -\frac{1}{2}(\gamma_3-\gamma_1) \\ \frac{1}{2}(\gamma_3+\gamma_1) & -\frac{1}{2}(\gamma_2+\gamma_3) & 0 & -\frac{1}{2}(\gamma_1-\gamma_2) \\ \frac{1}{2}(\gamma_2-\gamma_3) & \frac{1}{2}(\gamma_3-\gamma_1) & \frac{1}{2}(\gamma_1-\gamma_2) & 0 \end{pmatrix} \tag{4.4.14}$$

whereas for scalars

$$M_{\text{scalars}} \equiv C = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix} \tag{4.4.15}$$

the deformed $\mathcal{L}_{\mathcal{N}=4}$ takes the form

$$\mathcal{L}_{\mathcal{N}=4} = \mathrm{Tr}\left(-\frac{1}{2}f^{\alpha\beta}f_{\alpha\beta} + i\lambda\left[\nabla, \bar{\lambda}\right] + \bar{a}_i\Box a^i + i\psi^i\left[\nabla, \bar{\psi}_i\right]\right)$$
$$+ \; g^2\,\mathrm{Tr}\left(-\frac{1}{4}\left[a^i, \bar{a}_i\right]\left[a^j, \bar{a}_j\right] + \frac{1}{2}\left[a^i, a^j\right]_{C_{ij}}\left[\bar{a}_i, \bar{a}_j\right]_{C_{ij}}\right)$$
$$+ \; \left\{ig\,\mathrm{Tr}\left(\left[\bar{\psi}_i, \bar{\lambda}\right]_{B_{i4}} a^i + \frac{1}{2}\epsilon_{ijk}\left[\psi^i, \psi^j\right]_{B_{ij}} a^k\right) \; + h.c.\right\} \tag{4.4.16}$$

while the bosonic sector reads

$$
\begin{aligned}
\mathcal{L}_b &= \operatorname{tr}\Big(\bar{q}\,(\Box - m^2)\,q + \tilde{q}\,(\Box - m^2)\,\bar{\tilde{q}}\Big) - \frac{g^2}{4}\operatorname{tr}\Big(\bar{q}\,q\,\bar{q}\,q + \tilde{q}\,\bar{\tilde{q}}\,\tilde{q}\,\bar{\tilde{q}} - 2\bar{q}\,\bar{\tilde{q}}\,\tilde{q}\,q + 4\tilde{q}\,\bar{\tilde{q}}\,\bar{q}\,q\Big) \\
&+ \frac{g^2}{2}\operatorname{tr}\left(\tilde{q}\left[a^i,\bar{a}_i\right]\bar{\tilde{q}} - \bar{q}\left[a^i,\bar{a}_i\right]q + \bar{q}\bar{a}_1 a^1 q + \tilde{q}a^1\bar{a}_1\bar{\tilde{q}}\right) \\
&+ \Big\{ g^2\operatorname{tr}\left(\tilde{q}\left[\bar{a}_2,\bar{a}_3\right]_{C_{23}}q\right) - gm\operatorname{tr}\left(e^{-i\pi(\gamma_2-\gamma_3)}\bar{q}a^1 q + e^{i\pi(\gamma_2-\gamma_3)}\tilde{q}a^1\bar{\tilde{q}}\right) + h.c.\Big\}
\end{aligned}
\tag{4.4.17}
$$

and the fermionic one

$$
\mathcal{L}_f = i\operatorname{tr}\left(\bar{\chi}\overrightarrow{\nabla}\chi - \tilde{\chi}\overleftarrow{\nabla}\bar{\tilde{\chi}}\right) + \Big\{ im\operatorname{tr}\Big(\tilde{\chi}\chi\Big) + h.c.\Big\} \tag{4.4.18}
$$

Finally the boson–fermion interaction terms become

$$
\begin{aligned}
\mathcal{L}_{int} &= ig\operatorname{tr}\Big(e^{i\frac{\pi}{4}(\gamma_2-\gamma_3)}\bar{\chi}\bar{\lambda}q - e^{-i\frac{\pi}{4}(\gamma_2-\gamma_3)}\tilde{q}\bar{\lambda}\bar{\tilde{\chi}} \\
&\qquad + e^{i\frac{\pi}{4}(\gamma_2-\gamma_3)}\tilde{q}\psi^1\chi + e^{-i\frac{\pi}{4}(\gamma_2-\gamma_3)}\bar{\chi}\psi^1 q + \tilde{\chi}a^1\chi\Big) + h.c.
\end{aligned}
\tag{4.4.19}
$$

We observe that the fundamental fields q and $\tilde{q}$ experiment the γ_1–deformation only through the modified commutator $[\bar{a}_2,\bar{a}_3]_{C_{23}}$ in $\mathcal{L}_b$. Moreover, γ_2 and γ_3 are always present in the combination $(\gamma_2-\gamma_3)$ so that the corresponding phases disappear when $\gamma_2=\gamma_3$, in particular for supersymmetric deformations.

4.5 Summary

In this chapter we have studied the embedding of D7–branes in LM–Frolov backgrounds with the aim of finding the mesonic spectrum of the dual Yang–Mills theory with flavors, according to the gauge/gravity correspondence. Since these theories have $\mathcal{N}=1$ or no supersymmetry depending on the choice of the deformation parameters $\hat{\gamma}_i$, they provide an interesting playground in the study of generalizations of the AdS/CFT correspondence to more realistic models with less supersymmetry.

These geometries are smoothly related to the standard $\mathrm{AdS}_5\times\mathrm{S}^5$ from which they can be obtained by operating with TsT transformations. Therefore, if we consider D7–brane embeddings which closely mimic the ones of the undeformed case [164] we expect the flavor probes to share some properties with the probes of the undeformed case. Driven by this observation we have considered a spacetime filling D7–brane wrapped on a deformed three–sphere in the internal coordinates. We have found that for both the supersymmetric and the non–supersymmetric deformations a static configuration exists which is completely independent of the specific values of the deformation parameters $\hat{\gamma}_i$. As a consequence the D7–brane still lies at fixed values of its transverse directions and exhibits no quark condensate [164]. We remark that this shape is exact and stable in the supersymmetric as well as in the non–supersymmetric cases.

Although the shape of the brane does not feel the effects of the deformation, its fluctuations do. In fact, studying the scalar and vector fluctuations we have found that a non–trivial dependence on the $\hat{\gamma}_{2,3}$ parameters appears both in terms which correct the free dynamics of the modes and in terms which couple the $U(1)$ worldvolume gauge field to one of the scalars in the mutual orthogonal directions to the D3–D7 system. All the deformation–dependent contributions arise from the Dirac–Born–Infeld term in the D7–brane action, whereas the Wess–Zumino term does not feel the deformation. The $\hat{\gamma}_1$ parameter, associated to a TsT transformation along the torus inside the D7 worldvolume, never enters the equations of motion.

A smooth limit to the undeformed equations of motion exists for $\hat{\gamma}_i \to 0$. In this limit all the modes decouple and we are back to the undeformed solutions of [167]. The effect of the deformations becomes negligible also in the UV limit ($\rho \to \infty$). This is an expected result since the deformations involve tori in the internal space and in the UV limit the metric of the background reduces to flat four dimensional Minkowski spacetime.

On the other hand, the situation changes once we consider the general deformed equations. In fact, solving analytically these equations for elementary excitations of scalars and vectors we have found that the mass spectrum is still discrete and with a mass gap and the corresponding eigenstates match the one of the undeformed case. However, the mass eigenvalues acquire a non–trivial dependence on $\hat{\gamma}_{2,3}$. These new terms, being proportional to the $U(1) \times U(1)$ quantum numbers (m_2, m_3), induce a level spitting according to a Zeeman–like effect.

We have performed a detailed analysis of the level splitting and of the corresponding degeneracy. The situation turns out to be very different according to $\hat{\gamma}_2$ and $\hat{\gamma}_3$ being equal or not. In fact, for $\hat{\gamma}_2 \neq \hat{\gamma}_3$ the degeneracy is almost completely broken since only a residual degeneracy associated to the invariance of the mass under $(m_2, m_3) \to (-m_2, -m_3)$ survives. In particular, the breaking of $SO(4)$ is manifest. Instead, for $\hat{\gamma}_2 = \hat{\gamma}_3$ the mass levels split but for each value of the mass an accidental degeneracy survives which is remnant of the $\mathcal{N} = 2$ case. While in the supersymmetric case ($\hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}_3$) this allows to arrange mesons in massive $\mathcal{N} = 1$ multiplets according to the fact that our embedding preserves supersymmetry, this higher degree of degeneracy in the bosonic sector of the theory does not have a clear explanation at the moment. In order to make definite statements about the supersymmetry properties of the mesonic spectrum and supersymmetry breaking one should study the fermionic sector. A useful strategy could be the bottom–up approach described in [183]. We leave this interesting open problem for the future.

Our analysis shares some similarities with other cases considered in the literature.

First of all, we have found that a stable embedding of the probe brane can be realized which is static and independent of the deformation parameters. This feature has been already encountered for other brane configurations in deformed backgrounds. An example is given by particular dynamical probe D3–branes (giant gravitons) which have been first well understood in [192]. In fact, there it has been shown that giant gravitons exist and are stable even in the absence of supersymmetry and their dynamics turns out to be completely independent of the deformation parameters, being then equal to the one of the undeformed theory. Moreover, since the giants wrap the same cycle inside the internal deformed space as our D7–brane does, their bosonic fluctuations encode the same dependence on the deformation parameters observed in the mesonic spectrum coming from the D7.

A second similarity emerges with the case of flavors in non–commutative theories investigated in [177]. In fact, the non–trivial coupling between scalar and gauge modes that in our case is induced by the deformation resembles the one which appears in the case of D7–branes embedded in $AdS_5 \times S^5$ with a B field turned on along spacetime directions. This is not surprising since both theories can be obtained performing a TsT transformation of $AdS_5 \times S^5$: If the TsT is performed in AdS one obtains the dual of a non–commutative theory while the LM–Frolov picture is recovered if this transformation deforms the internal S^5.

The field theory dual to the (super)gravity picture we have considered can be obtained by deforming the standard action for $\mathcal{N} = 4$ super Yang–Mills coupled to massive $\mathcal{N} = 2$ hypermultiplets by the $*$–product prescription [66]. In principle, in the supergravity dual description this should correspond to performing a TsT deformation *after* the embedding of the probe brane. However, as we have discussed, adding the flavor brane in the deformed background or deforming the Karch–Katz D3–D7 configuration are commuting operations. Therefore, the prescription we propose on the field theory side is consistent with what we have done on the string theory side. It is important to stress that the choice of the embedding we have made is crucial for the above reasoning.

What we obtain is a deformed gauge field theory with massive fundamental matter parametrized by four real parameters γ_i and m. We can play with them in order to break global $U(1)$ symmetries, conformality and/or supersymmetry in a very controlled way. In fact, in the quenching approximation a non–vanishing mass parameter related to the location of the probe in the dual geometry breaks conformal invariance and one of the $U(1)$ global symmetries of the massless theory. On the other hand, the values of the deformation parameters γ_i determine the degree of supersymmetry of the theory, as already discussed. It is interesting to note that as we found on the gravity side, the three deformation parameters play different roles in the fundamental sector of the theory. In fact, $\gamma_{2,3}$ always appear in the combination $(\gamma_2 - \gamma_3)$, so that if $\gamma_2 = \gamma_3$ this sector gets deformed only by γ_1–dependent phases induced by the interaction with the adjoint matter. In the supersymmetric case this particular behavior is manifest when using superspace formalism since a non–trivial deformation appears only in the adjoint sector, whereas the flavor superpotential remains undeformed.

Let us conclude mentioning some directions in which our work could be extended. We have considered only the non–interacting mesonic sector. Expanding the D7–brane action beyond the second order in α' one can get informations on the interactions among the mesons and understand how the deformation enters the couplings. Moreover, one could extend our analysis to mesons with large spin in Minkowski, similarly to what has been done in the ordinary, undeformed case [167].

Finally it could be very interesting to study in detail the other embeddings proposed in [194] and in particular the one which seems to exhibit chiral symmetry breaking. Moreover, going beyond the quenching approximation has been representing an interesting subject since the recent efforts to study back–reacted models [182].

Appendix A

Color conventions and integrals in momentum space

In this Appendix we give our conventions and a series of useful identities involving the group generators. Moreover, we list the results for loop integrals that we have used along the calculations.

Our basic conventions to deal with color structures are the following. For a general simple Lie algebra we have:

$$[T_a, T_b] = i f_{abc} T^c \tag{A.0.1}$$

where T_a are the generators and f_{abc} the structure constants. The matrices T_a are normalized as

$$Tr\,(T_a T_b) = \delta_{ab} \tag{A.0.2}$$

We specialize to the case of $SU(N)$ Lie algebra whose generators T_a, $a = 1, \cdots, N^2 - 1$ are taken in the fundamental representation, i.e. they are $N \times N$ traceless matrices. The basic relation which allows to deal with products of T_a is the following

$$T^a_{ij} T^a kl = \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) \tag{A.0.3}$$

From this identities, we can easily obtain all the identities used to compute the color structures associated to the Feynmann diagrams relevant for the two point correlation functions. They are

$$f_{acd} f_{bcd} = 2N\delta_{ab} \tag{A.0.4}$$

$$f_{abm} f_{cdm} + f_{cbm} f_{dam} + f_{dbm} f_{acm} = 0. \tag{A.0.5}$$

$$Tr\,(T_a T_b T_c T_d)(T_a T_b T_c T_d) = \frac{1}{N^2}(N^2 - 1)(N^2 + 3) \tag{A.0.6}$$

$$Tr\,(T_a T_b T_c T_d)(T_a T_b T_d T_c) = -\frac{1}{N^2}(N^2 - 1)(N^2 - 3) \tag{A.0.7}$$

$$Tr\,(T_aT_bT_cT_d)(T_dT_cT_bT_a) = \frac{1}{N^2}(N^2-1)(N^4-3N^2+3) \tag{A.0.8}$$

and

$$Tr\,(T_cT_aT_dT_b)f_{cme}f_{dmf} = -(\delta_{ea}\delta_{fb} + \delta_{fa}\delta_{eb}) \tag{A.0.9}$$

$$Tr\,(T_cT_dT_aT_b)f_{cme}f_{dmf} = \delta_{ef}\delta_{ab} + N Tr\,(T_eT_fT_aT_b) \tag{A.0.10}$$

Now we focus on the main integrals used in the text. All the calculations are performed in n dimensions, with $n = 4 - 2\epsilon$ and in momentum space. We give the results as ϵ expansions.

We begin by considering the momentum integrals associated to the one–loop and two–loop diagrams in Fig. 2.1 for the perturbative corrections to the superpotential.

At one loop, after performing D–algebra, the diagram 2.1b) gives the standard triangle contribution [116]. Assigning external momenta p_i ($p_1 + p_2 + p_3 = 0$) we have

$$p_3^2 \int \frac{d^nq}{(2\pi)^n}\frac{1}{q^2(q-p_2)^2(q+p_1)^2} = \frac{1}{(4\pi)^2}\,\Phi^{(1)}\,(x,y) + \mathcal{O}(\epsilon) \tag{A.0.11}$$

where

$$x \equiv \frac{p_1^2}{p_3^2} \quad \text{and} \quad y \equiv \frac{p_2^2}{p_3^2} \tag{A.0.12}$$

The p_3^2 in front of the integral is produced by D–algebra. The function $\Phi^{(1)}(x,y)$ can be represented as a parametric integral

$$\Phi^{(1)}(x,y) = -\int_0^1 \frac{d\xi}{y\,\xi^2 + (1-x-y)\xi + x}\left(\log\frac{y}{x} + 2\log\xi\right) \tag{A.0.13}$$

Since we look for a local contribution to the superpotential we are interested in the result of the integral for external momenta set to zero. A consistent way [117] to take the limit of vanishing external momenta is to set $p_i^2 = m^2$ for any i so having $x, y = 1$ and let the IR cut–off m^2 going to zero at the end of the calculation. In the limit we obtain a finite local result [117]

$$-\int_0^1 d\xi \frac{\log\xi(1-\xi)}{1-\xi(1-\xi)} \tag{A.0.14}$$

At two loops two types of integrals appear. From diagrams 2.1c) and 2.1d) we have integrals of the form

$$\begin{aligned}(p_3^2)^2 \int \frac{d^nq\,d^nr}{(2\pi)^{2n}}\frac{1}{(r+p_1)^2(q+p_1)^2(r-p_2)^2(q-p_2)^2r^2(q-r)^2} &= \\ = \frac{1}{(4\pi)^4}\,\Phi^{(2)}\,(x,y) + \mathcal{O}(\epsilon)\end{aligned} \tag{A.0.15}$$

with x and y as in (A.0.12). The function $\Phi^{(2)}(x,y)$ is defined by [116]

$$\Phi^{(2)}(x,y) = -\frac{1}{2}\int_0^1 \frac{d\xi}{y\,\xi^2 + (1-x-y)\xi + x} \log\xi \left(\log\frac{y}{x} + \log\xi\right)\left(\log\frac{y}{x} + 2\log\xi\right) \qquad \text{(A.0.16)}$$

As in the one–loop case, the limit $x, y \to 1$ gives a finite local contribution to the effective superpotential.

From diagrams 2.1c)–g) this kind of integral also appears

$$p_3^2 \int \frac{d^n q\, d^n r}{(2\pi)^{2n}} \frac{1}{q^2\, r^2 (q-r)^2 (q-p_3)^2 (r-p_3)^2} = \frac{1}{(4\pi)^4}\, 6\zeta(3) + \mathcal{O}(\epsilon) \qquad \text{(A.0.17)}$$

where one of the external momenta has been already set to zero (in this case we can safely set one of the external momenta to zero from the very beginning since we do not introduce fake IR divergences). This is already the local finite contribution we obtain by setting also $p_3^2 = 0$.

When we deal with two-point correlation functions, at tree-level we have ($k = \Delta_0$ is the free scale dimension of the operators involved and p is the external momentum)

$$\begin{aligned} &\int \frac{d^n q_1 \ldots d^n q_{k-1}}{(2\pi)^{n(k-1)}} \frac{1}{q_1^2 (q_2-q_1)^2 (q_3-q_2)^2 \ldots (p-q_{k-1})^2} \\ &\qquad = \frac{1}{\epsilon}\left[\frac{1}{(4\pi)^2}\right]^{k-1} \frac{(-1)^k}{[(k-1)!]^2} (p^2)^{k-2-(k-1)\epsilon} + \mathcal{O}(1) \end{aligned} \qquad \text{(A.0.18)}$$

At two loops we are interested in the four diagrams listed in Fig. 2.3. From the graph 2.3a) we obtain

$$\begin{aligned} &\int \frac{d^n q_3 \ldots d^n q_{k-1}}{(2\pi)^{n(k+1)}} \frac{1}{(q_4-q_3)^2 \ldots (p-q_{k-1})^2} \times \\ &\int \frac{d^n k\, d^n l\, d^n r\, d^n s \quad r^2 (q_3-l)^2}{k^2 l^2 (k-l)^2 (r-k)^2 (r-l)^2 (s-l)^2 (r-s)^2 (q_3-r)^2 (q_3-s)^2} \\ &\quad = \frac{1}{\epsilon}\left[\frac{1}{(4\pi)^2}\right]^{k+1} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k+1)} [6\zeta(3) - 20\zeta(5)] (p^2)^{k-2-(k+1)\epsilon} + \mathcal{O}(1) \end{aligned} \qquad \text{(A.0.19)}$$

The momentum integral for the graph 2.3b) gives

$$\begin{aligned} &\int \frac{d^n q_3 \ldots d^n q_{k-1}}{(2\pi)^{n(k+1)}} \frac{-q_3^2}{(q_4-q_3)^2 \ldots (p-q_{k-1})^2} \times \\ &\int \frac{d^n k\, d^n l\, d^n r\, d^n s}{k^2 l^2 (k-l)^2 (r-k)^2 (s-l)^2 (r-s)^2 (q_3-r)^2 (q_3-s)^2} \\ &\quad = \frac{1}{\epsilon}\left[\frac{1}{(4\pi)^2}\right]^{k+1} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k+1)} 40\zeta(5) (p^2)^{k-2-(k+1)\epsilon} + \mathcal{O}(1) \end{aligned} \qquad \text{(A.0.20)}$$

Finally, the graphs 2.3c) and 2.3d) lead to the same contribution

$$\int \frac{d^n r\, d^n q_2 \dots d^n q_{k-1}}{(2\pi)^{n(k+1)}} \frac{1}{(q_2-r)^2(q_3-q_2)^2 \dots (p-q_{k-1})^2} \times$$
$$\int \frac{d^n k\, d^n l}{k^2 l^2 (k-l)^2 (r-k)^2 (r-l)^2} \tag{A.0.21}$$
$$= \frac{1}{\epsilon}\left[\frac{1}{(4\pi)^2}\right]^{k+1} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k+1)} 6\zeta(3)(p^2)^{k-2-(k+1)\epsilon} + \mathcal{O}(1)$$

Appendix B

Ladder color and ∇–algebra

In this Appendix we give some ingredients useful to better understand the computations performed in Section 2.4.1 and Section 2.4.2. We first report the full expression for the color of the four loop diagram depicted in Fig. 2.7:

$$\begin{aligned} K_4 &= \frac{1}{2}\Big[(|h_1|^2+|h_2|^2)^4+(|h_1|^2-|h_2|^2)^4\Big]+ \\ &+ \frac{4}{N^2}\Big[|h_3|^8-4|h_3|^6(|h_1|^2+|h_2|^2)+2|h_3|^4(3|h_1|^4+4|h_1|^2|h_2|^2+3|h_2|^4)+ \\ &- 2|h_3|^2(3|h_1|^6+5|h_1|^4|h_2|^2+5|h_1|^2|h_2|^4+3|h_2|^6)+ \\ &+ (|h_1|^8+8|h_1|^6|h_2|^2+6|h_1|^4|h_2|^4+8|h_1|^2|h_2|^6+|h_2|^8)\Big]+ \\ &- \frac{4}{N^4}\Big[5|h_3|^8-20|h_3|^6(|h_1|^2+|h_2|^2)+12|h_3|^4(|h_1|^4+|h_1|^2|h_2|^2+|h_2|^4)+ \\ &- 8|h_3|^2(|h_1|^6-|h_1|^4|h_2|^2-|h_1|^2|h_2|^4+|h_2|^6)\Big]+ \\ &- \frac{4}{N^6}\Big[10|h_3|^8+32|h_3|^6(|h_1|^2+|h_2|^2)-8|h_3|^4|h_1|^2|h_2|^2\Big]+\frac{256}{N^8}|h_3|^8 \end{aligned} \tag{B.0.1}$$

From this formula one can easily obtain the explicit value of the f function in (2.4.24):

$$\begin{aligned} f &= 8\Big[a_1^4+8a_1^3b_1+6a_1^2b_1^2+8a_1b_1^3+b_1^4-2(a_1+b_1)(3a_1^2+2a_1b_1+3b_1^2)c1+ \\ &+ 2(3a_1^2+4a_1b_1+3b_1^2)c_1^2-4(a_1+b_1)c_1^3+c_1^4\Big]+\frac{8}{N^2}\Big[8(a_1-b_1)^2(a_1+b_1)c_1+ \\ &- 12(a_1^2+a_1b_1+b_1^2)c_1^2+20(a_1+b_1)c_1^3-5c_1^4\Big]+\frac{8}{N^4}\Big[8a_1b_1c_1^2-32(a_1+b_1)c_1^3-10c_1^4\Big]+ \\ &+ \frac{512}{N^6}c_1^4 \end{aligned} \tag{B.0.2}$$

Here we give a detailed explanation of ∇–algebra operations used in Section 2.4.2 for the calculation of the gauge beta function. In what follows we denote

$$// \equiv \frac{\frac{1}{2}\nabla^a \nabla_a}{\Box} \;\rightarrow\; 1 \,-\, \frac{1}{2}\,\frac{\Gamma^a \Gamma_a}{\Box} \qquad\qquad \Box \equiv \frac{1}{2}\,\partial^a\,\partial_a \qquad\qquad \blacktriangleright \equiv \nabla_a = \partial_a - i\Gamma_a \tag{B.0.3}$$

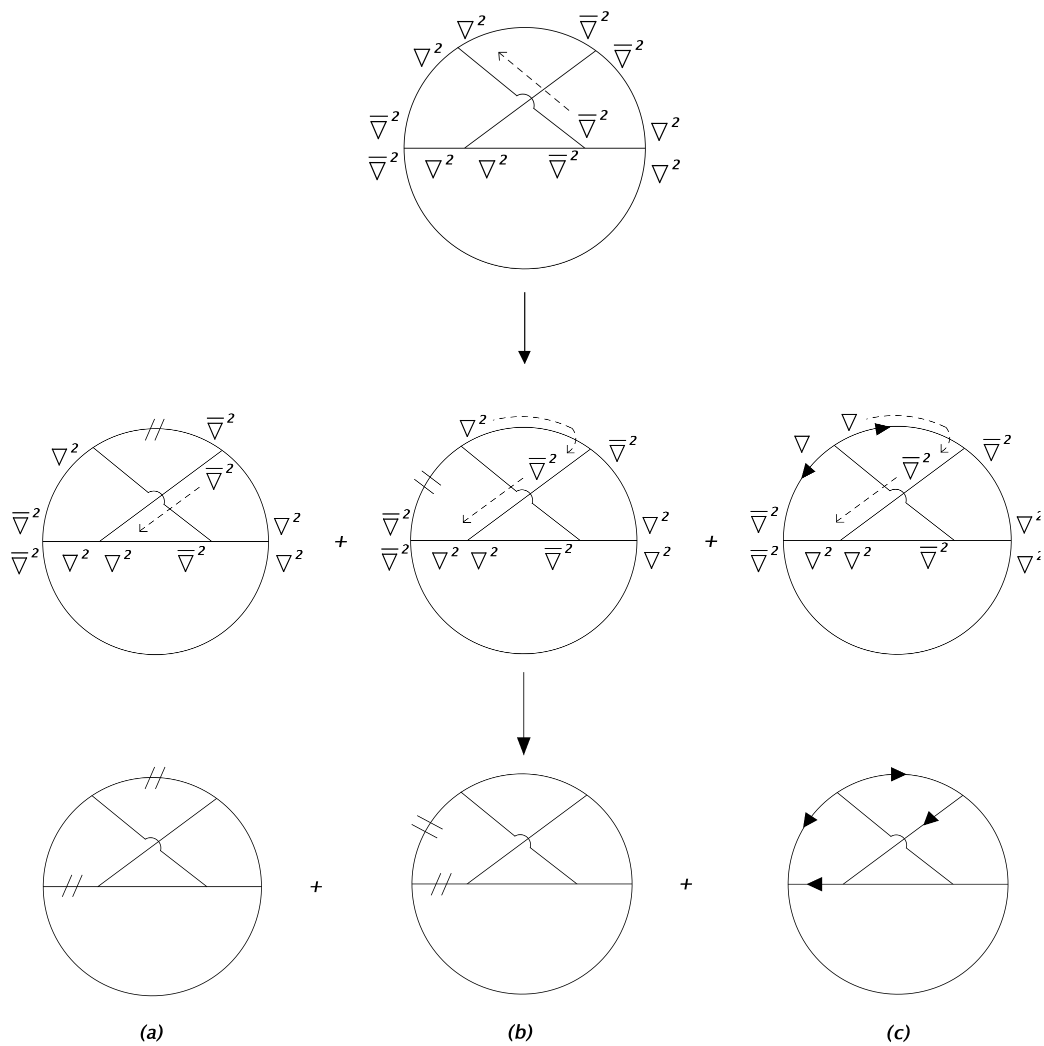

Figure B.1: ∇–algebra operations on four–loop vacuum diagram.

In particular, let us consider the contributions depicted in Fig. B.1. Starting from the top vacuum diagram and performing integration by parts we end up with three different graphs. Each of them gives rise to a single bosonic diagram: Fig. B.1 (a), (b), (c). Now we can expand the covariant propagators to extract tadpole–type contributions. It is easy to see that (a) and (b) diagrams in B.1 are equivalent and give rise to the tadpole graphs shown in Fig. B.2. Analogously the B.1(c) diagram can be expanded to give the relevant tadpole contributions as indicated in Fig. B.3.

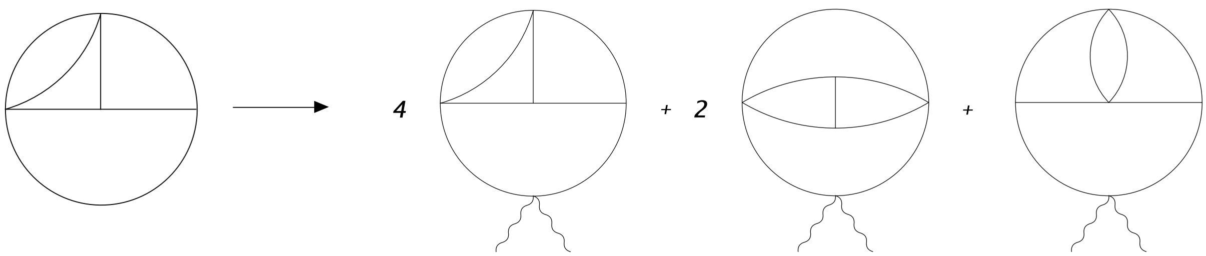

Figure B.2: Tadpole contributions from propagator expansions of diagrams B.1(a) and B.1(b).

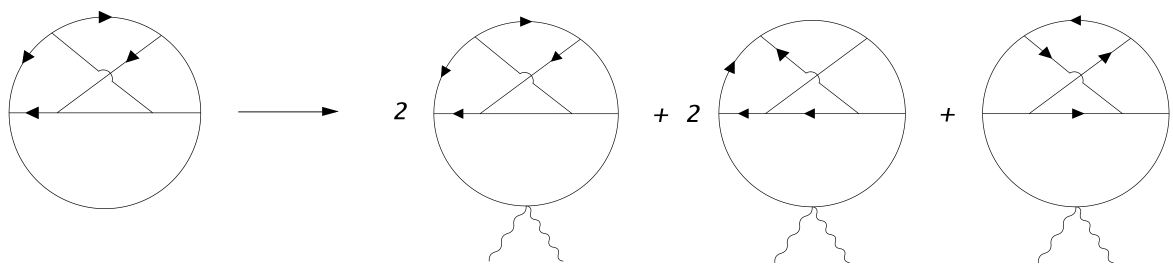

Figure B.3: Tadpole contributions from relevant propagator expansions of diagram B.1(c).

The integrals associated to diagrams in Fig. B.3 are much harder to compute because of the presence of four derivatives, indicated by the black arrows. However, after some proper integrations by parts, they can be reduced to simpler scalar integrals, as depicted in Fig. B.4. Notice that in the whole procedure we have neglected all tadpole graphs with $1/\epsilon$ divergences, which do not contribute to the four–loop effective action.

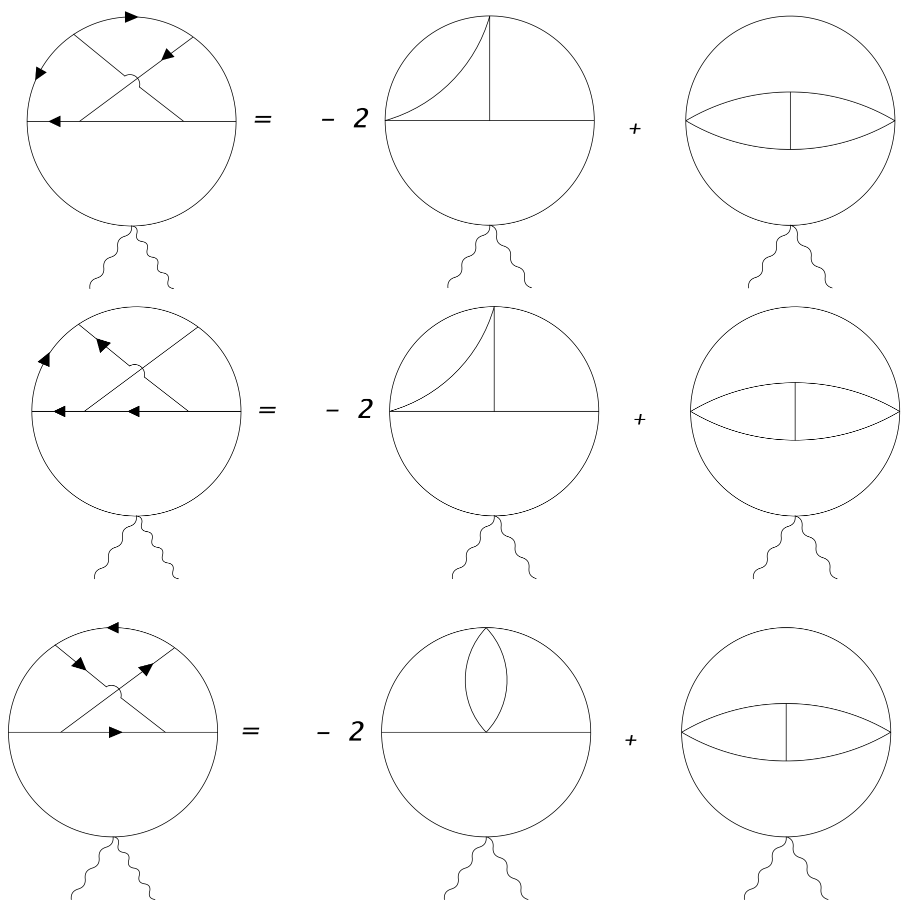

Figure B.4: Scalar reduction of integrals with derivatives.

Now we just need to sum up the various contributions generated by B.1(a), (b) and (c) diagrams. Actually there is no need to compute all these integrals explicitly thanks to a beautiful diagrammatic cancellation (see the main text in Section 2.4.2 for the result).

Appendix C

Worldvolume D–brane action

Within the framework of perturbative string theory, a Dp-brane is a $(p+1)$-dimensional extended surface in spacetime which supports the endpoints of open strings. The massless modes of this open string theory form a supersymmetric U(1) gauge theory with a vector A_a, $9-p$ real scalars Φ^i and their superpartner fermions, which are ignored throughout the following discussion. At leading order, the low-energy action corresponds to the dimensional reduction of that for ten-dimensional U(1) super-Yang-Mills theory. However, as usual in string theory, there are higher order $\alpha' = l_s^2$ corrections — l_s is the string length scale. For constant field strengths, these stringy corrections can be resummed to all orders, and the resulting action takes the Born-Infeld form [15]

$$S_{DBI} = -T_p \int_{\Sigma_{p+1}} d^{p+1}\sigma \; e^{-\phi} \sqrt{-det(P\left[G-B\right]_{ab} + 2\pi l_s^2 F_{ab})} \tag{C.0.1}$$

where T_p is the Dp-brane tension which in our conventions reads [132]

$$T_p = \frac{1}{l_s (2\pi l_s)^p} \tag{C.0.2}$$

This Born-Infeld action describes the couplings of the Dp-brane to the massless Neveu-Schwarz fields of the bulk closed string theory, *i.e.* the (string-frame) metric G_{MN}, dilaton ϕ and Kalb-Ramond two-form B_{MN}. The symbol $P[\ldots]$ denotes the pull-back of the bulk spacetime tensors to the D-brane world-volume. In particular, $P\left[G\right]_{ab} \equiv g_{ab} = G_{MN}\partial_a X^M \partial_b X^N$ is the pull-back of the spacetime metric on the world–volume[1] and $P\left[B\right]_{ab} \equiv b_{ab}$ is the pull-back to the worldvolume of the target NSNS two-form potential $b_{ab} = B_{MN}\partial_a X^M \partial_b X^N$. We can see that the one–form gauge potential A_a enters the action through the field strength $F_{ab} = \partial_a A_b - \partial_b A_a$. It is useful to introduce the modified field strength $\mathcal{F} = 2\pi l_s^2 F - B = 2\pi l_s^2 dA - B$ which is invariant under the combined gauge transformations

$$\delta B = d\Lambda \, , \qquad \delta A = \frac{\Lambda}{2\pi l_s^2} \, . \tag{C.0.3}$$

Dp-branes are charged under RR potentials and their action should contain a term coupling the brane to these fields. However, the RR fields are subject to gauge transformations and the coupling

[1]Here (a,b) are world–volume indices while indices (M,N) are target space indices.

should be gauge invariant [16]. In order to find a candidate for this coupling let us define C to be the formal sum of the background RR fields

$$C = \sum_{q=0}^{8} C_q \, , \tag{C.0.4}$$

where C_q is a differential form of degree q. The fields C_q are the RR gauge potentials of IIB (q even) supergravity. Their gauge transformation is

$$\delta_{RR} C = d\Lambda - H \wedge \Lambda + d\Lambda \wedge e^B \, , \tag{C.0.5}$$

where $H = dB$ is the NSNS 3-form field strength and Λ is a formal sum of arbitrary forms Λ_q with degree q

$$\Lambda = \sum_{q=0}^{7} \Lambda_q . \tag{C.0.6}$$

The field strengths of the RR fields, given by

$$\tilde{F} = dC - H \wedge C = \sum_{q=0}^{10} \tilde{F}_q \, , \tag{C.0.7}$$

are invariant under the above transformation. The form expansion of $\tilde{F}$ yields all the modified field strengths of the IIB potentials and their duals. To relate the potentials to their duals we impose

$$\tilde{F}_9 = \star \tilde{F}_1 \qquad \tilde{F}_7 = - \star \tilde{F}_3 \qquad \tilde{F}_5 = \star \tilde{F}_5 \tag{C.0.8}$$

where $\star$ is the Hodge dual in ten dimensions. Note that $\tilde{F}_5$ is self-dual. The RR field strengths given by (C.0.7) satisfy the Bianchi identity

$$d\tilde{F} = H \wedge \tilde{F}. \tag{C.0.9}$$

The requirement of target gauge invariance and the fact that D-branes possess electric charge determines the coupling to be of the form

$$\int_{\Sigma_{p+1}} P \left[C \wedge e^{\mathcal{F}} \right] \, , \tag{C.0.10}$$

where it is to be understood that one select the $(p+1)$–form in the expansion. The integral (C.0.10) over the worlvolume can be written as an integral over a $(p+2)$-dimensional manifold M_{p+2} whose boundary is the $(p+1)$-dimensional worldvolume Σ_{p+1}. Indeed it can be proved that

$$\tilde{F} \wedge e^{\mathcal{F}} = d \left[C \wedge e^{\mathcal{F}} \right] \, , \tag{C.0.11}$$

and then it follows that (C.0.10) is equivalent to

$$\int_{M_{p+2}} P \left[\tilde{F} \wedge e^{\mathcal{F}} \right] \, , \tag{C.0.12}$$

which is manifestly gauge invariant. Moreover, despite of the fact that $C \wedge e^{\mathcal{F}}$ is not invariant under the RR gauge transformations, its variation is a total derivative, thus making (C.0.10) gauge invariant. Actually,

$$\delta_{RR}[C \wedge e^{\mathcal{F}}] = d[\Lambda \wedge e^{\mathcal{F}}] \,. \tag{C.0.13}$$

The term (C.0.10) is a topological term of the Wess-Zumino type and it will be denoted by S_{WZ}. If the Dp-brane has RR charge μ_p the WZ action takes the final explicit form [17, 18, 19]

$$S_{WZ} = \mu_p \int_{\Sigma_{p+1}} P\left[\sum_q C_q \, e^{-B}\right] e^{2\pi l_s^2 F} \ . \tag{C.0.14}$$

Extremal branes satisfy the BPS bound $T_p = |\mu_p|$ and their action will be given by the compact

$$S \ = \ -T_p \int_{\Sigma_{p+1}} d^{p+1}\sigma \, e^{-\phi} \sqrt{-det(g+\mathcal{F})} \ \pm \ T_p \int_{\Sigma_{p+1}} P\left[C \wedge e^{\mathcal{F}}\right] \ , \tag{C.0.15}$$

where the $+$ sign is for branes an the $-$ for anti-branes.

Eq. (C.0.14) shows that a Dp-brane is naturally charged under the (p+1)-form RR potential with charge μ_p. If we consider the special case of the D0-brane (a point particle), the Born-Infeld action reduces to the familiar world-line action of a point particle, where the action is proportional to the proper length of the particle trajectory. Actually this string theoretic D0-brane action is not quite this simple geometric action, rather it is slightly embellished with the additional coupling to the dilaton which appears as a prefactor to the standard Lagrangian density. (Note, however, that the tensors B and F drop out of the action since the determinant is implicitly over a one-dimensional matrix.) Turning to the Wess-Zumino action, we see that a D0-brane couples to C_1 (a vector). Then eq. (C.0.14) reduces to the familiar coupling of a Maxwell field to the world-line of a point particle, *i.e.*

$$\mu_0 \int_{\Sigma_1} P\left[C_1\right] \simeq q \int A_\mu \frac{dX^\mu}{d\tau} d\tau \ . \tag{C.0.16}$$

Higher dimensional Dp-branes can also support a flux of $\mathcal{F}$, which complicates the world-volume actions above. From eq. (C.0.14), we see that such a flux allows a Dp-brane to act as a charge source for RR potentials with a lower form degree than p+1 [17].

The Born-Infeld action (C.0.1) has a geometric interpretation, *i.e.* it is essentially the proper volume swept out by the Dp-brane, which is indicative of the fact that D-branes are actually dynamical objects. This dynamics becomes more evident with an explanation of the static gauge choice implicit in constructing the above action. To begin, we employ spacetime diffeomorphisms to position the world-volume on a fiducial surface defined as $X^i = 0$ with $i = p+1, \ldots, 9$. With world-volume diffeomorphisms, we then match the world-volume coordinates with the remaining spacetime coordinates on this surface, $\sigma^a = X^a$ with $a = 0, 1, \ldots, p$. Now the world-volume scalars Φ^i play the role of describing the transverse displacements of the D-brane, through the identification

$$X^i(\sigma) = 2\pi l_s^2 \Phi^i(\sigma) \qquad \text{with } i = p+1, \ldots, 9. \tag{C.0.17}$$

With this identification and by defining $E_{MN} \equiv G_{MN} - B_{MN}$, the general formula for the pull-back reduces to

$$P[E]_{ab} \ = \ E_{MN} \partial_a X^M \partial_b X^N \tag{C.0.18}$$

$$= \quad E_{ab} + 2\pi l_s^2 \, E_{ai} \, \partial_b \Phi^i + 2\pi l_s^2 \, E_{ib} \, \partial_a \Phi^i + (2\pi l_s^2)^2 \, E_{ij} \partial_a \Phi^i \partial_b \Phi^j \ .$$

In this way, the expected kinetic terms for the scalars emerge to leading order in an expansion of the Born-Infeld action (C.0.1). Note that our conventions are such that both the gauge fields and world-volume scalars have the dimensions of $length^{-1}$ — hence the appearance of the string scale in eq. (C.0.17).

Although it was mentioned above, we want to stress that these world-volume actions are low energy effective actions for the massless states of the open and closed strings, which incorporate interactions from all disk amplitudes (all orders of tree level for the open strings). The Born-Infeld action was originally derived [15] using standard beta function techniques applied to world-sheets with a boundary [20]. In principle, they could also be derived from a study of open and closed string scattering amplitudes and it has been verified that this approach yields the same interactions to leading order [21, 22, 23]. As a low energy effective action then, eqs. (C.0.1) and (C.0.14) include an infinite number of stringy corrections, which essentially arise through integrating out the massive modes of the string — see the discussion in [27].

At this point, we should also note that the bulk supergravity fields appearing in eqs. (C.0.1) and (C.0.14) are in general functions of all of the spacetime coordinates, and so they are implicitly functionals of the world-volume scalars. In static gauge, the bulk fields are evaluated in terms of a Taylor series expansion around the fiducial surface $X^i = 0$. Hence the world-volume action implicitly incorporates an infinite class of higher dimension interactions involving derivatives of the bulk fields as well. However, beyond this class of interactions incorporated in eqs. (C.0.1) and (C.0.14), once again the full effective action includes other higher derivative bulk field corrections [19, 24, 25, 26]. It is probably fair to say that the precise domain of validity of the D-brane action from the point of view of the bulk fields is poorly understood.

Appendix D

Toroidal coordinates and spherical harmonics on S^3

This appendix is intended to review the construction of a complete set of scalar spherical harmonic on S^3, following [157]. Then we give the main properties of the vector spherical harmonics. For convenience, we have chosen to visualize the three–sphere with a toroidal coordinate system.

Let x, y, z, and w be the usual coordinates in $\mathbf{R}^4$, so the unit 3-sphere S^3 is defined by $x^2 + y^2 + z^2 + w^2 = 1$. The coordinates θ, ϕ_2, and ϕ_3 parameterize the 3-sphere as

$$\begin{aligned} x &= \cos\theta\,\cos\phi_2 \\ y &= \cos\theta\,\sin\phi_2 \\ z &= \sin\theta\,\cos\phi_3 \\ w &= \sin\theta\,\sin\phi_3 \end{aligned} \tag{D.0.1}$$

for

$$\begin{aligned} 0 \leq\ & \theta \leq \pi/2 \\ 0 \leq\ & \phi_2 \leq 2\pi \\ 0 \leq\ & \phi_3 \leq 2\pi. \end{aligned} \tag{D.0.2}$$

For each fixed value of $\theta \in [0, \pi/2]$, the ϕ_2 and ϕ_3 coordinates sweep out a torus. Taken together, these tori almost fill S^3. The exceptions occur at the endpoints $\theta = 0$ and $\theta = \pi/2$, where the stack of tori collapses to the circles $x^2 + y^2 = 1$ and $z^2 + w^2 = 1$, respectively. This is our framework expressed in toroidal coordinates.

The coordinates θ, ϕ_2, and ϕ_3 are everywhere orthogonal to each other. Thus the metric on the 3-sphere may be written as

$$ds^2 = d\theta^2 + c_\theta^2\, d\phi_2^2 + s_\theta^2\, d\phi_3^2 \tag{D.0.3}$$

where we have used the notation $c_\theta \equiv \cos\theta$ and $s_\theta \equiv \sin\theta$ for concision. The Laplacian in toroidal coordinates takes the explicit form

$$\nabla^2 = \frac{1}{c_\theta s_\theta}\left[\frac{\partial}{\partial\theta}\left(c_\theta s_\theta \frac{\partial}{\partial\theta}\right) + \frac{s_\theta}{c_\theta}\left(\frac{\partial}{\partial\phi_2}\right)^2 + \frac{c_\theta}{s_\theta}\left(\frac{\partial}{\partial\phi_3}\right)^2\right]. \tag{D.0.4}$$

The wave number l parameterizes the eigenmodes of the Laplacian on the 3-sphere S^3. Each integer wave number $l > 0$ corresponds to an eigenvalue $-l(l+2)$ with multiplicity $(l+1)^2$ [158, 159]. Hence, a scalar spherical harmonic $\mathcal{Y}$ on S^3 satisfies the Helmholtz equation

$$\nabla^2 \mathcal{Y} = -l(l+2)\mathcal{Y} \tag{D.0.5}$$

We will look for solutions that factor as

$$\mathcal{Y}(\theta, \phi_2, \phi_3) = X(\theta)\, P_2(\phi_2)\, P_3(\phi_3). \tag{D.0.6}$$

We have no *a priori* guarantee that all solutions must take this form, but we will see that the number of independent solutions of this form does indeed equal the dimension $(l+1)^2$ of the full eigenspace.

Substituting the expression (D.0.4) for ∇^2 and the factorization (D.0.6) of $\mathcal{Y}$ into the Helmholtz equation (D.0.5) gives

$$\frac{P_2 P_3}{c_\theta s_\theta} \frac{\partial}{\partial \theta} \left(c_\theta s_\theta \frac{\partial X}{\partial \theta} \right) + \frac{X P_3}{c_\theta^2} \frac{\partial^2 P_2}{\partial \phi_2^2} + \frac{X P_2}{s_\theta^2 \theta} \frac{\partial^2 P_3}{\partial \phi_3^2} = -l(l+2)\, X P_2 P_3. \tag{D.0.7}$$

Multiplying through by $c_\theta^2 s_\theta^2 / (X P_2 P_3)$ isolates the P_2 and P_3 factors

$$\frac{c_\theta s_\theta}{X} \frac{d}{d\theta} \left(c_\theta s_\theta \frac{dX}{d\theta} \right) + s_\theta^2 \left(\frac{1}{P_2} \frac{d^2 P_2}{d\phi_2^2} \right) + c_\theta^2 \left(\frac{1}{P_3} \frac{d^2 P_3}{d\phi_3^2} \right) = -l(l+2)\, c_\theta^2 s_\theta^2 \tag{D.0.8}$$

The expressions in P_2 and P_3 must each be constant, and to allow a periodic solution the constants must be negative,

$$\frac{1}{P_2} \frac{d^2 P_2}{d\phi_2} = -m_2^2 \tag{D.0.9}$$

$$\frac{1}{P_3} \frac{d^2 P_3}{d\phi_3^2} = -m_3^2. \tag{D.0.10}$$

The solutions are the usual

$$P_2^{m_2}(\phi_2) = e^{i m_2 \phi_2} \tag{D.0.11}$$

and

$$P_3^{m_3}(\phi_3) = e^{i m_3 \phi_3} \tag{D.0.12}$$

Substituting (D.0.9) and (D.0.10) into the Helmholtz equation (D.0.8) reduces it to a second order ordinary differential equation for X

$$\frac{c_\theta s_\theta}{X} \frac{d}{d\theta} \left(c_\theta s_\theta \frac{dX}{d\theta} \right) - m_2^2\, s_\theta^2 - m_3^2\, c_\theta^2 = -l(l+2)\, c_\theta^2 s_\theta^2. \tag{D.0.13}$$

For integers l, m_2, and m_3 satisfying $|m_2 \pm m_3| = l - 2k$ with $k = 0, 1, \cdots$, equation (D.0.13) is a close relative of the Jacobi equation and admits the solution

$$X_l^{m_2, m_3}(\theta) = c_\theta^{|m_2|}\, s_\theta^{|m_3|}\, P_d^{(|m_2|, |m_3|)}(c_{2\theta}) \tag{D.0.14}$$

where $P_d^{(|m_2|,|m_3|)}$ is the Jacobi polynomial

$$P_d^{(|m_2|,|m_3|)}(u) = \frac{1}{2^d} \sum_{i=0}^{d} \binom{|m_3|+d}{i} \binom{|m_2|+d}{d-i} (u+1)^i \, (u-1)^{d-i} \tag{D.0.15}$$

and

$$d = \frac{l - (|m_2| + |m_3|)}{2}. \tag{D.0.16}$$

Substituting the expressions for X, P_2, and P_3 from (D.0.14), (D.0.11), and (D.0.12) gives the eigenmode

$$\mathcal{Y}_l^{m_2,m_3}(\theta, \phi_2, \phi_3) = c_\theta^{|m_2|} \, s_\theta^{|m_3|} \, P_d^{(|m_2|,|m_3|)}(c_{2\theta}) \, e^{im_2\phi_2} \, e^{im_3\phi_3} \tag{D.0.17}$$

The Jacobi polynomial (D.0.15) may be expanded as a homogeneous polynomial of degree $2d$ in x, y, z, and w and this fact proves that the $\mathcal{Y}_l^{m_2,m_3}$ are smooth even along the circles $\theta = 0$ and $\theta = \pi/2$, where the toroidal coordinate system collapses.

For each l, the set of $\mathcal{Y}_l^{m_2,m_3}$ forms a basis for the space of eigenfunctions on S^3 with wave number l. More precisely, define the basis

$$B_l = \{\, \mathcal{Y}_l^{m_2,m_3} \quad | \quad |m_2 \pm m_3| = l - 2k \,\}. \tag{D.0.18}$$

To prove that B_l is a basis, we must show that the $\mathcal{Y}_l^{m_2,m_3}$ it contains are linearly independent and span the full eigenspace.

Linear independence. The inner product of two elements $\mathcal{Y}_l^{m_2,m_3}$ and $\mathcal{Y}_l^{m_2',m_3'}$ of B_l is

$$\begin{aligned}
&\langle \mathcal{Y}_l^{m_2,m_3}, \mathcal{Y}_l^{m_2',m_3'} \rangle \\
= &\int_{S^3} \mathcal{Y}_l^{m_2,m_3} \mathcal{Y}_l^{m_2',m_3'} \, dV \qquad \text{(D.0.19)}\\
= &\int_{\theta=0}^{\pi/2} \int_{\phi_2=0}^{2\pi} \int_{\phi_3=0}^{2\pi} (X_l^{m_2,m_3} P_2^{m_2} P_3^{m_3}) \, (X_l^{m_2',m_3'} P_2^{m_2'} P_3^{m_3'}) \, c_\theta \, s_\theta \, d\phi_3 \, d\phi_2 \, d\theta \\
= &\left(\int_{\theta=0}^{\pi/2} X_l^{m_2,m_3} X_l^{m_2',m_3'} \, c_\theta \, s_\theta \, d\theta \right) \left(\int_{\phi_2=0}^{2\pi} P_2^{m_2} P_2^{m_2'} \, d\phi_2 \right) \left(\int_{\phi_3=0}^{2\pi} P_3^{m_3} P_3^{m_3'} \, d\phi_3 \right).
\end{aligned}$$

If $m_2 \neq m_2'$ (resp. $m_3 \neq m_3'$), then the orthogonality of $\langle P_2^{m_2}, P_2^{m_2'} \rangle = 0$ (resp. $\langle P_3^{m_3}, P_3^{m_3'} \rangle = 0$) immediately implies $\langle \mathcal{Y}_l^{m_2,m_3}, \mathcal{Y}_l^{m_2',m_3'} \rangle = 0$, proving that $\mathcal{Y}_l^{m_2,m_3}$ and $\mathcal{Y}_l^{m_2',m_3'}$ are orthogonal. Because the $\mathcal{Y}_l^{m_2,m_3}$ in B_l are nonzero and pairwise orthogonal, they must be linearly independent.

Span. We have shown that the $\mathcal{Y}_l^{m_2,m_3}$ in B_l are linearly independent. To prove that they span the full eigenspace, it suffices to check that the number of elements of B_l equals the dimension of the full eigenspace, which is known to be $(l+1)^2$. The set $B_0 = \{\mathcal{Y}_0^{0,0}\}$ has $(0+1)^2 = 1$ element, and the set $B_1 = \{\mathcal{Y}_1^{+1,0}, \mathcal{Y}_1^{-1,0}, \mathcal{Y}_1^{0,+1}, \mathcal{Y}_1^{0,-1}\}$ has $(1+1)^2 = 4$ elements, as required. For the remaining B_l, with $l \geq 2$, we proceed by induction, assuming that the set B_{l-2} is already known to contain $((l-2)+1)^2 = (l-1)^2$ elements. Each element $\mathcal{Y}_{l-2}^{m_2,m_3} \in B_{l-2}$ corresponds to an element $\mathcal{Y}_l^{m_2,m_3} \in B_k$. The set B_l also contains the additional elements $\mathcal{Y}_l^{0,\pm l}, \mathcal{Y}_l^{\pm 1,\pm(l-1)}, \ldots, \mathcal{Y}_l^{\pm(l-1),\pm 1}, \mathcal{Y}_l^{\pm l,0}$. Taking into account the plus-or-minus signs, this gives $2 + 4 + \ldots + 4 + 2 = 2 + 4(l-1) + 2 = 4l$

additional elements. Adding these to the $(l-1)^2$ elements corresponding to B_{l-2}, we get a total of $(l-1)^2 + 4l = (l+1)^2$ elements, as required.

This completes the proof that B_l is a basis for the space of eigenfunctions on S^3 with wave number l.

Vector spherical harmonics come into three classes. Choosing them to be also eigenfunctions of $\frac{\partial}{\partial \phi_{2,3}}$ we have longitudinal harmonics $\mathcal{H}_i = \nabla_i \mathcal{Y}_l^{m_2,m_3}$, $l \geq 1$ which are in the $(\frac{l}{2}, \frac{l}{2})$ representation of $SO(4)$ with (m_2, m_3) ranging as before, namely $|m_2 \pm m_3| = l - 2k$. Transverse harmonics $\mathcal{M}_i^+ \equiv \mathcal{Y}_i^{(l,m_2,m_3);+}$ with $l \geq 1$ in the $\left(\frac{l-1}{2}, \frac{l+1}{2}\right)$ and $\mathcal{M}_i^- \equiv \mathcal{Y}_i^{(l,m_2,m_3);-}$ with $l \geq 1$ in the $\left(\frac{l+1}{2}, \frac{l-1}{2}\right)$. Their degeneracy is $l(l+2)$ and it is counted by $|m_2 + m_3| = l \pm 1 - 2k, |m_2 - m_3| = l \mp 1 - 2k$. These harmonics satisfy

$$\begin{aligned}
\nabla_i \nabla^i \mathcal{M}_j^\pm - R_j^k \mathcal{M}_k^\pm &= -(l+1)^2 \mathcal{M}_j^\pm \\
\epsilon_{ijk} \nabla^j \mathcal{M}^{\pm;k} &= \pm \sqrt{g}\,(l+1) \mathcal{M}_i^\pm \\
\nabla^i \mathcal{M}_i^\pm &= 0 \\
\frac{\partial}{\partial \phi_{2,3}} \mathcal{M}_i^\pm &= i m_{2,3} \mathcal{M}_i^\pm
\end{aligned} \tag{D.0.20}$$

where $\sqrt{g} = c_\theta s_\theta$ is the square root of the determinant of the metric on S^3, whereas $R_j^i = 2\delta_j^i$ is the Ricci tensor.

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