On Non-Commutative Integrals

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On Non-Commutative Integrals

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Par

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N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Résumé

Dans cette thèse, nous proposons une notion d'intégrales non-commutative pour les algèbres libres. L'organisation est la suivante :

- Comme premier exemple d'intégrale non-commutative, nous étudions en détail les intégrales de matrices hermitiennes. À partir de leur différentes propriétés, nous proposons des axiomes pour une première version d'intégrales non-commutative. Celle-ci prenne valeurs dans les nombres complexes et nous les appelons "intégrales à l'infini". Toujours motivé par l'étude de l'intégration matricielle, nous définissons une deuxième version d'intégrales non-commutative. Cette fois les intégrales prennent leurs valeurs dans l'anneau des séries formelles à coefficients complexes, c'est pourquoi nous les nommons "intégrales formelles".
- Inspiré par une présentation de Kontsevich, nous définissons une certaine algèbre de Batalin-Vilkovisky $\mathcal{C}$ dont la construction est similaire à une construction de Ginzburg. Pour ce faire, nous généralisons la notion de double crochet de Van den Bergh dans le but d'obtenir des crochets gradués de degré 1. À proprement parlé, nous construisons deux algèbres, une pour chaque type d'intégrales, mais l'une est obtenue à partir de l'autre en spécifiant la valeur d'un paramètre. Le lien avec les intégrales non-commutatives est donné par les fait que les intégrales correspondent à des morphismes de complexes de chaînes ayant pour source l'algèbre $\mathcal{C}$. Si cette description caractérise entièrement les intégrales à l'infini, la situation est bien moins claire pour les intégrales formelles.
- Le résultat décrit ci-dessus nous informe que les intégrales à l'infini sont classifiées par la cohomologie en degré 0 d'une certaine algèbre. Nous calculons cette cohomologie dans le cas d'une seule variable. Toujours pour une seule variable, nous trouvons ensuite un nombre fini de paramètres qui déterminent uniquement les intégrales formelles. Les techniques utilisées pour parvenir à ce résultat sont issues de la récurrence topologique.
- A la lumière du résultat précédent, nous conjecturons qu'une intégrale formelle est un morphisme d'algèbre de Batalin-Vilkovisky après avoir déformé de manière adéquate le produit de $\mathcal{C}$.

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Introduction

Integration is one of the crown jewels of mathematics. Its usefulness, both in theory and in application, cannot be overstated. It began as a method for computing areas under curves by approximating it with arbitrary small rectangles, an idea rigorously formalized by Riemann sums. Over time, this central idea expanded in many directions: from single-variable functions to multivariable ones, to a broader class of functions using Lebesgue measure, and eventually to manifolds and beyond.

At its core, integration can be thought of as the process of summing a function over a space to compute quantities such as areas, volume, or averages. In classical geometry, the space over which one is integrating is commutative, meaning that the algebra of functions on the said space is commutative. In non-commutative geometry, the space in question may not necessarily be, well, commutative anymore. While it is not clear what it would mean for a space to be non-commutative on the level on the space itself, it is a lot more clear what it should mean for its algebra of functions. Drawing on a principle from algebraic geometry - the understanding of a space is equivalent to the understanding of its algebra of functions - non-commutative geometry studies non-commutative algebras as algebras of functions of non-commutative spaces.

Many concepts from commutative geometry have counterparts in the non-commutative setting. For example, the role of a Poisson bracket is played by double Poisson brackets introduced by Van den Bergh [dB04].

In this thesis, we propose a notion of non-commutative integrals for free associative algebras. One reason for the restriction to free algebras is that the concept of volume element lying at the heart of integration theory turns out to be surprisingly hard to construct in the non-commutative setting. However, once a volume form has been fixed, there is a notion of divergence of a vector field, and it is possible to construct an analogue of divergence for free associative algebras. Important properties of integrals such as integration by parts, the divergence theorem, or more generally Stokes theorem can be stated using the concept of divergence. Non-commutative divergence might be thought of as the shadow of a non-commutative volume form.

As a motivating example of non-commutative integration, we study matrix integration or more precisely Gaussian matrix integration. In that context, Stokes theorem is known as "Loop equations" and takes a very peculiar form.

The plan is as follows. Fix $A(n)$ to be the free associative algebra on n generators. We begin by briefly presenting some elements of non-commutative calculus, necessary to later define non-commutative integrals. We then study in details Gaussian integration of $N \times N$ Hermitian matrices. Unsurprisingly, the different values of such integrals depends heavily on N. By considering the limit as N goes to infinity, we derive axioms for a first version of non-commutative integral as maps

$$Sym\big(A(n)/[A(n), A(n)]\big) \to \mathbb{C}.$$

We call those integrals "infinity integrals".

Coming back to the study of Gaussian integration, we see that those integrals can be interpreted as power series in N^{-2}, from which we derive axioms for yet another version of non-commutative integrals called "power series integrals". As the name suggests, this version of non-commutative integrals are maps

$$Sym\big(A(n)/[A(n), A(n)]\big) \otimes \mathbb{C}[[\hbar]] \to \mathbb{C}[[\hbar]]$$

and the "infinity integrals" described above are the leading order of power series integrals. Both infinity and power series integrals are actually defined with respect to a potential V. Different choices of V correspond to different volume forms.

Aiming towards a classification of integrals, we then reframe both versions of integrals in a cohomological setting. We motivate this idea by first presenting a reformulation of the de Rham complex in terms of polyvector fields. We have learned about this point of view in [Gwi12]. This reformulation indicates that the right algebraic framework for our purposes is the framework of BV-algebras (Batalin-Vilkovisky algebra). We recall the basic properties of these. Classical examples of BV algebra can be constructed from (graded) involutive Lie bialgebra. More generally Perry and Pulmann described in [PP24] sufficient and necessary conditions to construct a BV operator from a graded bracket and cobracket on a graded vector space. This is where we present a slightly different version of Van den Bergh double Gerstenhaber bracket [dB04], more suited to our goal. Usual double brackets are maps $A \otimes A \to A \otimes A$ where A is an algebra. They satisfy some axioms devised to make them induce Lie brackets on the space $A/[A, A]$. Our graded version of double brackets allows us to define easily graded bracket satisfying the conditions in [PP24]. In the original article of Van den Bergh, the graded setting was only mentioned and proofs were given only for non-graded double bracket. We adapt those proofs to the graded setting using diagram calculus and get the following theorem:

Theorem A. *Let A be a graded algebra and $\Pi(-,-) : A \otimes A \to A \otimes A$ a double bracket such that $\Pi(-,-,-) = 0$. Then the associate bracket*

$$[-,-]_\Pi := |A| \otimes |A| \to |A|$$

satisfies the following property

- *graded symmetry* $[-,-]_\Pi = [-,-]_\Pi \circ \tau$
- *graded Jacobi identity* $[-,-]_\Pi \circ ([-,-]_\Pi \otimes Id) \circ Cyc = 0$

More details about various notation and definitions can be found in Section 5.3.

Equipped with this toolbox, we define a graded analogue of a Lie bialgebra due to Schedler [Sch04]. The idea behind this construction is due to Ginzburg [Gin07].

Inspired by a talk of Kontsevich, we show that this BV algebra is closely related to our definitions of integrals. More precisely, we define two algebras $\mathcal{C}_\hbar(n)$ and $\mathcal{C}(n)$, the latter being obtained from the former by formally sending $\hbar$ to 0. While the definition of the algebra does not depend on the choice of the potential, the BV operator on $\mathcal{C}_\hbar(n)$ does. Furthermore, it becomes a differential when $\hbar = 0$ (but still depends on V). In that case, we have Theorem 5.50

Theorem B. *There is a one-to-one correspondence between infinity integrals with respect to the potential V and differential graded algebra morphisms*

$$(\mathcal{C}(n), \Delta_V) \to (\mathbb{C}, 0).$$

The situation is less clear for power series integrals. Nevertheless, we still have in Theorem 5.54

Theorem C. *Every power series integral with respect to the potential V induces a map of chain complex*

$$(\mathcal{C}_\hbar(n), \Delta_V) \to (\mathbb{C}[[\hbar]], 0)$$

Note that this results does not mention the BV structure whatsoever, it is only a result on the level of chain complexes. We'll come back to that.

Since the zeroth cohomology of $(\mathcal{C}(n), \Delta_V)$, classifies infinity integrals for the potential V, we compute it for $n = 1$ using the Homological Perturbation Lemma.

Theorem D. *For any potential V of degree $k+1$, the zeroth cohomology of the algebra $\mathcal{C}(1)$ is given by*

$$H^0(\mathcal{C}(1), \Delta_V) \cong Sym(\mathbb{C}[x]/(x^k)).$$

Using methods coming from topological recursion [CEO06][EO07], we then classify power series integrals in one variable. The idea is as follows. Starting with a power series integral, we define generating functions encoding the data of the power series integral. Using loop equations, we then promote these generating functions to differentials $\omega_{g.n}$ for $g \geqslant 0, n \geqslant 1$ on a Riemann surface Σ. The bridge with topological recursion is given by the following theorem

Theorem E. *For every power series integral in one variable $\varphi_\hbar$, the differential forms $\omega_{g,n}$ constructed from the coefficients of $\varphi_\hbar$ satisfy the topological recursion equations.*

The precise statement is given by Theorem 7.20. We stress that the forms $\omega_{g.n}$ are *not* defined by the topological recursion formula. However, we show that they satisfy the same equations as the one used to define the higher differentials in topological recursion. The meaning of this theorem is that these forms are uniquely determined by $\omega_{0,1}$ and $\omega_{0,2}$, which is far from obvious from their definition. This allows us to find a set of parameters uniquely fixing power series integrals (in one variable) in Theorem 7.21

Theorem F. *A power series integral in one variable $\varphi_\hbar$ for the potential*

$$V = \sum_{k=2}^{d} \frac{a_k}{k} x^k$$

is uniquely determined by the values of

$$\lim_{\hbar \to 0} \varphi_\hbar(x^i)$$

for $1 \leqslant i \leqslant d-2$. and

$$\lim_{\hbar \to 0} \frac{\varphi_\hbar\big((x^i)(x^j)\big) - \varphi_\hbar(x^i)\varphi_\hbar(x^j)}{\hbar}$$

for $1 \leqslant i, j \leqslant d-3$.

The outline of this strategy was indicated to us by Nicolas Orantin and is an adaptation of techniques used in random matrix theory and enumerative geometry [EO08].

Knowing that a finite set of parameters uniquely determines power series integrals, we then speculate that it might be fruitful to reinterpret power series integrals as BV maps from certain deformations of our initial BV algebra.

Conjecture G. *Integrals with respect to the potential V are equivalent to BV-algebra maps*

$$(\mathcal{C}_\hbar(1), *, \Delta_V) \to (\mathbb{C}[[\hbar]], 0)$$

where $*$ is a deformation of the product with respect to which Δ_V is a differential operator of order at most 2.

This speculation is motivated by the fact that this is the case for Gaussian integration, as seen in Proposition 8.18 which is a reinterpretation of Wick's Lemma. It also has similitude with star product in perturbative algebraic quantum field theory [FR15a][FR15b], even though the language used there is quite different.

Finally, we present different perspectives and how the study of non-commutative integrals fits with other parts of mathematics:

- Non-commutative integrals can be thought of as some kind of universal integrals for groups and Lie algebra. This point of view might be useful in getting a better understanding of Duflo's isomorphism $S(\mathfrak{g})^\mathfrak{g} \cong Z(U(\mathfrak{g}))$ for a Lie algebra $\mathfrak{g}$ [Duf77].

- Similarly, the volume of the moduli space of flat connections on a surface [Wit91] also involves integrals that have an universal form and thus a universal theory of integration for groups might be insightful.
- Related to the previous point, Magee and Puder [MP19][MP22] study integrals of product of traces over multiple copies of $U(N)$. They compute such quantities with formulae involving not only surfaces but also map from the boundaries of these surfaces to wedges of circles. Their construction might be an example of our definition of non-commutative integrals adapted to free groups.
- In [GGHZ21], Gwilliam and al. construct in a different manner the BV algebra we call $\mathcal{C}_\hbar(n)$, and relate it to some A_∞ algebras. It would be interesting to understand the role played by integrals in that context. In an other paper [GHZ22], they use ribbon graph to produce a quasi-isomorphism between their BV-algebra and power series. It seems likely that one can use those same ribbon graphs to deform the product on the BV-algebra in such a way that their quasi-isomorphism becomes a morphism of BV-algebra.

Contents

1 Notation

We group here various definitions that will be used throughout the text. Most of them are related to graded vector spaces. In the whole text, every construction is over a field $\mathbb{K}$ of characteristic 0.

Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ and $W = \bigoplus_{i\in\mathbb{Z}} W_i$ be graded vector spaces. Denote by $d(v)$ the degree of a homogeneous element $v \in V$. For simplicity purposes, when the degree is in a power of -1, we will write it with the same symbol as the element itself. That is, for $v \in V$ homogeneous of degree $d(v)$, $(-1)^v := (-1)^{d(v)}$.

The tensor product of V and W is the graded tensor space $V \otimes W$ where

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j.$$

The tensor swap map $\tau : V \otimes W \to W \otimes V$ is the map define by $\tau(a \otimes b) = (-1)^{ab} b \otimes a$ for a and b homogeneous and then extended linearly. The map $\xi : V \otimes V \otimes V \to V \otimes V \otimes V$ is the map that cyclically permutes tensor factors :

$$\xi(a \otimes b \otimes c) = (-1)^{a(b+c)} b \otimes c \otimes a$$

for a, b and c homogeneous. Let also $Cyc := 1 + \xi + \xi^2 : V \otimes V \otimes V \to V \otimes V \otimes V$. Those three maps will also be used in the non-graded setting by simply setting all the elements to be of degree 0.

Routinely, maps $[-,-] : V \otimes V \to V$ and $\delta : V \to V \otimes V$ will respectively be called a bracket and a cobracket, even though they might not satisfy the Lie (co)algebra axioms. Sometimes we shall use curly brackets $\{-,-\}$ for the bracket.

For a map $[-,-] : V \otimes V \to V$ of degree 1, we will denote by $[-,-]^{(2)}$ the map $V \otimes (V \otimes V) \to V \otimes V$ defined on homogeneous $a, b, c \in V$ by :

$$[a, b \otimes c]^{(2)} = [a, b] \otimes c + (-1)^{(a+1)b} b \otimes [a, c].$$

Be aware that the sign for the second term is non-standard for the bracket in of degree 1.

Given a graded algebra A, we will consider two different A-bimodules structures on the tensor product $A \otimes A$. The first one is the *outer module structure* given by

$$a(b \otimes c)d = ab \otimes cd$$

and the second one is the *inner bimodule structure* given by

$$a * (b \otimes c) * d = (-1)^{ab+cd+ad} bd \otimes ac.$$

2 Elements of non-commutative calculus

We begin by introducing some basic notions of non-commutative calculus.

2.1 Elements of non-commutative calculus

Commutative calculus on $\mathbb{R}^n$ deals with $\mathcal{C}^\infty(\mathbb{R}^n)$, the algebra of functions, and this is a commutative algebra. In non-commutative calculus, one replaces this algebra of functions by a non-commutative algebra A. In the case of $\mathbb{R}^n$, let

$$A := \mathbb{C}\langle\langle x_1, \ldots, x_n \rangle\rangle$$

be the algebra of non-commutative power series in n generators.

In the study of commutative calculus, one then defines the space of vector fields $\mathfrak{X}(\mathbb{R}^n)$. Quite quickly, one shows that they form a Lie algebra and that they act on $\mathcal{C}^\infty(\mathbb{R}^n)$ by derivations. Furthermore, every vector field V has the form

$$V = \sum_i f_i \frac{\partial}{\partial x_i}$$

for some $f_i \in \mathcal{C}^\infty(\mathbb{R}^n)$. All this information can be stated in a somewhat more conceptual manner:

Proposition 2.1. *The collection of partial derivatives $\frac{\partial}{\partial x_i}$ generate $\mathfrak{X}(\mathbb{R}^n)$ as a $\mathcal{C}^\infty(\mathbb{R}^n)$-module.*

Since we know what non-commutative functions are, Proposition 2.1 suggests that to understand non-commutative vector fields, one should first understand non-commutative partial derivative.

Let $f(x_1, \ldots, x_n)$ be a commutative function, that is an element of $\mathcal{C}^\infty(\mathbb{R}^n)$. The partial derivative $\frac{\partial}{\partial x_i} f$ is the coefficient of the linear part in X of $f(x_1, \ldots, x_i + X, \ldots, x_n)$. In the commutative setting, this coefficient is an other function. If one now plays the same game in the non-commutative setting, one finds that the linear part in X is an expression of the form

$$(\frac{\partial}{\partial x_i} f)' X (\frac{\partial}{\partial x_i} f)''$$

where both $(\frac{\partial}{\partial x_i} f)'$ and $(\frac{\partial}{\partial x_i} f)''$ are non-commutative power series. In particular, one needs to remember what is left of the symbol X and what is right of the symbol X. Thus, naturally, the partial derivative takes values in $A \otimes A$. Let us have a look at a couple of examples.

- Let $f(x_1, x_2) = x_1$, then $f(x_1 + X, x_2) = x_1 + X$ and $f(x_1, x_2 + X) = 0$. Hence $\frac{\partial}{\partial x_1} f = 1 \otimes 1$ and $\frac{\partial}{\partial x_2} f = 0$.
- Let $f(x_1, x_2) = x_1 x_2 x_1$, then $f(x_1 + X, x_2) = x_1 x_2 x_1 + x_1 x_2 X + X x_2 x_1 + X x_2 X$ and $f(x_1, x_2 + X) = x_1 x_2 x_1 + x_1 X x_1$. Hence $\frac{\partial}{\partial x_1} f = x_1 x_2 \otimes 1 + 1 \otimes x_2 x_1$ and $\frac{\partial}{\partial x_2} f = x_1 \otimes x_1$.

What about the Leibniz rule? An obvious computation shows that for any $f, g \in A$

$$\frac{\partial}{\partial x_i}(fg) = f \frac{\partial}{\partial x_i} g + (\frac{\partial}{\partial x_i} f) g.$$

In other words, the map $\frac{\partial}{\partial x_i}$ is a derivation of A with values in $A \otimes A$ equipped with the outer A-bimodule structure. This is the non-commutative analogue of the fact that vector fields act as derivations. In the non-commutative setting, we are dealing with bimodules for one needs to remember what was on the left and what was on the right. This motivates the following definition:

Definition 2.2. The partial derivative $\frac{\partial}{\partial x_i} : A \to A \otimes A$ is the derivation of A with values in $A \otimes A$ (equipped with the outer A-bimodule structure) defined on generators by

$$\frac{\partial}{\partial x_i} x_j = \delta_{ij} 1 \otimes 1.$$

More explicitly one has the following formula for $f = f_1 \dots f_k$ a monomial in A:

$$\frac{\partial}{\partial x_i}(f_1 \dots f_k) = \sum f_1 \dots f_{j-1} \otimes f_{j+1} \dots f_k$$

where the sum is taken over all the j such that $f_j = x_i$.

Remark 2.3. We will use Sweedler's notation and often write $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i}' \otimes \frac{\partial f}{\partial x_i}''$.

Is is then natural to interpret the space of all such derivations as the space of all vector fields:

Definition 2.4. Let $\mathbb{D}er(A) := Der(A, A \otimes A)$ where $A \otimes A$ is equipped with the outer bimodule structure.

Remark 2.5. Elements of $\mathbb{D}er(A)$ are often called *double derivations*

Using the *inner* bimodule structure on $A \otimes A$, one can, in turn, make $\mathbb{D}er(A)$ a A-bimodule. It is a good exercise to check that one indeed needs to use the inner bimodule structure of $A \otimes A$ to preserve the derivation property. It turns out that this space of noncommutative vector fields is also generated by the partial derivatives:

Proposition 2.6. *The collection of partial derivatives $\frac{\partial}{\partial x_i}$ generate $\mathbb{D}er(A)$ as a A-bimodule.*

Proof. Let d be a double derivations and write $d(x_i) = d(x_i)' \otimes d(x_i)''$ (remember Remark 2.3, we are using Sweedler's notations). Now define $\tilde{d} \in \mathbb{D}er(A)$ as

$$\tilde{d} = \sum_{i=1}^{n} d(x_i)'' \frac{\partial}{\partial x_i} d(x_i)'.$$

Since $d(x_i) = \tilde{d}(x_i)$ for every j and both maps are derivations, we have $d = \tilde{d}$. □

While we are on the topic of partial derivatives, let us record a useful identity relating derivations of A and partial derivatives :

Lemma 2.7. *Let $u \in Der(A)$. For any element f in A we have the following equality :*

$$u(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}' u(x_i) \frac{\partial f}{\partial x_i}''$$

Proof. By linearity, it is enough to show the result for $f = f_1 \dots f_k$ a monomial in A. One has:

$$\begin{aligned} u(f) &= \sum_{a=1}^{k} f_1 \dots u(f_a) \dots f_n \\ &= \sum_{i=1}^{n} \sum_{\{a | f_a = x_i\}} f_1 \dots u(f_a) \dots f_n \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}' u(x_i) \frac{\partial f}{\partial x_i}'' \end{aligned}$$

□

Although all of this seems to be a bit ad-hoc at first glance, it follows one guiding principle due to Kontsevich and Rosenberg [KR99]:

Definition 2.8. Let B be an associative algebra and V a vector space. The non-commutative version of a structure on B should induce the commutative version of the said structure on

$$Rep(B, End(V)).$$

Let us show how this works for functions. We said that B should be thought of as the space of non-commutative functions, so let us pick an element $b \in B$. It should somehow induce a function in $\mathcal{O}(RepB, End(V))$. To that end consider

$$\begin{aligned} Tr_b : Rep(B, End(v)) &\to \mathbb{C} \\ \rho &\longmapsto Tr(\rho(b)) \end{aligned}$$

Actually, the map Tr descends to a map $|B| \to \mathcal{O}(Rep(B, End(V))$, where $|B|$ is the quotient of B by the *subspace* of commutators. Furthermore, this map can be extended as a map of algebra to

$$Tr : Sym|B| \to \mathcal{O}(Rep(B, End(V)).$$

This indicates that one should not think of the algebra of non-commutative as just the algebra B, but one should rather think of $Sym|B|$ as the algebra of non-commutative functions.

Remark 2.9. It has been shown (c.f. [Kha12]) that the image of the map Tr is in the subalgebra of $GL(V)$ invariant functions $\mathcal{O}(Rep(B, End(V))^{GL(V)}$. It turns out that every invariant function can be constructed this way (c.f. [Pro87]).

Many more central notions of geometry such as differential forms, De Rham Cohomology, symplectic forms, or Poisson bracket have been developed in the non-commutative setting and they all satisfy the Kontsevich-Rosenberg principle. One can find a nice short exposition of those ideas in [Fer17]. However, a core notion for us will be a notion that does not quite fit in the framework of the Kontsevich-Rosenberg principle, the notion of divergence of a vector field.

Recall that in commutative calculus, the divergence is a map $Div : \mathfrak{X}(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$. Its expression in Cartesian coordinates is

$$Div\left(\sum f_i \frac{\partial}{\partial x_i}\right) = \sum \frac{\partial f_i}{\partial x_i}$$

In the case of the algebra $A = \mathbb{C}\langle\langle x_1, \ldots, x_n\rangle\rangle$ we have non-commutative analogues of partial derivatives, so one can mimic this definition.

Definition 2.10. The divergence map is the map $Div : Der(A) \to |A| \otimes |A|$ defined by

$$Div(u) = \sum_i |\frac{\partial}{\partial x_i} u(x_i)|$$

with the slight abuse of notation $|a \otimes b| = |a| \otimes |b|$.

Note that we decide to project everything onto $|A|$. An explanation could be that we know from our earlier discussion that functions should be a product of elements of $|A|$. Another one is that without this projection, the following fact proved in [AKKN23] would simply not be true.

Proposition 2.11. *The map $Div : Der(A) \to |A| \otimes |A|$ is a Lie 1-cocycle. In other words*

$$Div([u, v]) = u(Div(v)) - v(Div(u))$$

for every $u, v \in Der(A)$.

The above result is important for it is the analogue to the classical result stating that the commutative divergence is a Lie 1-cocycle. In the commutative case, the divergence is the only degree 0 such cocycle (here by degree we mean the number of x_i). This is clearly no longer true in the non-commutative setting. Indeed $\tau \circ Div : Der(A) \to |A| \otimes |A|$ where τ is the tensor swap map, is an other example of degree 0 Lie 1-cocycle. It was conjectured in [AKKN23] that Div and $\tau \circ Div$ span the space of such cocycles if $n \geqslant 2$.

This does not quite adhere to the Kontsevich-Rosenberg principle for non-commutative vector fields should be *double* derivations. However, it is not clear what it would mean for the divergence to be a Lie 1-cocycle as it is already not clear how to make the space of double derivations a Lie algebra.

Since we saw that functions are products of elements of $|A|$. We should slightly adapt the notion of partial derivative to cyclic words:

Definition 2.12. The map $\frac{\partial}{\partial x_i} : |A| \to A$ is defined for a cyclic word $f = |f_1 \dots f_k|$ in the alphabet made by the $x_i's$ by

$$\sum f_{j+1} \dots f_k f_1 \dots f_{j-1}$$

where the sum is taken over all the $j's$ such that $x_i = f_j$.

Remark 2.13. Let $\mu : A \otimes A \to A$ be the multiplication map and $\tau : A \otimes A \to A \otimes A$ be the tensor swap map that sends $a \otimes b$ to $b \otimes a$. One has for f a cyclic word and $\tilde{f}$ of representative of f

$$\frac{\partial}{\partial x_i} f = \mu \circ \tau \circ \frac{\partial}{\partial x_i}(\tilde{f})$$

which relates the cyclic partial derivative to the standard one. In Sweedler's notation this is written

$$\frac{\partial f}{\partial x_i} = \frac{\partial \tilde{f}}{\partial x_i}'' \frac{\partial \tilde{f}}{\partial x_i}'$$

Definition 2.14. For $1 \leqslant i, j \leqslant n$, define $\frac{\partial^2}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i} \circ \frac{\partial}{\partial x_j} : |A| \to A \otimes A$ where we first apply the cyclic partial derivative and then the standard one.

Lemma 2.15. *For every $1 \leqslant i, j \leqslant n$ we have the equality*

$$\frac{\partial^2}{\partial x_i \partial x_j} = \tau \circ \frac{\partial^2}{\partial x_j \partial x_i}.$$

Proof. Let f be a cyclic word in $|A|$. We may assume that f is of the form $|Ax_iBx_jC|$ with A, B, C having no x_i nor x_j.

Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial(CAx_iB)}{\partial x_i} = CA \otimes B$$

and

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial(Bx_jCA)}{\partial x_j} = B \otimes CA$$

□

2.2 Minus signs matter

All the constructions of the previous section also make sense in a graded setting, although one has to be careful with minus signs. The purpose of this section is precisely to be careful now, in order to be maybe a bit more sloppy later.

For $1 \leqslant i \leqslant n$ let α_i be of weight $d(\alpha_i)$ and denote by $\mathcal{A}$ the free graded associative algebra generated by the $\alpha_i's$.

Just as in the non-graded case, one considers the quotient of $\mathcal{A}$ by the subspace of *graded* commutators and denotes it by $|\mathcal{A}|$.

In the graded setting, the partial derivative $\frac{\partial}{\partial \alpha_i}$ is a derivation of degree $-d(\alpha_i)$:

Definition 2.16. The partial derivative $\frac{\partial}{\partial \alpha_i} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the derivation of degree $-d(\alpha_i)$ of $\mathcal{A}$ with values in $\mathcal{A} \otimes \mathcal{A}$ defined on generators by

$$\frac{\partial}{\partial \alpha_i} \alpha_j = \delta_{ij} 1 \otimes 1.$$

This means that this time there are some minus signs in the explicit formula:

$$\frac{\partial}{\partial \alpha_i}(f_1 \dots f_k) = \sum (-1)^{(f_1 + \dots + f_{j-1})\alpha_i} f_1 \dots f_{j-1} \otimes f_{j+1} \dots f_k$$

where, just as before, the sum is taken over the j such that $f_j = \alpha_i$

Similarly, some minus signs appear in the partial derivative of a cyclic word :

Definition 2.17. The map $\frac{\partial}{\partial x_i} : |\mathcal{A}| \to \mathcal{A}$ is defined for a cyclic word $f = |f_1 \dots f_k|$ in the alphabet made by the $\alpha_i's$ by

$$\sum (-1)^{(f_1 + \dots + f_{j-1})(f_j + \dots + f_n)} f_{j+1} \dots f_k f_1 \dots f_{j-1}$$

where the sum is taken over all the $j's$ such that $\alpha_i = f_j$.

Remark 2.18. If one understands the tensor swap map $\tau : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ in the graded sense, i.e. $\tau(a \otimes b) = (-1)^{ab} b \otimes a$ for a and b homogeneous element, one has once more the equality

$$\frac{\partial}{\partial \alpha_i} f = \mu \circ \tau \circ \frac{\partial}{\partial \alpha_i}(\bar{f})$$

for f a cyclic word and $\bar{f}$ any representative of f.

Of course, we can still define double partial derivatives :

Definition 2.19. For $1 \leqslant i, j \leqslant n$, define $\frac{\partial^2}{\partial \alpha_i \partial \alpha_j} := \frac{\partial}{\partial \alpha_i} \circ \frac{\partial}{\partial \alpha_j} : |\mathcal{A}| \to \mathcal{A} \otimes \mathcal{A}$ where we first apply the cyclic partial derivative and then the standard one.

Lemma 2.15 still holds if one understands everything in the graded sense:

Lemma 2.20. *For every $1 \leqslant i, j \leqslant n$ we have the equality*

$$\frac{\partial^2}{\partial \alpha_i \partial \alpha_j} = (-1)^{\alpha_i \alpha_j} \, \tau \circ \frac{\partial^2}{\partial \alpha_i \partial \alpha_j}.$$

Proof. Let f be a cyclic word in $|\mathcal{A}|$. We may assume that f is of the form $|A \alpha_i B \alpha_j C|$ with A, B, C having no α_i nor α_j.

Then

$$\begin{aligned}\frac{\partial^2 f}{\partial\alpha_i\partial\alpha_j} &= (-1)^{(A+\alpha_i+B)(\alpha_j+C)}\frac{\partial(CA\alpha_iB)}{\partial\alpha_i} \\ &= (-1)^{(A+\alpha_i+B)(x_j+C)+(A+C)\alpha_i}CA\otimes B\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial\alpha_j\partial\alpha_i} &= (-1)^{A(\alpha_i+B+x_j+C)}\frac{\partial(B\alpha_jCA)}{\partial x_j} \\ &= (-1)^{A(\alpha_i+B+x_j+C)+B\alpha_j}B\otimes CA.\end{aligned}$$

When you then apply τ to $\frac{\partial^2 f}{\partial\alpha_j\partial\alpha_i}$ you swap tensor factors and pick up the sign $(-1)^{B(A+C)}$. Now the two exponents of (-1) differ by exactly $\alpha_i\alpha_j$. $\square$

3 Integrals

Our aim is to construct a non-commutative version of integration. It is well known that for a n-manifold M, once a volume form ω has been chosen, integration defines a linear map

$$\int_M : \mathcal{C}^\infty(M) \to \mathbb{R}.$$

In the case of a n-dimensional vector space, there is a canonical volume form, namely $dx = dx_1 \wedge \cdots \wedge dx_n$. Every other volume form can then be expressed as

$$e^{-V} dx$$

for some function V.

We aim to construct a non-commutative version of integration starting with the non-commutative algebra $A = \mathbb{C}\langle x_1, \ldots, x_n\rangle$. In that setting, we have seen that it is natural to take $Sym|A|$ as the ring of functions. An integral should then be a map

$$\varphi : Sym|A| \to \mathbb{C}.$$

Of course integration is not just any old linear map, it has other properties. What should they be in the non-commutative setting? We will take our cue from matrix integration, more precisely Gaussian integration.

3.1 Gaussian matrix integration

As advertised, we shall have a look at Gaussian integration of matrices as a guideline.

Let $\mathcal{H}_N$ be the space of $N \times N$ Hermitian matrices. We are interested in the Gaussian expectation value $\langle - \rangle : \mathcal{C}^\infty(\mathcal{H}_N) \to \mathbb{C}$ defined by

$$f \mapsto \langle f \rangle := \frac{1}{Z} \int_{\mathcal{H}_N} dM f(M) e^{-N \frac{tr(M^2)}{2}}$$

where dM is the Lebesgue measure on $\mathcal{H}_N$ (identified with $\mathbb{R}^{N^2}$) and Z is a normalization constant such that $\langle 1 \rangle = 1$.

We are going to somewhat restrict the class of functions by considering only products of traces, namely functions f of the form

$$f(M) = \prod_{k=1}^{n} tr(M^{p_k}).$$

Remark 3.1. This restriction is not a big deal for our purpose. Indeed, functions of this form are exactly the functions obtained from the Kontsevich-Rosenberg principle.

Using Stokes Theorem we can derive the so called loop equations for those expectations values:

Proposition 3.2. *For every $k \in \mathbb{N}$*

$$N\big\langle tr(M^{k+1})\big\rangle = \sum_{l=0}^{k-1} \big\langle tr(M^{k-1-l})\, tr(M^l)\big\rangle$$

Proof. Since for matrices whose eigenvalues are sufficiently big, the integrand is arbitrarily small, Stokes' theorem tells us that

$$\sum_{ij}\int_{\mathcal{H}_N} dM \frac{\partial}{\partial M_{ij}}\Big((M^k)_{ij}\; e^{-N\frac{tr(M^2)}{2}}\Big) = 0.$$

An easy computation of the derivative gives

$$\sum_{ij}\int_{\mathcal{H}_N} dM\Big(\sum_{l=0}^{k-1}(M^{k-1-l})_{ii}(M^l)_{jj} - N(M^k)_{ij}(M)_{ji}\Big)e^{-N\frac{tr(M^2)}{2}} = 0$$

which in turn yields

$$N\langle tr(M^{k+1})\rangle = \sum_{l=0}^{k-1}\langle tr(M^{k-1-l})\; tr(M^l)\rangle$$

□

More generally, by considering the total derivative

$$\sum_{ij}\int_{\mathcal{H}_N} dM \frac{\partial}{\partial M_{ij}}\Big(\;(M^k)_{ij}\; tr(M^{a_1})\dots tr(M^{a_n})\; e^{-N\frac{tr(M^2)}{2}}\Big)$$

one gets the following result :

Proposition 3.3. *For every $k, n \in \mathbb{N}$ and $a_1, \dots a_n \in \mathbb{N}$*

$$\begin{aligned} N\langle tr(M^{k+1})\prod_{i=1}^{n} tr(M^{a_i})\rangle &= \sum_{l=0}^{k-1}\langle tr(M^{k-1-l})\; tr(M^l)\prod_{i=1}^{n} tr(M^{a_i})\rangle \\ &+ \sum_{j=1}^{n} a_j\langle tr(M^{k+a_j-1})\prod_{i\neq j} tr(M^{a_i})\rangle \end{aligned}$$

From these equations, one can compute the different expectation values recursively. A classical result also gives a method to compute them directly. Indeed, it is a well-known fact that integrals of product of traces can be computed by enumerating ribbon graphs, that is graphs that have a cyclic ordering of the half-edges at each vertex. This is the famous result of 't Hooft [Hoo74] :

Theorem 3.4.

$$\frac{1}{\prod n_j!}\Big\langle\prod_{k=1}^{n} N\frac{tr(M^{p_k})}{p_k}\Big\rangle = \sum_{graph\ G\ with\ vertices\ of\ valency\ p_k}\frac{N^{\chi(G)}}{|Aut(G)|}$$

where $\chi(G)$ is the Euler characteristic of the ribbon graph G and n_j is the number of k such that $p_k = j$.

Remark 3.5. From now on, when we use "graph" we always mean "ribbon graph". Since there will be no mention of standard graph, there will be no confusion.

We need to explain what we mean by an automorphism of a ribbon graph. To that end, we will consider a ribbon graph as the following data :

- A finite set even cardinality $H = \{h_1, \dots, h_{2l}\}$ whose elements are called half edges.

- A partition $V = \{V_1, ..., V_n\}$ of H whose element are called vertices
- A cyclic ordering of each V_i (the cyclic ordering at each vertex).
- A partition $E = \{E_1, \ldots, E_l\}$ into sets of size 2 of H whose elements are called edges.

An automorphism of the ribbon graph is then a permutation σ of the set H of half-vertices such that

- The permutation σ descends to a permutation of the set V (i.e. the image of a vertex is vertex).
- The permutation σ preserves the cycling ordering at each vertex.
- The permutation σ descends to permutation of E.

However, for the moment, the important information of this theorem is not so much what are the precise coefficients coming from the contribution of a graph, but just that it is proportional to an even power of N. Now the Euler characteristic of a connected graph is at most 2, thus the Euler characteristic of a graph G with C_G connected components is at most $2C_G$. An easy consequence is :

Proposition 3.6. *Let f be a product of l traces. The leading order in N of $\frac{1}{N^l}\langle f\rangle$ is 0. In other words, $\frac{1}{N^l}\langle f\rangle$ is a power series in N^{-2}.*

Proof. According to Theorem 3.4, the graph contributing to $\langle f\rangle$ are built on l vertices, thus have at most l components. Their Euler characteristic is then at most $2l$. Because each trace carries a factor $\frac{1}{N}$, the quantity we wish to compute is equal to N^{-2l} times the one in Theorem 3.4. This means that leading order contribution from a graph is of order at most $2l - 2l = 0$. □

Corollary 3.7. *For every $l \geqslant 1$ and $l_i \geqslant 0$, the numbers $c_{p_1,p_2,\ldots,p_l}$ defined by*

$$c_{p_1,p_2,\ldots,p_l} := \lim_{N\to\infty} \langle \prod_{i=1}^{l} \frac{1}{N} tr(x^{p_i})\rangle$$

are well defined.

Remark 3.8. The number c_{2k} is the k-th Catalan number $\frac{(2k)!}{(k+1)!k!}$.

Using the loop equations, we get

Corollary 3.9. *For every $k \geqslant 0, n \geqslant 0$ and $a_1, \ldots, a_n \geqslant 1$ we have the following equality*

$$c_{k+1,a_1,\ldots,a_n} = \sum_{l=0}^{k-1} c_{k-1-l,l,a_1,\ldots,a_n}$$

Proof. This is just a reformulation of Proposition 3.3, after dividing by N^{n+2} and taking the limit $N \to \infty$. □

Definition 3.10. Let $f = \prod_{i=1}^{n} \frac{tr(M^{a_i})}{N}$ be a product of traces. Set

$$c_f = c_{a_1,\ldots a_2}.$$

We now see that those numbers are multiplicative :

Proposition 3.11. *Let f, g be two product of traces, then*

$$c_{fg} = c_f c_g$$

Proof. For the sake of the argument, let us assume for the moment that $f = tr(M^a)/N$ for some $a \in \mathbb{N}^*$ and $g = tr(M^b)/N$ for some $b \in \mathbb{N}^*$.

For N fixed, let us then have a look at $\langle N^2 tr(M^a) tr(M^b) \rangle$ and $\langle N tr(M^a) \rangle \langle N tr(M^b) \rangle$. By 't Hooft theorem, the leading order of the expectation value $\langle N^2 tr(M^a) tr(M^b) \rangle$ is given by enumerating graphs of highest possible Euler characteristic built on two vertices of valency a and b. These are the graph with as many connected components as we have vertices, and each of these components are planar graph. This means that the leading order is N^4.

For each of $\langle N tr(M^a) \rangle$ and $\langle N tr(M^b) \rangle$, the leading order is N^2, corresponding to planar graphs (necessarily connected for we are only considering graphs built on a single vertex). Thus the leading order of the product is also N^4.

This means that the coefficient of the leading order of

$$\langle N^2 tr(M^a) tr(M^b) \rangle - \langle N tr(M^a) \rangle \langle N tr(M^b) \rangle \tag{1}$$

is given by :

$$\begin{cases} \sum_G \frac{ab}{|AutG|} - \sum_{(G_1,G_2)} \frac{a}{|AutG_1|} \frac{b}{|AutG_2|} & \text{if } a \neq b \\ \sum_G \frac{2a^2}{|AutG|} - \sum_{(G_1,G_2)} \frac{a}{|AutG_1|} \frac{a}{|AutG_2|} & \text{if } a = b \end{cases}$$

Both for $a \neq b$ and $a = b$, the first sum is over the set of all planar disconnected graph built on two vertices of valency respectively a and b and the second sum is on pairs of planar graph built on one vertices of valency respectively a and b.

If $a \neq b$, there is a bijection between the two sets over which the sums are taken, simply given by connected components. Furthermore, the automorphism group of a disconnected graph is given by the product of the automorphism group of its connected component. So in the end we get that actually the coefficient of N^4 is 0.

In the case where $a = b$, we are in the one of the following two situations. Either the two connected components of the graphs are the same or they are not. In the first case, the size of the automorphism group of the disconnected graph on two vertices is twice the product of the size of the automorphism groups of its connected components. Indeed there is an additional symmetry coming from exchanging the two vertices. In the other case, the graph appears twice in the expansion of the product $\langle N tr(M^a) \rangle \langle N tr(M^a) \rangle$. In both cases, the cases we get that the coefficient of N^4 in equation (1) is 0.

By definition we have :

$$c_{fg} = \lim_{N \to \infty} \frac{1}{N^4} \langle N^2 tr(M^a) tr(M^b) \rangle$$
$$c_f c_g = \lim_{N \to \infty} \frac{1}{N^2} \langle N tr(M^a) \rangle \frac{1}{N^2} \langle N tr(M^b) \rangle$$

and thus $c_{fg} - c_f \, c_g = 0$.

The general case where both f and g are product of k and l traces respectively is treated similarly. The leading order of the corresponding Gaussian expectation values corresponding to c_{fg} is given by the graph with $k + l$ planar connected components and all those graphs are product of graphs built on the vertices coming from f with graphs built on the vertices coming from g. At the leading order, all the contributions get cancelled, and after dividing by $N^{2(k+l)}$ and taking the limit $N \to \infty$ we get that $c_{fg} - c_f c_g = 0$. The proof in more details follows the same argument as the proof of Proposition 4.7, which is a stronger result.

□

Let us now reformulate all of this in terms of the algebra $A = \mathbb{C}\langle x \rangle$ and a map

$$\varphi : Sym|A| \to \mathbb{C}.$$

As we have already seen, an element $(x^{a_1}) \dots (x^{a_n}) \in Sym|A|$ should be thought of as a function on $\mathcal{H}_N$ given by

$$M \mapsto \prod_{i=1}^{n} tr(M^{a_i}).$$

We chose, however, to multiply each trace with a factor $\frac{1}{N}$.

In other words, for any (cyclic) word $W(x)$ we have a function Tr_W on the space of $N \times N$-Hermitian matrices $\mathcal{H}_N$ defined by

$$\begin{aligned} Tr_W : \mathcal{H}_N &\to \mathbb{C} \\ M &\mapsto \frac{1}{N} tr(W(M)). \end{aligned}$$

This defines a linear map $Tr : |A| \to Hom(\mathcal{H}_N, \mathbb{C})$, which can then be extended as map of algebra to $Tr : Sym|A| \to Hom(\mathcal{H}_N, \mathbb{C})$. The integral φ is the map

$$\varphi(f) = c_{Tr(f)}.$$

It is clear how to interpret Proposition 3.11: it just says that φ is a map of algebra.

The next question is how to interpret Corollary 3.9 purely in term of the map φ and the algebra A.

We know that derivations of A should correspond to vector fields on $\mathcal{H}_N$. Note that derivations of A can be extended to derivations of $Sym|A|$ by the Leibniz rule. Here we used that $|A| = A$, but in general any derivation on an algebra $\mathcal{A}$ descends to a map on $|\mathcal{A}|$. The following proposition is an easy computation.

Proposition 3.12. *For $k \geqslant 0$, let u be the unique derivation on A such that $u_k(x) = x^k$. The following diagram is commutative*

$$\begin{array}{ccc} Sym|A| & \xrightarrow{\quad Tr \quad} & \mathcal{C}^\infty(\mathcal{H}_N) \\ \big\downarrow{\scriptstyle u_k} & & \big\downarrow{\scriptstyle \sum_{i,j} \frac{\partial}{\partial M_{ij}} (M^k)_{ij}} \\ Sym|A| & \xrightarrow[\quad Tr \quad]{} & \mathcal{C}^\infty(\mathcal{H}_N) \end{array}$$

By linearity, we can obtain a loop equation like equality for any choice of product of traces and any vector field. Is is also clear that the term

$$\sum_{l=0}^{k-1} c_{k-1-l,l,a_1,\dots,a_n}$$

in Corollary 3.9 is equal to

$$\varphi(Div(u_k) f).$$

where we projected $Div(u_k)$ to $Sym^2|A| \subset Sym|A|$.

Knowing that, we can reinterpret the loop equations as

$$\varphi(-u(V)f + Div(u)f) = 0$$

where $u \in Der(A)$, $f \in Sym|A|$ and $V = \frac{1}{2}(x^2)$. We note that the value of V is now part of the data of our integral. It is called the potential.

Note that with our choice of normalization for a trace, $Tr_{x^0} = 1$ which implies that the map $\varphi : Sym|A| \to \mathbb{C}$ descends to a map $\varphi : Sym|A|/((x^0) - 1_{Sym}) \to \mathbb{C}$.

3.2 More matrix integration

The loop equations (Proposition 3.3) can actually be derived in a broader context, that is for more general potentials.

Set $\mathcal{V}(x) = \frac{x^2}{2} - \sum_{k=3}^{d} \frac{a_k}{k} x^k$ and let $\langle - \rangle_{\mathcal{V}} : \mathcal{C}^{\infty}(\mathcal{H}_N) \to \mathbb{C}$ be defined by

$$f \mapsto \langle f \rangle_{\mathcal{V}} := \frac{1}{Z_{\mathcal{V}}} \int_{\mathcal{H}_N} dM f(M) e^{-N\mathcal{V}(M)}$$

where $Z_{\mathcal{V}}$ is a normalization factor.

Of course, the meaning of such an expression is not quite clear for there might be some convergence issues. There are two options to fix this. The first one is to consider only potentials whose degree d is even and a_d negative, for in that case the integral is convergent.

The second option is to interpret these integrals as a perturbation of the Gaussian integral, and *define* $\langle f \rangle_{\mathcal{V}}$ as a power series in $a_3, \ldots, a_d$. What we mean by that is define

$$Z_{\mathcal{V}} := Z \sum_{n_1,\ldots,n_d \geqslant 0} \langle \prod_{k=3}^{d} (\frac{N a_k x^k}{k})^{n_k} \rangle$$

and then define

$$Z_{\mathcal{V}} \langle f \rangle_{\mathcal{V}} := \sum_{n_1,\ldots,n_d \geqslant 0} \langle f \prod_{k=3}^{d} (\frac{N a_k x^k}{k})^{n_k} \rangle$$

In words, we have interpreted $Z_{\mathcal{V}} \langle f \rangle_{\mathcal{V}}$ as $\langle f e^{N\mathcal{V}(M)} \rangle$ and then formally permute integral and exponential.

For some potentials, both definitions are valid at once. In those cases, denote momentarily the two different definitions $\langle - \rangle_{conv}$ and $\langle - \rangle_{formal}$. In general

$$\langle - \rangle_{conv} \neq \langle - \rangle_{formal}.$$

Furthermore, if denote by Z_{conf} and Z_{formal} the normalizing factors in the two different definitions, both are functions of the $\vec{a} = (a_2, \ldots, a_d)$ but

$$Z_{conv}(\vec{a}) \neq Z_{formal}(\vec{a}).$$

However, both definition satisfy loop equations

Proposition 3.13. *For every $k, n \in \mathbb{N}$ and $b_1, \dots b_n \in \mathbb{N}$*

$$N\langle tr(M^{k-1}\mathcal{V}'(M)) \prod_{i=1}^{n} tr(M^{b_i})\rangle_{\mathcal{V}} = \sum_{l=0}^{k-1}\langle tr(M^{k-1-l})\, tr(M^{l}) \prod_{i=1}^{n} tr(M^{b_i})\rangle_{\mathcal{V}} + \sum_{j=1}^{n} b_j\langle tr(M^{k+b_j-1}) \prod_{i\neq j} tr(M^{b_i})\rangle_{\mathcal{V}}$$

In the case of convergent integrals, it is the same proof as the one we had for Gaussian integration. In the formal interpretation, it follows from the Gaussian case, since loop equations are linear.

One can also compute such integral using ribbon graphs. We shall not delve into the full details, but rather refer to [Pol04] or [Eti24]. Let us still mention that in order the get the coefficient of $a_3^{n_3} \dots a_d^{n_d}$ in

$$\frac{1}{\prod n_j!}\left\langle \prod_{k=1}^{n} N\frac{tr(M^{p_k})}{p_k}\right\rangle_{\mathcal{V}}$$

one should consider graphs built on n_3 vertices of valency a_3, n_4 vertices of valency a_4 and so on together with *labelled* vertices of valency p_k for $k = 1, \dots, n$. One should, however, not consider all such graphs, but only those in which connected component contains at least a labelled vertex. This means that the Euler characteristic of the graphs we are considering is at most N^{2n}, and then every coefficient of

$$\frac{1}{\prod n_j!}\left\langle \prod_{k=1}^{n} \frac{tr(M^{p_k})}{Np_k}\right\rangle_{\mathcal{V}}$$

is a polynomial in N^{-2}.

The goal of this digression is not so much to give a precise account of formal matrix integrals but rather to reassure the reader that the phenomenon we described for Gaussian integration of matrices is not specific to the quadratic potential but a feature of matrix integration in a much broader sense. We refer the reader to [EKR18] for more details.

3.3 Axioms for integrals

We generalize the properties of φ of the previous sections to more variables and any choice of potential V.

Let $A := \mathbb{C}\langle x_1, \dots, x_n\rangle$ be the free associative algebra on n generators and denote by $|A|$ the quotient of A by the subspace of commutators. Finally consider $Sym|A|$, the symmetric algebra on the space $|A|$.

Let us fix once and for all a potential $V \in |A|$.

Definition 3.14. An infinity integral (with respect to the potential V) is a homomorphism of algebra $\varphi : Sym|A|/(|1| - 1_{Sym}) \to \mathbb{C}$ satisfying the loop equation

$$\varphi(-u(V)f + Div(u)f) = 0$$

for every $u \in Der(A)$ and $f \in Sym|A|$.

This definition deserves a bit of an explanation. Since u is a derivation of A, it descends to a well defined map $u : |A| \to |A|$. Now $u(V)$ is an element of $|A|$, hence of $Sym|A|$ and one can then multiply it with $f \in Sym|A|$. Similarly $Div(u)$ is by definition an element of $|A| \otimes |A|$ and can be projected to $Sym^2|A|$ and then multiplied with f. So the expression in the argument of φ is indeed an element of $Sym|A|$.

We have seen that Gaussian integration gives an example of an integral with respect to the potential $V = \frac{1}{2}|x^2|$. We now show that this is actually the *only* example for this potential.

Proposition 3.15. *There exists a unique infinity integral φ for the potential $V = \frac{1}{2}|x^2|$.*

Proof. Since φ is a map of algebra, one has that $\varphi(1) = 1$. We claim that $\varphi(|x^k|)$ is uniquely determined by the loop equations.

Indeed, suppose that all values of $\varphi(|x^i|)$ are uniquely determined by the loop equations for $0 \leqslant i \leqslant k$. We then have

$$\varphi\big(|x^{k+1}|\big) = \varphi\big(u_k(V)\big) = \sum_{l=0}^{k-1} \varphi\big(|x^l|\,|x^{k-1-l}|\big) = \varphi\big(|x^l|\big)\,\varphi\big(|x^{k-1-l}|\big)$$

where u_k is the unique derivation of A sending x to x^k and we have used the loop equation and the fact that φ is a map of algebra. □

4 Power series integrals

By revisiting the motivating example of Gaussian integration of matrices, we define an other version of integrals, this time with values in power series with coefficient in $\mathbb{C}$.

4.1 Back to Gaussian integration

Let us come back to Gaussian integration of Hermitian matrices introduced in Section 3.1. Remember that for any word $W(x) \in A = \mathbb{C}\langle x \rangle$, we associated a function Tr_W on the space of $N \times N$ Hermitian matrices by

$$\begin{aligned} Tr_W : \mathcal{H}_N &\to \mathbb{C} \\ M &\mapsto \frac{1}{N} tr(W(M)). \end{aligned}$$

and we denoted by Tr the linear map $Tr : Sym|A| \to Hom(\mathcal{H}_N, \mathbb{C})$ the map associating to a product of (cyclic) words the product of the corresponding functions. We then had a look at the Gaussian average of such functions. In other words, we were interested in the functional $\langle - \rangle : Sym|A| \to \mathbb{C}$ defined by

$$\langle Tr_f \rangle = \frac{1}{Z} \int_{\mathcal{H}_N} dM \; Tr_f(M) \; e^{-N\frac{tr(M^2)}{2}}$$

We saw in Section 3.1 that according to 't Hooft Theorem (Theorem 3.4) $\langle Tr(f) \rangle$ is given by a Laurent series in N^2, and in Proposition 3.6 we've seen that actually there are only negative powers of N^2. Let us then interpret those Gaussian averages as power series in N^{-2}:

Definition 4.1. Let $\varphi_{ps} : Sym|A| \to \mathbb{C}[[N^{-2}]]$ be the map that to a product of word associates the power series of the Gaussian average of the corresponding traces. In formula

$$\varphi_{ps}(f) := \langle Tr_f \rangle = \frac{1}{Z} \int_{\mathcal{H}_N} dM \; Tr_f(M) \; e^{-N\frac{tr(M^2)}{2}}$$

Let us us now reinterpret Proposition 3.3 in term of φ_{ps}. It was saying that for every $k, n \in \mathbb{N}$ and $a_1, \ldots a_n \in \mathbb{N}$

$$\begin{aligned} N\Big\langle tr(M^{k+1}) \prod_{i=1}^{n} tr(M^{a_i}) \Big\rangle = & \sum_{l=0}^{k-1} \Big\langle tr(M^{k-1-l}) \; tr(M^l) \prod_{i=1}^{n} tr(M^{a_i}) \Big\rangle \\ & + \sum_{j=1}^{n} a_j \Big\langle tr(M^{k+a_j-1}) \prod_{i \neq j} tr(M^{a_i}) \Big\rangle \end{aligned} \quad (2)$$

We saw that the different terms of Equation (2) could be identified as images of elements of $Sym|A|$ by the map φ_{ps}. Indeed, let u be the unique derivation of A sending x to x^k and let $V = \frac{1}{2}|x^2|$, we had

- $\big\langle tr(M^{k+1}) \; \prod_{i=1}^{n} tr(M^{a_i}) \big\rangle = \varphi_{ps}\Big(u(v) \prod_{i=1}^{n} |x^{a_i}| \Big)$
- $\sum_{l=0}^{k-1} \big\langle tr(M^{k-1-l}) \; tr(M^l) \; \prod_{i=1}^{n} tr(M^{a_i}) \big\rangle = \varphi_{ps}\Big(Div(u) \prod_{i=1}^{n} |x^{a_i}| \Big)$

For the last term, an easy computation shows that

- $\sum_{j=1}^n a_j \langle tr(M^{k+a_j-1}) \prod_{i\neq j} tr(M^{a_i})\rangle = \varphi_{ps}\Big(u\big(\prod_{i=1}^n |x^{a_i}|\big)\Big)$

By linearity, we have proved that the power series version of Gaussian integration satisfies an other version of the loop equation :

Proposition 4.2. *For every $u \in Der(A)$ and $f \in Sym|A|$, one has*

$$\varphi_{ps}(-u(V)f + Div(u)f + \frac{1}{N^2}u(f)) = 0$$

Remark 4.3. One can decompose φ_{ps} as

$$\varphi_{ps} = \sum_{k=0} \varphi_k N^{-2k}$$

with $\varphi_k : Sym|A|/(|1| - 1_{Sym}) \to \mathbb{C}$. We have $\varphi_0 = \varphi$ the integral constructed in Section 3.1

Of course it is totally unreasonable to expect φ_{ps} to be a map of algebra. For example,

$$\varphi_{ps}(|x|) = 0$$
$$\varphi_{ps}(|x|^2) = \frac{1}{N^2}$$

Nevertheless, it turns out that there are still relations between the different products. Those relations are best expressed through *cumulants*:

Definition 4.4. The n-cumulant is the map

$$\varphi_c : Sym^n|A| \to \mathbb{C}[[N^{-2}]]$$

defined recursively by the formula:

$$\varphi_{ps}\Big(|x^{a_1}|\dots|x^{a_n}|\Big) = \sum_\pi \prod_{j=1}^{l(\pi)} \varphi_c\big(\prod_{i\in B_j}|x^{a_i}|\big)$$

where π is a partition of $\{1,\dots,n\}$ into $l(\pi)$ subsets denoted by $B_1,\dots,B_{l(\pi)}$.

Example 4.5. For example, the first two cumulants are given by

$$\varphi_c(|x^a|) = \varphi_{ps}(|x^a|)$$
$$\varphi_c(|x^a|\,|x^b|) = \varphi_{ps}(|x^a|\,|x^b|) - \varphi_{ps}(|x^a|)\,\varphi_{ps}(|x^b|)$$

Remark 4.6. Alternatively, one could also define the n-cumulant as

$$\varphi_c\Big(|x^{a_1}|\dots|x^{a_n}|\Big) = \sum_\pi (-1)^{l(\pi)-1}(l(\pi)-1)!\prod_{j=1}^{l(\pi)} \varphi_{ps}\big(\prod_{i\in B_j}|x^{a_i}|\big)$$

Since the contribution a disconnected graph is given by the product of the contributions of its connected components, an argument in the same spirit as the one we used to prove that the leading order of Gaussian integration is multiplicative can be used to show that cumulants can be computed using 't Hooft Theorem by considering only *connected* graphs:

Proposition 4.7.

$$\frac{1}{\prod n_j!}\varphi_c\left(\prod_{k=1}^{n}\frac{|x^{a_k}|}{a_k}\right) = \frac{1}{N^{2n}} \sum_{\substack{\textit{connected graph } G\\ \textit{with vertices of valency } p_k}} \frac{N^{\chi(G)}}{|Aut(G)|}$$

where all the coefficient are the same as in 't Hooft Theorem (Theorem 3.4).

Proof. (Sketch) Since the sum in Definition 4.4 is over all possible partitions, the n-cumulant is a symmetric linear function in its n argument, it enough to understand the situation when all the a_k are equal to the same number a.

If $a_1 = a_2 \cdots = a_n = a$, we have the following equality of formal power series in t:

$$\sum_{n=1}^{\infty} \frac{\varphi_c(|x^a|^n)}{n!} t^n = log\Bigg(\varphi_{ps}\big(exp(t|x^a|)\big)\Bigg). \tag{3}$$

We shall now see that

$$\varphi_c\Bigg((|x^a|^n\Bigg) = \frac{a^n n!}{N^{2n}} \sum_{\substack{\text{connected graph G with}\\ n \text{ vertices of valency } a}} \frac{N^{\chi(G)}}{|Aut(G)|}$$

satisfies Equation (3).

This is easier seen by first applying *exp* to Equation 3. The coefficient of t^n in the left hand side is then given by :

$$\begin{aligned} &\sum_{l\geqslant 0} \sum_{k_1,\ldots,k_l} \sum_{\substack{n_1,\ldots,n_l\\ \sum k_i n_i = n}} \frac{1}{\prod_i k_i!}\left(\frac{\varphi_c(|x^a|^{n_i})}{n_i!}\right)^{k_i} \\ =&\sum_{l\geqslant 0} \sum_{k_1,\ldots,k_l} \sum_{\substack{n_1,\ldots,n_l\\ \sum k_i n_i = n}} \frac{1}{\prod_i k_i!}\left(\frac{a^{n_i}}{N^{2n_i}} \sum_{\substack{\text{connected graph G with}\\ n_i \text{ vertices of valency } a}} \frac{N^{\chi(G)}}{|Aut(G)|}\right)^{k_i} \end{aligned} \tag{4}$$

If we show that this equal to

$$\frac{a^n}{N^{2n}} \sum_{\substack{\text{graph G with}\\ n \text{ vertices of valency } a}} \frac{N^{\chi(G)}}{|Aut(G)|}$$

we are done. Clearly there is no issue with the prefactors given by powers of a and N; the real content is the part coming from the graphs.

Given a graph any G build of n vertices of valency a, say it has k_1^G connected components with n_1^G vertices, k_2^G connected components with n_2^G vertices, ... , $k_{l_G}^G$ connected components with $n_{l_G}^G$ vertices.

The contribution coming from this graph will be equal to the contribution coming by picking each connected component in the term of (4) corresponding to $l = l_G$, $n_i = n_i^G, k_i = k_i^G$

Indeed the power of N is the right one for the Euler characteristic of graph is given by the sum of the Euler characteristic of its connected components. Let us now have a look at the k_i^G

connected components of G with n_i^G vertices. Say there are g_1 of those connected components that are the same graph G_1, g_2 of those connected components are the same graph G_2,..., g_r of those connected components are the same graph G_r. This means that the product of graphs $G_1^{g_1} \dots G_r^{g_r}$ will appear

$$\binom{k_i}{g_1, \dots, g_r}$$

times in

$$\left(\sum_{\substack{\text{connected graph G with} \\ n_i \text{ vertices of valency } a}} \frac{N^{\chi(G)}}{|Aut(G)|} \right)^{k_i}.$$

But also, since you have connected components of G that are the same, the size of the automorphism group of the part of G coming from those connected components composed of n_i^G vertices is equal to

$$\prod_{j=1}^{r} g_r! |Aut(G_r)|^{g_r}$$

Doing this analysis for all the $1 \leqslant i \leqslant l_G$ we get that the factor

$$\prod_i \frac{1}{k_i^G!}$$

together with the contribution of all those product coming the connected components of G gives exactly

$$\frac{N^{\chi(G)}}{|Aut(G)|}$$

□

Corollary 4.8. *For every n and $a_1, \dots, a_n \geqslant 1$, the leading order in N^{-2} of the n-cumulant $\varphi_c(|a_1| \dots |a_n|)$ is $n-1$.*

Proof. Since all the graphs G contributing to the cumulant are connected, their Euler characteristic is at most 2. Together with the factor $1/N^{2n}$ in front, the leading order in N^2 is indeed $n-1$. □

Remark 4.9. Corollary 4.8 implies that the map φ of Section 3.1 is a map of algebra. This is actually a stronger result.

4.2 Power series valued integrals

Just as before, let $A = \mathbb{C}\langle x_1, \dots, x_n \rangle$. The example of Gaussian integration motivates the definition of a power series version of integrals. The parameter $\hbar$ should be thought of as N^{-2}. The definition will make use of the notion of cumulant. For the sake of clarity, we rewrite this definition here:

Definition 4.10. Let $\varphi_\hbar : Sym|A| \otimes \mathbb{C}[[\hbar]]$ be a $\mathbb{C}[[\hbar]]$-linear map. The k-cumulant of $\varphi_\hbar$ is the map $\varphi_c : Sym^k|A| \to \mathbb{C}[[\hbar]]$ defined by

$$\varphi_\hbar(f_1 \dots f_k) = \sum_\pi \prod_{j=1}^{l(\pi)} \varphi_c\big(\prod_{i\in B_j} f_i\big)$$

where π is a partition of $\{1, \dots, k\}$ into $l(\pi)$ subsets denoted by $B_1, \dots, B_{l(\pi)}$.

Definition 4.11. A power series integral (with respect to the potential V) is a $\mathbb{C}[[\hbar]]$ linear map $\varphi_\hbar : Sym|A|/(|1| - 1_{Sym}) \otimes \mathbb{C}[[\hbar]] \to \mathbb{C}[[\hbar]]$ satisfying the following conditions:

- $\varphi_\hbar(1) = 1$
- $\varphi_\hbar$ satisfies the loop equation :
$$\varphi_\hbar(-u(V)f + Div(u)f + \hbar\, u(f)) = 0 \tag{5}$$
for every $u \in Der(A)$ and $f \in Sym|A| \otimes \mathbb{C}[[\hbar]]$.
- For every $k \in \mathbb{N}$, the k-cumulant $\varphi_c : Sym^n|A| \to \mathbb{C}[[\hbar]]$ of $\varphi_\hbar$ factors through $\hbar^{k-1}\mathbb{C}[[\hbar]]$

Because of $\mathbb{C}[[\hbar]]$ linearity, a power series integral $\varphi_\hbar$ is entirely determined by its value on the subalgebra $Sym|A|/(|1| - 1_{Sym})$. We shall denote by the same symbol $\varphi_\hbar$ the power series integral restricted to $\mathbb{C}$ linear map from $Sym|A|/(|1| - 1_{Sym})$ to $\mathbb{C}[[\hbar]]$.

Thus, a power series integral $\varphi_\hbar$ can be decomposed as :

$$\varphi_\hbar = \varphi_0 + \varphi_1\, \hbar + \varphi_2\, \hbar^2 + \cdots$$

where $\varphi_i : Sym|A| \to \mathbb{C}$ is a $\mathbb{C}$ linear map. Unwinding the two defining conditions of a power series integral and looking at the coefficient of $\hbar^0$, one sees that φ_0 is a integral in the sense of Definition 3.14. In other words, for any power series integral $\varphi_\hbar$ the diagram

$$\begin{array}{ccc} Sym|A|/(|1| - 1_{Sym}) \otimes \mathbb{C}[[\hbar]] & \xrightarrow{\varphi_\hbar} & \mathbb{C}[[\hbar]] \\ \downarrow{\scriptstyle \hbar\mapsto 0} & & \downarrow{\scriptstyle \hbar\mapsto 0} \\ Sym|A|/(|1| - 1_{Sym}) & \xrightarrow[\varphi_0]{} & \mathbb{C} \end{array}$$

commute and φ_0 is an infinity integral when $\varphi_\hbar$ is a power series integral.

This begs the following question: given an infinity integral $\varphi_0 : Sym|A| \to \mathbb{C}$ can it always be extended to a power series integral? If yes, is the extension unique? How much do the answers to those questions depend on the potential V?

We will see that it will be fruitful to actually reinterpret the loop equations as the differential of a well chosen chain complex.

5 Cohomological reformulation of integrals

We wish to reinterpret both infinity and power series integrals in a cohomological setting. We begin by given some motivation coming from commutative integration. We then present the algebraic tools needed in the non-commutative setting and finally define a BV-algebra related to integrals.

5.1 Commutative integration cohomologically

As a motivation for what is to come, we explain how integration on manifolds can be understood in a cohomological setting. We have learned this point of view from [Gwi12].

Let M be a smooth, compact, closed, oriented manifold of dimension n. In this nice setting, we can integrate any top forms and thus have a linear map

$$\int_M : \Omega^n(M) \longrightarrow \mathbb{R}.$$

Stokes' theorem is then equivalent to saying that the linear map $\int_M$ descends to a map

$$\int_M : H^n_{dR}(M) \longrightarrow \mathbb{R}.$$

This suggest that we could view the space $H^n_{dR}(M)$ as the space of integrals on M. Fix now a volume form μ on M. Given a function $f \in \mathcal{C}^\infty(M)$, we can multiply it with μ to obtain an other top form $f\mu$. In other words, we have a map

$$\begin{aligned} m_\mu : \mathcal{C}^\infty(M) &\longrightarrow \Omega^n(M) \\ f &\longmapsto f\mu \end{aligned}$$

We can do a similar thing for polyvector fields

$$\begin{aligned} m_\mu : \bigwedge^k T_M &\longrightarrow \Omega^{n-k}(M) \\ X &\longmapsto \iota_X \mu \end{aligned}$$

where $\iota_X\mu$ denotes the contraction of the n form μ with the k polyvector fields X. This means that altogether we have a map

$$\begin{aligned} m_\mu : \bigwedge T_M &\longrightarrow \Omega^\bullet(M) \\ X &\longmapsto \iota_X \mu \end{aligned}$$

As μ is a volume form, it is nowhere vanishing and the map m_μ has an inverse m_μ^{-1} which allows us to transport the de Rham differential d on $\Omega^\bullet(M)$ to $\bigwedge T_M$, i.e. we define a map

$$\Delta_\mu := m_\mu^{-1} \circ d \circ m_\mu.$$

It is at least clear what Δ_μ does to vector field. Indeed, for a vector field $X \in T_M$, $\Delta_\mu(X)$ is the unique function such that

$$(\Delta_\mu(X))\mu = d \circ \iota_X \mu,$$

which is exactly the definition of $Div_\mu X$, the divergence of X with respect to the volume form μ. In short, we have a chain complex $(\bigwedge T_M, \Delta_\mu)$ isomorphic to the de Rham complex in which Δ_μ extends the divergence of vector fields to polyvector fields. Saying that

$$\bigwedge T_m = SymT_M[1]$$

is a mild reformulation that brings the chain complex that we constructed closer to the form of other construction that will appear in the non-commutative setting. Note that framed like this, everything is shifted to the left, and the complex is concentrated between degrees $-n$ and 0. The space of integrals should then be zeroth cohomology of that complex.

Finally, let us pay attention to the case $M = \mathbb{R}^n$. Here there is a preferred volume form, namely $\mu_0 := dx_1 \wedge \cdots \wedge dx_n$. To simplify notations, denote the vector field $\frac{\partial}{\partial x_i}$ by η_i. With this notation, the map m_{μ_0} is the map

$$\eta_{i_1} \wedge \cdots \wedge \eta_{i_k} \mapsto \pm dx_1 \wedge \cdots \wedge \widehat{dx}_{i_1} \wedge \cdots \wedge \widehat{dx}_{i_k} \wedge \cdots \wedge dx_n$$

and Δ_{μ_0} is given by

$$\Delta_{\mu_0}(f\ \eta_{i_1} \wedge \cdots \wedge \eta_{i_k}) = \sum_i \pm \frac{\partial f}{\partial x_i} \eta_{i_1} \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_{i_k}$$

This can be written in a much more economic manner as

$$\Delta_{\mu_0} = \sum_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial \eta_i}$$

(one should not worry about the signs once everything is understood in a graded setting). If we have an other arbitrary volume form μ, we can express it as

$$\mu = e^{-V} \mu_0$$

for some function V. Let us compute Δ_μ. We have

$$\begin{aligned} d \circ m_\mu(f\ \eta_{i_1} \wedge \cdots \wedge \eta_{i_k}) &= \pm d\big(f\ e^{-S} \mu_0 \backslash \{dx_{i_1}, \ldots, dx_{i_k}\}\big) \\ &= \sum_i \pm \big(\frac{\partial f}{\partial x_i} e^{-V} - f\ \frac{\partial V}{\partial x_i} e^{-V}\big) dx_i \wedge \mu_0 \backslash \{dx_{i_1}, \ldots, dx_{i_k}\} \end{aligned}$$

And thus

$$\Delta_\mu(f\ \eta_{i_1} \wedge \cdots \wedge \eta_{i_k}) = \sum_i \pm \big(\frac{\partial f}{\partial x_i} e^{-S} - f\ \frac{\partial V}{\partial x_i} e^{-S}\big) \eta_{i_1} \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_{i_k}$$

Written more elegantly, we have

$$\begin{aligned} \Delta_\mu &= \Delta_{\mu_0} - \sum_i \frac{\partial V}{\partial x_i} \frac{\partial}{\partial \eta_i} \\ &= \Delta_{\mu_0} - \{V, -\} \end{aligned}$$

where $\{-, -\}$ is the Poisson bracket on $SymT_M[1]$ given by

$$\{F, G\} := \sum_i \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial \eta_i} - \frac{\partial F}{\partial \eta_i} \frac{\partial G}{\partial x_i}$$

From this study in $M = \mathbb{R}^n$, or one should say in local coordinates, we see that Δ_μ is given by an universal differential operator of order 2 perturbed by a differential operator of order 1 itself coming from the order 2 operator. It is good to keep that in mind when we delve into the non-commutative analogue of this construction.

5.2 General facts about BV-algebra

The algebraic structure on polyvector fields described in the previous section is an example of a Batalin-Vilkovisky algebra (BV-algebra). Here is the general definition

Definition 5.1. A BV-algebra (B, Δ) is a graded commutative algebra V equipped with a degree 1, square zero map $\Delta : B \to B$ satisfying the so called 7 term equation:

$$\begin{aligned}\Delta(abc) - \Delta(ab)c - (-1)^{a(b+c)}\Delta(bc)a - (-1)^{c(a+b)}\Delta(ca)b \\ + \Delta(a)bc + (-1)^{a(b+c)}\Delta(b)ca + (-1)^{c(a+b)}\Delta(c)ab = 0\end{aligned}$$

for a, b, c homogeneous elements. Note that setting $a = b = c = 1$ in the above equation yields $\Delta(1) = 0$. The map Δ is called the *BV operator*.

Remark 5.2. Operator satisfying the 7 term equation are called "differential operator of order at most 2". For example any derivation satisfies the 7 term equation, and if u_1 and u_2 are two derivation of the algebra B, their composition $u_1 \circ u_2$ is also a differential operator of order at most 2.

From the BV operator Δ one can define a bracket on B by the formula

$$[a, b] = (-1)^a \Delta(ab) - (-1)^a \Delta(a)b - a\Delta(b).$$

This bracket has the following good properties. A proof can be found in [Get94]

Proposition 5.3. *Let (B, Δ) be a BV-algebra, the map*

$$a \otimes b \mapsto [a, b] = (-1)^a \Delta(ab) - (-1)^a \Delta(a)b - a\Delta(b)$$

for a, b homogeneous elements and extended linearly has the following properties :

- *The bracket $[-.-]$ is a map of degree 1.*
- *For a, b homogeneous, $[a, b] = -(-1)^{(a+1)(b+1)}[b, a]$.*
- *For a, b, c homogeneous*

$$[a, [b, c]] = [[b, a], c]] + (-1)^{(a+1)(b+1)}[b, [a, c]]$$

- *For a, b homogeneous $\Delta[a, b] = [\Delta(a), b] + (-1)^{a+1}[a, \Delta(b)]$*
- *For a, b, c homogeneous $[a, bc] = [a, b]c + (-1)^{(a+1)b}b[a, c]$.*

Remark 5.4. The first and last property in the previous proposition says that for every $a \in B$ homogeneous of degree k, the map $b \mapsto [a, b]$ is a derivation of degree $k + 1$.

Remark 5.5. Note that in general the operator Δ fails to be a derivation. The bracket associates to Δ encodes this failure. In particular, if Δ is a derivation the bracket is 0.

Proposition 5.6. *Let (B, Δ) be a BV-algebra and let $a \in B$ be a homogeneous element of degree 0 such that $[a, a] = 0$. Then the derivation $[a, -]$ is a degree 1 map that squares to zero, i.e. a differential.*

Proof. The map $[a, -]$ is of degree $+1$ by remark 5.4. Using the graded Jacobi identity of the bracket one has for $b \in B$:

$$\begin{aligned}[a,[a,b]] &= [[a,a],b] - [[a,[a,b]] \\ &= -[a,[a,b]]\end{aligned}$$

hence $[a,[a,b]] = 0$. That is $[a,-]^2 = 0$.

□

Proposition 5.7. *Let (B,Δ) be a BV-algebra and let $a \in B$ be a homogeneous element of degree 0 such that $[a,a] = 0$ and $\Delta(a) = 0$. Then the degree 1 map $\Delta + [a,-]$ is a BV operator.*

Proof. It is easy to check that any differential is a BV operator. By the above proposition, $[a,-]$ is a differential and since Δ is also a BV operator, their sum $\Delta + [a,-]$ satisfies the 7 term equation.

We now check that $\Delta + [a,-]$ squares to zero. We already know that both Δ and $[a,-]$ square to zero so we only have to check that $\Delta \circ [a,-] + [a,-] \circ \Delta = 0$ if $\Delta(a) = 0$. One has for every $b \in B$:

$$\begin{aligned}(\Delta \circ [a,-] + [a,-] \circ \Delta)(b) &= \Delta([a,b]) + [a,\Delta(b)] \\ &= [\Delta(a),b] - [a,\Delta(b)] + [a,\Delta(b)] \\ &= 0.\end{aligned}$$

□

Remark 5.8. Since $\Delta + [a,-]$ differs from Δ by a derivation, the associated bracket does not change.

Example 5.9. Let $(\mathfrak{g},[-,-],\delta)$ be a be an involutive Lie bialgebra. Then the exterior algebra $\Lambda\mathfrak{g}$ admits a BV-operator Δ defined by

$$\begin{aligned}\Delta(x_1 \ldots x_n) &= \sum_{i<j} (-1)^{i+j} [x_i,x_j] x_1 \ldots \hat{x_i} \ldots \hat{x_j} \ldots x_n \\ &+ \sum_i (-1)^i \delta(x_i) x_1 \ldots \hat{x_i} \ldots x_n.\end{aligned}$$

One can check that $[x,y]_{BV} = [x,y]$. See Proposition 5.13 for the proof of a more general statement.

This example can extended to a graded setting:

Example 5.10. Let $(\mathfrak{g},[-,-],\delta)$ be a be an involutive graded Lie bialgebra (in particular both maps $[-,-]$ and δ are of degree 0). Then the graded symmetric algebra on the shifted space $Sym(\mathfrak{g}[1])$ becomes a BV algebra with BV operator

$$\begin{aligned}\Delta(x_1 \ldots x_n) &= \sum_{i<j} (-1)^{(x_1+\cdots+x_{i-1})x_i + (x_1+\cdots+x_{j-1})x_j + x_i x_j + x_i} [x_i,x_j] x_1 \ldots \hat{x_i} \ldots \hat{x_j} \ldots x_n \\ &+ \sum_i (-1)^{(x_1+\cdots+x_{i-1})x_i + x_i} \delta(x_i) x_1 \ldots \hat{x_i} \ldots x_n.\end{aligned}$$

If $\mathfrak{g}$ is concentrated in degree 0, one recovers the previous example.

For later purposes, we wish to do the same when we have a graded vector space V equipped with a bracket and a cobracket both of degree 1. That is, build a BV operator on the graded symmetric algebra $SymV$ out of the (co-)bracket.

Definition 5.11. Let $(V, [-,-], \delta)$ be a graded vector space together with two degree 1 maps $[-,-] : V \otimes V \to V$ and $\delta : V \to V \otimes V$. We denote by br and $\tilde{\delta}$ the endomorphisms of $SymV$ defined on product of homogenous elements by

$$br(v_1 \dots v_n) = \sum_{i<j} (-1)^{(v_1+\dots+v_{i-1})v_i + (v_1+\dots+v_{j-1})v_j + v_i v_j} [v_i, v_j] v_1 \dots \hat{v_i} \dots \hat{v_j} \dots v_n$$
$$\tilde{\delta}(v_1 \dots v_n) = \sum_{i<j} (-1)^{(v_1+\dots+v_{i-1})v_i} \delta(v_i) v_1 \dots \hat{v_i} \dots v_n$$

The following proposition due to Perry and Pulman [PP24] explains what are the Lie bialgebra like conditions required for $\Delta = br + \tilde{\delta}$ to be a BV operator:

Proposition 5.12. *In the same setting as above, the map $\Delta = br + \tilde{\delta}$ is a BV operator if and only if the following conditions are fulfilled:*

- $[-,-] = [-,-] \circ \tau$
- $[-,-] \circ ([-,-] \otimes Id) \circ Cyc = 0$
- $\delta = \tau \circ \delta$
- $Cyc(\delta \otimes Id) \circ \delta = 0$
- $[-,-] \circ \delta = 0$
- $\delta \circ [-,-] = (-1)^{x+1}[x, \delta(y)]^{(2)} + (-1)^{(x+1)y+1}[y, \delta(x)]^{(2)}$

Proposition 5.13. *Let $[-,-]$ and δ be as above and denote by Δ the corresponding BV operator. Let us also denote momentarily by $[-,-]_{BV}$ the bracket on $SymV$ obtained from Δ. For $a, b \in V$ one has*

$$[a,b]_{BV} = (-1)^a [a,b].$$

Proof. This is a straightforward computation:

$$\begin{aligned} [a,b]_{BV} &= (-1)^a \Delta(ab) - (-1)^a \Delta(a) b - a\Delta(b) \\ &= (-1)^a [a,b] + (-1)^a \delta(a) b + (-1)^{a+ab} \delta(b) a - (-1)^a \delta(a) b - a\delta(b) \\ &= (-1^a)[a,b] \end{aligned}$$

where we used that $d(\delta(b)) = d(b) + 1$. □

5.3 Double bracket

In [dB04], Van den Bergh defines double bracket as map $A \otimes A \to A \otimes A$ for an associative algebra A satisfying some axioms. From a double bracket, one can obtain a Lie bracket on the space $|A|$. Furthermore, a double bracket induce Poisson bracket on the $Rep(A, End(V))$.

We present here a slight modification of the construction of double Gerstenhaber algebra found in that same article. This modification is more suited to our purpose for it will produce degree 1 brackets satisfying the properties described in Proposition 5.12

In this section fix once and for all a graded algebra A and we denote by μ its multiplication map.

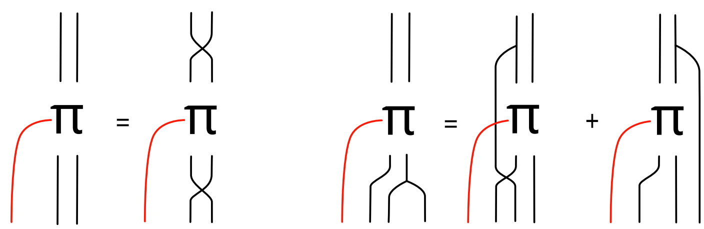
Definition 5.14. A double bracket on the graded algebra A is a degree one map

$$\Pi(-,-) : A \otimes A \to A \otimes A$$

such that

- $\Pi(a,b) = (-1)^{ab}\ \tau(\Pi(b,a))$
- $\Pi(a,bc) = (-1)^{(a+1)b}\ b\Pi(a,c) + \Pi(a,b)c$.

Throughout this section we will use graphical calculus to prove different identities. All diagrams go from bottom to top. The two properties of the map $\Pi(-,-)$ can then be represented by



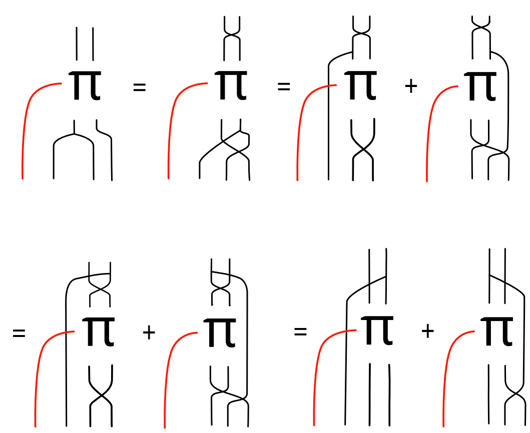
Note the red line which represents the fact that $\Pi(-,-)$ is a degree 1 map.

As a first exercise in graphical calculus we get :

Proposition 5.15. *Let $\Pi(-,-)$ be a double bracket. For a,b,c homogeneous element of A we have*

$$\Pi(ab,c) = (-1)^{a}\ a * \Pi(b,c) + (-1)^{bc}\ \Pi(a,c) * b$$

Proof. This is just the following simple graphical computation:



□

Putting together the two derivation like property of the double bracket, we get the following formula to compute the double bracket :

Proposition 5.16. *Let $a = a_1 \cdots a_n$ and $b = b_1 \cdots b_n$ be two elements of A. Then*

$$\Pi(a,b) = \sum_{i=1}^{n} \sum_{j=1}^{m} \pm\, a_1 \cdots a_{i-1} * (b_1 \cdots b_{j-1}\, \Pi(a_i, b_j)\, b_{j+1} \cdots b_m) * a_{i+1} \cdots a_n$$

where the sign is $(-1)^{(a_1+\cdots+a_{i-1})+(a_{i+1}+\cdots+a_n)b+(b_1+\cdots+b_{j-1})(a_i+1)}$.

Definition 5.17. Let $\Pi(-,-)$ be a double bracket and define the map

$$[-,-]_\Pi := \mu \circ \Pi(-,-) : A \otimes A \to A.$$

This map is the bracket associated to the double bracket $\Pi(-,-)$.

If the context is clear, we will simply write $[-,-]$ instead of $[-,-]_\Pi$

The following two lemmas follows from the definition of a double bracket and are totally straightforward:

Lemma 5.18. *Let $\Pi(-,-)$ be a double bracket and let $a \in A$ be an homogeneous element of degree $d(a)$. The associated bracket $[a,-] : A \to A$ is a derivation of degree $d(a)+1$*

Lemma 5.19. *For $a, b \in A$ we have the following equality*

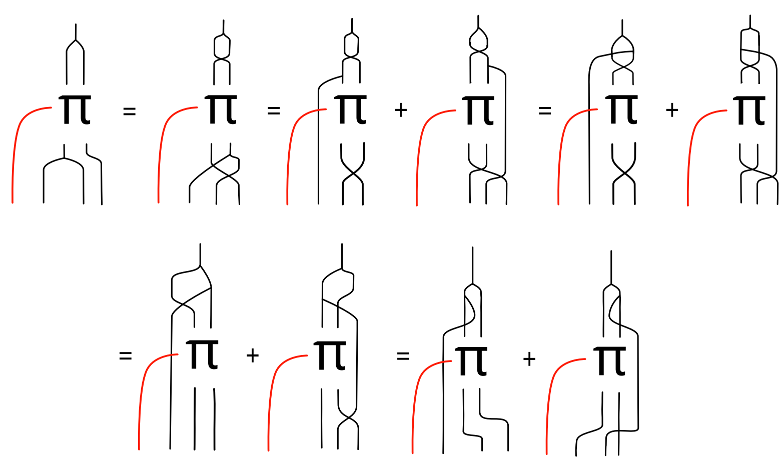
$$[a,b] = (-1)^{ab}\, [b,a] \text{ in } |A|.$$

We also need a last lemma which requires a little bit of work :

Lemma 5.20. *Let $\Pi(-,-)$ be a double bracket and let a, b, c be homogeneous element in A then we have the following equality for the associated bracket:*

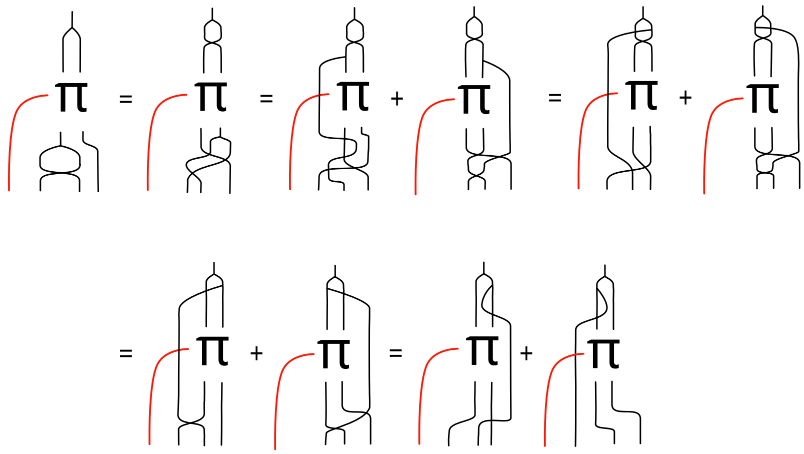
$$[ab, c] = (-1)^{ab}\, [ba, c].$$

Proof. We compute both side of the equation graphically. On the one hand, for the term $[ab, c]$ we get:



where the used the associativity of multiplication in the last equality.

On the other hand, for the term $(-1)^{ab}\, [ba, c]$ we get :



□

Proposition 5.21. *Let* $\Pi(-,-)$ *be a double bracket. Then the associated bracket* $[-,-]$ *descends to maps*

- $|A| \otimes A \to A$
- $|A| \otimes |A| \to |A|$.

Furthermore, if we also denote by $[-,-]$ *those two maps, we have the following equality for* $a, b \in A$

$$[|a|, |b|] = (-1)^{ab}\, [|b|, |a|].$$

Proof. The first map is well defined by 5.20. For the second map, Lemma 5.19 and 5.20 together give us

$$|[a, bc]| = (-1)^{bc}\, |[a, cb]|.$$

The graded symmetry of the second map follows from Lemma 5.19. □

We have seen what are sufficient conditions on the double bracket to get a graded symmetric bracket. We now have a look at the Jacobi identity.

Definition 5.22. Let $\Pi(-,-)$ be a double bracket. We define $\Pi(-,\Pi(-,-))_L$ (respectively $\Pi(-,\Pi(-,-))_R$) by the left (respectively right) diagram:

Definition 5.23. Given a double bracket $\Pi(-,-)$, define $\Pi(-,-,-) : A \otimes A \otimes A \to A \otimes A \otimes A$ by

$$\Pi(-,-,-) := \Pi(-,\Pi(-,-))_L + \xi \circ \Pi(-,\Pi(-,-))_L \circ \xi^{-1} + \xi^2 \circ \Pi(-,\Pi(-,-))_L \circ \xi^{-2}$$

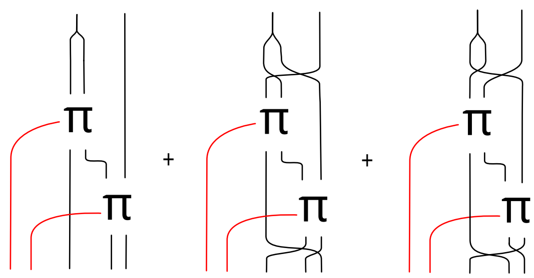
We have some kind of prototype of the Jacobi identity:

Proposition 5.24. *For $a, b, c \in A$ homogeneous we have the following equality in $A \otimes A$:*

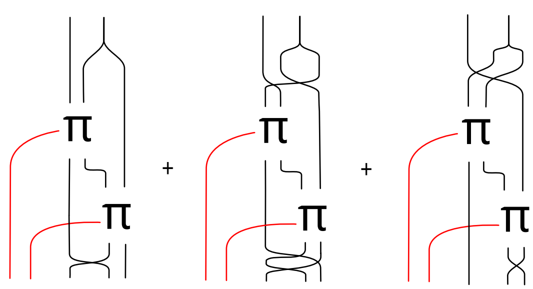
$$(-1)^a \; [a, \Pi(b,c)]^{(2)} + \Pi([a,b],c) + (-1)^{(a+1)b} \; \Pi(b,[a,c])$$
$$= (\mu \otimes Id) \; \Pi(a,b,c) + (-1)^{ab} \; (Id \otimes \mu) \; \Pi(b,a,c).$$

Proof. We first compute (graphically) the right hand side of the equation.

For the term $(\mu \otimes Id) \; \Pi(a,b,c)$ we get:

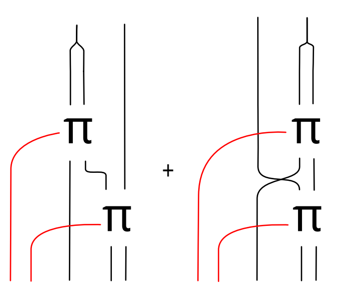


For the term $(-1)^{ab} \; (Id \otimes \mu) \; \Pi(b,a,c)$ we get :

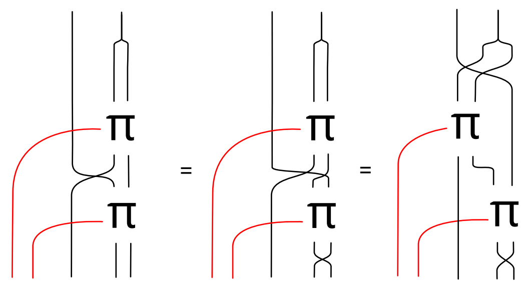


We now compute the term on the left hand side of the equation.

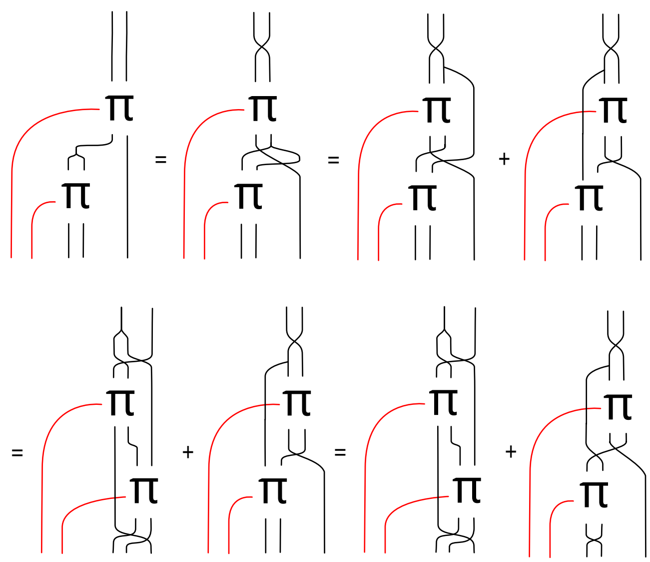
- For $(-1)^a \; [a, \Pi(b,c)]^{(2)}$ we get



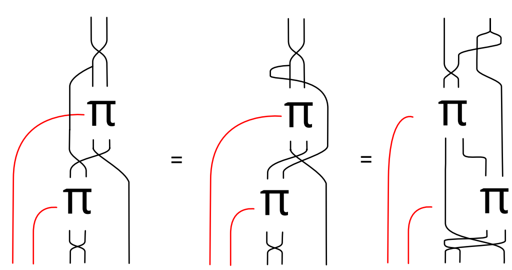
The first summand corresponds to the first summand of $(\mu \otimes Id) \; \Pi(a,b,c)$. The second summand is equal to the third summand of $(-1)^{ab} \; (Id \otimes \mu) \; \Pi(b,a,c)$. Indeed:



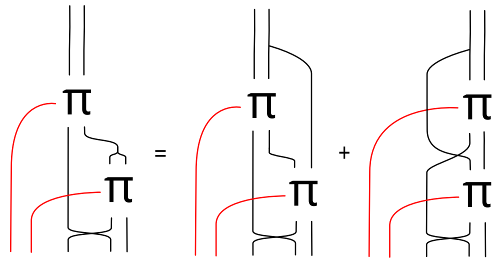
- For $\Pi([a,b],c)$ we get using the derivation like property :



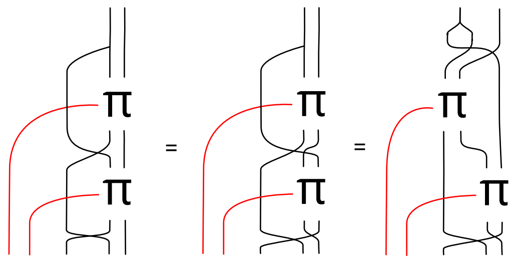
The term on the left is equal to the second term of $(\mu \otimes Id)\ \Pi(a,b,c)$, while a bit more work shows that the term on the right is equal to the second term of $(-1)^{ab}\ (Id \otimes \mu)\ \Pi(b,a,c)$:



- For $(-1)^{(a+1)b}\ \Pi(b,[a,c])$ we get using once more the derivation like property:



We see that the left term is equal to the first term in $(-1)^{ab}\,(Id \otimes \mu)\;\Pi(b,a,c)$. The rigth term is equal to the third term in $(\mu \otimes Id)\;\Pi(a,b,c)$:



All the terms on the left hand side of the equation have their counterpart on the right hand side, which finishes the proof. $\square$

Corollary 5.25. *Let $\Pi(-,-)$ be a double bracket with the property that $\Pi(-,-,-) = 0$. Then the associated bracket on $|A|$ satisfies the following version of the Jacobi identity:*

$$[-,-] \circ ([-,-] \otimes Id) \circ Cyc = 0.$$

Proof. It is enough to show that

$$[[|a|,|b|],|c|] + (-1)^{c(a+b)}\;[[|c|,|a|],|b|] + (-1)^{a(b+c)}\;[[|b|,|c|],|a|] = 0$$

for a,b,c homogeneous elements of A. Using that the associated bracket on $|A|$ is of degree one and graded symmetric, this is equivalent to

$$[[|a|,|b|],|c|] + (-1)^{b(a+1)}\;[|b|[|a|,|c|]] + (-1)^{a}\;[|a|,[|b|,|c|]] = 0.$$

This is just the multiplication map applied to Proposition 5.24 together with the hypothesis that $\Pi(-,-,-) = 0$. $\square$

A double bracket $\Pi(-,-)$ that satisfies $\Pi(-,-,-) = 0$ will be said to satisfy the *double Jacobi identity*, the name being motivated by the previous proposition.

To summarize, we have shown the following theorem

Theorem 5.26. *Let A be a graded algebra and $\Pi(-,-) : A \otimes A \to A \otimes A$ a double bracket such that $\Pi(-,-,-) = 0$. Then the associate bracket*

$$[-,-]_\Pi := |A| \otimes |A| \to |A|$$

satisfies the following property

- *graded symmetry* $[-,-]_\Pi = [-,-]_\Pi \circ \tau$
- *graded Jacobi identity* $[-,-]_\Pi \circ ([-,-]_\Pi \otimes Id) \circ Cyc = 0$

The main advantage of working with a double bracket on A instead of directly with the bracket on $|A|$ is that one can make use of the algebra structure of A. In particular, if A is a free algebra, one can define a map $\Pi(-,-) : A \otimes A \to A \otimes A$ on generators and then extend it uniquely as a double bracket.

The obvious question is: what are the sufficient conditions for the double bracket to satisfy the double Jacobi identity? To answer that question we prove some properties of $\Pi(-,-,-)$.

Lemma 5.27. *Given a double bracket $\Pi(-,-)$, we have the equality*

$$\Pi(-,-,-) = \xi \circ \Pi(-,-,-) \circ \xi^{-1}.$$

Proof. Straightforward from the definiton of $\Pi(-,-,-)$. □

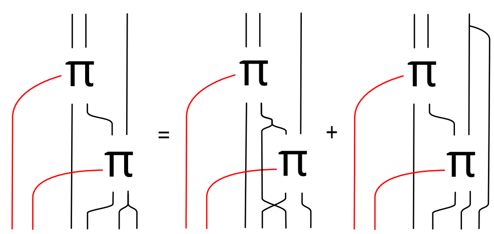
We also show that the map $\Pi(a,b,-) : A \to A \otimes A \otimes A$ is a derivation:

Proposition 5.28. *Given a double bracket $\Pi(-,-)$ and homogeneous elements $a, b \in A$, one has*

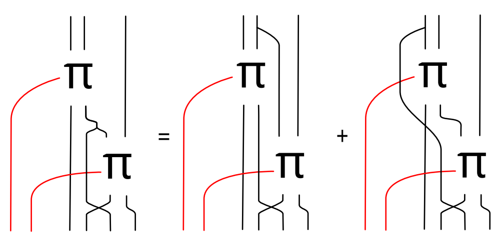
$$\Pi(a,b,cd) = (-1)^{c(a+b)}\, c\, \Pi(a,b,d) + \Pi(a,b,c)\, d$$

Proof. We shall just compute (graphically) the three terms in $\Pi(a,b,cd)$

- For $\Pi(a,\Pi(b,cd))_L$ we get :



While for the second term we recognize $\Pi(a,\Pi(b,c))_L\, d$, the first term requires a bit more work:

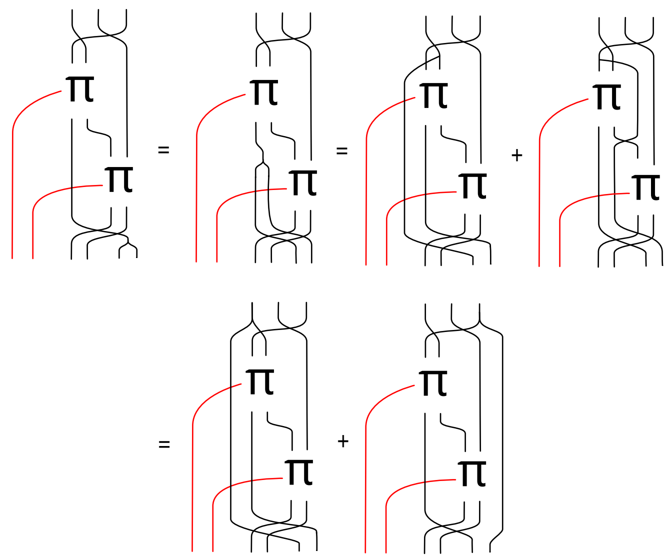


So the contribution from this part in total is equal to

$$(-1)^{c(a+b)}\, c\, \Pi(a,\Pi(b,c))_L + \Pi(a,\Pi(b,c))_L\, d + (-1)^{bc+a+c}\, \Pi(a,c)\Pi(b,d)$$

where we slightly abuse notations and write $(x \otimes y)(z \otimes t)$ for $x \otimes yz \otimes t$.

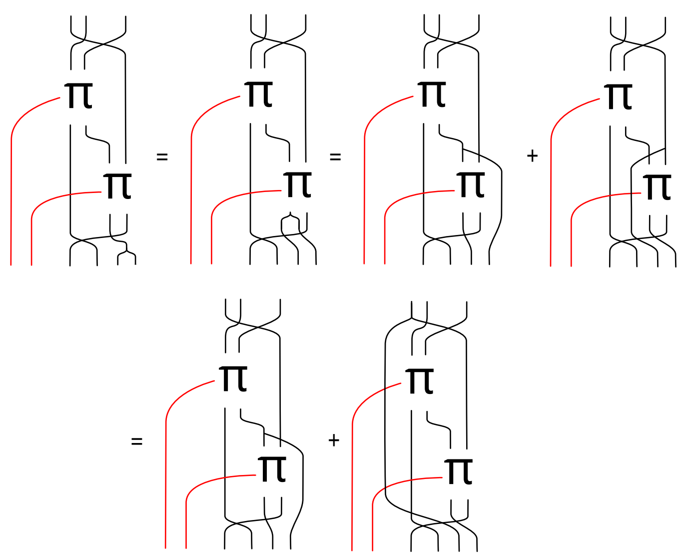
- For $\xi \circ \Pi(-,\Pi(-,-))_L \circ \xi^{-1}\ (a,b,cd)$ we have



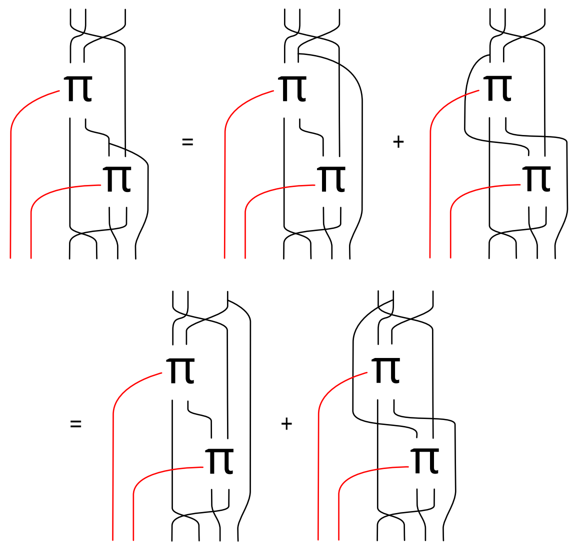
which is equal to

$$(-1)^{c(a+b)}\; c\; (\xi \circ \Pi(-,\Pi(-,-))_L \circ \xi^{-1}\; (a,b,d)) + (\xi \circ \Pi(-,\Pi(-,-))_L \circ \xi^{-1}\; (a,b,c))\; d.$$

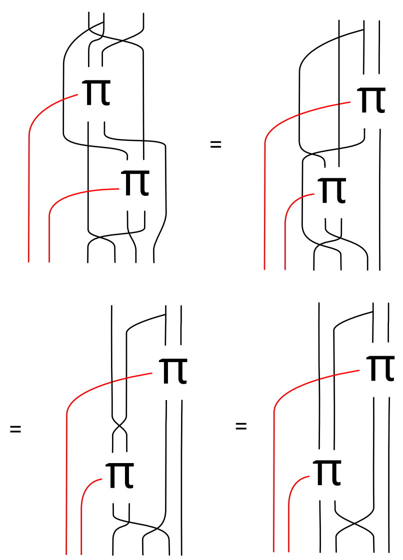
- Finally for $\xi^2 \circ \Pi(-,\Pi(-,-))_L \circ \xi^{-2}\; (a,b,cd)$ we have:



The second term is $(-1)^{c(a+b)}\; c\; (\xi^2 \circ \Pi(-,\Pi(-,-))_L \circ \xi^{-2}\; (a,b,d))$ and for the first term we keep going :



The first term is now equal to $(\xi^2 \circ \Pi(-, \Pi(-,-))_L \circ \xi^{-2}\ (a,b,c))\ d$ and the second one is $(-1)^{bc+a+c+1}\ \Pi(a,c)\Pi(b,d)$. Indeed



Adding all three contributions we get the desired result.

□

Putting together Lemma 5.27 and Proposition 5.28 we get :

Proposition 5.29. *Given a double bracket* $\Pi(-,-)$, $\Pi(-,-,-) = 0$ *if and only if* $\Pi(a,b,c) = 0$ *for every generators* $a, b, c \in A$.

5.4 Ginzburg's algebra

In order to reinterpret loop equations in a homological setting, let us remark that for $A = \mathbb{C}\langle x_1, \ldots, x_n\rangle$

$$\mathbb{D}er(A) \otimes_{A\otimes A^{op}} A \cong Der(A)$$

as A-bimodule (remember that the A bimodule structure on $\mathbb{D}er(A)$ is coming from the *inner* bimodule structure on $A \otimes A$). This seems to indicate that one should think of derivations of A as cyclic words in the symbols x_i and $\frac{\partial}{\partial x_i}$, motivating the following definition.

Definition 5.30. The Ginzburg algebra $\mathcal{G}(n)$ is the free graded algebra

$$\mathbb{C}\langle x_1, \ldots, x_n, \eta_1, \ldots, \eta_n, \rangle$$

where x_i is of degree 0 and η_i is of degree -1.

The goal of this section is to construct an BV algebra structure on the algebra

$$\mathcal{C}(n) := Sym|\mathcal{G}(n)|/(|1| - 1_{Sym}).$$

Actually, since we are also interested in power series integrals, we will define a BV algebra structure on the algebra $\mathcal{C}_\hbar(n) := \mathcal{C}(n) \otimes \mathbb{C}[[\hbar]]$. This will give rise to a BV algebra structure on $\mathcal{C}(n)$ by formally sending $\hbar$ to 0. In that case, the differential operator will be of order 1 and we will be dealing with a differential graded algebra.

We will do this by using all the machinery of the previous sections. First we will define a double bracket on $\mathcal{G}(n)$. As we have seen in Section 5.3, this equips $|\mathcal{G}(n)|$ with a bracket $\{-,-\}$. Furthermore, we will also define a cobracket δ such that $\{-,-\}$ and δ satisfy all the conditions of Proposition 5.12. Introducing $\hbar$ in some places we will finally get a BV operator on $\mathcal{C}_\hbar(n)$.

The story of this construction begins with Ginzburg in [Gin00] where given a quiver Q, he constructed a Lie bracket of the space of path of the double quiver $\bar{Q}$. Later, Schedler [Sch04] introduced a Lie cobracket on that same space of path, rendering it an involutive Lie bialgebra. Lately, Perry and Pulmann [PP24] considered a graded version coming with a BV algebra structure. Their construction of the bracket is slightly different than ours, but yields the same result. One might remark that all of sudden we are mentioning quivers. Our story corresponds to the quiver with only one vertex and n arrows.

As promised, let us start with the double bracket:

Definition 5.31. Let $\Pi(-,-) : \mathcal{G}(n) \otimes \mathcal{G}(n) \to \mathcal{G}(n) \otimes \mathcal{G}(n)$ be the map defined by

$$\Pi(x_i, \eta_j) = \delta_{ij} 1 \otimes 1$$

and extended as a (graded) double bracket.

Proposition 5.32. *The double bracket* $\Pi(-,-)$ *satisfies the double Jacobi identity.*

Proof. By Proposition 5.29, it is enough to check that $\Pi(a, b, c) = 0$ for a, b, c generators, i.e. elements of $\{x_1, \ldots, x_n, \eta_1, \ldots, \eta_n\}$. This is obviously true for $\Pi(b, c) = 0$ or $1 \otimes 1$, but in either case we then have $\Pi(a, \Pi(b, c))_L = 0$.

□

Definition 5.33. Define $\{-,-\} : |\mathcal{G}(n)| \otimes |\mathcal{G}(n)| \to |\mathcal{G}(n)|$ to be the bracket associated to $\Pi(-,-)$

By the results of Section 5.3, we have

Proposition 5.34. *The bracket* $\{-,-\}$ *satisfies*

- $\{-,-\} = \{-,-\} \circ \tau$
- $\{-,-\} \circ (\{-,-\} \otimes Id) \circ Cyc = 0$

At this point, more hands on formula for $\{-,-\}$ and δ are welcomed for concrete computations.

Proposition 5.35. *Let* f *and* g *be homogeneous element of* $|\mathcal{G}(n)|$. *We have the following equalities:*

$$\{f,g\} = \sum_{i=1}^{n} (-1)^f |\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \eta_i}| + |\frac{\partial f}{\partial \eta_i}\frac{\partial g}{\partial x_i}|$$

Proof. By linearity, may be assume that f and g are (equivalence classes of) monomials. Let $f = |f_1 \dots f_k|$ and $g = |g_1 \dots g_l|$.

Let us begin with the first equality. Both sides of the equality are of the following form: a sum over $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$ and the term indexed by i and j is obtained by erasing f_i and g_i, getting a coefficient for f_i and g_j and rearranging the remaining letters in some order to get a new word that we then consider up to commutators.

We show that on both the sides, the term that we obtain by deleting f_i and g_j is the same. To simplify notations, we write $f = |af_ib|$ and $g = |cg_jd|$ where $a, b, c, d \in \mathbb{C}\langle x_1, \dots x_n, \eta_1 \dots, \eta_n\rangle$.

We have different cases to consider, depending on what generators are f_i and g_j:

- Both f_i and g_j are one of the x (not necessarily the same). In that case the contribution on both side of the equation are 0.
- Both f_i and g_j are one of the η (not necessarily the same). In that case the contribution on both side of the equation are 0.
- $f_i = x_p$ and $g_j = \eta_p$. In that case, the contribution to the left hand side is (using Proposition 5.16)
$$(-1)^{a+gb+c+ac+bd+ab}|cbad|$$
while for the right hand side only the term $(-1)^f|\frac{\partial f}{\partial x_p}\frac{\partial g}{\partial \eta_p}|$ contributes and its contribution is equal to
$$(-1)^{f+ab+c(1+d)}|badc| = (-1)^{f+ab+c(1+d)+c(b+a+d)}|cbad|$$
It is straightforward to check that the two exponent of -1 are equal modulo 2.
- $f_i = \eta_p$ and $g_j = x_p$. In that case, the contribution to the left hand side is (using Proposition 5.16)
$$(-1)^{a+bg+ac+bd+ab}|cbad|$$
while for the right hand side only the term $|\frac{\partial f}{\partial \eta_p}\frac{\partial g}{\partial x_p}|$ contributes and its contribution is equal to
$$(-1)^{a(b+1)+cd}|badc| = (-1)^{ab+a+cd+bc+ac+cd}|cbad|$$
It is straightforward to check that the two exponent of -1 are equal modulo 2.
- $f_i = x_p$ and $g_j = \eta_p$ where $p \neq q$. In that case the contribution on both side of the equation are 0.
- $f_i = \eta_p$ and $g_j = x_q$ where $p \neq q$. In that case the contribution on both side of the equation are 0.

$\square$

We have the first half of our BV operator. Inspired by the previous Proposition we define a cobracket.

Definition 5.36. Let $\delta : |\mathcal{G}(n)| \to |\mathcal{G}(n)| \otimes |\mathcal{G}(n)|$ defined by

$$\delta := \frac{1}{2}\sum_{i=1}^{n}(| - | \otimes | - |) \circ (\frac{\partial^2}{\partial x_i \partial \eta_i} + \frac{\partial^2}{\partial \eta_i \partial x_i})$$

Proposition 5.37. *The map δ has the two following properties:*

- $\delta = \tau \circ \delta$
- $Cyc(\delta \otimes Id) \circ \delta = 0$

Proof. The first one is a direct consequence of Lemma 2.20. For the second part we refer the reader to [PP24] □

We also refer the reader to [PP24] for the proof of the cocycle condition:

Proposition 5.38. *We have the following equality relating $\{-,-\}$ and δ :*

$$\delta(\{f,g\}) = (-1)^{f+1}\{f,\delta(g)\}^{(2)} + (-1)^{(f+1)g+1}\{g,\delta(f)\}^{(2)}$$

for f and g homogeneous elements of $|\mathcal{G}(n)|$.

Putting together Proposition 5.34, Proposition 5.37, Proposition 5.38 and Proposition 5.12 we get a BV structure on the algebra $\mathcal{C}_{\hbar}^{nr}(n) = Sym|\mathcal{G}(n)| \otimes \mathbb{C}[[\hbar]]$ (here the "nr" stands for "non-reduced"):

Proposition 5.39. *The map $\Delta := \delta + \hbar br$ endows $\mathcal{C}_{\hbar}^{nr}(n)$ with the structure of a BV-algebra.*

Proof. It is clear that if δ and $\{-,-\}$ satisfy all the conditions of Proposition 5.12, then δ and $\hbar\{-,-\}$ also satisfy those conditions.

□

Remark 5.40. If we denotes by $\{-,-\}_{BV}$ the bracket associated to the BV operator Δ, we have the following equality: $\{-,-\}_{BV} = \hbar\{-,-\}$.

Lemma 5.41. *Let V be degree 0 element of $\mathcal{C}_{\hbar}^{nr}(n)$. Then*

$$\{V,V\} = 0$$

and

$$\Delta(V) = 0.$$

Proof. Since V is of degree 0, it is a (linear sum of) product of degree 0 cyclic words, that is of cyclic words where every letter is one of the x_i. The result is then obvious for both the bracket and the cobracket pair dual variables x_i and η_i. □

Proposition 5.42. *For any V degree 0 element of $\mathcal{C}_{\hbar}^{nr}(n)$, the map $\Delta_V = -\{V,-\} + \Delta$ is a BV operator on $\mathcal{C}_{\hbar}(n)$.*

Proof. This is just a consequence of Proposition 5.7 and Lemma 5.41. Strictly speaking, this shows that $\hbar\{V,-\} + \Delta$ is BV operator, but since everything is $\mathbb{C}[[\hbar]]$- linear we are done. □

Remark 5.43. By setting $\hbar \to 0$, we obtain a BV-algebra structure on $\mathcal{C}^{nr}$. Actually, since the part of the Δ which is of second order disappears, $(\mathcal{C}^{nr}, d_V + \delta)$ is in fact a dg-algebra.

We shall see that (a quotient of) the BV algebra $(\mathcal{C}^{nr}_{\hbar}(n), \Delta_V)$ is of importance to us for its zeroth cohomology is closely related to integrals. We begin by recasting degree -1 elements of $\mathcal{G}(n)$ as derivations of $A = \mathbb{C}\langle x_1, \ldots, x_n\rangle$.

Indeed, recall from Lemma 5.18 that for any degree -1 element f of $\mathcal{G}(n)$, the map $\{f, -\}$: $\mathcal{G}(n) \to \mathcal{G}(n)$ is a degree 0 derivation. Of course this map descends to the degree 0 component of $\mathcal{G}(n)$, namely A. we shall abuse notations and still denote by $\{f, -\}$ the map from A to A. Recall also from Lemma 5.21 that this map depends on f only up to commutators. Furthermore on the subspace $|\mathcal{G}(n)|_{-1}$ of degree -1 elements, the bracket is anti-symmetric and the graded Jacobi-like identity becomes the usual Jacobi identity. The vector space $|\mathcal{G}(n)|_{-1}$ equipped with the bracket $\{-, -\}$ is then a Lie algebra.

Proposition 5.44. *The map*

$$\begin{aligned} \chi : |\mathcal{G}(n)|_{-1} &\to Der(A) \\ f &\mapsto \chi(f) := \{f, -\} \end{aligned}$$

is an isomorphism of Lie algebra.

Proof. Let f be an element of degree -1 in $|\mathcal{G}(n)|$. We can write f as

$$f = \sum_{i=1}^{n} |\eta_i f_i(x_1, \ldots, x_n)|$$

with $f_i \in A$. we then have that

$$\{f, x_j\} = f_j.$$

From this it follows easily that if $\{f, -\}$ is the trivial derivation, $f = 0$ as a cyclic word to start with.

Furthermore, let u be a derivation of A that sends x_i to the associative polynomial u_i. Define the cyclic word

$$\tilde{u} = \sum_{i=1}^{n} |\eta_i u_i(x_1, \ldots, x_n)|.$$

Then $\{\tilde{u}, -\} = u$ as derivations of A for their value on generators are the same.

Consider now two degree -1 cyclic words $|\eta_i f_i(x_1, \ldots, x_n)|$ and $g = |\eta_j g_j(x_1, \ldots, x_n)|$. We compute their bracket :

$$\{f, g\} = -|\frac{\partial f}{\partial x_j}^{''} \eta_i \frac{\partial f}{\partial x_j}^{'} g| + |f \frac{\partial g}{\partial x_i}^{''} \eta_j \frac{\partial g}{\partial x_i}^{'}|$$

The derivation $\chi(\{f, g\})$ is then the derivation

$$\begin{aligned} x_i &\mapsto -\frac{\partial f}{\partial x_j}^{'} g \frac{\partial f}{\partial x_j}^{''} \\ x_j &\mapsto \frac{\partial g}{\partial x_i}^{'} f \frac{\partial g}{\partial x_i}^{''} \end{aligned}$$

and sends all other generators to 0.

On the other hand, the derivation $\chi(f)$ maps x_i to the polynomial f_i and all other generators to 0, while the second derivation $\chi(g)$ maps x_j to g_j and all other generators to 0. Using Lemma 2.7, we get

$$\chi(f) \circ \chi(g)(x_k) = \delta_{jk}\ \chi(f)(g_j) = \frac{\partial g}{\partial x_i}^{'}\ f\ \frac{\partial g}{\partial x_i}^{''}$$

$$\chi(g) \circ \chi(f)(x_k) = -\delta_{ik}\ \chi(g)(f_i) = \frac{\partial f}{\partial x_j}^{'}\ g\ \frac{\partial f}{\partial x_j}^{''}$$

Since the values of $\chi(\{f, g\})$ and $[\chi(f), \chi(g)]$ are equal on generators, this two derivations are equal. By linearity it follows that the map χ is a Lie algebra map. □

Remark 5.45. Note that $|\mathcal{G}(n)|_{-1} \cong \mathbb{D}er(A) \otimes_{A\otimes A^{op}} A \cong Der(A)$. We have just shown that bracket $\{-, -\}$ restricted to the subspace of degree -1 elements is the same as the one coming for the Lie algebra $Der(A)$.

Definition 5.46. Let $L : Der(A) \otimes Sym|A| \to Sym|A|$ be the linear map defined on pure tensors by :

$$Der(A) \otimes Sym|A| \ni u \otimes f \mapsto u(V)f + Div(u)f \in Sym|A|$$

Definition 5.47. Note that $\mathcal{C}^{nr}(n)_0 = Sym|A|$. Since

$$\mathcal{C}^{nr}(n)_{-1} \cong |\mathcal{G}(n)|_{-1} \otimes \mathcal{C}^{nr}(n)_0 = |\mathcal{G}(n)|_{-1} \otimes Sym|A|$$

, by tensoring the map χ with the identity map on $Sym|A|$ we obtain a map $\mathcal{C}(n)_{-1} \to Der(A) \otimes Sym|A|$. We shall also call this map χ

Proposition 5.48. *The two maps $d_V + \delta$ and $L \circ \chi$ from $\mathcal{C}(n)^{nr}_{-1}$ to $\mathcal{C}(n)^{nr}_0 = Sym|A|$ are equal.*

Proof. Both the source and the target of maps are $Sym|A|$ bimodule (with the obvious module structures).

The maps d_V and δ are both maps of bimodules for they are derivations and they vanishes on degree 0 elements. The maps L and χ are clearly bimodule maps by construction. By linearity, it is then enough to check the equality on elements of the form $|\eta_i f_0(x_1, \ldots, x_n)|$ where f_0 is an associative word.

On the one hand,

$$d_V + \delta(|\eta_i f_0(x_1, \ldots, x_n)|) = |\frac{\partial V}{\partial x_i} f_0| + |\frac{\partial f_0}{\partial x_i}^{'}|\,|\frac{\partial f_0}{\partial x_i}^{''}|$$

On the other hand, $u := \chi(|\eta_i f_0(x_1, \ldots, x_n)|)$ is the derivation that sends x_i to f_0 and all other generators to 0. Its divergence is thus equal to $|\ \frac{\partial}{\partial x_i} f_0\ | = |\frac{\partial f_0}{\partial x_i}^{'}|\,|\frac{\partial f_0}{\partial x_i}^{''}|$ and $u(V)$ is equal to $|\frac{\partial V}{\partial x_i} f_0|$. □

In summary, we have the following commutative diagram :

$$\begin{array}{ccc} \mathcal{C}^{nr}(n)_{-1} & \underset{\longleftarrow}{\overset{\chi}{\longrightarrow}} & Der(A) \otimes Sym|A| \\ \Big\downarrow{\scriptstyle d_V+\delta} & & \Big\downarrow{\scriptstyle L} \\ \mathcal{C}^{nr}(n) & \underset{=}{\longrightarrow} & Sym|A| \end{array}$$

Finally, we just have to encode that integrals are normalized, i.e. they vanish on the ideal $(|1| - 1_{Sym})$

Definition 5.49. Define the algebra $\mathcal{C}(n)$ by

$$\mathcal{C}(n) = \mathcal{C}^{nr}(n)/(|1| - 1_{Sym})$$

Clearly Δ_V descends to $\mathcal{C}(n)$.

Theorem 5.50. *There is a one-to-one correspondence between infinity integrals with respect to the potential V and differential graded algebra morphisms*

$$(\mathcal{C}(n), \Delta_V) \to (\mathbb{C}, 0).$$

Proof. Let $\varphi : Sym|A|/(|1| - 1_{Sym}) \to \mathbb{C}$ be an integral. It can easily be extended to a degree 0 map $\varphi : \mathcal{C}(n) \to \mathbb{C}$ by $\varphi|_{\mathcal{C}(n)_k} = 0$ for every $k \neq 0$. Clearly this is map of algebra for it is only non trivial in degree 0 and in degree 0 it is the integral with started with. The fact that this is a map of complexes is equivalent to the loop equations by Proposition 5.48.

Conversely, give a map of differential graded algebras, taking the degree 0 part gives an integral.

□

it follows from the above proposition that computing the zeroth cohomology of $\mathcal{C}$ gives a minimal set of parameters for infinity integrals.

In order to get a similar kind of statement for power series integrals, we now show a result similar to Proposition 5.48 for the part of Δ_V coming from the bracket $\{-,-\}$

Definition 5.51. Let $D : Der(A) \otimes Sym|A| \to Sym|A|$ be the linear map defined on pure tensors by :

$$Der(A) \otimes Sym|A| \ni u \otimes f \mapsto u(f).$$

Proposition 5.52. *The two maps br and $D \circ \chi$ from $C^{nr}(n)_{-1}$ to $C^{nr}(n)_0 = Sym|A|$ are equal.*

Proof. By linearity, it is enough to show the result for an element

$$F = |\eta_i f_0(x_1, \ldots, x_n)||f_1(x_1, \ldots, x_n)| \ldots |f_k(x_1, \ldots, x_n)| \in \mathcal{C}^{nr}(n)_{-1}$$

where for $0 \leqslant l \leqslant k$, $f_l(x_1, \ldots, x_n) \in A$.

On the one hand we have by definition of br

$$br(F) = \sum_{l=1}^{k} |f_0 \frac{\partial f_l}{\partial x_i}'' \frac{\partial f_l}{\partial x_i}' | \prod_{m \neq l} |f_m|$$

On the other hand, $\chi(|\eta_i f_0(x_1,\ldots,x_n)|$ is derivation sending x_i to f_0 and all generators to 0. Thus

$$D \circ \chi(F) = \sum_{l=1}^{k} |\frac{\partial f_l}{\partial x_i}' f_0 \frac{\partial f_l}{\partial x_i}'' | \prod_{m \neq l} |f_m|$$

□

Definition 5.53. Let $\mathcal{C}_\hbar(n)$ be the algebra defined by

$$\mathcal{C}_\hbar(n) = \mathcal{C}(n) \otimes \mathbb{C}[[\hbar]]$$

It also clear that Δ descends to $\mathcal{C}_\hbar(n)$ for both δ and br applied to $|1| - 1_{Sym}$ are equal to 0.

Theorem 5.54. *Every power series integral with respect to the potential V induces a map of chain complex*

$$(\mathcal{C}_\hbar(n), \Delta_V) \to (\mathbb{C}[[\hbar]], 0)$$

Unlike the $\hbar = 0$ case, we can't conclude that the space of power series integrals is equivalent to the zeroth cohomology of $\mathcal{C}_\hbar(n)$ because we don't know what to do of the condition involving cumulants.

6 Computations of the zeroth cohomology for some potentials

We now give an answer to the question of finding a minimal amount of parameters for infinity integrals (that is when $\hbar = 0$) for infinity integrals in one variable for any potential V. We do this by computing the zeroth cohomology of $(\mathcal{C}(1), \Delta_V)$.

The idea behind the argument is very close to the one we used to show that there is a unique infinity integral for the potential $V = \frac{1}{2}|x^2|$. The cohomological setting will make it easier to handle polynomials of higher degree and show that our minimal set of parameters is indeed minimal.

Most of the heavy lifting is made by the Homological Perturbation Lemma, so let us start with that.

6.1 The Homological perturbation Lemma

Say one is interested in the study of a chain complex $(A, d+\delta)$ where the differential is actually some kind of perturbation of a differential d by δ. The natural question is the following: knowing the cohomology of (A, d) can one compute the cohomology of $(A, d+\delta)$?

The Homological Perturbation Lemma gives an answer to that question but only if one has a lot of control on the unperturbed complex. Just knowing its cohomology is not quite enough. The following definition makes "having a lot of control" precise :

Definition 6.1. A strong deformation retract is the following data:

$$K \circlearrowright (A, d_A) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (B, d_B)$$

1. Two chain complexes (A, d_A) and (B, d_B).
2. Two maps of complexes $\pi : A \to B$ and $\iota : B \to A$.
3. A map of degree -1 $K : A \to A$.

with the following properties:

1. $\pi\iota = Id_B$, that is B is a retract of A.
2. $\iota\pi - Id_A = d_A K + K d_A$, i.e. K is a homotopy between $\iota\pi$ and the identity.
3. The side conditions $K^2 = K\iota = \pi K = 0$.

The Homological Perturbation Lemma allows you to add a "small" perturbation to the differential d_A of a strong deformation and get an other strong deformation retract.

Theorem 6.2. *(Homological Perturbation Lemma) Let*

$$K \circlearrowright (A, d_A) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (B, d_B)$$

be a strong deformation retract and $\delta : A \to A$ be a degree 1 map such that $(d_A + \delta)^2 = 0$ and $(Id - \delta K)$ is invertible. Then there is a strong deformation retract

$$\tilde{K} \circlearrowright (A, d_A + \delta) \underset{\tilde{\iota}}{\overset{\tilde{\pi}}{\rightleftarrows}} (B, d_B + \delta_B)$$

where

$$\delta_B = \pi(1 - \delta K)^{-1}\delta\iota$$
$$\tilde{\iota} = \iota + K(1 - \delta K)^{-1}\delta\iota$$
$$\tilde{\pi} = \pi + \pi(1 - \delta K)^{-1}\delta K$$
$$\tilde{K} = K + K(1 - \delta K)^{-1}\delta K$$

Typically, (A, d_A) is some complex for which we have a strong control over the cohomology and K is some kind of inverse operation to d_A. Then (B, d_B) is the said cohomology with differential 0 and the Homological Perturbation Lemma allows to compute the perturbed cohomology $d_A + \delta$ in a much smaller -and thus easier- complex. To check that $Id - \delta K$ is invertible, one can simply check that the geometric series $\sum_{j=0}^{\infty}(\delta K)^j$ is well defined on A. The proof of this theorem can be found in the very good exposition article by Crainic [Cra04]

Of course, the example that we have in the back of our mind this whole time is the case of the Ginzburg algebra $\mathcal{C}(1)$. In this case we could see the differential $\delta + d_V$ of $\mathcal{C}(1)$ as a perturbation of d_V. So if we find a strong deformation retract for $(\mathcal{C}(1), d_V)$ this might go a long way. Of course the question is now: how can find such a strong deformation retract? The good news is that now the whole dga structure is just coming from the extension to the symmetric algebra of a complex, namely $(|\mathbb{C}\langle x, \eta\rangle|, d_V)$. We will now see that in general a strong deformation retract can be extended to a strong deformation retract of the corresponding symmetric algebras (with differentials extended by the Leibniz rule). We've learned this from [Gwi12]

Let

$$K \circlearrowright (A, d_A) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (B, d_B)$$

be a strong deformation retract. Then the map π (respectively ι) can be extended as a map of algebra to $Sym\,\pi : SymA \to SymB$ (respectively $Sym\,\iota : SymB \to SymA$. Similarly, the map d_A (respectively d_B) can be extended as a derivation to a map $d_A : SymA \to SymA$ (respectively $d_B : SymB \to SymB$). The only map which does not admit a totally straightforward extension is the homotopy K.

The equality $\pi\iota = Id_B$ tells us that B can be identified with the subspace $\iota(B)$ of A and that the map $P := \iota\pi$ is a projection operator. We have an other projection operator $P^{\perp} = P - Id_A$ whose image will be denoted by $B^{\perp}$. We thus have the decomposition (as graded vector spaces for the moment) $A = B \oplus B^{\perp}$. Since π and ι are map of complexes, $B \subset A$ is actually a subcomplex. The side condition implies that the homotopy K also respects the decomposition.

Using the decomposition $A = B \oplus B^{\perp}$, one gets the isomorphism $SymA \cong SymB \otimes SymB^{\perp}$. Extend the map $P^{\perp}$ as a derivation to $SymA$ and denote it also by $P^{\perp}$. On an element

$a_1 a_2 \dots a_n$, the new map $P^\perp$ acts as the multiplication by the number of a_i that are in B. In other words, the subspace

$$E_n := SymB \otimes Sym^n B^\perp$$

is the eigenspace of $P^\perp$ for the eigenvalue n. Note that those are subcomplexes and clearly $SymA = \bigoplus_{n=0}^{\infty} E_n$.

We are finally ready to define an homotopy $SymK : SymA \to SymA$. First extend K to $SymA$ as a derivation (and still denote it by K). Then define a new map $SymK : SymA \to SymA$ by

$$SymK|_{E_n} = \begin{cases} \frac{1}{n} K & \text{if } n \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.3. *Let*

$$K \circlearrowright (A, d_A) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (B, d_B)$$

be a strong deformation retract. Then

$$SymK \circlearrowright (SymA, d_A) \underset{Sym\iota}{\overset{Sym\pi}{\rightleftarrows}} (SymB, d_B)$$

as defined above is a strong deformation retract.

6.2 The case of one variables and $V = \frac{1}{k+1}|x^{k+1}|$

Consider the Ginzburg algebra $\mathcal{C}(1)$ and potential $V = \frac{1}{k+1}|x^{k+1}|$ for $k \geqslant 1$. One can compute the zeroth cohomology $H^0(\mathcal{C}(1), V)$ of the algebra $(\mathcal{C}(1) = Sym\ |\mathbb{C}\langle x, \eta\rangle|, \delta + d_V)$ somewhat easily. The argument is in essence the same that the one we used to show that there is an unique infinity integral for a quadratic potential. However, the technique using the Homological Perturbation Lemma can then be generalized to more situations.

The idea is to use the fact that $\mathcal{C}(1)$ is built from a simple algebra (at least in degree 0 et -1) whose zeroth cohomology is easy to compute. Hopefully one can use this to get some idea of what the zeroth cohomology of $(\mathcal{C}(1) = Sym\ |\mathbb{C}\langle x, \eta\rangle|, d_V)$ is. Then we might use the Homological Perturbation Lemma to compute the cohomology of that same algebra but with differential now $\delta + d_V$. This makes our task significantly easier, we now just have to find a strong deformation retract for the easy complex $(|\mathbb{C}\langle x, \eta\rangle|, d_V)$.

Since we are working with only one pair of dual variable and we are only interested in the zeroth cohomology of $(\mathcal{C}(1), \delta + d_V)$, the starting complex $|\mathbb{C}\langle x, \eta\rangle|$ can be replaced by something much simpler:

- To compute the zeroth cohomology of $(\mathcal{C}(1) = Sym\ |\mathbb{C}\langle x, \eta\rangle|, \delta + d_V)$, one only needs the degree -1 and degree 0 elements. Since this is a symmetric algebra, such elements are product of degree -1 and degree 0 cyclic words.
- Since cyclic word and standard word in one generator are the same thing, an element of degree 0 is just a linear combination of monomials x^i.

- In degree -1, every cyclic word has to have exactly one η. Since the word is cyclic, one can place that η at the end of the word. Thus a degree -1 element is a linear combination of words of the form $x^i\eta$.
- Since we are only interested in the zeroth cohomology, we don't need to know the elements of degree strictly less than -1 and there is no element of positive degree.

In short we are left with the task a finding a strong deformation retract for the complex (A, d) given by

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{C}[x]\eta \xrightarrow{d} \mathbb{C}[x] \longrightarrow 0 \longrightarrow \cdots$$

where the only non zero spaces are in degree -1 and 0 and the d is the linear map

$$d : x^i\eta \mapsto x^{i+k}.$$

It easy now extremely easy to construct a strong deformation of the complex (A, d) onto its cohomology:

Proposition 6.4. *The data of*

$$K \circlearrowright (A, d) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (\mathbb{C}[x]/(x^k), 0)$$

where

- $\mathbb{C}[x]$ *is concentrated in degree* 0*.*
- π *is the projection to the quotient in degree* 0*.*
- ι *is the linear map that sends the class* $[x^i]$ *to the polynomial* x^i*.*
- K *defined by*

$$K(x^i) = \begin{cases} -x^{i-k}\eta & \text{if } i \geqslant k \\ 0 & \text{otherwise.} \end{cases}$$

is a strong deformation retract.

Proof. All the computations are as straightforward as it gets.

□

Remark 6.5. The number of front of $|x^{k+1}|$ in V is of no real importance. Indeed, if you multiply the potential by $\alpha \in \mathbb{C}^*$, the differential is then also multiplied by α and dividing the homotopy K by α still yields a strong deformation retract. We choose to put $1/(k+1)$, for then it disappears from all the expressions.

We now wish to apply Proposition 6.3. The only map whose definition is not totally clear from the start is the extension $SymK$ of the homotopy K. Let

$$x^{i_1} \otimes \cdots \otimes x^{i_m} \otimes \cdots \otimes x^{i_{m+n}}$$

be a product of monomials in $Sym\ \mathbb{C}[x]$ such that i_a is strictly smaller than k if and only if a is smaller or equal to m. Then its image by $SymK$ is

$$-\frac{1}{n}\sum_{l=1}^{n} x^{i_1}\otimes\cdots\otimes x^{i_m}\otimes\cdots\otimes x^{i_{m+l-1}}\otimes x^{i_{m+l}-k}\eta\otimes x^{i_{m+l+1}}\otimes\cdots\otimes x^{i_{m+n}}.$$

For the last step, we will abuse notations and still denote by δ the degree 1 map on $SymA$ coming from the BV-operator on $\mathcal{C}$ after all the earlier identifications. We wish to perturb the differential d by δ. We first need to check that the perturbation is "small".

Proposition 6.6. *The map $\sum_{j=0}^{\infty}(\delta\circ SymK)^j$ is well defined on $SymA$.*

Proof. To show that $\sum_{j=0}^{\infty}(\delta\circ SymK)^j$ is well defined it is enough to show that for any product of monomial

$$x_i := x^{i_1}\otimes\cdots\otimes x^{i_m}\otimes\cdots\otimes x^{i_{m+n}},$$

there is $j_0 \geqslant 0$ such that $(\delta\circ SymK)^{j_0}(x_i) = 0$. Here, as before, i_a is strictly smaller than k if and only if a is smaller or equal to m.

Define the *weight* of x_i to be the total number of $x's$ in x_i, that is

$$w(x_i) = i_1 + \cdots + i_{m+n}.$$

We shall prove the result by induction on the weight.

If the weight is strictly less than k, there is nothing to prove for the map $SymK$ already acts as 0. For weight k, either $x_i = x^k$ and then

$$\delta\circ SymK(x_i) = \delta(\eta) = 0$$

or x_i is the product of monomials of degree lower than k and $SymK$ acts as 0 on it.

Suppose now that the weight of x_i is bigger than k. We then have that

$$SymK(x_i) = -\frac{1}{n}\sum_{l=1}^{n} x^{i_1}\otimes\cdots\otimes x^{i_m}\otimes\cdots\otimes x^{i_{m+l-1}}\otimes x^{i_{m+l}-k}\eta\otimes x^{i_{m+l+1}}\otimes\cdots\otimes x^{i_{m+n}}.$$

Now δ applied to $SymK(x_i)$ only changes the $x^{i_{m+l}-k}\eta$ into

$$\sum_{a=0}^{i_{m+l}-k-1} x^a\otimes x^{i_{m+l}-k-1-a}$$

which is of lower weight of $x^{i_{m+l}}$ and we are done.

□

Proposition 6.7. *The cohomology of $(SymA, d+\delta)$ is concentrated in degree 0 and is isomorphic to $Sym(\mathbb{C}[x]/(x^k))$*

Proof. Just apply the Homological Perturbation Lemma to the strong deformation retract

$$SymK \circlearrowright (SymA, d) \underset{Sym\iota}{\overset{Sym\pi}{\rightleftarrows}} (Sym(\mathbb{C}[x]/(x^k)), 0)$$

with perturbation δ. Since the complex on the right is concentrated in degree 0 it stays the same after the application of the Homological Perturbation Lemma.

□

Which reformulated in terms in of the algebra $\mathcal{C}(1)$ becomes :

Corollary 6.8. *For the potential $V = \frac{1}{k+1}|x^{k+1}|$, the zeroth cohomology of the Ginzburg algebra in one pair of dual variables is given by*

$$H^0(\mathcal{C}(1), V) \cong Sym(\mathbb{C}[x]/(x^k)).$$

This result can be reformulated as follows: an integral with respect to the potential $V = \frac{1}{k+1}|x^{k+1}|$ is uniquely determined by its value on the monomials $|x^i|$ for $0 \leqslant i \leqslant k-1$. Once those values are specified, loop equations and the multiplicative condition impose the values of the integral on the whole of $Sym(\mathbb{C}[x])$.

We now tackle potentials of a slightly more complicated form, namely $V = \frac{1}{l+1}|x^{l+1}| + \frac{1}{k+1}|x^{k+1}|$ with $1 \leqslant l < k$.

The idea is simply to start by considering the differential d_V as a perturbation of the differential d_k coming from the higher degree monomial by the differential d_l coming from the lower degree monomial.

So we start by the considering the same strong deformation as before, namely

$$K \circlearrowright (A, d_k) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (\mathbb{C}[x]/(x^k), 0)$$

where

- $\mathbb{C}[x]$ is concentrated in degree 0.
- π is the projection to the quotient in degree 0.
- ι is the linear map that sends the class $[x^i]$ to the polynomial x^i.
- K defined by

$$K(x^i) = \begin{cases} -x^{i-k} & \text{if } i \geqslant k \\ 0 & \text{otherwise.} \end{cases}$$

with d_k defined by

$$d_k : x^i\eta \mapsto x^{i+k}.$$

If we want to perturb d_k by d_l, we have to check that the map $\sum_{j=0}^{\infty}(d_l \circ K)^j$ is well defined. This turns out to be really easy:

Proposition 6.9. *The map $\sum_{j=0}^{\infty}(d_l \circ K)^j : A \to A$ is well defined and its value on x^a is given by :*

$$\sum_{j=0}^{\infty}(d_l \circ K)^j(x^a) = x^a + \sum x^{a-j(k-l)}$$

where the sum on the right is taken over all $j \geqslant 1$ such that $a - (j-1)(k-l) \geqslant k$.

Proof. This follows from the easy computation

$$d_l \circ K(x^i) = \begin{cases} 0 \text{ if } i < k \\ x^{i-k+l} \text{ otherwise} \end{cases}$$

□

Applying the Homological Perturbation Lemma, we get

Proposition 6.10. *The data*

$$\tilde{K} \circlearrowright (A, d_l + d_k) \underset{\tilde{\imath}}{\overset{\tilde{\pi}}{\rightleftarrows}} (\mathbb{C}[x]/(x^k), 0)$$

where

- $\mathbb{C}[x]$ *is concentrated in degree* 0.
- $\tilde{\pi}$ *is the linear map defined by*

$$\tilde{\pi}(x^a) = \begin{cases} [x^a] \text{ if } a < k \\ [x^{a-j_0(k-l)}] \text{ with } j_0 = min\{j \geqslant 1 | a - j(k-l) < k\} \text{ otherwise} \end{cases}$$

- $\tilde{\imath}$ *is the linear map that sends the class* $[x^i]$ *to the polynomial* x^i.
- $\tilde{K}$ *defined by*

$$K(x^a) = \begin{cases} 0 \text{ if } i < k \\ \sum_{j=0}^{\lfloor \frac{a-k}{k-l} \rfloor} x^{a-(j+1)(k-l)}\eta \text{ otherwise} \end{cases}$$

is a strong deformation retract.

Using the same argument as when the potential was a single monomials, namely by induction on the total number of x's, one can show that the map $\delta \circ Sym\tilde{K}$ is well defined on $SymA$. In the end we get :

Proposition 6.11. *For the potential* $V = \frac{1}{l+1}|x^{l+1}| + \frac{1}{k+1}|x^{k+1}|$ *with* $1 \leqslant l < k$, *the zeroth cohomology of the Ginsburg algebra in one pair of dual variables is given by*

$$H^0(\mathcal{C}(1), V) \cong Sym(\mathbb{C}[x]/(x^k)).$$

It is now clear what is the zeroth cohomology of the algebra $(Sym\mathcal{C}(1), d_V + \delta)$ when V is any polynomial. One should first consider the complex $|\mathbb{C}\langle x, \eta\rangle|$ with the differential coming only from the highest degree term, and step by step perturb this differential with lower and lower degree terms of V. After a final number of steps we get a strong deformation retract relating $(|\mathbb{C}\langle x, \eta\rangle|, d_v)$ with $(\mathbb{C}[x]/(x^k), 0)$, k being the degree of V. The maps realizing this strong deformation retract are now somewhat complicated. The homotopy of this strong deformation retract applied to a word produces many words. However, all of those words have lower total number of x's than the word we started with. One can then perturb $(Sym\mathcal{C}(1), d_V)$ by δ and get finally to :

Theorem 6.12. *For any potential* V *of degree* $k+1$, *the zeroth cohomology of the algebra* $\mathcal{C}(1)$ *is given by*

$$H^0(\mathcal{C}(1), V) \cong Sym(\mathbb{C}[x]/(x^k)).$$

7 Topological recursion

Unlike infinity integrals, power series integrals are not fully determined by the cohomology of $(\mathcal{C}_\hbar(n), \Delta)$. Using topological recursion, we find a finite set of parameters which uniquely determine a power series integral, in the one variable case.

For the reminder of this section, let then $A = \mathbb{C}\langle x \rangle$ and fix a potential

$$V(x) = \sum_{k=2}^{d} \frac{a_k}{k} x^k \in A.$$

All integrals will be with respect to this potential. We can assume there is no constant term for the potential V appears in the loop equations as the argument of a derivation.

Let us fix once and for all a power series integral $\varphi_\hbar$ for this potential.

7.1 The strategy

Before diving into formulas, let us take a moment to explain the strategy; or at least where it is coming from.

Loosely speaking, topological recursion starts with a so-called "spectral curve", which is the following data:

- A Riemann surface Σ;
- A covering $t : \Sigma \to \mathbb{CP}^1$ of the Riemann sphere;
- A meromorphic 1-form $\omega_{0,1}$ on Σ;
- A symmetric meromorphic bidifferential form $\omega_{0,2}$ on $\Sigma \times \Sigma$, with only a double pole on the diagonal and no residue.

From this initial data, topological recursion produces recursively a whole family of differential forms $\omega_{g,n}$ on Σ^n. Here g and n are both non-negative integer. The definition of the form $\omega_{g,n}$ involves only characteristic of the Riemann surface Σ and uses only those $\omega_{g',n'}$ for which $2g' + n' < 2g + n$. This explains why the recursion is topological: the induction is on the "Euler characteristic number" $2g + n$.

To define a power series integral $\varphi_\hbar$, one has to specify a lot of information: for every product of cyclic words, one has to specify the expansion in $\hbar$ of the integral of this product. Of course, the loop equations tell us that we can not just do anything, there are some relations. Upon rearranging all this information into generating functions $W_{g,n}$, we find that the $W_{g,n}$ satisfy some equations equivalent to the loop equations. After closing one eye, one can recognize that those equations are somewhat similar to the ones defining the different $\omega_{g,n}$ of topological recursion (here the trick is to not close both eyes, otherwise one does not recognize anything).

The goal is then the following: define a Riemann surface covering the Riemann sphere together with differentials forms $\omega_{g,n}$ using the loop equations, and then show that these forms satisfy the topological recursion formula. This will mean that these are uniquely determined by $\omega_{0,1}$ and $\omega_{0,2}$.

One can learn more about topological recursion in [EO07], [Eyn14], or [Eyn16]. Different results about Riemann surfaces, and more particularly algebraic curves will be used. For the reader's convenience, they have been gathered in Appendix A. We will often multiply different symmetric multidifferential n-forms on n copies of a Riemann surface Σ. If ω_1 and ω_2 are multidifferential forms on n_1 (respectively n_2) copies of Σ, the product $\omega_1\omega_2$ is a multidifferential form on n_1+n_2 copies of Σ. We insist that it is *not* the wedge product of those forms.

We are thankful to Nicolas Orantin for taking the time to present us this strategy.

7.2 The generating functions $W_{g,n}$

As we said, the first step is to rearrange all the data of the integral $\varphi_\hbar$ into generating functions. Remember that by definition of $\varphi_\hbar$ being an power series integral, the leading order in $\hbar$ of the n-cumulant of $\varphi_\hbar$ is $n-1$. With that in mind, let us start with a bit of notation :

Definition 7.1. Let $T_{l_1,\dots,l_n}$ be the power series in $\hbar$ given by shifting the n-cumulant by $\hbar^{n-1}$

$$T_{l_1,\dots,l_n} = \hbar^{-(n-1)}\,\varphi_c\big((x^{l_1})\dots(x^{l_n})\big).$$

and let $T^{(g)}_{l_1,\dots,l_n}$ be the numbers defined by the expansion in $\hbar$ of $T_{l_1,\dots,l_n}$

$$T_{l_1,\dots,l_n} = \sum_{g=0}^{\infty} T^{(g)}_{l_1,\dots,l_n}\hbar^g$$

One can then collect all those numbers for g and n fixed into a single generating function:

Definition 7.2. For g and n fixed, let $W_{g,n}(t_1,\dots,t_n)$ be the function in n formal parameters given by

$$W_{g,n}(t_1,\dots,t_n) := \sum_{\vec{l}\in\mathbb{N}^n} T^{(g)}_{l_1\dots,l_n}\prod_{i=1}^{n} t_i^{-l_i-1}$$

Remark 7.3. As one can expect the notation $W_{g,n}$ is not random, those generating functions will be very closely related to the differentials $\omega_{g,n}$

Integrals being normalize as the following consequence

Proposition 7.4. *For any $n \geqslant 2$, if one of the $l_i = 0$ then $T_{l_1,\dots,l_n} = 0$.*

Proof. By symmetry we can suppose that $i = 1$. For any partition τ of the set $\{1,\dots,n\}$, set $Q(\tau)$ to be the partition of the $\{2,\dots,n\}$ obtained by forgetting in which block is 1. Any partition π of $\{2,\dots,n\}$ has $l(\pi)+1$ preimage by Q, for 1 could be added to any block of π or form a block on its own. For simplicity, given a partition τ of $\{1,\dots,n\}$, define

$$\varphi_\hbar(\tau) = \prod_{j=1}^{l(\tau)} \varphi_\hbar\Big(\prod_{i\in B_j}(x^{l_i})\Big)$$

Note that if $Q(\tau_1) = Q(\tau_2)$ then $\varphi_\hbar(\tau_1) = \varphi_\hbar(\tau_2)$ for it does not matter in which integral x^0 is by the normalization property. Using Remark 4.6, we have

$$\begin{aligned} T_{l_1,\dots,l_n} &= \sum_{\tau}(-1)^{l(\tau)-1}(l(\tau)-1)!\varphi_\hbar(\tau) \\ &= \sum_{\pi}\sum_{\tau\in Q^{-1}(\pi)}(-1)^{l(\tau)-1}(l(\tau)-1)!\varphi_\hbar(\tau) \end{aligned}$$

where the sum over τ is a sum over partition of $\{1,\dots,n\}$, whereas the sum over π is a sum over partition of $\{2,\dots,n\}$.

The preimages of π by Q are as follows: there are $l(\pi)$ preimages with the same number of blocks as π, and exactly one preimage with one extra block. From this we get

$$\begin{aligned}
&\sum_{\{\tau \in Q^{-1}(\pi)\}} (-1)^{l(\tau)+1}(l(\tau)-1)!\varphi_\hbar(\tau) \\
&= l(\pi)\ (-1)^{l(\pi)-1}\ (l(\pi)-1)!\ \varphi_\hbar(\tau_0) + (-1)^{l(\pi)}\ l(\pi)!\ \varphi_\hbar(\tau_0) \\
&= 0
\end{aligned}$$

where τ_0 is any preimage by Q of π. □

We know try to express the loop equations (5) in terms of the generating functions $W_{g,n}$.

As a warm up, let us look at the easiest instance of the loop equation. Using the notation of Equation (5), the easiest instance of the loop equation is when $u = x^l \frac{\partial}{\partial x}$ and $f = 0$. In that case the loop equation reads :

$$\sum_{j=0}^{l-1} \varphi_\hbar\big((x^j)\ (x^{l-1-j})\big) = \varphi_\hbar\big(V'(x)v^l\big)$$

which becomes

$$\sum_{j=0}^{l-1} \varphi_\hbar(x^j)\varphi_\hbar(x^{l-1-j}) + \varphi_c\big((x^j)\ (x^{l-1-j})\big) = \varphi_\hbar\big(V'(x)v^l\big)$$

after plugging in the definition of the 2-cumulant.

Rewriting this in terms of the different T yields:

$$\sum_{j=0}^{l-1} T_j T_{l-1-j} + \hbar T_{j,l-1-j} = \sum_{k=2}^{d} a_k T_{k+l-1}.$$

and looking at the power of $\hbar^g$ we have

$$\sum_{j=0}^{l-1}\Bigg[\sum_{h=0}^{g} T_j^{(h)} T_{l-1-j}^{(g-h)} + T_{j,l-1-j}^{(g-1)}\Bigg] = \sum_{k=2}^{d} a_k T_{k+l-1}^{(g)}. \tag{6}$$

Note that this is true for every $l \geqslant 0$. Thus after multiplying both sides of Equation (6) with x^{-l-1} and summing over all the $l \geqslant 0$ we finally get (g is still fixed) :

$$\sum_{h=0}^{g} W_{h,1}(t)W_{g-h,1}(t) + W_{g-1,2}(t,t) = \sum_{l=0}^{\infty}\sum_{k=2}^{d} a_k t^{k-1} T_{k+l-1}^{(g)} t^{-k-l} \tag{7}$$

One should note that the right hand side of Equation (7) is the principal part of the function $V'(t)W_{g,1}(t)$.

Finally, we get the following equation :

Proposition 7.5. *For every $g \geqslant 0$*

$$\sum_{h=0}^{g} W_{h,1}(t)W_{g-h,1}(t) + W_{g-1,2}(t,t) = V'(t)W_{g,1}(t) - P_{g,1}(t) \tag{8}$$

where $P_{g,1}(t)$ is the polynomial given by

$$P_{g,1}(t) := \sum_{k=2}^{d}\sum_{l=1}^{k-1} a_k t^{k-1} T^{(g)}_{k-l-1} t^{-k+l}$$

Now that the warm up is done, let us do the general case.

While the relation for the general case will involve only the different $W_{g,n}$, it is convenient to define some other generating functions for the intermediate steps.

Definition 7.6. For every $n \geqslant 1$, define the symmetric generating functions in n variables $F_n(t_1,\dots,t_n)$ and $C_n(t_1,\dots,t_n)$ by

$$\begin{aligned} F_n(t_1,\dots t_n) &= \sum_{\vec{l}\in\mathbb{N}^n} \varphi_\hbar(\prod_i (x^{l_i})) \prod_i t_i^{l_i-1} \\ C_n(t_1,\dots t_n) &= \sum_{\vec{l}\in\mathbb{N}^n} \varphi_c(\prod_i (x^{l_i})) \prod_i t_i^{l_i-1} \end{aligned}$$

The following lemma is obvious from the definitions of $W_{g,n}$ and C_n

Lemma 7.7. *For every $n \geqslant 1$,*

$$C_n(t_1,\dots t_n) = \hbar^{n-1} \sum_{g\geqslant 0} \hbar^g W_{g,n}(t_1,\dots,t_n)$$

And the next lemma is obvious from the definitions of C_n, F_n and cumulants

Lemma 7.8. *For every $n \geqslant 1$*

$$F_n(t_1,\dots t_n) = \sum_\pi \prod_{j=1}^{l(\pi)} C_{|B_j|}(B_j)$$

where π is a partition of $\{t_1,\dots,t_n\}$ into $l(\pi)$ subsets denoted by $B_1,\dots,B_{l(\pi)}$ and $C_{|B_j|}(B_j)$ is the functions whose variables are the t_i that are in B_j. (Since C_n is symmetric, the order does not matter.)

Proposition 7.9. *The loop equations for $\varphi_\hbar$ imply for every $n \geqslant 0$*

$$\begin{aligned} F_{n+1}(t_1,t_1,t_2,\dots,t_n) + \hbar \sum_{i\geqslant 2} \frac{\partial}{\partial t_i} \frac{F_{n-1}(t_1,t_2,\dots,\hat{t_i},\dots,t_n) - F_{n-1}(t_2,\dots,t_n)}{t_1 - t_i} \\ = \Big[V'(t_1)F_n(t_1,t_2,\dots,t_n)\Big]_{-,1}. \end{aligned}$$

By $[-]_{-,1}$ we mean the part with only strictly negative powers of t_1, the hat denotes omission and by convention $F_0 = 0$.

Proof. The case $n = 0$ follows from the warm up case, which is stronger since it is done degree by degree (in $\hbar$). For $n \geqslant 1$ the loop equation applied to u_{l_1} and $f = \prod_{i=2}^{n} x^{l_i}$ with $l_i \geqslant 0$ reads as

$$\begin{aligned}\sum_{j=0}^{l_1-1} \varphi_\hbar\big(x^j x^{l_1-1-j} \prod_{i=2}^{n} x^{l_i}\big) + \hbar \sum_{i=2}^{n} l_i \varphi_\hbar\big(x^{l_i-1-l_1} \prod_{\substack{j\neq i \\ j\geqslant 2}} x^{l_j}\big) \\ = \varphi_\hbar\big(x^{l_1+1} \prod_{i=2}^{n} x^{l_i}\big) + a_k \sum_{k=3}^{d} \varphi_\hbar\big(x^{l_1-1+k} \prod_{i=2}^{n} x^{l_i}\big)\end{aligned} \tag{9}$$

Let us multiply the whole equation by $\prod_{i=1}^{n} t_i^{-l_i-1}$ and summing over all $l_1, \dots l_n \geqslant 0$. From the first term of the left hand side of Equation (9) we get $F_{n+1}(t_1, t_1, t_2, \dots t_n)$.

The second term on the left hand side of (9) gives

$$\hbar \sum_{i=2}^{n} \frac{\partial}{\partial t_i} \frac{F_{n-1}(t_1, t_2, \dots, \hat{t_i}, \dots, t_n) - F_{n-1}(t_2, \dots, t_n)}{t_1 - t_i}.$$

Indeed, if we integrate the i-th summand with respect to t_i and then multiply by $(t_1 - t_i)$ we have (be aware that the term in $l_i = 0$ is equal to 0)

$$\begin{aligned}&\hbar \sum_{l_i\geqslant 1, l_j\geqslant 0} \varphi_\hbar\big(x^{l_i-1-l_1} \prod_{j\neq i} x^{l_j}\big) t_i^{-l_i+1} \prod_{j\neq i} t_j^{-l_j-1} \\ &- \hbar \sum_{l_i\geqslant 1, l_j\geqslant 0} \varphi_\hbar\big(x^{l_i-1-l_1} \prod_{j\neq i} x^{l_j}\big) t_1^{-l_1} t_i^{-l_i} \prod_{j\neq 1,i} t_j^{-l_j-1}\end{aligned}$$

By shifting the first sum both in l_1 and l_i this becomes

$$\begin{aligned}&\hbar \sum_{l_1\geqslant 1, l_j\geqslant 0} \varphi_\hbar\big(x^{l_i-1-l_1} \prod_{j\neq i} x^{l_j}\big) t_1^{-l_1} t_i^{-l_i} \prod_{j\neq i,1} t_j^{-l_j-1} \\ &- \hbar \sum_{l_i\geqslant 1, l_j\geqslant 0} \varphi_\hbar\big(x^{l_i-1-l_1} \prod_{j\neq i} x^{l_j}\big) t_1^{-l_1} t_i^{-l_i} \prod_{j\neq 1,i} t_j^{-l_j-1}\end{aligned}$$

Most of this cancels out, except the part in $l_i = 0$ form the first sum and the part in $l_1 = 0$ from the second sum. This can be seen to be exactly

$$\hbar F_{n-1}(t_1, t_2, \dots, \hat{t_i}, \dots, t_n) - \hbar F_{n-1}(t_2, \dots, t_n)$$

after shifting back the sums.

Finally the term on the right hand side of Equation (9) gives

$$\Big[V'(t_1) F_n(t_1, t_2, \dots, t_n)\Big]_{-,1}$$

$\square$

Reformulating this in terms of the C_n we obtain

Proposition 7.10.

$$C_{n+1}(t_1,t_1,t_2,\dots,t_n)+\sum_{A\sqcup B=\{t_2,\dots,t_n\}} C_{|A|+1}(t_1,A)C_{|B|+1}(t_1,B)$$
$$+\hbar\sum_{i\geqslant 2}\frac{\partial}{\partial t_i}\frac{C_{n-1}(t_1,t_2,\dots,\hat{t_i},\dots,t_n)-C_{n-1}(t_2,\dots,t_n)}{t_1-t_i} \tag{10}$$
$$=\Big[V'(t_1)C_n(t_1,t_2,\dots,t_n)\Big]_{-,1}$$

Proof. We shall do the proof by induction on n. The case $n=0$ has been done earlier.

The proof is quite simple, one simply needs to use Proposition 7.9 and Lemma 7.8 and group the terms coming from the different partitions the right way.

From Proposition 7.9 we have

$$F_{n+1}(t_1,t_1,t_2,\dots,t_n)+\hbar\sum_{i\geqslant 12}\frac{\partial}{\partial t_i}\frac{F_{n-1}(t_1,t_2,\dots,\hat{t_i},\dots,t_n)-F_{n-1}(t_2,\dots,t_n)}{t_1-t_i}$$
$$=\Big[V'(t_1)F_n(t_1,t_2,\dots,t_n)\Big]_{-,1}. \tag{11}$$

and Lemma 7.8 say that each term in this sum is itself a sum over partitions of different C_ks. The right hand side term will be a sum over partitions of the set

$$\{t_1,t_2,\dots t_n\}.$$

Let π a non trivial partition of this set into $l(\pi)\geqslant 2$ blocks $B_1,\dots,B_{l(\pi)}$. Without lose of generality let us assume that $t_1\in B_1$. The partition π induces partitions of the sets

$$\{t_1,t_2,\dots,\hat{t_i}\dots t_n\}$$

for $2\leqslant i\leqslant n$. Denote those partitions π^i. Consider also partitions τ of the *multiset*

$$\{t_1,t_1,t_2,\dots t_n\}$$

such that

- τ has either $l(\pi)$ or $l(\pi)+1$ block.
- If it has $l(\pi)$ blocks, all the blocks are the same as the blocks of π, with the exception that t_1 is doubled.
- If it has $l(\pi)+1$ blocks, it is obtained by splitting the block B_1 of π into two non empty blocks, and putting a second t_1 into the half of B_1 which does not have it already.

Denote by $P(\pi)$ the set of all such partitions τ.

By the induction hypothesis, the term coming from a non trivial π on the right hand side of Equation (11) will be cancelled by the terms in F_{n+1} corresponding to partitions in $P(\pi)$ together with the terms coming from

$$\sum_{i\geqslant 2, i\in B_1}\frac{\partial}{\partial t_i}\frac{F_{n-1}(t_1,t_2,\dots,\hat{t_i},\dots,t_n)-F_{n-1}(t_2,\dots,t_n)}{t_1-t_i}$$

corresponding to the partitions π^i. Indeed all those term will give Equation (10) for $n=|B_1|$ multiplied by $\prod_{j=2}^{l(\pi)}C_{|B_j|}(B_j)$.

We are thus left with exactly what we want.

□

Proposition 7.11. *Loop equations imply*

$$\begin{aligned} W_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n) + \sum_{h=0}^{g} \sum_{A\sqcup B=\{t_2,\dots,t_n\}} W_{h,|A|+1}(t_1,A)W_{g-h,|B|+1}(t_1,B) \\ + \sum_{j=2}^{n} \frac{\partial}{\partial t_j}\left(\frac{W_{g,n-1}(t_1,t_2,\dots,\hat{t_j}\dots,t_n) - W_{g,n-1}(t_2,\dots,t_n)}{t_1-t_j}\right) \\ = V'(t_1)W_{g,n}(t_1,\dots,t_n) - P_{g,n}(t_1,\dots,t_n) \end{aligned} \tag{12}$$

where the hat denotes omission and $P_{g,n}(t_1,\dots,t_n)$ *is the part of*

$$V'(t_1)W_{g,n}(t_1,\dots,t_n)$$

which is polynomial in t_1*.*

Proof. This follows at once from Lemma 7.7 by looking at the coefficient of $\hbar^{g+n-1}$ in Proposition 7.10 □

Note that those equations are recursive in $2g+n$. In particular for $(g,n)=(0,1)$, the equation for $W_{0,1}$ does not involve any other $W_{g,n}$ (but it involves $P_{0,1}$, we'll come to this). For everybody's comfort let us record once and for all what Equation (12) looks like when isolating $W_{g,n}$:

Corollary 7.12. *Loop equations imply*

$$\begin{aligned} \Big(2W_{0,1}(t_1) - V'(t_1)\Big)W_{g,n}(t_1,\dots,t_n) &= -W_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n) \\ &- \sum^{*}_{\substack{g_1+g_2=g \\ A\sqcup B=\{t_2,\dots,t_n\}}} W_{g_1,|A|+1}(t_1,A)W_{g_2,|B|+1}(t_1,B) \\ &- \sum_{j=2}^{n} \frac{\partial}{\partial t_j}\left(\frac{W_{g,n-1}(t_1,t_2,\dots,\hat{t_j}\dots,t_n) - W_{g,n-1}(t_2,\dots,t_n)}{t_1-t_j}\right) - P_{g,n}(t_1,\dots,t_n) \end{aligned}$$

where $\sum^{*}_{\substack{g_1+g_2=g \\ A\sqcup B=\{t_2,\dots,t_n\}}}$ *means that we don't consider the terms corresponding to* (g_1,A) *being equal to* $(0,\varnothing)$ *or* $(g,\{t_2,\dots,t_n\})$

7.3 The Riemann surface Σ and $\omega_{0,1}$

Looking at Equation (12), it seems that if one would hope to find recursively all the different $W_{g,n}$, one first has to compute all the $P_{g,n}$. This is where topological recursion comes to the rescue. Indeed, now that we have our generating functions, we can start defining the different objects needed to begin topological recursion. Of course the very first thing to construct has to be the Riemann surface Σ. We shall see below that in general, this can be constructed knowing only the leading order of $\varphi_\hbar(x^j)$ for $1\leqslant j\leqslant d-2$. Equation (12) for $g=0,n=1$, only involves one generating function, namely $W_{0,1}$. It is only natural to start with it.

Equation (8) applied to $g = 0$ (or Equation (12) for $(g, n) = (0, 1)$) yields the following equation for $W_{0,1}$:

$$W_{0,1}(t)^2 = V'(t)W_{0,1}(t) - P_{0,1}(t). \tag{13}$$

As we said earlier this is a closed form for $W_{0,1}$ once $P_{0,1}$ is known, and

$$P_{0,1}(t) = \sum_{k=2}^{d} \sum_{l=1}^{k-1} a_k t^{k-1} T^{(0)}_{k-l-1} t^{-k+l} \tag{14}$$

From the definition of $P_{0,1}$ we see that it only involves the values T_j for $0 \leqslant j \leqslant d-2$. Those numbers are the leading order of $\varphi_\hbar(x^j)$ for $0 \leqslant j \leqslant d-2$. According to our strategy we need to keep track what part of the data of $\varphi_\hbar$ we have used. Let us then make a mental note that we have used those leading orders.

Remark 7.13. It can actually be showed, for example in [EKR18], that $W_{0,1} = \frac{1}{2}(V' - M\sqrt{\sigma})$ where $M(t)$ and $\sigma(t)$ are polynomials such $(V')^2 - 4P_{0,1} = M^2\sigma$ and σ only has simple roots. Thus $W_{0,1}$ is a multivalued function on the complex plane.

Define now the function $s(t)$ on the Riemann sphere by

$$s(t) := W_{0,1}(t) - \frac{V'(t)}{2}.$$

By definition of $W_{0,1}$, this function is for the moment only defined around infinity. However, the loop equation (13) implies that $s(t)$ is solution to

$$s(t)^2 = \frac{V'(t)^2}{4} - P_{0,1}(t) \tag{15}$$

which also defines $s(t)$ has a multivalued function on the Riemann sphere (and also $W_{0,1}$ since it is s shifted by a polynomial). It is then natural to look at the algebraic curve defined by Equation (15). So let us consider the Riemann surface

$$\Sigma = \overline{\left\{(s,t) \in \mathbb{C}^2 |\ s^2 = \frac{V'(t)^2}{4} - P_{0,1}(t)\right\}}$$

The projection onto the t coordiantes realizes Σ as a double cover of the Riemann sphere. Since $V'(t)$ is a polynomial in t of degree $d-1$, the right hand side of Equation (15) is polynomial of degree $2d-2$. This has two consequences; the first one is that Σ is generically of genus $d-2$, and the second one is that ∞ is not a branch point. If we denote by $\infty_\pm$ the two points of Σ above $t = \infty$, we have

$$s(z) \approx \pm\frac{V'(t(z))}{2} \text{ as } z \to \infty_\pm$$

We shall suppose that

$$\frac{V'(t)^2}{4} - P_{0,1}(t) = \alpha \prod_{p=0}^{2d-2} (t - \alpha_p)$$

with α_p distinct numbers, i.e. it has only simple roots.

Define the meromorphic 1-form

$$\omega_{0,1} := s dt.$$

One can recover the data of the leading orders of $\varphi_\hbar(x^j)$ for any $j \geqslant 0$ by looking at the principal part of $\omega_{0,1}$ around ∞. Indeed, let t_∞ be a local parameter around ∞_-. Around this point the meromorphic 1-form $st^j dt$ is given by

$$st^j dt = \big(W_{0,1}(t_\infty^{-1}) + \frac{V'(t_\infty^{-1})}{2}\big)t_\infty^{-j} t_\infty^{-2} dt_\infty$$

Since V' is a polynomial, we have on the one hand that the powers of t_∞ coming from it are negative, and become strictly smaller than -1 after multiplying with t_∞^{-2}. On the other hand,

$$W_{0,1}(t_\infty^{-1})t_\infty^{-j}t_\infty^{-2}dt_\infty = \sum_{l\geqslant 0} T_l^{(0)} t_\infty^{l+1-j-2} dt_\infty$$

and the residue is then given by $T_j^{(0)}$. In short, we have for every $j \geqslant 0$:

$$T_j^{(0)} = Res(st^j dt; \infty_-).$$

In summary, we have shown that the knowledge of the leading order of the integral $\varphi_\hbar(x^j)$ for $0 \leqslant j \leqslant d-2$ is enough to know the leading orders of $\varphi_\hbar(x^k)$ for any k by the loop equations. This should not come as a surprise for we have seen that the leading order of a power series integral is itself an integral in the sense of Definition 3.14 and, in Section 6.2 we have seen that those are totally determined by their values on (x^j) for $0 \leqslant j \leqslant d-2$.

Said somewhat differently, the above discussion informs us that if we want to define to a power series integrals $\varphi_\hbar = \varphi_0 + \varphi_1 \hbar + \dots$, the first thing we do is say what are the values of $\varphi_0(x^j)$ for $0 \leqslant j \leqslant d-2$. This defines the polynomial $P_{0,1}(t)$. We can then build the Riemann surface Σ using Equation (15) and we read what are the values of $\varphi_0(x^k)$ in the principal part of the meromorphic 1-form sdt around ∞. What should we do form there? Well, that's the content of the next section.

7.4 The 2-form $\omega_{0,2}$

The last ingredient we need to start topological recursion is a symmetric meromorphic bidifferential with only a double pole on the diagonal and no residue. We don't want just any such, we want one whose behaviour at ∞ is given by $W_{0,2}$. We now come to the construction of this form $\omega_{0,2}$.

The first step is to make use of Corollary 7.12 for $g = 0, n = 2$. This can be rearranged to give

$$W_{0,2}(t_1,t_2) = \frac{-P_{0,2}(t_1,t_2) - \frac{\partial}{\partial t_2}\big(\frac{W_{0,1}(t_1)-W_{0,1}(t_2)}{t_1-t_2}\big)}{2s(t_1)}$$

using the definition of the function $s(t)$. Remember that $P_{0,2}(t_1,t_2)$ is by definition a polynomial in its first variable and an easy computation shows that, just like $P_{0,1}$, its degree in t_1 is $d-2$. More precisely, we have

$$P_{0,2}(t_1,t_2) = \sum_{j\geqslant 0}\sum_{k=2}^{d}\sum_{l=1}^{k-1} a_k T_{k-l-1,j}^{(0)}\ t_1^{l-1}t_2^{-j-1}. \quad (16)$$

In fact, the coefficient of the term in $t_1^{d-2}t_2^{-j-1}$ is $a_d T_{0,j}^{(0)}$ which is equal to 0 by Proposition 7.4. So $P_{0,2}(t_1,t_2)$ is actually a polynomial of degree $d-3$. This is important.

We would want to promote $W_{0,2}$ to a symmetric (meromorphic) function on $\Sigma \times \Sigma$, but it is not well defined away from $\infty_\pm$.

This means that we can't naively define our two form as

$$W_{0,2}(t(z_1), t(z_2))\ dt(z_1)dt(z_2)$$

for this is not globally defined. We can't neither use Equation (7.4) for the term in $P_{0,2}$ is not defined globally neither. What we can do however, is already promote the part

$$\frac{1}{2s(t_1)}\frac{\partial}{\partial t_2}\left(\frac{W_{0,1}(t_1) - W_{0,1}(t_2)}{t_1 - t_2}\right)$$

to a bidifferential form on $\Sigma \times \Sigma$ since the functions $W_{0,1}(z)$, $t(z)$ and $s(z)$ are well defined global functions on Σ. Set then $\tilde{\omega}(z_1, z_2)$ to be the form on $\Sigma \times \Sigma$ defined by

$$\tilde{\omega}(z_1, z_2) = \frac{1}{2s(z_1)} d_{z_2}\left(\frac{W_{0,1}(t(z_1)) - W_{0,1}(t(z_2))}{t(z_1) - t(z_2)}\right)dt(z_1)$$

where d_{z_2} is the de Rham differential applied to the second copy of Σ.

Let us now pay attention to the part coming from $P_{0,2}$. We know that it should correspond to a form whose behaviour as $t(z_i) \to \infty$ is given by

$$\frac{P_{0,2}(t(z_1), t(z_2))}{s(z_1)} dt(z_1)dt(z_2) \tag{17}$$

which is a holomorphic around ∞ since $P_{0,2}$ is a polynomial of degree at most $d-3$ in its first variable (c.f. Proposition A.1). To get a globally define form, we could try to find a holomorphic form on $\Sigma \times \Sigma$ whose behaviour around ∞ is given by Equation (17). Thankfully, we know a basis of holomorphic forms on $\Sigma \times \Sigma$, namely

$$\frac{t^a(z_1)t^b(z_2)}{s(z_1)s(z_2)} ds(z_1)ds(z_2)$$

for $1 \leqslant a, b \leqslant d-3$.

Around infinity, we wish to get the equality :

$$\frac{-P_{0,2}(t(z_1), t(z_2))}{s(z_1)} = \sum_{0 \leqslant a,b \leqslant d-3} \kappa_{a,b} \frac{t^a(z_1)t^b(z_2)}{s(z_1)s(z_2)}$$

for some numbers $\kappa_{a,b}$, or equivalently

$$-P_{0,2}(t(z_1), t(z_2))s(z_2) = \sum_{0 \leqslant a,b \leqslant d-3} \kappa_{a,b}\ t^a(z_1)t^b(z_2) \tag{18}$$

Around ∞, we know that $s(z_2)$ takes the form

$$\prod_{p=1}^{2d-2} \sqrt{t(z_2) - \alpha_p} = t(z_2)^{d-1} \prod_{p=1}^{2d-2} \sqrt{1 - \alpha_p/t(z_2)}.$$

Let us express this product of square roots as

$$\sum_{n \geqslant 0} R_n t(z_2)^{-n}$$

which together with Equation (16) means that the right hand side of Equation (18) is equal to

$$-\sum_{n\geqslant 0}\sum_{j\geqslant 0}\sum_{k=2}^{d}\sum_{l=1}^{k-1} a_k T^{(0)}_{k-l-1,j} R_n\, t(z_1)^{l-1} t(z_2)^{-j-1} t(z_2)^{d-1-n} \tag{19}$$

From there, we can read what should be the coefficient $\kappa_{a,b}$ for $0 \leqslant a,b \leqslant d-3$ in Equation (19), namely

$$\kappa_{a,b} := -\sum_{j=0}^{d-3-b}\sum_{k=a+2}^{d-1} a_k T^{(0)}_{k-a-2,j} R_{d-2-b-j}. \tag{20}$$

Observe that the only $T^{(0)}_{i,j}$ involved in those expressions are those with $0 \leqslant i,j \leqslant d-3$. This means that once we have fixed those coefficients we can *define* $\omega(z_1,z_2)$ by

$$\omega(z_1,z_2) := -\tilde{\omega}(z_1,z_2) + \sum_{0\leqslant a,b\leqslant d-3} \kappa_{a,b} \frac{t^a(z_1)t^b(z_2)}{s(z_1)s(z_2)} \tag{21}$$

with the $\kappa_{a,b}$ defined by Equation (20), and we have

$$\omega(z_1,z_2) = W_{0,2}(t(z_1),t(z_2))dt(z_1)dt(z_2)$$

as $t(z_i) \to \infty$. Notice that this implies that ω is symmetric by the Identity theorem. Indeed it is symmetric around ∞ since the generating function $W_{0,2}$ is symmetric.

Let us fix $z_2 \in \Sigma$ and consider $\omega(z_1,z_2)$ as a 1-form on Σ. What could be its pole?

- Naturally, there is no pole coming from the holomorphic part.
- There might be a pole when $s(z_1) = 0$, that is when z_1 is a branch point of the cover $t : \Sigma \to \mathbb{CP}^1$, but this is compensated by the rest of the expression since $W_{0,1}(z_1) = s(z_1) + \frac{V'(t(z_1))}{2}$ and $V'(t(z))/s(z)dt(z)$ is holomorphic around branch points.
- The term in $W_{0,1}$ might introduce a pole when $t(z_1)$ goes to infinity, but it is compensated by the pole of $s(z_1)$.
- There might be a pole when $t(z_1) = t(z_2)$. If z_1 and z_2 are in the same sheet, i.e. $z_1 = z_2$ we have $W_{0,1}(z_1) - W_{0,2}(z_2) = 0$ and of course there is no pole. If z_1 is not in the same sheet as z_2, i.e. $z_1 = \sigma(z_2)$ where σ is the hyperelliptic involution, then in that case there is truly a pole. Let us pick ξ to be a parameter around around $\sigma(z_2)$ (and $\xi \circ \sigma$ is then a parameter around z_2). In those coordinates, the non-constant part in z_1 of our form is given by
$$\frac{1}{s(\xi(z_1))}\frac{s(\xi(z_1)) + \frac{V'(\xi(z_1))}{2}}{(\xi(z_1)-\xi(z_2))^2} d\xi(z_1)d\xi(z_2) \approx \frac{d\xi(z_1)d\xi(z_2)}{(\xi(z_1)-\xi(z_2))^2}$$
as $z_1 \to \sigma(z_2)$, i.e. this pole is of order 2 and has no residue.

Proposition 7.14.

$$\omega(z_1,z_2) + \omega(\sigma(z_1),z_2)) = \frac{dt(z_1)dt(z_2)}{(t(z_1)-t(z_2))^2}.$$

Proof. Since $s(\sigma(z_1)) = -s(z_1)$ we have that the two holomorphic parts cancel each other and

$$d_{z_2}\frac{-W_{0,1}(t(z_2))}{t(z_1)-t(z_2)}\frac{1}{2s(z_1)} + d_{z_2}\frac{-W_{0,1}(t(z_2))}{t(\sigma(z_1))-t(z_2)}\frac{1}{2s(\sigma(z_1))} = 0$$

Finally,

$$\frac{d_{z_2}\frac{W_{0,1}(z_1)}{t(z_1)-t(z_2)}}{2s(z_1)} + \frac{d_{z_2}\frac{W_{0,1}(\sigma(z_1))}{t(\sigma(z_1))-t(z_2)}}{2s(\sigma(z_1))} = -\frac{2s(z_1)}{2s(z_1)}\frac{dz(z_2)}{(t(z_1)-t(z_2))^2}$$
$$= -\frac{dz(z_2)}{(t(z_1)-t(z_2))^2}$$

where we used that $W_{0,1}(z_1) = s(z_1) + \frac{V'(t(z_1))}{2}$ □

This means that the form $\omega(z_1, z_2) - \frac{dt(z_1)dt(z_2)}{(t(z_1)-t(z_2))^2}$ is a bidifferential form with only a double pole on the diagonal and no residue. This is the form want to continue with topological recursion, i.e. define

Definition 7.15. The 2-form $\omega_{0,2}$ is defined by

$$\omega_{0,2}(z_1, z_2) := \omega(z_1, z_2) - \frac{dt(z_1)dt(z_2)}{(t(z_1)-t(z_2))^2} = -\omega(\sigma(z_1), z_2).$$

We will also need a slightly different two form

Definition 7.16. The 2-form $\hat{\omega}_{0,2}$ is defined by

$$\hat{\omega}_{0,2}(z_1, z_2) := \omega(z_1, z_2) - \frac{1}{2}\frac{dt(z_1)dt(z_2)}{(t(z_1)-t(z_2))^2}.$$

While the use of the 2-form is not apparent yet, it will be useful to define higher forms $\omega_{g,n}$. An other upshot is that it is antisymmetric with respect to the hyperelliptic involution σ:

Proposition 7.17.

$$\hat{\omega}_{0,2}(\sigma(z_1), z_2) = -\hat{\omega}_{0,2}(z_1, z_2)$$

Proof. This follows from Proposition 7.14 and the fact that

$$\frac{1}{2}\frac{dt(z_1)dt(z_2)}{(t(z_1)-t(z_2))^2} = \frac{1}{2}\frac{dt(\sigma(z_1))dt(z_2)}{(t(\sigma(z_1))-t(z_2))^2}$$

□

Let us summarize what we have done in this section. We have defined a symmetric meromorphic form $\omega_{0,2}$ with only a double pole on the diagonal with no residue using only the knowledge of the subleading order in $\hbar$ of

$$\varphi_\hbar((x^i)(x^j)) - \varphi_\hbar((x^i))\varphi_\hbar((x^j))$$

for $0 \leqslant i, j \leqslant d-3$. All the other subleading orders of 2-cumulants can then be read in the residues of $\omega_{0,2}$.

In the next section, we shall see that all others $W_{g,n}$ are uniquely determined by the finite number of choices we have done so far.

7.5 Topological recursion

So far we have fixed a Riemann surface Σ together with a 1-form $\omega_{0,1}$ and a symmetric form $\omega_{0,2}$ on $\Sigma\times\Sigma$ from which we can read the different coefficients of the functions $W_{0,1}$ and $W_{0,2}$. To construct all of this data, we needed to know only finitely many parameters of $\varphi_\hbar$. We shall now construct all other differentials $\omega_{g,n}$ using the loop equations. From their definition, it will be obvious that their behaviour around ∞ will be related to the generating functions $W_{g,n}$. We will show that they satisfy the topological recursion formula, ensuring that they are actually fixed by $\omega_{0,1}$ and $\omega_{0,2}$.

Before defining the different $\omega_{g,n}$ we fix a basis of cycles $(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^{d-2}$ in $H^1(\Sigma,\mathbb{C})$ such that

$$\mathcal{A}_i\bigcap\mathcal{A}_j=\mathcal{B}_i\bigcap\mathcal{B}_j=0 \text{ and } \mathcal{A}_i\bigcap\mathcal{B}_j=\delta_{ij}$$

Here, $\bigcap$ denotes the intersection number of two cycles and such a basis always exists, c.f. Appendix A and references therein for more details.

We shall use Corollary 7.12 to define the forms $\omega_{g,n}$ recursively on $2g+n$. Indeed set first

$$\begin{aligned}\omega_{g,n}(z_1,\dots,z_n):=&\\ -\frac{\overline{\omega}_{g-1,n+1}(z_1,z_1,z_2,\dots,z_n)}{2\hat{\omega}_{0,1}(z_1)}&-\left[\sum^{*}_{\substack{g_1+g_2=g\\ A\sqcup B=\{z_2,\dots,z_n\}}}\hat{\omega}_{g_1,|A|+1}(z_1,A)\hat{\omega}_{g_2,|B|+1}(z_1,B)\right]\frac{1}{2\hat{\omega}_{0,1}(z_1)}\\ +\sum_{j=2}^{n}d_j\frac{\hat{\omega}_{g,n-1}(z_2,\dots,z_n)}{(t(z_1)-t(z_j))dt(z_j)}\frac{dt(z_1)}{2s(z_1)}&-\frac{P_{g,n}(t(z_1),\dots,t(z_n))\prod_{i=1}^{n}dt(z_n)}{2s(z_1)}\end{aligned}\tag{22}$$

where $\hat{\omega}_{g,n}=\omega_{g,n}$ if $(g,n)\neq(0,2)$ and $\hat{\omega}_{0,2}$ is given by Definition 7.16 and $\overline{\omega_{g,n}}=\omega_{g,n}$ if $(g,n)\neq(0,2)$ and $\overline{\omega}_{0,2}=\omega$ (see Equation 21)

Remark 7.18. The definition of $\overline{\omega}_{g,n}$ is just to make the definition of $\omega_{1,1}$ fit.

Remark 7.19. At first glance, when comparing with Corollary 7.12, it might seem that the terms of the form

$$d_j\frac{\omega_{g,n-1}}{t(z_1)-t(z_j)}$$

are missing. They have actually been absorbed by some terms in the sum over different $A\sqcup B=\{z_1,\dots,z_n\}$, more precisely when A or B is equal to $\{z_j\}$ because of the definition of $\hat{\omega}_{0,2}$.

Just as before, there is a small issue with defining

$$\frac{P_{g,n}(t(z_1),\dots,t(z_n))}{s(z_1)}\prod_{i=1}dt(z_i).$$

What we mean by that is that, just like in the case of $P_{0,2}$, we choose a product of holomorphic 1-forms whose development at ∞ is given by $P_{g,n}$.

Note that around infinity

$$\omega_{g,n}(z_1,\dots,z_n)=W_{g,n}(t(z_1),\dots,t(z_n))\,dt(z_1)\dots dt(z_n)$$

and thus $\omega_{g,n}$ is a symmetric meromorphic form by the Identity theorem.

We will see that unlike $P_{0,2}$, $P_{g,n}$ is fixed by our previous choices. To be more precise we have the following theorem:

Theorem 7.20. *The multidifferential forms $\omega_{g,n}$ satisfy the topological recursion formula*

$$\omega_{g,n+1}(z_0,\vec{z}) = \frac{1}{2}\sum_{i=1}^{2d-2} \operatorname*{Res}_{z\to b_i} \frac{\left[\int_{p=\sigma(z)}^{z} \omega_{0,2}(z_0,p)\right]}{2\omega_{0,1}(z)} \times \left[-\omega_{g-1,n+2}(z,\sigma(z),\vec{z}) - \sum_{\substack{g_1+g_2=g \\ A\sqcup B=\vec{z}}}^{*} \omega_{g_1,|A|+1}(z,A)\,\omega_{g_2,|B|+1}(\sigma(z),B)\right]$$

where $\vec{z} := \{z_1,\ldots,z_n\}$ and $b_1,\ldots,b_{2d-2}$ are the branch points of the cover $t:\Sigma\to\mathbb{CP}^1$.

Notice that this theorem states that once $\omega_{0,1}$ and $\omega_{0,2}$ have been fixed, then the other $\omega_{g,n}$ are uniquely determined. In particular, the different $P_{g,n}$ for $2g+n>2$ do not enter the formulas. In other words, the parameters of $\varphi_\hbar$ we have used to construct Σ, $\omega_{0,1}$ and $\omega_{0,2}$ together with the loop equations are enough to uniquely determine the power series integral $\varphi_\hbar$. This can summarized in the following theorem

Theorem 7.21. *A power series integral in one variable $\varphi_\hbar$ for the potential*

$$V = \sum_{k=2}^{d} \frac{a_k}{k} x^k$$

is uniquely determined by the values of

$$\lim_{\hbar\to 0} \varphi_\hbar(x^i)$$

for $1 \leqslant i \leqslant d-2$. and

$$\lim_{\hbar\to 0} \frac{\varphi_\hbar\big((x^i)(x^j)\big) - \varphi_\hbar(x^i)\varphi_\hbar(x^j)}{\hbar}$$

for $1 \leqslant i,j \leqslant d-3$.

The rest of this section is dedicated to the proof of Theorem 7.20. The proof relies on several classical yet powerful results about Riemann surfaces. The reader is invited to have a look at Appendix A and references therein for more details about those results.

Before we get started we record two easy observations:

Proposition 7.22. *For $2g+n-2>0$, the differential forms $\omega_{g,n}$ have poles only at the branch points b_i of the cover $t:\Sigma\to\mathbb{CP}^1$*

Proof. This follows at once by induction. Indeed by the induction hypothesis, all the numerators can only have poles at the branch points, and dividing by $\omega_{0,1}(z_1)$ can only introduce poles when that quantity is equal to 0. But this can happen only at the branch points. ☐

Proposition 7.23. *For every (g,n) and $1 \leqslant i \leqslant n$*

$$\omega_{g,n}(z_1,\ldots\sigma(z_i),\ldots,z_n) = -\omega_{g,n}(z_1,\ldots,z_i\ldots,z_n)$$

Proof. It is clearly true for $\omega_{0,1}$, and it is true for $\hat{\omega}_{0,2}$ by Proposition 7.17 and symmetry. It also follows from the same proposition for $\omega_{0,3}$. It then follows easily by induction that the result is true for any $w_{g,n}$ when considering the first variable, and then any variable by symmetry. □

Proof. (of Theorem 7.20)

For z_0 fixed, consider

$$\int_{p=o}^{z} \omega_{0,2}(z_0, p)$$

where the integration path lies inside the fundamental domain of Σ. In the variable z, it only has a pole at $z = z_0$. This allows us to write the Cauchy formula

$$\omega_{g,n+1}(z_0, \vec{z}) = \operatorname*{Res}_{z \to z_0} \left[\int_{p=o}^{z} \omega_{0,2}(z_0, p) \right] \omega_{g,n+1}(z, \vec{z}) \tag{23}$$

Riemann bilinear identity (Proposition A.10) together with Proposition A.4 informs us that

$$\begin{aligned} &\operatorname*{Res}_{z \to \text{ all poles}} \left[\int_{p=o}^{z} \omega_{0,2}(z_0, p) \right] \omega_{g,n+1}(z, \vec{z}) \\ &= \sum_{i=1}^{2d-2} \oint_{z \in \mathcal{A}_i} \omega_{0,2}(z_0, z) \oint_{z \in \mathcal{B}_i} \omega_{g,n+1}(z, \vec{z}) - \oint_{z \in \mathcal{B}_i} \omega_{0,2}(z_0, z) \oint_{z \in \mathcal{A}_i} \omega_{g,n+1}(z, \vec{z}) \end{aligned} \tag{24}$$

Here by all poles we mean z_0 (the unique pole of $\int_{p=o}^{z} \omega_{0,2}(z_0, p)$) and all the branch points b_i (the poles of $\omega_{g,n+1}$). Putting Equations (23) and (24) together, we get

$$\begin{aligned} \omega_{g,n+1}(z_0, \vec{z}) &= - \sum_{i=1}^{2d-2} \operatorname*{Res}_{z \to b_i} \left[\int_{p=o}^{z} \omega_{0,2}(z_0, p) \right] \omega_{g,n+1}(z, \vec{z}) \\ &+ \sum_{i=1}^{2d-2} \oint_{z \in \mathcal{A}_i} \omega_{0,2}(z_0, z) \oint_{z \in \mathcal{B}_i} \omega_{g,n+1}(z, \vec{z}) - \oint_{z \in \mathcal{B}_i} \omega_{0,2}(z_0, z) \oint_{z \in \mathcal{A}_i} \omega_{g,n+1}(z, \vec{z}) \end{aligned} \tag{25}$$

However, we have that the integrals over $\mathcal{A}$ cycles of $\omega_{0,2}$ in Equation (25) vanishes, leaving us with

$$\begin{aligned} \omega_{g,n+1}(z_0, \vec{z}) = &- \sum_{i=1}^{2d-2} \operatorname*{Res}_{z \to b_i} \left[\int_{p=o}^{z} \omega_{0,2}(z_0, p) \right] \omega_{g,n+1}(z, \vec{z}) \\ &- \oint_{z \in \mathcal{B}_i} \omega_{0,2}(z_0, z) \oint_{z \in \mathcal{A}_i} \omega_{g,n+1}(z, \vec{z}) \end{aligned} \tag{26}$$

Plugging in Equation (22) in the term with the residue, we have to compute the residues of different contributions. Notice that

$$\frac{\left[\int_{p=o}^{z} \omega_{0,2}(z_0, p) \right] dt(z)}{2s(z)} \left[\sum_{j=2}^{n} d_j \frac{\hat{\omega}_{g,n}(z_2, \ldots, z_n)}{(t(z) - t(z_j)) dt(z_j)} - P_{g,n+1}(t(z), t(z_1), \ldots, t(z_n)) \prod_{i=1}^{n} dt(z_n) \right]$$

is holomorphic around b_i. Indeed we are looking at something that has no pole in b_i for $dt(z)/s(z)$ has no pole at the branch point b_i. So after taking residue, this term's contribution is 0.

Performing the change of variable $z \to \sigma(z)$ together with Proposition 7.23, we have that

$$- \operatorname*{Res}_{z\to b_i} \left[\int_{p=o}^{z} \omega_{0,2}(z_0, p) \right] \frac{\omega_{g-1,n+1}(z, z, \vec{z})}{2\omega_{0,1}(z)} = \operatorname*{Res}_{z\to b_i} \left[\int_{p=\sigma(z)}^{o} \omega_{0,2}(z_0, p) \right] \frac{\omega_{g-1,n+1}(z, \sigma(z), \vec{z})}{2\omega_{0,1}(z)} \tag{27}$$

and by Proposition 7.23 we also have

$$- \operatorname*{Res}_{z\to b_i} \left[\int_{p=o}^{z} \omega_{0,2}(z_0, p) \right] \frac{\omega_{g-1,n+1}(z, z, \vec{z})}{2\omega_{0,1}(z)} = \operatorname*{Res}_{z\to b_i} \left[\int_{p=o}^{z} \omega_{0,2}(z_0, p) \right] \frac{\omega_{g-1,n+1}(z, \sigma(z), \vec{z})}{2\omega_{0,1}(z)} \tag{28}$$

and thus taking the average of Equations (27) and (28) we get

$$- \operatorname*{Res}_{z\to b_i} \left[\int_{p=o}^{z} \omega_{0,2}(z_0, p) \right] \frac{\omega_{g-1,n+1}(z, z, \vec{z})}{2\omega_{0,1}(z)} = \frac{1}{2} \operatorname*{Res}_{z\to b_i} \left[\int_{p=\sigma(z)}^{z} \omega_{0,2}(z_0, p) \right] \frac{\omega_{g-1,n+1}(z, \sigma(z), \vec{z})}{2\omega_{0,1}(z)} \tag{29}$$

Similarly, we have

$$\begin{aligned} &- \operatorname*{Res}_{z\to b_i} \frac{\left[\int_{p=0}^{z} \omega_{0,2}(z_0, p)\right]}{2\hat{\omega}_{0,1}(z)} \sum_{\substack{g_1+g_2=g \\ A\sqcup B=\vec{z}}}^{*} \hat{\omega}_{g_1,|A|+1}(z, A)\, \hat{\omega}_{g_2,|B|+1}(z, B) \\ =&\frac{1}{2} \operatorname*{Res}_{z\to b_i} \frac{\left[\int_{p=\sigma(z)}^{z} \omega_{0,2}(z_0, p)\right]}{2\hat{\omega}_{0,1}(z)} \sum_{\substack{g_1+g_2=g \\ A\sqcup B=\vec{z}}}^{*} \hat{\omega}_{g_1,|A|+1}(z, A)\, \hat{\omega}_{g_2,|B|+1}(\sigma(z), B) \end{aligned} \tag{30}$$

In this expression, all the $\hat{\omega}_{g,n}$ can be replaced by $\omega_{g,n}$. Indeed,

$$\begin{aligned} &\hat{\omega}_{g_1,|A|+1}(z, A)\, \hat{\omega}_{g_2,|B|+1}(\sigma(z), B) \;+\; \hat{\omega}_{g_2,|B|+1}(z, B)\, \hat{\omega}_{g_1,|A|+1}(\sigma(z), A) \\ =&\omega_{g_1,|A|+1}(z, A)\, \omega_{g_2,|B|+1}(\sigma(z), B) \;+\; \omega_{g_2,|B|+1}(z, B)\, \omega_{g_1,|A|+1}(\sigma(z), A) \end{aligned}$$

by Proposition 7.23 together with

$$\frac{dt(z_1)dt(z_2)}{(t(z_1) - t(z_2))^2} = \frac{dt(\sigma(z_1))dt(z_2)}{(t(\sigma(z_1)) - t(z_2))^2}.$$

Thus

$$\begin{aligned} &\sum_{\substack{g_1+g_2=g \\ A\sqcup B=\vec{z}}}^{*} \hat{\omega}_{g_1,|A|+1}(z, A)\, \hat{\omega}_{g_2,|B|+1}(\sigma(z), B) \\ =&\sum_{\substack{g_1+g_2=g \\ A\sqcup B=\vec{z}}}^{*} \omega_{g_1,|A|+1}(z, A)\, \omega_{g_2,|B|+1}(\sigma(z), B) \end{aligned} \tag{31}$$

Naturally, $\hat{\omega}_{0,1} = \omega_{0,1}$.

Putting this together with Equation (29) we obtain

$$\begin{aligned}
&\omega_{g,n+1}(z_0, \vec{z}) = \\
&\frac{1}{2}\sum_{i=1}^{2d-2} \operatorname*{Res}_{z\to b_i} \frac{\left[\int_{p=\sigma(z)}^{z} \omega_{0,2}(z_0,p)\right]}{2\omega_{0,1}(z)} \left[-\omega_{g-1,n+2}(z,\sigma(z),\vec{z}) - \sum_{\substack{g_1+g_2=g\\ A\sqcup B=\vec{z}}}^{*} \omega_{g_1,|A|+1}(z,A)\,\omega_{g_2,|B|+1}(\sigma(z),B)\right] \\
&- \int_{z\in\mathcal{B}_i} \omega_{0,2}(z_0,z) \int_{z\in\mathcal{A}_i} \omega_{g,n+1}(z,\vec{z}).
\end{aligned} \tag{32}$$

This is almost the result we wish to get, we just have one extra term that we somehow need to get rid of, namely the term in product of integrals.

It is a standard result of topological recursion that the integral over $z_0 \in \mathcal{A}_i$ of the first term in the right hand side of (32) is equal to 0 (see for example [EO07]). Note also that

$$\alpha(z_0, z_1, \dots z_n) := \oint_{z\in\mathcal{B}_i} \omega_{0,2}(z_0,z) \oint_{z\in\mathcal{A}_i} \omega_{g,n+1}(z,\vec{z})$$

is a $n+1$ form. Since $\oint_{z_0\in\mathcal{A}_i} \omega_{0,2}(z_0,z) = 0$ we have

$$\oint_{z_0\in\mathcal{A}_i} \alpha(z_0, z_1, \dots z_n) = 0$$

Those two facts put together show that

$$\oint_{z_0\in\mathcal{A}_i} \omega_{g,n+1}(z_0,\vec{z}) = 0$$

and thus the second term in (32) is equal to 0, which proves the topological recursion formula. □

7.6 Two examples

Using the machinery developed above, we can extend the result of Proposition 3.15 to power series integrals

Proposition 7.24. *There is a unique power series integral for the potential $V(x) = \frac{x^2}{2}$.*

Proof. For this potential, there is no choice for what the polynomial $P_{0,1}(t)$ can be. Indeed, from (14) we have

$$P_{0,1}(t) = T_0^{(0)}$$

and

$$T_0^{(0)} = 1$$

by the normalization property. We then have that the equation defining the Riemann surface Σ is

$$s^2(t) = \frac{t^2}{4} - 1 = \frac{1}{4}(t - \frac{1}{2})(t + \frac{1}{2})$$

from which we get that Σ is of genus 0, i.e. it is the Riemann sphere. Since there is no holomorphic 1-form on the Riemann sphere, there is only one bidifferential form with only poles on the diagonal with no residue, namely

$$\frac{dt(z_1)dt(z_2)}{(t(z_1) - t(z_2))^2}$$

and we thus have no choice as to what $\omega_{0,2}$ could be. □

Proposition 7.25. *Let V be a potential of degree 3. An integral $\varphi_\hbar$ with respect to V is uniquely determined by φ_0.*

Proof. Since V is of degree 3, $\frac{(V'(t))^2}{4} - P_{0,1}(t)$ is of degree 4, no matter the choice of $P_{0,1}$. The Riemann surface Σ is then at most of genus 1. From there, the 2-form $\omega_{0,2}$ depends only on the choice of $T^{(0)}_{0,0}$, but this number is equal to 0 by Proposition 7.4. □

8 Deformation

We have seen that both infinity integrals and power series integrals can be understood as maps from the zeroth cohomology of a certain chain complex. In the case of infinity integrals, this chain complex is actually a differential graded algebra, and integrals are maps of algebra. This allowed us to find a finite amount of parameters that would uniquely determine an infinity integrals.

In the case of power series integrals, it is a priori not clear at all that one can also find a finite amount of parameters which determine the integral. However, at least in the case of one variables, power series integrals are uniquely determine by numbers

$$\lim_{\hbar\to 0} \varphi_\hbar(x^i) = \varphi_0(x^i)$$

and

$$\lim_{\hbar\to 0} \frac{\varphi_\hbar((x^i)(x^j)) - \varphi_\hbar(x^i)\varphi_\hbar(x^j)}{\hbar}$$

for finitely many i, j. However, it is not clear how to interpret that in the cohomological setting. One issue is that the number of parameters is not even fixed: different choices of $P_{0,1}$ (c.f. notation of Section 7) might produce spectral curves with different genera.

By examining one last time Gaussian integration, we propose a direction to answer this problem. It has to be said that this section is much more speculative in nature.

8.1 A new product

We look at Gaussian integration in a slightly broader context than Gaussian integration of Hermitian matrices. Let $V = \mathbb{R}^d$ be a real vector space of dimension d together with a $d \times d$ invertible symmetric matrix B and denote by $(-,-)$ the usual scalar product. Denote also by $\mathcal{O}_d = \mathbb{R}[x_1, \ldots x_d]$ the ring of commutative polynomials in d variables. In what follows, for $1 \leqslant i \leqslant d$ denote by ∂_{x_i} the partial derivative with respect to x_i. To make notations a bit less cumbersome, we define

$$B^{ij} := (B^{-1})_{ij}$$

and we use Einstein convention throughout.

Definition 8.1. For any $f \in \mathcal{O}_d$ define

$$\langle f \rangle := \frac{1}{Z} \int_{\mathbb{R}^d} dx f(x) e^{-\frac{(Bx,x)}{2}}$$

where dx is the Lebesgue measure and Z is a number such that $\langle 1 \rangle = 1$.

Those expectations value can be computed using Wick's theorem.

Theorem 8.2. *(Wick) For any $f \in \mathcal{O}_d$,*

$$\langle f \rangle = f(\partial_1, \ldots, \partial_d)\, e^{\frac{(B^{-1}x,x)}{2}}{}_{|x=0} = e^{\frac{B^{ij}\partial_{x_i}\partial_{x_j}}{2}} f(x)_{|x=0}$$

Remark 8.3. Theorem 3.4 introducing ribbon graphs to compute Gaussian expectation values of Hermitian matrices is actually a consequence of Wick's theorem in that context.

We shall see that this formula allows us to define a new product $*$ on $\mathcal{O}_d$ such that $\langle f * g \rangle = \langle f \rangle \langle g \rangle$.

We may assume that B is a diagonal matrix. Indeed, since B is symmetric, we may perform a linear change of variables to bring it to a diagonal form.

Definition 8.4. Let $D : \mathcal{O}_d \otimes \mathcal{O}_d \to \mathcal{O}_d \otimes \mathcal{O}_d$ be the operator defined by

$$e^{-B^{ii}\,\partial_{x_i}\otimes\partial_{x_i}}$$

which is well defined on tensor products on polynomials for given two polynomials f and g, only finitely many term of the exponential are non-zero.

Definition 8.5. For two polynomials f and g in $\mathcal{O}_d$ define

$$f * g := \mu \circ D(f \otimes g)$$

where μ is the usual product of $\mathcal{O}_d$.

Proposition 8.6. $\langle f * g\rangle = \langle f\rangle \langle g\rangle$

Proof. We notice that

$$e^{B^{ii}\frac{\partial^2_{x_i}}{2}} \circ \mu = \mu \circ e^{B^{ii}\frac{\partial^2_{x_i}\otimes 1 + 2\partial_{x_i}\otimes\partial_{x_i} + 1\otimes\partial^2_{x_i}}{2}}$$

for every partial derivative in the left hand side is applied to the function coming either from the first factor of the tensor product or from the second factor.

We then have

$$e^{B^{ii}\frac{\partial^2_{x_i}}{2}} \circ \mu \circ D = \mu \circ e^{B^{ii}\frac{\partial^2_{x_i}\otimes 1 + 2\partial_{x_i}\otimes\partial_{x_i} + 1\otimes\partial^2_{x_i}}{2}} \circ D = \mu \circ e^{B^{ii}\frac{\partial^2_{x_i}\otimes 1 + 1\otimes\partial^2_{x_i}}{2}}$$

By Theorem 8.2, evaluating at $x = 0$ for f and g in $\mathcal{O}_d$ gives us $\langle f * g\rangle = \langle f\rangle \langle g\rangle$. $\square$

In section 5, we saw that the commutative analogue of the algebra $\mathcal{C}(1)$ was given by polyvector fields (actually it was even the motivation for the construction of $\mathcal{C}(1)$). We know see how the product $*$ fits into that framework.

Definition 8.7. Let $\mathcal{T} := \mathbb{R}[x_1, \dots, x_d, \eta^1, \dots, \eta^d]$ be the graded commutative algebra of polynomials in variables x_i and η^i, the former being of degree 0 and the latter of degree -1.

Note that in the commutative setting there is no need to quotient out by commutators and then take the symmetric algebra on the resulting vector space.

Remark 8.8. The map $\langle - \rangle : \mathcal{O}_d \to \mathbb{C}$ can be reinterpreted as a degree 0 map $\mathcal{T} \to \mathbb{C}$.

Remark 8.9. The operator D defined earlier extends to the $\mathcal{T}$ and thus the new multiplication $f * g = \mu \circ D(f \otimes g)$ also extends to $\mathcal{T}$. Note that this new product is still graded commutative for D is a map of degree 0.

In this commutative setting, the differential operator of order at most two $\Delta : \mathcal{T} \to \mathcal{T}$ is

$$\Delta := \partial_{x_i}\partial_{\eta^i} - B^{ii} x_i \partial_{\eta^i}$$

Proposition 8.10. *Let $f \in \mathcal{T}$ be a homogeneous of degree -1. Then $\langle \Delta(f)\rangle = 0$.*

Proof. This is true if $d = 1$ where it is just integration by parts. Since B is diagonal and we are only dealing with polynomials, the general case is follows by Fubini. $\square$

Clearly, Δ is a differential operator of order at most 2 on $\mathcal{T}$ (with respect to the usual product of polynomials). We wish to show that this statement stays true if we consider the new product $*$ instead.

To be more precise we wish to show that

$$\Delta(a*b*c) - \Delta(a*b)*c + \Delta(a)*b*c - (-1)^a a*\Delta(b*c) - (-1)^{(a+1)b} b*\Delta(a*c) \\ + (-1)^a a*\Delta(b)*c + (-1)^{a+b} a*b*\Delta(c) = 0.$$

To prove it, we start by recording the following straightforward facts:

Lemma 8.11. *Let A be an algebra. An operator Δ is a differential operator of order at most 2 if and only if for every homogeneous element $a \in A$ the map $[a,-]_\Delta : A \to A$ defined by*

$$b \mapsto [a,b]_\Delta := (-1)^a \Delta(a*b) - (-1)^a \Delta(a)*b - a*\Delta(b)$$

is a derivation of degree $|a|+1$

Lemma 8.12. *Let A be a graded algebra and let d be a derivation of degree 1. Then d is a differential operator of order at most 2 and for every homogeneous $a \in A$ we have $[a,-]_d = 0$.*

Lemma 8.13. *Let A be a graded algebra and let d_1, d_2 be two derivations. The composition $d_1 \circ d_2$ is a differential operator of order at most 2.*

Lemma 8.14. *Let A be a graded commutative algebra and let d_1 and d_2 be two derivations of A (not necessarily of the same degree). For every homogenous $a \in A$, the map*

$$b \mapsto d_1(a)d_2(b)$$

is a derivation of degree $|a|+|d_1|+|d_2|$

Lemma 8.15. *The maps $\partial_{x_i}, \partial_{\eta_i} : \mathcal{T} \to \mathcal{T}$ are derivations for the product $*$.*

Proof. The fact that ∂_{x_i} is a derivation for the usual product of polynomials μ can be rewritten as

$$\partial_{x_i} \circ \mu = \mu \circ (\partial_{x_i} \otimes 1 + 1 \otimes \partial_{x_i}).$$

Since partial derivatives commute, both the operators $\partial_{x_i} \otimes 1$ and $1 \otimes \partial_{x_i}$ commute with D.

We then have

$$\partial_{x_i} \circ \mu \circ D = \mu \circ (\partial_{x_i} \otimes 1 + 1 \otimes \partial_{x_i}) \circ D = \mu \circ D \circ (\partial_{x_i} \otimes 1 + 1 \otimes \partial_{x_i})$$

which precisely mean that ∂_{x_i} is a derivation for the product $* = \mu \circ D$.

The proof is the same for ∂_{η_i}, one just needs to understand the tensor products $\partial_{\eta_i} \otimes 1$ and $1 \otimes \partial_{\eta_i}$ as tensor products of graded maps (that is $(\alpha \otimes \beta)\ (a \otimes b) = (-1)^{\beta a}\ \alpha(a) \otimes \beta(b)$ for homogeneous maps α, β and homogeneous elements a, b).

□

Putting all this together one can show

Proposition 8.16. *The operator $\Delta := \partial_{x_i}\partial_{\eta^i} - B^{ii} x_i \partial_{\eta_i}$ is a degree 1 differential operator of order at most 2 for the algebra $(\mathcal{T}, *)$, i.e. $(\mathcal{T}, *, \Delta)$ is a BV-algebra.*

Proof. From lemmas 8.13 and 8.15, $\partial_{x_i}\partial_{\eta_i}$ is a differential operator of order at most 2. We are thus left to show that $x_i\partial_{\eta_i}$ is a differential operator of order at most 2. We shall use Lemma 8.11 and we just need to compute $[f,g]_{x_i\partial_{\eta_i}}$ which is equal to

$$(-1)^f x_i\partial_{\eta_i} \circ \mu \circ D(f \otimes g) - (-1)^f \mu \circ D \circ (x_i\partial_{\eta_i} \otimes 1)(f \otimes g) - (-1)^f \mu \circ D \circ (1 \otimes x_i\partial_{\eta_i})(f \otimes g)$$

Note the factor $(-1)^f$ in front of the last term which comes from the tensor product of graded maps. To compute this, let us record some commutations relations.

As before, we have $x_i\partial_{\eta_i} \circ \mu = \mu \circ (x_i\partial_{\eta_i} \otimes 1 + 1 \otimes x_i\partial_{\eta_i})$. Let us now compute the commutator of $x_i\partial_{\eta_i} \otimes 1$ and D.

On the one hand we have

$$(x_i\partial_{\eta_i} \otimes 1) \circ D = \sum_k \frac{(-1)^k}{k!} \, x_i\partial_{\eta_i}(B^{ll}\partial_{x_l} \otimes \partial_{x_l})^k$$

On the other hand we have

$$\begin{aligned} D \circ (x_i\partial_{\eta_i} \otimes 1) &= \sum_k \frac{(-1)^k}{k!} \, (B^{ll}\partial_{x_l} \otimes \partial_{x_l})^k \circ (x_i\partial_{\eta_i} \otimes 1) \\ &= \sum_k \frac{(-1)^k}{k!} \, x_i\partial_{\eta_i}(B^{ll}\partial_{x_l} \otimes \partial_{x_l})^k - \sum_k \frac{(-1)^k}{k!} \, B^{ii}(B^{ll}\partial_{x_l} \otimes \partial_{x_l})^k \circ (\partial_{\eta_i} \otimes \partial_{x_i}) \\ &= (x_i\partial_{\eta_i} \otimes 1) \circ D - B^{ii} D \circ (\partial_n \otimes \partial_{x_i}) \end{aligned}$$

we then have for f and g homogeneous element of $\mathcal{T}$

$$\begin{aligned} &(-1)^f x_i\partial_{\eta_i} \circ \mu \circ D(f \otimes g) - (-1)^f \mu \circ D \circ (x_i\partial_{\eta_i} \otimes 1)(f \otimes g) - (-1)^f \mu \circ D \circ (1 \otimes x_i\partial_{\eta_i})(f \otimes g) \\ &= (-1)^f \mu \circ (x_i\partial_{\eta_i} \otimes 1 + 1 \otimes x_i\partial_{\eta_i}) \circ D(f \otimes g) - (-1)^f \mu \circ D \circ (x_i\partial_{\eta_i} \otimes 1)(f \otimes g) \\ &- (-1)^f \mu \circ D \circ (1 \otimes x_i\partial_{\eta_i})(f \otimes g) \\ &= (-1)^f \mu \circ D \circ (x_i\partial_{\eta_i} \otimes 1 + 1 \otimes x_i\partial_{\eta_i} + B^{ii}\partial_{x_i} \otimes \partial_{\eta_i} + B^{ii}\partial_{\eta_i} \otimes \partial_{x_i})(f \otimes g) \\ &- (-1)^f \mu \circ D \circ (x_i\partial_{\eta_i} \otimes 1)(f \otimes g) - (-1)^f \mu \circ D \circ (1 \otimes x_i\partial_{\eta_i})(f \otimes g) \\ &= (-1)^f \mu \circ D \circ (B^{ii}\partial_{x_i} \otimes \partial_{\eta_i} + B^{ii}\partial_{\eta_i} \otimes \partial_{x_i})(f \otimes g) \end{aligned}$$

Which is the sum of two derivations of degree $|f| + 1$ by Lemma 8.14.

□

It turns out that the product is a trivial deformation:

Proposition 8.17. *The following identity holds as map from $\mathcal{O}_d \otimes \mathcal{O}_d \to \mathcal{O}_d$*

$$\mu \circ D = e^{-B^{ii}\frac{\partial^2_{x_i}}{2}} \circ \mu \circ (e^{B^{ii}\frac{\partial^2_{x_i}}{2}} \otimes e^{B^{ii}\frac{\partial^2_{x_i}}{2}})$$

Proof. Using the previously recorded identity

$$e^{B^{ii}\frac{\partial^2_{x_i}}{2}} \circ \mu = \mu \circ e^{B^{ii}\frac{\partial^2_{x_i} \otimes 1 + 2\partial_{x_i} \otimes \partial_{x_i} + 1 \otimes \partial^2_{x_i}}{2}}$$

we get

$$\begin{aligned} e^{B^{ii}\frac{\partial^2_{x_i}}{2}} \circ \mu \circ D &= \mu \circ e^{B^{ii}\frac{\partial^2_{x_i} \otimes 1 + 2\partial_{x_i} \otimes \partial_{x_i} + 1 \otimes \partial^2_{x_i}}{2}} \circ D \\ &= \mu \circ e^{B^{ii}\frac{\partial^2_{x_i} \otimes 1 + 1 \otimes \partial^2_{x_i}}{2}} \\ &= \mu \circ (e^{B^{ii}\frac{\partial^2_{x_i}}{2}} \otimes e^{B^{ii}\frac{\partial^2_{x_i}}{2}}) \end{aligned}$$

□

In the end, the content of this section can be summarized in the following statement

Proposition 8.18. *For every symmetric matrix B, the map $\langle - \rangle : (\mathcal{T}, *, \Delta) \to \mathbb{C}$ given by Gaussian integration is a map of BV algebras.*

8.2 Speculations

The results of the previous section might suggest the following point of view for power series integrals. One should maybe consider integrals as maps of BV algebras

$$\varphi_\hbar : (\mathcal{C}_\hbar(1), *, \Delta_V) \longrightarrow \mathbb{C}[[\hbar]]$$

where $* = \sum_{i\leqslant 0} \hbar^i \mu_i$ is a deformation of the product with respect to which Δ_V is a still a differential operator of order at most 2.

If $\varphi_\hbar$ is a map of BV algebra, the value of

$$\lim_{\hbar\to 0} \frac{\varphi_\hbar(fg) - \varphi_\hbar(f)\varphi_\hbar(g)}{\hbar}$$

has to be equal to $-\varphi_0(\mu_1(f,g))$ and then the whole integral $\varphi_\hbar$ is fully determined by φ_0.

It is not clear that such an approach is possible, though, for a power series integral also has to satisfy the cumulant condition. The first question is then what kind of constraint does the cumulant condition for power series integrals impose on the different μ_i? It is obvious that 2-cumulants of integrals coming from deformed products are multiples of $\hbar$. For higher cumulants, the situation is not quite as clear. Let us have a look at the 3-cumulant :

Proposition 8.19. *In the setting described above, let a, b, c be three cyclic words. The coefficient in $\hbar$ of $\varphi_c(abc)$ is equal to*

$$\varphi_0\big(-\mu_1(ab, c) + a\mu_1(b, c) + b\mu_1(a, c)\big)$$

Proof. This is just an explicit computation using the fact that $\varphi_\hbar$ is a map of algebra for the product $*$. ☐

We can go one step further, indeed $\mathcal{C}_\hbar(1)$ is a free algebra, all deformations of the product are trivial. We can then assume that $*$ is of the form

$$* = \psi^{-1} \circ \mu \circ (\psi \otimes \psi)$$

with $\psi = 1 + \sum_{i\geqslant 1} \hbar^i \psi_i$ an automorphism of $\mathcal{C}_\hbar(1)$. We then know that

$$\mu_1(a, b) = -\psi_1(a, b) + \psi_1(a)b + a\psi_1(b)$$

Plugging that in Proposition 8.19, we get

Proposition 8.20. *If ψ_1 is a differential operator of order at most 2, then the coefficient in $\hbar$ of $\varphi_c(abc)$ vanishes.*

For higher cumulants, the situation becomes quite involved, and a more conceptual approach would be necessary.

A second question is what are the constraints imposed by requiring the BV operator Δ_V to still be of order at most 2 with respect to the new product. Note that Δ_V is of order at most 2 with respect to $*$ if and only if $\tilde{\Delta}_V := \psi \circ \Delta_V \circ \psi^{-1}$ is of order at most 2 with respect to the initial product. From there one can see that, at least infinitesimally, if ψ_1 is a derivation, then $\tilde{\Delta}_V$ is of order at most 2. Note that requiring ψ to $\tilde{\Delta}_V$ to be of order 2 implies to that the 3-cumulant is a multiple of $\hbar^2$.

Assume for the moment that one manages to answer the two previous questions and that it is indeed possible to define power series integrals by first deforming the product. The next natural question is whether all integrals can be obtained in this way? Maybe topological recursion could help fix ψ_i for $i \geqslant 2$ from only the knowledge of ψ_1?

Conjecture 8.21. *Integrals with respect to the potential V are equivalent to BV-algebra maps*

$$(\mathcal{C}_\hbar(1), *, \Delta_V) \to (\mathbb{C}[[\hbar]], 0)$$

where $*$ is a deformation of the product with respect to which Δ_V is a differential operator of order at most 2.

9 Outlook

We conclude by presenting a couple of outlooks about how non-commutative integrals fit in the broader mathematical landscape.

9.1 Duflo Isomorphism

Consider a finite dimensional Lie $\mathfrak{g}$. From there, one can construct two different algebras: the symmetric algebra $Sym(\mathfrak{g})$ and the universal enveloping algebras $U(\mathfrak{g})$. The so-called PBW theorem states that the symmetrization map

$$\begin{aligned} S : Sym(\mathfrak{g}) &\longrightarrow U(\mathfrak{g}) \\ x_1 \dots x_n &\longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)} \end{aligned}$$

is an isomorphism of vector spaces. Of course, this map has no chance to be a map of algebras, for in general the universal enveloping algebra is not a commutative algebra. However, both spaces are naturally $\mathfrak{g}$-modules and the symmetrization map S is equivariant with respect to both actions. Thus, one can restrict it to the subspace of invariant and get an isomorphism of vector spaces

$$Sym(\mathfrak{g})^{\mathfrak{g}} \cong U(\mathfrak{g})^{\mathfrak{g}} = Z(U(\mathfrak{g}))$$

While $Z(U(\mathfrak{g}))$ is now a commutative algebra, the map S is still not quite a morphism of algebras yet.

In [Duf77], Duflo defines an element $J(x)$ belonging to (a completion of) the symmetric algebra $\widehat{Sym}(\mathfrak{g}^*)$ as

$$J(x) := det(\frac{1 - e^{-ad_x}}{ad_x})$$

and proves the following theorem

Theorem 9.1. *The map $S \circ J^{\frac{1}{2}}$ defines an isomorphism of algebras*

$$Sym(\mathfrak{g})^{\mathfrak{g}} \to Z(U(\mathfrak{g}))$$

One downside of Duflo's proof is that it heavily relies on classification results of Lie algebras, even though both $Sym(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g}^{\mathfrak{g}})$ are constructed only knowing that $\mathfrak{g}$ is a Lie algebra, not which Lie algebra it is.

To make a connection with non-commutative integrals, we have to give a more geometric interpretation of Duflo's theorem. Suppose that the Lie algebra $\mathfrak{g}$ is coming from a Lie group G. The algebra $Sym(\mathfrak{g})$ can be understood as the algebra $\mathcal{D}_0(\mathfrak{g})$ of distribution on $\mathfrak{g}$ supported at 0, with multiplication given by the convolution with respect to the addition in $\mathfrak{g}$. In other words, given two such distributions f, h, their product is given by

$$(f * h)(x) = \int_{\mathfrak{g}} f(y)g(x - y)dy$$

Similarly, the algebra $U(\mathfrak{g})$ can be understood as the algebra $\mathcal{D}_e(G)$ of distributions on the group G supported at the identity; and this time multiplication is given by convolution with respect to the group law. That is, for F, H two such distributions,

$$(F * H)(g) = \int_G F(u)H(gu^{-1})du$$

In this context, the symmetrization map S corresponds to the map

$$\mathcal{D}_0(\mathfrak{g}) \to \mathcal{D}_e(G)$$

given by precomposing with the exponential map $exp : \mathfrak{g} \to G$, and Duflo's theorem states that

$$\begin{aligned} Duf : \mathcal{D}_0(\mathfrak{g})^G &\longrightarrow \mathcal{D}_e(G)^G \\ f &\longmapsto \big(g = exp(x) \mapsto J^{\frac{1}{2}}(x)f(x)\big) \end{aligned}$$

is an isomorphism of algebras.

The hope would be that if one has a universal theory of integration for Lie algebras and groups, one could give a universal proof of Duflo's theorem. By a "universal theory", we mean a theory which relies only on structural maps of Lie algebras or groups. In other words, a theory of integration for the free Lie algebra (group), from which one could get integration for any specific Lie algebra $\mathfrak{g}$ (group G) by interpreting elements of the free Lie algebra (free group) as formulae.

Integrals on the free associative algebra might be useful for both the "Lie" world and the "group" world meet in the "associative" world. Let us explain.

On the one hand, consider the free group

$$\pi := \langle X_1, \ldots, X_n \rangle$$

on n generators. There are multiple isomorphisms of vector spaces between the completion of the group algebra $\widehat{\mathbb{C}\pi}$ and the free algebra $A := \mathbb{C}\langle\langle x_1, \ldots, x_n \rangle\rangle$. One example is the exponential map

$$\begin{aligned} \widehat{\mathbb{C}\pi} &\longrightarrow A \\ X_i &\longmapsto \sum_k \frac{1}{k!} x_i^k. \end{aligned}$$

An other one is the so-called Magnus map

$$\begin{aligned} \widehat{\mathbb{C}\pi} &\longrightarrow A \\ X_i &\longmapsto 1 + x_i. \end{aligned}$$

On the other hand, the free Lie algebra $\mathbb{L} = Lie(x_1, \ldots, x_n)$ can also be found in A. Indeed, A admits the structure of a Hopf algebra with comultiplication Δ, counit ϵ and antipode S defined by

$$\begin{aligned} \Delta(x_i) &:= x_i \otimes 1 + 1 \otimes x_i \\ S(x_i) &:= -x_i \\ \epsilon(x_i) &:= 0 \end{aligned}$$

The free Lie algebra $\mathbb{L}$ can then be identified with the space of primitive elements of (A, Δ, ϵ, S).

In both cases, there is a suitable concept of divergence related to the divergence of A, making it possible to define integrals in those two worlds. Hopefully, associative integrals give examples of such integrals. However, it seems that in order to get involutions, one would need to go at least one step further and develop a theory of integration with free variables.

9.2 Volume of moduli spaces of flat connections

Let G be a Lie group with corresponding Lie algebra $\mathfrak{g}$ and let Σ be an orientable surface. One can look at the space of of $\mathfrak{g}$-valued connections on Σ, that is elements

$$A \in \Omega^1(\Sigma) \otimes \mathfrak{g}$$

such that

$$dA + \frac{1}{2}[A, A] = 0.$$

The *Gauge group* $\mathcal{G}^\Sigma$ of all maps $g : \Sigma \to G$ acts on the space of flat connections by $g \cdot A = g^{-1}Ag + g^{-1}dg$. One can then define the moduli space of flat connections as

$$\mathcal{M}(\Sigma, G) := \{A \in \Omega^1(\Sigma, \mathfrak{g}) \mid dA + \frac{1}{2}[A, A] = 0\}/\mathcal{G}^\Sigma.$$

In [AB82], Atiyah and Bott construct a Poisson structure on $\mathcal{M}(\Sigma, G)$. When Σ is a closed surface, this Poisson structure is actually symplectic. The moduli space can also be described as

$$Hom(\pi_1(\Sigma), G)/G$$

where the action of G is by conjugation. The isomorphism

$$\mathcal{M}(\Sigma, G) \cong Hom(\pi_1(\Sigma), G)/G$$

is given by looking at monodromies of the generators of the fundamental group $\pi_1(\Sigma)$. With this description, it also has a Poisson structure due to Goldman [Gol86].

When G is compact, for example $G = U(N)$, the volume of the moduli space

$$Vol(\mathcal{M}(\Sigma, G))$$

has been computed by Witten [Wit91].

In that case the volume is proportional to an expression of the form

$$\int_{G^{2g}} \delta_e(\prod[a_i, b_i]) \prod_i da_i db_i$$

where g is the genus of Σ and and $[-, -]$ is the group commutator. Notice that this expression is also "universal", and an universal integration theory might be insightful.

9.3 Words measure on free groups

In [MP19] and [MP22] Magee and Puder study words measures on free groups. Their construction is as follows.

Consider the free group F_r on r generators. Given a word w of F_r, they interpret it as a map

$$w : U(N)^r \to U(N)$$

and consider the pushforward of the Haar measure on $U(N)^r$ to $U(N)$, calling it the w-measure.

Given l words $w_1, \ldots, w_l \in F_r$, they are interested in computing integrals of the form

$$Tr_{w_1,\ldots,w_l}(N) := \int_{A_1,\ldots,A_r \in U(N)} \prod_{i=1}^{l} tr(w_i(A_1, \ldots, A_l))d\mu$$

as functions of N.

While it was known from Weingarten calculus (c.f. [CMN22] for an introduction) that for N large enough, $Tr_{w_1,\ldots,w_l}(N) \in \mathbb{Q}(N)$, they come up with a method to compute it by considering surfaces with l boundary components coming from matching of letters. As we have seen, it is not surprising that one can use combinatorics of surfaces to compute matrix integrals. It is somewhat more surprising that one also needs to take into account maps from the boundary of the surface S to $\bigvee_{i=1}^{r} S^1$ satisfying some condition. Loosely speaking their main result looks like this

$$Tr_{w_1,\ldots,w_l}(N) = \sum_{(S,f)} C(f) N^{\chi(S)}$$

where $C(f)$ is a number depending on the map f and $\chi(S)$ is the Euler characteristic of S.

The situation is similar to Gaussian integration, and thus to non-commutative integrals, but in a group setting. There is also a striking difference in the appearance of maps from surfaces to "bouquet" of circles as Magee and Puder call them.

Note that this story is not unrelated to moduli space of flat connection for the integral can be thought of as an integral over $Hom(F_r, U(N))$ and the fundamental group of surface with boundary is a free group.

9.4 Other constructions of $\mathcal{C}_\hbar(n)$

The algebra $\mathcal{C}_\hbar(n)$ constructed in Section 5.4 was also constructed in a slightly different context. In [GGHZ21], Ginot, Gwilliam Hamilton and Zeinalian start with a graded vector space V over $\mathbb{K}$ equipped with a symplectic form $\langle -, - \rangle$ of odd degree. From there, they define a Lie bracket δ and a lie cobracket ∇, both of odd degree on

$$H[V] := \prod_{k=0}^{\infty} \left[(V^*)^{\otimes k} \right]_{\mathbb{Z}/k\mathbb{Z}}$$

using the dual of the symplectic form on V. Using this bracket and cobracket, they get a BV operator $\Delta = \nabla + \gamma\delta$ on

$$\hat{P}^{\wedge nc}_{\gamma,\nu}[V] := \mathbb{K}[[\gamma]] \hat{\otimes} \left(\prod_{i=1}^{\infty} \left[H[V]^{\otimes i} \right]_{S_i} \right)$$

Their notation puts more emphasis on the trivial cyclic word which they denote by ν. The slightly confusing thing for us is that they use δ for the *bracket* (recall that we used δ for the *cobracket*).

Of course, if we pick a basis $\mathcal{X} = \{x_1, \ldots, x_n\}$ of V, $H(V)$ is nothing but the space of (graded) cyclic words in the alphabet $\mathcal{X}$. If $V = W \oplus W[1]$ for a vector space W of dimension n concentrated in degree 0 then $P^{\wedge nc}_{\gamma,\nu}[V]$ is the same thing as what we called $\mathcal{C}_\hbar(n)$. It turns out that their bracket and cobracket are the same as ours.

While their construction is a bit more general, it also puts more emphasis of the vector space V. Using the BV-structure, they consider $\hat{P}^{\wedge nc}_{\gamma,\nu}[V]$ as an odd graded differential Lie algebra and define a filtration

$$\hat{P}^{\wedge nc}_{\gamma,\nu}[V] = F_0 \hat{P}^{\wedge nc}_{\gamma,\nu}[V] \supset F_1 \hat{P}^{\wedge nc}_{\gamma,\nu}[V] \supset \ldots$$

The solution of the Maurer-Cartan equation in $\hat{P}^{\wedge nc}_{\gamma,\nu}[V]/F_1\hat{P}^{\wedge nc}_{\gamma,\nu}[V]$ are precisely the cyclic A_∞-structures on (a suspension of) V. Cyclic means that it A_∞-structure is compatible with the symplectic form $\langle -, -\rangle$. Such solutions are linear combination of cyclic words and a word of length k corresponds to the operation of arity $k-1$. Since those solutions are linear combination of cyclic words, they correspond to potential in our setting. It would be interesting to understand what is the role played by integrals in this context.

In a subsequent paper [GHZ22], they apply this construction to the following very simple cyclic A_∞ algebra $\mathcal{A}$: the graded space $\mathcal{A}$ has generators a and b in degree zero and one respectively, the symplectic form is given by

$$\langle b, a\rangle = 1 = -\langle a, b\rangle$$

and the only non trivial operation is the operation of arity 0, i.e. the differential d and it is given by $da = b$. They show that

$$(\hat{P}^{\wedge nc}_{\gamma,\nu}[\mathcal{A}], d + \Delta)$$

is closely related to Gaussian integration of Hermitian matrices. Indeed the differential d corresponds to the cyclic words $|(a^*)^2|$ and on $(\hat{P}^{\wedge nc}_{\gamma,\nu}[\mathcal{A}], d + \Delta)$, $d = \{(a^*)^2, -\}$. They also produce a quasi-isomorphism of complexes

$$(\hat{P}^{\wedge nc}_{\gamma,\nu}[\mathcal{A}], d + \Delta) \longrightarrow \mathbb{C}[\gamma, \nu]$$

by means of ribbon graphs. It seems extremely likely that one can deform the product on $\hat{P}^{\wedge nc}_{\gamma,\nu}[\mathcal{A}]$ using those same ribbon graphs in order to make the above quasi-isomorphism a morphism of BV-algebra.

A Riemann Surfaces

We gather here different results about Riemann surfaces. Good references on the subject are plenty, but we recommend the book of W. Schlag [Sch14].

Fix once and for all a complex polynomial $P(z)$ of degree $2g + 2$ with simple roots and and consider the Riemann surface

$$\Sigma := \overline{\{(x, y) \in \mathbb{C} \mid y^2 = P(x)\}}.$$

Proposition A.1. *As a $\mathbb{C}$-vector space, the space of holomorphic differential of Σ is generated by*

$$\frac{dx}{y}; \frac{x^2 dx}{y}; \ldots; \frac{x^{g-1} dx}{y}$$

Proposition A.2. *Let ω be a meromorphic form and let p be a pole of ω. The order of the pole p $ord(\omega, p)$ and the residue $Res(\omega, p)$ are well defined, i.e. do not depend on the coordinates.*

Proposition A.3. *The meromorphic 1-form on Σ are of the form*

$$\omega = (\rho_1(x) + \rho_2(x)y)dx$$

There is a residue theorem for Riemann surfaces (we are not quite precise with what we mean by integration region, a more precise description can be found in [Sch14])

Proposition A.4. *Let ω be a meromorphic form and $N \subset \Sigma$ an integration region such that there is no pole of ω on ∂N. Then*

$$\frac{1}{2\pi i} \int_{\partial_N} = \sum_{p \in N} Res(\omega, p)$$

A thorough discussion of what follows can be found in [Eyn18]

Proposition A.5. *Given a symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ of cycles of $H_1(\Sigma, \mathbb{Z})$, there exists a unique bilinear meromorphic form $B(z_1, z_2)$ such that as a 1-form in its first argument, it has only a double pole with no residue when $z_1 = z_2$ and is normalized such that*

$$B(z_1, z_2) \approx \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2} + holo.$$

as $z_1 \to z_2$, and

$$\oint_{z_1 \in \mathcal{A}_i} B(z_1, z_2) = 0$$

This form is called the fundamental form of the second kind, or sometimes the "Bergman Kernel". From it one can construct a basis of holomorphic differentials on Σ:

Proposition A.6. *In the same context as the above Proposition, for $1 \leqslant i \leqslant g$, the differential forms*

$$\omega_i(z) := \frac{1}{2\pi i} \oint_{z_1 \in \mathcal{B}_i} B(z_1, z)$$

form a basis of holomorphic form on Σ satisfying

$$\oint_{\mathcal{A}_i} \omega_j(z) = \delta_{ij}.$$

Definition A.7. In the above setting, define numbers $\tau_{i,j}$ by

$$\tau_{i,j} := \oint_{\mathcal{B}_j} \omega_i$$

Proposition A.8. *Let ω be a symmetric meromorphic bidifferential form $\Sigma\times\Sigma$ with only double poles with no residue on the diagonal. Then it has the form*

$$\omega = B + \sum_{ij} \kappa_{ij}\omega_i\omega_j$$

.

Furthermore, for every $1 \leqslant i \leqslant g$ the elements $\mathcal{A}_i^\kappa$ and $\mathcal{B}_i^\kappa$ be of $H_1(\Sigma, \mathbb{C})$ defined by

$$\mathcal{A}_i^\kappa = \mathcal{A}_i - \sum_{j=1}^{d-2} \kappa_{ij}\left(\mathcal{B}_j - \sum_{l=1}^{d-2} \tau_{jl}\mathcal{A}_l\right)$$

and

$$\mathcal{B}_i^\kappa = \mathcal{B}_i - \sum_{j=1}^{d-2} \tau_{ij}\mathcal{A}_l$$

are such that

$$\mathcal{A}_i^\kappa \bigcap \mathcal{A}_j^\kappa = 0 = \mathcal{B}_i^\kappa \bigcap \mathcal{B}_j^\kappa$$

$$\mathcal{A}_i^\kappa \bigcap \mathcal{B}_j^\kappa = \delta_{ij}$$

$$\oint_{z_1\in\mathcal{A}_i^\kappa} \omega(z_1, z_2) = 0 \text{ and } \oint_{z_1\in\mathcal{B}_i^\kappa} \omega(z_1, z_2) = 2\pi i\omega_i(z_2)$$

The cycles $\mathcal{A}_i^\kappa$ and $\mathcal{B}_i^\kappa$ are called the modified cycles.

Definition A.9. Given a symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ of cycles of $H_1(\Sigma, \mathbb{Z})$, the fundamental domain of Σ is

$$\Sigma_0 := \Sigma\backslash\bigcup_i \mathcal{A}_i \cup \mathcal{B}_i$$

Proposition A.10. *(Riemann Bilinear Identity) Let $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ of cycles of $H_1(\Sigma, \mathbb{Z})$ and let f be a meromorphic function on Σ, holomorphic on a neighbourhood of $\partial\Sigma_0$ and let ω be a closed meromorphic 1-form. Then*

$$\int_{\partial\Sigma_0} f\omega = \sum_{i=1}^g \int_{\mathcal{A}_i} df \int_{\mathcal{B}_i} \omega - \int_{\mathcal{B}_i} df \int_{\mathcal{A}_i} \omega$$

The same result holds when replacing $\mathcal{A}_i$ and $\mathcal{B}_i$ by the modified cycles $\mathcal{A}_i^\kappa$ and $\mathcal{B}_i^\kappa$

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