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# AVERAGE MULTIPLICITY, SECONDARY TRAJECTORY, AND MUELLER ANALYSIS*

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Abstract

The correction to the average multiplicity is calculated in a general Mueller analysis when a secondary Regge trajectory with intercept 1/2 is also included. $\langle n \rangle$ is then approximately given by $\left[A \ell n\, s + B + C\, s^{-1/2}\right]$, which fits well all existing multiplicity data from $s = 25\ \mathrm{GeV}^2$ to $2800\ \mathrm{GeV}^2$. We show that part of the coefficient C may be estimated from the $90^0$ production data. We comment on the sensitivity (or insensitivity) of $\langle n \rangle$ as a test of production models.

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From general Regge behavior for the six-line connected part Mueller derived[1] the $(A\ln s + B)$ formula for the average multiplicity by keeping only the leading Regge contribution. Recent multiplicity data[2] seem to suggest that the low laboratory energy data may not lie on a straight line with the new ISR data when average multiplicity is plotted against $\ln s$. The purpose of this paper is to calculate the correction to the $(A\ln s + B)$ formula when a secondary Regge trajectory with intercept 1/2 is also included in Mueller's analysis. We find that the average multiplicity is approximately given by

$$\langle n\rangle \approx A\ln s + B + C\, s^{-1/2} \quad , \tag{1}$$

where C depends on the type of incident particles. We find Eq. (1) fits very well all existing multiplicity data from $s = 25\ \mathrm{GeV}^2$ to $2800\ \mathrm{GeV}^2$. We also show that for reactions which show early scaling in the fragmentation region, part of C may be estimated from the wide-angle production data at $90^\circ$.

For simplicity, we consider only the case where the two incident particles are identical. One can easily generalize in a straightforward manner to the unidentical case. We start with the formula relating $\langle n\rangle$ to the single particle inclusive cross section

$$\langle n\rangle\, \sigma_T = \int \left(E\frac{d\sigma}{d^3\underset{\sim}{q}}\right) \frac{d^3\underset{\sim}{q}}{E} \quad . \tag{2}$$

We divide the C.M. $q_{\parallel}$ phase space into three regions: $[-p, -\epsilon p] \cup [-\epsilon p, \epsilon p] \cup [\epsilon p, p]$, where $p \equiv \frac{\sqrt{s}}{2}$.[3] As explained later, $\epsilon$ is suitably chosen so that regions I and III are the fragmentation regions, and region II is the pionization region. Following Mueller,[1] this means that we can make a single O(2,1) expansion for $E\frac{d\sigma}{d^3\underset{\sim}{q}}$ in regions I and III and a double O(2,1) expansion in region II.

Denoting fragmentation and pionization by the subscripts of f and p, we have

$$\left(E\frac{d\sigma}{d^3\underset{\sim}{q}}\right)_f = \sum_i \frac{\gamma_i(\underset{\sim}{q})}{s}\left(2p_a\cdot p_b\right)^{\alpha_i} , \tag{3}$$

$$\left(E\frac{d\sigma}{d^3\underset{\sim}{q}}\right)_p = \sum_{i,j} \frac{\gamma_{ij}\left(q_T^2\right)}{s}\left(2p_a\cdot q\right)^{\alpha_i}\left(2p_b\cdot q\right)^{\alpha_j} , \tag{4}$$

where $p_a$ and $p_b$ are the incident momenta and q is the momentum of the detected particle. In the following, we denote by m and $\mu$ the mass of a nucleon and a pion, respectively. We consider the sum of the contributions from a Pomeranchuk trajectory with $\alpha_P = 1$ and a Regge trajectory with $\alpha_R = 1/2$. At high incident energy,

$$2p_a\cdot q \approx \frac{s}{2}\left(\sqrt{x^2 + \frac{4\mu_T^2}{s}} - x\right) , \tag{5a}$$

$$2p_b\cdot q \approx \frac{s}{2}\left(\sqrt{x^2 + \frac{4\mu_T^2}{s}} + x\right) , \tag{5b}$$

$$2p_a\cdot p_b \approx s , \tag{5c}$$

where $x \equiv \frac{2q_{\parallel}}{\sqrt{s}}$ , $\mu_T \equiv \sqrt{q_T^2 + \mu^2}$, and particle a is defined to be along the + z-axis.

We first consider the fragmentation contribution. By symmetry, the total contribution from the two fragmentation regions is equal to twice the contribution from either fragmentation region. Substituting Eq. (5c) into Eq. (3), we have

$$\left(E \frac{d\sigma}{d^3 \underset{\sim}{q}}\right)_f \approx \gamma_P(\underset{\sim}{q}) + s^{-1/2}\,\gamma_R(\underset{\sim}{q})\,, \tag{6}$$

when $\epsilon p \le q_{\parallel} \le p$.

Therefore,

$$\langle n\rangle_f \approx \frac{2}{\sigma_T}\int d^2\underset{\sim}{q}_T \int_{\epsilon p}^{p} \frac{dq_{\parallel}}{\sqrt{q_{\parallel}^2 + \mu_T^2}} \left[\gamma_P(\underset{\sim}{q}) + s^{-1/2}\,\gamma_R(\underset{\sim}{q})\right]. \tag{7}$$

Following Mueller,[1] when we transform from C.M. to the rest frame of a, because of the invariance of $\left(E \frac{d\sigma}{d^3 \underset{\sim}{q}}\right)_f$, Eq. (7) can be rewritten as

$$\langle n\rangle_f = \frac{2}{\sigma_T}\int d^2\underset{\sim}{q}_T \int_{\frac{m\epsilon}{2} - \frac{\mu_T^2}{2m\epsilon}}^{\frac{m}{2} - \frac{\mu_T^2}{2m}} \frac{dq_{\parallel}}{\sqrt{q_{\parallel}^2 + \mu_T^2}} \left[\gamma_P(\underset{\sim}{q}) + s^{-1/2}\gamma_R(\underset{\sim}{q})\right]$$

$$= B_f(\epsilon) + s^{-1/2}\, C_f(\epsilon)\,, \tag{8}$$

where

$$B_f(\epsilon) \equiv \frac{2}{\sigma_T}\int d^2\underset{\sim}{q}_T \int_{\frac{m\epsilon}{2} - \frac{\mu_T^2}{2m\epsilon}}^{\frac{m}{2} - \frac{\mu_T^2}{2m}} \frac{dq_{\parallel}}{\sqrt{q_{\parallel}^2 + \mu_T^2}}\, \gamma_P(\underset{\sim}{q})\,, \tag{9a}$$

$$C_f(\epsilon) \equiv \frac{2}{\sigma_T} \int d^2 \underset{\sim}{q}_T \int_{\frac{m\epsilon}{2} - \frac{\mu_T^2}{2m\epsilon}}^{\frac{m}{2} - \frac{\mu_T^2}{2m}} \frac{dq_{\parallel}}{\sqrt{q_{\parallel}^2 + \mu_T^2}} \gamma_R(\underset{\sim}{q}) \quad . \tag{9b}$$

We now calculate the pionization contribution. First notice that when $q_{\parallel} = \epsilon p$ (i.e., at the boundary of the pionization region and the fragmentation region of a), Eq. (5a) implies

$$\left(2 p_a \cdot q\right)_{q_{\parallel} = \epsilon p} \approx \frac{\mu_T^2}{\epsilon} \quad , \tag{10}$$

where we have assumed

$$\epsilon^2 \gg \frac{4 \mu_T^2}{s} \quad , \tag{11a}$$

a condition that is later needed to show that the leading term in $\langle n \rangle$ is $A \ln s$. Equation (10) implies that for the double O(2,1) expansion to be valid near $q_{\parallel} \lesssim \epsilon p$, $\epsilon$ must be chosen to satisfy

$$\epsilon \ll \bar{\epsilon} \equiv \frac{\mu_T^2}{m^2} \quad . \tag{11b}$$

If $|\underset{\sim}{q}_T| \approx 350$ MeV/c, then $\bar{\epsilon} \approx 0.16$. Equations (11a) and (11b) thus define the region in which $\epsilon$ is chosen.

Substituting Eq. (5a) and (5b) into Eq. (4) and (2), we have

$$\langle n \rangle_p \approx \frac{1}{\sigma_T} \int d^2 \underset{\sim}{q}_T \int_{-\epsilon}^{\epsilon} \frac{dx}{\sqrt{x^2 + \frac{4\mu_T^2}{s}}} \left[ \mu_T^2 \gamma_{PP}\left(q_T^2\right) + \sqrt{2}\, \gamma_{PR}\left(q_T^2\right) \mu_T \left( \sqrt{x^2 + \frac{4\mu_T^2}{s}} - x \right)^{\frac{1}{2}} \right.$$

$$\left. + s^{-1/2} \mu_T \gamma_{RR}\left(q_T^2\right) \right] . \tag{12}$$

The first and third terms in Eq. (12) are trivial integrals, and the second term can be evaluated with the substitution

$$y^2 = \sqrt{x^2 + \frac{4\mu_T^2}{s}} - x.$$

Keeping terms up to $s^{-1/2}$ and using Eq. (11a), we find

$$\langle n \rangle_p \approx A \ln s + B_p + C_p s^{-1/2} , \tag{13}$$

where

$$A \equiv \frac{1}{\sigma_T} \int d^2 \underset{\sim}{q}_T\, \mu_T^2\, \gamma_{PP}\left(q_T^2\right) , \tag{14a}$$

$$B_p(\epsilon) \equiv \frac{1}{\sigma_T} \left\{ \int d^2 \underset{\sim}{q}_T \mu_T^2 \ln\left(\frac{\epsilon^2}{\mu_T^2}\right) \gamma_{PP}\left(q_T^2\right) + 4\sqrt{\epsilon} \int d^2 \underset{\sim}{q}_T\, \mu_T \gamma_{PR}\left(q_T^2\right) \right\} , \tag{14b}$$

$$C_p(\epsilon) \equiv \frac{1}{\sigma_T} \left\{ -\frac{4}{\sqrt{\epsilon}} \int d^2 \underset{\sim}{q}_T\, \mu_T^2 \gamma_{PR}\left(q_T^2\right) + \int d^2 \underset{\sim}{q}_T\, \mu_T \ln\left(\frac{s\,\epsilon^2}{\mu_T^2}\right) \gamma_{RR}\left(q_T^2\right) \right\} . \tag{14c}$$

Notice that the $C_p s^{-1/2}$ term actually contains a term of the form $\left(C_p' \ln s\right) s^{-1/2}$. However, since the $s^{-1/2}$ terms are negligible for very large s, phenomenologically we may neglect the $\ln s$ variation and group the two terms together under

$C_p s^{-1/2}$. Equation (1)[4] then follows from (8) and (13), with[5]

$$B \equiv B_f(\epsilon) + B_p(\epsilon) \quad , \tag{15a}$$

$$C \equiv C_f(\epsilon) + C_p(\epsilon) \quad . \tag{15b}$$

We want to remark that Eq. (1) also follows from any multiperipheral model when a secondary output pole at 1/2 is also kept.

Equations (9), (14), and (15) seem to imply that B and C may depend on $\epsilon$, i.e., our choice of the pionization and fragmentation boundary, which should obviously not be the case. We can explicitly show that B and C are independent of $\epsilon$ by the following method. Near the pionization and fragmentation boundary, Eq. (4),(5a), and (5b) imply

$$\left(E\frac{d\sigma}{d^3\underset{\sim}{q}}\right)_p \underset{x\approx\epsilon}{\approx} \mu_T^2\gamma_{PP}\left(q_T^2\right) + \mu_T\sqrt{x}\,\gamma_{PR}\left(q_T^2\right) + s^{-1/2}\left[\frac{\mu_T^2}{\sqrt{x}}\,\gamma_{PR}\left(q_T^2\right) + \mu_T\gamma_{RR}\left(q_T^2\right)\right] \tag{16}$$

where we have used Eq. (11a). Comparing Eq. (16) with (6), we can conclude

$$\gamma_P(\underset{\sim}{q}) \underset{x\approx\epsilon}{\approx} \mu_T^2\,\gamma_{PP}\left(q_T^2\right) + \mu_T\sqrt{x}\,\gamma_{PR}\left(q_T^2\right) \quad , \tag{17a}$$

$$\gamma_R(\underset{\sim}{q}) \underset{x\approx\epsilon}{\approx} \frac{\mu_T^2}{\sqrt{x}}\,\gamma_{PR}\left(q_T^2\right) + \mu_T\,\gamma_{RR}\left(q_T^2\right) \quad . \tag{17b}$$

Substituting Eq.(17) into Eq. (9) and using (14) and (15), we can then show[6] that $\frac{dB}{d\epsilon} = \frac{dC}{d\epsilon} = 0$ when $\epsilon$ is in the region defined by Eq. (11).

We can fit very well with Eq. (1) all existing multiplicity data.[2] The least $\chi^2$ fit is shown in Fig. 1, giving A = 2.0, B = -4.8, C = 10.0 with $\chi^2 = 5.6$ for 13 degrees of freedom. However, we also have obtained good fits with $(A\ell n\, s + B)$, giving A = 1.4, B = -1.1 with $\chi^2 = 14.3$ for 14 degrees of freedom, and with

$(C\,s^{1/4})$, giving C = 1.7 with $\chi^2 = 16$ for 15 degrees of freedom. These fits are also shown in Fig. 1. The conclusion that we can draw is that with present large error bars at the ISR energies, we cannot differentiate any of these three forms. However, when these error bars are reduced, we may be able to eliminate $(A \ln s + B)$, but probably still cannot differentiate $(A \ln s + B + C\,s^{-1/2})$ from $C\,s^{1/4}$ even at ISR energies.

Since the fragmentation region contributes to B and C, B and C are not universal constants as in the case of A, i.e., they depend on the type of incident particles. We now show that within certain approximations, for reactions where there is early scaling in the fragmentation region[7] (as in proton-proton interactions with a detected pion), we can give a rough estimate of part of C by relating it to the $90^\circ$ production data. Equations (11b) and (10) tell us that those x which satisfy $\bar{\epsilon} \leq x \leq 1$ are definitely in the fragmentation region. Since the approach to scaling in the fragmentation region is governed by $\gamma_R(\underset{\sim}{q})$, early scaling then suggests $\gamma_R(\underset{\sim}{q})$ for this range of x is negligible and thus by Eq. (9b), $C_f(\bar{\epsilon})$ is also negligible. When $x \approx \epsilon$, $\gamma_R(\underset{\sim}{q})$ is no longer negligible and is given by Eq. (17b). We now make the approximation of replacing this smooth transition for $\gamma_R(\underset{\sim}{q})$ for x near $\bar{\epsilon}$ by a sharp transition at $x = \bar{\epsilon}$, so that $\gamma_R(\underset{\sim}{q}) = 0$ for $\bar{\epsilon} \leq x \leq 1$ and $\gamma_R(\underset{\sim}{q})$ is given by Eq. (17b) for $x < \bar{\epsilon}$ (the latter means one can make a double O(2,1) expansion for $x < \bar{\epsilon}$). If we define $C'$ to be C minus the R-R contribution, then[8]

$$C' \approx -\frac{4}{\sigma_T \sqrt{\bar{\epsilon}}} \int d^2 \underset{\sim}{q}_T \, \mu_T^2 \, \gamma_{PR}\left(q_T^2\right) . \qquad (18)$$

We want to relate $C'$ to the $90^\circ$ production data. These data show that

$\left(\frac{d\sigma}{d\Omega}\right)_{\theta=90^\circ}$ approaches its asymptotic value from below. The rate of this approach is governed by $\gamma_{PR}\left(q_T^2\right)$ appearing in the leading non-scaling term, i.e., the $s^{-1/4}$ term.[9] At 90°, x = 0 and $|\underset{\sim}{q}| = |\underset{\sim}{q}_T|$, we therefore have

$$\frac{1}{\sigma_T}\left(\frac{d\sigma}{d\Omega}\right)_{\theta=90^\circ} = \frac{1}{2\pi\sigma_T}\int\left(E\frac{d\sigma}{d^3\underset{\sim}{q}}\right)_{x=0}\frac{|\underset{\sim}{q}_T|}{\mu_T}\, d^2\underset{\sim}{q}_T \quad , \tag{19}$$

where we have used azimuthal symmetry. Equations (4) and (5) allow us to write (19) as

$$\frac{1}{\sigma_T}\left(\frac{d\sigma}{d\Omega}\right)_{\theta=90^\circ} \approx a_0 + a_1 s^{-1/4} \quad , \tag{20}$$

where

$$a_1 \equiv \frac{1}{\pi\sigma_T}\int d^2\underset{\sim}{q}_T \sqrt{\mu_T}\, |\underset{\sim}{q}_T|\, \gamma_{PR}\left(q_T^2\right) \quad , \tag{21}$$

and an analogous expression for $a_0$. Because $|\underset{\sim}{q}_T|$ of secondaries is distributed in a small neighborhood about its mean value of $\langle|\underset{\sim}{q}_T|\rangle \approx 350$ MeV/c, factors of $\mu_T$ and $|\underset{\sim}{q}_T|$ may be replaced by their mean values and taken out of the integrals in Eq. (18) and (21). Comparing (18) and (21), we conclude

$$C' \approx -\left(\frac{4\pi\langle\mu_T\rangle^{3/2}}{\sqrt{\epsilon}\,\langle|\underset{\sim}{q}_T|\rangle}\right) a_1 \quad . \tag{22}$$

Thus, if $a_1 < 0$, then $C' > 0$. The present 90° production data have very large error bars. But taking it at its face value, $a_1 \approx -0.6\ \text{GeV}^{1/2}$.[10] This gives $C' \approx 12$.[11] By comparing this value of $C'$ with the best fit value of C, we can estimate the sign and the approximate magnitude of the RR contribution to C, and therefore obtain some knowledge about $\gamma_{RR}\left(q_T^2\right)$. However, with the large uncertainties in present data, it is inadvisable to make this estimate now.

In conclusion, we have calculated in a general Mueller analysis the correction to the $(A\ell n\, s + B)$ formula by including a secondary trajectory with intercept 1/2. The correction term is approximately $C\, s^{-1/2}$, and therefore can explain why the low energy multiplicity data should lie slightly higher than the straight line extrapolated from the high-energy multiplicity data. However, it may be difficult even at ISR energies to differentiate this formula from the formula $C\, s^{1/4}$ as predicted by statistical[12] or hydrodynamical[13] models. This means that measuring $\langle n \rangle$ may not be a sensitive test of some production models. We also show how part of C may be roughly estimated from the $90^{\circ}$ production data for reactions where there is early scaling in the fragmentation region.

Acknowledgments

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References and Footnotes

1. A. H. Mueller, Phys. Rev. D2, 2963 (1970).

2. For a review, see J. C. Sens, invited paper presented at the Fourth International Conference on High Energy Collisions, Oxford, U.K., April 5-7, 1972. See also E. L. Berger, Argonne report ANL/HEP 7229 (1972).

3. Here we have ignored the fact that $(q_{\parallel})_{max}$ of a pion may be slightly larger than $\frac{\sqrt{s}}{2}$ because of its smaller mass than the nucleon.

4. For an alternative derivation of Eq. (1) (and also in terms of the rapidity variable), see R. Cahn, Berkeley report LBL 1007 and SLAC report, to be published.

5. If we had also included a lower trajectory with intercept 0, then the ($\alpha_i = 1$, $\alpha_j = 0$) combination and ($\alpha_i = 1/2$, $\alpha_j = 0$) combination contribute respectively a term to B and to C. In keeping with the spirit of Regge phenomenology, we keep only leading trajectories until there is evidence of discrepancy.

6. In this proof, it is more convenient to transform back to the C.M. frame so that the upper and lower limits in Eq. (9) are p and $\epsilon p$ respectively.

7. By early scaling in the fragmentation region, we mean scaling at incident energy of about 12 GeV/c so that the reason for a small correction term is due to a small residue and not due to $s^{-1/2}$ damping.

8. What would happen if we had chosen another $\bar{\epsilon}$, say $\bar{\epsilon}' < \bar{\epsilon}$? Then the fragmentation contribution from the region $\bar{\epsilon}' \leq x < \bar{\epsilon}$ is no longer negligible because there is no early scaling in this region. Because of Eq. (17b), this fragmentation contribution has two terms: a P-R term and an R-R term. The sum of this P-R term and $C'(\bar{\epsilon}')$ is then equal to $C'(\bar{\epsilon})$. An analogous statement can be made about the R-R contribution. This is the meaning of $\epsilon$-independence.

9. Here we assume that the dominate correction to scaling of the $90^{o}$ production data is due to the $s^{-1/4}$ term, and not the $s^{-1/2}$ term. This is, of course, reasonable unless for some mysterious reason $\gamma_{PR}\left(q_T^2\right) \ll \gamma_{RR}\left(q_T^2\right)$.

10. The data given in Ref. 2 is actually $\frac{1}{\sigma_{inelastic}}\left(\frac{d\sigma}{d\Omega}\right)$. Here we have approximated $\sigma_{inelastic} \approx \sigma_T$.

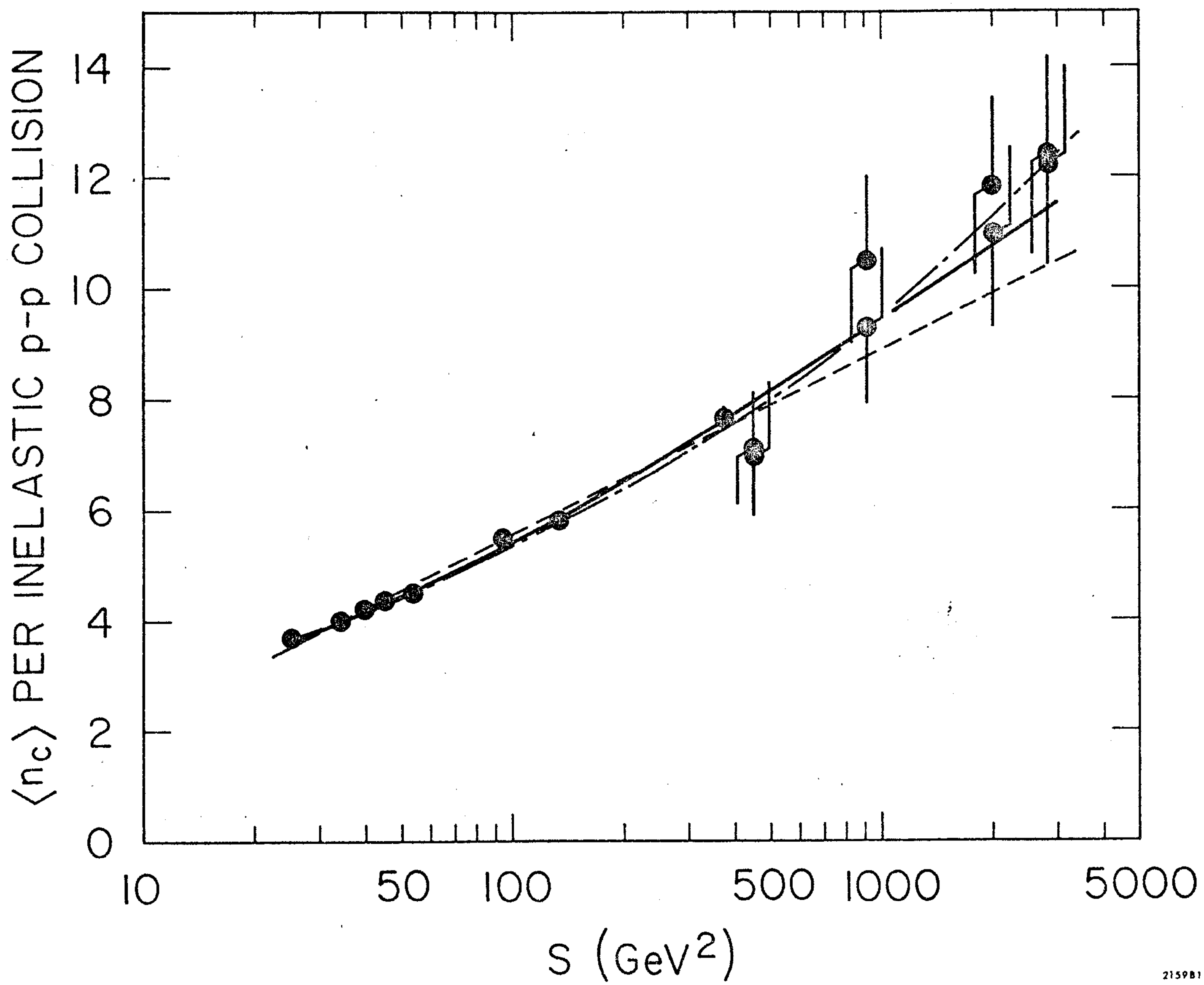
11. Because of the approximations used in making this estimate, we should consider this method as capable of giving only a rough estimate of $C'$.

12. E. Fermi, Progr. Theor. Phys. (Kyoto) 5, 570 (1950).

13. Collected Papers of L. D. Landau, ed. D. Ter Haar (Gordon and Breach), N. Y., 1965), Papers 74 and 88; see also P. Carruthers and M. Duong-van, Cornell report CLNS-188 (1972).


## Figure Caption

1. Average charged multiplicity plotted against s. The data is taken from Berger (Ref. 2). The fits are with $A \ln s + B + C s^{-1/2}$ (solid curve), $A \ln s + B$ (dashed curve), and $C s^{1/4}$ (dotted-dashed curve).



⟨$n_c$⟩ PER INELASTIC p-p COLLISION
0
2
4
6
8
10
12
14
10
50
100
500
1000
5000
S (GeV²)
2159B1


Fig. 1