PLOS ONE

RESEARCH ARTICLE

Stability analysis of nonlinear localized modes in the coupled Gross-Pitaevskii equations with $\mathcal{PT}$ -symmetric Scarf-II potential

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Citation: Zhang J-R, Wang X (2023) Stability analysis of nonlinear localized modes in the coupled Gross-Pitaevskii equations with P T -symmetric Scarf-II potential. PLoS ONE 18(11): e0294790. https://doi.org/10.1371/journal.pone.0294790

Editor: Boris Malomed, Tel Aviv University, ISRAEL

Received: June 5, 2023

Accepted: November 7, 2023

Published: November 27, 2023

Data Availability Statement: This article does not report data and the data availability policy is not applicable to this article.

Funding: This work was supported by the National Training Program of Innovation (Grant numbers 202210019045). The funding body plays an important role in the design of the study, in analysis, calculation, and in writing of the manuscript.

Competing interests: The authors have declared that no competing interests exist.

Abstract

We study the nonlinear localized modes in two-component Bose-Einstein condensates with parity-time-symmetric Scarf-II potential, which can be described by the coupled Gross-Pitaevskii equations. Firstly, we investigate the linear stability of the nonlinear modes in the focusing and defocusing cases, and get the stable and unstable domains of nonlinear localized modes. Then we validate the results by evolving them with 5% perturbations as an initial condition. Finally, we get stable solitons by considering excitations of the soliton via adiabatical change of system parameters. These findings of nonlinear modes can be potentially applied to physical experiments of matter waves in Bose-Einstein condensates.

1 Introduction

Bose-Einstein condensate (BEC) [1, 2], as one of the important physical phenomena, has attracted the attention of researchers. The successful observation of solitons in BECs has become one of the research focuses in the fields of condensed matter physics and atom optics [3–5]. Compared with the single-component ones, the multi-component BECs possess the inter-component interactions and have complicated quantum phases and properties [6–13]. Many novel phenomena have been discovered in multi-component BECs [14–22], including symbiotic solitons, soliton trains, soliton pairs, multi-domain walls, and multi-mode collective excitations. As one kind of multi-component BECs, the two-component BECs trapped in a quasi-one-dimensional harmonic potential at zero temperature can be described by the following coupled Gross-Pitaevskii equations [23, 24]:

$$i\hbar\frac{\partial\Psi_1}{\partial t} = \left(-\frac{\hbar^2}{2M}\nabla^2 + V_1(\boldsymbol{r}) + g_{11}|\Psi_1|^2 + g_{12}|\Psi_2|^2\right)\Psi_1, \tag{1a}$$

$$i\hbar\frac{\partial\Psi_2}{\partial t} = \left(-\frac{\hbar^2}{2M}\nabla^2 + V_2(\boldsymbol{r}) + g_{21}|\Psi_1|^2 + g_{22}|\Psi_2|^2\right)\Psi_2, \tag{1b}$$

where Ψ_j is the two-component parameter, $\boldsymbol{r} = (x, y, z)$, $\hbar$ is the Planck constant, M is the atomic mass, ∇^2 is the Laplacian, $V_j(\boldsymbol{r}) = [\omega_{jx}^2 x^2/2 + \omega_{j\perp}^2(y^2+z^2)]M/2$ stands for the harmonic potentials, g_{jj} is the interactions between atoms, and $g_{j,3-j}$ describes the inter-component interactions ($j = 1, 2$) [25–28].

If the trap frequencies in the radial directions $\omega_{j\perp}$ are larger than the axial directions ω_{jx}, Eq (1) becomes a quasi-one-dimensional system along the x direction. Through the normalization and transformation $\Psi_j \rightarrow \psi_j$, $x \rightarrow \sqrt{\hbar/(M\omega_{1\perp})}x$, $t \rightarrow (2\pi/\omega_{1\perp})t$, Eq (1) can be written in the form [25–28]:

$$i\frac{\partial \psi_1}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\lambda_1^2}{2}x^2 + b_{11}|\psi_1|^2 + b_{12}|\psi_2|^2\right)\psi_1, \tag{2a}$$

$$i\frac{\partial \psi_2}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\lambda_2^2}{2}x^2 + b_{21}|\psi_1|^2 + b_{22}|\psi_2|^2\right)\psi_2, \tag{2b}$$

where $\lambda_j = \omega_{jx}/\omega_{j\perp}$, b_{jj} and $b_{j,3-j}$ are related to $\int |\psi_j|^2$x, trap frequencies in the radial directions $\omega_{j\perp}$, and interactions between atoms g_{jj} or inter-component $g_{j,3-j}$ ($j = 1, 2$). For the harmonic potentials, the soliton states were studied widely [25–28]. In this paper, we consider the two-component BECs trapped in another potential.

Put forward by Bender and his coworker in 1998 [29–31], parity-time- ($\mathcal{PT}$-) symmetry behaviors have attracted much attention in both non-Hermitian Hamiltonian systems and nonlinear wave systems [32–34], which make systems with complex potentials possibly support fully-real linear spectra [35] and stable nonlinear modes [36–39]. That is, the potential function $U(x) = V(x) + iW(x)$ satisfies $V(x) = V(-x)$ and $W(-x) = -W(x)$ [29–31]. Over the past few years, various $\mathcal{PT}$-symmetric potentials have been introduced into the nonlinear Schrödinger equation and the existence of different nonlinear local modes is analytically and numerically investigated [40–51]. To better investigate physical phenomena, it is meaningful to introduce new forms of $\mathcal{PT}$-symmetric potentials in nonlinear systems.

In this paper, we investigate the coupled Gross-Pitaevskii equations with complex $\mathcal{PT}$-symmetric potentials [28]:

$$i\frac{\partial \psi_1}{\partial t} = \left(-\frac{\partial^2}{\partial x^2} - a_1(|\psi_1|^2 + |\psi_2|^2) - U_1(x)\right)\psi_1, \tag{3a}$$

$$i\frac{\partial \psi_2}{\partial t} = \left(-\frac{\partial^2}{\partial x^2} - a_2(|\psi_1|^2 + |\psi_2|^2) - U_2(x)\right)\psi_2, \tag{3b}$$

where a_j represent the intra-component and inter-component interactions while the interactions take equal values when $a_1 = a_2$, $U_j(x)$ are the complex $\mathcal{PT}$-symmetric potentials, and the imaginary parts of $U_j(x)$ stand for the gain or loss term from the thermal clouds ($j = 1, 2$).

The present paper is built up as follows. In Sect. 2, we consider the analytic bright-soliton solution in the coupled Gross-Pitaevskii equations with complex $\mathcal{PT}$-symmetric Scarf-II potentials; the linear stability analysis and the numerical evolution results corroborating the analytical solitons are presented in Sect. 3; In Sect. 4, we perform numerical simulations for the excitation and evolution of nonlinear modes via adiabatical change of system parameters; Finally, conclusions and discussions are given in Sect. 5.

2 Localized modes in coupled Gross-Pitaevskii equations

We concentrate on stationary solutions of Eq (3) in the form

$$\psi_j(x,t) = \phi_j(x)e^{iv_jt}, \quad j = 1, 2\ , \tag{4}$$

where ν_j are the real propagation constant. The complex solutions $\phi_j(x)$ satisfy the following condition

$$\left[\frac{d^2}{dx^2} + a_1(|\phi_1|^2 + |\phi_2|^2) + U_1(x)\right]\phi_1 = \nu_1\phi_1 \ , \tag{5a}$$

$$\left[\frac{d^2}{dx^2} + a_2(|\phi_1|^2 + |\phi_2|^2) + U_2(x)\right]\phi_2 = \nu_2\phi_2 \ , \tag{5b}$$

which can be solved for the given potentials $U_j(x)$ and real propagation constant ν_j.

For the $\mathcal{PT}$-symmetric potentials $U_j(x)$ are all chosen as the well-known Scarf-II potentials as

$$U_j(x) = V_j \operatorname{sech}^2(x) + iW_j \operatorname{sech}(x)\tanh(x). \tag{6}$$

We have the analytic bright-soliton solution as [37]

$$\phi_j(x) = A_j \operatorname{sech}(x)\, \mathrm{e}^{i\varphi_j}, \quad j = 1, 2 \ , \tag{7}$$

with the phases being

$$\phi_j = \frac{W_j}{3} \arctan[\sinh(x)], \quad j = 1, 2 \ , \tag{8}$$

under the constraints of

$$\frac{18 + W_j^2 - 9V_j}{9a_j} = \sum_{n=1,2} A_n^2, \quad j = 1, 2 \ , \tag{9}$$

and $\nu_j = 1$.

For the nonlinear modes given in Eq (7), the power of the solutions is defined as $P_j = \int_{-\infty}^{\infty} |\phi_j(x)|^2 dx$, $P = P_1 + P_2$, while the Poynting vector $S_j = \frac{i}{2}(\phi_j\phi_{jx}^* - \phi_j^*\phi_{jx}) = A_j^2 W_j/3 \operatorname{sech}^3(x)$. The power flows from left (right) to right (left) at x_0 when $S(x_0) > 0$ ($S(x_0) < 0$).

3 Linear stability analysis

In this section, we investigate the linear stability of the nonlinear modes, which is a standard protocol to show the stability of nonlinear localized modes. We consider the perturbed solution $\psi_j(x, t)$, in the form

$$\psi_j(x,t) = \phi_j(x)\, \mathrm{e}^{i\nu_j t} + \epsilon\, [f_j(x)\, \mathrm{e}^{i\delta t} + g_j^*(x)\, \mathrm{e}^{-i\delta^* t}]\, \mathrm{e}^{i\nu_j t} \tag{10}$$

where $\epsilon \ll 1$, which is the small perturbation on the solution. $f_j(x)$ and $g_j(x)$ are the perturbation eigenfunctions of the linearized eigenvalue problem. By substituting Eq (10) into Eq (5) and linearizing with respect to ϵ, we can drive the following linear eigenvalue problem:

$$\begin{pmatrix} L_1 & a_1\phi_1^2 & a_1\phi_1\phi_2^* & a_1\phi_1\phi_2 \\ -a_1\phi_1^{*2} & -L_1^* & -a_1\phi_1^*\phi_2^* & -a_1\phi_1^*\phi_2 \\ a_2\phi_1^*\phi_2 & a_2\phi_1\phi_2 & L_2 & a_2\phi_2^2 \\ -a_2\phi_1^*\phi_2^* & -a_2\phi_1\phi_2^* & -a_2\phi_2^{*2} & -L_2^* \end{pmatrix} \begin{pmatrix} f_1 \\ g_1 \\ f_2 \\ g_2 \end{pmatrix} = \delta \begin{pmatrix} f_1 \\ g_1 \\ f_2 \\ g_2 \end{pmatrix}, \tag{11}$$

where

$$L_1 = -v_1 + \partial_x^2 + U_1 + 2a_1\phi_1\phi_1^* + a_1\phi_2\phi_2^* \tag{12a}$$

$$L_2 = -v_2 + \partial_x^2 + U_2 + a_2\phi_1\phi_1^* + 2a_2\phi_2\phi_2^* \tag{12b}$$

The imaginary part of δ measures the growth rate of the perturbation instability. If $|\mathrm{Im}(\delta)| > 0$, then the perturbation will grow exponentially with t, and the solutions are unstable; otherwise, the solutions are stable. In our numerical simulation, we use the Fourier collocation method to discretize the associated differential operator as a matrix to solve the eigenvalue problem [52]. To further verify the stability of the solitons, we numerically investigate the stability by evolving them with 5% perturbations as the initial condition to simulate the random white noise (i.e., $\psi(x, 0) = \phi(x)(1 + \xi)$ and ξ represents 5% perturbations). In our numerical simulations, the second-order spatial differential is carried out by using Fourier spectral collocation method, and the integration in time is carried out by using the explicit fourth-order Runge-Kutta method [53].

Firstly, under the constraint of $A_1 = 0.5$, $V_1 = V_2$, $W_1 = W_2$, we consider the focusing case $a_1 = a_2 = 1$ and the defocusing case $a_1 = a_2 = -1$, respectively. Then we get the stable (blue) and unstable (red) domains of nonlinear localized modes in (V_1, W_1) space [see Fig 1]. They are determined by the maximum absolute value of imaginary parts of the linearized eigenvalue δ in Eq (11). We find that solitons tend to be unstable with the increase of $|W_1|$ in the focusing case. It is worth noting that when $|W_1| = 3$, solitons are stable in the defocusing case.

Since the above situations are obtained in the case of $A_1 = 0.5$, and A_2 is obtained by Eq 9. Next, we consider the case of $A_1 = A_2$. The relationships between P_2 and the parameter W_1 are shown in Fig 2. We can find that when other parameters are fixed, P_2 and $|W_1|$ are positively correlated in the focusing case, while they are negatively correlated in the defocusing case. In addition, for both cases $A_1 = 0.5$ and $A_1 = A_2$, the intervals of W_1 have no difference when the solutions are stable.

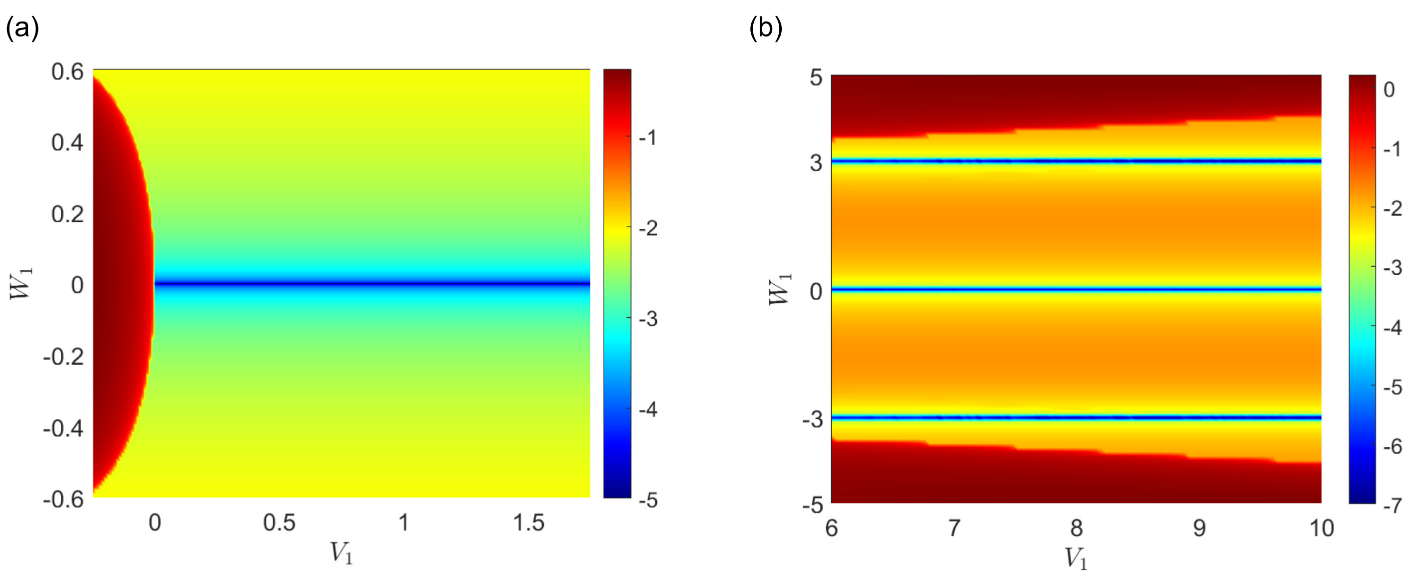

Fig 1. Maximal imaginary part of the linearization eigenvalue δ in the (V_1, W_1)-space (common logarithmic scale), under the constraint of $A_1 = 0.5$, $V_1 = V_2$, $W_1 = W_2$ and (a) $a_1 = 1$; (b) $a_1 = -1$.

https://doi.org/10.1371/journal.pone.0294790.g001

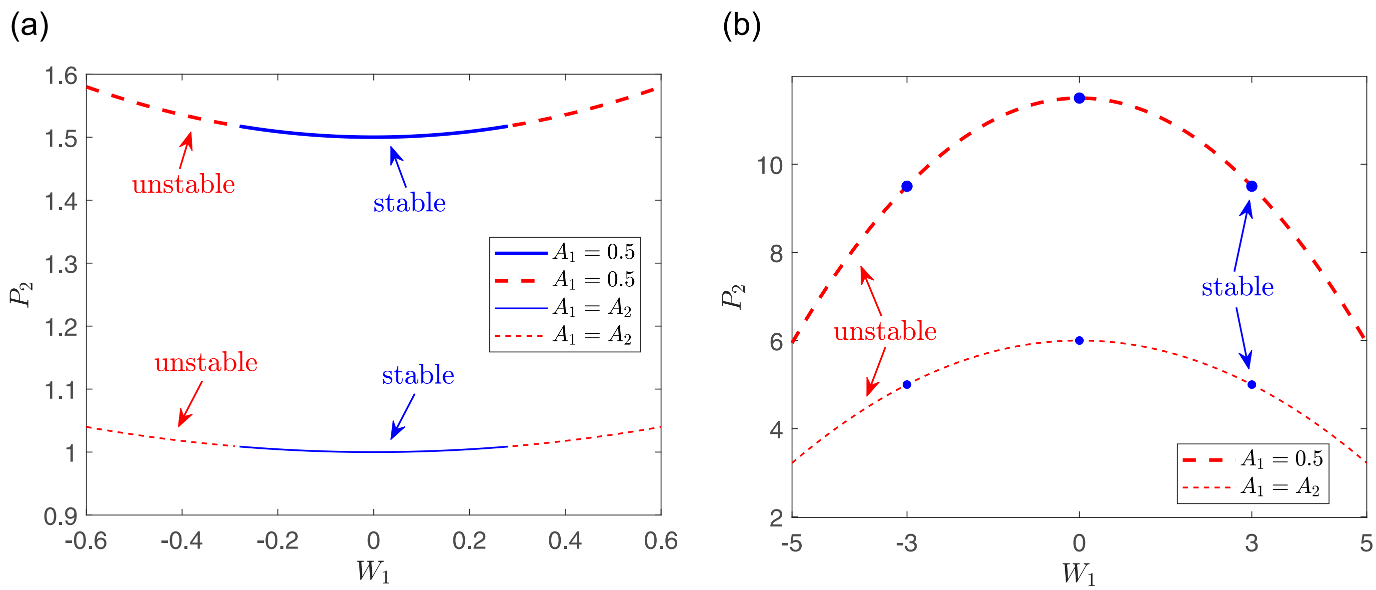

Fig 2. The relationship between power of nonlinear mode ϕ_2 and W_1. The parameters are chosen as: $W_1 = W_2$, and (a) $V_1 = V_2 = 1$, $a_1 = 1$; (b) $V_1 = V_2 = 8$, $a_1 = -1$.

https://doi.org/10.1371/journal.pone.0294790.g002

In particular, for the fixed parameters $a_1 = a_2 = 1$, $A_1 = 0.5$, $V_1 = V_2 = 1$, Fig 3(a)–3(c) display the stable soliton for $W_1 = W_2 = 0.25$ while Fig 3(d)–3(f) display the unstable soliton for $W_1 = W_2 = 0.55$; for the fixed parameters $a_1 = a_2 = -1$, $A_1 = 0.5$, $V_1 = V_2 = 8$, Fig 4(a)–4(c) display the stable soliton for $W_1 = W_2 = 3$ while Fig 4(d)–4(f) display the unstable soliton for $W_1 = W_2 = 2$.

Furthermore, in the focusing case, the amplitude of the nonlinear mode is periodically oscillating when V_1 and W_1 are sufficiently small, and it experiences more than 5 periods within $1200 \leq t \leq 1500$ [see Fig 5].

4 Adiabatic excitation for the nonlinear modes

In this section, we consider excitations of the above-mentioned solitons via adiabatical changes of system parameters. We change the parameters as the functions of t. To modulate the system parameters smoothly, we consider the following "switch-on" function:

$$\zeta(t) = \begin{cases} \zeta^{(\mathrm{ini})}, & t = 0, \\ \dfrac{\zeta^{(\mathrm{end})} - \zeta^{(\mathrm{ini})}}{2}\left[1 + \sin\left(\dfrac{\pi t}{500} - \dfrac{\pi}{2}\right)\right] + \zeta^{(\mathrm{ini})}, & 0 < t < 500, \\ \zeta^{(\mathrm{end})}, & 500 \leq t \leq 1500, \end{cases} \tag{13}$$

where $\zeta^{(\mathrm{ini})}$, $\zeta^{(\mathrm{end})}$ respectively represent the real initial-state and final-state parameters [38, 45, 54]. Adiabatic excitation includes two stages: excitation stage ($0 < t < 500$) and propagation stage ($500 \leq t \leq 1500$). We consider two cases of excitations by setting a_1, V_1 and V_2 to be functions of t, that is $a_1 \to a_1(t)$, $V_1 \to V_1(t)$ and $V_2 \to V_2(t)$. To facilitate the display of power changes over time, we set $A_1 = A_2$, $W_1 = W_2 = 0.55$, $V_1^{(\mathrm{ini})} = 1$, $V_1^{(\mathrm{end})} = 2$, $a_1^{(\mathrm{ini})} = 0.1$, $a_1^{(\mathrm{end})} = 1$. Firstly, we set $V_2^{(\mathrm{ini})} = 1$, $V_2^{(\mathrm{end})} = 2$, $a_2 = 0.1$, and the power of nonlinear modes is

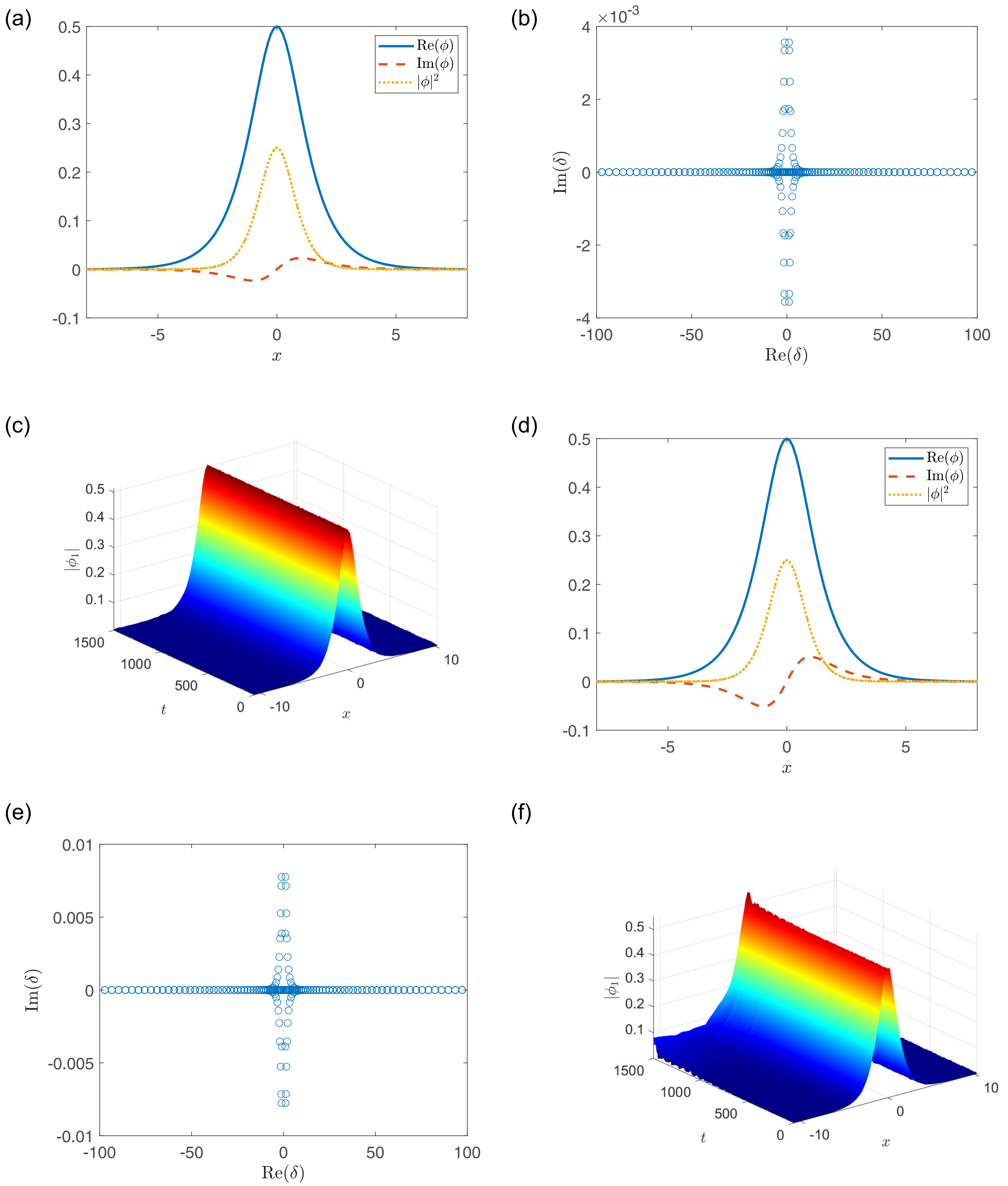

Fig 3. (a, d) The soliton solutions. (b, e) Linear stability eigenvalues. (c, f) Stable or unstable propagations of nonlinear modes. The parameters are chosen as: $a_1 = 1$, $A_1 = 0.5$, $V_1 = V_2 = 1$, and (a-c) $W_1 = W_2 = 0.25$; (d-f) $W_1 = W_2 = 0.55$.

https://doi.org/10.1371/journal.pone.0294790.g003

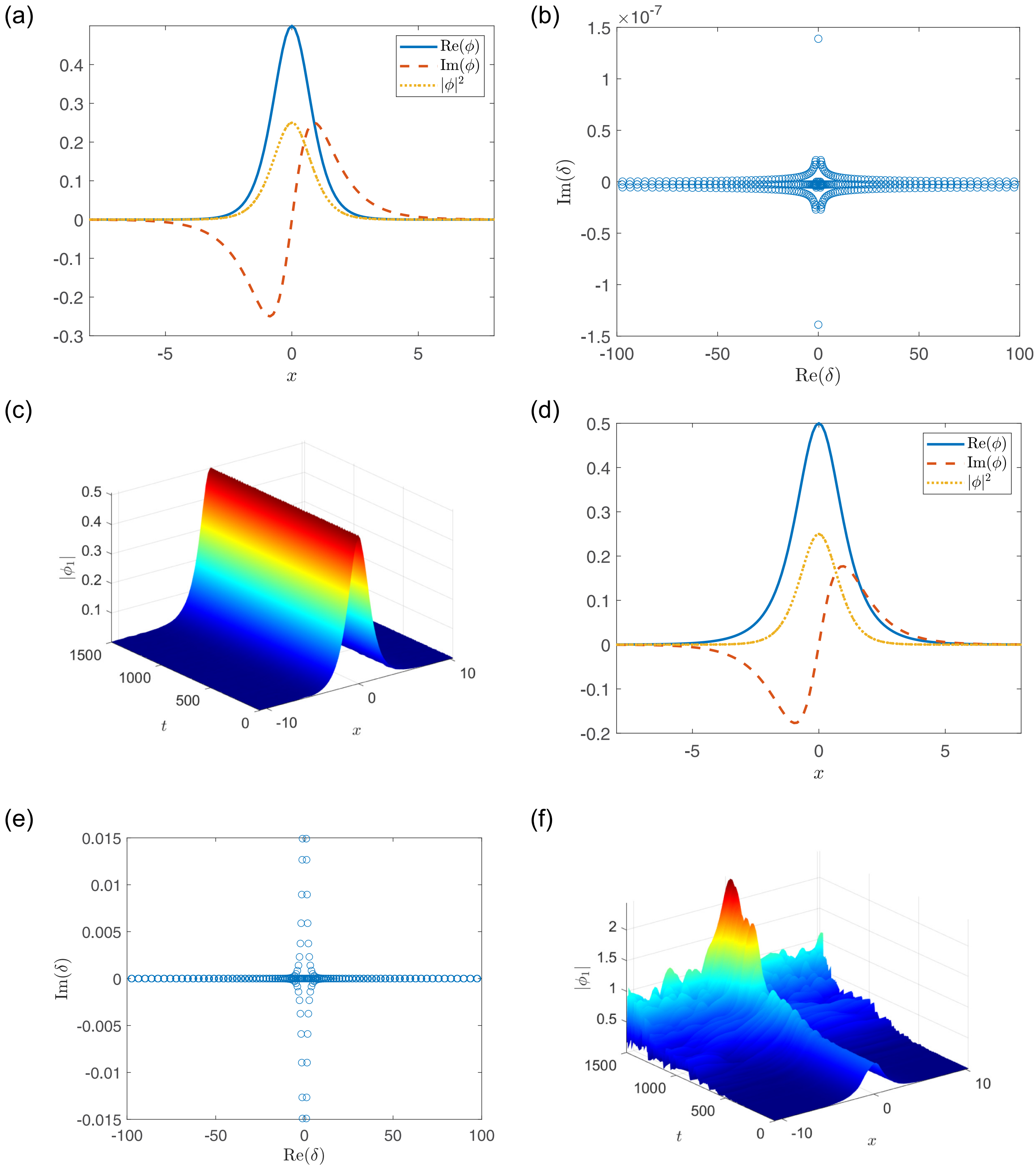

Fig 4. (a, d) The soliton solutions. (b, e) Linear stability eigenvalues. (c, f) Stable or unstable propagations of nonlinear modes. The parameters are chosen as: $a_1 = -1$, $A_1 = 0.5$, $V_1 = V_2 = 8$, and (a-c) $W_1 = W_2 = 3$; (d-f) $W_1 = W_2 = 2$.

https://doi.org/10.1371/journal.pone.0294790.g004

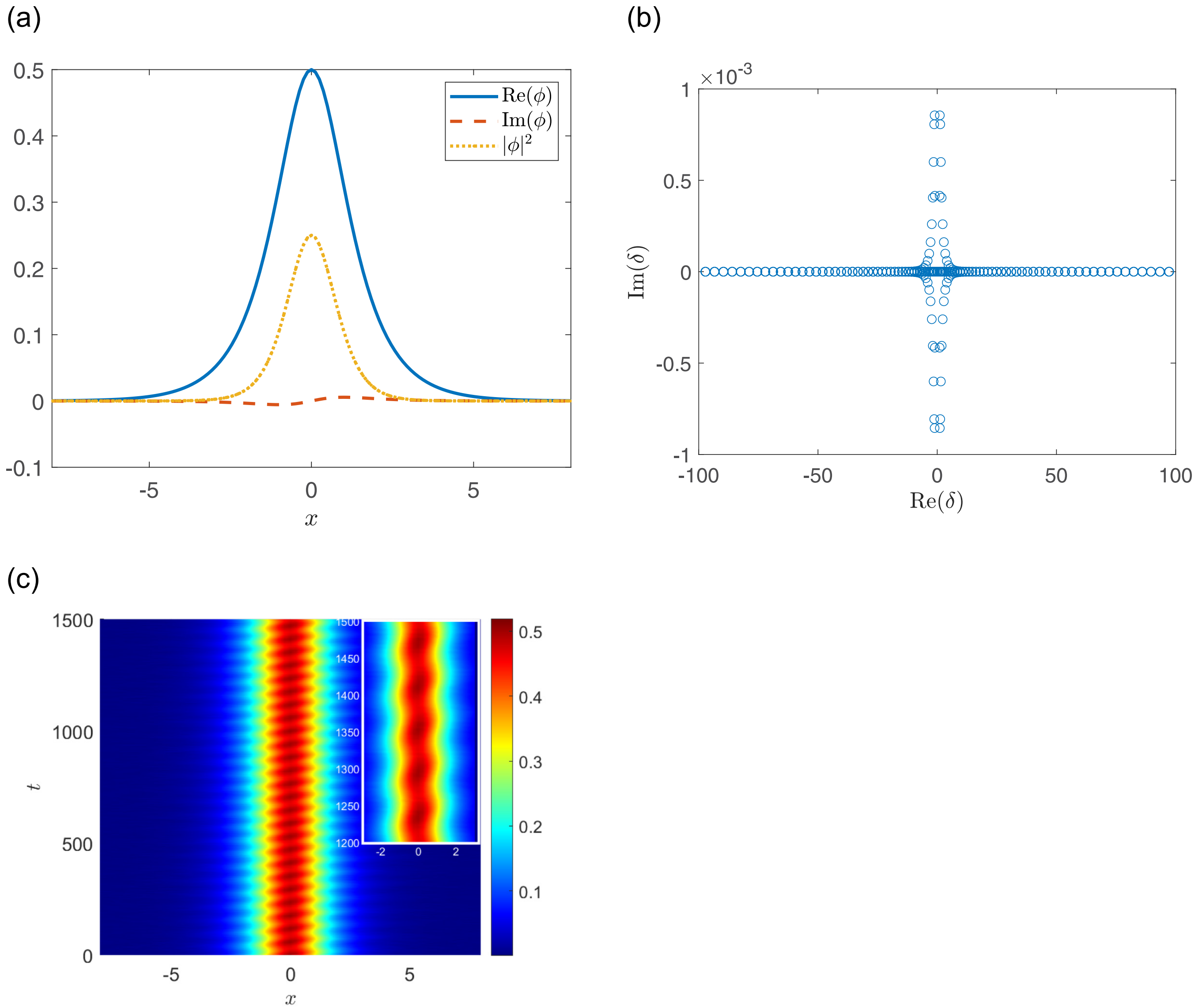

Fig 5. (a) The soliton solutions. (b) Linear stability eigenvalues. (c) Stable propagations of nonlinear modes. The parameters are chosen as: $a_1 = 1$, $A_1 = 0.5$, $V_1 = V_2 = 0.01$, $W_1 = W_2 = 0.06$.

https://doi.org/10.1371/journal.pone.0294790.g005

reduced [see Fig 6a–6c]. Then, we set $V_2^{(\mathrm{ini})} = 2$, $V_2^{(\mathrm{end})} = 1$, $a_2 = 0.0033$, and the total power of nonlinear modes is to decrease and then increase [see Fig 6d–6f]. The above results mean that the power is not conserved during adiabatic excitations, and it has a correlation with the initial and final state potential parameters.

5 Conclusion

In conclusion, we study the nonlinear modes in two-component Bose-Einstein condensates with $\mathcal{PT}$-symmetric Scarf-II potential, which can be described by the coupled Gross-Pitaevskii equations. We investigate the linear stability of the nonlinear modes and validate the results by evolving them with 5% perturbations as an initial condition. We find that solitons tend to be

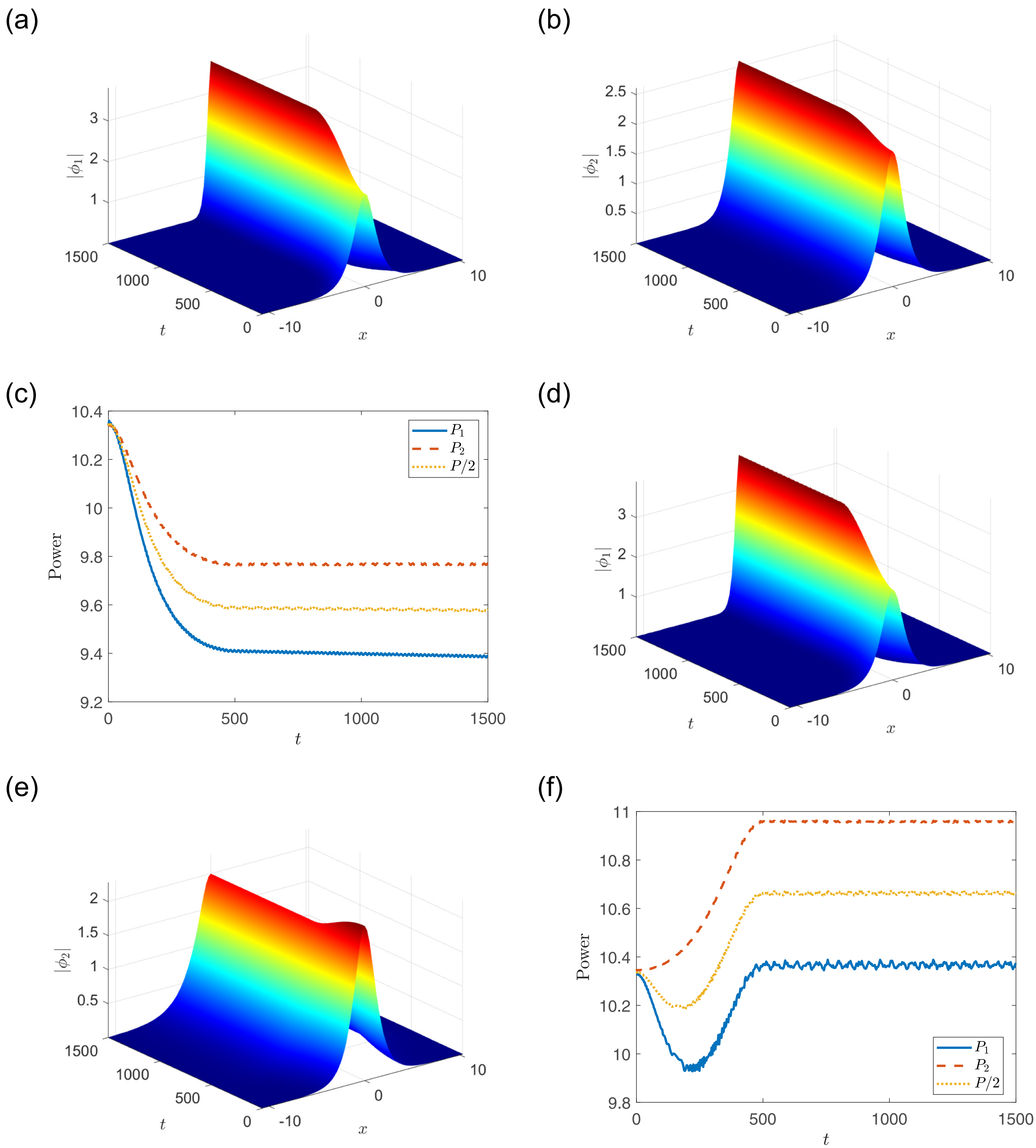

Fig 6. Adiabatic excitation of nonlinear mode and its evolution. The parameters are chosen as: $A_1 = A_2 = 2.2733$, $W_1 = W_2 = 0.55$, $V_1^{(\mathrm{ini})} = 1$, $V_1^{(\mathrm{end})} = 2$, $a_1^{(\mathrm{ini})} = 0.1$, $a_1^{(\mathrm{end})} = 1$ and (a-c) $V_2^{(\mathrm{ini})} = 1$, $V_2^{(\mathrm{end})} = 2$, $a_2 = 0.1$; (d-f) $V_2^{(\mathrm{ini})} = 2$, $V_2^{(\mathrm{end})} = 1$, $a_2 = 0.0033$.

https://doi.org/10.1371/journal.pone.0294790.g006

unstable with the increase of $|W_1|$ in the focusing case. It is worth noting that when $|W_1| = 3$, solitons are stable in the defocusing case. In the focusing case, the amplitude of the nonlinear mode is periodically oscillating when V_1 and W_1 are sufficiently small. Finally, we consider excitations of the solitons via adiabatical changes of system parameters, then we find that the power is not conserved during this adiabatic excitation. These findings of nonlinear modes can be potentially applied to physical experiments of matter waves in Bose-Einstein condensates.

In addition, we can consider other $\mathcal{PT}$-symmetric potentials in the coupled Gross-Pitaevskii equations. Due to the limitations of the parameters in this model, the amplitudes of the two solutions are constrained by a certain relationship. Therefore, we can also consider the case of unequal intra-component and inter-component interactions.

Acknowledgments

We express our sincere thanks to the editor, referees and all the members of our discussion group for their valuable comments.

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Validation: Jia-Rui Zhang.

Visualization: Jia-Rui Zhang, Xia Wang.

Writing – original draft: Jia-Rui Zhang.

Writing – review & editing: Jia-Rui Zhang, Xia Wang.

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