Generalised U-duality in M-theory

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Contents

Abstract

Dualities play a very important role in connecting different theoretical physics models to each other. In the framework of string theory, dualities build links between 5 types of strings, highlighting the unified origin reflected by M-theory. In this thesis we discuss how to define and utilise a special type of dualities - generalised dualities. These are an extended notion of dualities that applies to a broader variety of dual spaces with no background isometries. We study new types of generalised U-dualities, and use these to construct and analyse new dual solutions in M-theory.

In chapter 1 we start with reviewing different types of dualities in string and M theory, revisiting main algebraic and physics aspects of dualities and their mechanisms. We also go through a brief overview of string theory, defining different types of strings and their main characteristics, and presenting the dimensional reduction mechanism that we utilise later in generating dual solutions.

Chapter 2 is dedicated to exploring geometries with Exceptional Drinfeld Algebra (EDA) structure applied to studying the generalised U-duality - a special type of dualities in M theory. We have also provided classification of different "three-algebra geometries" that represents a specially chosen case of EDA, and studied in more depth examples that resulted in novel uplifts for special gaugings of seven dimensional maximal supergravity.

In chapter 3 we discuss the notion of generalised U-duality as a solution generating technique in supergravity. Using the exceptional geometry technique, we demonstrate how to generate a new solution in 11-dimensional supergravity starting with type IIA supergravity. We further analyse the features of newly generated solution and explore its AdS limit and charges. We end the chapter by solving the Killing spinor equation in the extremal AdS limit case, and finding a $\frac{1}{2}$-BPS solution. This provides us with new interesting insights about the nature of U-dual solutions.

In chapter 4 we continue the investigation of the same solution generating technique using exceptional geometry, expanding our analysis to initial solutions with more complicated geometrical structure, generalising the results obtained in the previous chapter. The newly generated solutions are now described by an underlying 6-algebra structure, generalising the 3-algebra structure in the previous case. Using exceptional geometry techniques we provide an 11-dimensional uplift of the 4-dimensional gauged supergravity. Similarly to the examples generated in the previous chapter, we construct a new $\frac{1}{2}$-BPS solution and elaborate on the properties of the new dual solution.

We conclude in chapter 5 with some final thoughts, summarising the results of previous chapters and highlighting the contribution of the work we presented in understanding the nature of generalised dualities in physics and how they serve as solution generating techniques, expanding our understanding of connections in M-theory and supergravity. We indicate a few further directions that could be interesting for further investigation.

Abstract

Dualiteiten spelen een zeer belangrijke rol in het verbinden van verschillende theoretisch natuurkundige modellen met elkaar. In het kader van de snaartheorie leggen dualiteiten verbanden tussen 5 soorten snaren en benadrukken ze de verenigde oorsprong die wordt weerspiegeld door M-theorie. In de context van dit werk bespreken we hoe we een speciaal type dualiteiten - veralgemeende dualiteiten - kunnen definiëren en gebruiken. Dit is een uitgebreide notie van dualiteiten die van toepassing is op een bredere variëteit van duale ruimten zonder achtergrondisometrieën. We bestuderen nieuwe soorten veralgemeende U-dualiteiten en gebruiken deze om nieuwe duale oplossingen in M-theorie te construeren en te analyseren.

In hoofdstuk 1 beginnen we met een overzicht van verschillende soorten dualiteiten in snaar- en M-theorie, waarbij we de belangrijkste algebraïsche en fysische aspecten van dualiteiten en hun mechanismen bespreken. We geven ook een kort overzicht van snaartheorie, definiëren verschillende soorten snaren en hun belangrijkste kenmerken, en presenteren het standaard dimensionale reductiemechanisme dat we later gebruiken bij het genereren van duale oplossingen.

Hoofdstuk 2 is gewijd aan het verkennen van geometrieën met Exceptional Drinfeld Algebra (EDA) structuur toegepast op het bestuderen van de veralgemeende U dualiteit - een speciaal type van dualiteiten in M theorie. We hebben ook een classificatie gegeven van verschillende "drie-algebra geometrieën" die een speciaal gekozen geval van EDA vertegenwoordigt, en we zijn dieper ingegaan op voorbeelden die hebben geleid tot nieuwe uplifts voor speciale gaugings van zevendimensionale maximale superzwaartekracht.

In hoofdstuk 3 bespreken we het begrip veralgemeende U-dualiteit als oplossingsgeneratietechniek in superzwaartekracht. Met behulp van de uitzonderlijke meetkunde techniek laten we zien hoe we een nieuwe oplossing kunnen genereren in 11-dimensionale superzwaartekracht, beginnend met een speciaal type oplossing in type IIA superzwaartekracht. We analyseren verder de eigenschappen van de nieuw gegenereerde oplossing en onderzoeken zijn AdS-limiet en ladingen. We eindigen het hoofdstuk met het oplossen van de Killing-spinorvergelijking in de extreme AdS-limiet en het vinden van een $\frac{1}{2}$-BPS oplossing.

In hoofdstuk 4 gaan we verder met het onderzoek van dezelfde oplossingsgeneratietechniek met behulp van uitzonderlijke meetkunde, waarbij we onze analyse uitbreiden naar initiële oplossingen met een gecompliceerdere geometrische structuur en de resultaten uit het vorige hoofdstuk veralgemenen. Met behulp van uitzonderlijke meetkunde technieken geven we een 11-dimensionale uplift van de 4-dimensionale gauged superzwaartekracht. Net als de voorbeelden uit het vorige hoofdstuk construeren we een nieuwe $\frac{1}{2}$-BPS oplossing en gaan we dieper in op zijn eigenschappen.

We sluiten af in hoofdstuk 5 met enkele laatste gedachten, waarin we de resultaten van de vorige hoofdstukken samenvatten.

List of publications

1. Exploring Exceptional Drinfeld Geometries, Chris D. A. Blair, Daniel C. Thompson, Sofia Zhidkova, JHEP 09 (2020) 151 (Q1 journal) [87], corresponds to chapter 2.

2. Generalised U-dual Solutions in Supergravity, Chris D. A. Blair, Sofia Zhidkova, JHEP (2022) 81 (Q1 journal) [113], corresponds to chapter 3.

3. Generalised U-dual Solutions via ISO(7) gauged supergravity, Chris D. A. Blair, Sofia Zhidkova, JHEP (2022) 93 (Q1 journal) [117], corresponds to chapter 4.

Chapter 1

Introduction

1.1 String and M-theory

Finding a theory that would unify all the existing interactions in nature under one model is an eternal challenge of theoretical physics. A unification that would provide a sole origin to all existing theories on different energy scales and levels is a challenge that physicists try to answer. At small scales, the Standard Model of elementary particle physics gives a self-consistent explanation of the original principles of physics that matches the results obtained experimentally in the recent scientific observations. On the opposite side of scale level, Einstein's theory of General Relativity provides a reliable prediction of the behaviour of gravitational effects throughout the universe, which was recently again confirmed via the first direct detections of gravitational waves.

However, neither of these theories are complete. Whilst there is no known way to deduce the masses of the elementary particles (and other parameters in the theory) from first principles in the Standard Model, in the General Relativity applying a standard quantisation procedure will result in a non-renormalisable theory (the problem of black hole solutions indicates the existence of problems on a quantum theory level). We do not have a clear method on how to combine the two theories, since the Standard Model is renormalisable on quantum level, while General Relativity is not. At the highest energies and smallest possible scales, both quantum and gravitational effects are important, and we need a theory of "quantum gravity". With the hope that this would be the fundamental theory of nature, that enables to provide a precise explanation for the masses of the elementary particles and other theoretical challenges (such as the origin of dark matter and dark energy). Unifying quantum mechanics and gravity has proven to be one of the hardest challenges in theoretical physics.

One of the best and most well supported mathematically candidates for a theory of quantum gravity is string theory. The initial idea of this theory is that instead of elementary *particles*, we should consider elementary *strings*. Compared to particles, the fundamental distinction is that strings are extended objects, with additional degrees of freedom, leading to various consequences.

A collection of multiple states can be found via string quantisation procedure, corresponding to different vibrations of the string, which will appear as particles when working at scales larger than the string length. The behaviour of these particle-like states of the string resembles the interaction appearing in the Standard Model and Gravity in the same time. Moreover, one of the states can be identified with the hypothetical Graviton particle spectrum – the carrier of the gravitational interaction.

This provides an additional supporting point to the candidature of the String theory taking the role of the quantum gravity, with many strong features, such as the lack of numerous free parameters, instead having only one free parameter – the string length, unlike the Standard Model, which appears to have a whole range of unspecified parameters.

The consistency requirements of string theory are remarkably stringent, and it only works in 10 dimensions. (Properties of the four-dimensional physics we experience can be encoded in the structure of the *geometry* of the other six dimensions, although finding the appropriate construction to match our observable universe is an open problem.) At first, these consistency conditions seemed to allow five distinct string theories, despite the hope that any fundamental quantum gravity should be unique. However, a further compelling and fascinating property of string theory is the presence of *dualities* in the theory, leading to a unified theoretical description, in which the 5 possible solutions can be viewed as special cases, resulting from the same parent theory – the M-theory.

M-theory was originally introduced as the strong coupling limit of type IIA superstring theory [1]. M-theory is required to reduce to 11 dimensional supergravity - a theory combining the nature of general relativity and supersymmetry - at energies significantly smaller than the inverse Planck length $1/l_P$ in the same way as the type IIA or IIB superstring theory reduces to type IIA or IIB supergravity at energies significantly smaller than the inverse string length $1/l_s$. The reduction of M-theory on a circle will result in type IIA string theory, and reduction of 11-dimensional supergravity on a circle provides us with the 10-dimensional type IIA supergravity, highlighting a clear connection between the theories.

1.2 Dualities in string and M-theory

String theory provides a unified framework that encompasses both quantum mechanics and general relativity. Among the phenomena within string theory, duality stands out as a powerful concept that has greatly deepened our insights into the underlying structure of spacetime and quantum field theories. Dualities relate seemingly distinct string backgrounds and play a crucial role in unravelling the nonperturbative and perturbative nature of string theory.

In theoretical physics one of the simplest examples of such a duality appears in classical electromagnetism in the case of Maxwell's equations in vacuum, where the theory reveals a symmetry under the interchange of the electric and magnetic fields. Another great example of dualities in physics is the Ising temperature duality in thermodynamics, where the lattice Ising model expresses similar behaviour below and above a critical point, making the model at low temperatures dual to the model at high temperatures.

The simplest string duality follows from the fact that strings have length. Particles only see the world one point at a time, but strings are extended objects. As a result they experience geometry very differently. For instance, a string does not distinguish between *winding* around large circle of radius R and *moving* around a small circle of radius $1/R$ (or vice versa). We say that the large and small circle descriptions of the geometry are *dual*. This is known as T-duality.

Another type of dualities - S-duality - involves the coupling g determining the strength of string-string interactions (this is not an independent parameter, but is determined dynamically within the theory). Some string theories at strong coupling, g large, are the same as others at weak coupling, g small. This is a remarkable equivalence, as it allows us to understand strong coupling physics (difficult) using weak coupling physics (easier). Even more surprisingly, in one "dual" description, type IIA string theory at strong coupling in 10 dimensions turns out to "grow" an extra spacetime dimension and is described in terms of an 11-dimensional theory.

Combining these basic dualities leads to a fascinating web of connections that tells us we should think of all different string theories as being different dual descriptions of a single underlying more fundamental theory [1, 2]. This theory has become known as M-theory. However, despite much progress in many areas, the final formulation of this M-theory remains mysterious.

We will review the notion of T-duality and its generalisations. The usual T-duality symmetry is present when one has a background with commuting i.e. abelian geometric symmetries, represented by the Abelian Killing vectors of the solution. This is called Abelian T-duality. Abelian T-duality has an extremely important implementation in relating type IIA string theory to type IIB string theory by compactifying each theory on a circle with inverse radii. Abelian T-duality is an exact symmetry of string theory.

Non-Abelian T-duality, extends the Abelian T-duality, unveiling connections between diverse string backgrounds. Unlike its Abelian counterpart, which arises in the presence of Abelian Killing vectors, non-Abelian T-duality arises when considering compactifications of string theory on spaces with non-Abelian isometries [3], [4].

Non-Abelian T-duality is not an exact symmetry of string theory. For Non-Abelian isometry groups certain anomalies can arise when performing the dual transformation [5]. Even though the quantum status of non-Abelian T-duality as a genuine duality of the full string theory is less clear (though there is some recent evidence in favour of this [6]), it has proven an efficient classical solution generating mechanism, in particular for holographic backgrounds with interesting dual field theories [7–9]. There is a further generalisation known as Poisson-Lie duality in which there are no isometries present on either side of the duality [10]. In Poisson-Lie duality, there is an equivalence between two different classical sigma models [10], which can be proven using a single sigma model whose target space is doubled [11]. Both these dualities can be used to describe string theory in interesting settings such as backgrounds where the sigma model is integrable (for a review, see [12]).

Unlike the Abelian T-duality, the non-Abelian T-duality is not completely obvious on the level of higher quantum corrections. While in the Abelian T-duality the mapping between 2 dual solution in a sigma model can be extended in all orders of the quantum α' correction, in the case of its non-Abelian counterpart, this can only be done to a certain orders [13]. However, the interest of considering various generalised versions of T and afterwards U dualities is still viable, and is motivated by the idea of generating new solutions and building a bigger picture of the M-theory structure, revealing a web of connections between different solutions.

U-duality is a combination of S and T dualities applied one after another, it is a new "unified" type of dualities in M-theory. Unlike the T-duality it wasn't very clear how to generalise U-duality in the case of non-Abelian isometries (represented by non-Abelian Killing vectors). While T-duality is an exact perturbative symmetry of string theory at each order in the string coupling constant g, S-duality and U-duality are non-perturbative in the coupling constant orders.

U-duality has several important implementations in understanding the M-theory and building connections between different solutions. It provides a helpful mechanism of studying compactifications of superstring theories, alongside other types of dualities.

The natural way of observing U-duality comes from studying dualities of string theories and supergravity. Type IIA string theory reduced on a circle is T-dual to type IIB string theory reduced on a circle of an inverted radius. Type IIB theory contains an internal global SL(2,$\mathbf{Z}$) symmetry, reflected in S-duality preserving the symmetry of the theory. T-duality between type IIA and IIB string theories combined together with the duality SL(2) symmetry group of the $\mathbf{T}^2$ torus (or more generally the symmetry of the d-torus) on which we performed a compactification of M-theory to obtain a lower-dimensional effective theory (M-theory on 2-torus or d-torus more generally) forms the U-duality symmetry, generally given by the $E_{d(d)}$ groups.

In the following section we will briefly review the string theory models and provide an overview of the main field in the spectrum to proceed further with the explanation of dualities.

1.3 Overview of string theory

In the string theory there are 5 consistent types of strings that can be all unified under a bigger 11-dimensional theory - the M-theory, as special cases. The set of five types of string is built of: type I strings, in which strings are unoriented and the spectrum is composed of open and closed strings with $\mathcal{N} = 1$ supersymmetry, type II strings, consisting of oriented closed strings with a maximal amount of supersymmetry $\mathcal{N} = 2$, with type IIA as its non-chiral version, and type IIB as its chiral version, and two types of heterotic (also with $\mathcal{N} = 1$ supersymmetry, made by mixing the left-moving sector of the bosonic string with the right-moving sector of the superstring) - one with the SO(32) gauge group and the second with the $\mathrm{E}_8\times\mathrm{E}_8$ gauge group.

These different types of strings are all special cases of the M-theory and are connected to each other with a web of dualities. Type IIA and type IIB strings are related to each other via T-duality, as are the two types of heterotic strings. S-duality connects type I strings to the heterotic strings with the SO(32) gauge group, and type IIB to itself again (type IIB is self dual by S-duality). Moreover, type IIA string theory in the infinitely strong coupling constant limit becomes equivalent to a bigger theory - the M-theory: meaning there is an S-duality between type IIA strings and the M-theory. Similarly, S duality is present between the heterotic string with the $E_8\times E_8$ gauge group and the M-theory. Thus all the 5 types of string theories are connected to each other via dualities, and are special cases of a bigger picture - the M-theory. At the same time 11-dimensional supergravity is a low energy limit of the 11-dimensional M-theory.

Let us start with reviewing the bosonic sector of string theory and introduce the Polyakov action defined on the worldvolume Σ - the space-time that a string sweeps out as it moves through spacetime, that is representing the basic fundamental action of bosonic strings on the worldsheet:

$$S_P = \frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \sqrt{|h|}h^{ab}g_{\mu\nu}(X)\partial_a X^\mu(\tau,\sigma)\partial_b X^\nu(\tau,\sigma) \tag{1.3.1}$$

where the structure $\frac{1}{4\pi\alpha'}$ has the meaning of the string tension T, the coordinates τ, σ are the worldsheet coordinates, X^μ are the coordinates of the target manifold, and the metric $g_{\mu\nu}$ is the metric of the target space (26 dimensions or 10 in supersymmetric version), h_{ab} is the worldsheet space metric.

For the case of flat Minkowski metric ($g_{\mu\nu}(X) = \eta_{\mu\nu}$) the action is invariant under space-time translation and Lorentz transformations (Poincare symmetry) in the target space (global symmetry), as well as under diffeomorphisms and Weyl transformation (metric rescaling) in the worldsheet space (local symmetries).

Starting with open strings - type of strings with free endpoints and a topology of a line, the boundary conditions can divided into:

1. Neumann boundary conditions (N)

$$n^a\partial_a X^\mu|_{\partial\Sigma} = 0 \tag{1.3.2}$$

where n^a is a unit normal vector to the boundary $\partial\Sigma$. This condition in the case of a free open string propagation reduces to a simple $\partial_\sigma X^\mu|_{\partial\Sigma} = 0$ on the surface boundary $\partial\Sigma$.

2. Dirichlet boundary conditions (D)

$$t^a\partial_a X^\mu|_{\partial\Sigma} = 0 \tag{1.3.3}$$

where t^a is a unit tangent vector to the boundary $\partial\Sigma$. In a case of the simple free string propagating in a worldsheet - a manifold describing the embedding of the string in the spacetime - this reduces to $\partial_\tau X^\mu|_{\partial\Sigma} = 0$, which can be rewritten in the following way:

$$X^\mu|_{\partial\Sigma} = C^\mu \tag{1.3.4}$$

for a constant C^μ, meaning that the string is fixed on the end point. In the case of propagating in an p-dimensional hypersurface, Dirichlet boundary condition will restrict $d-p$ coordinates, breaking translation invariance in these directions. This is associated to dynamical $p+1$-dimensional objects in the hypersurface called D-branes - objects where the endpoints of open strings lie (for Dirichlet boundary conditions).

In the case of closed strings - circle topology, the strings can be divided into periodic and anti-periodic. Anti-periodic boundary conditions are not used for bosons, and only appear in the fermionic sector.

In addition to the classification of strings to open and closed, they also can be oriented and unoriented, with oriented strings corresponding to oriented worldsheet surfaces, and unoriented strings - to unoriented worldsheet surfaces.

Speaking about the field spectrum of the bosonic sector theory, based on the string type and their degrees of freedom in 26 dimensional space-time we have the following classification of the massless fields:

- Closed oriented strings $\rightarrow$ massless fields of the spectrum $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ
- Closed unoriented string $\rightarrow$ massless fields of the spectrum $g_{\mu\nu}$, ϕ
- Open and closed oriented strings $\rightarrow$ massless fields of the spectrum $g_{\mu\nu}$, $B_{\mu\nu}$ V_μ^{IJ}, ϕ
- Open and closed unoriented strings $\rightarrow$ massless fields of the spectrum $g_{\mu\nu}$, $V_\mu^{[IJ]}$, ϕ

where the field V_μ^{IJ} are the $U(n)$ vectors and the indices $I, J = 1, 2$ indicate to which D-brane each string endpoint is attached, $B_{\mu\nu}$ is a 2-form field and ϕ is a scalar field.

The low energy limit of the string theory is the limit in which the particle theory can be recovered and the string length is set to zero, the string constant $\alpha' \rightarrow 0$, and only the massless modes become relevant. In this limit we also obtain the corresponding effective field theory.

The effective low energy action of the string theory in the string frame in d dimensions ($d = 26$ in the bosonic string theory to preserve Poincare invariance, or in the supersymmetric theory $d = 10$) can be described by the action (where the strung coupling constant $g_s \equiv e^{\phi_0}$ with ϕ_0 the vacuum expectation value of the dilaton ϕ has to be small for the low energy limit to take place) [14]

$$S = \int d^d x \sqrt{|g|} e^{-2\phi} \left[R - 4(\partial\phi)^2 + \frac{1}{12} H^2\right] \tag{1.3.5}$$

with the space-time metric $g_{\mu\nu}$, ϕ is the dilaton, and the Kalb-Ramond 2-form $B_{\mu\nu}$, with the totally anti-symmetric field strength composed from it

$$H = 3\partial B \tag{1.3.6}$$

with antisymmetrized indices (hidden). This action is derived by requiring conformal invariance in quantum theory after renormalisation procedure (see section 3.2 of [15]).

Now, briefly touching the fermionic - supersymmetric part of the string theory (worldsheet supersymmetry), we add fermionic fields describing the internal spin degrees of freedom. For this consider anticommuting variables $\psi^\mu(\tau,\sigma)$, as well as a gravitino field χ, which after integrating out the graviton results in the so-called Ramond-Neveu-Schwarz (RNS) model

$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma[\eta^{ab}\partial_a X^\mu \partial_b X_\mu - i\bar{\psi}\partial\!\!\!/\psi_\mu] \tag{1.3.7}$$

For open superstrings, the endpoints $\sigma = 0, 2\pi l$ are free, and both Neuman and Dirichlet boundary conditions can be chosen for X^μ, while for ψ^μ we have left and right moving modes - ψ^μ_- and ψ^μ_+ For superstrings with closed bosonic and open fermionic parts, in combination we have 4 types of boundary conditions for the fermionic part: RR, NSNS, RNS, NSR, built on the following blocks:

- Ramond boundary conditions (R)

$$\psi^\mu_+(\sigma=0) = \psi^\mu_-(\sigma=0), \quad \psi^\mu_+(\sigma=2\pi l) = \psi^\mu_-(\sigma=2\pi l), \tag{1.3.8}$$

- Neveu-Schwarz boundary condition (NS)

$$\psi^\mu_+(\sigma=0) = \psi^\mu_-(\sigma=0), \quad \psi^\mu_+(\sigma=2\pi l) = -\psi^\mu_-(\sigma=2\pi l), \tag{1.3.9}$$

Since for closed superstrings $\sigma \sim \sigma + 2\pi l$ for each component ψ^μ_+ and ψ^μ_- independently we can have

- Ramond boundary conditions (R) - periodic

$$\psi^\mu_\pm(\sigma=0) = \psi^\mu_\pm(\sigma=2\pi l), \tag{1.3.10}$$

- Neveu-Schwarz boundary condition (NS) - antiperiodic

$$\psi^\mu_\pm(\sigma=0) = -\psi^\mu_\pm(\sigma=2\pi l), \tag{1.3.11}$$

In the case of superstrings we have a larger spectrum of massless fields in the 10 dimensional supergravity. The full classification is give in section 20.2 of [14], where the bosonic sector is augmented by the additional $C^{(1)}$, $C^{(3)}$ fields in the type IIA case, additional $C^{(0)}$, $C^{(2)}$, $C^{(4)}$ (where $C^{(4)}$ is self-dual) in the type IIB case, and an additional $C^{(2)}$ field in the type I case.

The massless modes of type II superstrings compose the supergravity multiplets of the maximal 10-dimensional supergravity theory, while type I strings correspond to the N=1 supergravity spectrum in the low energy limit. We will use these basic description indirectly in studying supersymmetric solutions (effective actions) in the case of supergravity models we use to generate U-dual solutions in further chapters.

Apart from string dualities, we have dualities between extended objects of M-theory - Dp-branes (D is for Dirichlet boundary conditions), appearing from compactifications of 11-dimensional theory with 2 fundamental objects: M2 branes and their dual M5 branes. String theories also contain branes apart of strings: type IIA string theory contains even dimensional branes - Dp-branes coupled to odd dimensional $p+1$ forms C_{p+1}, while type IIB string theory consists of odd dimensional branes coupled to even C_p forms. The action of a p-dimensional brane on the space-time coordinates $X^\mu(\xi)$, with $\mu = 0, ..., d-1$ and the worldsheet coordinates ξ^a with $a = 0, ..., p$ with the pullback metric $h_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ is represented by the so-called generalised Nambu-Goto action:

$$S_{NG}^{(p)} = -T_{(p)} \int d^{p+1}\xi \sqrt{|h_{ab}|} \tag{1.3.12}$$

with the brane tension $T_{(p)}$, and is proportional to the volume captured by the p-brane.

Various supergravity theories are related to each other via a compactification on a circle (or a double compactification). It is worth noting that the decompactification limit of 11-dimensional theory coincides with the strong coupling limit of the type IIA theory. We can also spot some connection between the spectra of the 11-dimensional model and the reduced 10 dimensional theory. The Kaluza-Klein scalar contributes to the appearance of the dilaton field in the reduced model, and the Kaluza-Klein vector gives the RR 1-form, while the 3-form gives a 2-form and a 3-form.The 1-form is associated with the D0-brane, the 3-form - with D2-brane: since there are D0- and D2-branes associated with the RR 1-form and 3-form respectively, we find that they originate from the 11-dimensional graviton moving in the compact direction and from a two-dimensional object - the M2-brane, that couples to the 11-dimensional 3-form. This M2-brane gives rise to the type IIA string when it is wrapped around the compact dimension (coupled to $B_{\mu\nu}$), and once it is not wrapped around the compact dimension, then it gives rise to the 2-dimensional brane - the D2-brane (via the form $C^{(3)}_{\mu\nu\rho}$ to which this D2 brane is coupled).

On a broader scale dimensional reduction from 11 to 10 dimensional theory relate various objects to each other: p-branes, M-branes, D-branes, strings, waves, Kaluza-Klein monopoles, etc. Accompanied with T and S dualities these transformations build a web of connections between different objects in M-theory and string theories (see [14] for more details).

In order to study connections between different supergravity models, let's demonstrate an example of reduction from the 11-dimensional bosonic sector of Supergravity theory (corresponding to the low energy limit of M-theory) to the bosonic sector of a 10-dimensional supergravity (corresponding to the low energy limit of (super) string theory, type IIA supergravity). This mechanism will provide us with the basic techniques used for dimensional reductions to be implemented further in the thesis. We start with the 11-dimensional action:

$$S_{11} = \frac{1}{16\pi G_{(11)}} \int d^{11}\hat{x}\sqrt{|\hat{g}|}\Big[\hat{R} - \frac{1}{2\cdot 4!}\hat{G}^2 - \frac{1}{(144)^2}\frac{1}{\sqrt{\hat{g}}}\epsilon^{(11)}\hat{G}\hat{G}\hat{C}\Big] \tag{1.3.13}$$

where $G_{(11)}$ is the coupling constant in the 11-dimensional theory, $\hat{R}$ is the 11-dimensional curvature, and all the hatted fields are in 11-dimensions, $\hat{G} = 4\partial\hat{C}$.

Assuming that all the fields are independent of the coordinate $z \equiv \hat{x}^{10}$, we perform a reduction from 11 to 10 dimensional theory, with a spectrum of 10 dimensional fields

$$\{\phi, g_{\mu\nu}, B_{\mu\nu}, C^{(3)}_{\mu\nu\rho}, C^{(1)}_{\mu}\} \tag{1.3.14}$$

where the metric, the Kalb-Ramond 2 -form and the dilaton - are the NSNS sector fields, while the vector and the 3-form are RR sector fields.

Then we use the Kaluza-Klein (or a special case of Scherk–Schwarz [16]) procedure to re-write the 11-dimensional fields in terms of 10-dimensional fields (in String frame and rescaled by the dilaton):

$$\begin{aligned} \hat{g}_{\mu\nu} &= e^{-\frac{2}{3}\phi}g_{\mu\nu} - e^{\frac{4}{3}\phi}C^{(1)}_{\mu}C^{(1)}_{\nu}, \quad \hat{C}_{\mu\nu\rho} = C^{(3)}_{\mu\nu\rho} \\ \hat{g}_{\mu z} &= -e^{\frac{4}{3}\phi}C^{(1)}_{\mu} \quad \hat{C}_{\mu\nu z} = B_{\mu\nu} \quad \hat{g}_{zz} = -e^{\frac{4}{3}\phi} \end{aligned} \tag{1.3.15}$$

then proceeding with each term of the 11-dimensional action separately (for detailed procedure see section 22 of [14]), we finally arrive to the following 10-dimensional action

$$\begin{aligned} S_{10} = \frac{g_A^2}{16\pi G_{(10)}} \int d^{10}x\sqrt{|g|}\Big[& e^{-2\phi}\big(R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2\big) \\ -\frac{1}{4}\big((G^{(2)})^2 + \frac{1}{2\cdot 3!}(G^{(4)})^2\big) &- \frac{1}{144\sqrt{|g|}}\epsilon^{(10)}\partial C^{(3)}\partial C^{(3)}B\Big] \end{aligned} \tag{1.3.16}$$

where the constant $g_A = e^{\phi_0}$ absorbs the asymptotic value of the dilaton $\phi \to \phi_0$ in the action, and

$$\frac{1}{G_{(10)}} = \frac{1}{G_{(11)}}\int dz = \frac{2\pi R_{11}}{G_{(11)}} \tag{1.3.17}$$

and the field strengths

$$H = 3\partial B, \quad G^{(2)} = 2\partial C^{(1)}, \quad G^{(4)} = 4\big(\partial C^{(3)} - HC^{(1)}\big) \tag{1.3.18}$$

and we have taken the 11th coordinate z to be on a circle of Planck length radius rescaled by g_A: $R_{11} = g_A^{2/3}l_{Planck}$ - the only scale available in the 11-dimensional supergravity. The dilaton representing the string coupling constant, shows us that the strong-coupling limit of the type IIA string theory corresponds to Large radius decompactification limit of the 11 dimensional theory.

1.4 Dualities

1.4.1 Why dualities are interesting?

A duality is an equivalence between two seemingly different physical theories or descriptions.

As was highlighted previously, dualities do not only appear in the string and M-theory, they have taken place in the framework of classical physics, playing a significant role in building connections and interpretations of different solutions and physics models. Dualities are important in M-theory and supergravity as a solution generating technique and a connecting mechanism for seemingly diferent solutions, revealing the link between these solutions.

In order to understand better the origins of the dualities in superstring and M-theory, we have to go back to the original examples obtained via simple compactification on a circle, and the dualities generated by inverting parameters such as the circle radius and the coupling constant, that keeps the mass spectrum unchanged.

The action of dualities adds to the obvious symmetries of the theory under consideration, such as non-linear sigma model, where the manifest symmetries are diffeomorphisms, Weyl transformations of the metrics and gauge symmetries of the gauge fields, a series of less obvious symmetries, such as the duality of the target space obtained via compactifications on circles or tori, replacing the radius by the inverted radius. The precise technique of the elementary T-duality will be explained in the following sections.

In the previous section we had a quick overview of the main types of strings and the field spectra in different conditions, now we will focus on the symmetries and connections generated by dualities in the NS-NS and RR sectors of the string theories.

T-duality is in particular interesting in connecting type IIA and type IIB string theories compactified on circles of opposite radii (R and α'/R) resulting in the same NS-NS bosonic sector, forming a $\mathbf{Z}_2$ part of the full $O(d, d, \mathbf{Z})$ T-duality group.

T-duality forms an exact symmetry of string theory at every level of perturbation of string coupling constant g, while S and similarly U dualities do not possess this feature. However, analysing various types of dualities and their generalisations, allows us to build a web of connections between various supergravity solution, thus, giving a broader understanding of duality connections in M-theory.

1.4.2 Abelian T-duality in examples

Let us demonstrate how Abelian T-duality appears in string theory. Consider a closed Bosonic string moving in a target 26 dimensional space $M = \mathbb{R}^{1,24} \times S^1$ with a periodicity constraints on one of the coordinate components $X(\tau, \sigma) = (X^0, X^1, ..., X^{24}, X^{25})$ on a two-dimensional world sheet Σ with the string coordinates (τ, σ)

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi m R \tag{1.4.1}$$

with a winding number $m \in \mathbb{Z}$ and string radius R.

The corresponding Polyakov action for this theory is

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d\sigma d\tau \big(h^{ab}\sqrt{|h|} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu\big) \tag{1.4.2}$$

with the metric of the target manifold $g_{\mu\nu}$ and the worldsheet metric h_{ab}, and the constant α', where the combination $T \equiv \frac{1}{4\pi\alpha'}$ has the meaning of the intrinsic string tension. The equation of motion for this action reveals the following solutions, including the right and left moving components

$$X_L^\mu = \frac{1}{2}X_0^\mu + \frac{1}{2}\alpha' P_L^\mu(\tau + \sigma) + \text{oscillator terms} \tag{1.4.3}$$

$$X_R^\mu = \frac{1}{2}X_0^\mu + \frac{1}{2}\alpha' P_R^\mu(\tau - \sigma) + \text{oscillator terms} \tag{1.4.4}$$

with the expression for the oscillator terms as $\sim \frac{1}{m} \exp im(\tau + \sigma)$ and $\sim \frac{1}{m} \exp im(\tau - \sigma)$ for the left and right modes correspondingly, and the momenta are quantised in the compact 25th direction $(n \in \mathbb{Z})$

$$P_L^{25} = \frac{n}{R} + \frac{mR}{\alpha'} \tag{1.4.5}$$

$$P_R^{25} = \frac{n}{R} - \frac{mR}{\alpha'} \tag{1.4.6}$$

Calculation of the theory spectrum reveals

$$M^2 \equiv -\partial_\tau X^\mu \partial_\tau X_\nu = \frac{n^2}{R^2} + \frac{m^2 R^2}{(\alpha')^2} + \frac{2}{\alpha'}(N_L + N_R - 2) \tag{1.4.7}$$

where N_L, N_R are the numbers of left and right moving modes correspondingly. This spectrum is invariant under the transformation $R \to \frac{\alpha'}{R}$, $n \to m$, which also corresponds to $X_L \to X_L$ and $X_R \to -X_R$. This means that the mass spectrum for a string moving in one background $M = \mathbb{R}^{1,24} \times S^1$ is exactly the same as the mass spectrum of a string moving in a different background $M' = \mathbb{R}^{1,24} \times S'^1$ with S'^1 having an inverse radius to S^1 (up to the constant α'). And this characterises the simplest example of T-duality.

In the framework of non-linear sigma model Abelian T-duality reveals an interesting feature connecting seemingly different actions to each other. As an example, a non-linear sigma model defined on a d-dimensional manifold M

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi [\sqrt{|h|} h^{ab} g_{\mu\nu} \partial_a x^\mu \partial_b x^\nu + \epsilon^{ab} b_{\mu\nu} \partial_a x^\mu \partial_b x^\nu] \tag{1.4.8}$$

with a target space metric $g_{\mu\nu}$, the torsion $b_{\mu\nu}$ the dilaton ϕ and the worldsheet metric h_{ab}. With an abelian isometry in one of the directions θ of the coordinate set $\{\theta, x^\alpha\}$ where $\alpha = 1, ..., d-1$, meaning that there's no dependence on θ of the theory fields. And this theory can be then obtained from a $d+1$ dimensional sigma model with an additional variable $\tilde{\theta}$ acting as a Lagrange multiplier and an extra 1-form V defined on the manifold M

$$S_{d+1} = \frac{1}{4\pi\alpha'} \int d^2\xi [\sqrt{|h|} h^{ab} (g_{00} V_a V_b + 2 g_{0\alpha} V_a \partial_b x^\alpha + g_{\alpha\beta} \partial_a x^\alpha \partial_b x^\beta) \tag{1.4.9}$$

$$+ i\epsilon^{ab} (2 b_{0\alpha} V_a \partial_b x^\alpha + b_{\alpha\beta} \partial_a x^\alpha \partial_b x^\beta) + 2i\epsilon^{ab} \tilde{\theta} \partial_a V_b] \tag{1.4.10}$$

where the equation of motion for $\tilde{\theta}$ gives us $\epsilon^{ab} \partial_a V_b = 0$, which in the case of a trivial worldsheet vector $V_a = \partial_a \theta$ would give us the original theory. Solving for V_a in this theory gives

$$V_a = -\frac{1}{g_{00}} (g_{0\alpha} \partial_a x^\alpha + i \frac{\epsilon_a{}^b}{\sqrt{h}} (b_{0\alpha} \partial_b x^\alpha + \partial_b \tilde{\theta})) \tag{1.4.11}$$

integrating over this V_a field gives us a dual action with a different geometry

$$\tilde{S} = \frac{1}{4\pi\alpha'} \int d^2\xi [\sqrt{|h|} h^{\mu\nu} (\tilde{g}_{00} \partial_a \tilde{\theta} \partial_b \tilde{\theta} + 2 \tilde{g}_{0\alpha} \partial_a \tilde{\theta} \partial_b x^\alpha + \tilde{g}_{\alpha\beta} \partial_a x^\alpha \partial_b x^\beta)$$
$$+ i\epsilon^{ab} (2 \tilde{b}_{0\alpha} \partial_a \tilde{\theta} \partial_b x^\alpha + \tilde{b}_{\alpha\beta} \partial_a x^\alpha \partial_b x^\beta)] \tag{1.4.12}$$

where the connection between the dual fields and the original fields is given by a set of so-called Buscher's rules [18]

$$\tilde{g}_{00} = \frac{1}{g_{00}} \tag{1.4.13}$$

$$\tilde{g}_{0\alpha} = \frac{b_{0\alpha}}{g_{00}}, \quad \tilde{b}_{0\alpha} = \frac{g_{0\alpha}}{g_{00}} \tag{1.4.14}$$

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha} g_{0\beta} - b_{0\alpha} b_{0\beta}}{g_{00}} \tag{1.4.15}$$

$$\tilde{b}_{\alpha\beta} = b_{\alpha\beta} - \frac{g_{0\alpha} b_{0\beta} - g_{0\beta} b_{0\alpha}}{g_{00}} \tag{1.4.16}$$

As was shown by Buscher in [18] the T-dual action will remain conformally invariant if the dilaton field (not included previously, and appears as higher α' correction in the theory) transforms as

$$\tilde{\phi} = \phi - \frac{1}{2} \log g_{00} \tag{1.4.17}$$

This dual sigma model defined by $(\tilde{g}, \tilde{b}, \tilde{\phi})$ and independent of $\tilde{\theta}$ variable, is obtained from the original model by performing the duality transformation with respect to shifts on $\tilde{\theta}$. This is a prominent example of Abelian T-duality applied to a sigma model.

In the example above, the Buscher procedure was applied to the target space-time with an abelian Isometry group $U(1)$. However, the Buscher procedure can be generalised to the case of a target space with a non-Abelian isometry group, where the gauge fields are valued in the algebra of this isometry group, mapping one solution of supergravity to another. A simple example of non-Abelian T-duality is considered in the next paragraph. In the case of extended objects - branes duality a double reduction has to be performed for the $D(p+1)$-brane and a single reduction for the Dp-brane, then duality connects the two reduced actions to one another.

1.4.3 Non-Abelian T-duality in examples

Original studies on the non-Abelian T-duality commenced in [3]. These dualities originate from non-Abelian isometries of sigma models in string theory, and the dual solution is generated by integrating out the gauge fields.

Buscher's rules and procedure can be generalised to the case of sigma model with non-Abelian isometries into the so-called generalised Buscher's rules, as we will show in what follows.

We will start with the non-gauged non-linear sigma model

$$S = \int d^2z(g_{\mu\nu} + B_{\mu\nu})\partial X^\mu \bar{\partial} X^\nu \tag{1.4.18}$$

where the partial derivatives act as $\partial X^\mu \equiv \partial_z X^\mu$ and $\bar{\partial} X^\mu \equiv \partial_{\bar{z}} X^\mu$ and $z \equiv \tau + i\sigma$, $\bar{z} \equiv \tau - i\sigma$. The target space is a compact non-Abelian group G with the left-invariant vector fields v_a, to which we will introduce dual 1-forms l, called the Maurer-Cartan forms with the components l^a

$$\langle v_a, l^b \rangle = \delta_a^b \tag{1.4.19}$$

while the vector fields v_a obey the bracket equation defining the structure constants of a Lie algebra $f_{bc}{}^{a}$

$$[v_a, v_b] = f_{ab}{}^{c} v_c \tag{1.4.20}$$

the Maurer-Cartan forms satisfy the Maurer-Cartan equation

$$dl^a = -\frac{1}{2} f_{bc}{}^{a} l^b \wedge l^c \tag{1.4.21}$$

A metric on G can be defined in terms of the Maurer-Cartan forms as

$$ds^2 = g_{ab} l^a l^b \tag{1.4.22}$$

Alternatively, we can introduce a G-invariant matrix E of the dimension $dim(G) \times dim(G)$, where $\times$ is a direct product, and decompose this matrix into its symmetric and anti-symmetric forms defining the metric g and the B-field of the theory model

$$E_{ab} = g_{ab} + B_{ab} \tag{1.4.23}$$

where

$$g_{ab} = \frac{1}{2}(E_{ab} + E_{ba}), \quad B_{ab} = \frac{1}{2}(E_{ab} - E_{ba}) \tag{1.4.24}$$

The Maurer-Cartan forms corresponding to the left invariant vector fields are invariant under the action of the right invariant vector fields, that in mathematical terms can be demonstrated in the equation

$$\mathcal{L}_{r_a} l^b = 0 \tag{1.4.25}$$

from where it follows that $\mathcal{L}_{r_a} E = 0$, meaning that the action is invariant under the infinitesimal action of right-invariant vector fields. The theory possesses a global symmetry $g \to gh$ (for the Lie group element g) generated by the invariance under the infinitesimal action of the right-invariant fields

$$\delta_\epsilon X^\mu = r_a^\mu \epsilon^a, \tag{1.4.26}$$

where the right vector fields r_a here form a non-Abelian Lie algebra with structure constants ${f_{bc}}^a$ in the non-Abelian T-duality case. Gauging this symmetry by introducing non-abelian gauge fields A^a, minimally coupled to the action. This involves the upgrade of the derivative to a covariant derivative

$$dX^\mu \to DX^\mu \equiv dX^\mu - r_a^\mu A^a \tag{1.4.27}$$

which makes the theory invariant under the infinitesimal local gauge transformation.

Splitting the coordinates X into gauged and ungauged, or in other words, into coordinates parametrising the base X^μ - ungauged, and the coordinates X^a parametrising the non-abelian Lie group G fiber - gauged. Thus we get for the metric in terms of the Maurer-Cartan forms of the group

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu + 2g_{\mu a} dX^\mu l^a + g_{ab} l^a l^b \tag{1.4.28}$$

and similarly for the B-field

$$B = B_{\mu\nu} dX^\mu \wedge dX^\nu + 2B_{\mu a} dX^\mu \wedge l^a + B_{ab} l^a \wedge l^b \tag{1.4.29}$$

After applying the gauging, and adding the Lagrange multiplier term in a form of field strength F, similarly to the Abelian T-duality case, the action of the non-linear gauged sigma model will be of the form

$$S_g = \int d^2z[E_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + E_{a\nu}\partial X^a \bar{\partial} X^\nu + E_{\mu b}\partial X^\mu \bar{D} X^b + E_{ab} D X^a \bar{D} X^b + \chi_a(\partial \bar{A}^a - \bar{\partial} A^a + f^a{}_{bc} A^b \bar{A}^c)] \tag{1.4.30}$$

where the fields of the sigma model: metric g and the B-field are unified under the matrix E:

$$E = g + B \tag{1.4.31}$$

and

$$DX^a = \partial X^a - A^a \tag{1.4.32}$$

$$\bar{D} X^a = \bar{\partial} X^a - \bar{A}^a \tag{1.4.33}$$

with the non-Abelian gauge fields $A, \bar{A}$.

We will not dive deep into the description of the procedure of obtaining the non-Abelian T-duality rules in what is given below, but will rather describe the short procedure. The full analysis is given in [3].

The solution of equation of motion for the gauge fields A^a and $\bar{A}^a$ gives

$$A^a = E_{bc}(M^{-1})^{ca}\partial X^b + E_{\mu b}(M^{-1})^{ba}\partial X^\mu + (M^{-1})^{ba}\partial \chi_a \tag{1.4.34}$$

$$\bar{A}^a = E_{bc}(M^{-1})^{ab}\bar{\partial} X^c + E_{b\mu}(M^{-1})^{ab}\bar{\partial} X^\mu - (M^{-1})^{ab}\bar{\partial}\chi_b \tag{1.4.35}$$

where we defined M as

$$M_{ab} = E_{mn} + \chi_c f^c{}_{ab} \tag{1.4.36}$$

Integrating out the gauge fields by parts in the action (1.4.30) after submitting the solution found above, then fixing the gauge due to the gauge invariance of the action under the right-invariant vector field, choosing $\partial X^a = 0$ and $\bar{\partial} X^a = 0$ (matching the degrees of freedom we're allowed to fix due to the action symmetry), and introducing dual fields $\tilde{E}_{\mu\nu}$, $\tilde{E}_{a\nu}$, $\tilde{E}_{\mu b}$, $\tilde{E}_{ab}$ as

$$\tilde{E}_{\mu\nu} = E_{\mu\nu} - E_{\mu a}(M^{-1})^{ab}E_{b\nu}, \quad \tilde{E}_{a\nu} = -(M^{-1})_{ab}E^b{}_\nu, \quad \tilde{E}_{\mu a} = E_{\mu b}(M^{-1})^b{}_a, \quad \tilde{E}_{ab} = (M^{-1})_{ab} \tag{1.4.37}$$

corresponding to the non-Abelian Buscher's rules [3], giving the T-dual action

$$S = \int d^2z\big[\tilde{E}_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + \tilde{E}_{a\nu}\partial \chi^a \bar{\partial} X^\nu + \tilde{E}_{\mu b}\partial X^\mu \bar{\partial}\chi^b + \tilde{E}_{ab}\partial\chi^a\bar{\partial}\chi^b\big] \tag{1.4.38}$$

and this dual action can be interpreted as a non-linear sigma model with coordinates $\tilde{X} = \{X^\mu, \chi^a\}$. This was the mechanism of non-Abelian T-duality action on a sigma model. This Buscher procedure provides an efficient algorithm of obtaining T-dual sigma models via integrating out the gauge fields (gauging the isometry).

1.4.4 Poisson-Lie T-duality

Non-Abelian T-duality admits the action of the non-Abelian isometry group. For constructing the Poisson-Lie T-duality we require the notion of the Drinfeld double algebra, which is simply even-dimensional real algebra $\mathfrak{d}$ with generators T_A obeying

$$[T_A, T_B] = F_{AB}{}^C T_C \tag{1.4.39}$$

equipped with a symmetric split-signature-invariant pairing $\eta(\cdot,\cdot)$ such that $\mathfrak{d}$ admits at least one decomposition $\mathfrak{d} = \mathfrak{g} + \tilde{\mathfrak{g}}$ with $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ sub-algebras that are maximally isotropic with respect to $\mathfrak{d}$: i.e. the inner product is trivial for any two elements that belongs to the subalgebra, and this feature is maximal (there's no bigger subalgebra that contains the former subalgebra satisfying the same property). In the case of Abelian T-duality, the generators T_A commute, and the structure constants $F_{AB}{}^C$ are trivial. This corresponds to the isometries and the Abelian nature of the underlying algebra. In the case of the non-Abelian T-duality (NATD) the structure constants $F_{AB}{}^C$ do not vanish. For non-Abelian T-duality to take place the dualised background does not necessarily have to possess isometries.

For the Poisson-Lie T-duality to take place, the structure of $F_{AB}{}^C$ has to satisfy the Drinfeld double structure

$$F_{ab}{}^c = f_{ab}{}^c, \quad F_{abc} = 0, \quad F_a{}^{bc} = \tilde{f}^{bc}{}_a, \quad F_a^b{}_c = -f_{ac}{}^b, \quad F^{ab}{}_c = \tilde{f}^{ab}{}_c, \quad F^{abc} = 0 \tag{1.4.40}$$

The decomposition in terms of generators of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ - t_a and $\tilde{t}^a$ - in a way that $T_A \equiv (t_a, \tilde{t}^a)$ leads to the following double algebra:

$$[t_a, t_b] = f_{ab}{}^c t_c, \quad [t_a, \tilde{t}^b] = \tilde{f}^{bc}{}_a t_c - f_{ac}{}^b \tilde{t}_c, \quad [\tilde{t}^a, \tilde{t}^b] = \tilde{f}^{ab}{}_c \tilde{t}^c \tag{1.4.41}$$

Then, according to [11], we can introduce the Poisson-Lie bivector field constructed from the adjoint action of an element g of the group G constructed by the exponentiation of the subalgebra g in the following way

$$g t_a g^{-1} = (a_g)_a^b t_b, \quad g\tilde{t}^a g^{-1} = (b_g)^{ab} t_b + (a_{g^{-1}})_b^a \tilde{t}^b \tag{1.4.42}$$

so, that

$$\Pi_g^{ab} \equiv (b_g)^{ac} (a_{g^{-1}})_c^b \tag{1.4.43}$$

and according to this definition, as was shown in [27], this Poisson-Lie bivector field satisfies a set of features, as a consequence of the adjoint action, such as

$$\Pi_g^{ab} = -\Pi_g^{ba}, \quad \Pi_e = 0, \quad \Pi_{hg} = \Pi_g + (a_{g^{-1}} \times a_{g^{-1}})\Pi_h \tag{1.4.44}$$

as well as a differential equation, after introducing the left-invariant 1-form $l = l^a t_a = g^{-1}dg$, so that

$$d\Pi_g^{ab} = -l^c \tilde{f}^{ab}{}_c - 2l^c f_{cd}{}^{[a}\Pi^{b]d} \tag{1.4.45}$$

and can be included in the Lie derivative action of the algebra $\mathfrak{d}$ on its dual vector field v_a (which is dual to the left invariant 1-form $l \equiv l^a t_a$), using $\pi^a \equiv \Pi_g^{ab} v_b$ - a set of vector fields built from the PL bi-vector Π_g^{ab}, as follows

$$L_{v_a} v_b = -f_{ab}{}^c v_c, \quad L_{v_a}\pi^b = f_{ac}{}^b \pi^c - \tilde{f}^{bc}{}_a v_c, \quad L_{\pi^a}\pi^b = -\tilde{f}^{ab}{}_c \pi^c \tag{1.4.46}$$

This Lie derivative can be further generalised to the so-called Dorfman (or the generalised Lie) derivative acting on a set of generalised vectors in order to incorporate the gauge symmetries, unified into a generalised frame field $E_A = (E_a, \tilde{E}^a)$ in the following manner

$$E_a = v_a \qquad \tilde{E}^a = \pi^a + l^a \tag{1.4.47}$$

Then, the Lie derivative that was previously acting separately on the components l^a, v_a and π^a, can be now generalised into a generalised Lie or Dorfman derivative, with its action on the generalised vectors $U = u^i\partial_i + \mu_i dx^i$ and $V = v^i\partial_i + \nu_i dx^i$ as

$$\mathcal{L}_U V = L_u v + L_u \nu - \iota_v d\mu \tag{1.4.48}$$

which in terms of the generalised frame field can be repackaged into [24]

$$\mathcal{L}_{E_A} E_B = -F_{AB}^{\ \ C} E_C \tag{1.4.49}$$

furnishing the algebra of the Drinfeld double $\mathfrak{d}$ [12].

In the type II supergravity, the metric and the B-field can be packaged into a structure that is called the generalised metric H_{MN}, respecting the $O(d,d)$ structure which is associated with the string toroidal compactifications on T^n [28]

$$H_{MN} = \begin{pmatrix} g_{mn} - B_{mp}g^{pq}B_{qn} & -B_{mp}g^{pn} \\ g^{mp}B_{pn} & g^{mn} \end{pmatrix} \tag{1.4.50}$$

Where, in the case of an Abelian T-duality this generalised metric is required to be constant, and the twisting it with constant $O(D,D)$ matrices $C_M^{\ P}$ transforms one constant solution to another constant solution [24]

$$H'_{MN} = C_M{}^P C_N{}^Q H_{PQ} \tag{1.4.51}$$

For the case of non-Abelian T-duality the generalised metric can be twisted by non-constant frame fields [29]

$$H_{MN}(x) = E_M{}^A(x) E_N{}^B(x) H_{AB} \tag{1.4.52}$$

Poisson-Lie T-duality, maps one set of structure constants to another, and gives rise to new frame fields, that can be used to twist the generalised metric, giving a dual generalised metric, which is constructed by the action of a constant $O(D,D)$ transformation, with matrices $C_A{}^B$

$$H'_{AB} = C_A{}^C C_B{}^D H_{CD}, \quad T'_A = C_A{}^B T_B, \quad F'_{ABC} = C_A{}^E C_B{}^F C_C{}^G F_{EFG} \tag{1.4.53}$$

where the matrices $C_A{}^B$ have to be chosen in a way that the structure constants $F'_{AB}{}^C$ have the form of Drinfeld double 1.4.40. This new dual solution remains a Supergravity solution since it is mapped by the $O(D,D)$ symmetry of the formulation of the gauged supergravity, obtainable by reduction (taking into account the transformation of the gaugings F).

This gives rise to a novel T-dual algebra

$$[T'_A, T'_B] = F'_{AB}{}^C T_C, \quad \mathcal{L}_{E'_A} E'_B{}^M = -F'_{AB}{}^C E'_C{}^M \tag{1.4.54}$$

To deal with the generalised metric in the form of an $O(D,D)$ object 1.4.50, it is useful to define the frame field in the matrix representation, based on its definition 1.4.47. For this we will need to use the left and right invariant 1-forms - l and r of the group element g as follows

$$l \equiv l^a_m T_a dx^m \equiv g^{-1}dg, \quad r \equiv r^a_m T_a dx^m \equiv dgg^{-1} \tag{1.4.55}$$

where the previous elements used in 1.4.47 are related to the 1-forms as follows $l^a_m v^m_b = \delta^a_b$. Then the frame field can be represented in the $O(D,D)$ matrix form as

$$E_M{}^A = \begin{pmatrix} l^a_m & 0 \\ v^m_b \Pi^{ab} & v^m_a \end{pmatrix} \tag{1.4.56}$$

So, we can review the Poisson-Lie (PL) T-duality as a constant $O(D,D)$ rotation, with the generalised frame field satisfying 1.4.49 with the structure constants $F_{AB}{}^C$ of the Drinfeld double. A further extension of the non-Abelian T-duality was found in [10], [20], and it can be applied to a more general class of target spaces.

For PL duality transformation, we can combine the metric and B field into a matrix E

$$E_{mn} \equiv g_{mn} + B_{mn} \tag{1.4.57}$$

Poisson-Lie symmetry will require the following condition on E_{mn} [10]

$$L_{v_a} E_{mn} = \tilde{f}^{bc}{}_a v_b^p v_c^q E_{pm} E_{qn} \tag{1.4.58}$$

where $\tilde{f}^{bc}{}_a$ are structure constants of the dual algebra, and v_a^i are the left-invariant vector fields corresponding to the right action of the group G. A similar requirement is applicable to the dual matrix $\tilde{E}_{mn}$

$$L_{\tilde{v}_a} \tilde{E}_{mn} = f^{bc}{}_a \tilde{v}_b^p \tilde{v}_c^q \tilde{E}_{pm} \tilde{E}_{qn} \tag{1.4.59}$$

Solution of these equations can be found via integration to take the following form [10] expressed in terms of the original and dual bivectors Π and $\tilde{\Pi}$ defined above via the adjoint action of dual groups elements in terms of the inverse of E_{mn} and $\tilde{E}_{mn}$:

$$E^{mn} = \big((E^0)^{ab} + \Pi^{ab}\big) v_a^m v_b^n \tag{1.4.60}$$

$$\tilde{E}^{mn} = \big(((E^0)^{-1})^{ab} + \tilde{\Pi}^{ab}\big) \tilde{v}_a^m \tilde{v}_b^n \tag{1.4.61}$$

where $(E^0)^{ab}$ is a constant matrix, inverted in the dual algebra case according to the initial conditions and connection between the dual algebra at the unit element of the double [23]. Using the $O(d,d)$ embedding of the metric g and the 2-form B into the generalised metric (1.4.50) and splitting E^0_{ab} into a symmetric and an anti-symmetric part $E^0_{ab} = g^0_{ab} + B^0_{ab}$ building H_{AB} with these elements g^0_{ab} and B^0_{ab} as in (1.4.50) with flat indices, we can recover the same results (1.4.60) and (1.4.61) for the elements g_{mn} and B_{mn} dual sigma models (with E^0_{ab} replaced by $(E^0)^{-1}_{ab}$ in the dual case) and using (1.4.52). We will demonstrate this explicitly when E^0_{ab} is fully symmetric and diagonal on a simple example below.

An example of PL duality will be briefly described in section (2.4.2) of chapter 2 in this thesis, where the solution 2.4.27 is PL dual to the initial solution 2.4.24. Another simple example of a PL duality can be taken from [30].

The example considers Poisson-Lie T-duality for Drinfeld double algebras represented by $su(2)$ and e_3 (this is not the algebra of the Euclidean group in 3 dimensions, and is just a name we use here to denote the algebra with brackets as below) with the generators T_a and $\tilde{T}^a$ correspondingly, with the following brackets (we denote $a,b = 1,2,3$ and $i,j = 1,2$):

$$[T_a, T_b] = \epsilon_{abc} T_c, \quad [\tilde{T}^3, \tilde{T}^i] = \tilde{T}^i, \quad [\tilde{T}^i, \tilde{T}^j] = 0 \tag{1.4.62}$$

and the mixed brackets:

$$[T_i, \tilde{T}^j] = \epsilon_{ij} \tilde{T}^3 - \delta_{ij} T_3, \quad [T_3, \tilde{T}^i] = \epsilon_{ij} \tilde{T}^j, \quad [\tilde{T}^3, T_i] = \epsilon_{ij} \tilde{T}^j - T_i \tag{1.4.63}$$

These algebras are PL T-dual to each other as it can be seen from the relationship between their generators.

Starting with the left-invariant Maurer-Cartan forms corresponding to $su(2)$

$$\begin{aligned} l_1 &= \cos\psi \sin\theta d\phi - \sin\psi d\theta \\ l_2 &= \sin\psi \sin\theta d\phi + \cos\psi d\theta \\ l_3 &= d\psi + \cos\theta d\phi \end{aligned} \tag{1.4.64}$$

we can find the expression for the PL bivector satisfying (1.4.45) to be of the form

$$\Pi^{ab} = -\epsilon^{abc} A_c, \quad \text{with } \vec{A} = (\cos\psi\sin\theta, \sin\psi\sin\theta, \cos\theta - 1) \tag{1.4.65}$$

Now choosing the constant matrix E^0_{ab} to be of the form $(E^0_{ab})^{-1} = diag(\lambda_1, \lambda_2, \lambda_3)$, using (1.4.60) we find for the generalised metric components g_{ab} and B_{ab} of $su(2)$ gauged sigma-model

$$ds^2 = \frac{1}{V}\big(A_a A_b + \frac{\lambda_1\lambda_2\lambda_3}{\lambda_a}\delta_{ab}\big) l^a l^b \tag{1.4.66}$$

$$B = \frac{1}{V}\epsilon_{abc}\lambda_c A_c l^a \wedge l^b \tag{1.4.67}$$

with

$$V \equiv \lambda_1\lambda_2\lambda_3 + \lambda_a A_a^2 \tag{1.4.68}$$

Then studying the dual e_3 algebra, we use the expression for left-invariant Maurer-Cartan forms

$$\begin{aligned} \tilde{l}_1 &= e^{-\chi} dy_1 \\ \tilde{l}_2 &= e^{-\chi} dy_2 \\ \tilde{l}_3 &= d\chi \end{aligned} \tag{1.4.69}$$

While the PL bivector that satisfies (1.4.45) for these left invariant forms and dual structure constants is

$$\tilde{\Pi}^{ab} = -\epsilon^{abc}\tilde{A}_c, \quad \text{with } \vec{\tilde{A}} = (y_1 e^{-\chi}, y_2 e^{-\chi}, \sinh\chi e^{-\chi} - \frac{1}{2}(y_1^2 + y_2^2)e^{-2\chi}) \tag{1.4.70}$$

from where using the form of E^0_{ab} and the relation for the dual generalised metric (1.4.61) we get the following results for the components $\tilde{g}_{ab}$ and $\tilde{B}_{ab}$ of the PL T-dual sigma-model

$$\begin{aligned} d\tilde{s}^2 &= \frac{1}{\tilde{V}}\big(\tilde{A}_a\tilde{A}_b + \frac{\lambda_a}{\lambda_1\lambda_2\lambda_3}\delta_{ab}\big)\tilde{l}^a\tilde{l}^b \\ \tilde{B} &= \frac{1}{\tilde{V}}\epsilon_{abc}\frac{1}{\lambda_c}\tilde{A}_c\tilde{l}^a \wedge \tilde{l}^b \end{aligned} \tag{1.4.71}$$

with

$$\tilde{V} \equiv \frac{1}{\lambda_1\lambda_2\lambda_3} + \frac{\tilde{A}_a^2}{\lambda_a} \tag{1.4.72}$$

this reflects one of the simplest examples of PL T-duality.

Another way to derive these results is using the $O(d,d)$ embedding in a form of (1.4.50) where in H_{AB} according to choice of E^0 we have $g^0_{ab} = \frac{\delta_{ab}}{\lambda_a}$ and $B^0_{ab} = 0$ resulting in

$$H_{AB} = \begin{pmatrix} \frac{\delta_{ab}}{\lambda_a} & 0 \\ 0 & \delta_{ab}\lambda_a \end{pmatrix} \tag{1.4.73}$$

and for the dual model

$$\tilde{H}_{AB} = \begin{pmatrix} \delta_{ab}\lambda_a & 0 \\ 0 & \frac{\delta_{ab}}{\lambda_a} \end{pmatrix} \tag{1.4.74}$$

then using (1.4.52) with the frame fields corresponding to $su(2)$ in the form of (1.4.56) with Π^{ab} as defined above, and l^a_m and their inverse v^m_a

$$E_M{}^A = \begin{pmatrix} l^a_m & 0 \\ -v^m_b\epsilon^{abc}A_c & v^m_a \end{pmatrix} \tag{1.4.75}$$

it is easy to find the above result for the metric g_{ab} and the field B_{ab} as in (1.4.66), and similarly for the dual components as in (1.4.71), in which we see a similar structure with the factor $\lambda_1\lambda_2\lambda_3$ inverted in the dual case, due to the inverse choice of factors in H_{AB} and $\tilde{H}_{AB}$.

1.4.5 Defining U-duality: generalisation approach

In order to move toward U-duality, which is a combination of T and S dualities, we first will have to give a general understanding of the S dualities principles.

In type IIB supergravity a global $Sl(2,\mathbf{Z})$ exists corresponding to the S-duality symmetry that acts on the scalar field combination of the dilaton ϕ and the axion χ combined into one complex field ρ:

$$\rho \equiv \chi + ie^{-\phi} \tag{1.4.76}$$

Under the action of the S-duality the metric and the 4-form potential fields (bosonic sector) remain invariant, while the pair of the 2-form potentials transform as doublet, and the action on the complex field ρ can be described via an action of $Sl(2,\mathbf{Z})$ group with integer elements a, b, c, d, such that $ad - bc = 1$, in the way

$$\rho \to \frac{a\rho + b}{c\rho + d} \tag{1.4.77}$$

Compactifying on a circle and identifying with eleven-dimensional supergravity compactified on a torus implies that the modulus of the IIB theory should be equated to the modular parameter of the torus - ρ [31]. Within U-duality this $\mathcal{T}^2$ torus can be further generalised to a d-torus, corresponding to an exceptional symmetry group $E_{d(d)}$

The mathematical foundations of generalized U-duality are rooted in the exceptional algebra, which plays a central role in describing the symmetries and transformations among different M-theory backgrounds.

Generalised U-duality is a symmetry that arises in the context of M-theory, enabling a solution generating technique that gives rise to new supergravity examples. Like the previously discussed non-Abelian or PL T-duality, generalised U-duality extends the dualisation approach to a broader range of transformations, i.e. spaces without abelian isometries.

In the case of generalised U-duality in order to search for non-perturbative analogues of non-abelian T-duality and PL duality, we extend the simple Drinfeld Double Algebra to Exceptional Drinfeld Algebra (EDA) which is associated with the non-Abelian U-duality. This has been proposed in [26, 27], by generalising the Drinfeld double Lie algebra to a new algebraic structure - the EDA. Interestingly, this does *not* form a Lie algebra, but a more general structure known as a Leibniz algebra (technically, the bracket of two algebra elements is not antisymmetric).

In the case of the generalised T-duality (a term used to refer to non-Anelian and PL T-dualities in a unified description), the base was a double Lie algebra. A transition to a more generalised version of the Lie algebra - the so-called Leibniz algebra in which the bilinear product is generalise to a bracket that can be not anti-symmetric, and satisfies the Leibniz identity for the elements of the algebra $g_1,\ g_2,\ g_3 \in G$

$$[[g_1, g_2], g_3] + [[g_2, g_3], g_1] + [[g_3, g_1], g_2] = 0 \tag{1.4.78}$$

Our studies will focus around the E_n exceptional algebra with $n = 4$ corresponding to the generalised vector parametrisation that transforms under the extended symmetry group of the theory, which includes both the spacetime diffeomorphisms and the internal symmetries associated with the exceptional group.:

$$V^I = \begin{pmatrix} v^i \\ v_{i_1 i_2} \end{pmatrix} \tag{1.4.79}$$

Generally speaking, the PL version of U-duality is a sort of extension of PL T-duality with the Lie algebra being generalised, and the Exceptional Drinfeld algebra is systematically extended with the similar structure of equation

$$\mathcal{L}_{E_A} {E_B}^M = -{F_{AB}}^C {E_C}^M \tag{1.4.80}$$

where the structure constants ${F_{AB}}^C$ possess the Drinfeld double features, and the Lie derivative is generalised to the Leibniz derivative.

On the example of SL(5) exceptional algebra, where we will use 5-dimensional indices $\mathcal{M}, \mathcal{N} =$

$1, ...5$, while the generalised vector is 10 dimensional: comprising of 4 dimensions of the simple vector and six components of the 2-form. The SL(5) group has 2 totally antisymmetric invariants: $\epsilon^{\mathcal{MNPQK}}$ and $\epsilon_{\mathcal{MNPQK}}$ to preserve which under generalised diffeomorphisms we will have to define the generalised Lie derivative as follows (see [34], [37] for details):

$$\mathcal{L}_\Lambda V^{\mathcal{M}} = \frac{1}{2}\Lambda^{\mathcal{PQ}}\partial_{\mathcal{PQ}}V^{\mathcal{M}} - V^{\mathcal{P}}\partial_{\mathcal{PQ}}\Lambda^{\mathcal{MQ}} + \frac{1}{5}\partial_{\mathcal{PQ}}\Lambda^{\mathcal{PQ}}V^{\mathcal{M}} \tag{1.4.81}$$

where the antisymmetric derivative $\partial_{\mathcal{MN}}$ is defined via its components ($i = 1, ..., 4$ stands as 4 dimensional index) $\partial_{i5} = \partial_i$ and $\partial_{ij} = \frac{1}{2}\epsilon_{ijkl}\tilde{\partial}_{kl}$, where the partial derivative $\tilde{\partial}_{kl}$ is taken with respect to the dual coordinates $\tilde{x}^{kl}$ that together with the original 4-dimensional coordinates x^i generalise the notion of diffeomorphisms (incorporating simple diffeomorphisms and gauge transformation), and can be further grouped into a set of antisymmetric coordinates $X^{\mathcal{MN}}$ with components $X^{i5} = x^i$, $X^{ij} = \frac{1}{2}\epsilon^{ijkl}\tilde{x}^{kl}$.

Combining this with the generalised Lie derivative action on the generalised vector $V^{\mathcal{MN}}$, the general rule can be rewritten in the form of 10-dimensional index $M = [\mathcal{MN}]$ on the generalised vector $V^M = V^{\mathcal{MN}}$ with respect to $\Lambda^M = \Lambda^{\mathcal{MN}}$

$$\mathcal{L}_\Lambda V^M = \Lambda^N\partial_N V^M - V^N\partial_N\Lambda^M + \epsilon^{MN\mathcal{K}}\epsilon_{PQ\mathcal{K}}\partial_N\Lambda^P V^Q - \frac{1}{5}\partial_N\Lambda^N V^M \tag{1.4.82}$$

The closure of the algebra generated by this is achieved by demanding the so-called section condition to take place for any two elements Ψ, Ψ'

$$\epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{MN}}\partial_{\mathcal{PQ}}\Psi = 0, \quad \epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{MN}}\Psi\partial_{\mathcal{PQ}}\Psi' = 0 \tag{1.4.83}$$

solution of which can be found by requiring $\tilde{\partial}^{ij} = 0$. This reduces the generalised Lie derivative for the generalised vectors $U^M = (u, \lambda_{(2)})$, $V^M = (v, \eta_{(2)})$ to the following expression (L is the simple Lie derivative)

$$\mathcal{L}_U V = (L_u v, L_u\eta_{(2)} - \iota_v d\lambda_{(2)}) \tag{1.4.84}$$

which can be seen as the generalisation of the expression (1.4.48) used in the PL T-duality case.

In order to fulfil relation (1.4.80) in the case of EDA now, we can choose the following parametrisation for the frame fields (M - stands for the curved index, A - for the flat one)

$${E_A}^M = \begin{pmatrix} {E_a}^M \\ E^{a_1a_2M} \end{pmatrix} \tag{1.4.85}$$

and generalising the Π bivector in the PL T-duality case to a 3-vector λ in the U-duality case, we can introduce further a fully anti-symmetric $\lambda^{a_1a_2a_3} \equiv \lambda^{[a_1a_2a_3]}$, so

$$E^M{}_A = \begin{pmatrix} E^M_a \\ E^{a_1a_2M} \end{pmatrix} = \begin{pmatrix} v^m_a & 0 \\ \lambda^{a_1a_2b}v^m_b & l^{[a_1}_{[m_1}l^{a_2]}_{m_2]} \end{pmatrix} \tag{1.4.86}$$

with

$$E_a = (v_a, 0), \quad E^{a_1a_2} = (\lambda^{a_1a_2a_3}v_{a_3}, l^{a_1} \wedge l^{a_2}) \tag{1.4.87}$$

and the elements satisfying

$$L_{v_a}v^m_b = -f_{ab}{}^c v^m_c, \quad dl^a = \frac{1}{2}f_{bc}{}^a l^b \wedge l^c, \quad v_a l^b = \delta^b_a \tag{1.4.88}$$

Where the trivector $\lambda^{a_1a_2a_3}$ defined via the scalar matrix $K_A{}^B(x)$ obtained via the adjoint action of the group elements $g(x) \equiv e^{x^aT_a}$ as follows

$$g^{-1}(x) \circ T_A \equiv K_A{}^B(x)T_B \tag{1.4.89}$$

with the parametrization of $K_A{}^B(x)$ as follows (here we use the adjoint element of the Leibniz algebra $a_b{}^a$ defined as $l^a_m = a_b{}^a r^b_m$)

$$K_A{}^B \equiv \begin{pmatrix} a_a{}^b & 0 \\ -\lambda^{a_1a_2c}a_c{}^b & (a^{-1})_{[b_1}{}^{a_1}(a^{-1})_{b_2]}{}^{a_2} \end{pmatrix} \tag{1.4.90}$$

While the action of the generalised Lie derivative on the generalised vector W^M

$$W^M = \begin{pmatrix} w^m \\ w_{m_1m_2} \end{pmatrix} \tag{1.4.91}$$

is defined by

$$\mathcal{L}_V W^M = \begin{pmatrix} L_v w^m \\ (L_v w_{(2)} - \iota_w dv_{(2)})_{m_1m_2} \end{pmatrix} \tag{1.4.92}$$

In particular, we will focus further on considering the case of $SL(5)$ and the E_7, which can be seen in a way as the dual counterpart of $SL(5)$ in the dimensional classification.

Now, using the definition of the generalised Lie derivative 1.4.92 on the components of the frame fields 1.4.86 with the assumption of $\lambda^{a_1a_2a_3} = 0$ at the point $x = 0$ according to section 3 of [26] we get the following set of non-trivial structure constants elements of ${F_{AB}}^C$

$$F_{ab}{}^{c} = f_{ab}{}^{c}, \tag{1.4.93}$$
$$F_{a}{}^{b_1b_2c} = e_a^m\partial_m\lambda^{b_1b_2c}, \tag{1.4.94}$$
$$F_{a}{}^{b_1b_2}{}_{c_1c_2} = 4f_{ad}{}^{e}\delta^{b_1b_2}_{ef}\delta^{fd}_{c_1c_2}, \tag{1.4.95}$$
$$F^{a_1a_2}{}_{b}{}^{c} = -F_{b}{}^{a_1a_2c} = -e_b^m\partial_m\lambda^{a_1a_2c}, \tag{1.4.96}$$
$$F^{a_1a_2}{}_{bc_1c_2} = 6f_{[bc_1}{}^{[a_1}\delta^{a_2]}_{c_2]}, \tag{1.4.97}$$
$$F^{a_1a_2b_1b_2}{}_{c_1c_2} = -4e_d^m\partial_m\lambda^{a_1a_2[b_1}\delta^{b_2]d}_{c_1c_2} \tag{1.4.98}$$

Now, let's review the properties of the generators in the case of Leibniz algebra and rebuild the algebra for special cases based on the structure constants we obtained above. We will rewrite the elements ${F_{AB}}^C$ to be constant. However, now ${F_{AB}}^C \neq F_{[AB]}^{C}$, since the Leibniz algebra is an extended version of the Lie algebra, and the generators satisfy

$$T_A \circ T_B = {F_{AB}}^C T_C \tag{1.4.99}$$

this bilinear form is not anti-symmetric, but the elements of the Leibniz algebra satisfy the equation

$$T_A \circ (T_B \circ T_C) = (T_A \circ T_B) \circ T_C - T_A \circ (T_C \circ T_B) \tag{1.4.100}$$

then the corresponding Leibniz algebra corresponding to (1.4.93 - 1.4.98) in terms of the generators will be

$$T_a \circ T_b = {f_{ab}}^c T_c, \tag{1.4.101}$$
$$T_a \circ T^{b_1b_2} = f_{a}{}^{b_1b_2c}T_c + 2f_{ac}{}^{[b_1}T^{b_2]c}, \tag{1.4.102}$$
$$T^{a_1a_2} \circ T_b = -f_{b}{}^{a_1a_2c}T_c + 3f_{[c_1c_2}{}^{[a_1}\delta^{a_2]}_{b]}T^{c_1c_2}, \tag{1.4.103}$$
$$T^{a_1a_2} \circ T^{b_1b_2} = -2f_{d}{}^{a_1a_2[b_1}T^{b_2]d} \tag{1.4.104}$$

where the elements f are anti-symmetric only in part of indices (${f_{ab}}^c = f_{[ab]}^{c}$ and $f_{a}{}^{b_1b_2b_3} = f_{a}{}^{[b_1b_2b_3]}$), and satisfy several identities as a consequence of the Leibniz identity, as shown in [26]

$$f_{[ab}{}^{e}f_{c]e}{}^{d]} = 0, \tag{1.4.105}$$
$$f_{bc}{}^{e}f_{e}{}^{a_1a_2d} = -6f_{e[b}{}^{[d}f_{c]}{}^{a_1a_2]e}, \tag{1.4.106}$$
$$f_{d_1d_2}{}^{[a_1}\delta^{a_2]}_{b}f_{c}{}^{a_1a_2]e} = 0, \tag{1.4.107}$$
$$3f_{[d_1d_2}{}^{[a_1}\delta^{a_2]}_{e]}f_{c}{}^{eb_1b_2} = -4f_{ef}{}^{[a_1}f_{c}{}^{a_2]e[b_1}\delta^{b_2]f}_{d_1d_2}, \tag{1.4.108}$$
$$f_{c}{}^{ea_1a_2}f_{e}{}^{db_1b_2} = 3f_{c}{}^{e[b_1b_2}f_{e}{}^{d]a_1a_2} \tag{1.4.109}$$

Now, let us give an explicit form of the generators T_A based on the definition (1.4.99) and the above-mentioned properties. We rewrite

$$T_A \circ T_B = (F_A)_B{}^C T_C \tag{1.4.110}$$

which in its turn gives

$$(F_a)_B{}^C = \begin{pmatrix} f_{ab}{}^c & 0 \\ f_a{}^{b_1 b_2 c} & -2f_{a[c_1}{}^{[b_1}\delta^{b_2]}_{c_2]} \end{pmatrix} \tag{1.4.111}$$

$$(F^{a_1 a_2})_B{}^C = \begin{pmatrix} -f_b{}^{a_1 a_2 c} & 6f_{[c_1 c_2}{}^{[a_1}\delta^{a_2]}_{b]} \\ 0 & -2f_d{}^{a_1 a_2 [b_1}\delta^{b_2]d}_{c_1 c_2} \end{pmatrix} \tag{1.4.112}$$

These features will be important in what follows in the thesis.

Differentiating the definition of the scalar matrix (1.4.89), we deduce the differential equation on the trivector $\Pi^{a_1 a_2 a_3}$, starting with

$$\partial_m g^{-1}(x) \circ T_A = \partial_m K_A{}^B(x) T_B \tag{1.4.113}$$

Then expanding the left hand side of the equation

$$\partial_m g^{-1}(x) \circ T_A = -g^{-1} \circ \partial_m g \circ g^{-1} \circ T_A = -(l^d_m T_d) \circ (K_A{}^B T_B) = -l^d_m K_A{}^B (F_d)_B{}^C T_C \tag{1.4.114}$$

which, after using the explicit form of $K_A{}^B$ in (1.4.89) gives the following differential equations on $a_a{}^b$ and $\lambda^{a_1 a_2 a_3}$

$$e^m_c \partial_m a_a{}^b = a_a{}^d a_c{}^e f_{de}{}^b \tag{1.4.115}$$

$$e^m_c \partial_m \lambda^{a_1 a_2 a_3} = (a^{-1})_{b_1}{}^{a_1} (a^{-1})_{b_2}{}^{a_2} (a^{-1})_{b_3}{}^{a_3} a_c{}^d f_d{}^{b_1 b_2 b_3} \tag{1.4.116}$$

A few other properties that we will use further in this thesis can be derived from the Leibniz identity in a form of its consequence

$$(g \circ T_A) \circ (g \circ T_B) = g \circ (T_A \circ T_B) \tag{1.4.117}$$

which, decomposed into different components of $(F_A)_B{}^C$ gives the following properties (as it was carefully derived in [26])

$$(a^{-1})_a{}^e (a^{-1})_b{}^f a_g{}^c f_{ef}{}^g = f_{ab}{}^c \tag{1.4.118}$$

$$a_a{}^e (a^{-1})_{f_1}{}^{b_1} (a^{-1})_{f_2}{}^{b_2} (a^{-1})_{f_3}{}^{b_3} f_e{}^{f_1 f_2 f_3} = f_a{}^{b_1 b_2 b_3} + 3 f_{ac}{}^{[b_1} \lambda^{b_2 b_3]c} \tag{1.4.119}$$

$$f_{ab}{}^c \lambda^{abd} = 0 \tag{1.4.120}$$

$$3(f_{e[c}{}^{a_1} \delta^{[a_2}_{d]} \lambda^{b_1 b_2]e} - f_{e[c}{}^{a_2} \delta^{[a_1}_{d]} \lambda^{b_1 b_2]e}) + f_{cd}{}^{[a_1} \lambda^{a_2]b_1 b_2} = 0 \tag{1.4.121}$$

$$f_d{}^{b_1b_2c}\lambda^{a_1a_2d} - 3f_d{}^{a_1a_2[b_1}\lambda^{b_2c]d} = 3f_{de}{}^{[c}\lambda^{b_1b_2]d}\lambda^{a_1a_2e} - 4f_{de}{}^{[a_1}\lambda^{a_2]d[b_1}\lambda^{b_2]ec} \tag{1.4.122}$$

these properties will be used in the future analysis of the special cases to be considered in the main part of the thesis, restraining the set of supergravity solutions falling under this classification (spaces where we can apply our dualisation procedure).

Let's provide a brief description of how the generalised U-duality works. The EDA consisting of two subalgebras (in the ideal scenario we always use subalgebras that are maximally isotropic - there does not exist another subalgebra that is isotropic and contain the former subalgebra - but, as we will show further in Chapter 2, it is possible to use a set of maximally isotropic generators, that are not always subalgebras) $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are related by an exceptional algebra transformation $\mathcal{T}_M{}^N$ (in the example considered in the next chapter this will correspond to an SL(5) transformation), and with the corresponding frame fields E_M and E'_M. Then we introduce the generalised Scherk-Schwarz reduction (see section (2.2.3) for details and definition) of 11-dimensional supergravity to a reduced dimension (7-dimensional in the SL(5) case) maximal gauged supergravity, which will be then uplifted and dualised to a new supergravity solution.

The generalised metric, unifying the bosonic fields of the theory, can be parametrised via frame fields and a scalar matrix M_{AB} constant on the internal space coordinates

$$H_{MN} = E_M{}^A E_N{}^B M_{AB} \tag{1.4.123}$$

Then, the exceptional group transformation (SL(5) transformation in our following example in Chapter 2) on the fields of the reduced dimensional (7-dimensional) supergravity acts as

$$M'_{AB} = \mathcal{T}_A{}^C \mathcal{T}_B{}^D M_{CD} \tag{1.4.124}$$

producing a new dual scalar matrix M'_{AB}, which we then lift back to 11-dimensional maximal supergravity using the frame fields $E'_M{}^A$ producing the new dual scalar fields, corresponding to the generalised U-dual supergravity solution with the generalised metric

$$H'_{MN} = E'_M{}^A E'_N{}^B M'_{AB} \tag{1.4.125}$$

providing us with a new supergravity solution. We will demonstrate this procedure in examples in the following 2 chapters and present it as a solution generating technique in supergravity and M-theory.

1.5 Brief overview of Exceptional Field Theory (ExFT)

Exceptional Field Theory (ExFT) is a theory describing spacetimes with exceptional symmetry groups, that incorporate exceptional Lie algebra symmetries ($SL(5)$, E_6, E_7 and E_8). The field components in ExFT consist of the metric g_{mn} 3-form C_{mnp} and a 6-form C_{klmnpq} for larger dimensions, transformation of which (d-dimensional diffeomorphisms and gauge transformations) can be described via a generalised vector parameter Λ^M unifying in itself the vector field Λ, 2 and 5 -forms λ_{mn}, λ_{mnklp} corresponding to the transformation of each field:

$$\begin{aligned}\delta g_{mn} &= L_\Lambda g_{mn} \\ \delta C_{mnp} &= L_\Lambda C_{mnp} + 3\partial_{[m}\lambda_{np]} \\ \delta C_{klmnpq} &= L_\Lambda C_{klmnpq} + 6\partial_{[k}\lambda_{lmnpq]} + 30C_{[klm}\partial_n\lambda_{pq]}\end{aligned} \tag{1.5.1}$$

where the gauge parameters are unified in the generalised vector $\Lambda^M = (\Lambda^m, \lambda_{mn}, \lambda_{mnlpq})$, forming representations of the exceptional groups $E_{d(d)}$. This can be expressed in the generalised Lie derivative (special case of Leibniz derivative). In a more general form for the $E_{d(d)}$ generalised Lie derivative with respect to coordinates Y^M lying in the R_1 representation - a representation of $E_{d(d)}$ where the generalised vector appears, the generalised Lie derivative acting on a generalised vector V^M of a weight λ_V reads as

$$\mathcal{L}_U V^M = U^N\partial_N V^M - \alpha\mathbf{P}^M{}_N{}^P{}_Q\partial_P U^Q V^N + \lambda_V\partial_N U^N V^M \tag{1.5.2}$$

where $\mathbf{P}^M{}_N{}^P{}_Q$ is a projector projecting from the $R_1 \times R_1$ to the adjoint representation, and α is a constant depending on the group under consideration. The closure of the algebra generated by this generalised Lie derivative will result in a section condition applied to any fields or gauge parameters Ψ, Ψ' in the theory

$$Y^{MN}_{PQ}\partial_M\Psi\partial_N\Psi' = 0, \quad Y^{MN}_{PQ}\partial_M\partial_N\Psi = 0 \tag{1.5.3}$$

where the Y tensor is defined as

$$Y^{MN}_{PQ} = -\alpha\mathbf{P}^M{}_Q{}^N{}_P + \delta^M_P\delta^N_Q + \frac{1}{n-2}\delta^M_Q\delta^N_P \tag{1.5.4}$$

which in the case of $SL(5)$ reduces to the condition 1.4.83 introduced in the previous section.

The action of ExFT is constructed from the fields: n-dimensional metric $g_{\mu\nu} \in GL(n)/SO(1,n-1)$ which transforms as a scalar of a weight $\frac{2}{n-2}$ under the generalised diffeomorphisms, the generalised metric $H_{MN} \in E_{d(d)}/H_d$ - a tensor of weight zero, where H_d is a maximal compact subgroup of $E_{d(d)}$, and a set of gauge fields: A_μ, ... $C_{\mu_1...\mu_{9-d}}$, where each p-form belongs to the representation R_p of $E_{d(d)}$ and transforms with the weight $\frac{p}{n-2}$. The strength field composed of these gauge fields do not transform tensorially under the generalised Lie derivative, and require special compensator fields to be added, defining the tensor hierarchy in ExFT [38].

Consider an example of $SL(5)$: the field content consists of the following elements

$$\{g_{\mu\nu}, H_{\mathcal{MN},\mathcal{PQ}}, A_\mu^{\mathcal{MN}}, B_{\mu\nu\mathcal{M}}, C_{\mu\nu\rho}{}^{\mathcal{M}}\} \tag{1.5.5}$$

where greek indices are 7-dimensional indices related to the "external" 7-dimensional space coordinates x^μ, and $\mathcal{M}$, $\mathcal{N} = 1, ..., 5$ denote five-dimensional fundamental indices constructed of the 10-dimensional indices $M = 1, ..., 10$ of R_1: $M = [\mathcal{MN}]$. Field strengths tensors of the gauge fields $A_\mu^{\mathcal{MN}}$, $B_{\mu\nu\mathcal{M}}$, and $C_{\mu\nu\rho}{}^{\mathcal{M}}$ are $F_{\mu\nu}{}^{\mathcal{MN}}$, $H_{\mu\nu\rho\mathcal{M}}$ and $J_{\mu\nu\rho\sigma}{}^{\mathcal{M}}$ correspondingly. The generalised metric $H_{\mathcal{MN},\mathcal{PQ}}$ parametrises the coset space SL(5)/SO(5), and can be decomposed as

$$H_{\mathcal{MN},\mathcal{PQ}} = m_{\mathcal{MP}} m_{\mathcal{QN}} - m_{\mathcal{MQ}} m_{\mathcal{PN}} \tag{1.5.6}$$

with $m_{\mathcal{MN}} = m_{\mathcal{NM}}$ and $\det(m_{\mathcal{MN}}) = 1$.

The SL(5) action invariant under generalised diffeomorphisms (in each term) and external diffeomorphisms (the latter requirement fixes the coefficients between terms in the action) is found to be of the form:

$$\begin{aligned} S_{SL(5)} = \int dx^7 dY \sqrt{|g|} \big(R(g) + \frac{1}{4} D_\mu m^{\mathcal{MN}} D^\mu m_{\mathcal{MN}} - \frac{1}{8} m_{\mathcal{MP}} m_{\mathcal{NQ}} F_{\mu\nu}{}^{\mathcal{MN}} F^{\mu\nu\mathcal{PQ}} \\ - \frac{1}{12} m^{\mathcal{MN}} H_{\mu\nu\rho\mathcal{M}} H^{\mu\nu\rho}{}_N + L_{int}(m, g) + (\sqrt{|g|})^{-1} L_{top} \big) \end{aligned} \tag{1.5.7}$$

This action matches the decomposition of 11-dimensional supergravity action (1.3.13): as it has been shown, the 11-dimensional action decomposed properly into terms match the SL(5) ExFT action 1.5.7 in every term, after performing the decompositions via a Kaluza-Klein reduction and further uplift of the 11-dimensional supergravity action (see sections (3.2) and (4.6.2) of [49]). In addition to that, different solutions of the section condition in ExFT (defined back in (1.4.83)) result in different type IIB supergravity solutions, establishing a strong connection between supergravity and ExFT.

Using a parametrisation of SL(5) in terms of the generalised metric: the 4-dimensional internal metric ϕ_{mn} and the 3-form represented by $C^m = \frac{1}{3!} \epsilon^{mnkl} C_{nkl}$

$$m_{\mathcal{MN}} = \phi^{-2/5} \begin{pmatrix} \phi_{mn} & \phi_{mk} C^k \\ -\phi_{nk} C^k & \phi + \phi_{kl} C^k C^l \end{pmatrix} \tag{1.5.8}$$

and for the gauge field strengths according to the tensor hierarchy

$$F_{\mu\nu}{}^{\mathcal{MN}} = \begin{pmatrix} F^m_{\mu\nu} \\ \frac{1}{2} \epsilon^{mnkl} (F_{\mu\nu kl} - F^p_{\mu\nu} C_{klp}) \end{pmatrix} \tag{1.5.9}$$

$$H_{\mu\nu\rho m} = -F_{\mu\nu\rho m}, \quad J_{\mu\nu\rho\sigma}{}^5 = -F_{\mu\nu\rho\sigma} \tag{1.5.10}$$

while $H_{\mu\nu\rho 5}$ and $J_{\mu\nu\rho\sigma}{}^i$ can be derived from the duality relation between $H_{\mathcal{M}}$ and $J^{\mathcal{M}}$ resulting from the equation of motion and resulting in the relation $m^{\mathcal{MN}} H_{\mathcal{N}} \sim \star J^{\mathcal{M}}$. Plugging these fields in the SL(5) action (1.5.7) one can easily verify that the 11-dimensional SUGRA action (1.3.13) can be recovered in every term matching its SL(5) ExFT decomposition (into 4 "internal" and 7 "external" coordinates). This highlights an important role of ExFT in supergravity, and we will be using this technique further in the thesis in constructing new solutions via dualising the original supergravity solutions decomposed according to the ExFT structure.

Chapter 2

Exceptional Drinfeld Geometries

2.1 Introduction

The textbook T-duality symmetry of string theory that applies in backgrounds with Abelian isometries is a cornerstone of the duality web that ultimately leads to M-theory [1, 2]. Less standard is the application of T-duality to backgrounds whose isometry group is non-Abelian [3]. While its status as a precise duality in either α' and g_s expansions is not fully resolved, at the very least non-Abelian T-duality (NATD) is a useful tool as a solution generating symmetry of Type II supergravity (for a review see [12]). More exotic still are applications of T-duality to backgrounds which have no isometries at all. Poisson-Lie (PL) T-duality, introduced by Klimčík and Severa [10, 11], provides situations where such a non-isometric duality can be realised. This is made possible when the target spaces have a certain Poisson-Lie symmetry property giving rise to an unexpectedly rich algebraic structure encoded by a Drinfeld double, $\mathfrak{d}$ [19].[1] Despite this lack of isometry, the corresponding non-linear sigma models can actually exhibit classical (and quantum) integrability [39]. Close connections between integrability and Poisson-Lie duality have come under renewed focus with holographic motivation following the development of the integrable η [39] and related λ [40] deformations applied to the $AdS_5 \times S^5$ superstring in [41] and [42] respectively.

Poisson-Lie geometries (i.e. those for which PL T-duality can be realised) can at first sight seem convoluted, especially when presented in terms of the regular geometric data consisting of the metric and Kalb-Ramond two-form. However, when viewed using generalised geometry the situation is radically improved; the PL property of the target space is encapsulated [23] by a generalised parallelisation [43, 44]. This consists of a set of generalised frame fields that span the generalised tangent bundle, $TM \oplus T^\star M$, and which furnish the Drinfeld double algebra under the generalised Lie derivative. Moreover there is a natural candidate for the extended target space that appears in

[1] The Drinfeld double $\mathfrak{d}$ is an even-dimensional Lie algebra that can be decomposed into two sub-algebras $\mathfrak{d} = \mathfrak{g} + \tilde{\mathfrak{g}}$ that are maximally isotropic with respect to an ad-invariant inner product of split signature. The Jacobi identity of $\mathfrak{d}$ enforces a cocycle compatibility condition between $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$.

both the world-sheet doubled sigma-model [45,46] and in the Double Field Theory approach [23,24], namely the group $D = \exp \mathfrak{d}$.[2]

The U-duality symmetry of M-theory can also be viewed as a generalisation of T-duality, arising when one combines the perturbative T-duality symmetry with non-perturbative S-dualities. Until recently, there has been no hint of whether U-duality admits non-Abelian or generalised versions. A proposal for the algebraic structure that would underlie such dualities has been introduced in [26,27] and called the Exceptional Drinfeld Algebra (EDA).

Roughly an EDA is an algebra $\mathfrak{d}_n$, defined by a bracket, $[\bullet,\bullet] : \mathfrak{d}_n \otimes \mathfrak{d}_n \to \mathfrak{d}_n$, which does not need to be antisymmetric but obeys the Leibniz identity, and admits a Lie subalgebra $\mathfrak{g}$, of dimensions n or $n-1$. Moreover $\mathfrak{g}$ can be considered a maximally isotropic subalgebra in a sense we shall make more precise later. For the case of $n \leq 4$, that shall be our concern here, the data of an EDA can be interpreted as consisting of a Lie-algebra $\mathfrak{g}$ together with a three-algebra $\tilde{\mathfrak{g}}$ that are restricted to obey a cocycle compatibility condition. A key point of [26,27] was that the EDA can be realised by a generalised Leibniz parallelisation for the exceptional tangent bundle $TG \oplus \wedge^2 T^\star G$ thus echoing the set up of Poisson-Lie T-duality and allowing this framework to be used to generate solutions using the ideas of generalised Scherk-Schwarz reductions. Some features of the geometry, and the membrane interpretation, were then given in [35], while a classification of all possible EDAs for the case of $n = 3$ was made in [36].

We shall explore the geometry associated to this new M-theoretic algebraic structure in a number of explicit examples. These examples reveal intriguing connections to several topics. We study geometries which encode the structure constants of three-algebras, which naturally show up amongst the structure constants of the Exceptional Drinfeld Algebra. Here we can also connect with a class of CSO gaugings of 7-dimensional maximal supergravity. Hence, we get for free out of our construction some simple new uplifts for these gaugings. These uplifts could be regarded as "non-Abelian U-duals", in some sense, of spheres with flux. We will also describe the embedding of Poisson-Lie T-duality into this set-up in some detail, revealing a construction whereby the Exceptional Drinfeld Algebra involves augmenting the Drinfeld double with a spinor representation. Making a frequent usage of some technical results within Exceptional Field Theory which, to allow for completeness but avoid distraction, have been included as appendix material here. (For a detailed review, see [49].)

2.2 The $SL(5)$ Exceptional Drinfeld Algebra

2.2.1 The algebra

We begin by specifying the Exceptional Drinfeld Algebra in the case of the group $E_{4(4)} = SL(5)$. We introduce five-dimensional fundamental $SL(5)$ indices $\mathcal{A}, \mathcal{B} = 1 \ldots, 5$. The generators of the

[2] The discussion here is adapted to the case where the physical target space M is a group manifold $M = G \cong D/\tilde{G}$ with $G = \exp \mathfrak{g}$ and $\widetilde{G} = \exp \tilde{\mathfrak{g}}$. However, when M can be constructed as a double coset, $M = H \backslash D / \widetilde{G}$, similar ideas apply both from the world-sheet [47] and target space [48] perspectives.

Exceptional Drinfeld Algebra live in the ten-dimensional antisymmetric representation, and we can label these with a pair of antisymmetric five-dimensional indices, $T_{\mathcal{AB}} = -T_{\mathcal{BA}}$. The brackets of the generators are

$$[T_{\mathcal{AB}}, T_{\mathcal{CD}}] = \frac{1}{2} F_{\mathcal{AB},\mathcal{CD}}{}^{\mathcal{EF}} T_{\mathcal{EF}} \,, \tag{2.2.1}$$

(where the factor of 1/2 is inserted to avoid overcounting) and these need not be antisymmetric. We do require the Leibniz identity

$$\big[T_{\mathcal{BB}'}, [T_{\mathcal{CC}'}, T_{\mathcal{DD}'}]\big] = \big[[T_{\mathcal{BB}'}, T_{\mathcal{CC}'}], T_{\mathcal{DD}'}\big] + \big[T_{\mathcal{CC}'}, [T_{\mathcal{BB}'}, T_{\mathcal{DD}'}]\big] \,, \tag{2.2.2}$$

which in terms of the structure constants leads to

$$\frac{1}{2} F_{\mathcal{BB}',\mathcal{EE}'}{}^{\mathcal{AA}'} F_{\mathcal{CC}',\mathcal{DD}'}{}^{\mathcal{EE}'} - \frac{1}{2} F_{\mathcal{CC}',\mathcal{EE}'}{}^{\mathcal{AA}'} F_{\mathcal{BB}',\mathcal{DD}'}{}^{\mathcal{EE}'} = \frac{1}{2} F_{\mathcal{BB}',\mathcal{CC}'}{}^{\mathcal{EE}'} F_{\mathcal{EE}',\mathcal{DD}'}{}^{\mathcal{AA}'} \,. \tag{2.2.3}$$

If the bracket is antisymmetric, this reduces to the usual Jacobi identity.

More generally, the constraint (2.2.3) is the same as the quadratic constraint of gauged supergravity. This link – or equivalently the fact that we are restricting to Leibniz algebras which can arise from a generalised parallelisation of SL(5) exceptional geometry – also motivates the assumption that the structure constants can be decomposed into irreducible representations as

$$F_{\mathcal{AB},\mathcal{CD}}{}^{\mathcal{EF}} = 4 F_{\mathcal{AB}[\mathcal{C}}{}^{[\mathcal{E}} \delta^{\mathcal{F}]}_{\mathcal{D}]} \,, \quad F_{\mathcal{ABC}}{}^{\mathcal{D}} = Z_{\mathcal{ABC}}{}^{\mathcal{D}} + \frac{1}{2} \delta^{\mathcal{D}}_{[\mathcal{A}} S_{\mathcal{B}]\mathcal{C}} - \frac{1}{6} \tau_{\mathcal{AB}} \delta^{\mathcal{D}}_{\mathcal{C}} - \frac{1}{3} \delta^{\mathcal{D}}_{[\mathcal{A}} \tau_{\mathcal{B}]\mathcal{C}} \,, \tag{2.2.4}$$

where $\tau_{\mathcal{AB}} = -\tau_{\mathcal{BA}}$, $S_{\mathcal{AB}} = S_{\mathcal{BA}}$ and $Z_{\mathcal{ABC}}{}^{\mathcal{D}} = Z_{[\mathcal{ABC}]}{}^{\mathcal{D}}$, $Z_{\mathcal{ABC}}{}^{\mathcal{C}} = 0$. This means that the only SL(5) irreducible representations appearing in the structure constants of our Leibniz algebra are those specified by the linear constraint of gauged maximal supergravity in seven-dimensions [50].

Now we impose the further conditions that make this SL(5) Leibniz algebra into an Exceptional Drinfeld Algebra. We require that there is a Lie subalgebra $\mathfrak{g} \subset \mathfrak{d}_4$ which is isotropic in the sense that[3]

$$\epsilon^{\mathcal{ABCDE}} T_{\mathcal{AB}} \otimes T_{\mathcal{CD}} \Big|_{\mathfrak{g}} = 0 \,, \tag{2.2.5}$$

and we further require this subalgebra to be *maximal* in the sense that appending any extra generator to $\mathfrak{g}$ will violate (2.2.5). This means that it will have either dimension 4 or 3, and so can be interpreted (borrowing terminology from Exceptional Field Theory) as the *physical* subalgebra in either an M-theory or type IIB background, respectively. To articulate this condition in a more invariant fashion we can say that alongside $\mathfrak{d}_n$ we must specify a "pure spinor" Λ in an appropriate representation[4] of $E_{n(n)}$ which acts linearly on the $\mathfrak{d}_n$ vector space schematically as $\Lambda \bullet T$. We then

[3]Note that a systematic construction of generalised frames corresponding to a given set of generalised fluxes was set out in [51] in which a similar condition plays a necessary role: it really just ensures that the section condition of Exceptional Field Theory is satisfied.

[4]In DFT this would actually be a spinor representation, in ExFT it is not generically spinorial but will obey a purity constraint projecting out certain representations in the tensor product of Λ with itself.

demand that the kernel of this action, $\mathfrak{g} = \ker(\Lambda)$ be a Lie subalgebra. There are different choices for Λ that will result in a subalgebra $\mathfrak{g}$ of dimension n, which we call an M-theory section, and dimension $n-1$ which we shall call a IIB-theory section. This pure spinor approach is essentially the same as that used to define solutions to the so-called section condition of Exceptional Field Theory [34, 52].

For the case of SL(5), in the IIB-theory section the pure spinor Λ is in the $\overline{\mathbf{10}}$ and the purity condition is that $\Lambda^{[\mathcal{AB}}\Lambda^{\mathcal{CD}]} = 0$. The linear action is defined by

$$\Lambda \bullet T := \Lambda^{\mathcal{AC}} T_{\mathcal{CB}} - \frac{1}{5}\Lambda^{\mathcal{CD}} T_{\mathcal{CD}} \delta^{\mathcal{A}}{}_{\mathcal{B}}.$$

As an example consider $\Lambda^{45} = -\Lambda^{54} = 1$ with the other components zero. Evidently this is pure and it is such that it defines

$$\ker(\Lambda) = \mathrm{span}\{T_{12}, T_{13}, T_{23}\}\,. \tag{2.2.6}$$

In the M-theory section the pure spinor Λ is in the $\mathbf{5}$, the purity constraint is automatic and no further conditions are placed on Λ. The action on generators is

$$\Lambda \bullet T := \Lambda_{[\mathcal{A}} T_{\mathcal{BC}]}\,. \tag{2.2.7}$$

Consider taking $\Lambda_{\mathcal{A}} = \delta_{\mathcal{A},5}$, in which case

$$\ker(\Lambda) = \mathrm{span}\{T_{a5} | a = 1 \ldots 4\}\,. \tag{2.2.8}$$

We will continue now in this M-theory section, and decompose indices as $\mathcal{A} = (a, 5)$, where $a = 1, \ldots, 4$ such that the physical subalgebra is generated by the generators $t_a \equiv T_{a5}$, with Lie algebra structure constants $f_{ab}{}^c$.

In terms of the irreducible representations, the Exceptional Drinfeld Algebra is wholly defined in terms of the Lie algebra structure constants $f_{ab}{}^c$ along with S_{ab}, τ_{ab} and τ_{a5}, with:

$$\begin{gathered} S_{55} = 0\,, \quad Z_{abc}{}^5 = 0\,, \quad Z_{ab5}{}^5 = \frac{2}{3}\tau_{ab}\,, \quad Z_{abc}{}^d = -\tau_{[ab}\delta^d_{c]}\,, \\ S_{a5} = -\frac{2}{3}\tau_{a5} - \frac{4}{3} f_{ab}{}^b\,, \quad Z_{ab5}{}^c = -f_{ab}{}^c - \frac{2}{3}\delta^c_{[a} f_{b]d}{}^d\,. \end{gathered} \tag{2.2.9}$$

To write down the algebra explicitly, we combine S_{ab} and τ_{ab} into a "dual" structure constant with three upper antisymmetric indices given by

$$\tilde{f}^{abc}{}_d = \frac{1}{4}\epsilon^{abce}(S_{de} + 2\tau_{de})\,. \tag{2.2.10}$$

If we further define the "dual" generators $\tilde{t}^{ab} \equiv \frac{1}{2}\epsilon^{abcd}T_{cd}$, then the Exceptional Drinfeld Algebra can then be written as

$$
\begin{aligned}
[t_a, t_b] &= f_{ab}{}^c t_c\,, \\
[t_a, \tilde{t}^{bc}] &= 2 f_{ad}{}^{[b}\tilde{t}^{c]d} - \tilde{f}^{bcd}{}_a t_d - \frac{1}{3}\mathfrak{L}_a \tilde{t}^{bc}\,, \\
[\tilde{t}^{bc}, t_a] &= 3 f_{[de}{}^{[b}\delta^{c]}_{a]}\tilde{t}^{de} + \tilde{f}^{bcd}{}_a t_d + \mathfrak{L}_d \delta^{[b}_a \tilde{t}^{cd]}\,, \\
[\tilde{t}^{ab}, \tilde{t}^{cd}] &= 2\tilde{f}^{ab[c}{}_e \tilde{t}^{d]e}\,,
\end{aligned}
\tag{2.2.11}
$$

in which we introduced the combination $\mathfrak{L}_a = \tau_{a5} - f_{ad}{}^d$. With $\mathfrak{L}_a = 0$ this presentation closely resembles the structure of a Drinfeld double. However crucially this bracket has a symmetric part that vanishes if and only if

$$
\frac{2}{3}\mathfrak{L}_{[d}\delta^c_{e]} + f_{de}{}^c = 0\,, \quad \tau_{ab} = 0\,. \tag{2.2.12}
$$

In addition to the Jacobi identity on $\mathfrak{g}$, the Leibniz closure conditions (2.2.3) enforce that the dual structure constants obey the fundamental identity of a three-algebra

$$
\tilde{f}^{abg}{}_c \tilde{f}^{def}{}_g - 3\tilde{f}^{g[de}{}_c \tilde{f}^{f]ab}{}_g = 0\,. \tag{2.2.13}
$$

There are also a set of compatibility equations between $\tilde{f}^{abc}{}_d$ and $f_{ab}{}^c$ which include in particular a condition

$$
6 f_{f[a}{}^{[c}\tilde{f}^{de]f}{}_{b]} + f_{ab}{}^f \tilde{f}^{cde}{}_f + \frac{2}{3}\tilde{f}^{cde}{}_{[a}\mathfrak{L}_{b]} = 0. \tag{2.2.14}
$$

When $\mathfrak{L}_a = 0$ this last condition states that the dual structure constants, viewed as a map $\tilde{f} : \mathfrak{g} \to \wedge^3\mathfrak{g}$ define a $\wedge^3\mathfrak{g}^*$ valued one-cochain.

2.2.2 The generalised geometry realisation

A geometric realisation of this algebra can be achieved using as data the left-invariant forms l^a and dual vector fields v_a, obeying $\iota_{v_a} l^b = \delta^b_a$, of a group manifold G, together with a trivector λ^{abc} and a scalar α that are required to obey differential conditions:

$$
dl^a = \frac{1}{2} f_{bc}{}^a l^b \wedge l^c\,, \quad L_{v_a} v_b = -f_{ab}{}^c v_c\,, \tag{2.2.15}
$$

$$
d\lambda^{abc} = \tilde{f}^{abc}{}_d l^d + 3 f_{ed}{}^{[a}\lambda^{bc]d} l^e + \frac{1}{3}\lambda^{abc}\mathfrak{L}_d l^d\,, \tag{2.2.16}
$$

$$
L_{v_a} \ln\alpha = \frac{1}{3}\mathfrak{L}_a \equiv \frac{1}{3}(\tau_{a5} - f_{af}{}^f)\,. \tag{2.2.17}
$$

Below, we will often write the trivector λ^{abc} in its dualised form

$$
\lambda^{abc} = \epsilon^{abcd}\lambda_d\,, \quad \lambda_a = \frac{1}{3!}\epsilon_{bcda}\lambda^{bcd}\,. \tag{2.2.18}
$$

These data can be naturally understood in terms of a generalised frame field using SL(5) exceptional generalised geometry or SL(5) exceptional field theory [37, 53–56]. We provide the necessary background material in appendix A.1, and will only summarise the key details here. A generalised frame is a section of the generalised tangent bundle $TM \oplus \Lambda^2 T^*M$, where M denotes a four-dimensional manifold, and so we can write $E_{\mathcal{AB}} = (e_{\mathcal{AB}}, \omega_{(2)\mathcal{AB}})$ in terms of vector field $e_{\mathcal{AB}}$ and a two-form $\omega_{(2)\mathcal{AB}}$. Under the generalised Lie derivative (for more see appendix A.1.1) which acts as

$$\mathcal{L}_{E_{\mathcal{AB}}} E_{\mathcal{CD}} = (L_{e_{\mathcal{AB}}} e_{\mathcal{CD}}, L_{e_{\mathcal{AB}}} \omega_{(2)\mathcal{CD}} - \iota_{e_{\mathcal{CD}}} d\omega_{(2)\mathcal{AB}})\,, \tag{2.2.19}$$

the frames are constructed such that they obey

$$\mathcal{L}_{E_{\mathcal{AB}}} E_{\mathcal{CD}} = -\frac{1}{2} F_{\mathcal{AB},\mathcal{CD}}{}^{\mathcal{EF}} E_{\mathcal{EF}}\,, \tag{2.2.20}$$

where in general the quantities $F_{\mathcal{AB},\mathcal{CD}}{}^{\mathcal{EF}}$ give non-constant "generalised fluxes" defined as in appendix A.1. We are interested in the case where a set of frames can be found with constant fluxes, in which case their generalised Lie derivatives (2.2.20) furnish a geometric realisation of a Leibniz algebra.

We can achieve such a realisation of our Exceptional Drinfeld Algebra. First, we decompose our 10-dimensional generalised frame as

$$E_a \equiv E_{a5}\,, \quad E^{ab} \equiv \frac{1}{2}\epsilon^{abcd} E_{cd}\,, \tag{2.2.21}$$

and specify that, in terms of pairs of vectors and two-forms, these are given by

$$E_a = (v_a, 0)\,, \quad E^{ab} = (\lambda^{abc} v_c, \alpha l^a \wedge l^b)\,. \tag{2.2.22}$$

The differential conditions (A.3.21), (A.3.22) and (2.2.17) ensure that the algebra of frames (2.2.20) reproduces the Exceptional Drinfeld Algebra (A.3.25) subject to the imposition of some algebraic constraints which take the form:

$$0 = f_{[ab}{}^d \lambda_{c]} + 6\lambda_{[a} \mathfrak{L}_b \delta^d_{c]}\,, \quad 0 = \tau_{[ab}\lambda_{c]}\,. \tag{2.2.23}$$

These constraints ensure that the structure constants of the EDA are invariant under an adjoint action of $G = \exp \mathfrak{g}$ [26, 27]. They are also what is needed to ensure that the structure constants are indeed constant.

In what follows, it will be convenient to package the same data into a frame field $\tilde{E}_{\mathcal{A}}$ in the **5** representation i.e. as sections of the bundle $(\mathbb{R} \oplus \Lambda^3 T^*M) \otimes (\det T^*M)^{-3/10}$. Here the weight factor is such that the frame has unit determinant when viewed as a five-by-five matrix (see appendix A.1

for more details). This matrix is given by

$$\tilde{E}^{\mathcal{M}}{}_{\mathcal{A}} = \Delta^{-\frac{1}{2}} \begin{pmatrix} l^{\frac{1}{2}}\alpha^{\frac{1}{2}}v^i{}_a & 0 \\ l^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}\lambda_a & l^{-\frac{1}{2}}\alpha^{-\frac{1}{2}} \end{pmatrix}, \tag{2.2.24}$$

where $l \equiv \det l^a{}_i$ and $\Delta = \alpha^{\frac{3}{5}} l^{\frac{1}{5}}$ is a corrective weight whose interpretation in terms of the determinant of the external 7-dimensional metric is explained in appendix A.1.

2.2.3 The geometry

In the $E_{n(n)}$ Exceptional Generalised Geometry (EGG) / Exceptional Field Theory (ExFT) approach to supergravity an artificial splitting is made into n internal directions (coordinates of which we denote x) and $D = 11-n$ external directions (coordinates of which we denote X). This splitting allows the field content[5] of the supergravity to be reassembled into appropriate representations of the $E_{n(n)}$.

In the case at hand, $n = 4$, the degrees of freedom associated to the "internal" four-dimensional metric, g_{ij}, and three-form, C_{ijk}, parametrise the coset SL(5)/SO(5). This coset can be described using a generalised frame or equivalently a SO(5)-invariant matrix $m_{\mathcal{MN}}$ called the generalised metric. The technical details of how to extract the conventional geometric data from a generalised metric are presented in the appendix. In particular note that we have one extra piece of geometric data, namely the scalar $\Delta \equiv \Delta(x)$ (or equivalently α), which is related to the determinant of the external metric.

Here we will consider generalised metrics admitting a particular factorised form using the generalised frame field (2.2.24), such that

$$m_{\mathcal{MN}}(X, x) = \tilde{E}^{\mathcal{A}}{}_{\mathcal{M}}(x)\tilde{E}^{\mathcal{B}}{}_{\mathcal{N}}(x)\bar{m}_{\mathcal{AB}}(X), \tag{2.2.25}$$

where $\bar{m}_{\mathcal{AB}}(X)$ denotes an SL(5)/SO(5) coset element depending only on the external coordinates X. This factorised form of eq. (2.2.25) is known as a generalised Scherk-Schwarz reduction ansatz. It is now well-established that, starting with EGG/ExFT, such an ansatz gives rise to lower-dimensional maximal gauged supergravities [57, 58] (this idea was pioneered in the half-maximal case in DFT in [59–61]). The structure constants of the Exceptional Drinfeld Algebra are interpreted as the embedding tensor which specifies the gauging of this theory, and the matrix $\bar{m}_{\mathcal{AB}}$ contains the scalars of the gauged supergravity.

One can regard two separate generalised frames $E^{\mathcal{A}}$ and $E'^{\mathcal{A}}$ producing the same Exceptional Drinfeld Algebra, up to some SL(5) transformation acting on the indices $\mathcal{A}$, but possibly depending on different choices of the physical coordinates, as being generalised U-dual in the sense that they

[5] More precisely the bosonic field content is packaged into representations of $E_{n(n)}$ while the fermions (which play no role in the discussion here) form representations of the maximal compact subgroup.

will both reduce to the same 7-dimensional theory.

A key point here is that to complete the geometries given by the EDA frame fields as fully-fledged solutions of 11-dimensional supergravity one needs to determine the external sector by solving the equations of the resulting lower dimensional gauged supergravity. Conversely, given a solution of the gauged supergravity whose embedding tensor matches the form of an EDA, then the ansatz (2.2.25) provides an uplift. Our immediate aim however is not to construct full supergravity solutions, instead we wish simply to gain some intuition for the sort of geometries that arise when the generalised frame fields of the EDA are used to construct the internal metric. To this end let us simply set $\bar{m}_{\mathcal{AB}}(X) = \delta_{\mathcal{AB}}$ and set to zero off-diagonal components of fields i.e. those with mixed four-dimensional and seven-dimensional indices. Using the dictionary reproduced in full in appendix A.1.3, we can, as in [35], work out the geometry giving rise to the Exceptional Drinfeld Algebra

$$\begin{aligned} ds_{11}^2 &= \alpha^{2/3}(1+\lambda_c\lambda^c)^{1/3}\left(ds_7^2 + \frac{1}{1+\lambda_c\lambda^c}(\delta_{ab}+\lambda_a\lambda_b)l^a\otimes l^b\right) \\ &= \alpha^{2/3}(1+\lambda_c\lambda^c)^{1/3}ds_7^2 + ds_4^2\,, \\ C_{(3)} &= -\frac{1}{6}\frac{\alpha}{1+\lambda_c\lambda^c}\lambda_{bcd}l^b\wedge l^c\wedge l^d\,, \end{aligned} \tag{2.2.26}$$

where we use δ_{ab} to contract Lie algebra indices.

2.3 Three-algebra geometries

We will start by exploring geometries with

$$f_{ab}{}^c = 0\,, \quad \tilde{f}^{abc}{}_d \neq 0\,, \tag{2.3.1}$$

which we shall refer to as three-algebra geometries. The analogue of such cases in terms of non-Abelian T-duality would be the geometries that one obtains *after* dualising from a geometry with a group manifold symmetry, $f_{ab}{}^c \neq 0$, $\tilde{f}^{ab}{}_c = 0$.

The corresponding Exceptional Drinfeld Algebra is most transparently expressed in terms of the undualised generators

$$\begin{aligned} [T_{a5}, T_{b5}] &= 0\,, \\ [T_{a5}, T_{bc}] &= \frac{1}{2}(S_{a[b} + 2\tau_{a[b})T_{c]5} = -[T_{bc}, T_{a5}]\,, \\ [T_{ab}, T_{cd}] &= -\tau_{ab}T_{cd} + (S_{c][b} + 2\tau_{c][b})T_{a][d}\,. \end{aligned} \tag{2.3.2}$$

When $\tau_{ab} = 0$, this is the Lie algebra $CSO(p,q,r+1)$, $p+q+r=4$, as is clear from diagonalising S_{ab} such that $S_{\mathcal{AB}} \sim \mathrm{diag}(\underbrace{+1,\cdots+1}_{p},\underbrace{-1,\ldots,-1}_{q},\underbrace{0,\ldots,0}_{r+1})$. When $\tau_{ab} \neq 0$ we have a genuine Leibniz algebra. The conditions for closure are

$$S_{a[b}\tau_{cd]} = 0\,, \quad \tau_{[ab}\tau_{cd]} = 0\,, \tag{2.3.3}$$

which are also what are required for the final equation of (2.2.23) to hold. The only solutions can be organised according to the rank of S_{ab} assuming the latter has been diagonalised:[6]

- S_{ab} has rank 4 or 3, then $\tau_{ab} = 0$,
- S_{ab} has rank 2, say $S_{11} \neq 0$, $S_{22} \neq 0$, then we can have $\tau_{12} \neq 0$,
- S_{ab} has rank 1, say $S_{11} \neq 0$, then we can have $\tau_{12}, \tau_{13}, \tau_{14} \neq 0$,
- S_{ab} has rank 0, then we can have either $\tau_{12}, \tau_{13}, \tau_{14} \neq 0$ or $\tau_{12}, \tau_{13}, \tau_{23} \neq 0$ (or other choices related by relabellings of the indices).

In order to realise this algebra using a generalised frame, we introduce 4-dimensional coordinates x^i and take

$$l^a{}_i = \delta^a_i\,, \quad \lambda^{abc} = \tilde{f}^{abc}{}_d x^d\,, \quad \alpha = \text{constant}\,, \tag{2.3.4}$$

(where $x^a \equiv \delta^a_i x^i$). To extract the geometry, we note that

$$\lambda_a = \frac{1}{6}\epsilon_{bcda}\tilde{f}^{bcd}{}_e x^e = \frac{1}{4}(S_{ab} - 2\tau_{ab})x^b\,, \tag{2.3.5}$$

which we can use in the general formulae (2.2.26).

If we choose the coordinates x^i to be periodic, then this corresponds to a U-fold, as to make the space globally well-defined we have to patch via a shift of the trivector. This is a non-geometric U-duality transformation, and we can then further view the flux $\tilde{f}^{abc}{}_d$ as an M-theory non-geometrc Q-flux [62]. This is the generalisation of the interpretation of non-Abelian T-dual geometries as T-folds [63].

We note that the paper [26] considered an example where $\tilde{f}^{234}{}_1, \tilde{f}^{234}{}_2, \tilde{f}^{134}{}_1$, $\tilde{f}^{134}{}_2$ are all non-zero, in which case S_{ab} has rank two (but is not diagonal in this basis), while for τ_{ab} only $\tau_{12} \neq 0$. For $\tilde{f}^{234}{}_1 = \tilde{f}^{234}{}_2 = 0$ this allowed other isotropic subalgebras corresponding to the embedding of the non-Abelian T-dual of the Bianchi VI algebra.

2.3.1 Non-Abelian T-duality revisited and $CSO(3,0,2)$

As a first example, let's consider $CSO(3,0,2)$, for which we set

$$S_{ab} = 4\,\mathrm{diag}(1,1,1,0)\,, \quad \tau_{ab} = 0\,. \tag{2.3.6}$$

[6] If S_{ab} is not diagonal then the constraints on τ_{ab} will be different, as will the form of the algebra, but this will be related by a similarity transform.

We will show now how this set up actually provides an embedding for the non-Abelian T-dual (NATD) of the three-sphere S^3 with respect to an $SU(2)_L$ isometry sub-group. In the M-theory section the four-dimensional geometry with coordinates (x^i, x^4), $i = 1,2,3$, is given by

$$\begin{aligned} ds_4^2 &= (1+\delta_{mn}x^m x^n)^{-2/3}\left((\delta_{ij}+x_i x_j)dx^i dx^j + (dx^4)^2\right) , \\ C_{(3)} &= -\frac{1}{2!}\frac{\epsilon_{ijk4}x^k}{1+\delta_{mn}x^m x^n}dx^i\wedge dx^j \wedge dx^4 . \end{aligned} \tag{2.3.7}$$

With x^4 taken to be periodic and identified with the M-theory circle, we can reduce to give a IIA configuration for which the 3-dimensional internal part is:

$$\begin{aligned} ds_3^2 &= \frac{1}{1+\delta_{mn}x^m x^n}\left(\delta_{ij}+x_i x_j\right)dx^i dx^j , \\ B_{(2)} &= -\frac{1}{2!}\frac{\epsilon_{ijk}x^k}{1+\delta_{mn}x^m x^n}dx^i\wedge dx^j , \\ e^{\Phi} &= (1+\delta_{mn}x^m x^n)^{-1/2} . \end{aligned} \tag{2.3.8}$$

This is indeed the aforementioned NATD geometry.

This prompts the obvious question: how does the geometry prior to T-dualisation (i.e. that of the S^3 with round metric) manifest itself within the EDA setting? To address this we will need to consider the EDA in the IIB-theory section.[7]

To see this, let's look at the Exceptional Drinfeld Algebra more closely. Let's relabel our indices such that now $a = 1,2,3$. Then the only non-zero components of the three-algebra structure constants in this case are

$$\tilde{f}^{ab4}{}_c = -\epsilon^{ab}{}_c \tag{2.3.9}$$

where $\epsilon^{ab}{}_c \equiv \epsilon^{abd}\delta_{dc}$.

Adapted to this we assemble the generators of the EDA as $t_a \equiv T_{a5}$, $t_4 \equiv t_{45}$, $\tilde{t}^a \equiv \frac{1}{2}\epsilon^{abc}T_{bc}$ and $s_a = T_{a4}$ such that the algebra is given by

$$[t_a, t_b] = 0 , \quad [\tilde{t}^a, \tilde{t}^b] = -\epsilon^{ab}{}_c\tilde{t}^c , \quad [t_a, \tilde{t}^b] = -\epsilon^{bc}{}_a t_c , \tag{2.3.10}$$

$$0 = [t_4, t_a] = [t_4, s_a] = [t_4, \tilde{t}^b] , \tag{2.3.11}$$

$$[t_a, s_b] = +\delta_{ab}t_4 , \quad [s_a, s_b] = 0 , \quad [s_a, \tilde{t}^b] = -2\epsilon_a{}^{bc}s_c , \tag{2.3.12}$$

The original M-theory section physical subalgebra is $U(1)^4$ generated by t_a, t_4. In IIA, we have a $U(1)^3$ generated by t_a. In this presentation we now see an additional $SU(2)$ subalgebra generated by $\tilde{t}^{a4} \equiv \frac{1}{2}\epsilon^{abc}T_{bc}$. This non-Abelian algebra is indeed a maximal isotropic in the IIB-theory section specified by the pure spinor with non-zero components $\Lambda^{45} = -\Lambda^{54} = 1$.

[7] This is natural; non-Abelian T-duality will change the chirality from IIB to IIA if three isometry generators are dualised as is the case for $SU(2)$.

Working now in this IIB-theory section it is easy to establish a set of generalised frame fields that realise this EDA. As detailed in the appendix, here the relevant generalised tangent bundle is $E = TM \oplus T^\star M \oplus T^\star M \oplus \Lambda^3 T^\star M$ and we use the notation $A = (a, \alpha_{(1)}, \tilde{\alpha}_{(1)}, \alpha_{(3)})$ to denote its sections (the generalised vectors). Using the type IIB generalised Lie derivative (A.1.10), this algebra can be realised using the following generalised frame:

$$\begin{aligned} E^a = \frac{1}{2}\epsilon^{abc} E_{bc} &= (v^a, 0, 0, 0)\,, \\ E_a = E_{a5} &= (0, l_a, 0, 0)\,, \\ E_{a4} &= (0, 0, l_a, 0)\,, \\ E_{45} &= (0, 0, 0, \mathrm{vol})\,, \end{aligned} \tag{2.3.13}$$

where l_a are the left-invariant one-forms on SU(2), v^a the dual vector fields, and vol is the corresponding volume form.

Here we see that there is a natural block diagonal decomposition of the generalised frame field. Let us consider the top left block i.e. the projections of E^a and E_a to the $O(3,3)$ generalised tangent bundle $TM \oplus T^\star M$. These are exactly of the form of the generalised frames for Poisson-Lie duality [23] in the case that the Drinfeld double is semi-Abelian of the form given in eq. (2.3.10). This is precisely what is required to realise non-Abelian T-duality starting with the round metric on the S^3.[8] The bottom right block, i.e. the projections of E_{a5} and E_{45} to $T^\star M \oplus \Lambda^3 T^\star M$ can be understood as defining a spinor representation of the $O(3,3)$ generalised frame field given by the top left block. We shall discuss this feature in more detail when we return to the full Poisson-Lie duality context.

Relationship to Hohm-Samtleben frame

We would like now to relate the EDA generalised frame described above to previous constructions of SL(5) generalised frames realising the same $CSO(3,0,2)$ gaugings. A particular class of generalised frames realising $CSO(p,q,r)$ gaugings were constructed by Hohm and Samtleben in [57]. For $q = 0$, this frame depends on the coordinates $y^{\underline{i}}$, where $\underline{i} = 1, \ldots, p-1$, which are coordinates on an S^{p-1},[9] and we let $u \equiv \delta_{\underline{i}\underline{j}} y^{\underline{i}} y^{\underline{j}}$. Then, the frame involves both a three-form and a trivector

$$E_a = (u_a, -\iota_{u_a} C_{(3)})\,, \quad E^{ab} = (0, \alpha u^a \wedge u^b) + \lambda^{abc} E_c\,, \tag{2.3.14}$$

[8] What is used here is only an $SU(2)_L$ isometry group, so the considerations here do not directly impose the bi-invariant metric on S^3. This comes about because of the assumption made earlier in the generalised Scherk-Schwarz ansatz that $\bar{m}_{\mathcal{AB}} = \delta_{\mathcal{AB}}$. Choosing other constant $\bar{m}_{\mathcal{AB}}$ will give non-Abelian T-duals and their lifts of the S^3 equipped with metric $ds^2 = g^{ab} l_a \otimes l_b$ and two-form $B = b^{ab} l_a \wedge l_b$ with g^{ab} and b_{ab} constant.

[9] Generalised frames describing sphere reductions in general have been constructed [44] and can be checked also to involve both a three-form and a trivector.

with a vielbein $u^i{}_a \equiv (1-u)^{1/2}\delta^i_a$, a function $\alpha = (1-u)^{1/6}$, and (writing the dualised forms) both a trivector and three-form, given by

$$\lambda_a = ((1-u)^{-1/2}\delta_{\underline{ik}}y^{\underline{k}}, 0)\,, \quad C^i = ((1-u)^{-1/2}y^{\underline{i}}K(u), 0)\,. \tag{2.3.15}$$

For $p = 3, q = 0, r = 2$, $K(u)$ obeys the differential equation $2(1-u)u\partial_u K = (-2+u)K - 1$, and the solution is $K(u) = -1/u$.

For $CSO(3,0,2)$, the four-dimensional physical geometry encoded in this frame is $\mathbb{R}^2 \times S^2$ equipped with

$$\begin{aligned} ds_4^2 &= (dy^3)^2 + (dy^4)^2 + \left(\delta_{\underline{ij}} + \frac{y_{\underline{i}}y_{\underline{j}}}{1-u}\right) dy^{\underline{j}}dy^{\underline{j}}\,, \\ C_{(3)} &= -\epsilon_{\underline{ik}}y^{\underline{k}}(1-u)^{-1/2}(1-\frac{1}{u})dy^{\underline{i}} \wedge dy^3 \wedge dy^4\,. \end{aligned} \tag{2.3.16}$$

Although the three-form looks rather complicated, the field strength is just $F_{(4)} = \mathrm{Vol}(S^2) \wedge dy^3 \wedge dy^4$.

Compactifying the coordinates y^3, y^4, this trivially reduces (on y^4, say) to a IIA configuration with $S^1 \times S^2$ internal space

$$\begin{aligned} ds_3^2 &= (dy^3)^2 + \left(\delta_{\underline{ij}} + \frac{y_{\underline{i}}y_{\underline{j}}}{1-u}\right) dy^{\underline{j}}dy^{\underline{j}}\,, \\ B_{(2)} &= -\epsilon_{\underline{ik}}y^{\underline{k}}(1-u)^{-1/2}(1-\frac{1}{u})dy^{\underline{i}} \wedge dy^3\,, \end{aligned} \tag{2.3.17}$$

and a constant dilaton. This can be T-dualised on y^3, in order to produce a solely metric configuration:

$$ds_3^2 = (d\tilde{y}^3 + \frac{1}{u}(1-u)^{+1/2}\epsilon_{\underline{ij}}y^{\underline{j}}dy^{\underline{i}})^2 + \left(\delta_{\underline{ij}} + \frac{y_{\underline{i}}y_{\underline{j}}}{1-u}\right) dy^{\underline{j}}dy^{\underline{j}}\,. \tag{2.3.18}$$

Taking our sphere coordinates to be $y^1 = \sin\theta\cos\phi$, $y^2 = \sin\theta\sin\phi$, where $\theta \in (0,\pi)$, $\phi \in (0,2\pi)$, then $u = \sin^2\theta$, $1-u = \cos^2\theta$, and $dy^1y^2 - dy^2y^1 = -\sin^2\theta d\phi$. As a result, the geometry becomes

$$ds_3^2 = (d\tilde{y}^3 - \cos\theta d\phi)^2 + d\Omega_2^2\,. \tag{2.3.19}$$

This is the three-sphere S^3 described as a Hopf fibration.

All these backgrounds produce seven-dimensional gaugings which are equivalent up to global SL(5) transformations acting on the generalised fluxes. The complete duality chain between the Hohm-Samtleben frame (2.3.14) and our EDA frame (2.3.8) consists of: reduction from M-theory to IIA, T-duality on the Hopf fibre to IIB, non-Abelian T-duality on S^3 back to IIA, followed by uplift to M-theory. This can be interpreted as a "generalised U-duality" however one that consists of a chain of ordinary plus non-Abelian T-dualities. Part of this duality chain takes place entirely within the EDA setting, but that involving the frame (2.3.14) uses a different construction of generalised frames. We depict the relationships between these geometries and different SL(5) frames in figure 2.2.

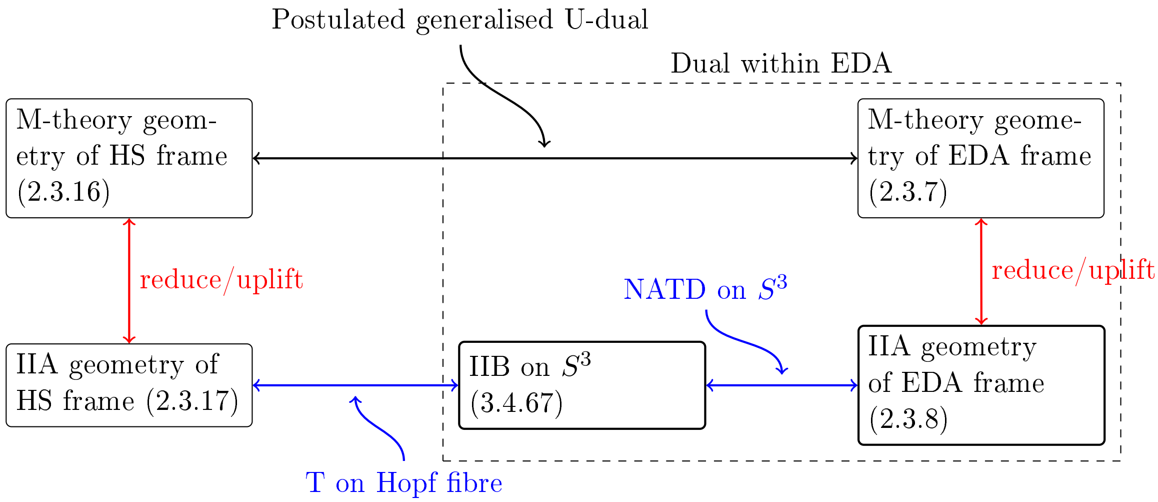

Figure 2.1: Duality chains involving the NATD of S^3 and alternative $CSO(3,0,2)$ frames

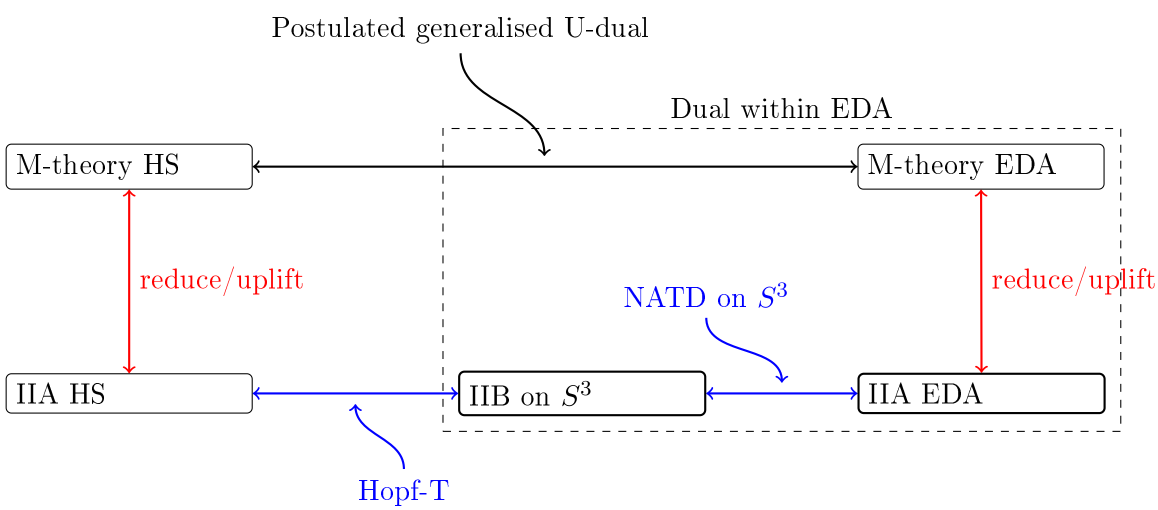

Figure 2.2: Duality chains involving the NATD of S^3 and alternative $CSO(3,0,2)$ frames

Non-metric 3-algebras

A variant of the situation above is to consider the non-metric 3-algebras considered in [64–66] for which

$$\tilde{f}^{ab4}{}_c = \tilde{f}^{ab}{}_c\,, \quad \tilde{f}^{abc}{}_d = \tilde{f}^{ab4}{}_4 = \tilde{f}^{abc}{}_4 = 0\,, \tag{2.3.20}$$

with $\tilde{f}^{ab}{}_c$ the structure constants of a Lie algebra. In terms of the embedding tensor components we have equivalently

$$S_{44} = S_{4c} = \tau_{4c} = 0\,, \quad S_{ab} = -2\epsilon_{cd(a}\tilde{f}^{cd}{}_{b)}\,, \quad \tau_{ab} = -\epsilon_{abc}\tilde{f}^{cd}{}_d\,, \tag{2.3.21}$$

which for $\tau_{ab} = 0$ requires that $\tilde{f}$ define a uni-modular algebra. In this case the EDA is as in (2.3.10)-(2.3.12) after the replacement of $\epsilon^{ab}{}_c \to -\tilde{f}^{ab}{}_c$, and the construction of the IIB-theory section generalised frames goes hrough unchanged. This then provides an EDA embedding of non-Abelian T-duality of uni-modular group manifolds G with respect to a G_L isometry.

For instance, with $S_{ab} = \mathrm{diag}(1,1,-1,0)$, such that we describe $CSO(2,1,2)$ gaugings, we have that the non-metric three algebra is built from $SL(2)$, and that the story above will go through. Recall that we are using δ_{ab} to contract algebra indices (i.e. not the indefinite Killing form) and hence the IIB NATD geometry above will be based on H_3 rather than S^3.

2.3.2 Euclidean 3-algebra and $CSO(4,0,1)$

We now consider the case where S_{ab} is of maximal rank:

$$S_{ab} = 4\,\mathrm{diag}(1,1,1,1)\,, \quad \tau_{ab} = 0\,. \tag{2.3.22}$$

The corresponding three-algebra structure constants are totally anti-symmetric

$$\tilde{f}^{abcd} \equiv \tilde{f}^{abc}{}_e\delta^{ed} = \epsilon^{abcd}\,. \tag{2.3.23}$$

This is well known as the unique solution of the fundamental identity for three-algebra structure constants for Euclidean three-algebras.

The four-dimensional geometry in this case is, with $x^i = (x^1, x^2, x^3, x^4)$,

$$\begin{aligned} ds_4^2 &= (1+\delta_{mn}x^m x^n)^{-2/3}(\delta_{ij} + x_i x_j)dx^i dx^j\,, \\ C_{(3)} &= -\frac{1}{3!}\frac{1}{1+\delta_{mn}x^m x^n}\epsilon_{ijkl}x^l dx^i \wedge dx^j \wedge dx^k\,. \end{aligned} \tag{2.3.24}$$

The field strength is:

$$\begin{aligned} F_{(4)} &= -\frac{1}{4!}\frac{4+2\delta_{mn}x^m x^n}{(1+\delta_{mn}x^m x^n)^2}\epsilon_{ijkl}dx^i \wedge dx^j \wedge dx^k \wedge dx^l\,, \\ &= -(4+2\delta_{mn}x^m x^n)(1+\delta_{mn}x^m x^n)^{-7/6}\mathrm{Vol}_{(4)}\,. \end{aligned} \tag{2.3.25}$$

If we assume our coordinates are non-compact, we can write $x^i = r\hat{x}^i$ with $\hat{x}^i\hat{x}^j\delta_{ij} = 1$ parametrising a three-sphere, hence

$$\begin{aligned} ds_4^2 &= (1+r^2)^{1/3}\Big[dr^2 + \frac{r^2}{1+r^2} d\Omega_3^2\Big]\,, \\ F_{(4)} &= -\frac{4+2r^2}{(1+r^2)^2} r^3 dr \wedge \mathrm{Vol}(S^3)\,. \end{aligned} \tag{2.3.26}$$

Observe that the form of this geometry is very similar to that of the NATD geometry (2.3.7), except now as seen in spherical coordinates we have an $SO(4)$ rather than $SO(3)$ isometry.

Algebra and IIB isotropics

Relabelling such that $a = 1, 2, 3$ as before, we have

$$\tilde{f}^{abc}{}_4 = \epsilon^{abc}\,, \quad \tilde{f}^{ab4}{}_c = -\epsilon^{ab}{}_c\,. \tag{2.3.27}$$

The Exceptional Drinfeld Algebra is given explicitly by the following antisymmetric brackets which indeed describe the algebra $CSO(4,0,1)$ (i.e. ISO(4)):

$$[t_a, t_b] = 0 = [t_a, t_4]\,, \tag{2.3.28}$$

$$[t_a, \tilde{t}^{bc}] = +\epsilon^{bc}{}_a t_4\,, \quad [t_a, \tilde{t}^{b4}] = -\epsilon^{bc}{}_a t_c\,, \quad [t_4, \tilde{t}^{bc}] = -\epsilon^{bcd} t_d\,, \quad [t_4, \tilde{t}^{b4}] = 0\,, \tag{2.3.29}$$

$$[\tilde{t}^{ab}, \tilde{t}^{cd}] = -2\epsilon^{cd[a}\tilde{t}^{b]4}\,, \quad [\tilde{t}^{ab}, \tilde{t}^{c4}] = -2\epsilon^{c[a}{}_d \tilde{t}^{b]d}\,, \quad [\tilde{t}^{a4}, \tilde{t}^{b4}] = -\epsilon^{ab}{}_c \tilde{t}^{c4}\,, \tag{2.3.30}$$

We now want to find all four- and three-dimensional subalgebras of this algebra, and check which of these are isotropic in the sense of (2.2.5). For the Poincaré group in four-dimensions, the classification of all subalgebras was done in [67]. From their results we can extract that the only real isotropic subalgebras of ISO(4) (up to relabelling of the indices) turn out to be the four-dimensional Abelian subalgebra generated by t_a, along with the following three-dimensional subalgebras: SU(2) generated by $\tilde{t}^{a4}$, and ISO(2) generated either by $t_a, t_b, \tilde{t}^{c4}$ with $a \neq b \neq c$ or by t_a, t_4 and $\tilde{t}^{bc}$ with $a \neq b \neq c$. In terms of the undualised generators, these correspond to $\{T_{12}, T_{13}, T_{23}\}$, $\{T_{a5}, T_{b5}, T_{ab}\}$ and $\{T_{a5}, T_{45}, T_{a4}\}$ respectively. All of these are IIB isotropics.

Now we encounter a puzzling feature; there are no geometric IIB uplifts of this $CSO(4,0,1)$ gauging [68]. So it seems that despite the presence of a IIB isotropic we are unable to geometrically furnish this EDA within type IIB exceptional generalised geometry. This does not preclude the possibility of there being *non-geometric* gaugings i.e. ones which depend on both the IIB coordinates and their duals as mentioned in [68]. If this is the case, this suggests the natural home for a "dual" version of this frame would be in some "deformed" version of IIB. This may be analogous to, or perhaps coincide with, the so-called generalised IIB theory [69, 70], which necessarily arises when carrying out certain generalised T-dualities, and which can be realised in double or exceptional field theory by introducing explicit dual coordinate dependence [71, 72], for instance see the DFT implementation of such dualities in [22, 25]. Although this would be interesting to develop further, we would prefer to first understand the possibility of generalised U-duality transformations between the usual 10- and 11-dimensional theories, so we leave this for future work.

Relationship to IIA on S^3

Instead, let us investigate the relationship to the known $CSO(4, 0, 1)$ gauging arising from reduction of type IIA on S^3, or 11-dimensional supergravity on $\mathbb{R} \times S^3$ [73]. Again, the idea is that any alternative frame giving rise to the same gaugings ought to provide a version of generalised U-duality.

Let us again focus on the general $CSO(p, q, r)$ frame of [57], which we wrote down in the previous subsection in (2.3.14) and (2.3.15). For the case $p = 4$, $q = 0$, $r = 1$ we have coordinates $y^i = (y^{\underline{i}}, y^z)$ where $\underline{i} = 1, 2, 3$, and we again define $u \equiv \delta_{\underline{ij}} y^{\underline{i}} y^{\underline{j}}$. The function $K(u)$ appearing in the three-form (2.3.15) is now

$$K = -{}_2F_1[1, 1; 1/2; 1 - u] = -u^{-3/2}(u^{1/2} + (1 - u)^{1/2} \arcsin(1 - u)^{1/2}) \tag{2.3.31}$$

obeying

$$2(1 - u)u\partial_u K = (-3 + 2u)K - 1\,. \tag{2.3.32}$$

This corresponds to the following four-dimensional geometry:

$$\begin{aligned} ds_4^2 &= (dy^z)^2 + \left(\delta_{\underline{ij}} + \frac{y_{\underline{i}} y_{\underline{j}}}{1 - u}\right) dy^i dy^j\,, \\ C_{(3)} &= \frac{1}{2}\epsilon_{\underline{ijk}} y^k (1 - u)^{-1/2}(1 + K(u)) dy^{\underline{i}} \wedge dy^{\underline{j}} \wedge dy^z\,, \end{aligned} \tag{2.3.33}$$

The coordinates $y^{\underline{i}}$ are now seen to parametrise the three-sphere S^3, while the isometry direction y^z parametrises $\mathbb{R}$ (or S^1 if compact). Thanks to the equation (2.3.32) we can show that the four-form flux is constant, and this background is:

$$\begin{aligned} ds_4^2 &= (dy^z)^2 + d\Omega_3^2\,, \\ F_{(4)} &= 2\,\mathrm{Vol}(S^3) \wedge dy^z\,, \end{aligned} \tag{2.3.34}$$

where $d\Omega_3^2$ is the metric on S^3. If one reduces on y^z, this gives IIA on S^3 with H-flux.

We therefore have two constructions of $CSO(4, 0, 1)$ frames. The one based on the Exceptional Drinfeld Algebra corresponds to the geometry (2.3.24). This generalised frame consists of a trivial four-dimensional vielbein and a linear trivector. This geometry therefore has an alternative description as $\mathbb{R}^4$ (or T^4 if compact) carrying M-theory Q-flux, $Q_a{}^{bcd} \sim \tilde{f}^{bcd}{}_a$. The second construction is based on the geometry (2.3.34), that is $\mathbb{R} \times S^3$ (or $S^1 \times S^3$) carrying flux of the four-form. Unlike the case of the $CSO(3,0,2)$ gauging discussed above, there does not appear to be any easy duality chain involving conventional dualities and non-Abelian T-dualities (as in Figure 2.2) that relates the two. Hence we believe them to be related by a novel sort of generalised U-duality transformation.

2.3.3 A Leibniz geometry: $\tau_{ab} \neq 0$

For an example where the EDA is not a Lie algebra, take the non-zero components of τ_{ab} to be

$$\tau_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} n^\gamma \,, \quad \alpha = 1,2,3 \,. \tag{2.3.35}$$

The geometry is easily seen to be

$$\begin{aligned} ds_4^2 &= \left(1 + \frac{1}{4}(n^2x^2 - (n\cdot x)^2)\right)^{-2/3} \left((dx^4)^2 + \delta_{ij}dx^i dx^j + \frac{1}{4}(\epsilon_{ijk}n^i x^j dx^k)^2\right) \,, \\ C_{(3)} &= \frac{1}{2}\frac{1}{1+\frac{1}{4}(n^2x^2 - (n\cdot x)^2)} n_i x_j dx^i \wedge dx^j \wedge dx^4 \,, \end{aligned} \tag{2.3.36}$$

where $n^i \equiv \delta^i_\alpha n^\alpha$, $i = 1,2,3$, $n^2 \equiv \delta_{ij}n^i n^j$, $x^2 \equiv \delta_{ij}x^i x^j$, $n\cdot x \equiv \delta_{ij}n^i x^j$. This three-form is pure gauge.

To explore the algebra, we define $u_\alpha \equiv \epsilon_{\alpha\beta\gamma}\tilde{t}^{\beta\gamma}$, $v^\alpha \equiv \tilde{t}^{\alpha 4}$, $w_\alpha \equiv t_\alpha$ and $\phi \equiv t_4$. In this basis the M-theory section isotropic that we are considering (specified by the pure spinor $\Lambda_{\mathcal{A}} = \delta_{\mathcal{A},5}$) is the subgroup generated by w_α and ϕ with u_α and v^α the 'dual' generators. The algebra is

$$[u_\alpha, u_\beta] = 0 = [w_\alpha, w_\beta] = [\phi, \mathfrak{d}] = [\mathfrak{d}, \phi] \,, \quad [v^\alpha, v^\beta] = v^{[\alpha} n^{\beta]} \,, \tag{2.3.37}$$

$$[w_\alpha, v^\beta] = -[v^\beta, w_\alpha] = \frac{1}{2}(\delta^\beta_\alpha n^\gamma w_\gamma - n^\beta w_\alpha) \,, \quad [w_\alpha, u_\beta] = \frac{1}{2}\epsilon_{\alpha\beta\gamma}n^\gamma \phi \,, \tag{2.3.38}$$

$$[u_\alpha, v^\beta] = -\frac{1}{2}(\delta^\beta_\alpha n^\gamma u_\gamma - n^\beta u_\alpha) \,, \quad [v^\beta, u_\alpha] = -\frac{1}{2}(\delta^\beta_\alpha n^\gamma u_\gamma + n^\beta u_\alpha) \,. \tag{2.3.39}$$

Notice the non-skew (i.e. Leibniz) nature of the algebra is contained entirely in the $[u,v]$ and $[v,u]$ relations, with $[u_\alpha, v^\beta] + [v^\beta, u_\alpha] = -\delta^\beta_\alpha n^\gamma u_\gamma$.

A second M-theory section isotropic sub-algebra is generated by u_α and ϕ, which is again Abelian (this isotropic is that specified by the pure spinor $\Lambda_{\mathcal{A}} = \delta_{\mathcal{A},4}$). Although this simply implements interchange of the 4 and 5 directions, there is no way that this new isotropic can qualify as an EDA. To see this consider the fluxes (A.3.23) which imply

$$Z_{\alpha\beta 4}{}^4 = -\frac{1}{3}\tau_{\alpha\beta} \,, \quad Z_{\alpha\beta 5}{}^5 = \frac{2}{3}\tau_{\alpha\beta} \;. \tag{2.3.40}$$

To interpret this new isotropic as an EDA we must be able to find a $\tau'_{\alpha\beta}$ such that

$$Z_{\alpha\beta 4}{}^4 = \frac{2}{3}\tau'_{\alpha\beta} \,, \quad Z_{\alpha\beta 5}{}^5 = -\frac{1}{3}\tau'_{\alpha\beta} \,, \tag{2.3.41}$$

and there is no such $\tau'_{\alpha\beta}$. This can be traced to the fact that the $[w,v]$ bracket is skew whilst the $[u,v]$ is not. The fact that we can find M-theory isotropics for which the EDA conditions are not satisfied seems to point towards a possible relaxation of some of the constraints of EDA.

The sub-algebra given by v^i and ϕ does not correspond to an M-theory section isotropic but that given by the v^i alone does correspond to a IIB-theory section isotropic.

2.4 Embedding Drinfeld doubles

2.4.1 Decomposing the Exceptional Drinfeld Algebra

The embedding of Drinfeld doubles inside the exceptional Drinfeld algebra has been outlined already in [26]. Here we expand on the discussion in that paper by systematically explaining how the Drinfeld double algebra is extended using a spinor representation, including the explicit form of the generalised frames and constraints that are needed to realise this in generalised geometry. Then, we describe explicitly how this works for the example of the Bianchi II - Bianchi V Drinfeld double, which in [27] was found to be a solution to a coboundary ansatz in the EDA. This realises an explict example where both $f_{ab}{}^c$ and $\tilde{f}^{abc}{}_d$ are non-zero, and demonstrates as well one useful feature of the EDA approach which is that it geometrises the dilaton of Poisson-Lie duality.

We can describe the embedding of Drinfeld doubles by restricting to four-dimensional algebras containing a three-dimensional Lie subalgebra such that, setting $a = 1, 2, 3$,

$$[T_{a5}, T_{b5}] = f_{ab}{}^c T_{c5}\,, \quad [T_{a5}, T_{45}] = f_{a4}{}^4 T_{45}\,, \tag{2.4.1}$$

and by further restricting

$$\tilde{f}^{ab4}{}_c \equiv \tilde{f}^{ab}{}_c \neq 0\,, \quad \tilde{f}^{abc}{}_d = \tilde{f}^{abc}{}_4 = 0 = \tilde{f}^{ab4}{}_4\,, \quad \tau_{45} = 0\,. \tag{2.4.2}$$

Geometrically, we assume that v_a and l^a obey the defining group manifold relations with the three-dimensional structure constants $f_{ab}{}^c$, while we take

$$\lambda^{ab4} = -\pi^{ab}\,, \quad \lambda^{abc} = 0\,, \quad v_4 = \alpha \partial_4\,, \quad l^4 = \alpha^{-1} dx^4\,, \tag{2.4.3}$$

where we now require that α be a function of the three-dimensional coordinates x^i such that $L_{v_a} \ln \alpha \equiv -f_{a4}{}^4$ which ensures starting with (A.3.22) that π^{ab} obeys the condition satisfied by the Poisson-Lie bivector:

$$d\pi^{ab} = -\tilde{f}^{ab}{}_c l^c - 2 l^c f_{cd}{}^{[a} \pi^{b]d}\,. \tag{2.4.4}$$

Starting from (2.2.26), the above restrictions lead to the following NSNS sector geometry:

$$\begin{aligned} ds_{10}^2 &= ds_7^2 + \frac{1}{1+\lambda_c\lambda^c}(\delta_{ab}+\lambda_a\lambda_b) l^a \otimes l^b \,, \\ B_{(2)} &= \frac{1}{2}\frac{1}{1+\lambda_c\lambda^c}\epsilon_{abc}\lambda^a l^b \wedge l^c \,, \\ e^\phi &= \alpha^{-1}(1+\lambda_c\lambda^c)^{-1/2} \,. \end{aligned} \tag{2.4.5}$$

Extracting G_{ab} and B_{ab}, the coefficients of the left-invariant forms, it is quick to check that

$$[(G-B)^{-1}]^{ab} = \delta^{ab} + \pi^{ab} \,, \tag{2.4.6}$$

which is exactly the form required for a Poisson-Lie geometry [11]. (Again, we could extend this beyond the case $g_{ab} = \delta_{ab}$ by taking a more general matrix $\bar{m}_{\mathcal{AB}}$ in (2.2.25).)

We now turn to the decomposition of the exceptional Drinfeld algebra (A.3.25). We group the generators as $t_A = (t_a, t^{a4})$, $\hat{t}^\alpha = (t_4, t^{ab})$. In terms of $O(3,3)$ representations, the set t_A form a vector and the set $\hat{t}^\alpha$ form a Majorana-Weyl spinor. The isotropy condition (2.2.5) is equivalent to:

$$\eta^{AB} t_A t_B\big|_{\mathfrak{g}} = 0 \,, \quad \Gamma^A{}_{\alpha\beta} t_A t^\beta\big|_{\mathfrak{g}} = 0 \,, \tag{2.4.7}$$

where η_{AB} is the usual $O(3,3)$ metric with components $\eta_a{}^b = \eta^b{}_a = \delta^b_a$, $\eta_{ab} = \eta^{ab} = 0$, and Γ_A is an $O(3,3)$ gamma matrix, see appendix A.2.3.

After decomposing the EDA brackets (A.3.23) using (2.4.1) and (2.4.2) (see the explicit details in appendix A.2.3), and regrouping into $SO(3,3)$ covariant quantities, we find the algebra is

$$\begin{aligned} [t_A, t_B] &= F_{AB}{}^C t_C \,, \\ [t_A, \hat{t}^\alpha] &= \frac{1}{4} F_{AB}{}^C (\Gamma^B{}_C)^\alpha{}_\beta \hat{t}^\beta - \frac{1}{2}\tau_A \hat{t}^\alpha \,, \\ [\hat{t}^\alpha, t_A] &= -[t_A, \hat{t}^\alpha] + \frac{1}{4}\left(\frac{1}{6} F_{BCD}(\Gamma_A \Gamma^{BCD})^\alpha{}_\beta - (\Gamma_A\Gamma^B)^\alpha{}_\beta \tau_B\right)\hat{t}^\beta \,, \\ [\hat{t}^\alpha, \hat{t}^\beta] &= 0 \,, \end{aligned} \tag{2.4.8}$$

where the Drinfeld double structure constants $F_{AB}{}^C$, which obey $F_{ABC} \equiv F_{AB}{}^D \eta_{CD} = F_{[ABC]}$, have the expected non-zero components

$$F_{ab}{}^c = f_{ab}{}^c \,, \quad F^{ab}{}_c = \tilde{f}^{ab}{}_c \,, \tag{2.4.9}$$

and we also have[10]

$$\tau_a = -2 f_{a4}{}^4 + f_{ac}{}^c \,, \quad \tau^a = -\tilde{f}^{ac}{}_c \,. \tag{2.4.10}$$

[10] This corresponds to the usual $O(d,d)$ trombone defined using the generalised dilaton d via $\tau_A = E^M{}_A \partial_M(-2d) + \partial_M E^M{}_A$, where $E^M{}_A$ is the $O(d,d)$ generalised vielbein (corresponding to (2.4.15)). For us, $e^{-2d} = \alpha^2 \det l$.

Observe that in the second line of (2.4.8) we have the natural action of the Drinfeld double generators in the spinor representation. Then in the third line we have a novel action of the spinor representation on the algebra generators t_A, which makes this extension of the Drinfeld double into a Leibniz algebra in general.

This is not always possible due to the closure condition, as already noted in this context in [26], which requires

$$f_{ab}{}^c \tilde{f}^{ab}{}_d = 0 \,. \tag{2.4.11}$$

This also follows from the general condition for a half-maximal gauging to admit an uplift to the maximal theory [74], see appendix A.2.1.

Next, we can write down the corresponding generalised frames. Formally, we should decompose the exceptional tangent bundle into IIA language. Letting M denote the three-dimensional manifold, we introduce the doubled tangent bundle $\mathcal{E} \cong TM \oplus T^*M$, whose sections pair vectors and one-forms, plus a bundle $\mathcal{S} \cong \mathbb{R} \oplus \Lambda^2 T^*M$, whose sections pair functions and two-forms. The former bundle gives the $O(3,3)$ vector representation while the latter gives a four-dimensional spinor representation. These appear in the decomposition $\mathbf{10} = \mathbf{6} \oplus \mathbf{4}$ of the antisymmetric representation of SL(5).

Given $V = (v, \lambda_{(1)}) \in \mathcal{E}$ and $S = (\sigma_{(0)}, \sigma_{(2)}) \in \mathcal{S}$ the generalised Lie derivative inherited from the exceptional geometry is:

$$\mathcal{L}_V V' = (L_v v', L_v \lambda'_{(1)} - \iota_{v'} d\lambda_{(1)}) \in \mathcal{E} \,, \tag{2.4.12}$$

$$\mathcal{L}_V S = (L_v \sigma_{(0)}, L_v \sigma_{(2)} + d\lambda_{(1)} \sigma_{(0)}) \in \mathcal{S} \,, \tag{2.4.13}$$

$$\mathcal{L}_S V = (-L_v \sigma_{(0)}, -\iota_v d\sigma_{(2)} - \lambda_{(1)} \wedge d\sigma_{(0)}) \in \mathcal{S} \,, \tag{2.4.14}$$

while $\mathcal{L}_S S' = 0$.

We now reorganise our SL(5) frame $E_{\mathcal{AB}}$ into an $O(d,d)$-vector valued frame $E_A = (E_a, E^a)$, where $E^a = \frac{1}{2}\epsilon^{abc} E_{bc}$, and a spinor-valued frame, $\hat{E}^\alpha = (\hat{E}^0, \hat{E}^{ab})$, where $\hat{E}^0 \equiv E_{45}$, $\hat{E}^{ab} \equiv \frac{1}{2}\epsilon^{abc} E_{c4}$.

The vector-valued frame E_A gives as sections of $TM \oplus T^*M$

$$E_a = (v_a, 0) \,, \quad E^a = (\pi^{ab} v_b, l^a) \,, \tag{2.4.15}$$

which is what we expect for the Drinfeld double [23], while the spinor frame gives as sections of $\mathbb{R} \oplus \Lambda^2 T^*M$

$$\hat{E}^0 = \alpha(1,0) \,, \quad \hat{E}^{ab} = \alpha(\pi^{ab}, l^a \wedge l^b) \,. \tag{2.4.16}$$

In the IIB case, the only change we need to make is to take the spinors to have opposite chirality, i.e. the spinor bundle now consists of odd p-forms, $\bar{\mathcal{S}} \cong T^*M \oplus \Lambda^3 T^*M$. Given $S = (\sigma_{(1)}, \sigma_{(3)}) \in \bar{\mathcal{S}}$

the corresponding generalised Lie derivatives are (inherited from (A.1.10)):

$$\mathcal{L}_V S = (L_v\sigma_{(1)}, L_v\sigma_{(3)} - d\lambda_{(1)} \wedge \sigma_{(1)}) \in \bar{\mathcal{S}}\,, \tag{2.4.17}$$

$$\mathcal{L}_S V = (-\iota_v d\sigma_{(1)}, d\sigma_{(1)} \wedge \lambda_{(1)}) \in \bar{\mathcal{S}}\,, \tag{2.4.18}$$

and again $\mathcal{L}_S S' = 0$. The IIB spinor frame is then

$$\hat{E}^a = \alpha(l^a, 0)\,, \quad \hat{E}^{abc} = \alpha(3\pi^{[ab}l^{c]}, l^a \wedge l^b \wedge l^c)\,. \tag{2.4.19}$$

Although we can always construct the vector and spinor frames for a given Drinfeld double, they will not always obey the Leibniz algebra (2.4.8). Indeed, we have to ensure that the algebra generates constant structure constants, which leads to constraints:

$$\pi^{[ab}\tilde{f}^{c]d}{}_d = 0\,, \quad f_{bc}{}^a\pi^{bc} + 2f_{b4}{}^4\pi^{ab} = 0\,, \tag{2.4.20}$$

which also follow from the constraints (2.2.23) from the point of view of the Exceptional Drinfeld Algebra. In addition, the closure condition (2.4.11) must hold.

In this way we have also recovered a result directly from an M-theory perspective that the RR fields compatible with PL T-duality are essentially constant $O(d,d)$ spinors dressed by the spinor representation of the generalised frame field. This was seen from a DFT perspective in [23, 24] and from a Courant algebroid approach [75].

2.4.2 Example: Bianchi II and V

Bianchi II + U(1) in M-theory

This example of an Exceptional Drinfeld Algebra was found in [27] by requiring the three-algebra structure constants to be determined as a coboundary ansatz. This gives an M-theory solution where the physical subalgebra is Bianchi II + $U(1)$. The Bianchi II algebra, or Heisenberg algebra, can be described in a basis $\{t_1, t_2, t_3\}$ where the single non-vanishing structure constant is $f_{23}{}^1 = 1$. The corresponding group data, including the trivial $U(1)$ factor with generator t_4, and $\alpha = 1$, is:

$$l^a = (dx^1 - x^3dx^2, dx^2, dx^3, dx^4)\,, \quad v_a = (\partial_1, \partial_2 + x_3\partial_1, \partial_3, \partial_4)\,. \tag{2.4.21}$$

A trivector obeying (A.3.22) is

$$\lambda_a = (0, x^3, -x^2, 0)\,, \tag{2.4.22}$$

with $\tilde{f}^{124}{}_2 = \tilde{f}^{134}{}_3 = 1$. From the above, this describes an embedding of a dual three-dimensional subalgebra with structure constants $\tilde{f}^{12}{}_2 = \tilde{f}^{13}{}_3 = 1$, corresponding to the known Bianchi II / Bianchi V Drinfeld double (see [76] for a classification of six dimensional doubles).

The M-theory geometry is

$$
\begin{aligned}
ds_4^2 &= \frac{1}{(1+(x^2)^2+(x^3)^2)^{2/3}}\big((dx^1 - x^3 dx^2)^2 + (1+(x^3)^2)(dx^2)^2 + (1+(x^2)^2)(dx^3)^2 \\
&\qquad - 2x^2x^3 dx^2 dx^3 + (dx^4)^2\big)\,, \\
C_{(3)} &= \frac{1}{1+(x^2)^2+(x^3)^2}\left(\frac{1}{2}d((x^2)^2+(x^3)^2)\wedge dx^1\wedge dx^4 + (x^3)^2 dx^2\wedge dx^3\wedge dx^4\right)\,,
\end{aligned}
\tag{2.4.23}
$$

where $dC_{(3)} = 0$. Reducing on the $U(1)$ direction gives a IIA geometry with

$$
\begin{aligned}
ds_3^2 &= \frac{1}{1+(x^2)^2+(x^3)^2}\big((dx^1 - x^3 dx^2)^2 + (1+(x^3)^2)(dx^2)^2 + (1+(x^2)^2)(dx^3)^2 \\
&\qquad - 2x^2x^3 dx^2 dx^3\big)\,, \\
H_{(3)} &= 0\,, \\
e^{\phi} &= (1+(x^2)^2+(x^3)^2)^{-1/2}\,,
\end{aligned}
\tag{2.4.24}
$$

which matches the known geometry of a Drinfeld double based on the groups Bianchi II and Bianchi V. It is worth remarking that the physical dilaton that arises here was implicitly constrained by the EDA. In conventional T-duality the Buscher procedure can be used to ascertain the form of the dilaton (from the determinant produced by Gaussian elimination of gauge fields). However there is no similar technique for PL duality, and determining the form of the dilaton requires either some heavy work [77] or DFT techniques [24]. The answer here was mandated by the EDA and is in agreement with these approaches.

Bianchi V in IIB

We now have to supply the embedding of the dual Bianchi V description, in type IIB. Now the dual structure constants are $\tilde{f}^{23}{}_1 = 1$ while the physical ones are $f_{12}{}^2 = f_{13}{}^3 = 1$. A choice of group data is

$$
l^a = (d\tilde{x}^1, e^{\tilde{x}^1}d\tilde{x}^2, e^{\tilde{x}^1}d\tilde{x}^3)\,, \quad v_a = (\partial_1, e^{-\tilde{x}^1}\partial_2, e^{-\tilde{x}^1}\partial_3)\,. \tag{2.4.25}
$$

We have to pick a bivector that not only satisfies the usual Poisson-Lie condition (2.4.4) but also the conditions (2.4.20) that ensure the IIB vector plus spinor frame embeds into the Exceptional Drinfeld Algebra. With $f_{a4}{}^4 = 0$, this requires that $\pi^{12} = \pi^{13} = 0$. Then from (2.4.4) we find that π^{23} must obey $d\pi^{23} = (-1 + 2\pi^{23})l^1$, and the solution vanishing at the origin is

$$
\pi^{23} = \frac{1}{2}(1 - e^{2\tilde{x}^1})\,. \tag{2.4.26}
$$

The corresponding physical geometry with string frame metric is

$$\begin{aligned} ds_3^2 &= (d\tilde{x}^1)^2 + \frac{e^{2\tilde{x}^1}}{1+(\pi^{23})^2}\left((d\tilde{x}^2)^2 + (d\tilde{x}^3)^2\right) , \\ B_{(2)} &= -\frac{\pi^{23}e^{2\tilde{x}^1}}{1+(\pi^{23})^2} d\tilde{x}^2 \wedge d\tilde{x}^3 , \\ e^{\phi} &= (1+(\pi^{23})^2)^{-1/2} . \end{aligned} \tag{2.4.27}$$

2.5 Discussion

The goal of this chapter was to make geometrically concrete the algebraic structures introduced in [26, 27]. These "exceptional Drinfeld geometries" provide generalised parallelisable spaces with a non-trivial relationship between the more complicated geometry and the simpler generalised frame based on a group manifold and the trivector. We have now developed an interesting first set of examples where the exceptional Drinfeld algebra can be explicitly connected to geometries.

A primary motivation for the introduction of the Exceptional Drinfeld Algebras was to generalise the Drinfeld double algebras that appear in generalised T-duality. As a confidence-building measure, we have described in detail how to embed $O(3,3)$ Drinfeld doubles and Poisson-Lie T-duality into the SL(5) Drinfeld algebras. We saw that not all Drinfeld doubles can be embedded; that there are constraints that must be obeyed by their structure constants and by the explicit choice of Poisson-Lie bivector; and furthermore that the extension of the Drinfeld double requires introducing a "spinor" representative of the Drinfeld double and defining a non-trivial Leibniz algebra in which this acts in turn on the vector representation.

We also studied simple EDA examples where we only allowed the three-algebra structure constants to be non-zero, $\tilde{f}^{abc}{}_d$. These can all be realised by a simple trivector ansatz, linear in the coordinates. In some sense, these geometries are the analogues of what should be obtained after non-Abelian T-duality, and indeed here we could reproduce the usual non-Abelian T-dual pair involving an S^3.

In addition, this class of geometries can be seen to produce $CSO(p,q,r)$ gaugings of seven-dimensional maximal supergravities (with $r \geq 1$, due to the fact that at least one component of the symmetric gauging vanishes thanks to the definition of the EDA, $S_{55} = 0$). Thus we have in effect a very simple construction of new uplifts for such gaugings. We saw how in the $CSO(3,0,2)$ case, there was a duality chain relating our geometry to the alternative uplift due to [57], involving Hopf T-duality, non-Abelian T-duality, and M-theory uplifts. In the $CSO(4,0,1)$ case, there appears not to be such a chain using existing notions of generalised T-dualities.

We therefore have in this example a novel four-dimensional geometry, which encodes the Euclidean 3-algebra with $\tilde{f}^{abc}{}_d = \epsilon^{abc}{}_d$, and which we propose to identify as a generalised U-dual of M-theory on $\mathbb{R} \times S^3$. The form of this background is strikingly similar to that of the usual non-

Abelian T-dual of S^3, suggesting that the various subtleties with the construction (for instance, how do we determine the range of the coordinates? Should we regard it as U-fold?) can be interpreted similarly as in this familiar case.

The structure of the Exceptional Drinfeld Algebra is based on the existence of isotropic sub-algebras. We had hoped to find examples in which multiple four-dimensional isotropics would be present, which could then be used as the basis for M-theory to M-theory generalised U-dualities within the EDA set-up. Unfortunately, in the cases we have looked at, the conditions of the EDA appear to be very restrictive. Not only does one have to have an isotropic subalgebra (and our experience shows that they are limited in number), the whole EDA is further constrained exactly such that it admits a geometric realisation in terms of just a trivector. The example of section 2.3.3 shows that even when there can be multiple M-theory isotropics, not all of them can be compatible with an EDA. Equally we saw in the $CSO(4,0,1)$ example that one can find dual IIB isotropics that do not appear to admit a geometric generalised frame description

Note that from the IIB perspective, we have not systematically reproduced the EDA from the IIB side but starting with M-theory examples considered IIB descriptions only for those cases. One therefore needs to interpret the full set of EDA structure constants in terms of a IIB construction and check whether all are geometrically realisable using a three-dimensional group manifold plus bivectors, or whether additional geometric ingredients are needed. (Similarly one might also wonder whether any information is lost in going from M-theory to IIA.)

Perhaps ultimately it may be fruitful to consider relaxing some of the axioms we used to define the EDA. By comparison, the relaxation of the Drinfeld double (which we recall has two isotropic sub-algebras) to having only one isotropic subalgebra is vital to describe certain models with H-flux including the λ-deformed WZW [78]. It is likely one can also here find interesting algebras by either relaxing the group structure on $\mathfrak{g}$ or the three-algebra structure on dual generators.

Another limitation we may have been dealing with was simply our choice of dimension. When one goes beyond SL(5) to higher-rank groups (the $E_{6(6)}$ case has been studied in [79]), it is likely that the number of possible constructions and transformations will be much greater. Other restrictions that we would hope to relax in the future would be to consider cases corresponding to less SUSY and to generalise to coset spaces rather than group manifolds.

There are also open questions related to the mathematical description of exponentiation of an EDA, when not a Lie algebra, and the precise formulation of the extended geometry in these cases. This would likely make contact with the approach of [80] in which the physical space is identified with the quotient of an enlarged group manifold by a subgroup.

The algebraic structure of the exceptional Drinfeld algebra necessitated the introduction of a trivector in the generalised parallelisation. It would be interesting to compare this with some other approaches in the literature. For instance, given that the idea of generalised U-duality relies on relating alternative frames giving rise to equivalent gaugings, it would be interesting to compare to the approach of [51] which provides a systematic method for constructing frames given a set of generalised fluxes. This might also provide a method to carry out some of the generalisations mentioned above. Further, it would be interesting to compare this construction with that of [81, 82] where the trivector is viewed as a deformation of a pre-existing geometry.

This study paves the way to understanding the specific features and requirements one needs to define an Exceptional Drinfeld Algebra (EDA), including dimensional limitation and isotropy conditions. This helps us to consider several examples in the following chapters, and provides an indicative classification for further research in the direction of EDA and U-dual solutions.

Chapter 3

Generalised U-dual solutions in supergravity

3.1 Introduction

In this chapter we illustrate a method to take solutions of type IIA supergravity on a three-sphere, with NSNS flux, to new solutions of 11-dimensional supergravity on a four-dimensional space with particular properties. Principal amongst these properties is that the geometry of this space is secretly controlled by an underlying algebraic structure incorporating the structure constants of a three-algebra symmetry. This structure generalises that found in solutions generated by non-abelian T-duality, which produces geometries controlled by an underlying Lie algebra symmetry. We focus on an example where we start with the F1-NS5 near horizon solution of type IIA supergravity, and construct a new 11-dimensional solution involving M2-M5-M5' charges.

The context for our work is the question of how to formulate and use *generalised dualities* in M-theory. The classic formulation of a string or M-theory duality is in terms of an equivalence between theory 1 on space X_1 and theory 2 on space X_2. Conventional (abelian) T-duality corresponds to the case when theory 1 is type IIA string theory, theory 2 is type IIB string theory, and X_1 and X_2 are circles of inverse radius. U-duality can be stated as an equivalence between M-theory on dual d-dimensional tori, or type II theory on $(d-1)$-dimensional tori.

In supergravity, these dualities can be rephrased as expressing the fact that a dimensional reduction or consistent truncation of supergravity 1 on X_1 gives the same lower-dimensional theory as a reduction of supergravity 2 on X_2. This allows duality to be used as a solution generating technique, where solutions of supergravity 1 of the form $M \times X_1$ can be mapped to solutions of supergravity 2 of the form $M \times X_2$, by reducing and uplifting.

Generalised T- and U-duality extend this notion of duality to special classes of dual spaces X_1 and X_2, which are not tori. At a minimum, this is a solution generating method: given a supergravity solution meeting particular conditions, a generalised duality will produce a second supergravity solution related in a particular manner to the first. Whether this extends to a genuine duality of the full (quantum) string or M-theory is far from guaranteed, even in T-duality examples where worldsheet methods can be used to formulate aspects of the duality. However, these techniques have proven their value in supergravity alone as a source of new solutions with applications to holography, integrability and other areas (see [12] for a review and further references). It is perhaps also worth remembering that what is now known as U-duality first appeared – almost accidentally – in supergravity [83], long before the idea of M-theory was developed [1,2].

The most well-appreciated generalisation of T- or U-duality is non-abelian T-duality (NATD) [3]. This has a worldsheet derivation, at least for the transformation of the NSNS sector fields. The basic structure of this duality is that it takes a space with non-abelian isometries, for example a group manifold, to a space with fewer isometries. The dual solution is characterised by an underlying algebraic structure controlled by 'dual' structure constants $\tilde{f}^{ab}{}_c \neq 0$ inherited from the Lie algebra of the original non-abelian symmetry.

Unlike abelian T-duality, the worldsheet path integral derivation of the dual background does not lead to global information, in particular about the range or periodicity of the dual coordinates [4]. It is however possible to find various arguments to globally 'complete' the supergravity solution. For instance, combined with the correct transformations for the RR sector [7], non-abelian T-duality has been extensively applied to generate AdS solutions with interesting CFT duals. A common approach for NATD solutions with an AdS factor is to find a holographic completion by embedding the NATD solution into a supergravity solution with a well-defined holographic interpretation, usually in terms of a quiver field theory stemming from an underlying Hanany-Witten brane configuration [84]. Alternatively, as pointed out in [63,85], non-abelian T-dual solutions could be viewed globally as T-folds.

Both abelian and non-abelian T-duality are special cases of Poisson-Lie T-duality [10,11]. This applies to d-dimensional backgrounds which may in general lack isometries, but which geometrically encode data associated to a $2d$-dimensional Lie algebra called the Drinfeld double. This can be made manifest by adopting a generalised geometric (or double field theory) description [23,24]. For backgrounds admitting Poisson-Lie T-duality there exists a *generalised parallelisation* [43,44] providing a consistent truncation to a lower dimensional gauged supergravity. In general, two inequivalent higher-dimensional solutions admitting consistent truncations to the same lower dimensional theory can be viewed as dual in the sense we are considering. (Indeed, NATD was expressed in terms of consistent truncations [86] some years prior to its doubled geometry formulation [22–25]).

The generalised geometry approach opens the door to the study of new variants of U-duality, by using the exceptional generalised geometry (or exceptional field theory) description of 11-dimensional supergravity. This led to the proposals for Poisson-Lie U-duality and an associated 'exceptional Drinfeld algebra' (EDA) introduced in [26, 27] and further studied from a variety of angles in [35,79,87–91].

Whereas the Drinfeld double naturally encodes a pair of ordinary Lie subalgebras, the content of the EDA is more exotic. The EDA itself is generically a Leibniz rather than a Lie algebra. For M-theory backgrounds, the structure constants of the EDA are assembled from those of a Lie algebra $f_{ab}{}^{c}$ and a 'dual' 3-algebra with structure constants $\tilde{f}^{abc}{}_{d}$ (as well as other n-algebra structure constants if the dimension of the algebra is large enough).

In our paper [87], cases where $\tilde{f}^{abc}{}_{d} \neq 0$ but $f_{ab}{}^{c} = 0$ were studied. These should underlie backgrounds (termed 'three-algebra geometries' in [87]) analogous to those which are generated by non-abelian T-duality. A particularly simple example is the Euclidean 3-algebra in four-dimensions, $\tilde{f}^{abc}{}_{d} \sim \epsilon^{abc}{}_{d}$. The EDA in this case is the Lie algebra CSO$(4,0,1)$, and the generalised geometry construction gives a consistent truncation to seven-dimensional CSO$(4,0,1)$ gauged supergravity. An alternative consistent truncation in this case is provided by type IIA on S^3 with NSNS flux [73]. This gives a solution generating mechanism, whereby type IIA solutions of this form can be consistently truncated to solutions of the seven-dimensional CSO$(4,0,1)$ gauged supergravity, and then uplifted to new solutions of 11-dimensional supergravity using the generalised geometric formulation of [26, 27, 87].

In this chapter, we apply this logic to produce a new 11-dimensional solution starting with a non-extremal pp-F1-NS5 solution of type IIA, after taking the five-brane near horizon limit. Our new 11-dimensional solution has the following properties:

- Just as for non-abelian T-duality, the global properties of the new solution are a priori unknown. It can be described using a non-geometric generalised frame involving a trivector linear in the new four-dimensional dual coordinates, and so one possible global interpretation is as a U-fold. *(See section 3.4.1.)*

- The new solution can be viewed as carrying M2 and M5 brane charges. *(See section 3.4.2.)*

- In the extremal case, it admits a limit in which it becomes $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ foliated over an interval. This solution fits into the general class of M-theory AdS_3 solutions derived in [92]. These solutions are directly inspired by solutions generated by non-abelian T-duality, and provide a global completion of our solution (in this AdS limit), with a known holographic dual and brane interpretation. This is exactly analogous to NATD solutions. *(See section 3.4.3.)*

- The full extremal solution can be viewed as a deformation of the AdS_3 limit generated by a six-vector deformation parameter valued in $\mathrm{E}_{6(6)}$. This deformation is inherited from an $SO(2,2)$ T-duality-valued bivector deformation of the extremal F1-NS5 near horizon solution, which describes the interpolation from the AdS_3 near horizon region to an asymptotic linear dilaton spacetime. In that case, the deformation has been identified as being dual to (a variant of) the $T\bar{T}$ deformation of the dual CFT [93]. This identifies the task of understanding a corresponding field theory deformation dual to our full solution as an interesting open question. *(See section 3.4.4.)*

- The AdS limit of our solution admits a $\frac{1}{2}$-BPS solution of the 11-dimensional Killing spinor equation. *(See section 3.4.5.)*

- Finally, our solution can be used to generate new type IIA solutions by dimensional reduction (and hence other type II solutions by standard dualities). *(See section 3.4.6.)*

3.2 Generalised T- and U-duality

3.2.1 Duality and generalised geometry

We study notions of generalised duality which can be cleanly expressed using techniques from generalised geometry and double/exceptional field theory. Here we give a brief description of the necessary methods. For the d-dimensional 'internal space' X_1 we work with the generalised tangent bundle $TX_1 \oplus \Lambda^{(p)}T^*X_1$. Sections of this are known as generalised vectors and consist of a pair $V = (v, \omega)$ of a vector v and p-form ω. We only need the cases $p = 1$, corresponding to $O(d, d)$ generalised geometry relevant for discussing generalised T-duality in type II supergravity, and $p = 2$, allowing us to describe the SL(5) exceptional generalised geometry relevant for discussion of 11-dimensional supergravity when X_1 is four-dimensional. In both these cases, there is a common formula for the generalised Lie derivative of generalised vectors:

$$\mathcal{L}_V V' = (L_v v', L_v \omega' - \iota_{v'} d\omega) \,. \tag{3.2.1}$$

This captures the local symmetries of X_1, namely diffeomorphisms and gauge transformations of a $(p + 1)$-form. The geometry in the guise of the metric and this $(p + 1)$-form is encoded in a generalised metric, denoted $\mathcal{M}_{MN}$. This can be factorised in terms of a generalised vielbein, $\mathcal{M}_{MN} = E_M{}^A \Delta_{AB} E_N{}^B$. If we are just interested in describing the geometry of X_1 then we may take $\Delta_{AB} = \delta_{AB}$, but in particular solutions on $M \times X_1$ then Δ_{AB} may depend on the coordinates of M and describe scalar fields in the lower dimensional theory on M obtained by reducing on X_1. The inverse generalised vielbein gives a generalised frame E_A, providing a basis for generalised vectors. This frame will generate an algebra under generalised Lie derivatives:

$$\mathcal{L}_{E_A} E_B = -F_{AB}{}^C E_C \,. \tag{3.2.2}$$

If $F_{AB}{}^C$ are constant, then E_A provides a *generalised parallelisation*, which allows for a consistent truncation to a lower-dimensional supergravity.

A second (dual) consistent truncation then corresponds to the existence of an alternative generalised parallelisation built using a frame $\tilde{E}_A$ describing the generalised geometry on X_2. This frame should obey the *same* algebra (3.2.2) (possibly up to some change of basis corresponding to a constant $O(d, d)$ or E_d rotation on the indices A). This allows one to translate the problem of finding inequivalent dual consistent truncations to the problem of finding algebras admitting multiple solutions to the differential equations encoded in (3.2.2). As we will review below, in known variants of generalised or Poisson-Lie T- and U-duality, this can be done algorithmically within certain classes of algebras.

3.2.2 Non-abelian T-duality

The prototypical example of a generalised duality is non-abelian T-duality [3]. This applies to spacetimes with non-abelian isometries. A simple example is to consider a spacetime with an S^3 factor (equipped with the round metric), regarded as the group manifold SU(2). Starting with the worldsheet sigma model, we can gauge the (left) action of the group on itself and (assuming no other fields are turned on) arrive at the following dual background:

$$ds^2 = \frac{\delta_{ij} + x_i x_j}{1 + x^k x_k} \mathsf{d}x^i \mathsf{d}x^j \,, \quad B_{ij} = \frac{\epsilon_{ijk} x^k}{1 + x^m x_m} \,, \quad e^{-2\varphi} = 1 + x^k x_k \,. \tag{3.2.3}$$

The new dual coordinates x^i, $i = 1, 2, 3$ originally appear in the dualisation procedure as Lagrange multipliers imposing the flatness of the gauge field gauging the non-abelian isometry. Unlike in abelian T-duality, path integral arguments do not constrain the periodicity or range of these coordinates [4]: we will discuss two different methods to specify the global completion of NATD solutions below.

Underlying this duality is a pair of generalised frames for the $O(d, d)$ generalised geometry. (We describe this now with reference to the specific SU(2) example, with $d = 3$, but the essential features apply to d-dimensional group manifolds and their duals.) The first describes the consistent truncation on the $\mathrm{S}^3 \cong \mathrm{SU}(2)$ group manifold. It makes use of the following geometric data: the left-invariant forms l^a and dual vectors v_a obeying

$$dl^a = \tfrac{1}{2} f_{bc}{}^a l^b \wedge l^c \,, \quad L_{v_a} v_b = -f_{ab}{}^c v_c \,, \tag{3.2.4}$$

where for SU(2) the algebra index is three-dimensional, $a = 1, 2, 3$, and the structure constants are $f_{ab}{}^c = \epsilon_{ab}{}^c$. The generalised frame $E_A = \{E_a, E^a\}$ gives a basis for sections of $T(\mathrm{S}^3) \oplus T^*(\mathrm{S}^3)$ with

$$E_a = (v_a, 0) \,, \quad E^a = (0, l^a) \,. \tag{3.2.5}$$

Under generalised Lie derivatives, we have the algebra (3.2.2) with

$$F_{AB}{}^C \to \{F_{ab}{}^c = f_{ab}{}^c \,, F^{ab}{}_c = 0 \,, F_{abc} = F^{abc} = 0\} \,. \tag{3.2.6}$$

The second generalised frame describes the dual consistent truncation on the NATD geometry (3.2.3). This is not a group manifold, but it can be described in terms of an underlying Poisson-Lie group structure associated to the group $U(1)^3$ (or $\mathbb{R}^3$) with a non-trivial Poisson-Lie bivector, π^{ab}. The latter here obeys $d\pi^{ab} = -\tilde{f}^{ab}{}_c\tilde{l}^c$, where $\tilde{l}^a$ are trivial left-invariant one-forms, $\tilde{l}^a{}_i = \delta^c_i$ (with dual vectors $\tilde{v}_a{}^i = \delta^i_a$) and $\tilde{f}^{ab}{}_c$ are dual structure constants. For the NATD of SU(2), these also describe the $\mathfrak{su}(2)$ Lie algebra with $\tilde{f}^{ab}{}_c = \epsilon^{ab}{}_c$. We can therefore take a bivector linear in the coordinates $\pi^{ab} = -\epsilon^{ab}{}_c x^c$. The generalised frame $\tilde{E}_A = \{\tilde{E}_a, \tilde{E}^a\}$ gives a basis for sections of the extended tangent bundle of the dual geometry, with

$$E_a = (\tilde{v}_a, 0)\,, \quad E^a = (\pi^{ab}\tilde{v}_b, \tilde{l}^a)\,. \tag{3.2.7}$$

Under generalised Lie derivatives, we have the algebra (3.2.2) with

$$F_{AB}{}^C \to \{F_{ab}{}^c = 0\,, F^{ab}{}_c = \tilde{f}^{ab}{}_c\,, F_{abc} = F^{abc} = 0\}\,. \tag{3.2.8}$$

The use of the generalised frame (3.2.7) allows for a non-geometric interpretation of the global properties of the NATD geometry. As pointed out in [63, 85], if we take the coordinates x^i to be periodic, then under $x^i \sim x^i + \text{constant}$ the bivector π^{ab} shifts by a constant. Such a bivector shift can be viewed as a non-geometric $O(3,3)$ transformation. If we patch the dual solution by such a transformation, it must be regarded as a T-fold.

It is however more common to construct global completions of NATD solutions by leveraging information about brane charges and – for cases where there is an AdS factor in the full spacetime – holographic duals. To illustrate how this works, consider the example of the IIB D1-D5 near horizon solution, for which the spacetime is $\mathrm{AdS}_3 \times \mathrm{T}^4 \times \mathrm{S}^3$, supported by RR flux. The NATD dual geometry is a solution of massive IIA supergravity, with:

$$\mathsf{d}s^2 = \mathsf{d}s^2_{\mathrm{AdS}_3} + \mathsf{d}s^2_{\mathrm{T}^4} + \mathsf{d}\varrho^2 + \tfrac{\varrho^2}{1+\varrho^2}\mathsf{d}s^2_{\mathrm{S}^2}\,, \quad B = \tfrac{\varrho^3}{1+\varrho^2}\mathrm{Vol}_{\mathrm{S}^2}\,, \quad e^{-2\varphi} = 1 + \varrho^2\,, \tag{3.2.9}$$

along with dual RR fields [7]. Here we have adopted spherical coordinates $x^i \to (\varrho, \theta, \phi)$. The issue of the non-compactness of dual coordinates is then concentrated in determining the range of ϱ. This can be done by embedding the NATD solution into a global completion with a well-defined holographic dual and brane interpretation. For the NATD of $\mathrm{AdS}_5 \times \mathrm{S}^5$ obtained in [7] this method was demonstrated in [84], and has since been applied to many examples. For the solution (3.2.9), the requisite completion is provided by the construction and analysis [94–97] of a general class of massive IIA $\mathrm{AdS}_3 \times \mathrm{S}^2$ solutions with $3d$ $\mathcal{N} = (0,4)$ supersymmetry and an SU(2) structure. The NSNS fields take the form:

$$\begin{aligned} ds^2 &= \tfrac{u}{\sqrt{h_4 h_8}}\left(ds^2_{\mathrm{AdS}_3} + \tfrac{h_8 h_4}{4h_8h_4 + u'^2}ds^2_{\mathrm{S}^2}\right) + \sqrt{\tfrac{h_4}{h_8}}ds^2_{\mathrm{T}^4} + \sqrt{\tfrac{h_4 h_8}{u}}d\varrho^2\,, \\ B &= \tfrac{1}{2}\left(-\varrho + \tfrac{uu'}{4h_8h_4 + u'^2} + 2n\pi\right)\mathrm{Vol}_{\mathrm{S}^2}\,, \end{aligned} \tag{3.2.10}$$

This solution exhibits the following general features found in global completions of NATD AdS solutions: The coordinate ϱ takes values in a finite interval which is further divided into subintervals $\varrho \in [\varrho_n, \varrho_{n+1}]$. The functions determining the solution (u, h_4 and h_8) are linear in ϱ. They may however only be piecewise linear, and their slopes can jump from subinterval to subinterval. The 2-form B is modified by a large gauge transformation as one crosses each subinterval. There is a (flat space) dual brane configuration, with some branes wrapping the ϱ direction and others orthogonal and located at the endpoints of the subintervals. This dual brane configuration allows for the identification of a dual quiver field theory. The NATD solution (3.2.9) can be regarded as giving the more general solution in the first subinterval, with $\varrho \in [0, \varrho_1]$, and $u \sim h_4 \sim h_8 \sim \varrho$.

Restricting to the case of vanishing Romans mass, the solutions of [94–97] give ordinary IIA solutions which can be uplifted to M-theory [92], giving a class of 11-dimensional AdS_3 solutions which we will re-encounter later.

3.2.3 Poisson-Lie T- and Poisson-Lie U-duality

Poisson-Lie T-duality Non-abelian T-duality can be viewed as a special case of Poisson-Lie T-duality [10, 11], which applies to spacetimes which may lack isometries. They instead admit an underlying Poisson-Lie group structure, involving a group G equipped not only with left-invariant forms and vectors, but with a Poisson-Lie bivector. Altogether these data obey:

$$dl^a = \tfrac{1}{2} f_{bc}{}^a l^b \wedge l^c \,, \quad L_{v_a} v_b = -f_{ab}{}^c v_c \,, \quad d\pi^{ab} = -\tilde{f}^{ab}{}_c l^c - 2 l^c f_{cd}{}^{[a} \pi^{b]d} \,, \tag{3.2.11}$$

involving simultaneously structure constants for both a Lie algebra $\mathfrak{g}$ and a 'dual' Lie algebra $\tilde{\mathfrak{g}}$. The corresponding spacetime geometry is very efficiently described by a generalised frame with: [23, 24]

$$E_a = (v_a, 0) \,, \quad E^a = (\pi^{ab} v_b, l^a) \,, \quad F_{AB}{}^C \to \{F_{ab}{}^c = f_{ab}{}^c \,, F^{ab}{}_c = \tilde{f}^{ab}{}_c \,, F_{abc} = F^{abc} = 0\} \,. \tag{3.2.12}$$

The case of a standard non-abelian group manifold then has $f_{ab}{}^c \neq 0$, $\tilde{f}^{ab}{}_c = 0$, while the NATD has the reverse. The full doubled Lie algebra (with structure constants $F_{AB}{}^C$) here is known as the Drinfeld algebra. Introducing generators $T_A = \{T_a, \tilde{T}^a\}$ obeying $[T_A, T_B] = F_{AB}{}^C$, we have

$$[T_a, T_b] = f_{ab}{}^c T_c \,, \quad [T_a, \tilde{T}^b] = \tilde{f}^{bc}{}_a T_c - f_{ac}{}^b \tilde{T}^c \,, \quad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{ab}{}_c \tilde{T}^c \tag{3.2.13}$$

The algebra is further equipped with an invariant bilinear form defined by $\eta(T_a, \tilde{T}^b) = \delta_a^b$, and otherwise zero. The subalgebras $\mathfrak{g} = \{T_a\}$ and $\tilde{\mathfrak{g}} = \{\tilde{T}^a\}$ are maximally isotropic with respect to this bilinear form, and duality at the level of the algebra involves changing one maximally isotropic subalgebra for another. This is upgraded to a duality at the level of geometry by constructing a dual generalised frame now built using the left-invariant forms and vectors of $\tilde{G} = \exp \tilde{\mathfrak{g}}$ (hence the frame generates the new maximally isotropic subalgebra as its vector part), together with the corresponding Poisson-Lie bivector encoding the structure constants for $\mathfrak{g}$.

Poisson-Lie U-duality A proposal was made in [26, 27] for the algebra and generalised frames which should describe a notion of Poisson-Lie U-duality. Let us concentrate on the case of $d = 4$, for which the U-duality group is SL(5). The proposal is to consider the natural generalisation of the Poisson-Lie group to the case where the bivector is replaced by a trivector. We then specify left-invariant forms and vectors and this trivector to obey[1]

$$dl^a = \tfrac{1}{2} f_{bc}{}^a l^b \wedge l^c \,, \quad L_{v_a} v_b = -f_{ab}{}^c v_c \,, \quad d\pi^{abc} = \tilde{f}^{abc}{}_d l^d + 3 f_{ed}{}^{[a} \pi^{bc]d} l^e \,, \tag{3.2.14}$$

where now $a, b = 1, \ldots, 4$. This introduces dual structure constants $\tilde{f}^{abc}{}_d$ which can be viewed as defining an antisymmetric three-bracket, associated to a 3-algebra rather than an ordinary Lie algebra.

These can be used to construct a generalised frame for SL(5) generalised geometry. A generalised vector in this case is a pair of a vector and a two-form, and lies in the ten-dimensional (antisymmetric) representation of SL(5). We pick a generalised frame $E_A = (E_a, E^{ab})$, where $E^{ab} = -E^{ba}$, given by

$$E_a = (v_a, 0) \,, \quad E^{ab} = (\pi^{abc} v_c, l^a \wedge l^b) \,. \tag{3.2.15}$$

Computing the algebra of generalised Lie derivatives (3.2.2) one finds an algebra dubbed the exceptional Drinfeld algebra (EDA). In terms of generators $T_A = (T_a, \tilde{T}^{ab})$, this algebra is

$$\begin{aligned} [T_a, T_b] &= f_{ab}{}^c T_c \,, & [\tilde{T}^{ab}, \tilde{T}^{cd}] &= 2 \tilde{f}^{ab[c}{}_e \tilde{T}^{d]e} \,, \\ [T_a, \tilde{T}^{bc}] &= 2 f_{ad}{}^{[b} \tilde{T}^{c]d} - \tilde{f}^{bcd}{}_a T_d \,, & [\tilde{T}^{bc}, T_a] &= 3 f_{[de}{}^{[b} \delta^{c]}_{a]} \tilde{T}^{de} + \tilde{f}^{bcd}{}_a T_d \,. \end{aligned} \tag{3.2.16}$$

Note that these brackets are generically not antisymmetric: the EDA is generically an example of Leibniz rather than a Lie algebra. Closure of the algebra imposes the Jacobi condition for the Lie algebra with structure constants $f_{ab}{}^c$, a cocycle condition involving both $f_{ab}{}^c$ and $\tilde{f}^{abc}{}_d$, and the fundamental identity for three-algebras involving just $\tilde{f}^{abc}{}_d$.

A notion of isotropic subalgebra exists, using now not a bilinear form but a bilinear map $\eta : \mathbf{10} \otimes_{\text{sym}} \mathbf{10} \to \bar{\mathbf{5}}$. The subalgebra $\mathfrak{g} = \{T_a\}$ is isotropic with respect to this definition. However, unlike in the case of the Drinfeld double, we are not guaranteed the existence of a second, dual maximal isotropic subalgebra. Note as well that the 'symmetry' between f and $\tilde{f}$ is now broken, and there are now more dual generators $\tilde{T}^{ab}$ than physical ones T_a.

One could nonetheless proceed to interrogate the notion of non-abelian U-duality, by starting with solutions defined by $f_{ab}{}^c \neq 0$, $\tilde{f}^{abc}{}_d = 0$, and dualising these, as for instance in [88]. However, an alternative goal is to inverse the usual order, and instead look at solutions with $f_{ab}{}^c = 0$, $\tilde{f}^{abc}{}_d \neq 0$.

[1] For simplicity, these formulae assume that $f_{ac}{}^c = 0$ and that an additional scalar present in the generalised frame is constant, as is the case for the example we will study. See appendix A.3.4 for more general formulae.

3.2.4 Dual three-algebras and beyond Poisson-Lie U-duality

The logic of focusing on solutions with ${f_{ab}}^c = 0$, ${\tilde{f}^{abc}}_d \neq 0$ is that they should be in some sense similar to the solutions generated by NATD. Our goal is therefore to construct examples of such solutions, verify whether they are actually 'dual' to known solutions, and verify to what extent this really resembles NATD. Furthermore, such solutions will encode three-algebra structure constants and so are perhaps intrinsically interesting as examples of a relationship between geometry and a non-standard algebraic structure.

In [87], examples of this kind were studied, and a first look at the corresponding '3-algebra geometries' was taken, but without constructing full supergravity solutions. A particularly interesting example is to take:

$$\tilde{f}^{abc}{}_d \propto \epsilon^{abc}{}_d \,. \tag{3.2.17}$$

This is the unique Euclidean 3-algebra. It can be viewed as the direct generalisation of the NATD of SU(2), for which we had $\tilde{f}^{ab}{}_c = \epsilon^{ab}{}_c$. The conditions (3.2.14) can be solved by taking $l^a{}_i = \delta^a_i$, $v_a{}^i = \delta^i_a$ and a linear trivector, $\pi^{abc} \propto \epsilon^{abc}{}_d x^d$, introducing coordinates x^i, $i = 1, \ldots, 4$. The EDA (3.2.16) in this case turns out to be the Lie algebra $\mathrm{CSO}(4,0,1)$.

However, it turned out that it is not possible to find valid dual isotropic subalgebras within this EDA [87]. This precludes using the Poisson-Lie U-duality framework of [26, 27] to construct a dual configuration. As noted in [87], this suggests simply that this framework may just be more restrictive than the T-duality case. In particular, we could relax the condition that the dual isotropic be a subalgebra. For example, we could allow ourselves to consider alternative bases (for the same overall algebra) but for which the selected physical generators T_a obey

$$[T_a, T_b] = \tfrac{1}{2} F_{abcd} \tilde{T}^{cd} \,. \tag{3.2.18}$$

This would be the starting point for defining a "quasi'-EDA.[2]

Equivalently, we may forget about specific algebraic interpretations. The EDA construction allows us to construct a generalised frame realising a consistent truncation from 11-dimensional SUGRA to 7-dimensional $\mathrm{CSO}(4,0,1)$ gauged SUGRA. This consistent truncation is on a non-trivial background geometry, resulting from the generalised frame with the trivector. However, it is already known that this gauged SUGRA can be obtained using a consistent truncation of type IIA on an S^3 with NSNS flux [73]. Viewing this as M-theory on $\mathrm{S}^3 \times I$, we have constant four-form flux, in line with the commutation relation (3.2.18).[3] Hence, we can alternatively find 'generalised U-dual' solutions by starting with solutions of type IIA supergravity to which this consistent truncation can be applied, reducing these to 7 dimensions, and then uplifting them using our EDA generalised

[2] In the case of T-duality, it is possible to relax the condition that the Drinfeld double has two isotropic subalgebras, allowing to describe models with H-flux, such as those studied in the context of certain integrable deformations in [78].

[3] This algebra would be explicitly realised by generalised geometric constructions of this consistent truncation [44, 57] – see [87] for a comparison with the generalised frames of [57] in particular.

frame for this gauging. We will now adopt this procedure and show what it leads to for a simple brane intersecting solution.

3.3 11-dimensional solution from exceptional Drinfeld algebra uplift

3.3.1 Type IIA pp-F1-NS5 and reduction to 7 dimensions

We begin our solution generating procedure by taking as our original solution the non-extremal pp-F1-NS5 solution of type IIA supergravity. After taking the five-brane decoupling limit (as reviewed in appendix A.3.1) to go to the near horizon limit of the five-branes, this solution becomes:

$$\begin{aligned} ds_s^2 &= f_1^{-1}(-f_n^{-1}W\mathsf{d}t^2 + f_n(\mathsf{d}z + \tfrac{1}{2}\tfrac{r_0^2\sinh 2\alpha_n}{f_n r^2}\mathsf{d}t)^2) + R^2W^{-1}\frac{\mathsf{d}r^2}{r^2} + R^2\mathsf{d}s^2_{\mathrm{S}^3} + \mathsf{d}s^2_{\mathrm{T}^4}\,, \\ H_{(3)} &= r_0^2\sinh 2\alpha_1\frac{1}{r^3f_1^2}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r + 2R^2\mathrm{Vol}_{\mathrm{S}^3}\,, \quad e^{-2\varphi} = \tfrac{r^2}{R^2}f_1\,, \end{aligned} \tag{3.3.1}$$

where $W = 1-\frac{r_0^2}{r^2}$, $R^2\equiv N_5 l_s^2$ and

$$f_1 = 1+\tfrac{r_0^2\sinh^2\alpha_1}{r^2}\,,\quad f_n = 1+\tfrac{r_0^2\sinh^2\alpha_n}{r^2}\,,\quad \sinh 2\alpha_1 = \tfrac{2N_1l_s^2}{v}\tfrac{1}{r_0^2}\,,\quad \sinh 2\alpha_n = \tfrac{2N_nl_s^4}{R_x^2 v}\tfrac{1}{r_0^2}\,. \tag{3.3.2}$$

Here N_1 is the number of F1s, N_5 the number of NS5s, N_n the number of units of pp-wave momentum, and the four-dimensional transverse space is taken to be a torus of volume $(2\pi l_s)^4 v$.

We will be particularly interested in the extremal limit. Turning off the pp-wave contribution ($N_n = 0$) the solution in this limit is

$$\begin{aligned} ds_s^2 &= f_1^{-1}(-\mathsf{d}t^2+\mathsf{d}z^2) + R^2\frac{\mathsf{d}r^2}{r^2} + R^2\mathsf{d}s^2_{\mathrm{S}^3} + \mathsf{d}s^2_{\mathrm{T}^4}\,, \\ H_{(3)} &= \frac{2r_1^2}{r^3f_1^2}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r + 2R^2\mathrm{Vol}_{\mathrm{S}^3}\,,\quad e^{-2\varphi} = \tfrac{r^2}{R^2}f_1\,, \end{aligned} \tag{3.3.3}$$

with $f_1 = 1+\frac{r_1^2}{r^2}$, $r_1^2 = N_1l_s^2/v$. This exhibits an interpolation from the near horizon region of the F1 to an asymptotic linear dilaton background. The former corresponds to taking $f_1 = \frac{r_1^2}{r^2}$ and the solution has the form

$$\begin{aligned} ds_s^2 &= \frac{r^2}{r_1^2}(-\mathsf{d}t^2+\mathsf{d}z^2) + R^2\frac{\mathsf{d}r^2}{r^2} + R^2\mathsf{d}s^2_{\mathrm{S}^3} + \mathsf{d}s^2_{\mathrm{T}^4}\,, \\ H_{(3)} &= \frac{2r}{r_1^2}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r + 2R^2\mathrm{Vol}_{\mathrm{S}^3}\,,\quad e^{-2\varphi} = \tfrac{r_1^2}{R^2}\,, \end{aligned} \tag{3.3.4}$$

with the metric being $\mathrm{AdS}_3\times\mathrm{T}^4\times\mathrm{S}^3$. Asymptotically, setting $f_1 = 1$ and defining a coordinate U

by $r = Re^{U/R}$ the solution approaches the pure NS5 near horizon solution:

$$ds_s^2 = -\mathsf{d}t^2 + \mathsf{d}z^2 + dU^2 + R^2 \mathsf{d}s^2_{\mathrm{S}^3} + \mathsf{d}s^2_{\mathrm{T}^4}\,, \quad H_{(3)} = 2R^2 \mathrm{Vol}_{\mathrm{S}^3}\,, \quad e^{-2\varphi} = e^{2U/R}\,, \tag{3.3.5}$$

with a flat metric and a linear dilaton. We will discuss later how this interpolating behaviour is inherited by our new 11-dimensional solution.

Owing to the presence of the S^3 factor with accompanying NSNS flux, the background (3.3.1) can be reduced to a solution of seven-dimensional CSO(4, 0, 1) gauged maximal supergravity using the ansatz of [73]. The necessary part of the truncation ansatz that we need is summarised in appendix A.3.2. Applying this to the solution (3.3.1) gives the seven-dimensional metric, scalars M_{ab} and Φ, and a three-form field strength $\tilde{F}_{(3)}$:

$$\begin{aligned} ds_7^2 &= (r/R)^{4/5} f_1^{2/5} \left(f_1^{-1}(-f_n^{-1} W \mathsf{d}t^2 + f_n(\mathsf{d}z + \tfrac{1}{2}\tfrac{r_0^2 \sinh 2\alpha_n}{f_n r^2}\mathsf{d}t)^2) + R^2 W^{-1}\frac{\mathsf{d}r^2}{r^2} + \mathsf{d}s^2_{\mathrm{T}^4} \right)\,, \\ M_{ab} &= \delta_{ab}\,, \quad \Phi = f_1^{-4/5}(r/R)^{-8/5}\,, \quad \tilde{F}_{(3)} = r_0^2 \sinh 2\alpha_1 \frac{1}{f_1^2 r^3} \mathsf{d}t \wedge \mathsf{d}z \wedge \mathsf{d}r\,. \end{aligned} \tag{3.3.6}$$

All other fields in the ansatz are vanishing. We next identify the data of (3.3.6) with the appropriate SL(5) covariant fields of the CSO(4, 0, 1) gauged supergravity. Take $\mathcal{A} = (a, 5)$ to be a five-dimensional fundamental SL(5) index, and let A denote a ten-dimensional index for the antisymmetric representation. The SL(5) covariant fields are: the SL(5)-invariant metric ds_7^2, a scalar matrix $\mathcal{M}_{\mathcal{AB}}$ parametrising the coset SL(5)/SO(5), and gauge fields in SL(5) representations. The latter include a one-form $\mathcal{A}_\mu{}^A$, in the 10-dimensional representation and a two-form $\mathcal{B}_{\mu\nu\mathcal{A}}$ in the five-dimensional representation, with corresponding field strengths $\mathcal{F}_{\mu\nu}{}^A$ and $\mathcal{H}_{\mu\nu\rho\mathcal{A}}$. The fields (3.3.6) provide a non-trivial scalar matrix and three-form field strength:

$$\mathcal{M}_{\mathcal{AB}} = \begin{pmatrix} \Phi^{-\frac{1}{4}}\delta_{ab} & 0 \\ 0 & \Phi \end{pmatrix}\,, \quad \mathcal{H}_{(3)\mathcal{A}} = (0, \tilde{F}_{(3)})\,. \tag{3.3.7}$$

3.3.2 11-dimensional uplift via exceptional field theory

Having mapped our solution to seven-dimensional gauged supergravity, we now uplift it to a *different* higher-dimensional solution using a distinct consistent truncation corresponding to the exceptional Drinfeld algebra realisation of the CSO(4, 0, 1) algebra [87]. This makes use of the SL(5) covariant reformulation of supergravity provided by SL(5) exceptional field theory (ExFT). To describe this uplift, let y^μ denote seven-dimensional coordinates describing the solution (3.3.6). We introduce an SL(5)-valued *generalised frame field* denoted by $\tilde{E}^M{}_A(x)$ in the ten-dimensional representation or by $\tilde{E}^{\mathcal{M}}{}_{\mathcal{A}}(x)$ in the five-dimensional representation, as well as a scalar function $\Delta(x)$. These depend on a set of four-dimensional coordinates x^i, $i = 1, \ldots, 4$, which will describe the internal space of the new eleven-dimensional solution. The new eleven-dimensional solution has a simple SL(5) covariant construction: we define the ExFT external metric, generalised metric and field strengths by

$$\begin{aligned} g_{\mu\nu}(y,x) &= \Delta^2(x) g_{\mu\nu}(y)\,, \quad \mathcal{M}_{\mathcal{MN}}(y,x) = \tilde{E}^{\mathcal{A}}{}_{\mathcal{M}}(x)\tilde{E}^{\mathcal{B}}{}_{\mathcal{N}}(x)\mathcal{M}_{AB}(y)\,, \\ \mathcal{F}_{(2)}{}^{M}(y,x) &= \Delta(x)\tilde{E}^{M}{}_{A}(x)\mathcal{F}_{(2)}{}^{A}(y)\,, \quad \mathcal{H}_{(3)\mathcal{M}}(y,x) = \Delta^2(x)\tilde{E}^{\mathcal{A}}{}_{\mathcal{M}}(x)\mathcal{H}_{(3)\mathcal{A}}(y)\,. \end{aligned} \tag{3.3.8}$$

It is in fact the combination $E^M{}_A \equiv \Delta \tilde{E}^M{}_A$ that must be used to construct the generalised frame (3.2.15) obeying the generalised parallelisation condition (3.2.2). To realise the CSO(4, 0, 1) algebra we take trivial left-invariant forms and vectors, $l^a{}_i = \delta^a_i$, $v_a{}^i = \delta^i_a$, and a trivector linear in the coordinates x^i. The choice $\pi^{abc} = \frac{1}{R}\epsilon^{abc}{}_d x^d$ reproduces the CSO(4, 0, 1) algebra and the scalar potential arising from the truncation of type IIA on an S^3 of radius R (see appendix A.3.4). Note here we can use δ^a_i to identify curved and flat indices here, for convenience. In terms of the five-dimensional representation of SL(5), this gives a generalised frame:

$$\tilde{E}^{\mathcal{A}}{}_{\mathcal{M}} = \begin{pmatrix} \delta^a_m & 0 \\ -\frac{x_m}{R} & 1 \end{pmatrix}\,, \quad \Delta = 1\,. \tag{3.3.9}$$

Using (3.3.9) and (3.3.8) applied to the background arising from the pp-F1-NS5 solution, we obtain a generalised metric and three-form of the form

$$\mathcal{M}_{\mathcal{MN}} = \begin{pmatrix} \Phi^{-\frac{1}{4}}\delta_{mn} + \Phi\frac{1}{R^2}x_m x_n & -\Phi\frac{1}{R}x_m \\ -\Phi\frac{1}{R}x_n & \Phi \end{pmatrix}\,, \quad \mathcal{H}_{(3)\mathcal{M}} = (-\tfrac{x_m}{R}\tilde{F}_{(3)}, \tilde{F}_{(3)})\,, \tag{3.3.10}$$

while the seven-dimensional ExFT external metric is unchanged. It is then a straightforward matter to convert this to a standard description in terms of the eleven-dimensional metric and four-form field strength using the known ExFT dictionary (see for instance the review [49]), summarised in appendix A.3.3.

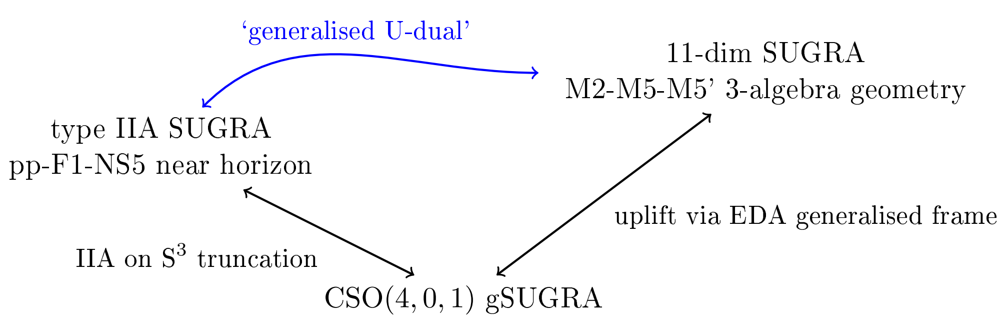

Figure 3.1: The relationship between our solutions

3.3.3 Resulting solution

Using equation (A.3.15) for the parametrisation of the generalised metric allows one to obtain the new internal four-dimensional metric and three-form, with the latter given by

$$C_{ijk} = -\frac{\epsilon_{ijkl} R x^l}{r^2 f_1 + x_m x^m} \,. \tag{3.3.11}$$

As there is no ExFT one-form present, the Kaluza-Klein vector ${A_\mu}^i$ vanishes, and using (A.3.12) one obtains the full 11-dimensional metric

$$\mathsf{d}s^2_{11} = (r^2 f_1 + x_k x^k)^{1/3} \left[\frac{(r^2 f_1)^{1/3}}{R^{4/3}} \left(ds^2_{M_3} + \mathsf{d}s^2_{\mathrm{T}^4} \right) + R^{2/3} (r^2 f_1)^{1/3} \frac{\left(\delta_{ij} + \frac{x_i x_j}{r^2 f_1}\right)}{r^2 f_1 + x^k x_k} \mathsf{d}x^i \mathsf{d}x^j \right] \tag{3.3.12}$$

where

$$ds^2_{M_3} = f_1^{-1}(-f_n^{-1} W \mathsf{d}t^2 + f_n(\mathsf{d}z + \tfrac{1}{2}\tfrac{r_0^2 \sinh 2\alpha_n}{f_n r^2}\mathsf{d}t)^2) + \tfrac{R^2 \mathsf{d}r^2}{r^2 W} \,. \tag{3.3.13}$$

The three-form (3.3.11) and the new four-dimensional part of the metric in equation (3.3.12) closely resemble the two-form and metric appearing in the NATD of S^3 (3.2.3), but now in one dimension higher (this is easiest to see by setting $r^2 f_1 = 1$).

To complete the solution, we use (A.3.16) to extract the remaining components of the four-form field strength (via a dualisation, as $\mathcal{H}_{\mu\nu\rho 5}$ directly gives components of the seven-form field strength). This gives a total four-form field strength:

$$\begin{aligned} F_{(4)} = {} & \frac{r_0^2 \sinh 2\alpha_1}{(r^2 f_1)^2} \frac{r x_i}{R} \mathsf{d}t \wedge \mathsf{d}z \wedge \mathsf{d}r \wedge \mathsf{d}x^i - \frac{r_0^2 \sinh 2\alpha_1}{R^3} \mathrm{Vol}_{\mathrm{T}^4} \\ & + \frac{R \frac{1}{4!} \epsilon_{ijkl} \mathsf{d}x^i \wedge \mathsf{d}x^j \wedge \mathsf{d}x^k}{(r^2 f_1 + x_p x^p)^2} \wedge \left((4 r^2 f_1 + 2 x_q x^q) \mathsf{d}x^l - 4 x^l \partial_r (r^2 f_1) \mathsf{d}r \right) \,. \end{aligned} \tag{3.3.14}$$

The dual seven-form field strength is[4]

$$\begin{aligned} \star F_{(4)} = {} & \frac{r_0^2 \sinh 2\alpha_1}{r^2 f_1 + x_p x^p} \frac{\epsilon_{ijkl} x^l}{R^2} \tfrac{1}{3!} \mathsf{d}x^i \wedge \mathsf{d}x^j \wedge \mathsf{d}x^k \wedge \mathrm{Vol}_{\mathrm{T}^4} \\ & - \frac{r_0^2 \sinh 2\alpha_1}{r f_1 (r^2 f + x_p x^p)} \tfrac{1}{4!} \epsilon_{ijkl} \mathsf{d}t \wedge \mathsf{d}z \wedge \mathsf{d}r \wedge \mathsf{d}x^i \wedge \mathsf{d}x^j \wedge \mathsf{d}x^k \wedge \mathsf{d}x^l \\ & + \frac{2r}{R^4} (2 r^2 f_1 + x_k x^k) \mathsf{d}t \wedge \mathsf{d}z \wedge \mathsf{d}r \wedge \mathrm{Vol}_{\mathrm{T}^4} + \frac{r^2 W}{R^3 r f_1} \frac{x_i}{R} \partial_r (r^2 f_1) \mathsf{d}t \wedge \mathsf{d}z \wedge \mathsf{d}x^i \wedge \mathrm{Vol}_{\mathrm{T}^4} \,. \end{aligned} \tag{3.3.16}$$

[4] We define the Hodge dual of a p-form F via

$$(\star F)_{\mu_1 \dots \mu_{D-p}} = \frac{1}{p!} \sqrt{|g|} \epsilon_{\mu_1 \dots \mu_{D-p} \nu_1 \dots \nu_p} g^{\nu_1 \rho_1} \dots g^{\nu_p \rho_p} F_{\rho_1 \dots \rho_p} \,, \tag{3.3.15}$$

where $\epsilon_{\mu_1 \dots \mu_D}$ denotes the Levi-Civita symbol $\epsilon_{01 \dots D-1} = +1$. This obeys $\star \star F = (-1)(-1)^{p(D-p)} F$.

Note that $(\star F_{(4)})_{ijky^1\dots y^4} = +C_{ijk}F_{y^1\dots y^4}$. We have $\mathsf{d}\star F_{(4)} = +\frac{1}{2}F_{(4)}\wedge F_{(4)}$.

3.4 Analysis of the extremal 11-dimensional solution

We now restrict to the extremal limit and set the pp-wave contribution to zero, making the replacements $W\to 1$, $f_1\to 1+\frac{r_1^2}{r^2}$, $r_0^2\sinh 2\alpha \to 2r_1^2$, $\alpha_n\to 0$. We can also simplify the form of our solution by appropriately rescaling the coordinates as well as the metric and three-form so as to effectively set the constants r_1 and R equal to 1.[5]

3.4.1 Solution as a U-fold

Having made these simplifications, we henceforth study the following solution of 11-dimensional supergravity:

$$\begin{aligned}
\mathsf{d}s^2_{11} &= (r^2 f_1 + x_k x^k)^{1/3}(r^2 f_1)^{1/3}\Big(f_1^{-1}(-\mathsf{d}t^2+\mathsf{d}z^2) + \frac{\mathsf{d}r^2}{r^2} + \mathsf{d}s^2_{\mathrm{T}^4}\Big)\\
&\quad + (r^2 f_1 + x_k x^k)^{-2/3}(r^2 f_1)^{1/3}\left(\delta_{ij} + \frac{x_i x_j}{r^2 f_1}\right)\mathsf{d}x^i\mathsf{d}x^j\\
F_{(4)} &= \frac{2r x_i}{(r^2 f_1)^2}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r\wedge\mathsf{d}x^i - 2\mathrm{Vol}_{\mathrm{T}^4} + \frac{(4r^2 f_1 + 2x_q x^q)}{(r^2 f_1 + x_p x^p)^2}\tfrac{1}{4!}\epsilon_{ijkl}\mathsf{d}x^i\wedge\mathsf{d}x^j\wedge\mathsf{d}x^k\wedge\mathsf{d}x^l\\
&\quad + \frac{x^l\partial_r(r^2 f_1)}{(r^2 f_1 + x_p x^p)^2}\tfrac{1}{3!}\epsilon_{ijkl}\mathsf{d}r\wedge\mathsf{d}x^i\wedge\mathsf{d}x^j\wedge\mathsf{d}x^k\,.
\end{aligned} \tag{3.4.1}$$

with $f_1 = 1+\frac{1}{r^2}$.

If we take the x^i coordinates to be periodic, this should be identified as a U-fold. This is analogous to the interpretation of NATD solutions as T-folds suggested in [63, 85]. For our solution, this U-fold interpretation follows from the form of the EDA frame, which features a trivector depending linearly on the coordinates x^i. The patching for $x^i\sim x^i+$ constant amounts therefore to a shift of this trivector, which is a non-trivial non-geometric U-duality transformation. From (3.3.9) we have

$$\tilde{E}^{\mathcal{A}}{}_{\mathcal{M}}(x^i + Rn^i) = \tilde{E}^{\mathcal{A}}{}_{\mathcal{N}}U^{\mathcal{N}}{}_{\mathcal{M}}\,,\quad U^{\mathcal{N}}{}_{\mathcal{M}} = \begin{pmatrix}\delta^n_m & 0\\ -n_m & 1\end{pmatrix}\,. \tag{3.4.2}$$

If $n_i = \delta_{ij}n^j$ are integers the matrix defines an $\mathrm{SL}(5;\mathbb{Z})$ U-duality transformation. We can describe its action on the four-dimensional internal geometry with metric ϕ_{ij} and three-form C_{ijk} using the generalised metric $\mathcal{M}_{\mathcal{MN}}$, which is a symmetric unit determinant five-by-five matrix, parametrised by the metric and three-form as in (A.3.15). Under $U\in\mathrm{SL}(5)$, this transforms as $\mathcal{M}_{\mathcal{MN}}\to$

[5] To be precise: this involves setting $(t,z,y^I) = R(\tilde{t},\tilde{z},\tilde{y}^I)$ and $(r,x^i) = r_1(\tilde{r},\tilde{x}^i)$, such that $\mathsf{d}s^2_{11} = R^{2/3}r_1^{4/3}\widetilde{\mathsf{d}s}^2_{11}$, $F_{(4)} = Rr_1^2\widetilde{F}_{(4)}$. We then work with $\widetilde{\mathsf{d}s}^2_{11}$ and $\widetilde{F}_{(4)}$, in which no dimensionful constants appear (and drop tildes). This scaling of the metric and gauge field is a symmetry of the equations of motion (the trombone). We can also introduce this scaling directly into the ExFT frame by introducing a constant parameter α as in appendix A.3.4.

$U^{\mathcal{P}}{}_{\mathcal{M}}U^{\mathcal{Q}}{}_{\mathcal{N}}\mathcal{M}_{\mathcal{PQ}}$. In the present case, we factorise $\mathcal{M}_{\mathcal{MN}}(y,x) = \tilde{E}^{\mathcal{A}}{}_{\mathcal{M}}(x)\mathcal{M}_{AB}(y)\tilde{E}^{\mathcal{B}}{}_{\mathcal{N}}(x)$, where as above y denotes 7-dimensional coordinates. This manifestly shows that the generalised metric and hence four-dimensional metric and three-form together transform under the U-duality transformation, or monodromy, in (3.4.2), for periodic x^i.

Associated to this U-fold interpretation is the fact that one can interpret the three-algebra structure constants as (non-geometric) M-theory Q-flux [62]. This is here defined by $Q_a{}^{bcd} \sim \partial_a \pi^{bcd} \sim \tilde{f}^{bcd}{}_a$.

We will not further pursue this U-fold interpretation, but now focus on ordinary geometric properties of the solution (3.4.1).

3.4.2 Solution in spherical coordinates and brane charges

We can rewrite the solution (3.4.1) by changing to spherical coordinates, letting $x^i = \rho\mu^i$ with $\mu^i\mu^j\delta_{ij} = 1$. This is what is usually done for solutions obtained via non-abelian T-duality. The possible non-compactness of the solution will now be determined by the range of ρ. In these coordinates, the metric and field strength of (3.4.1) have the form

$$\begin{aligned}
\mathsf{d}s^2_{11} &= (r^2 f_1 + \rho^2)^{1/3}(r^2 f_1)^{1/3}\left(\frac{1}{f_1}(-\mathsf{d}t^2 + \mathsf{d}z^2) + \frac{\mathsf{d}r^2}{r^2} + \mathsf{d}s^2_{\mathrm{T}^4} + \frac{\mathsf{d}\rho^2}{r^2 f_1}\right) \\
&\quad + (r^2 f_1 + \rho^2)^{-2/3}(r^2 f_1)^{1/3}\rho^2 \mathsf{d}s^2_{\mathrm{S}^3}\,, \\
F_{(4)} &= \frac{2r\rho}{(r^2 f_1)^2}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r\wedge\mathsf{d}\rho - 2\mathrm{Vol}_{\mathrm{T}^4} \\
&\quad + \frac{(4r^2 f_1 + 2\rho^2)}{(r^2 f_1 + \rho^2)^2}\rho^3 \mathsf{d}\rho\wedge\mathrm{Vol}_{\mathrm{S}^3} - \frac{\rho^4\partial_r(r^2 f_1)}{(r^2 f_1 + \rho^2)^2}\mathsf{d}r\wedge\mathrm{Vol}_{\mathrm{S}^3}\,.
\end{aligned} \tag{3.4.3}$$

The dual field strength is

$$\begin{aligned}
\star F_{(4)} &= -\frac{2\rho^4}{r^2 f_1 + \rho^2}\mathrm{Vol}_{\mathrm{S}^3}\wedge\mathrm{Vol}_{\mathrm{T}^4} - \frac{2\rho^3}{r f_1(r^2 f_1 + \rho^2)}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r\wedge\mathsf{d}\rho\wedge\mathrm{Vol}_{\mathrm{S}^3} \\
&\quad + 2r(2r^2 f_1 + \rho^2)\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r\wedge\mathrm{Vol}_{\mathrm{T}^4} + \frac{r\rho}{f_1}\partial_r(r^2 f_1)\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}\rho\wedge\mathrm{Vol}_{\mathrm{T}^4}\,.
\end{aligned} \tag{3.4.4}$$

We can discuss the possible M2 and M5 brane charges carried by this solution. These will be given by integrals[6]

$$q_{M2} = \int J_{\mathrm{Page}}\,,\quad q_{M5} = \int F_{(4)}\,, \tag{3.4.5}$$

where the Page charge density for M2 branes is $J_{\mathrm{Page}} = \star F_{(4)} - \frac{1}{2}C_{(3)}\wedge F_{(4)}$. Let us consider the latter. Let C_{sphere} and C_{torus} denote the restriction of the three-form to the sphere and torus

[6] It is possible to make this more exact and to in particular require quantised charges: we defer this discussion to appendix A.4.

respectively. We have

$$C_{\text{torus}} \wedge \mathsf{d}C_{\text{sphere}} + C_{\text{sphere}} \wedge \mathsf{d}C_{\text{torus}} = \mathsf{d}(C_{\text{torus}} \wedge C_{\text{sphere}}) + 2C_{\text{sphere}} \wedge \mathsf{d}C_{\text{torus}} \,. \tag{3.4.6}$$

An explicit choice of potential is:

$$C_{(3)} = \frac{\rho}{f_1}\mathsf{d}t \wedge \mathsf{d}z \wedge \mathsf{d}\rho - 2c_{(3)} + \frac{\rho^4}{r^2 f_1 + \rho^2}\mathrm{Vol}_{\mathrm{S}^3} \,, \tag{3.4.7}$$

where $\mathsf{d}c_{(3)} = \mathrm{Vol}_{\mathrm{T}^4}$. For this potential, the second term in (3.4.6) cancels with the contribution from $\star F_{(4)}$ such that $J_{\text{Page}} = -\mathsf{d}\left(c_{(3)} \wedge \frac{\rho^4}{r^2 f_1+\rho^2}\mathrm{Vol}_{\mathrm{S}^3}\right)$ and therefore is a total derivative. Hence the M2 charge vanishes up to large gauge transformations. In particular we can consider a large gauge transformation given by

$$C_{(3)} \to C_{(3)} + 4\pi j \mathrm{Vol}_{\mathrm{S}^3} \tag{3.4.8}$$

such that $T_{M2}\int C_{(3)} \to T_{M2}\int C_{(3)} + 2\pi j$, with $j \in \mathbb{Z}$. Using (3.4.6) this means

$$J_{\text{Page}} \to 8\pi j \mathrm{Vol}_{\mathrm{S}^3} \wedge \mathrm{Vol}_{\mathrm{T}^4} \,, \tag{3.4.9}$$

which generates a non-trivial M2 charge.

Next we consider the possible M5 brane charge. We firstly have a non-trivial M5 charge given by integrating $F_{(4)}$ against the torus. The M2 charge generated by the above large gauge transformation will be proportional to this M5 charge.

A further M5 charge, denoted M5', could be obtained by integrating $F_{(4)}$ over a four-cycle involving r, ρ and the sphere directions. Following closely the analysis of NATD solutions in [98], we look for a path in the (r, ρ) directions such that the three-sphere shrinks to zero size at beginning and end of the path, giving a closed four-cycle. This happens at $\rho = 0$; suppose it also happens for some value of $r = r_s$. Then a possible integration is to integrate from $\rho = 0$ to $\rho = \bar\rho$ at fixed $r = \bar r$, and then integrate at fixed $\bar\rho$ from $\bar r$ to $r = r_s$. Letting $C(\rho, r) = \frac{\rho^4}{r^2 f_1+\rho^2}$ we would then have

$$\begin{aligned} \int_{\rho=0}^{\rho=\bar\rho} F_{(4)}\Big|_{r=\bar r} + \int_{r=\bar r}^{r=r_s} F_{(4)}\Big|_{\rho=\bar\rho} &= 2\pi^2\left(C(\bar\rho,\bar r) - C(0,\bar r) + C(\bar\rho, r_s) - C(\bar\rho,\bar r)\right) \\ &= 2\pi^2(C(\bar\rho, r_s) - C(0,\bar r)) = \frac{2\pi^2\bar\rho^4}{r_s^2 f_1(r_s) + \bar\rho^2} \,. \end{aligned} \tag{3.4.10}$$

This is independent of $\bar r$. The issue is now whether one can find a closed four-cycle with the above properties. This issue is linked to the question of finding a global completion of the solution (3.4.3). Indeed, for the full metric (3.4.3) there is no way to close the cycle to give a non-zero value for the above integration. This is a signal that one needs additional ingredients, such as will be discussed in the next subsection at least for the AdS limit.

For the solution with $f_1 = 1$, that we would obtain by starting with the pure NS5 near horizon solution (3.3.5), extra ingredients are not needed. Our new 11-dimensional solution in this case has the form:

$$\begin{aligned} \mathrm{d}s_{11}^2 &= (r^2+\rho^2)^{1/3} r^{2/3}\left(-\mathrm{d}t^2+\mathrm{d}z^2+\frac{\mathrm{d}r^2}{r^2}+\mathrm{d}s_{\mathrm{T}^4}^2+\frac{\mathrm{d}\rho^2}{r^2}\right)+(r^2+\rho^2)^{-2/3} r^{2/3}\rho^2 \mathrm{d}s_{\mathrm{S}^3}^2\,,\\ F_{(4)} &= \mathrm{d}\left(\frac{\rho^4}{r^2+\rho^2}\mathrm{Vol}_{\mathrm{S}^3}\right)\,. \end{aligned} \tag{3.4.11}$$

A valid choice for the above four-cycle is to take $r_s = 0$ for which

$$q_{M5} = 2\pi^2\bar{\rho}^2\,. \tag{3.4.12}$$

Restoring dimensionful constants and requiring this to give a quantised brane charge provides one possible way to determine the range of ρ, fixing it to lie in the finite interval $\rho \in [0, \bar{\rho}]$.

3.4.3 AdS limit and holographic completion

The AdS limit amounts to setting $r^2 f_1 = 1$ in the solution (3.4.3):

$$\begin{aligned} \mathrm{d}s_{11}^2 &= (1+\rho^2)^{1/3}\left(\mathrm{d}s_{\mathrm{AdS}_3}^2+\mathrm{d}\rho^2+\mathrm{d}s_{\mathrm{T}^4}^2\right)+(1+\rho^2)^{-2/3}\rho^2\mathrm{d}s_{\mathrm{S}^3}^2\,,\\ F_{(4)} &= 2\rho\mathrm{Vol}_{\mathrm{AdS}_3}\wedge\mathrm{d}\rho-2\mathrm{Vol}_{\mathrm{T}^4}+\frac{(4+2\rho^2)}{(1+\rho^2)^2}\rho^3\mathrm{d}\rho\wedge\mathrm{Vol}_{\mathrm{S}^3}\,. \end{aligned} \tag{3.4.13}$$

In terms of the original F1-NS5 solution (3.3.3), this corresponds to going to the near horizon region also of the F1.

The solution (3.4.13) fits into a general class of M-theory AdS_3 solutions constructed in [92]. These solutions are of the form $\mathrm{AdS}_3 \times \mathrm{S}^3/\mathbb{Z}_k \times \mathrm{CY}_2$ foliated over an interval. They are closely related to the $\mathrm{AdS}_3 \times \mathrm{S}^2$ solutions (3.2.10) in massive IIA which provide a way to complete the NATD of $\mathrm{AdS}_3 \times \mathrm{T}^4 \times \mathrm{S}^3$. Restricting this class of solutions to ordinary IIA (by setting h_8 constant) allows for an uplift to M-theory. The resulting solutions presented in [92] read as follows:

$$\begin{aligned} \mathrm{d}s_{11}^2 &= \Delta\left(\frac{u}{\sqrt{\hat{h}_4 h_8}}\mathrm{d}s_{\mathrm{AdS}_3}^2+\sqrt{\frac{\hat{h}_4}{h_8}}\mathrm{d}s_{\mathrm{CY}_2}^2+\frac{\sqrt{\hat{h}_4 h_8}}{u}\mathrm{d}\varrho^2\right)+\frac{h_8^2}{\Delta^2}\mathrm{d}s_{\mathrm{S}^3/\mathbb{Z}_k}^2\,, \qquad \Delta=\frac{h_8^{1/2}\left(\hat{h}_4 h_8+\frac{1}{4}u'^2\right)^{1/3}}{\hat{h}_4^{1/6}u^{1/3}}\,,\\ F_{(4)} &= -\mathrm{d}\left(\frac{uu'}{2\hat{h}_4}+2\varrho h_8\right)\wedge\mathrm{Vol}_{\mathrm{AdS}_3}-\partial_\varrho\hat{h}_4\,\mathrm{Vol}_{\mathrm{CY}_2}+2h_8\,\mathrm{d}\left(-\varrho+\frac{uu'}{4\hat{h}_4 h_8+u'^2}\right)\wedge\mathrm{Vol}_{\mathrm{S}^3/\mathbb{Z}_k}\,, \end{aligned} \tag{3.4.14}$$

where the quotiented 3-sphere is written as an S^1 Hopf fibration over an S^2

$$\mathrm{d}s_{\mathrm{S}^3/\mathbb{Z}_k}^2=\frac{1}{4}\left[\left(\frac{\mathrm{d}\psi}{k}+\eta\right)^2+\mathrm{d}s_{\mathrm{S}^2}^2\right]\,, \quad \mathrm{d}\eta=\mathrm{Vol}_{\mathrm{S}^2}\,. \tag{3.4.15}$$

The functions u and $\hat{h}_4$ are again linear functions of ϱ, but h_8 is given by $h_8 = k$ an integer.

To match this to our solution (3.4.13), we relate our radial spherical coordinate ρ to the coordinate ϱ appearing in (3.4.14) via:

$$\rho^2 = 2\varrho\,. \tag{3.4.16}$$

This allows us to write (3.4.13) as

$$\begin{aligned} \mathrm{d}s^2_{11} &= (1+2\varrho)^{1/3}\left(\mathrm{d}s^2_{\mathrm{AdS}_3} + \tfrac{\mathrm{d}\varrho^2}{2\varrho} + \mathrm{d}s^2_{\mathrm{T}^4}\right) + (1+2\varrho)^{-2/3}2\varrho\mathrm{d}s^2_{\mathrm{S}^3}\,,\\ F_{(4)} &= 2\mathrm{Vol}_{\mathrm{AdS}_3}\wedge \mathrm{d}\varrho - 2\mathrm{Vol}_{\mathrm{T}^4} + \frac{8(1+\varrho)}{(1+2\varrho)^2}\varrho\mathrm{d}\varrho\wedge\mathrm{Vol}_{\mathrm{S}^3}\,. \end{aligned} \tag{3.4.17}$$

It is straightforward to confirm that the solution (3.4.17) is included in the class of solutions (3.4.14) for:[7]

$$k=1\,,\quad u(\varrho) = \hat{h}_4(\varrho) = 2\varrho\,, \tag{3.4.18}$$

giving $\Delta = (1+2\varrho)^{1/3}/(2\varrho)^{1/2}$, and taking the CY_2 to correspond to T^4 specifically (we could equally well have considered our solution on either T^4 or K3 from the beginning).

The general class of solutions (3.4.14) then has the necessary properties needed to provide a global completion and holographic dual of the AdS limit of our solution. As specified in [92], one considers the following set-up. The coordinate ϱ takes values in a finite interval $\varrho\in[0,2\pi(P+1)]$, which is divided into subintervals $\varrho\in[2\pi j, 2\pi(j+1)]$ for $j=0,\dots P$. The function u is linear in ϱ, while $\hat{h}_4$ is piecewise linear, with its slope jumping from subinterval to subinterval. It further is taken to obey $\hat{h}_4(0)=\hat{h}_4(2\pi(P+1))=0$, which has the effect of 'ending' the space at the endpoints of the interval (and allows for the computation of M5' brane charge by integrating the four-form flux on the full ρ interval and S^3). The 3-form $C_{(3)}$ is modified by a large gauge transformation (of the form (3.4.8)) as one crosses the endpoints of each subinterval. There is a (flat space) underlying brane configuration, involving M5 branes wrapping the (t,z,r) and S^3 directions, M5' branes wrapping the (t,z) and torus directions, and positioned at $\varrho = 2\pi j$, and M2 branes wrapping the (t,z,ϱ) directions stretched between these M5 branes. This dual brane configuration allows for the identification of a dual quiver field theory, described in [92]. Our solution (3.4.13) can be regarded as giving the more general solution only in the first subinterval, with $\varrho\in[0,2\pi]$. This is exactly analogous to the situation with NATD solutions, and shows that our solution based on dual three-algebra rather than Lie algebra structure constants admits a similar holographic interpretation.

3.4.4 Full solution as a six-vector deformation of AdS limit

We now return to the full solution (3.4.3), in order to explain how it can be viewed as a particular interpolation away from, or deformation of, its AdS_3 limit. To show this, it is helpful (though not

[7]To match precisely, we need to take into account some freedom to change signs of components of our four-form field strength, e.g. the overall sign $C_{(3)}\to -C_{(3)}$ is a matter of convention/orientation, we may also flip the sign of a torus coordinate, or change the sign of the electric B-field components of the original F1-NS5 solution.

strictly necessary) to introduce a dimensionless parameter λ by rescaling the AdS coordinates as

$$t \to \lambda^{-1/2} t\,, \quad z \to \lambda^{-1/2} z\,, \quad r \to \lambda^{+1/2} r\,. \tag{3.4.19}$$

The parameter λ now serves as a book-keeping device for describing the deformation of the AdS limit, which corresponds to $\lambda = 0$. The function f_1 is now $f_1 = 1 + \frac{1}{\lambda r^2}$ and hence the $\lambda \to 0$ limit picks out the near horizon region where one drops the constant term. Evidently for $\lambda = 0$ the rescaling (3.4.19) is singular, but nonetheless the metric and field strength are well-defined. Explicitly, one has:

$$\begin{aligned} \mathsf{d}s_{11}^2 &= (1+\rho^2+\lambda r^2)^{1/3}(1+\lambda r^2)^{-2/3}\left(r^2(-\mathsf{d}t^2+\mathsf{d}z^2)+d\rho^2\right) \\ &\quad + (1+\rho^2+\lambda r^2)^{1/3}(1+\lambda r^2)^{1/3}\left(\frac{\mathsf{d}r^2}{r^2}+\mathsf{d}s_{\mathrm{T}^4}^2\right) \\ &\quad + (1+\rho^2+\lambda r^2)^{-2/3}(1+\lambda r^2)^{1/3}\rho^2 \mathsf{d}s_{\mathrm{S}^3}^2\,, \\ F_{(4)} &= \frac{2r\rho}{(1+\lambda r^2)^2}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r\wedge\mathsf{d}\rho - 2\mathrm{Vol}_{\mathrm{T}^4} + d\left(\frac{\rho^4}{1+\lambda r^2+\rho^2}\mathrm{Vol}_{\mathrm{S}^3}\right)\,. \end{aligned} \tag{3.4.20}$$

This indeed reduces to the AdS limit (3.4.13) for $\lambda = 0$. For $\lambda \neq 0$ one has the full solution (in which we can always undo the rescaling by setting $\lambda = 1$).

The solution (3.4.20) with finite λ can be expressed as an $\mathrm{E}_{6(6)}$-valued deformation of the $\lambda = 0$ limit. This involves an action of $\mathrm{E}_{6(6)}$ on the t, z, ρ and S^3 directions. This $\mathrm{E}_{6(6)}$ transformation should be viewed as a solution generating transformation rather than a U-duality. It may at first seem highly mysterious that the group $\mathrm{E}_{6(6)}$ should appear rather than the SL(5) we used to generate the solution: this can be explained by tracing the origin of this deformation back to an SO(2, 2) T-duality transformation acting just on the (t, z) directions of the original F1-NS5 solution. Our full solution therefore inherits non-trivial structure associated to the action of 'duality' transformations in $2 + 4 = 6$ directions, which singles out $\mathrm{E}_{6(6)}$. We will explain this further below.

An $\mathrm{E}_{6(6)}$ transformation non-trivially mixes the metric with the three-form and six-form potentials, which can be explicitly introduced as:

$$\begin{aligned} C_{(3)} &= \frac{r^2\rho}{1+\lambda r^2}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}\rho + \frac{\rho^4}{1+\lambda r^2+\rho^2}\mathrm{Vol}_{\mathrm{S}^3}\,, \\ C_{(6)} &= -\frac{r^2\rho^3}{2}\left(\frac{1}{1+\lambda r^2}+\frac{1}{1+\lambda r^2+\rho^2}\right)\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}\rho\wedge\mathrm{Vol}_{\mathrm{S}^3}\,. \end{aligned} \tag{3.4.21}$$

The remaining components of $C_{(3)}$ and $C_{(6)}$, which have components along the torus, are electromagnetically dual to those written here. The relevant component of the dual field strength leading to the six-form potential is

$$\star F_{(4)} \supset -\frac{2\rho^3 r\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r\wedge\mathsf{d}\rho\wedge\mathrm{Vol}_{\mathrm{S}^3}}{(1+\lambda r^2)(1+\lambda r^2+\rho^2)} \tag{3.4.22}$$

As $d\star F_{(4)} - \frac{1}{2} F_{(4)} \wedge F_{(4)} = 0$ we then define $C_{(6)}$ by $dC_{(6)} = \star F_{(4)} - \frac{1}{2} C_{(3)} \wedge F_{(4)}$. The gauge choice for $C_{(6)}$ has been chosen so that it is finite for $\lambda \to 0$.

To describe the action of $\mathrm{E}_{6(6)}$, we make a $(6+5)$-dimensional split of the coordinates. Let $x^{\mathsf{i}} = (t, z, \rho, \theta^\alpha)$, where θ^α denote the coordinates on the unit sphere, and let $x^\mu = (r, y^1, \ldots, y^4)$ with the y^i corresponding to the torus coordinates. We decompose the metric as

$$ds^2 = \phi_{\mathsf{ij}} \mathsf{d}x^{\mathsf{i}} \mathsf{d}x^{\mathsf{j}} + |\phi|^{-1/3} g_{\mu\nu} \mathsf{d}x^\mu \mathsf{d}x^\nu \,, \tag{3.4.23}$$

such that the metric $g_{\mu\nu}$ is an $\mathrm{E}_{6(6)}$ invariant given by

$$g_{\mu\nu} \mathsf{d}x^\mu \mathsf{d}x^\nu = r^{4/3} \rho^2 (\det g_{\mathrm{S}^3})^{1/3} \left(\frac{\mathsf{d}r^2}{r^2} + \mathsf{d}s^2_{\mathrm{T}^4} \right) . \tag{3.4.24}$$

In particular, it is independent of λ.

The metric ϕ_{ij} transforms alongside the three-form components C_{ijk} and the six-form component $C_{\mathsf{ijklmn}} \equiv C \epsilon_{\mathsf{ijklmn}}$. The $\mathrm{E}_{6(6)}$ covariant object containing these fields is a 27×27 generalised metric. This can be written as [99, 100]

$$\mathcal{M}_{MN}(\phi, C_{(3)}, C_{(6)}) = U_M{}^K \bar{\mathcal{M}}_{KL} U_N{}^L \,, \quad \bar{\mathcal{M}}_{MN} = |\phi|^{1/3} \begin{pmatrix} \phi_{\mathsf{ij}} & 0 & 0 \\ 0 & 2\phi^{\mathsf{i}[\mathsf{j}} \phi^{\mathsf{j}']\mathsf{i}'} & 0 \\ 0 & 0 & (\det \phi)^{-1} \phi_{\mathsf{ij}} \end{pmatrix} , \tag{3.4.25}$$

$$U_M{}^N = \begin{pmatrix} \delta_{\mathsf{i}}{}^{\mathsf{j}} & -C_{\mathsf{ijj}'} & +\delta_{\mathsf{i}}{}^{\mathsf{j}} C + \frac{1}{4!} \epsilon^{\mathsf{jk}_1 \ldots \mathsf{k}_5} C_{\mathsf{ik}_1\mathsf{k}_2} C_{\mathsf{k}_3\mathsf{k}_4\mathsf{k}_5} \\ 0 & 2\delta^{\mathsf{ii}'}_{\mathsf{jj}'} & -\frac{1}{3!} \epsilon^{\mathsf{ii}'\mathsf{jk}_1\mathsf{k}_2\mathsf{k}_3} C_{\mathsf{k}_1\mathsf{k}_2\mathsf{k}_3} \\ 0 & 0 & \delta_{\mathsf{i}}{}^{\mathsf{j}} \end{pmatrix} . \tag{3.4.26}$$

Here the 27-dimensional $\mathrm{E}_{6(6)}$ fundamental index decomposes as $V^M = (V^{\mathsf{i}}, V_{\mathsf{ii}'}, V^{\bar{\mathsf{i}}})$ where $V_{\mathsf{ii}'} = -V_{\mathsf{i}'\mathsf{i}}$ and $V^M W_M \equiv V^{\mathsf{i}} W_{\mathsf{i}} + \frac{1}{2} V_{\mathsf{ii}'} W^{\mathsf{ii}'} + V^{\bar{\mathsf{i}}} W_{\bar{\mathsf{i}}}$. There are thus two six-dimensional vector indices: the second one can be viewed as coming from a dualisation of five-form indices $V^{\bar{\mathsf{i}}} \equiv \frac{1}{5!} \epsilon^{\mathsf{ij}_1 \ldots \mathsf{j}_5} V_{\mathsf{j}_1 \ldots \mathsf{j}_5}$.[8]

It is straightforward to evaluate the generalised metric for the six-dimensional metric and form-fields obtained from (3.4.20). Some general formulae applicable to situations where the six-dimensional metric and form-fields admit a (3+3)-dimensional decomposition are recorded in appendix A.3.5. One finds that the generalised metric depends *linearly* on λ, and furthermore that the λ dependence can be factorised via an $\mathrm{E}_{6(6)}$-valued transformation involving a six-vector parameter. Generally, we can introduce an $\mathrm{E}_{6(6)}$-valued matrix describing deformations involving a trivector

[8]Here both $\epsilon^{012345} = \epsilon_{012345} = +1$ are Levi-Civita symbols defined without relative minus signs for convenience.

Ω^{ijk} and a six-vector $\Omega^{\mathsf{ijklmn}} \equiv \Omega\epsilon^{\mathsf{ijklmn}}$, such that [100]

$$\widetilde{U}_M{}^N = \begin{pmatrix} \delta_{\mathsf{i}}{}^{\mathsf{j}} & 0 & 0 \\ -\Omega^{\mathsf{ii'j}} & 2\delta^{\mathsf{ii'}}_{\mathsf{jj'}} & 0 \\ \delta_{\mathsf{i}}{}^{\mathsf{j}}\Omega + \frac{1}{4!}\epsilon_{\mathsf{ik_1\dots k_5}}\Omega^{j\mathsf{k_1k_2}}\Omega^{\mathsf{k_3k_4k_5}} & -\frac{1}{3!}\epsilon_{\mathsf{ijj'k_1k_2k_3}}\Omega^{\mathsf{k_1k_2k_3}} & \delta_{\mathsf{i}}{}^{\mathsf{j}} \end{pmatrix} . \tag{3.4.27}$$

Again using the formulae in appendix A.3.5, it can be straightforwardly checked that the generalised metric describing the background (3.4.20) admits a factorisation

$$\mathcal{M}_{MN}(\lambda) = \widetilde{U}_M{}^K(\lambda)\mathcal{M}_{KL}(\lambda=0)\widetilde{U}_N{}^L(\lambda) \tag{3.4.28}$$

where $\widetilde{U}_M{}^N(\lambda)$ has the form of (3.4.27) with

$$\Omega^{\mathsf{ijk}} = 0\,, \quad \Omega = -\frac{\lambda}{2\rho^3\sqrt{\det g_{\mathrm{S}^3}}}\,, \tag{3.4.29}$$

where $\sqrt{\det g_{\mathrm{S}^3}}$ denotes the volume element on the unit three-sphere. Hence the factorisation (3.4.28) demonstrates that the full solution (3.4.20) is a six-vector deformation of the $\lambda = 0$ background corresponding to the AdS limit.

The fact that the deformation parameter is non-constant can be understood by viewing this form of the deformation as involving a change of coordinates as well as a constant $\mathrm{E}_{6(6)}$ transformation. This change of coordinates is just that which defines Cartesian coordinates x^i in place of the 'spherical' coordinates (ρ, θ^α). In terms of the Cartesian coordinates one has simply:

$$\Omega^{tzijkl} = -\frac{\lambda}{2}\epsilon^{ijkl}\,. \tag{3.4.30}$$

It is still non-trivial that this is a solution generating transformation, as the full solution depends on the x^i coordinates, and so we are not in a situation with isometries to which we would automatically be entitled to apply U-duality transformations. The six-vector deformation however commutes with the EDA generalised frame containing the trivector $\Omega^{ijk} \sim \epsilon^{ijkl}x_l$. Prior to applying the EDA generalised frame, what we have is an 11-dimensional configuration (that is not a solution) which already admits the six-vector factorisation.

This follows directly from the properties of the original F1-NS5 extremal solution. Using the same coordinate redefinition that introduces the parameter λ, the F1-NS5 extremal solution (3.3.3) can be written as[9]

$$ds_s^2 = \frac{r^2}{1+\lambda r^2}(-\mathsf{d}t^2 + \mathsf{d}z^2) + \frac{\mathsf{d}r^2}{r^2} + \mathsf{d}s^2_{\mathrm{S}^3} + \mathsf{d}s^2_{\mathrm{T}^4}\,, \quad B_{tz} = \frac{r^2}{1+\lambda r^2}\,, \quad e^{-2\varphi} = 1 + \lambda r^2\,. \tag{3.4.31}$$

The λ dependence now corresponds to an SO(2,2) T-duality deformation acting on the (t,z) direc-

[9]This rewriting is inspired by [93, 101].

tions. This is seen by passing to the appropriate SO(2,2) covariant description via a generalised metric

$$\mathcal{H}_{MN}(\lambda) = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} = \begin{pmatrix} 0 & Z \\ Z & (r^{-2}+\lambda)\eta \end{pmatrix} , \quad Z \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \eta \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{3.4.32}$$

factorising as

$$\mathcal{H}_{MN}(\lambda) = U_M{}^K(\lambda)\mathcal{H}_{KL}(\lambda=0)U_N{}^L(\lambda) , \quad U_M{}^N = \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} , \quad \beta \equiv \frac{\lambda}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{3.4.33}$$

The deformation matrix β has an interpretation as a bivector β^{ij}. (This can alternatively be seen as a TsT transformation.) In addition, the SO(2,2) invariant generalised dilaton is $e^{-2\varphi}\sqrt{|\det(g)|} = r^2$ and is independent of λ.

When we apply the reduction ansatz for type IIA on S^3 to the F1-NS5 background, the field strength component $H_{tzr} = \partial_r B_{tz}$ becomes the $\mathcal{A} = 5$ component of the SL(5) covariant field strength $\mathcal{H}_{(3)\mathcal{A}}$. On uplifting to an eleven-dimensional solution (using the coordinates x^i), this leads to the identification $F_{tzrijkl} \sim H_{tzr}\epsilon_{ijkl}$ giving a non-trivial dual seven-form field strength. Hence the B-field component B_{tz} induces the component C_{tzijkl} of the eleven-dimensional dual six-form. Accordingly, the bivector deformation β^{tz} becomes the six-vector deformation $\Omega^{tzijkl} = \beta^{tz}\epsilon^{ijkl}$. The smallest U-duality group capable of describing such a deformation is $E_{6(6)}$, and this provides the exact explanation for why $E_{6(6)}$ appears.

The structure of the F1-NS5 solution appearing here is associated to some intriguing physics. The solution can be viewed as interpolating from an AdS_3 geometry to a linear dilaton spacetime, holographically dual to Little String Theory [102, 103]. This interpolation, realised above via the bivector deformation, has been argued to correspond to a single-trace $T\bar{T}$ deformation of the dual CFT_2 [93], and has a worldsheet interpretation as a marginal current-current coupling. We might therefore expect that our full solution captures again a deformation related to $T\bar{T}$ of the CFTs dual to the AdS_3 limit of our solution (these are the quiver field theories described in [92]). Making this precise would be interesting future work.

A final comment here is that deformations of the form (3.4.27) generically lead to terms quadratic in the six-vector deformation unless the upper left block of the generalised metric vanishes, $\mathcal{M}_{\mathsf{ij}} = 0$. This block is of the form $\mathcal{M}_{\mathsf{ij}} \sim (\phi + C_{(3)}^2 + (C_{(6)} + C_{(3)}^2)^2)_{\mathsf{ij}}$ and so involves terms quadratic $C_{(6)}$ as well as both quadratic and quartic in $C_{(3)}$. Rather remarkably the gauge choice made above for the three- and six-form is such that here $\mathcal{M}_{\mathsf{ij}} = 0$.

3.4.5 Supersymmetry

In this section we discuss the supersymmetry of the AdS_3 limit (3.4.13) of our solution. The Killing spinor equation in our conventions[10]

$$\delta_\epsilon \psi_\mu = 2D_\mu \epsilon + \tfrac{i}{144}(\Gamma^{\nu\rho\sigma\lambda}{}_\mu - 8\Gamma^{\rho\sigma\lambda}\delta^\nu_\mu)\epsilon F_{\nu\rho\sigma\lambda} = 0\,. \tag{3.4.34}$$

We will proceed to solve this explicitly, finding a $\frac{1}{2}$-BPS solution (3.4.61). We denote the AdS coordinates by (t, z, r), the torus coordinates by y^i, $i = 1, \dots, 4$ and the (standard) three-sphere coordinates by (χ, θ, φ). Unless otherwise indicated, in the below equations the indices on the gamma matrices should be assumed to be flat.

We first assume that ϵ is independent of the torus coordinates y^i. Then the $\mu = y^i$ components of (3.4.34) provide an algebraic condition on ϵ:

$$\Big[\rho(1+\rho^2)^{-1}\Gamma_\rho - \frac{i}{2}(1+\rho^2)^{-1/2}\Big(2\rho\Gamma^{tzr\rho} - 4\Gamma^{y_1\dots y_4} + 4(1+\frac{1}{2}\rho^2)(1+\rho^2)^{-1/2}\Gamma^{\rho\chi\theta\varphi}\Big)\Big]\epsilon = 0\,. \tag{3.4.35}$$

The AdS components of (3.4.34) give differential equations

$$D_{\hat{m}}\epsilon + \tfrac{1}{6}\Gamma_{\hat{m}} X \epsilon = 0\,, \tag{3.4.36}$$

where

$$X = \Big(-(1+\rho^2)^{-1}\rho\Gamma_\rho + i(1+\rho^2)^{-1/2}\left(-2\rho\Gamma^{tzr\rho} - \Gamma^{y_1\dots y_4} + 2(1+\tfrac{1}{2}\rho^2)(1+\rho^2)^{-1/2}\Gamma^{\rho\chi\theta\varphi}\right)\Big)\,. \tag{3.4.37}$$

In (3.4.36) $\hat{m}$ denote curved AdS indices. The spin connection components are $D_{\hat{r}}\epsilon = \partial_r\epsilon$ and $D_{\hat{a}}\epsilon = \partial_a\epsilon - \frac{1}{2}\Gamma_{ar}\epsilon$, with $\hat{a}$ labelling the t and z directions, and $\Gamma_{\hat{r}} = r^{-1}\Gamma_r$, $\Gamma_{\hat{a}} = r\Gamma_a$ where Γ_r and Γ_a are the gamma matrices with flat indices. The form of the r-dependence of the $\hat{m} = r$ equation implies that the r-dependence of ϵ has to be of the form r^β, with a matrix β to be determined later, leading to a further algebraic condition on ϵ. Indeed, letting explicitly $\epsilon = r^\beta\tilde{\epsilon}$, where $\tilde{\epsilon}$ depends on t, z and the other spacetime coordinates, we get an equation

$$(\beta + \tfrac{1}{6}\Gamma_r X)\epsilon = 0\,. \tag{3.4.38}$$

It follows that $D_{\hat{m}}\epsilon = -\Gamma_{\hat{m}}\Gamma_r\beta\epsilon$. For the (t, z) components we get

$$\partial_a\epsilon = \Gamma_{ar}(-\beta + \tfrac{1}{2})\epsilon \Rightarrow \partial_a\tilde{\epsilon} = r^{-\beta}\Gamma_{ar}r^\beta\tfrac{1}{2}(1-2\beta)\tilde{\epsilon}\,. \tag{3.4.39}$$

We have an r-independent expression on the left hand side, and so by our assumptions the right hand side of has to be r-independent as well, thus, differentiating the right hand side with respect

[10] We follow [14] so that $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$ with η_{ab} having mostly minus signature.

to r we end up requiring the following expression to vanish:

$$r^{-\beta}\Big(\Gamma_{ar} - [\beta, \Gamma_{ar}]\Big)(1-2\beta)r^{\beta}\tilde{\epsilon} = 0\,, \tag{3.4.40}$$

which can be achieved if

$$\Big(\Gamma_{ar} - [\beta, \Gamma_{ar}]\Big)(1-2\beta) = 0\,. \tag{3.4.41}$$

If β commutes with Γ_{ar} then the only solution is $\beta = \frac{1}{2}I$. Alternatively, if β anticommutes with Γ_{ar}, then we can extract Γ_{ar} from the equation again leading to

$$\Big((2\beta)^2 - 1\Big) = 0\,, \tag{3.4.42}$$

which tells us 2β should square to a unit matrix. This condition and that of anticommuting with Γ_{tr} and Γ_{zr} is compatible with multiple choices for β, for instance $2\beta = \pm\Gamma_{tz}$, $2\beta = \pm i\Gamma_r$, $2\beta = \pm\Gamma_{tzy_1\dots y_4}$. However, not all options will lead to a non-trivial solution for $\tilde{\epsilon}$, and some of them have fewer supersymmetries than others, as we will see shortly.

Now let's assemble and make sense of the algebraic conditions on ϵ. We can rewrite (3.4.38) as

$$\Big[2\Gamma_r\beta + \frac{1}{3}\rho(1+\rho^2)^{-1}\Gamma_\rho - \frac{i}{6}(1+\rho^2)^{-1/2}\Big(4(1+\frac{1}{2}\rho^2)(1+\rho^2)^{-1/2}\Gamma^{\rho\chi\theta\varphi} - 2\Gamma^{y_1\dots y_4} - 4\rho\Gamma^{tzr\rho}\Big)\Big]\epsilon = 0 \tag{3.4.43}$$

Subtracting $\frac{1}{3}$ (3.4.35) from (3.4.43) we get:

$$\Big[2\Gamma_r\beta + i(1+\rho^2)^{-1/2}\big(\rho\Gamma^{tzr\rho} + \Gamma^{y_1\dots y_4}\big)\Big]\epsilon = 0 \tag{3.4.44}$$

This (for suitable β) will provide a coordinate-dependent projector condition on ϵ, similar to that appearing in non-abelian T-dual solutions [7]. We can also deduce a second projector condition. Let's first split the $\Gamma^{\rho\chi\theta\varphi}$ and Γ_r parts of (3.4.43) as

$$\begin{aligned}&\frac{1}{3}\Big[2\Gamma_r\beta + i(1+\rho^2)^{-1/2}\Big(\rho\Gamma^{tzr\rho} + \Gamma^{y_1\dots y_4}\Big)\Big]\epsilon \\ &\quad + \frac{2}{3}\Big[2\Gamma_r\beta + \rho(1+\rho^2)^{-1}\Gamma_\rho - i\Gamma^{\rho\chi\theta\varphi} - i(1+\rho^2)^{-1}\Gamma^{\rho\chi\theta\varphi} - i(1+\rho^2)^{-1/2}\Gamma^{y_1\dots y_4}\Big]\epsilon = 0\end{aligned} \tag{3.4.45}$$

the first line of which is exactly (3.4.44) thus vanishes. We can write the second line as

$$\Big[2\Gamma_r\beta - i\Gamma^{\rho\chi\theta\varphi} + \Gamma^{tzr}(1+\rho^2)^{-1}\Big(\rho\Gamma^{tzr\rho} + i\Gamma^{tzr\rho\chi\theta\varphi}\Big) - i(1+\rho^2)^{-1/2}\Gamma^{y_1\dots y_4}\Big]\epsilon = 0 \tag{3.4.46}$$

then using the fact that the product of all gamma matrices is (in our conventions) $-i$, we can rewrite $\Gamma^{tzr\rho\chi\theta\varphi} = i\Gamma^{y_1\dots y_4}$, and use (3.4.44) again to obtain

$$\Big[2\Gamma_r\beta - i\Gamma^{\rho\chi\theta\varphi} + i(1+\rho^2)^{-1/2}\Big(\Gamma^{tz} - \Gamma^{y_1\dots y_4}\Big)\Big]\epsilon = 0 \tag{3.4.47}$$

and then again rewriting $\Gamma^{y_1\dots y_4} = i\Gamma^{tzr}\Gamma^{\rho\chi\theta\varphi}$, and $\Gamma^{tz} = -\Gamma^{tzr}\Gamma^{r}$, we finally extract a common factor

$$\left(1 + i(1+\rho^2)^{-1/2}\Gamma^{tzr}\right)\left[2\Gamma_r\beta - i\Gamma^{\rho\chi\theta\varphi}\right]\epsilon = 0 \tag{3.4.48}$$

multiplying this by $\left(1 - i(1+\rho^2)^{-1/2}\Gamma^{tzr}\right)$ and extracting the non-negative resulting ρ^2 we arrive at the second projector condition on ϵ:

$$\left[2\Gamma_r\beta - i\Gamma^{\rho\chi\theta\varphi}\right]\epsilon = 0\,. \tag{3.4.49}$$

As we want our solution to be as supersymmetric as possible, we want to choose a β that will cancel some of the algebraic conditions on ϵ. Looking at (3.4.49) and keeping in mind that $\Gamma^{tzry_1\dots y_4} = -i\Gamma^{\rho\chi\theta\varphi}$, we immediately see that the choice $2\beta = \Gamma^{tzy_1\dots y_4}$ will turn this condition into a trivial one! Thus, we can conclude the choice $2\beta = \Gamma^{tzy_1\dots y_4}$ corresponds to a most supersymmetric solution; other choices would impose (3.4.49) and lead to a solution with fewer supersymmetries.

Now let us look at the full AdS part of the solution that corresponds to $2\beta = \Gamma^{tzy_1\dots y_4}$ and then come back to the remaining equations. We will write our solution in the form

$$\epsilon = \epsilon_{\rm AdS}\epsilon_\rho\epsilon_{\rm S^3}\epsilon_0 \tag{3.4.50}$$

with ϵ_0 is a constant spinor and the other factors are matrices depending on the AdS, ρ and sphere coordinates respectively.

The differential equation (3.4.39) on $\tilde\epsilon$ becomes

$$\partial_a\tilde\epsilon = \frac{1}{2}\Gamma_{ar}(1-2\beta)\tilde\epsilon \tag{3.4.51}$$

with the solution

$$\tilde\epsilon = \exp\left[\frac{1}{2}x^a\Gamma_{ar}(1-2\beta)\right]\bar\epsilon = \left(1 + \frac{1}{2}x^a\Gamma_{ar}(1-2\beta)\right)\bar\epsilon \tag{3.4.52}$$

where in the second equality we take into account our previous assumption that β anticommutes with Γ_{ar} and $(2\beta)^2 = I$ so that we can make an expansion of the exponent to the linear term. Here $\bar\epsilon = \epsilon_\rho\epsilon_{\rm S^3}\epsilon_0$. Hence the full factor $\epsilon_{\rm AdS}$ is

$$\epsilon_{\rm AdS} = r^{\frac{1}{2}\Gamma^{tzy_1\dots y_4}}\left(1 + \frac{1}{2}x^a\Gamma_{ar}(1-\Gamma^{tzy_1\dots y_4})\right). \tag{3.4.53}$$

Expanding the r exponent, this can be seen to match the form of the AdS solutions obtained in [104].

Now we consider the remaining differential equations on ϵ. We start with the case corresponding to the ρ coordinate:

$$\partial_\rho\epsilon - \frac{i}{6}(1+\rho^2)^{-1/2}\Gamma_\rho\left[\Gamma^{y_1\dots y_4} + 2\rho\Gamma^{tzr\rho} + 4(1+\frac{1}{2}\rho^2)(1+\rho^2)^{-1/2}\Gamma^{\rho\chi\theta\varphi}\right]\epsilon = 0 \tag{3.4.54}$$

Using the projector conditions (3.4.44) and (3.4.49) (the latter of course now an identity given the form of β), as well as gamma matrix identities, we can simplify this to

$$\partial_\rho \epsilon - \frac{1}{6}\rho(1+\rho^2)^{-1}\epsilon + \Gamma_{r\rho}\beta(1+\rho^2)^{-1}\epsilon = 0\,. \tag{3.4.55}$$

and now the solution for ϵ_ρ will depend on how $\epsilon_{\rm AdS}$ permutes with β. For our choice of β, all the matrices in $\epsilon_{\rm AdS}$ commute with $\Gamma_{r\rho}\beta$, and we can simply move $\epsilon_{\rm AdS}$ to the left of each term in the equation. We then end up with a differential equation for ϵ_ρ with the following solution:

$$\epsilon_\rho = (1+\rho^2)^{1/12}\exp\Big[\frac{1}{2}\tan^{-1}\rho\Gamma_{tzry_1\ldots y_4\rho}\Big] \tag{3.4.56}$$

We move on to the sphere components of the Killing spinor equation. We let $\epsilon_{\rm S^3} = \epsilon_\chi(\chi)\epsilon_\theta(\theta)\epsilon_\varphi(\varphi)$. The χ equation becomes after similar simplifications using the projector conditions

$$\partial_\chi\epsilon + \frac{1}{2}(1+\rho^2)^{-1/2}\Gamma_{\rho\chi}\Big[1+\rho\Gamma_{\rho tzry_1\ldots y_4}\Big]\epsilon = 0\,, \tag{3.4.57}$$

or

$$\partial_\chi\epsilon + \frac{1}{2}\exp[\Gamma_{tzry_1\ldots y_4\rho}\tan^{-1}\rho]\Gamma_{\rho\chi}\epsilon = 0\,. \tag{3.4.58}$$

Permuting $\Gamma_{\rho\chi}$ in the second term in this equation with ϵ_ρ we change the sign in the exponent of ϵ_ρ from equation (3.4.56), which combined with the exponential of this equation gives the same ϵ_ρ finally in the second term on the left. Thus, after extracting ϵ_ρ from the both terms of the equation to the left, we have the simple equation

$$\partial_\chi\epsilon_\chi + \frac{1}{2}\Gamma_{\rho\chi}\epsilon_\chi = 0 \Rightarrow \epsilon_\chi = \exp\Big[-\frac{1}{2}\Gamma_{\rho\chi}\chi\Big]\,. \tag{3.4.59}$$

The same technique can be applied to obtain ϵ_θ and ϵ_φ parts of the solution, which end up being

$$\epsilon_\theta = \exp\Big[-\frac{1}{2}\Gamma_{\chi\theta}\theta\Big]\,,\quad \epsilon_\varphi = \exp\Big[-\frac{1}{2}\Gamma_{\theta\varphi}\varphi\Big]\,. \tag{3.4.60}$$

The full solution we have obtained can therefore be written as

$$\epsilon = (1+\rho^2)^{1/12}r^\beta\Big(1+\frac{1}{2}x^a\Gamma_{ar}(1-2\beta)\Big)\exp\Big[-\beta\Gamma_{r\rho}\tan^{-1}\rho\Big]\epsilon_\Omega\epsilon_0\,, \tag{3.4.61}$$

with $\beta = \frac{1}{2}\Gamma^{tzy_1\ldots y_4}$, $\epsilon_\Omega = \epsilon_\chi\epsilon_\theta\epsilon_\varphi$. In addition we have the projector condition (3.4.44), which we can rewrite as

$$\left(1+\frac{i}{\sqrt{1+\rho^2}}\left(\Gamma^{tzr} - \rho\Gamma^{y_1\ldots y_4\rho}\right)\right)\epsilon = 0\,. \tag{3.4.62}$$

This can be shown to reduce to a single projector condition on the constant spinor ϵ_0. To show this, we apply the projector condition in its original form (3.4.44) to (3.4.61) and proceed as follows. We first permute the exponential in ϵ_ρ with $\Gamma_r\beta$ from (3.4.44). After then factoring out a common ϵ_ρ we can use the identities $\sin\tan^{-1}\rho = \rho(1+\rho^2)^{-1/2}$, $\cos\tan^{-1}\rho = (1+\rho^2)^{-1/2}$ to rewrite (3.4.44) applied to (3.4.61) as

$$(1+\rho^2)^{-1}\Big[\big(1+\rho\Gamma_{tzry_1\dots y_4\rho}\big)2\Gamma_r\beta + i\big(\rho\Gamma^{tzr\rho}+\Gamma^{y_1\dots y_4}\big)\Big]\epsilon_\chi\epsilon_\theta\epsilon_\varphi\epsilon_0 = 0\,. \tag{3.4.63}$$

Then permuting with ϵ_χ, the terms linear in ρ give different signs in the exponent containing Γ_ρ, leading to 2 equations:

$$(2\Gamma_r\beta + i\Gamma^{y_1\dots y_4})\epsilon_0 = 0\,, \quad (\Gamma_\rho + i\Gamma^{tzr\rho})\epsilon_0 = 0\,. \tag{3.4.64}$$

However these are actually equivalent and give the single condition:

$$(1+i\Gamma^{tzr})\epsilon_0 = 0\,. \tag{3.4.65}$$

Therefore we have 1 condition on ϵ_0, reducing the degrees of freedom by $\frac{1}{2}$, so this is a $\frac{1}{2}$-BPS solution. This is the same amount of supersymetry as the original F1-NS5 solution in its AdS_3 limit. Away from this limit we expect our full solution (3.4.3) is $\frac{1}{4}$-BPS. It is worth noting that the solutions of [92] are generically $\frac{1}{4}$-BPS, suggesting that our solution allows for an enhancement, likely due to the special case $k=1$. We note that a similar explicit Killing spinor solution was found in [105].

3.4.6 IIA reductions

Finally, let us record the expressions for different solutions of type IIA supergravity which can be obtained by reducing the solution (3.4.3) in different ways. All these solutions could further be T-dualised in multiple ways to give solutions of type IIB supergravity.

Reduction on T^4 direction Reducing on one of the T^4 directions we obtain

$$\begin{aligned}
\mathsf{d}s^2_{10} &= (r^2 f_1+\rho^2)^{1/2}(r^2 f_1)^{1/2}\left(\frac{1}{f_1}(-\mathsf{d}t^2+\mathsf{d}z^2)+\frac{\mathsf{d}r^2}{r^2}+\frac{\mathsf{d}\rho^2}{r^2 f_1}+\mathsf{d}s^2_{\mathrm{T}^3}\right)\\
&\quad + (r^2 f_1+\rho^2)^{-1/2}(r^2 f_1)^{1/2}\rho^2\mathsf{d}s^2_{\mathrm{S}^3}\,,\\
H_{(3)} &= -2\mathrm{Vol}_{\mathrm{T}^3}\,,\quad e^{-2\varphi} = (r^2 f_1+\rho^2)^{-1/2}(r^2 f_1)^{-1/2}\,,\quad F_{(2)}=0\,,\\
F_{(4)} &= \frac{2r\rho}{(r^2 f_1)^2}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r\wedge\mathsf{d}\rho + \frac{(4r^2 f_1+2\rho^2)}{(r^2 f_1+\rho^2)^2}\rho^3\mathsf{d}\rho\wedge\mathrm{Vol}_{\mathrm{S}^3} - \frac{\rho^4\partial_r(r^2 f_1)}{(r^2 f_1+\rho^2)^2}\mathsf{d}r\wedge\mathrm{Vol}_{\mathrm{S}^3}\,.
\end{aligned} \tag{3.4.66}$$

This still has an AdS_3 near horizon limit, and the full solution is a six-vector deformation of this. The six-vector is now associated to the NSNS six-form.

Reduction on Hopf fibre Writing the metric on the three-sphere as

$$\mathrm{d}s^2_{\mathrm{S}^3} = \frac{1}{4}\left((\mathrm{d}\psi+\eta)^2 + \mathrm{d}s^2_{\mathrm{S}^2}\right) , \quad \mathrm{d}\eta = \mathrm{Vol}_{\mathrm{S}^2} . \tag{3.4.67}$$

and reducing on the Hopf fibre direction parametrised by ψ we obtain

$$\begin{aligned}
\mathrm{d}s^2_{10} &= (r^2 f_1)^{1/2}\frac{\rho}{2}\left(\frac{1}{f_1}(-\mathrm{d}t^2+\mathrm{d}z^2) + \frac{\mathrm{d}r^2}{r^2} + \frac{\mathrm{d}\rho^2}{r^2 f_1} + \mathrm{d}s^2_{\mathrm{T}^4}\right) \\
&\quad + (r^2 f_1+\rho^2)^{-1}(r^2 f_1)^{1/2}\left(\frac{\rho}{2}\right)^3 \mathrm{d}s^2_{\mathrm{S}^2} , \\
H_{(3)} &= \tfrac{1}{8}\frac{(4r^2 f_1 + 2\rho^2)}{(r^2 f_1+\rho^2)^2}\rho^3 \mathrm{d}\rho\wedge\mathrm{Vol}_{\mathrm{S}^2} - \tfrac{1}{8}\frac{\rho^4\partial_r(r^2 f_1)}{(r^2 f_1+\rho^2)^2}\mathrm{d}r\wedge\mathrm{Vol}_{\mathrm{S}^2} , \\
e^{-2\varphi} &= (r^2 f_1+\rho^2)(r^2 f_1)^{-1/2}\left(\tfrac{\rho}{2}\right)^{-3} , \\
F_{(2)} &= \mathrm{Vol}_{\mathrm{S}^2} , \quad F_{(4)} = \frac{2r\rho}{(r^2 f_1)^2}\mathrm{d}t\wedge\mathrm{d}z\wedge\mathrm{d}r\wedge\mathrm{d}\rho - 2\mathrm{Vol}_{\mathrm{T}^4} .
\end{aligned} \tag{3.4.68}$$

This still has an AdS_3 near horizon limit, and the full solution is a five-vector deformation of this, with the five-vector associated to the RR five-form. As the M-theory $\mathrm{AdS}_3\times\mathrm{S}^3$ solutions of [92] were obtained by uplifting the $\mathrm{AdS}_3\times\mathrm{S}^2$ IIA solutions constructed in [94–97] on a Hopf fibre, the solution (3.4.68) can be interpreted using the latter.

Reduction on AdS direction Reducing on the z direction we obtain

$$\begin{aligned}
\mathrm{d}s^2_{11} &= (r^2 f_1+\rho^2)^{1/2} r\left(-\frac{1}{f_1}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \frac{\mathrm{d}\rho^2}{r^2 f_1} + \mathrm{d}s^2_{\mathrm{T}^4}\right) + (r^2 f_1+\rho^2)^{-1/2} r\rho^2 \mathrm{d}s^2_{\mathrm{S}^3} , \\
H_{(3)} &= \frac{2r\rho}{(r^2 f_1)^2}\mathrm{d}t\wedge\wedge\mathrm{d}r\wedge\mathrm{d}\rho , \quad e^{-2\varphi} = (r^2 f_1+\rho^2)^{-1/2}\tfrac{f_1}{r} , \quad F_{(2)} = 0 , \\
F_{(4)} &= -2\mathrm{Vol}_{\mathrm{T}^4} + \frac{(4r^2 f_1+2\rho^2)}{(r^2 f_1+\rho^2)^2}\rho^3\mathrm{d}\rho\wedge\mathrm{Vol}_{\mathrm{S}^3} - \frac{\rho^4\partial_r(r^2 f_1)}{(r^2 f_1+\rho^2)^2}\mathrm{d}r\wedge\mathrm{Vol}_{\mathrm{S}^3} .
\end{aligned} \tag{3.4.69}$$

This now has an AdS_2 near horizon limit, and the full solution is a five-vector deformation of this. The five-vector is associated to the RR five-form.

3.5 Discussion

In this chapter we first discussed the idea of generalised T- and U-dualities, viewed as a solution generating technique in supergravity. We reviewed how these generalised dualities can be linked to special classes of algebras, which are efficiently geometrically encoded using generalised parallelisations in generalised geometry. Building on our previous paper [87], we focused on an example in 11-dimensional supergravity characterised by non-vanishing dual 3-algebra structure constants in the underlying exceptional Drinfeld algebra introduced to control Poisson-Lie U-duality in [26, 27].

To produce a new supergravity solution, we had to step slightly outside the confines of the EDA set-up. We used the fact that our EDA generalised frame incorporating the Euclidean 3-algebra solution provided a consistent truncation to CSO$(4,0,1)$ maximal gauged supergravity in 7 dimensions. We were able to use this pragmatically to produce dual pairs of solutions by starting with the known truncation of type IIA on S^3 leading to the same gauged supergravity, reducing solutions of the latter form, and then uplifting with our EDA frame. Algebraically, this alternative starting point can be viewed as relaxing the requirement that one has to pick an isotropic set of dual generators forming a subalgebra. It would be interesting to complete this observation by formulating a more precise understanding of which families of generalised frames produce the EDA with the subalgebra requirement relaxed (the systematic approach of [51] would likely be useful here). This would allow our construction to be viewed in terms of a slightly enlarged notion of Poisson-Lie U-duality than that initially suggested in [26, 27].

The example described in this paper can be viewed as a proof-of-concept for the idea that it is possible to generate new supergravity solutions by formulating generalised notions of U-duality. It would be beneficial to develop a more systematic approach. For instance, it is very clear which spacetimes admit non-abelian T-duals: those with non-abelian isometries. It is not clear what spacetimes admit generalised U-duals characterised by non-vanishing dual 3-algebra structure constants. It is also not clear what role, if any, is played by an actual 3-algebra symmetry in such spacetimes.

Generalising to higher dimensions will also lead to higher-rank polyvectors and n-algebra symmetries. It would appear that solutions characterised by an ansatz involving polyvectors linear in the coordinates have notable properties. They describe not only the plethora of known NATD solutions, but also solutions such as the one constructed in this paper, which as we saw shared many features with solutions generated by NATD, including the general properties of its holographic completion. Classifying and understanding the types of solutions of this form, and the possible dual solutions they may arise from, would not only help establish generalised U-duality as a useful technique on a par with non-abelian T-duality but help elucidate the general structure.

Here it would also be important to develop an understanding of which properties (supersymmetry, brane charges) of such solutions are induced by the initial solution. For non-abelian T-duality, for example, one can precisely discuss which supersymmetries are preserved in terms of whether the action of the initial non-abelian isometries preserve the Killing spinor [7, 86, 106]. Generically one finds a reduced amount of supersymmetry in the dual solution as a result. In our example, in the AdS limit, we found our new solution had as many supersymmetries as the original near horizon F1-NS5 solution. It would be useful to understand from a general viewpoint why this was the case. This might be best formulated using exceptional field theory as a master formalism.

It would be possible to generate further examples by focusing on specific solutions of the gauged supergravities that appear in these polyvector constructions. For the CSO$(4,0,1)$ supergravity, numerous solutions were found in [107–109], all of which can be used to generate dual solutions by uplifting to type IIA on S^3 and to 11-dimensional supergravity via our EDA generalised frame.

Turning now to the specific example studied in this paper, this exhibits numerous interesting features linked to deformations and holographic duality. We argued that a holographic completion of the AdS_3 limit of our solution can be obtained from the class of solutions obtained in [92], which have well-defined quiver field theory duals. We showed that our full solution can be viewed as a six-vector deformation away from this AdS_3 limit. This deformation was inherited from the interpolation of the original F1-NS5 solution from its AdS_3 limit (in the near horizon region of the F1s) to the asymptotic linear dilaton spacetime associated to the pure NS5 near horizon limit. This interpolation has been argued to correspond to a 'single-trace' variant of the $T\bar{T}$ deformation in the CFT_2 dual of the AdS_3 limit [93] (the CFT dual (to the long string sector) of string theory on AdS_3 is a symmetric product $\mathcal{M}^{N_1}/S_{N_1}$ and the $T\bar{T}$ deformation of [93] deforms the block CFT $\mathcal{M}\to\mathcal{M}_{T\bar{T}}$).

The immediate question is whether there is an analogous interpretation applicable to our six-vector deformation of our AdS_3 limit in terms of a deformation of the CFT duals of [92]. This is not to necessarily suggest that this deformation will again be describable as a $T\bar{T}$ deformation, but it may have similar properties. In general, we would expect generalised U-duality, as for non-abelian T-duality, to produce backgrounds with different CFT duals. However, we can at least say that our solution generating technique preserved the fact that there *is* a deformation, encoded geometrically, and suggest that this may turn out to have a relationship to $T\bar{T}$.

A further comment is that in the F1-NS5 case, the endpoint of the deformation can be viewed as a vacuum of the Little String Theory [102, 103] dual to the asymptotic linear dilaton spacetime: for our solution, the latter spacetime maps to the 11-dimensional solution (3.4.11) (not an AdS geometry) which may accordingly itself have a similar holographic interpretation in terms of a dual M5 brane theory.

It may be therefore be interesting to study the deformation of the general class of geometries (3.4.14) of [92]. If we define

$$\begin{aligned} g_{ab}\mathrm{d}x^a\mathrm{d}x^b &= \Delta\left(\frac{u}{\sqrt{\hat{h}_4 h_8}}r^2(-\mathrm{d}t^2+\mathrm{d}z^2)+\frac{\sqrt{\hat{h}_4 h_8}}{u}\mathrm{d}\varrho^2\right)\,, \quad h_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta = \frac{h_8^2}{\Delta^2}\mathrm{d}s^2_{\mathrm{S}^3/\mathbb{Z}_k}\,,\\ G_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu &= \Delta\left(\frac{u}{\sqrt{\hat{h}_4 h_8}}\frac{\mathrm{d}r^2}{r^2}+\sqrt{\frac{\hat{h}_4}{h_8}}\mathrm{d}s^2_{\mathrm{CY}_2}\right)\,, \end{aligned} \tag{3.5.1}$$

and make the naturally analogous gauge choice

$$\begin{aligned} C_1 \equiv C_{tz\rho} &= \frac{r^2}{2}\partial_\varrho\left(\frac{uu'}{2\hat{h}_4}+2\varrho h_8\right)\,, \quad C_2\equiv C_{\psi\theta\phi} = 2h_8\left(-\varrho+\frac{uu'}{4\hat{h}_4 h_8+u'^2}\right)\sqrt{g_{\mathrm{S}^3/\mathbb{Z}_k}}\,,\\ C_6 &= -\frac{r^2}{2}\frac{4h_8^2u^2\hat{h}_4'^2}{h(4h_8\hat{h}_4+u'^2)}\sqrt{g_{\mathrm{S}^3/\mathbb{Z}_k}}+\tfrac{1}{2}C_1C_2\,, \end{aligned} \tag{3.5.2}$$

then we can immediately read off a deformed background from the expressions in appendix A.3.5.

This requires choosing a deformation parameter which produces a new solution: this is not guaranteed. Note that generically the $E_{6(6)}$ generalised metric block $\mathcal{M}_{\mathsf{ij}}$ is non-zero for the metric and potentials picked here. This means that the deformed metric will depend quadratically on λ instead of just linearly. This is not necessarily a problem, however it is possible that situations with vanishing $\mathcal{M}_{\mathsf{ij}}$ are special.

Other deformations of the AdS_3 limit of the F1-NS5 solution correspond to single-trace $J\bar{T}/\bar{J}T$ deformations of the dual CFT_2, see for instance [110,111]. These again have a straightforward worldsheet interpretation as TsT i.e. $O(d,d)$ transformations, and modify the bulk geometry. Focusing on deformations which preserve the ansatz for type IIA on S^3, it would be possible to map the corresponding backgrounds to new 11-dimensional geometries using our methodology, and to examine how the deformations are inherited by the new solution, as trivector deformations for example.

It may also be productive to explore these deformations algebraically in the context of the EDA proposal. For instance, embedding our SL(5)-valued trivector into $E_{6(6)}$ and combining with the six-vector deformation discussed in section 3.4.4, could be viewed through the lens of the $E_{6(6)}$ EDA [79]. This may connect to related work on polyvector deformations, including in the context of the EDA construction, such as [112].

We have provided new examples of implementing U-duality in constructing connections between seemingly different solutions, and discovered some interesting features of the new solution. In the general scope, this extends our understanding of special internal properties of the solutions generated via generalised U-duality, which in a way extends the framework of generalised T-duality and exhibit novel supersymmetric features. That gives a better hint of how different supergravity models are interconnected.

Chapter 4

Generalised U-dual solutions via ISO(7) gauged supergravity

4.1 Introduction

The T- and U-duality symmetries of supergravity act on spacetimes with abelian isometries. A first version of a generalised duality is non-Abelian T-duality (NATD) [3], which provides a mechanism that dualises a space with non-Abelian isometries to a space with fewer isometries. Both abelian and non-abelian T-duality are special cases of the Poisson-Lie T-duality [10, 11], which can be applied to backgrounds lacking isometries, and which are characterised by an underlying double algebra structure called the Drinfeld double algebra. Further extension of these dualities leads to notions of generalised U-duality, originally proposed using a generalised geometric approach (building on [23, 24] in the T-duality case) to describe the background, and generalises the Drinfeld double algebra to the so-called exceptional Drinfeld algebra (EDA), that generically is a Leibniz algebra instead of a Lie algebra [26, 27, 79, 91].

In our earlier papers [87, 113] we used this approach to study an attractive example of a generalised U-duality solution generating construction based on the $Sl(5)$ U-duality group acting in four dimensions. The relevant exceptional Drinfeld algebra was the Lie algebra ISO(4). The generalised U-duality map took solutions of type IIA supergravity on a three-sphere with NSNS flux to new solutions of eleven-dimensional supergravity: a basic example was provided starting with the near horizon NS5 brane.

In this paper, we revisit the generalised U-duality on another example based on the E_7 U-duality group acting in seven dimensions, with the relevant EDA now being an extension of the ISO(7) Lie algebra. We take a near horizon D2 brane solution as a test example, and show how to transform this into a new supergravity solution in eleven dimensions.

The appearance of ISO(4) and ISO(7) algebras is not a choice made a priori but a consequence of choosing to study particular natural algebraic structures, which appear in the definition of the underlying Drinfeld double algebra. First of all, we were motivated by the fact that in solutions obtained by NATD, the breaking of translational isometries in the new dual directions can be linked to the appearance of 'dual' Lie algebra structure constants $\tilde{f}^{ab}{}_c$.[1] For the dual of SU(2) i.e. NATD on S^3, these are $\tilde{f}^{ab}{}_c = \epsilon^{ab}{}_c$.

In the exceptional Drinfeld algebra [26, 27, 79, 91] these dual structure constants are generalised to 3- and 6-algebra structure constants, $\tilde{f}^{abc}{}_d$ and $\tilde{f}^{abcdef}{}_g$.[2] In the four-dimensional $Sl(5)$ case only the former appear. Choosing $\tilde{f}^{abc}{}_d = \epsilon^{abc}{}_d$ ($a = 1, \ldots, 4$) produced the ISO(4) algebra studied in [87, 113]. The solutions obtained could be seen to directly generalise many of the properties of the solutions resulting from NATD.

In this paper we generalise to the seven-dimensional case, where in principle we can have both the 3- and 6-algebra structures. We choose $\tilde{f}^{abc}{}_d = 0$ and take $\tilde{f}^{abcdef}{}_g = \epsilon^{abcdef}{}_g$, which as we explicitly show corresponds to the ISO(7) algebra.

This ISO(7) example can be viewed as being a sort of electromagnetic dual of our previous ISO(4) case. This is reflected in the replacement of the 3-algebra structure constants with 6-algebra structure constants, explicitly linked numerologically to the three-form and its magnetic dual six-form, and in the natural choices of NS5 brane (M5 brane on a circle) and D2 brane (M2 brane) as starting points for the construction.

Our approach to constructing new solutions relies on the fact that the generalised geometric realisation of the exceptional Drinfeld algebra provides a mechanism for carrying out a consistent truncation from 10- or 11-dimensional supergravity to a lower-dimensional gauged supergravity. Such truncations allow for both reduction and uplift of solutions. The algebra that is gauged is exactly the EDA. When a different consistent truncation is known leading to the same lower-dimensional theory, we can apply 'generalised U-duality' by mapping solutions to solutions by reducing via one consistent truncation and uplifting via the other. The example of [113] gave a consistent truncation of eleven-dimensional supergravity to seven-dimensional ISO(4) gauged maximal supergravity, distinct from the previously known origin of this theory via consistent truncation of type IIA supergravity on a three-sphere with NSNS flux.

In this paper, we will play the same game using reduction and uplift by inequivalent consistent truncations leading to the four-dimensional ISO(7) gauged maximal supergravity. The first known consistent truncation in this case is provided by type IIA SUGRA on S^6 [114–116]. We apply our solution generating technique by taking any solution of type IIA fitting into the appropriate reduction ansatz, consistently truncating it to a 4-dimensional solution, and then uplifting it to a new 11-dimensional SUGRA solution using the E_7 generalised geometry formulation based on the EDA [91]. It follows that this method gives an alternative consistent truncation, starting with eleven-dimensional supergravity and leading to ISO(7) gauged supergravity in four dimensions.

[1] In Poisson-Lie T-duality more generally, these can be interpreted as a coycle of a physical Lie algebra, which in this case is trivial.

[2] With interpretations as n-cocycles of a physical Lie algebra, which again will be trivial in our examples.

In fact, this alternative consistent truncation was identified in the paper [51] (which indeed demonstrated the existence of inequivalent consistent truncations for CSO gaugings more generally). Here we extend, or use, the observation of [51] in the following ways. Firstly we demonstrate explicitly how to use these inequivalent consistent truncations to perform a generalised U-duality, and explicitly produce a new 11-dimensional supergravity solution using this approach. We further highlight the algebraic interpretation of the second consistent truncation, by concretely connecting it to the EDA proposal with accompanying n-algebra structure, and by comparison to our previous papers [87, 113] we demonstrate how this all fits into the pattern of generalised dualities naturally extending non-abelian T-duality of a three-sphere.

In this chapter we specifically apply the uplift procedure to produce a new 11-dimensional solution starting with an extremal D2 brane solution after taking the near horizon limit. Then we analyse the properties of the new 11-dimensional solution, which turn out to be as follows:

- The new solution can be described by using the generalised geometry techniques with a 6-vector linear in the dual 4-dimensional coordinates. *(See sections 4.2.2 and 4.3.2.)*
- The new solution can be viewed as carrying an electric (M2) charge. *(See section A.4.)*
- The new solution can be viewed as a warped product of AdS_4, S^6 and an interval, and it possesses a $\frac{1}{2}$-BPS solution of the 11-dimensional Killing spinor equation. *(See section 4.3.3.)*

In section 4.2.1 we review the ISO(7) subalgebra of the E_7 Drinfeld algebra that we will use in our solution. In section 4.2.2 we construct the frame fields of E_7 Drinfeld subalgebra. Then, in section 4.3 we show an example of how to obtain a new 11-dimensional solution using this technology. In subsection 4.3.1 we start with the initial $D2$ brane solution that we use as an example of non-vacuum type IIA SUGRA solution. After that, in subsection 4.3.2 we write down the scalar matrix that we take to uplift the initial $D2$ brane solution and construct the new uplifted 11-dimensional SUGRA solution. Then, in sections A.4 and 4.3.3 we describe the properties of the uplifted solution, its charges, local vs global nature, and the amount of supersymmetry it possesses. We conclude with some brief discussion in section 4.4.

4.2 ISO(7) exceptional Drinfeld algebra and generalised frame

4.2.1 The algebra

The whole E_7 exceptional Drinfeld algebra was described in [91]. The 56 generators of the E_7 exceptional Drinfeld algebra are denoted $T_A = (T_a,\ T^{a_1 a_2},\ T^{a_1 \dots a_5},\ T^{a_1 \dots a_7, a'})$, where the Latin indices run from 1 to 7 and sets of multiple indices $a_1 \dots a_p$ are understood to be antisymmetric. The (generically non-antisymmetric) brackets of these generators can be written generally as:

$$[T_A, T_B] = X_{AB}{}^C T_C\,. \tag{4.2.1}$$

The EDA structure constants $X_{AB}{}^C$ are specified in terms of structure constants $f_{ab}{}^c$, $f^{a_1\ldots a_3}{}_b$, $f^{a_1\ldots a_6}{}_b$ and Z_a. The former three can be formally associated with Lie algebra, 3-algebra and 6-algebra structures. In this paper, we focus on non-zero 6-algebra structure constants only, $f^{a_1\ldots a_6}{}_b \neq 0$, in which case the algebra is given by the following non-zero brackets:

$$[T_a, T^{b_1\ldots b_5}] = -f^{b_1\ldots b_5 c}{}_a T_c, \quad [T_a, T^{b_1\ldots b_7,b'}] = 7 f^{[b_1\ldots b_6}{}_a T^{b_7]b'} \tag{4.2.2}$$

$$[T^{a_1\ldots a_5}, T_b] = f^{a_1\ldots a_5 c}{}_b T_c, \quad [T^{a_1\ldots a_5}, T^{b_1 b_2}] = 2 f^{a_1\ldots a_5[b_1}{}_c T^{b_2]c} \tag{4.2.3}$$

$$[T^{a_1\ldots a_5}, T^{b_1\ldots b_5}] = -5 f^{a_1\ldots a_5[b_1}{}_c T^{b_2\ldots b_5]c} \tag{4.2.4}$$

$$[T^{a_1\ldots a_5}, T^{b_1\ldots b_7,b'}] = -7 f^{a_1\ldots a_5[b_1}{}_c T^{b_2\ldots b_7]c,b'} - f^{a_1\ldots a_5 b'}{}_c T^{b_1\ldots b_7,c} \tag{4.2.5}$$

$$[T^{a_1\ldots a_7,a'}, T_b] = -21 f^{[a_1\ldots a_6}{}_c \delta^{a_7]a'c}_{bd_1d_2} T^{d_1d_2}, \quad [T^{a_1\ldots a_7,a'}, T^{b_1b_2}] = 7 f^{[a_1\ldots a_6}{}_c T^{a_7]a'cb_1b_2} \tag{4.2.6}$$

$$[T^{a_1\ldots a_7,a'}, T^{b_1\ldots b_5}] = 21 f^{[a_1\ldots a_6}{}_c \delta^{a_7]a'c}_{d_1d_2e} T^{b_1\ldots b_5 d_1 d_2,e} \tag{4.2.7}$$

In the absence of the other structure constants, the 6-algebra structure constants must obey the identity

$$f^{da_1\ldots a_5}{}_c f^{b_1\ldots b_6}{}_d - 6 f^{a_1\ldots a_5[b_1}{}_d f^{b_2\ldots b_6]d}{}_c = 0\,, \tag{4.2.8}$$

ensuring closure of the algebra. This can be viewed as a generalisation of the Jacobi identity for Lie algebras and the fundamental identity for 3-algebras.

We now further restrict to the following special case:

$$f^{b_1\ldots b_6}{}_a = \epsilon^{b_1\ldots b_6 c}\delta_{ac} \tag{4.2.9}$$

where $\epsilon^{b_1\ldots b_6 c}$ is a 7-dimensional Levi-Civita symbol and δ_{ab} is seven-dimensional identity matrix. This is easily verified to obey (4.2.8). After defining the dualised notations

$$\tilde{T}^a = \frac{1}{7!}\epsilon_{a_1\ldots a_7} T^{a_1\ldots a_7,a}, \quad \tilde{T}_{bc} = \frac{1}{5!}\epsilon_{bca_1\ldots a_5} T^{a_1\ldots a_5}\,, \tag{4.2.10}$$

the non-trivial brackets of the algebra then simplify to

$$\begin{aligned} [T_a, \tilde{T}_{bc}] &= 2\delta_{a[b}T_{c]}, \quad [T_a, \tilde{T}^b] = -\delta_{ac}T^{bc}\,, \\ [\tilde{T}_{bc}, T_a] &= -2\delta_{a[b}T_{c]}, \quad [\tilde{T}_{ab}, T^{cd}] = -4\delta_{e[a}\delta^{[c}_{b]}T^{d]e}\,, \\ [\tilde{T}_{ab}, \tilde{T}_{cd}] &= 4\delta_{\left[a[c\right.}\tilde{T}_{\left.d]b\right]}, \quad [\tilde{T}_{ab}, \tilde{T}^c] = 2\delta_{d[a}\tilde{T}^d\delta^c_{b]}\,. \end{aligned} \tag{4.2.11}$$

The generators $(T_a, \tilde{T}_{bc})$ generate the ISO(7) Lie algebra.[3] The other brackets (note that these

[3]This can be generalised by replacing δ_{ab} in (4.2.9) by a symmetric matrix of indefinite signature, which would correspond to the algebra of the CSO$(p, q, r+1)$ gaugings with $p+q+r=7$; replacing δ_{ab} by a matrix with both

are not antisymmetric and e.g. $[\tilde{T}^a, T_b] = 0$) match those specified by the ISO(7) gauging of four-dimensional maximal supergravity (for example, compare with appendix C of [118] where the full structure constants $X_{AB}{}^C$ appearing in (4.2.1) are given).

4.2.2 The generalised frame

Given any exceptional Drinfeld algebra, a generalised frame can be constructed realising the algebra under the generalised Lie derivative of the appropriate exceptional generalised geometry. This explicit construction is described in [26, 27, 79, 91]. The data that enters the generalised frame consists of a (left- or right-)invariant vielbein $e^a{}_m$, obeying the Maurer-Cartan equation with Lie algebra structure constants $f_{ab}{}^c$, a 3-vector $\pi^{b_1 b_2 b_3}$ and a 6-vector $\pi^{b_1 \dots b_6}$, as well as a scalar function Δ. The vielbein is linked to a group manifold and the n-vectors and scalar obey equations of the form:

$$\begin{aligned} D_a \pi^{b_1 b_2 b_3} &= f^{b_1 b_2 b_3}{}_a + \dots \,, \\ D_a \pi^{b_1 \dots b_6} &= f^{b_1 \dots b_6}{}_a - 10 f^{[b_1 b_2 b_3}{}_a \pi^{b_4 b_5 b_6]} + \dots \,, \\ D_a \Delta &= Z_a \,, \end{aligned} \tag{4.2.12}$$

where $D_a \equiv e_a{}^i \partial_i$ and the $\dots$ corresponds to the terms with Lie algebra structure constants, which are absent in our case.

Now let's construct the necessary data and generalised frame fields for the E_7 subalgebra with only the six-algebra structure constants $f^{b_1 \dots b_6}{}_a$ non-trivial. The above differential equations then yield $e^a_m = \delta^a_m$, $\pi^{b_1 b_2 b_3} = 0$, $\Delta = 1$ and allow for a six-vector linear in the coordinates, $\pi^{b_1 \dots b_6} = x^i \delta^a_i f^{b_1 \dots b_6}{}_a$. Then, referring to eq. (5.34) of [91], we can construct the generalised frame, which will by definition obey

$$\mathcal{L}_{E_A} E_B = -X_{AB}{}^C E_C \tag{4.2.13}$$

under the E_7 generalised Lie derivative, thereby realising the algebra of the ISO(7) gauging. A generalised frame for the E_7 generalised geometry gives a basis $E_A{}^M$ for generalised vectors, which correspond to vectors, two-forms, five-forms and seven-forms tensored with one-forms. In form notation, the EDA generalised frame describing the ISO(7) algebra has the following elements:

$$\begin{aligned} E_a &= (e_a, 0, 0, 0) \,, \\ E^{a_1 a_2} &= (0, e^{a_1} \wedge e^{a_2}, 0, 0) \,, \\ E^{a_1 \dots a_5} &= (-\pi^{b a_1 \dots a_5} e_b, 0, e^{a_1} \wedge \dots \wedge e^{a_5}, 0) \,, \\ E^{a_1 \dots a_7, a'} &= (0, -7\pi^{[a_1 \dots a_6} e^{a_7]} \wedge e^{a'}, 0, (e^{a_1} \wedge \dots \wedge e^{a_7}) \otimes e^{a'}) \,, \end{aligned} \tag{4.2.14}$$

symmetric and antisymmetric parts would give something more exotic in which the 28-dimensional 'electric' algebra is no longer Lie.

where in particular the vielbein e_a and one-form e^a have trivial components, ${e_a}^i = \delta^i_a$, ${e^a}_i = \delta^a_i$, and $\pi^{a_1\dots a_6} = x_b \epsilon^{a_1\dots a_6 b}$.

It is useful to record an explicit expression for this frame as a 56×56 E_7 valued matrix. The natural decomposition of the generalised vector index is $V^M = (V^m, V_{m_1m_2}, V_{m_1\dots m_5}, V_{m_1\dots m_7,m'})$ but it is convenient to dualise the five-form and mixed symmetry components (as with the algebra generators above) such that the seven-dimensional decomposition used is $V^M = (V^m, V_{m_1m_2}, V^{m_1m_2}, V_{m'})$. Using this convention for both M and A indices we can write the ISO(7) exceptional Drinfeld algebra generalised frame, or rather its inverse which is more useful for our purposes below, as

$$ {E_M}^A = \begin{pmatrix} \delta^a_m & 0 & 0 & 0 \\ 0 & 2\delta^{m_1}_{[a_1}\delta^{m_2}_{a_2]} & 0 & 0 \\ 2x_{[m_1}\delta^a_{m_2]} & 0 & 2\delta^{[a_1}_{m_1}\delta^{a_2]}_{m_2} & 0 \\ 0 & 2x_{[a_1}\delta^m_{a_2]} & 0 & \delta^m_a \end{pmatrix} . \tag{4.2.15} $$

This generalised frame (which could also have been constructed using the results of [51]) can be used to construct solutions of 11-dimensional supergravity by uplifting solutions of ISO(7) gauged supergravity. Given such a solution, depending on four-dimensional coordinates y, and given in terms of the four-dimensional metric $g_{\mu\nu}(y)$, the scalar matrix $M_{AB}(y)$, and one-form ${\mathcal{A}_\mu}^A(y)$, a solution to eleven-dimensional supergravity can be constructed by computing the following quantities:

$$ g_{\mu\nu}(y,x) = g_{\mu\nu}(y)\,, \quad \mathcal{M}_{MN}(y,x) = {E_M}^A(x){E_N}^B(x)M_{AB}(y)\,, \quad {\mathcal{A}_\mu}^M(y,x) = {E_A}^M(x){\mathcal{A}_\mu}^A(y)\,, \tag{4.2.16} $$

which correspond to the external metric, generalised metric and external one-form of the E_7 exceptional field theory/exceptional generalised geometry description of 11-dimensional supergravity in a $4+7$ split [33, 53]. Using the known dictionary between this formulation and the standard variables of 11-dimensional supergravity, the uplifted solution can be extracted. Conversely the ansatz (4.2.16) with the generalised frame (4.2.15) specifies the general form (again on making use of the exceptional geometry dictionary) of a consistent truncation from 11-dimensional supergravity to the ISO(7) gauged supergravity. This is a standard application of exceptional geometric techniques (see e.g. [57, 58]).

Rather than slavishly work out the full explicit details (which we defer for future work), we will illustrate how this uplift mechanism works on an explicit example, in keeping with our motivation in terms of generalised dualities. A question which needs to be addressed at this point is how to find examples of solutions which we can feed in to this mechanism. The ISO(7) gauged supergravity has no known vacua, so we need to consider other sorts of solutions. A natural candidate is that obtained by the near horizon limit of the D2 brane, which gives a domain wall solution in four dimensions [119]. We now turn to this solution and its transformation to a new eleven-dimensional solution.

4.3 New 11-dimensional solution

4.3.1 The initial D2 brane solution

In our previous study [113] of the ISO(4) exceptional Drinfeld algebra, we constructed an example of generalised U-duality where we started with the near horizon NS5 solution in type IIA, reduced to seven-dimensional ISO(4) gauged supergravity and uplifted using an $Sl(5)$ exceptional Drinfeld algebra frame to eleven dimensions. Here we will start with the D2 brane solution in type IIA instead, whose near horizon geometry has the appropriate form for the ISO(7) consistent truncation. Lifting everything to 11-dimensions, this D2 comes from the M2 while the previously considered NS5 comes from the M5. Swapping M5 for M2 reflects the fact that on switching from ISO(4) to ISO(7) we exchange a trivector for a six-vector, mirroring the exchange of the role of the eleven-dimensional three- and six-forms in the M2 and M5 solutions. In other words, we are applying electromagnetic duality to the entirety of our previous generalised U-duality described in [113].

The D2 brane solution in the string frame is:

$$ds_S^2 = H^{-1/2}[-dt^2 + dy_1^2 + dy_2^2] + H^{1/2}[dr^2 + r^2 d\Omega_{(6)}^2]\,, \tag{4.3.1}$$

$$e^{-2\phi} = H^{-1/2}\,, \quad C_{ty^1y^2} = H^{-1} - 1\,, \tag{4.3.2}$$

with $H = 1 + \frac{1}{r^5}$.[4] The Einstein frame metric is:

$$ds_E^2 = H^{-5/8}[-dt^2 + dy_1^2 + dy_2^2] + H^{3/8}[dr^2 + r^2 d\Omega_{(6)}^2]\,. \tag{4.3.3}$$

To perform the reduction to a four-dimensional solution, we use the ansatz as in [115] for a consistent truncation of type IIA SUGRA on S^6 in the Einstein frame. Assuming all the vector fields $A_{(1)}$ and $B_{(2)}$ appearing in the ansatz are turned off, for the metric and dilaton this ansatz has the form:

$$ds_E^2 = \Delta^{-1} ds_4^2 + g_{mn} dy^m dy^n\,, \quad e^{-\frac{3}{2}\phi} = \Delta \mu_a \mu_b M^{a8,b8} \tag{4.3.4}$$

where

$$\mu^a \mu^b \delta_{ab} = 1 \text{ in } \mathbb{R}^7, \quad \Delta^2 = \det g_{mn} / \det \hat{g}_{mn} \tag{4.3.5}$$

and $\hat{g}_{mn}$ is the round $SO(7)$ symmetric metric on S^6. The matrix $M^{a8,b8}$ represents a block of the scalar matrix of the four-dimensional theory. For the D2 solution, we can use the simplified ansatz

$$M^{a8,b8} \equiv \delta^{ab} M\,. \tag{4.3.6}$$

[4] Assuming that by choice of units and rescaling of coordinates we can set all constants to 1.

Then comparing the dilaton forms we find

$$\Delta M = H^{-3/8}\,, \tag{4.3.7}$$

and comparing the metric ansatz we deduce

$$g_{mn} = H^{3/8} r^2 \hat{g}_{mn}, \quad \Delta = r^6 H^{9/8}, \quad M = r^{-6} H^{-3/2}\,, \tag{4.3.8}$$

and the 4-dimensional metric is then

$$ds_4^2 = r^6 H^{1/2}\Big[- dt^2 + dy_1^2 + dy_2^2 + H dr^2 \Big]\,. \tag{4.3.9}$$

Since in the D2 brane solution we have a 3-form with all external components, we have to match it with a non-trivial external 3-form of the type IIA gauged SUGRA ansatz on $S^{(6)}$. This ansatz is:

$$C_{(3)} = \mu_I \mu_J \mathcal{C}^{IJ}, \quad \text{where } \mathcal{C}^{IJ} = C_{ty_1y_2}{}^{IJ} dt \wedge dy^1 \wedge dy^2 \tag{4.3.10}$$

thus

$$C_{ty^1y^2} = \mu_I \mu_J \mathcal{C}_{ty_1y_2}{}^{IJ} \tag{4.3.11}$$

Comparing with the D2 solution, it's not hard to see that

$$\mathcal{C}_{ty_1y_2}{}^{IJ} = \delta^{IJ}(H^{-1} - 1)\,. \tag{4.3.12}$$

Although this three-form appears in the tensor hierarchy of the gauged supergravity, it does not constitute part of the degrees of freedom of the theory which will be uplifted to eleven dimensions. In 4 dimensions the field strength of this potential is dual to a scalar (which would therefore require a -1 form potential) and in fact this field strength can be related to the scalar potential of the theory [115, 116]. It thus serves as part of the definition of the gauged supergravity and not an independent field within it.

4.3.2 Uplifting the scalar matrix and obtaining the new solution

Let's construct the full 56×56 scalar matrix M_{AB} (the flat index $A = (ab, a8)$, where a runs from 1 to 7):

$$M_{AB} = \begin{pmatrix} M_{a8,b8} & M_{a8}{}^{cd} & M_{a8,cd} & M_{a8}{}^{c8} \\ M^{ab}{}_{cd} & M^{ab,cd} & M^{ab}{}_{c8} & M^{ab,c8} \\ M_{ab,c8} & M_{ab}{}^{cd} & M_{ab,cd} & M_{ab}{}^{c8} \\ M^{a8}{}_{cd} & M^{a8,cd} & M^{a8}{}_{b8} & M^{a8,b8} \end{pmatrix}\,, \tag{4.3.13}$$

from which the generalised metric of the eleven-dimensional uplift is constructed as follows

$$\mathcal{M}_{MN} = E_M{}^A M_{AB} E^B{}_N \,. \tag{4.3.14}$$

In order to construct the M_{AB} matrix we refer to the dictionary described in [115], from where, comparing with the form of the $D2$ brane solution of the previous section

$$M_{AB} = \begin{pmatrix} r^{-4}H^{-1/2}\delta_{ab} & 0 & 0 & 0 \\ 0 & r^{-8}H^{-3/2}\delta^{a_1[b_1}\delta^{b_2]a_2} & 0 & 0 \\ 0 & 0 & r^{-2}H^{-1/2}\delta_{a_3[b_3}\delta_{b_4]a_4} & 0 \\ 0 & 0 & 0 & r^{-6}H^{-3/2}\delta^{a_5b_5} \end{pmatrix} \tag{4.3.15}$$

Here to meet the requirement of det$M$= 1 we have to impose the near-horizon limit of the D2 brane solution by setting $H = \frac{1}{r^5}$.

The generalised metric describing the new uplifted solution is, after using the generalised frame (4.2.15)

$$\mathcal{M}_{MN} = \begin{pmatrix} r^{-3/2}\delta_{mn} & 0 & 2r^{-3/2}\delta_{m[n_2}x_{n_1]} & 0 \\ 0 & 2r^{-1/2}\delta^{m_1[n_1}\delta^{n_2]m_2} & 0 & 2r^{-1/2}x^{[m_1}\delta^{m_2]n} \\ 2r^{-3/2}\delta_{n[m_2}x_{m_1]} & 0 & r^{1/2}K_{m_1m_2,n_1n_2} & 0 \\ 0 & 2r^{-1/2}\delta^{m[n_2}x^{n_1]} & 0 & r^{3/2}K^{mn} \end{pmatrix} \tag{4.3.16}$$

where

$$K_{m_1m_2,n_1n_2} = 2\delta_{m_1[n_1}\delta_{n_2]m_2} + 4r^{-2}x_{[m_2}\delta_{m_1][n_1}x_{n_2]}, \quad K^{mn} = \delta^{mn}(1+r^{-2}x_ax^a) - r^{-2}x^mx^n \,, \tag{4.3.17}$$

We need to compare this with the expression for the parametrisation of the E_7 generalised metric in terms of the internal seven-dimensional components of the metric ϕ_{mn}, three-form and six-form. Referring for example to [99], we see that (4.3.16) corresponds to a generalised metric with vanishing three-form but non-trivial six-form. The precise parametrisation of the generalised metric that we need (taking care to follow the conventions of [91] which we used to construct the EDA generalised frame) then has the form:

$$\mathcal{M}_{MN} = \begin{pmatrix} \phi^{\frac{1}{2}}L_{mn} & 0 & 2\phi_{m[n_2}U_{n_1]} & 0 \\ 0 & \phi^{\frac{1}{2}}(2\phi^{m_1[n_1}\phi^{n_2]m_2} + 4U^{[m_1}\phi^{m_2][n_1}U^{n_2]}) & 0 & 2\phi^{n[m_2}U^{m_1]} \\ 2\phi_{n[m_2}U_{m_1]} & 0 & 2\phi^{-\frac{1}{2}}\phi_{m_1[n_1}\phi_{n_2]m_2} & 0 \\ 0 & 2\phi^{m[n_2}U^{n_1]} & 0 & \phi^{-\frac{1}{2}}\phi^{mn} \end{pmatrix} \tag{4.3.18}$$

where $\phi = \det(\phi_{mn})$,

$$U^m = \frac{1}{6!}\phi^{-1/2}\epsilon^{mn_1\dots n_6}C_{n_1\dots n_6}\,, \quad L_{mn} \equiv \phi_{mn}(1+U_pU^p) - U_mU_n\,, \tag{4.3.19}$$

and $U_m = \phi_{mn}U^n$, where here ϵ denotes the alternating symbol.

Comparing the two expressions we find that the seven-dimensional internal metric is:

$$\phi_{mn} = r^{-1/3}(1+r^{-2}x_px^p)^{-1/3}\big[\delta_{mn} + r^{-2}x_mx_n\big] \tag{4.3.20}$$

and that the six-form is:

$$C_{m_1\dots m_6} = \epsilon_{m_1\dots m_6 n}x^n r^{-2}(1+r^{-2}x_px^p)^{-1}\,. \tag{4.3.21}$$

The latter gives rise to the field strength components

$$F_{m_1\dots m_7} = \epsilon_{m_1\dots m_7}r^{-2}(1+r^{-2}x_nx^n)^{-2}\big[7+5r^{-2}x_px^p\big]\,, \tag{4.3.22}$$

$$F_{rm_1\dots m_6} = -2\epsilon_{m_1\dots m_6 n}x^n r^{-3}(1+r^{-2}x_px^p)^{-2}\,. \tag{4.3.23}$$

Now using the ExFT construction we can build the full new 11-dimensional solution. The 11-dimensional metric is:

$$\hat{g}_{\hat\mu\hat\nu} = \begin{pmatrix} |\phi|^\omega g^{ExFT}_{\mu\nu} + A_\mu{}^kA_\nu{}^l\phi_{kl} & A^k_\mu\phi_{kn} \\ A^k_\nu\phi_{km} & \phi_{mn}\end{pmatrix}\,, \tag{4.3.24}$$

where $\omega = -\frac{1}{n-2} = -\frac{1}{2}$ in our case of $n = 11-d = 4$. The 4-dimensional ExFT metric is that extracted in (4.3.9) from the D2 brane solution, in the near horizon limit:

$$(ds^2)^{ExFT} = r^{7/2}[-dt^2 + dy_1^2 + dy_2^2 + r^{-5}dr^2]\,, \tag{4.3.25}$$

and as there is no one-form present we have $A^k_\mu = 0$. Thus, the new 11-dimensional metric is

$$\hat{ds}^2_{11} = r^{-1/3}(1+r^{-2}x_kx^k)^{-1/3}\Big[r^5(1+r^{-2}x_px^p)[-dt^2+dy_1^2+dy_2^2+r^{-5}dr^2]+(\delta_{mn}+r^{-2}x_mx_n)dx^mdx^n\Big] \tag{4.3.26}$$

The only gauge field components present are those of the six-form given in (4.3.21). We can rewrite our solution in different coordinate systems. We can pass to spherical coordinates in place of the x^i, in terms of which we we can rewrite the new 11-dimensional metric as

$$\hat{ds}^2_{11} = r^{1/3}(r^2+\rho^2)^{2/3}\Big[r^3\big(-dt^2+dy_1^2+dy_2^2+r^{-5}dr^2\big)+r^{-2}d\rho^2\Big]+r^{1/3}(r^2+\rho^2)^{-1/3}\rho^2d\Omega^2_{(6)} \tag{4.3.27}$$

where $\rho^2 \equiv x_ix^i$ and $d\Omega^2_{(6)}$ denotes the metric on the unit six-sphere. The six-form potential and its field strength are:

$$C_{(6)} = \frac{\rho^7}{r^2+\rho^2}\text{Vol}_{S^6}\,, \quad F_{(7)} = -\frac{2\rho^7}{(r^2+\rho^2)^2} r dr \wedge \text{Vol}_{S^6} + \frac{7r^2+5\rho^2}{(r^2+\rho^2)^2}\rho^6 d\rho \wedge \text{Vol}_{S^6}\,. \tag{4.3.28}$$

The four-form field strength obtained by Hodge dualisation is

$$F_{(4)} = r^4(7r^2+5\rho^2)dt\wedge dy^1 \wedge dy^2 \wedge dr + 2r^5 \rho dt \wedge dy^1 \wedge dy^2 \wedge d\rho\,. \tag{4.3.29}$$

A further coordinate change relates the 4-dimensional part of the metric to a familiar form of the metric on AdS_4. This is a property inherited from the original D2 solution, whose near horizon string frame metric is a function of the radial coordinate times $AdS_4 \times S^6$ (in a dual frame [119] the metric is exactly $AdS_4 \times S^6$). By introducing a new coordinate

$$\tilde{r} \equiv \frac{2}{3} r^{3/2} \tag{4.3.30}$$

then the 4-dimensional bit of the solution can be shown to involve an AdS_4 metric in the Poincare patch, using the fact that

$$r^3[-dt^2+dy_1^2+dy_2^2+r^{-5}dr^2] = \mathcal{R}^{-2}\tilde{r}^2[-dt^2+dy_1^2+dy_2^2] + \mathcal{R}^2\frac{d\tilde{r}^2}{\tilde{r}^2} \tag{4.3.31}$$

where $\mathcal{R} = 2/3$ is the AdS radius.

We can finally comment on the behaviour of our metric as $r \to 0$. The Ricci scalar is

$$R = -\tfrac{1}{6} r^{-1/3}(49r^2+25\rho^2)(r^2+\rho^2)^{-5/3} \tag{4.3.32}$$

and so the solution is singular for $r \to 0$. This is also a feature of the D2 brane near horizon solution.

4.3.3 Properties of the new solution

Charges and global properties

The solution that we have obtained is a local solution: we have not yet specified the range of the coordinates x^i, or alternatively that of ρ if we change to spherical coordinates. The situation is entirely analogous to that found when obtaining solutions via non-abelian T-duality, and to our previous generalised U-duality construction [113]. If the x^i are to be regarded as periodically identified then our solution can be regarded as a non-geometric background, globally identified up to a non-trivial E_7 transformation acting as a constant shift of the six-vector used in constructing the solution, as noted in [51] and similar to examples in [63, 85, 113, 120]. Alternatively, we can work in the spherical coordinates and attempt to fix the range of ρ by requiring the solution carry well-defined brane charges.

Accordingly, let's consider the charges of the new uplifted solution. It only carries electric M2 charge, namely

$$Q_{M2} \sim \int \star F_4 = \int dC_6 \,, \tag{4.3.33}$$

where from above $C_6 = \rho^7/(r^2+\rho^2)\mathrm{Vol}_{S^6}$. We could try to specify a seven-cycle to evaluate this charge (generalising the argument of [98] for non-abelian T-dual solutions) by integrating from $\rho = \rho_0$ to some value $\rho = \rho_1$ at a fixed value of $r = r_0$, and then integrate from $r = r_0$ to $r = r_1$ at fixed $\rho = \rho_1$, such that the six-sphere part of the solution vanishes at $\rho = \rho_0$ and $r = r_1$. The result is independent of r_0, and gives $16\pi^3\rho_1^7/15(r_1^2+\rho_1^2)$. Choosing $\rho_0 = 0$ and $r_1 = 0$ would give an electric charge $Q_{M2} \sim 16\pi^3\rho_1^5/15$, which on properly reinserting dimensionful constants could be argued to fix ρ_1 by requiring the charge is an integer times the M2 charge.

Note that this M2 charge is analogous to the M5 charge appearing in our earlier solution [113], hence in this "dual" example the electric and magnetic charges are swapped, mirroring the swap of trivector and six-vector we noted earlier. To be more specific, the relevant M5 charge of [113] is that which is present when the initial solution there is solely the near horizon NS5 brane. It was also possible in [113] to start with an F1-NS5 intersection. The resulting new 11-dimensional solution then required a different global completion which was possible at least for its AdS_3 limit. This limit fit into a class of solutions [92] in a manner reminiscent of AdS solutions obtained via non-abelian T-duality. This involved a linear function of ρ^2, defined on a series of subintervals with jumps in slope across each subinterval. It is unclear if it is possible to apply similar thinking to our example in this chapter (which has a more complicated functional dependence on the r coordinate alongside ρ), or to find or classify other solutions built using the ISO(7) generalised frame.

SUSY analysis

Let us now look at the solution of the Killing spinor equation and find out how many supersymmetries the new uplifted solution has. The Killing spinor equation we need to solve is[5]

$$\delta_\epsilon \psi_\mu = 2\partial_\mu \epsilon - \frac{1}{2}\omega_\mu{}^{ab}\Gamma_{ab}\epsilon + \frac{i}{144}(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\Gamma^{\beta\gamma\delta}\eta^\alpha_\mu)\epsilon F_{\alpha\beta\gamma\delta} = 0 \tag{4.3.34}$$

where the Greek indices are the curved coordinates, and Latin indices are the flat ones. For the t-component (and similarly for y^1 and y^2), using the hatted indices for the curved coordinates, and unhatted for the flat ones, we explicitly have:

$$\Gamma^t\partial_{\hat{t}}\epsilon + \frac{1}{6}r^{1/2}(1+r^{-2}\rho^2)^{-1}\Big[2\rho\Gamma_\rho + 7r(1+\frac{5}{7}r^{-2}\rho^2)\Gamma_r - i\Big(2\rho\Gamma^{ty^1y^2\rho} + 7r(1+\frac{5}{7}r^{-2}\rho^2)\Gamma^{ty^1y^2r}\Big)\Big]\epsilon = 0\,. \tag{4.3.35}$$

Assuming that ϵ is t-independent (similarly y^1 and y^2 independent), and looking at the similar coordinate dependence in front of the same gamma-matrix combination, we can extract the following

[5]We follow the conventions we used in [113], in particular $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$ with η_{ab} having mostly minus signature.

projection condition on ϵ

$$(1 + i\Gamma^{ty^1y^2})\epsilon = 0 \tag{4.3.36}$$

which we can use in solving the rest of the equations.

The r and ρ equations become

$$\partial_{\hat{r}}\epsilon = r^{-1}(1 + r^{-2}\rho^2)^{-1}\Big[1 + \frac{1}{6}(1 + 5r^{-2}\rho^2) - \frac{1}{2}r^{-1}\rho\Gamma_{r\rho}\Big]\epsilon \tag{4.3.37}$$

$$\partial_{\hat{\rho}}\epsilon = \frac{1}{6}r^{-1}(1 + r^{-2}\rho^2)^{-1}[2r^{-1}\rho + 3\Gamma_{r\rho}]\epsilon \tag{4.3.38}$$

with the common solution

$$\epsilon = \epsilon_{r\rho}\bar{\epsilon}\,, \quad \epsilon_{r\rho} = r^{7/6}(1 + r^{-2}\rho^2)^{1/6}\exp\left[-\frac{1}{2}\Gamma_{r\rho}\tan^{-1}(\frac{r}{\rho})\right]\,, \tag{4.3.39}$$

where $\bar{\epsilon}$ depends on the S^6 coordinates only. Now, working in round spherical coordinates $(\chi, \theta_1, \ldots, \theta_5)$ on S^6, we can find a solution of the form $\bar{\epsilon} = \epsilon_\chi\epsilon_{\theta_1}\ldots\epsilon_{\theta_5}\epsilon_0$ with ϵ_0 a constant spinor. Indeed, we firstly have the equation

$$\partial_{\hat{\chi}}\epsilon = \frac{1}{2}(1 + r^{-2}\rho^2)^{-1/2}[\Gamma_{\rho\chi} + r^{-1}\rho\Gamma_{r\chi}]\epsilon \tag{4.3.40}$$

where we can commute the gamma matrices from the $\epsilon_{r\rho}$ part, moving it to the left of both sides of the equation, and end up solving for ϵ_χ

$$\epsilon_\chi = \exp\big[\frac{1}{2}\Gamma_{r\chi}\chi\big] \tag{4.3.41}$$

and in a similar manner for the rest of the 5 angles $\theta_1 \ldots\ldots \theta_5$ we find

$$\epsilon_{\theta_1} = \exp\big[\frac{1}{2}\Gamma_{\chi\theta_1}\theta_1\big], \quad \epsilon_{\theta_2} = \exp\big[\frac{1}{2}\Gamma_{\theta_1\theta_2}\theta_2\big], \ \text{etc...} \tag{4.3.42}$$

so the final solution is of the form

$$\epsilon = \epsilon_{r\rho}\epsilon_\chi\epsilon_{\theta_1}\cdots\epsilon_{\theta_5}\epsilon_0 \tag{4.3.43}$$

where after applying the condition (4.3.36) ϵ_0 is a constant spinor satisfying

$$(1 + i\Gamma^{ty^1y^2})\epsilon_0 = 0 \tag{4.3.44}$$

which kills a half of the total degrees of freedom, thus, our solution is $\frac{1}{2}$-BPS. This is consistent with the supersymmetry of the initial D2 solution and with the supersymmetry preservation of our previous example of 'generalised U-duality' [113].

4.4 Conclusion

In this chapter we discussed another example of a solution generating mechanism which can be viewed as a generalised U-duality transformation. We used a special case of the E_7 exceptional Drinfeld algebra, describing the four-dimensional ISO(7) gauging, and used this to construct a new 11-dimensional solution starting with the near horizon limit of the $D2$ brane solution of type IIA SUGRA. This can be seen as a "dual" construction (in the electromagnetic sense) of our previous example, based on the $Sl(5)$ exceptional Drinfeld algebra corresponding to the seven-dimensional ISO(4) gauging [113]. Together these examples generalise, in a particular manner, features of non-abelian T-duality to the 11-dimensional setting (see table 4.1), using the natural exceptional Drinfeld algebra cases with either non-trivial 3- and 6-algebra structure constants, and hence non-trivial tri- and six-vectors.

Solution obtained by:	Algebraic structure	Generalised frame
Non-abelian T-duality of S^3	$\tilde{f}^{ab}{}_c = \epsilon^{ab}{}_c$	bivector
$Sl(5)$ generalised U-duality of S^3 (w/NSNS flux)	$\tilde{f}^{abc}{}_d = \epsilon^{abc}{}_d$	trivector
E_7 generalised U-duality of S^6 (w/RR flux)	$\tilde{f}^{abcdef}{}_g = \epsilon^{abcdef}{}_g$	six-vector

Table 4.1: Properties of generalised dualities

We have so far only considered the M-theory realisation of the EDA, but it would be interesting to systematically explore similar features in its IIA and IIB decompositions. Here we would expect to construct a variety of other generalised frames involving n-vectors with a linear coordinate dependence, and identify the lower-dimensional gaugings these capture.

The usefulness of these constructions depends on whether the choice of EDA allows one to access gauged supergravities with either interesting known solutions or known alternative origins as consistent truncations from 10- and 11-dimensions. In this paper and in [113] we used the latter approach to identify brane solutions at the 10-dimensional level to which we could apply reduction and uplift. The ISO(7) example of this paper is a case where there are in fact no known vacua (the D2 brane solution reducing to a domain wall solution). We have made choices for the EDA which seemed algebraically 'natural' and to some extent gotten lucky in finding that these corresponded to uplifts of known gauged supergravities in fact corresponding to consistent truncations already identified from a different, though related, perspective in [51]. It would be good to extend and improve this search strategy, including to situations with simultaneously non-trivial 3- and 6-algebra structure constants, and more broadly to try to understand exactly what is the common feature (spheres with flux?) of the initial solutions 'dual' to the solutions built using these EDA generalised frames, and how the n-algebra symmetry manifests in these background (if at all).

A natural question about the ISO(7) case concerns whether we can do anything with the *dyonic* ISO(7) gaugings [121–123], which have a richer vacuum structure. These gaugings can be obtained by a consistent truncation from the massive type IIA theory on S^6 [115,116]: we have been using this consistent truncation in the massless limit for the ISO(7) gauging. It can be quickly checked that this gauging modifies the algebra (4.2.11) with additional non-zero brackets including $[\tilde{T}^a, \tilde{T}^b] \sim T^{ab}$. This bracket is however always zero in the EDA construction [91]. Hence the dyonic ISO(7) algebra is not an EDA – if it were we would immediately know how to construct a [geometric] generalised frame realising it. Indeed we have been informed by Y. Sakatani that making this bracket non-zero in an extension of the EDA always requires locally non-geometric R-fluxes, in agreement with the statement of [51] implying the dyonic ISO(7) gauging does not admit a locally geometric uplift.

Chapter 5

Conclusion

Summarizing the thesis, we have made an insightful overview of the supergravity and M-theory studying the dualities connecting different solutions within these theories. Leveraging the techniques of the Leibniz algebra and the generalised geometry, we have constructed several examples of new solutions generated with the ExFT formalism. The examples discussed clearly demonstrate the capabilities of the generalised U-duality in building liaisons between seemingly different solutions in supergravity (M-theory in general), and provides new hints towards the nature of the complicated web of M-theory solutions.

We saw that the structure of the Exceptional Drinfeld Algebra (EDA) is based on the existing of isotropic subalgebras, with a very restrictive conditions in order for the isotropies to appear, using the fact that the EDA generalised frame admits a consistent truncation to a lower dimensional theory. We have seen that not all of the M-theory isotropies are compatible with the EDA, and not all of the isotropies turn to admit the geometric generalised frame description.

Generalised U-duality can be used as a solution generating technique in supergravity, as we successfully demonstrated in chapter 3, using a consistent truncation from an initial theory to a maximally gauged 7-dimensional supergravity. We used this procedure to uplift the resulting gauged 7-dimensional model to a new dual solution using the EDA frame. We were able to generate dual pairs of solutions starting from the known type IIA supergravities on S^3 and uplifting it with the EDA frame.

It is, however, not clear whether the generalisation to other dimensions will express a similar behaviour with the ability of developing a generalised set of rules to describe the technique of generating dual solutions. For example, in the case of the generalised T-duality it is clear that the space-time with non-abelian isometries admit the generalise T-dual solution. In the case of the generalised U-duality a similar identifying criterium is lacking. The open questions are still the behaviour of the theories in supergravities with other duality groups. A preliminary study of the $E_{6(6)}$ gauged supergravity was given in [79]. Extending the classification to higher non-abelian group would be helpful to view the generalised U-duality as an algorithmic solution generating technique, that will enable develop a better way to reveal the characteristics of the nature of dualities in M-theory.

Another question open for a future research is the preservation of supersymmetry in the U-dual solution. Up to this point, we were able to spot that the number of supersymmetries remained preserved in the examples of $ISO(4)$ and $ISO(7)$ gauged theories. However, these were not the most general cases that we considered, and the question of the general proof is still open for further investigation. In the framework of this research we demonstrated on a few examples the conservation of the number of supersymmetries in the case of the CSO(4,0,1) supergravity. For the non-abelian T-duality the number of supersymmetries in the dual solution turn out to be reduced compared to the original solution. While, in the examples we reviewed in the thesis, for the case of generalised U-duality we observed the preservation of the number of supersymmetries after applying the duality transformation. Here, we must highlight that the examples we considered were viewed in the AdS limit.

After a proper consideration of the example with the CSO(4,0,1) gauged supergravity, we turned towards studying a special case of the E_7 gauged supergravity. This choice wasn't spontaneous – it reflects the duality in 11 dimensional M-theory between 4 and 7 dimensions, corresponding to the numeration of the gauging exceptional algebras, as well as reflecting the dualities of the supergravity models. We studied a special case of E_7 Exceptional Drinfeld algebra - the ISO(7) gauging case. This example also has a dual electromagnetic interpretation to the previous CSO(4,0,1) case. In terms of non-abelian Drinfeld algebra, these 2 examples also has a dual interpretation with either non-trivial 3 or 6 - algebra structure constants, corresponding to either tri-vectors or six-vectors. The open question here is generalisation of the suggested technique to other dimensions, correspondingly generating new set of n-vectors that describe the generalised frame of the n-algebra. The possibility of constructing such examples in other dimensions will depend on whether the choice of EDA will admit a gauged supergravity solution known previously or a consistently truncated from a known 11-dimensional solution. A good future direction would be to understand the common characteristics of the solution that can generate dual solutions using the EDA frames and the n-algebra behaviour in these backgrounds.

An open questions for future endeavours remains the investigation of the dyonic gauging special case in the ISO(7) gauging, which will involve an additional bracket relation in the algebra, removing it from the EDA algebras class. Our expectation is getting similar results to the SO(8) dyonic case, with a generalisation of the no-go theorem for this dyonic case.

Via analysis made in this chapter we, in a way, confirmed the general pattern between different gauging groups applied to the dualisation technique - we spotted similar features for the so-called electromagnetic dual case of ISO(7) gauging, comparing to its ISO(4) gauging counterpart discussed in the previous chapter. This study helped revealing more properties of the mechanisms of generalised U-duality and how different dual solution share similar features or possess differences compared to each other (supersymmetry, charges, etc..). This provides an insightful description of the principles taking place in the generalised U-duality procedure and the novel solutions generated through it, emphasising an interesting connectivity between different classes of gauging groups and dual solutions.

Appendix A

Appendix

A.1 SL(5) exceptional geometry

A.1.1 Generalised Lie derivative and generalised frames

Here we describe some of the technology of SL(5) exceptional generalised geometry / exceptional field theory [37,53–56]. We will use capital calligraphic indices $\mathcal{M}, \mathcal{N}, \dots = 1, \dots 5$ to label quantities transforming in the **5**, and use antisymmetric pairs of such indices to label quantities transforming in the **10**.

We start with the definition of the generalised Lie derivative, which captures the bosonic local symmetries (diffeomorphisms and gauge transformations) of supergravity. Let $A \in \mathbf{10}$ be a generalised vector of weight λ_A and $\Lambda \in \mathbf{10}$ be a generalised vector of weight $\lambda_\Lambda = -\omega \equiv 1/5$. The generalised Lie derivative of V with respect to Λ is

$$\mathcal{L}_\Lambda A^{\mathcal{M}\mathcal{N}} = \frac{1}{2}\Lambda^{\mathcal{P}\mathcal{Q}}\partial_{\mathcal{P}\mathcal{Q}}A^{\mathcal{M}\mathcal{N}} + 2\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{P}[\mathcal{M}}A^{\mathcal{N}]\mathcal{Q}} + \frac{1}{2}(1+\lambda_A+\omega)\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{P}\mathcal{Q}}A^{\mathcal{M}\mathcal{N}}\,. \tag{A.1.1}$$

Meanwhile a generalised tensor $C \in \mathbf{5}$ of weight λ_C has generalised Lie derivative

$$\mathcal{L}_\Lambda C^{\mathcal{M}} = \frac{1}{2}\Lambda^{\mathcal{P}\mathcal{Q}}\partial_{\mathcal{P}\mathcal{Q}}C^{\mathcal{M}} - C^{\mathcal{P}}\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{M}\mathcal{Q}} + \frac{1}{2}(\lambda_C+1+3\omega)\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{P}\mathcal{Q}}C^{\mathcal{M}}\,. \tag{A.1.2}$$

The actual coordinate dependence of all quantities in the theory is restricted by the formally SL(5) covariant section condition

$$\partial_{[\mathcal{M}\mathcal{N}} \otimes \partial_{\mathcal{K}\mathcal{L}]} = 0\,, \tag{A.1.3}$$

which has independent "solutions" [125] that break SL(5) covariance and correspond to underlying M-theory, type IIA or type IIB geometries.

M-theory generalised geometry

For the M-theory solution of the section condition, we label the SL(5) indices as $\mathcal{M} = (i, 5)$, with $i = 1, \ldots, 4$, and impose that $\partial_{ij} = 0$ acting on all quantities in the theory. Then in terms of the underlying M-theory generalised geometry we find that quantities in the **10** decompose as a pair consisting of a vector and a two-form, which are sections of (perhaps weighted) generalised tangent bundles

$$\Lambda = (v, \lambda_{(2)}) \in TM \oplus \Lambda^2 T^* M \,, \tag{A.1.4}$$

$$A = (a, \alpha_{(2)}) \in (TM \oplus \Lambda^2 T^* M) \otimes (\det T^* M)^{(\lambda_A + \omega)/2} \,, \tag{A.1.5}$$

and the generalised Lie derivative acts as:

$$\mathcal{L}_\Lambda A = (L_v a, L_v \alpha_{(2)} - \iota_a d\lambda_{(2)}) \,, \tag{A.1.6}$$

where the ordinary Lie derivative L_v acts on the vector v and two-form $\alpha_{(2)}$ which are of weight $\lambda_A + \omega$.

Meanwhile, a generalised tensor C in the fundamental corresponds to a scalar plus a three-form:

$$C = (c_{(0)}, c_{(3)}) \in (\mathbb{R} \oplus \Lambda^3 T^* M) \otimes (\det T^* M)^{(\lambda_C + 3\omega)/2} \,, \tag{A.1.7}$$

and

$$\mathcal{L}_\Lambda C = (L_v c_{(0)}, L_v c_{(3)} + d\lambda_{(2)} c_{(0)}) \tag{A.1.8}$$

in which the ordinary Lie derivative acts on the scalar $c_{(0)}$ and three-form $c_{(3)}$ which are of weight $\lambda_C + 3\omega$.

Type IIB generalised geometry

The type IIB solution of the section condition splits $\mathcal{M} = (i, \alpha)$ with $i = 1, 2, 3$ the spacetime index and $\alpha = 4, 5$ an SL(2) S-duality index. We impose $\partial_{i\alpha} = \partial_{\alpha\beta} = 0$ acting on all fields in the theory, and identify the natural derivatives with respect to the spacetime coordinates as $\partial^i \equiv \frac{1}{2}\epsilon^{ijk}\partial_k$. The positions of spacetime indices therefore naturally come out reversed.

A generalised vector A of weight λ_A can now be decomposed in terms of vectors, a doublet of one-forms and a three-form:

$$A = (a, \alpha_{(1)}, \tilde{\alpha}_{(1)}, \alpha_{(3)}) \in (TM \oplus T^* M \oplus T^* M \oplus \Lambda^3 T^* M) \otimes (\det T^* M)^{(\lambda_A + \omega)/2} \tag{A.1.9}$$

and with $\Lambda = (v, \lambda_{(1)}, \tilde{\lambda}_{(1)}, \lambda_{(3)})$ of weight $\lambda_\Lambda = 1/5$, the generalised Lie derivative acts as

$$\mathcal{L}_\Lambda A = (L_v a, L_v \alpha_{(1)} - \iota_a d\lambda_{(1)}, L_v \tilde{\alpha}_{(1)} - \iota_a d\tilde{\lambda}_{(1)}, L_v \alpha_{(3)} - d\lambda_{(1)} \wedge \tilde{\alpha}_{(1)} + d\tilde{\lambda}_{(1)} \wedge \alpha_{(1)}) \,, \tag{A.1.10}$$

with the spacetime Lie derivative L_v acting on the tensors here which are of spacetime weight $\lambda_A+\omega$.

A generalised tensor C of weight λ_C in the fundamental is equivalent to a one-form and a doublet of three-forms, all of spacetime weight $\lambda_C + 3\omega$:

$$C = (c_{(1)}, c_{(3)}, \tilde{c}_{(3)}) \in (T^*M \oplus \Lambda^3 T^*M \oplus \Lambda^3 T^*M) \otimes (\det T^*M)^{(\lambda_C+3\omega)/2} \tag{A.1.11}$$

with

$$L_\Lambda C = (L_v c_{(1)}, L_v c_{(3)} - c_{(1)} \wedge d\lambda_{(1)}, L_v \tilde{c}_{(3)} - c_{(1)} \wedge d\tilde{\lambda}_{(1)}) \,. \tag{A.1.12}$$

A.1.2 Generalised frames and their algebra

The physical fields describing the geometry live in the coset SL(5)/SO(5), which is parametrised by a unit determinant (inverse) generalised vielbein $\tilde{E}^{\mathcal{M}\mathcal{N}}{}_{\mathcal{A}\mathcal{B}} = 2\tilde{E}^{[\mathcal{M}}{}_{\mathcal{A}}\tilde{E}^{\mathcal{N}]}{}_{\mathcal{B}}$. The generalised vielbein $\tilde{E}^{\mathcal{M}}{}_{\mathcal{A}}$ in the **5** and that $\tilde{E}^{\mathcal{M}\mathcal{N}}{}_{\mathcal{A}\mathcal{B}}$ in the **10** have weight 0. In order to construct the algebra of frame fields, we have to instead use a generalised vielbein $E^{\mathcal{M}\mathcal{N}}{}_{\mathcal{A}\mathcal{B}}$ of weight $-\omega = 1/5$. This parametrises the coset $\mathbb{R}^+ \times \mathrm{SL}(5)/\mathrm{SO}(5)$. TO describe the $\mathbb{R}^+$ factor, we introduce a scalar Δ of weight 1/5:

$$\mathcal{L}_\Lambda \Delta = \frac{1}{2}\Lambda^{\mathcal{P}\mathcal{Q}}\partial_{\mathcal{P}\mathcal{Q}}\Delta + \frac{1}{2}\frac{1}{5}\partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{P}\mathcal{Q}}\Delta \tag{A.1.13}$$

and define

$$E^{\mathcal{M}}{}_{\mathcal{A}} = \Delta^{1/2}\tilde{E}^{\mathcal{M}}{}_{\mathcal{A}} \quad E^{\mathcal{M}\mathcal{N}}{}_{\mathcal{A}\mathcal{B}} = 2E^{[\mathcal{M}}{}_{\mathcal{A}}E^{\mathcal{N}]}{}_{\mathcal{B}} = \Delta\tilde{E}^{\mathcal{M}\mathcal{N}}{}_{\mathcal{A}\mathcal{B}} \,. \tag{A.1.14}$$

Hence $E^{\mathcal{M}}{}_{\mathcal{A}}$ is a set of 5 generalised tensors of weight $\lambda_{E_\mathcal{A}} = 1/10$, so $\lambda_{E_\mathcal{A}} + 3\omega = -1/2$. Using these quantities, the algebra of generalised frames under the generalised Lie derivative can be written

$$\mathcal{L}_{E_{\mathcal{A}\mathcal{B}}} E^{\mathcal{M}}{}_{\mathcal{C}} = -F_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}} E^{\mathcal{M}}{}_{\mathcal{D}} \,, \tag{A.1.15}$$

hence

$$\mathcal{L}_{E_{\mathcal{A}\mathcal{B}}} E^{\mathcal{M}\mathcal{N}}{}_{\mathcal{C}\mathcal{D}} = -\frac{1}{2}F_{\mathcal{A}\mathcal{B},\mathcal{C}\mathcal{D}}{}^{\mathcal{E}\mathcal{F}} E^{\mathcal{M}\mathcal{N}}{}_{\mathcal{E}\mathcal{F}} = 2F_{\mathcal{A}\mathcal{B}[\mathcal{C}}{}^{\mathcal{E}} E_{\mathcal{D}]E} \,, \tag{A.1.16}$$

where

$$F_{\mathcal{A}\mathcal{B},\mathcal{C}\mathcal{D}}{}^{\mathcal{E}\mathcal{F}} = 4F_{\mathcal{A}\mathcal{B}[\mathcal{C}}{}^{[\mathcal{E}}\delta^{\mathcal{F}]}_{\mathcal{D}]} \,. \tag{A.1.17}$$

The form of the generalised Lie derivative means that the generalised flux $F_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}}$ can be decomposed in terms of irreducible representations of SL(5)

$$F_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}} = X_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}} - \frac{1}{6}\tau_{\mathcal{A}\mathcal{B}}\delta^{\mathcal{D}}_{\mathcal{C}} - \frac{1}{3}\delta^{\mathcal{D}}_{[\mathcal{A}}\tau_{\mathcal{B}]\mathcal{C}} \tag{A.1.18}$$

with

$$X_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}} = Z_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}} + \frac{1}{2}\delta^{\mathcal{D}}_{[\mathcal{A}}S_{\mathcal{B}]\mathcal{C}} \,. \tag{A.1.19}$$

Here $\tau_{\mathcal{AB}} \in \overline{\mathbf{10}}$ is the so-called trombone gauging [126], $S_{\mathcal{AB}} \in \overline{\mathbf{15}}$ and $Z_{\mathcal{ABC}}{}^{\mathcal{D}} \in \mathbf{40}$ obeys $Z_{\mathcal{ABC}}{}^{\mathcal{D}} = Z_{[\mathcal{ABC}]}{}^{\mathcal{D}}$, $Z_{\mathcal{ABC}}{}^{\mathcal{C}} = 0$. Explicit expressions in terms of the unweighted and weighted vielbeins are:

$$\begin{aligned} \tau_{\mathcal{AB}} &= \Delta\left(6\tilde{E}^{\mathcal{M}}{}_{\mathcal{A}}\tilde{E}^{\mathcal{N}}{}_{\mathcal{B}}\partial_{\mathcal{MN}}\ln\Delta + \partial_{\mathcal{MN}}(\tilde{E}^{\mathcal{M}}{}_{\mathcal{A}}\tilde{E}^{\mathcal{N}}{}_{\mathcal{B}})\right) \\ &= 5E^{\mathcal{M}}{}_{\mathcal{A}}E^{\mathcal{N}}{}_{\mathcal{B}}\partial_{\mathcal{MN}}\ln\Delta + \partial_{\mathcal{MN}}(E^{\mathcal{M}}{}_{\mathcal{A}}E^{\mathcal{N}}{}_{\mathcal{B}}) \end{aligned} \tag{A.1.20}$$

$$S_{\mathcal{AB}} = 4\Delta\tilde{E}^{\mathcal{M}}{}_{(\mathcal{A}|}\partial_{\mathcal{MN}}\tilde{E}^{\mathcal{N}}{}_{|\mathcal{B})} = 4E^{\mathcal{M}}{}_{(\mathcal{A}|}\partial_{\mathcal{MN}}E^{\mathcal{N}}{}_{|\mathcal{B})} \tag{A.1.21}$$

$$\begin{aligned} Z_{\mathcal{ABC}}{}^{\mathcal{D}} &= \Delta\left(3\tilde{E}^{\mathcal{M}}{}_{[\mathcal{A}}\tilde{E}^{\mathcal{N}}{}_{\mathcal{B}}\tilde{E}^{\mathcal{P}}{}_{\mathcal{C}]}\partial_{\mathcal{MN}}\tilde{E}^{\mathcal{D}}{}_{\mathcal{P}} - 2\delta^{\mathcal{D}}_{[\mathcal{A}|}\partial_{\mathcal{MN}}\tilde{E}^{\mathcal{M}}{}_{|\mathcal{B}}\tilde{E}^{\mathcal{N}}{}_{\mathcal{C}]}\right) \\ &= 3\left(E^{\mathcal{M}}{}_{[\mathcal{A}}E^{\mathcal{N}}{}_{\mathcal{B}}E^{\mathcal{P}}{}_{\mathcal{C}]}\partial_{\mathcal{MN}}E^{\mathcal{D}}{}_{\mathcal{P}} - \frac{1}{2}\delta^{\mathcal{D}}_{[\mathcal{A}}\partial_{|\mathcal{MN}|}(E^{\mathcal{M}}{}_{\mathcal{B}}E^{\mathcal{N}}{}_{\mathcal{C}]})\right) + \frac{1}{2}\delta^{\mathcal{D}}_{[\mathcal{A}}\tau_{\mathcal{BC}]}\,. \end{aligned} \tag{A.1.22}$$

A.1.3 Dictionary to 11- and 10-dimensional geometries

The SL(5) generalised geometry splits the full 11- or 10-dimensional geometry into a seven-dimensional "external" part and a four-dimensional "internal" part. The 11- or 10-dimensional Einstein frame metric is decomposed as:

$$ds^2_{11} = g^{-1/5}G_{\mu\nu}dX^\mu dX^\nu + g_{ij}(dx^i + A_\mu{}^i dX^\mu)(dx^j + A_\nu{}^j dX^\nu)\,, \tag{A.1.23}$$

where $G_{\mu\nu}$, $\mu,\nu = 0,\dots,6$, corresponds to a seven-dimensional Einstein frame U-duality invariant metric, and has weight 2/5 under generalised Lie derivatives. It is consistent to then identify

$$\Delta = (\det G_{\mu\nu})^{1/14}\,. \tag{A.1.24}$$

The fields carrying both external and internal indices (such as the Kaluza-Klein vector $A_\mu{}^i$) appear in the SL(5) ExFT as n-dimensional p-forms in various representations of SL(5). However, we will assume that these all vanish in our set-up. We therefore have just to describe the internal metric and three-form, which together parametrise the afore-mentioned coset SL(5)/SO(5).

M-theory parametrisation

Start with the M-theory solution of the section condition, with physical coordinates $x^i \equiv x^{i5}$. A conventional representation of the SL(5)/SO(5) coset in terms of a (unit determinant) generalised vielbein, consistent with the diffeomorphism and gauge transformations generated by the generalised Lie derivative, is

$$\tilde{E}^{\mathcal{A}}{}_{\mathcal{M}} = g^{1/20}\begin{pmatrix} g^{-1/4}e^a{}_m & -g^{-1/4}e^a{}_nC^n \\ 0 & g^{1/4} \end{pmatrix}\,, \tag{A.1.25}$$

leading to a generalised metric $m_{\mathcal{M}\mathcal{N}} = \tilde{E}^{\mathcal{A}}{}_{\mathcal{M}} \tilde{E}^{\mathcal{B}}{}_{\mathcal{N}} \delta_{\mathcal{A}\mathcal{B}}$ in a five-dimensional representation

$$m_{\mathcal{M}\mathcal{N}} = g^{1/10} \begin{pmatrix} g^{-1/2} g_{mn} & -g^{-1/2} g_{mp} C^p \\ -g^{-1/2} g_{np} C^p & g^{1/2} + g^{-1/2} g_{pq} C^p C^q \end{pmatrix} , \tag{A.1.26}$$

where the four-dimensional metric is written as $g_{mn} = e^a{}_m e^b{}_n \delta_{ab}$ and the three-form $C^m = \frac{1}{6} \epsilon^{mnpq} C_{npq}$, where $\epsilon^{1234} = 1$ is the alternating symbol.

IIB parametrisation

The IIB solution of the section condition identifies the three-dimensional coordinates as $\tilde{x}_i \equiv \frac{1}{2} \epsilon_{ijk} x^{jk}$. In this case, denote the (Einstein frame) spacetime metric by g^{ij}, the vielbein by $e_a{}^i$, and their determinants by $g \equiv \det(g^{ij})$, $e \equiv \det(e_a{}^i)$. The alternating symbol in spacetime is ϵ^{ijk}, and has weight -1, and ϵ_{ijk} has weight $+1$. Also let $h^{\bar{\alpha}}{}_\alpha$ denote a vielbein for the coset SL(2)/SO(2) parametrised by the axio-dilaton, with $\mathcal{H}_{\alpha\beta} = h^{\bar{\alpha}}{}_\alpha h^{\bar{\alpha}}{}_\alpha \delta_{\bar{\alpha}\bar{\beta}}$. Then the IIB geometric parametrisation takes

$$E^{\mathcal{A}}{}_{\mathcal{M}} = e^{1/10} \begin{pmatrix} e^{1/2} e_i{}^a & 0 \\ e^{-1/2} h^{\bar{\alpha}}{}_\alpha C_i^\alpha & e^{-1/2} h^{\bar{\alpha}}{}_\alpha \end{pmatrix} , \quad h^{\bar{\alpha}}{}_\alpha = e^{\Phi/2} \begin{pmatrix} 1 & C_0 \\ 0 & e^{-\Phi} \end{pmatrix} , \tag{A.1.27}$$

$$m_{\mathcal{M}\mathcal{N}} = g^{1/10} \begin{pmatrix} g^{1/2} g_{ij} + g^{-1/2} \mathcal{H}_{\alpha\beta} C_i^\alpha C_j^\beta & g^{-1/2} \mathcal{H}_{\beta\gamma} C_i^\gamma \\ g^{-1/2} \mathcal{H}_{\alpha\gamma} C_j^\gamma & g^{-1/2} \mathcal{H}_{\alpha\beta} \end{pmatrix} , \tag{A.1.28}$$

with

$$C_i^\alpha = \frac{1}{2} \epsilon_{ijk} (C^{jk}, B^{jk}) , \quad \mathcal{H}_{\alpha\beta} = e^\Phi \begin{pmatrix} 1 & C_0 \\ C_0 & C_0^2 + e^{-2\Phi} \end{pmatrix} . \tag{A.1.29}$$

A.2 Embedding Drinfeld doubles in SL(5)

A.2.1 Half-maximal truncation

In order to describe an embedding of a Drinfeld double, we can truncate the Exceptional Drinfeld Algebra. This means reducing from SL(5) to $SO(3,3)$, along the lines of [37, 68]. The **5** of SL(5) produces one of the four-dimensional Majorana-Weyl spinor representations of (the double cover of) $SO(3,3)$ plus a singlet. In terms of the five-dimensional indices, we write $\mathcal{M} = (I, 4)$ where $I = 1, 2, 3, 5$ is the spinorial index. We break $\partial_{\mathcal{M}\mathcal{N}} = (\partial_{IJ}, \partial_{I4})$ and impose $\partial_{I4} = 0$. The bispinorial derivative ∂_{IJ} in fact transforms in the vector representation **6** of $SO(3,3)$.

We can compute the $O(3,3)$ generalised Lie derivative acting on the $\mathbf{5} = \mathbf{4} \oplus \mathbf{1}$, using (A.1.2). The singlet component transforms as a scalar of weight $\lambda_C + 1 + 3\omega$ under $O(3,3)$ diffeomorphisms with parameter Λ^{IJ}

$$\mathcal{L}_\Lambda C^4 = \frac{1}{2} \Lambda^{IJ} \partial_{IJ} C^4 + \frac{1}{2} (\lambda_C + 1 + 3\omega) \partial_{IJ} \Lambda^{IJ} C^4 . \tag{A.2.1}$$

The spinor in the **4** transforms as:

$$\mathcal{L}_\Lambda C^I = \frac{1}{2}\Lambda^{JK}\partial_{JK}C^I + \frac{1}{2}(\lambda_C + 1 + 3\omega)\partial_{JK}\Lambda^{JK}C^I - C^J\partial_{JK}\Lambda^{IK}\,, \tag{A.2.2}$$

defining an $SO(3,3)$ spinorial generalised Lie derivative [37]. Now, the generalised frame field $E^{\mathcal{M}}{}_{\mathcal{A}}$ has weight $\lambda_{E_{\mathcal{A}}} = 1/10$. Hence $E^4{}_{\mathcal{A}}$ gives $SO(3,3)$ scalars of weight 1/2, and $E^I{}_{\mathcal{A}}$ gives $SO(3,3)$ spinors. After truncating out the RR sector (by projecting out all components of the generalised vielbein carrying a single index $\mathcal{M} = 4$ or $\mathcal{A} = 4$), we are left with:

$$E^{\mathcal{M}}{}_{\mathcal{A}} = \begin{pmatrix} E^I{}_\alpha & 0 \\ 0 & e^{-d} \end{pmatrix}\,, \tag{A.2.3}$$

where $E^I{}_\alpha$ is an $SO(3,3)/SO(3)\times SO(3)$ coset element in the Majorana-Weyl spinor representation (and so has unit determinant), and e^{-2d} denotes the $SO(3,3)$ generalised dilaton, which is a scalar of weight 1.

We can now compute the algebra (A.3.17) of generalised frames of the form (A.2.3) and interpret these in $O(3,3)$ terms. The non-zero components of $F_{\mathcal{ABC}}{}^{\mathcal{D}}$ turn out to be:

$$F_{\alpha\beta\gamma}{}^\delta = \tilde{M}_{\alpha\beta\gamma}{}^\delta + \frac{1}{2}\delta^\delta_{[\alpha}S_{\beta]\gamma}\,, \quad F_{\alpha\beta 4}{}^4 = -\frac{1}{2}\tau_{\alpha\beta}\,, \quad F_{\alpha 4\beta}{}^4 = \frac{1}{2}\tau_{\alpha\beta} - \frac{1}{4}S_{\alpha\beta}\,, \tag{A.2.4}$$

where the irreducible fluxes have decomposed to give non-vanishing components:

$$\tau_{\alpha\beta} = E^I{}_\alpha E^J{}_\beta\partial_{IJ}(-2d) + \partial_{IJ}(E^I{}_\alpha E^J{}_\beta)\,, \quad S_{\alpha\beta} = 4E^I{}_{(\alpha|}\partial_{IJ}E^J{}_{|\beta)}\,, \tag{A.2.5}$$

$$Z_{\alpha\beta\gamma}{}^\delta = \tilde{M}_{\alpha\beta\gamma}{}^\delta + \frac{1}{2}\delta^\delta_{[\alpha}\tau_{\beta\gamma]}\,, \quad Z_{\alpha\beta 4}{}^4 = -\frac{1}{3}\tau_{\alpha\beta}\,, \tag{A.2.6}$$

with an $SO(3,3)$ irreducible representation

$$\tilde{M}_{\alpha\beta\gamma}{}^\delta = 3\left(E^I{}_{[\alpha}E^J{}_\beta E^K{}_{\gamma]}\partial_{JK}E^\delta{}_I - \frac{1}{2}\partial_{JK}(E^J{}_{[\alpha}E^K{}_\beta)\delta^\delta_{\gamma]}\right)\,, \tag{A.2.7}$$

obeying $M_{\alpha\beta\gamma}{}^\gamma = 0$. We can more conveniently define

$$\tilde{M}^{\alpha\beta} = \frac{1}{3!}\epsilon^{\gamma\delta\epsilon\alpha}M_{\gamma\delta\epsilon}{}^\beta = \frac{1}{2}\epsilon^{IJKL}\partial_{IJ}E^{(\alpha}{}_K E^{\beta)}{}_L \tag{A.2.8}$$

which is symmetric.

The two irreducible symmetric representations $S_{\alpha\beta}$ and $\tilde{M}^{\alpha\beta}$ can be related to the self-dual and anti-self-dual parts of the usual $SO(3,3)$ generalised flux f_{IJK} [74] (using gamma matrices or equivalently 't Hooft symbols), and a half-maximal theory uplifts to the maximal theory if [74]

$$S_{\alpha\beta}\tilde{M}^{\alpha\beta} = 0\,. \tag{A.2.9}$$

A.2.2 Drinfeld doubles

So far this is a standard exercise in determining the particular fluxes of the half-maximal theory. Now let's specialise to Drinfeld doubles. We break up our indices further as $I = (i, 5)$ and $\alpha = (a, 5)$.

Drinfeld double: IIA frame

To describe type IIA we take $\partial_{i5} \neq 0$ and $\partial_{ij} = 0$. Our data are the group manifold vector fields v_a, one-forms l^a and the Poisson-Lie bivector π^{ab}. We also define $\lambda_a \equiv \frac{1}{2}\epsilon_{abc}\pi^{bc}$. Then a type IIA choice of spinorial frame and generalised dilaton is:

$$E^I{}_\alpha = \begin{pmatrix} (\det l)^{1/2} v^i{}_a & 0 \\ (\det l)^{-1/2}\lambda_a & (\det l)^{-1/2} \end{pmatrix} , \quad e^{-2d} = e^{-2\tilde{\Phi}} \det l \,. \tag{A.2.10}$$

It can be checked that the following flux components are turned on:

$$\begin{aligned} \tau_{ab} &= \epsilon_{cd[a}\tilde{f}^{cd}{}_{b]} \,, \quad \tau_{a5} = -2\partial_a\tilde{\Phi} + f_{ac}{}^c \,, \\ S_{ab} &= -2\epsilon_{cd(a}\tilde{f}^{cd}{}_{b)} \,, \quad S_{a5} = -2f_{ac}{}^c \,, \\ \tilde{M}^{ab} &= \frac{1}{2}\epsilon^{cd(a}f_{cd}{}^{b)} \,, \quad \tilde{M}^{a5} = \frac{1}{2}\tilde{f}^{ac}{}_c \,. \end{aligned} \tag{A.2.11}$$

(This requires using the constraints (2.4.20), and taking the "dilaton" $\tilde{\Phi}$ to obey $\partial_a\tilde{\Phi} = f_{a4}{}^4$. This is not the physical dilaton but should be thought of as an extra function appearing in the definition of the frame (A.2.10). To match with section 2.4, take $\alpha = e^{-\tilde{\Phi}}$, and in (2.4.10) we have $\tau_a \equiv \tau_{a5}$ and $\tau^a \equiv \frac{1}{2}\epsilon^{abc}t_{bc}$.)

The SL(5) frame in the **10** consists of a part in **6** and a part in the **4** of $SO(3,3)$. The part in the **6** is obtained from the antisymmetrisation of the spinorial frame, $E^M{}_A \equiv 2E^I{}_{[\alpha}E^J{}_{\beta]}$. The part in the **4** is just the spinor frame weighted by e^{-d}. Let's denote this by $\hat{E}^I{}_\alpha \equiv e^{-d}E^I{}_\alpha$. Translating these into differential form language leads to the expressions (2.4.15) and (2.4.16).

Drinfeld double: IIB frame

To describe type IIB we take: $\partial_{i5} = 0$, $\partial_{ij} \neq 0$. The natural partial derivatives are thus $\partial^i = \frac{1}{2}\epsilon^{ijk}\partial_{jk}$. Our data are now vector fields v^a, one-forms l_a and Poisson-Lie bivector π_{ab}, with all indices in the opposite positions. A type IIB choice of spinorial frame and generalised dilaton is:

$$E^I{}_\alpha = \begin{pmatrix} (\det l)^{-1/2} l_a{}^i & -(\det l)^{-1/2} l_b{}^i \lambda^b \\ 0 & (\det l)^{1/2} \end{pmatrix} , \quad e^{-2d} = e^{-2\tilde{\Phi}} \det l \tag{A.2.12}$$

where $\lambda^a = \frac{1}{2}\epsilon^{abc}\pi_{bc}$. It can be checked that the following flux components are turned on:

$$\begin{aligned} \tau_{ab} &= \epsilon_{abc}(-2\partial^c\tilde{\Phi} + f^{cd}{}_d)\,, \quad \tau_{a5} = -\tilde{f}_{ab}{}^b\,, \\ S_{ab} &= -2\epsilon_{cd(a}f^{cd}{}_{b)}\,, \quad S_{a5} = -2\tilde{f}_{ac}{}^c\,, \\ \tilde{M}^{ab} &= \frac{1}{2}\epsilon^{cd(a}\tilde{f}_{cd}{}^{b)} \quad \tilde{M}^{a5} = \frac{1}{2}f^{ac}{}_c\,. \end{aligned} \tag{A.2.13}$$

(Again this used the constraints (2.4.20).)

We can again translate the frame into differential form language, leading to the expressions (2.4.15) and (2.4.19) (with indices in the opposite placement).

Uplift condition

The condition $S_{\alpha\beta}\tilde{M}^{\alpha\beta} = 0$ can be easily seen to imply that a Drinfeld double uplifts to an Exceptional Drinfeld Algebra only if:

$$\tilde{f}^{ab}{}_c f_{ab}{}^c = 0\,, \tag{A.2.14}$$

which is indeed the condition found in [26] by checking closure.

A.2.3 Spinors and gamma matrices

Let e^a denote a vielbein basis of one-forms, and e_a the inverse. We can represent an $O(d,d)$ spinor as a polyform, $C = \sum_p C_{(p)}$ and the gamma matrices using the wedge and interior products:

$$\Gamma^a = \sqrt{2}e^a\wedge\,, \quad \Gamma_a = \sqrt{2}\iota_{e_a}\,, \tag{A.2.15}$$

obeying the $O(d,d)$ Clifford algebra $\{\Gamma_a, \Gamma^b\} = 2\delta_a^b$, $\{\Gamma_a, \Gamma_b\} = 0$, $\{\Gamma^a, \Gamma^b\} = 0$.

The Majorana-Weyl representations correspond to even and odd polyforms. For $d = 3$, we can write these as:

$$C_{\text{even}} = C_0 + \frac{1}{2}C_{ab}e^a \wedge e^b\,, \quad C_{\text{odd}} = \frac{1}{6}\epsilon_{abc}(C^0 e^a \wedge e^b \wedge e^c + 3C^{ab}e^c)\,, \tag{A.2.16}$$

or in index notation $C_\alpha = (C_0, C_{ab})$, $C^\alpha = (C^0, C^{ab})$. Acting with a single gamma matrix maps between these representations. Acting with two gamma matrices on C_{even} we obtain the antisymmetric combination $(\Gamma_{AB})_\alpha{}^\beta$ with non-zero components

$$\begin{aligned} (\Gamma_{ab})_0{}^{cd} &= -4\delta_a^{[c}\delta_b^{d]}\,, \quad (\Gamma^{ab})_{cd}{}^0 = +4\delta_c^{[a}\delta_d^{b]}\,, \\ (\Gamma_a{}^b)_0{}^0 &= \delta_a^b\,, \quad (\Gamma_a{}^b)_{cd}{}^{ef} = 2\delta_a^b\delta_c^{[e}\delta_d^{f]} + 8\delta_{[c}^b\delta_{d]}^{[e}\delta_a^{f]}\,. \end{aligned} \tag{A.2.17}$$

Similarly, acting on C_{odd} we obtain the components of $(\Gamma_{AB})^{\alpha}{}_{\beta}$:

$$\begin{aligned}(\Gamma_{ab})^{cd}{}_0 &= -4\delta^c_{[a}\delta^d_{b]}\,, \quad (\Gamma^{ab})^0{}_{cd} = +4\delta^a_{[c}\delta^b_{d]}\,,\\ (\Gamma_a{}^b)^0{}_0 &= -\delta^b_a\,, \quad (\Gamma_a{}^b)^{cd}{}_{ef} = -2\delta^b_a\delta^{[c}_e\delta^{d]}_f - 8\delta^b_{[e}\delta^{[c}_{f]}\delta^{d]}_a\,.\end{aligned} \tag{A.2.18}$$

For convenience, let us record here also the reduction of the EDA relations that can be encoded in the algebra (2.4.8) using these gamma matrices. We have vector on vector brackets

$$\begin{aligned}[t_a, t_b] &= f_{ab}{}^c t_c\,, \quad [t^{a4}, t^{b4}] = \tilde{f}^{ab}{}_c t^{c4}\\ [t_a, t^{b4}] &= (-f_{ac}{}^b t^{c4} + \tilde{f}^{bc}{}_a t_c) = -[t^{b4}, t_a]\,,\end{aligned} \tag{A.2.19}$$

vector on spinor brackets

$$\begin{aligned}[t_a, t_4] &= f_{a4}{}^4 t_4\,, \quad [t_a, t^{bc}] = (2f_{ad}{}^{[b}t^{c]d} - \tilde{f}^{bc}{}_a t_4 + f_{a4}{}^4 t^{bc})\,,\\ [t^{a4}, t_4] &= \frac{1}{2} f_{bc}{}^a t^{bc}\,, \quad [t^{a4}, t^{bc}] = -2\tilde{f}^{a[b}{}_d \tilde{t}^{c]d}\,,\end{aligned} \tag{A.2.20}$$

and the spinor on vector brackets

$$\begin{aligned}[t_4, t_a] &= -f_{a4}{}^4 t_4\,, \quad [t^{bc}, t_a] = (3f_{[de}{}^{[b}\delta^{c]}_{a]} t^{de} + \tilde{f}^{bc}{}_a t_4 - 3f_{d4}{}^4\delta^{[b}_a t^{cd]})\,,\\ [t_4, t^{a4}] &= f_{b4}{}^4 \tilde{t}^{ab}\,, \quad [t^{bc}, t^{a4}] = -\tilde{f}^{bc}{}_d t^{ad}\,,\end{aligned} \tag{A.2.21}$$

while the spinor on spinor brackets vanish.

A.3 Ingredients

A.3.1 Five-brane near horizon limit of pp-F1-NS5

Initial solution We adapt the notation of [14, 127]. The non-extremal pp-F1-NS5 solution is

$$\begin{aligned}ds^2_s &= f_1^{-1}(-f_n^{-1}W\mathsf{d}t^2 + f_n(\mathsf{d}z + \tfrac{1}{2}\tfrac{r_0^2\sinh 2\alpha_n}{f_n r^2}\mathsf{d}t)^2) + f_5(W^{-1}\mathsf{d}r^2 + r^2\mathsf{d}s^2_{\mathrm{S}^3}) + \mathsf{d}s^2_{\mathrm{T}^4}\,,\\ B_{tz} &= -\tfrac{1}{2}\tfrac{r_0^2\sinh 2\alpha_1}{f_1 r^2}\,, \quad B_{tz1\ldots4} = -g_s^{-2}\tfrac{1}{2}\tfrac{r_0^2\sinh 2\alpha_5}{f_5 r^2}\,, \quad e^{-2\varphi} = g_s^{-2} f_1 f_5^{-1}\,,\end{aligned} \tag{A.3.1}$$

where

$$\begin{aligned}f_n &= 1 + \tfrac{r_n^2}{r^2}\,, \quad f_1 = 1 + \tfrac{r_1^2}{r^2}\,, \quad f_5 = 1 + \tfrac{r_5^2}{r^2}\,, \quad W = 1 - \tfrac{r_0^2}{r^2}\,,\\ r_1^2 &= r_0^2\sinh^2\alpha_1\,, \quad r_5^2 = r_0^2\sinh^2\alpha_5\,, \quad r_n^2 = r_0^2\sinh^2\alpha_n\,,\end{aligned} \tag{A.3.2}$$

and in terms of the numbers N_1, N_5, N_n of strings, five-branes and pp-waves, as well as the (dimensionless) volume parameter v of the T^4, we have

$$\sinh 2\alpha_1 = \tfrac{2N_1 l_s^2}{v}\tfrac{g_s^2}{r_0^2}\,, \quad \sinh 2\alpha_5 = \tfrac{2N_5 l_s^2}{r_0^2}\,, \quad \sinh 2\alpha_n = \tfrac{2N_n l_s^4}{R_x^2 v}\tfrac{g_s^2}{r_0^2}\,. \tag{A.3.3}$$

The extremal limit sends $r_0 \to 0$ and $\alpha_1, \alpha_5, \alpha_n \to \infty$ such that $r_0^2 \sinh 2\alpha_1$, $r_0^2 \sinh 2\alpha_5$ and $r_0^2 \sinh 2\alpha_n$ are constant and given by (A.3.3). Then $\sinh \alpha_a^2 \approx \frac{1}{2}\sinh 2\alpha_a$ and so

$$r_1^2 = \tfrac{N_1 l_s^2 g_s^2}{v}\,, \quad r_5^2 = N_5 l_s^2\,, \quad r_n^2 = \tfrac{N_n l_s^4 g_s^2}{R_x^2 v} \tag{A.3.4}$$

NS5 near horizon limit To obtain a solution we can apply our reduction and uplift procedure to, we need to go to the NS5 near horizon limit. This limit can be taken by sending the string coupling to zero such that

$$g_s \to 0\,, \quad \tfrac{r_0}{l_s g_s} \text{ fixed}\,. \tag{A.3.5}$$

This is the Little String Theory (LST) limit [102, 103].In this limit, α_1 and α_n are fixed, but

$$\sinh 2\alpha_5 \approx \tfrac{2N_5 l_s^2}{r_0^2} \to \infty \Rightarrow f_5 \to \tfrac{N_5 l_s^2}{r^2}\,. \tag{A.3.6}$$

If we define $u = \frac{r}{l_s g_s}$, $u_0 = \frac{r_0}{l_s g_s}$, then the three-charge background then becomes in the limit

$$\begin{aligned} ds_s^2 &= f_1^{-1}(-f_n^{-1} W \mathsf{d}t^2 + f_n(\mathsf{d}z + \tfrac{1}{2}\tfrac{u_0^2 \sinh 2\alpha_n}{f_n u^2}\mathsf{d}t)^2) + N_5 l_s^2 W^{-1}\frac{du^2}{u^2} + N_5 l_s^2 \mathsf{d}s^2_{\mathrm{S}^3} + \mathsf{d}s^2_{\mathrm{T}^4}\,, \\ H_3 &= -\frac{u_0^2 \sinh 2\alpha_1}{2} d(\frac{1}{f_1 u^2}) \wedge \mathsf{d}t \wedge \mathsf{d}x + 2N_5 l_s^2 \mathrm{Vol}(\mathrm{S}^3)\,, \\ e^{-2\varphi} &= N_5^{-1} u^2 f_1\,, \end{aligned} \tag{A.3.7}$$

with

$$f_1 = 1 + \tfrac{u_0^2 \sinh^2 \alpha_1}{u^2}\,, \quad f_n = 1 + \tfrac{u_0^2 \sinh^2 \alpha_n}{u^2}\,, \quad W = 1 - \tfrac{u_0^2}{u^2}\,. \tag{A.3.8}$$

Redefining $u = r'/l_s$, $u_0 = r'_0/l_s$ and immediately d ropping the primes we obtain the background in the form (3.3.2). In effect this is just the original three-charge background with the "1 + " d ropped from f_5 and g_s set to 1.

A.3.2 CSO(4, 0, 1) from IIA on S^3

This gauging is known to result from a warped reduction of IIA SUGRA on S^3 [73, 128]. For the pp-F1-NS5 solution, we only need to make use of the NSNS sector reduction ansatz. Here we need to introduce μ^a, $a = 1, \ldots, 4$ as constrained coordinates on the S^3, $\delta_{ab}\mu^a\mu^b = 1$, a unit determinant symmetric matrix[1] M_{ab} with inverse M^{ab}, and define

$$U = 2M^{ab}M^{bc}\mu^a\mu^c - \Delta M^{aa}\,, \quad \Delta = M^{ab}\mu^a\mu^b\,. \tag{A.3.9}$$

Then the ansatz is

$$\begin{aligned}
ds_s^2 &= \Phi^{1/2}ds_7^2 + \frac{1}{g^2}\Delta^{-1}M^{-1}_{ab}D\mu^a D\mu^b\,, \quad e^{2\varphi} = \Delta^{-1}\Phi^{5/4}\,,\\
H_3 &= \widetilde{F}_{(3)} - \tfrac{1}{2}\epsilon_{a_1a_2a_3a_4}g^{-1}\Delta^{-1}F^{a_1a_2}_{(2)}\wedge D\mu^{a_3}M^{a_4b}\mu^b\\
&\quad - \tfrac{1}{6}\epsilon_{a_1a_2a_3a_4}g^{-2}\Delta^{-2}\big(U\mu^{a_1}D\mu^{a_2}\wedge D\mu^{a_3}\wedge D\mu^{a_4} + 3D\mu^{a_1}\wedge D\mu^{a_2}\wedge DM^{a_3b}M^{a_4c}\mu^b\mu^c\big)\,,
\end{aligned} \tag{A.3.10}$$

where $D\mu^a \equiv d\mu^a + gA_{(1)}{}^{ab}\mu^b$, $DM^{ab} = dM^{ab} + 2gA_{(1)}{}^{(a|c}M^{c|b)}$ and $F_{(2)}{}^{ab} = dA_{(1)}{}^{ab} + gA_{(1)}{}^{ac}\wedge A_{(1)}{}^{cb}$. However these Kaluza-Klein gauge potentials will play no role in the cases we consider. Although we only write here the ansatz in the NSNS sector, we do need to make use of the full ansatz of [73] to identify the SL(5) covariant multiplets that result. For instance, the ansatz for the RR four-form field strength introduces a further four three-forms. These combine with the single three-form $\widetilde{F}_{(3)}$ in (A.3.10) to form the five-dimensional representation of SL(5). Similarly the scalars M_{ab} and Φ are joined by four additional scalar fields from the RR sector in order to obtain the full scalar coset SL(5)/SO(5). With the RR contribution set to zero, the SL(5) covariant scalar matrix $\mathcal{M}_{\mathcal{AB}}$, and accompanying scalar potential V, are given by:

$$\mathcal{M}_{\mathcal{AB}} = \begin{pmatrix} \Phi^{-1/4}M_{ab} & 0 \\ 0 & \Phi \end{pmatrix}\,, \quad V = \tfrac{1}{2}g^2\Phi^{1/2}(2M^{ab}\delta_{bc}M^{cd}\delta_{ad} - (M^{ab}\delta_{ab})^2)\,. \tag{A.3.11}$$

A.3.3 Exceptional field theory dictionary

Exceptional field theory (see the review [49]) describes 11-dimensional supergravity backgrounds after splitting into a d-dimensional internal part, with coordinates x^i, and $(11-d)$-dimensional external part, with coordinates X^μ. Fixing the 11-dimensional Lorentz symmetry we write the metric as

$$ds_{11}^2 = \phi^{-\frac{1}{9-d}}g_{\mu\nu}dX^\mu dX^\nu + \phi_{ij}Dx^iDx^j\,, \quad Dx^i \equiv \mathsf{d}x^i + A_\mu{}^i dX^\mu\,, \tag{A.3.12}$$

where $\phi \equiv \det\phi_{ij}$. The three-form and its four-form field strength are decomposed as follows:

$$C_{(3)} = \mathbf{C}_{(3)} + \mathbf{C}_{(2)i}Dx^i + \tfrac{1}{2}\mathbf{C}_{(1)ij}Dx^iDx^j + \tfrac{1}{3!}\mathbf{C}_{ijk}Dx^iDx^jDx^k\,, \tag{A.3.13}$$

$$F_{(4)} = \mathbf{F}_{(4)} + \mathbf{F}_{(3)i}Dx^i + \tfrac{1}{2}\mathbf{F}_{(2)ij}Dx^iDx^j + \tfrac{1}{3!}\mathbf{F}_{(1)ijk}Dx^iDx^jDx^k + \tfrac{1}{4!}\mathbf{F}_{ijkl}Dx^iDx^jDx^kDx^l\,, \tag{A.3.14}$$

where the (p) subscript denotes an n-dimensional p-form and all wedge products are implicit.

[1]Note that what we call M_{ab} is denoted $M^{-1}_{\alpha\beta}$ in [73].

The fields carrying purely internal indices enter a generalised metric parametrising a coset $\mathrm{E}_{d(d)}/H_d$, while those carrying external indices (asides from the external metric, $g_{\mu\nu}$) are treated as components of $(11-d)$-dimensional forms in a tensor hierarchy. For instance, one has $\mathcal{A}_\mu{}^M \sim (A_\mu{}^i, \mathbf{C}_{\mu ij}, \dots)$. Here one has to eventually include components of the dual six-form (and putative dualisations of the metric). In this way, each p-form gives a representation of $\mathrm{E}_{d(d)}$.

For $d=4$, we have $E_{4(4)} = \mathrm{SL}(5)$. Let $\mathcal{M} = 1, \dots, 5$ denote a fundamental index of SL(5). The generalised metric is represented by a five-by-five unit determinant symmetric matrix:

$$\mathcal{M}_{\mathcal{M}\mathcal{N}} = \phi^{\frac{1}{10}} \begin{pmatrix} \phi^{-\frac{1}{2}}\phi_{ij} & -\phi^{-\frac{1}{2}}\phi_{ik}C^k \\ -\phi^{-\frac{1}{2}}\phi_{ik}C^k & \phi^{\frac{1}{2}} + \phi^{-\frac{1}{2}}\phi_{kl}C^kC^l \end{pmatrix} , \tag{A.3.15}$$

where $C^i \equiv \frac{1}{6}\epsilon^{ijkl}\mathbf{C}_{jkl}$, $\mathbf{C}_{ijk} = -\epsilon_{ijkl}C^l$. The relevant part of the SL(5) tensor hierarchy consists of gauge fields $\mathcal{A}_\mu{}^{\mathcal{M}\mathcal{N}} = -\mathcal{A}_\mu{}^{\mathcal{N}\mathcal{M}}$, $\mathcal{B}_{\mu\nu\mathcal{M}}$, $\mathcal{C}_{\mu\nu\rho}{}^{\mathcal{M}}$, with field strengths $\mathcal{F}_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$, $\mathcal{H}_{\mu\nu\rho\mathcal{M}}$, $\mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{M}}$. These field strengths can be identified with components of the eleven-dimensional four-form and its seven-form dual as follows:

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^{i5} &= F_{\mu\nu}{}^i , & \mathcal{F}_{\mu\nu}{}^{ij} &= \tfrac{1}{2}\epsilon^{ijkl}(\mathbf{F}_{\mu\nu kl} - \mathbf{C}_{klm}\mathbf{F}_{\mu\nu}{}^m) , \\ \mathcal{H}_{\mu\nu\rho i} &= -\mathbf{F}_{\mu\nu\rho i} , & \mathcal{H}_{\mu\nu\rho 5} &= \tfrac{1}{4!}\epsilon^{ijkl}(-\mathbf{F}_{\mu\nu\rho ijkl} + 4\mathbf{F}_{\mu\nu\rho i}\mathbf{C}_{jkl}) , \\ \mathcal{J}_{\mu\nu\rho\sigma}{}^5 &= -\mathbf{F}_{\mu\nu\rho\sigma} , & \mathcal{J}_{\mu\nu\rho\sigma}{}^i &= \tfrac{1}{3!}\epsilon^{ijkl}(+\mathbf{F}_{\mu\nu\rho\sigma jkl} - \mathbf{C}_{jkl}\mathbf{F}_{\mu\nu\rho\sigma}) . \end{aligned} \tag{A.3.16}$$

The bare three-forms appear here as these field strengths transform covariantly under generalised diffeomorphisms. The minus signs are fixed such that the Bianchi identities of ExFT in the conventions used reproduce those of 11-dimensional supergravity, with $d\hat{F}_7 - \frac{1}{2}\hat{F}_4 \wedge \hat{F}_4 = 0$.

A.3.4 Exceptional Drinfeld algebra frame

Generalised frames A generalised frame in the SL(5) ExFT can be represented in the 10- or 5-dimensional representations. However we can only take the generalised Lie derivative with respect to generalised frames $E_{\mathcal{A}\mathcal{B}}$ in the former. The algebra of generalised frames is

$$\mathcal{L}_{E_{\mathcal{A}\mathcal{B}}} E^{\mathcal{M}}{}_{\mathcal{C}} = -F_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}} E^{\mathcal{M}}{}_{\mathcal{D}} , \tag{A.3.17}$$

or

$$\mathcal{L}_{E_{\mathcal{A}\mathcal{B}}} E^{\mathcal{M}\mathcal{N}}{}_{\mathcal{C}\mathcal{D}} = -\tfrac{1}{2}F_{\mathcal{A}\mathcal{B},\mathcal{C}\mathcal{D}}{}^{\mathcal{E}\mathcal{F}} E^{\mathcal{M}\mathcal{N}}{}_{\mathcal{E}\mathcal{F}} , \quad F_{\mathcal{A}\mathcal{B},\mathcal{C}\mathcal{D}}{}^{\mathcal{E}\mathcal{F}} = 4F_{\mathcal{A}\mathcal{B}[\mathcal{C}}{}^{[\mathcal{E}}\delta^{\mathcal{F}]}_{\mathcal{D}]} . \tag{A.3.18}$$

The gauging $F_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}}$ can be decomposed in terms of irreducible representations of SL(5)

$$F_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}} = Z_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}} + \tfrac{1}{2}\delta^{\mathcal{D}}_{[\mathcal{A}}S_{\mathcal{B}]\mathcal{C}} - \tfrac{1}{6}\tau_{\mathcal{A}\mathcal{B}}\delta^{\mathcal{D}}_{\mathcal{C}} - \tfrac{1}{3}\delta^{\mathcal{D}}_{[\mathcal{A}}\tau_{\mathcal{B}]\mathcal{C}} . \tag{A.3.19}$$

Here $\tau_{\mathcal{AB}} \in \overline{\mathbf{10}}$ is the so-called trombone gauging, $S_{\mathcal{AB}} \in \overline{\mathbf{15}}$ and $Z_{\mathcal{ABC}}{}^{\mathcal{D}} \in \mathbf{40}$ obeys $Z_{\mathcal{ABC}}{}^{\mathcal{D}} = Z_{[\mathcal{ABC}]}{}^{\mathcal{D}}$, $Z_{\mathcal{ABC}}{}^{\mathcal{C}} = 0$.

Exceptional Drinfeld algebra frame For the exceptional Drinfeld algebra introduced in [26,27] one has

$$\tilde{E}^{\mathcal{M}}{}_{\mathcal{A}} = \Delta^{-\frac{1}{2}} \begin{pmatrix} l^{\frac{1}{2}}\alpha^{\frac{1}{2}} v^i{}_a & 0 \\ l^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}\pi_a & l^{-\frac{1}{2}}\alpha^{-\frac{1}{2}} \end{pmatrix}, \quad \Delta \equiv \alpha^{\frac{3}{5}} l^{\frac{1}{5}}, \tag{A.3.20}$$

in terms of data $(\alpha, l^a{}_i, v^i{}_a, \pi_a = \frac{1}{3!}\epsilon_{bcda}\pi^{bcd})$ describing a particular group manifold with left-invariant frames $l^a{}_i$ and a trivector π^{abc}, obeying certain compatibility and differential conditions, in particular

$$dl^a = \tfrac{1}{2} f_{bc}{}^a l^b \wedge l^c, \quad L_{v_a} v_b = -f_{ab}{}^c v_c, \quad L_{v_a} \ln \alpha = \tfrac{1}{3}\mathfrak{L}_a \equiv \tfrac{1}{3}(\tau_{a5} - f_{af}{}^f), \tag{A.3.21}$$

$$d\pi^{abc} = \tilde{f}^{abc}{}_d l^d + 3 f_{ed}{}^{[a}\pi^{bc]d} l^e + \tfrac{1}{3}\pi^{abc}\mathfrak{L}_d l^d. \tag{A.3.22}$$

These imply that the components of the gaugings are

$$\begin{gathered} S_{55} = 0, \quad Z_{abc}{}^5 = 0, \quad Z_{ab5}{}^5 = \tfrac{2}{3}\tau_{ab}, \quad Z_{abc}{}^d = -\tau_{[ab}\delta^d_{c]}, \\ S_{a5} = -\tfrac{2}{3}\tau_{a5} - \tfrac{4}{3} f_{ab}{}^b, \quad Z_{ab5}{}^c = -f_{ab}{}^c - \tfrac{2}{3}\delta^c_{[a} f_{b]d}{}^d. \end{gathered} \tag{A.3.23}$$

while S_{ab} and τ_{ab} are defined via the "dual" structure constant with three upper antisymmetric indices

$$\tilde{f}^{abc}{}_d = \frac{1}{4}\epsilon^{abce}(S_{de} + 2\tau_{de}). \tag{A.3.24}$$

In terms of generators $T_{\mathcal{AB}}$ obeying $[T_{\mathcal{AB}}, T_{\mathcal{CD}}] = \frac{1}{2}F_{\mathcal{AB},\mathcal{CD}}{}^{\mathcal{EF}}T_{\mathcal{EF}}$ the algebra can be written in a compact form reminiscent of the Drinfeld double algebra if we let $T_a \equiv T_{a5}$, $\tilde{T}^{ab} \equiv \frac{1}{2}\epsilon^{abcd}T_{cd}$. The brackets are:

$$\begin{gathered} [T_a, T_b] = f_{ab}{}^c T_c, \qquad [\tilde{T}^{ab}, \tilde{T}^{cd}] = 2\tilde{f}^{ab[c}{}_e \tilde{T}^{d]e}, \\ [T_a, \tilde{T}^{bc}] = 2 f_{ad}{}^{[b}\tilde{T}^{c]d} - \tilde{f}^{bcd}{}_a T_d - \tfrac{1}{3}\mathfrak{L}_a \tilde{T}^{bc}, \quad [\tilde{T}^{bc}, T_a] = 3 f_{[de}{}^{[b}\delta^{c]}_{a]}\tilde{T}^{de} + \tilde{f}^{bcd}{}_a T_d + \mathfrak{L}_d \delta^{[b}_a \tilde{T}^{cd]}. \end{gathered} \tag{A.3.25}$$

CSO(4,0,1) **frame and scalar potential** This frame has $\alpha = 1$, $v^i{}_a = \delta^i_a$ and $\pi^{abc} = g\epsilon^{abcd}x_d$ [87] (where we use δ^i_a to identify the curved and flat indices on x^i and δ_{ab} to raise/lower). This results in $\tilde{f}^{abc}{}_d = g\epsilon^{abc}{}_d$ or equivalently $S_{ab} = 4g\delta_{ab}$, with the other structure constants components all vanishing.

When $S_{\mathcal{AB}} \neq 0$ is the only non-vanishing SL(5) gauging, the scalar potential resulting from

ExFT is in our conventions

$$V = \frac{1}{32}\left(2\mathcal{M}^{\mathcal{AB}}S_{\mathcal{BC}}\mathcal{M}^{\mathcal{CD}}S_{\mathcal{DA}} - (\mathcal{M}^{\mathcal{AB}}S_{\mathcal{AB}})^2\right) . \tag{A.3.26}$$

For the CSO(4, 0, 1) case with the scalar matrix as in (A.3.11) and the gauging $S_{\mathcal{AB}}$ resulting from the EDA frame, this exactly matches the scalar potential of (A.3.11).

A.3.5 $\mathrm{E}_{6(6)}$ generalised metric for a 3+3 split and six-vector deformation

Components Write the six-dimensional index as $\mathsf{i} = (a, \alpha)$, where both a and α are three-dimensional. Consider the case where

$$\phi_{\mathsf{ij}} = \begin{pmatrix} g_{ab} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix} , \quad C_{\mathsf{ijk}} \to (C_1\epsilon_{abc}, C_2\epsilon_{\alpha\beta\gamma}) , \quad \epsilon_{abc\alpha\beta\gamma} = \epsilon_{abc}\epsilon_{\alpha\beta\gamma} , \tag{A.3.27}$$

and $C_{\mathsf{i}_1\ldots\mathsf{i}_6} = C_6\epsilon_{\mathsf{i}_1\ldots\mathsf{i}_6}$. Let t denote the number of timelike directions of the metric ϕ_{ij}, and let $g \equiv \det(g_{ab})$, $h \equiv \det(h_{\alpha\beta})$. The components of the $\mathrm{E}_{6(6)}$ generalised metric defined by (3.4.25) can then be computed block-by-block to be

$$\begin{aligned} \mathcal{M}_{ab} &= |\phi|^{1/3} g_{ab}\left(1 + \tfrac{1}{gh}(hC_1^2 + (C_6 + \tfrac{1}{2}C_1C_2)^2)\right) , \\ \mathcal{M}_{\alpha\beta} &= |\phi|^{1/3} h_{\alpha\beta}\left(1 + \tfrac{1}{gh}(gC_2^2 + (C_6 - \tfrac{1}{2}C_1C_2)^2)\right) , \quad \mathcal{M}_{a\alpha} = 0 , \end{aligned} \tag{A.3.28}$$

$$\begin{aligned} \mathcal{M}_a{}^{bc} &= -(-1)^t|\phi|^{-2/3} g_{ad}\epsilon^{bcd}(hC_1 + C_2(C_6 + \tfrac{1}{2}C_1C_2)) , \\ \mathcal{M}_\alpha{}^{\beta\gamma} &= -(-1)^t|\phi|^{-2/3} h_{\alpha\delta}\epsilon^{\beta\gamma\delta}(gC_2 - C_1(C_6 - \tfrac{1}{2}C_1C_2)) , \\ \mathcal{M}_a{}^{\beta\gamma} &= 0 = \mathcal{M}_\alpha{}^{bc} = \mathcal{M}_b{}^{a\alpha} = \mathcal{M}_\beta{}^{a\alpha} , \end{aligned} \tag{A.3.29}$$

$$\begin{aligned} &\mathcal{M}_{a\bar{b}} = (-1)^t|\phi|^{-2/3} g_{ab}(C_6 + \tfrac{1}{2}C_1C_2) , \quad \mathcal{M}_{\alpha\bar{\beta}} = (-1)^t|\phi|^{-2/3} h_{\alpha\beta}(C_6 - \tfrac{1}{2}C_1C_2) , \\ &\mathcal{M}_{a\bar{\alpha}} = \mathcal{M}_{\alpha\bar{a}} = 0 , \end{aligned} \tag{A.3.30}$$

$$\begin{aligned} &\mathcal{M}^{ab}{}_{\bar{c}} = -(-1)^t|\phi|^{-2/3} g_{cd}\epsilon^{dab}C_2 , \quad \mathcal{M}^{\alpha\beta}{}_{\bar{\gamma}} = (-1)^t|\phi|^{1/3} h_{\gamma\delta}\epsilon^{\delta\alpha\beta}C_1 , \\ &\mathcal{M}^{a\alpha}{}_{\bar{b}} = \mathcal{M}^{a\alpha}{}_{\bar{\beta}} = \mathcal{M}^{ab}{}_{\bar{\alpha}} = \mathcal{M}^{\alpha\beta}{}_{\bar{a}} = 0 \end{aligned} \tag{A.3.31}$$

$$\mathcal{M}_{\bar{a}\bar{b}} = (-1)^t|\phi|^{-1/3} g_{ab} , \quad \mathcal{M}_{\bar{\alpha}\bar{\beta}} = (-1)^t|\phi|^{1/3} h_{\alpha\beta} , \quad \mathcal{M}_{\bar{a}\bar{\alpha}} = 0 , \tag{A.3.32}$$

Six-vector deformation Using (3.4.27), one sees that the six-vector deformation has the relatively simple effect of

$$\mathcal{M}_{\mathsf{i}\bar{\mathsf{j}}} \to \mathcal{M}_{\mathsf{i}\bar{\mathsf{j}}} + \tilde{\Omega}\mathcal{M}_{\mathsf{ij}} , \quad \mathcal{M}^{\mathsf{ii}'}{}_{\bar{\mathsf{j}}} \to \mathcal{M}^{\mathsf{ii}'}{}_{\bar{\mathsf{j}}} + \tilde{\Omega}\mathcal{M}^{\mathsf{ii}'}{}_{\mathsf{j}} , \quad \mathcal{M}_{\bar{\mathsf{i}}\bar{\mathsf{j}}} \to \mathcal{M}_{\bar{\mathsf{i}}\bar{\mathsf{j}}} + \tilde{\Omega}(\mathcal{M}_{\mathsf{i}\bar{\mathsf{j}}} + \mathcal{M}_{\mathsf{j}\bar{\mathsf{i}}}) + \tilde{\Omega}^2\mathcal{M}_{\mathsf{ij}} \tag{A.3.33}$$

leaving other blocks invariant. Then given a configuration with

$$ds^2_{11} = g_{ab}\mathsf{d}x^a\mathsf{d}x^b + h_{\alpha\beta}\mathsf{d}x^\alpha\mathsf{d}x^\beta + G_{\mu\nu}\mathsf{d}x^\mu\mathsf{d}x^\nu \tag{A.3.34}$$

and gauge field components C_1 and C_2 and C_6 as above, the effect of a six-vector deformation is to produce the following metric and gauge fields:

$$\begin{aligned}\widetilde{ds^2_{11}} &= (1+\Theta_1)^{1/3}(1+\Theta_2)^{-2/3}g_{ab}\mathsf{d}x^a\mathsf{d}x^b + (1+\Theta_1)^{-2/3}(1+\Theta_2)^{1/3}h_{\alpha\beta}\mathsf{d}x^\alpha\mathsf{d}x^\beta \\ &\quad + (1+\Theta_1)^{1/3}(1+\Theta_2)^{1/3}G_{\mu\nu}\mathsf{d}x^\mu\mathsf{d}x^\nu\,,\end{aligned} \tag{A.3.35}$$

$$\begin{aligned}\widetilde{C}_1 &= \frac{1}{1+\Theta_2}\left(C_1 - \Omega(gC_2 - C_1(C_6 - \tfrac{1}{2}C_1C_2))\right)\,,\\ \widetilde{C}_2 &= \frac{1}{1+\Theta_1}\left(C_2 + \Omega(hC_1 + C_2(C_6 + \tfrac{1}{2}C_1C_2))\right)\,,\\ \tilde{C}_6 &= \frac{1}{2}\frac{1}{1+\Theta_1}(C_6 + \tfrac{1}{2}C_1C_2 + \Omega(gh + hC_1^2 + (C_6 + \tfrac{1}{2}C_1C_2)^2))\\ &\quad + \frac{1}{2}\frac{1}{1+\Theta_2}(C_6 - \tfrac{1}{2}C_1C_2 + \Omega(gh + gC_2^2 + (C_6 - \tfrac{1}{2}C_1C_2)^2))\end{aligned} \tag{A.3.36}$$

where

$$\begin{aligned}\Theta_1 &= 2\Omega(C_6 + \tfrac{1}{2}C_1C_2) + \Omega^2(gh + hC_1^2 + (C_6 + \tfrac{1}{2}C_1C_2)^2)\,,\\ \Theta_2 &= 2\Omega(C_6 - \tfrac{1}{2}C_1C_2) + \Omega^2(gh + gC_2^2 + (C_6 - \tfrac{1}{2}C_1C_2)^2)\,.\end{aligned} \tag{A.3.37}$$

A.4 Charge quantisation

In this appendix we consider the requirement of brane charge quantisation for our new solution. We therefore reinstate the dimensionful constants r_1 and R inherited from the original F1-NS5 solution. We also note that we can include a constant α (assumed dimensionless) in the EDA frame corresponding to the trombone rescaling of the 11-dimensional solution. Including this, the extremal solution in spherical coordinates would be:

$$\begin{aligned}\mathsf{d}s^2_{11} &= \alpha^{2/3}(r^2f_1+\rho^2)^{1/3}R^{-4/3}(r^2f_1)^{1/3}\left(\frac{1}{f_1}(-\mathsf{d}t^2+\mathsf{d}z^2) + \frac{R^2\mathsf{d}\rho^2}{r^2f_1} + \frac{R^2\mathsf{d}r^2}{r^2} + \mathsf{d}s^2_{\mathrm{T}^4}\right)\\ &\quad + \alpha^{2/3}(r^2f_1+\rho^2)^{-2/3}R^{2/3}(r^2f_1)^{1/3}\rho^2\mathsf{d}s^2_{\mathrm{S}^3}\,,\\ F_{(4)} &= \alpha\frac{2r_1^2}{(r^2f_1)^2}\frac{r\rho}{R}\mathsf{d}t\wedge\mathsf{d}z\wedge\mathsf{d}r\wedge\mathsf{d}\rho - \alpha\frac{2r_1^2}{R^3}\mathrm{Vol}_{\mathrm{T}^4}\\ &\quad + \alpha\frac{R(4r^2f_1+2\rho^2)}{(r^2f_1+\rho^2)^2}\rho^3\mathsf{d}\rho\wedge\mathrm{Vol}_{\mathrm{S}^3} - \alpha\frac{R\rho^4}{(r^2f_1+\rho^2)^2}\partial_r(r^2f_1)\mathsf{d}r\wedge\mathrm{Vol}_{\mathrm{S}^3}\,.\end{aligned} \tag{A.4.1}$$

The dual field strength is

$$\star F_{(4)} = -\alpha^2 \frac{2r_1^2}{r^2 f_1 + \rho^2} \frac{\rho^4}{R^2} \mathrm{Vol}_{\mathrm{S}^3} \wedge \mathrm{Vol}_{\mathrm{T}^4} - \alpha^2 \frac{2r_1^2}{r f_1 (r^2 f_1 + \rho^2)} \rho^3 \mathsf{d}t \wedge \mathsf{d}z \wedge \mathsf{d}r \wedge \mathsf{d}\rho \wedge \mathrm{Vol}_{\mathrm{S}^3}$$
$$+ \alpha^2 \frac{2r}{R^4}(2r^2 f_1 + \rho^2) \mathsf{d}t \wedge \mathsf{d}z \wedge \mathsf{d}r \wedge \mathrm{Vol}_{\mathrm{T}^4} + \alpha^2 \frac{r\rho}{R^4 f_1} \partial_r (r^2 f_1) \mathsf{d}t \wedge \mathsf{d}z \wedge \mathsf{d}\rho \wedge \mathrm{Vol}_{\mathrm{T}^4} \,. \tag{A.4.2}$$

The number of membranes and fivebranes will be determined by

$$N_{M2} = \tfrac{1}{(2\pi)^6 l_p^6} \int J_{\mathrm{Page}} \,, \quad N_{M5} = \tfrac{1}{(2\pi)^3 l_p^3} \int F_{(4)} \tag{A.4.3}$$

As discussed in section 3.4.2, J_{Page} vanishes up to large gauge transformations of the form $C_{(3)} \to C_{(3)} + 4\pi j l_p^3 \mathrm{Vol}_{\mathrm{S}^3}$, $j \in \mathbb{Z}$, which shift $J_{\mathrm{Page}} \to J_{\mathrm{Page}} + 4\pi j l_p^3 \alpha \frac{2r_1^2}{R^3} \mathrm{Vol}_{\mathrm{S}^3} \wedge \mathrm{Vol}_{\mathrm{T}^4}$. Hence

$$N_{M2} = N_1 4\pi j \tfrac{l_s^6}{l_p^3} \tfrac{\alpha}{R^3} \,. \tag{A.4.4}$$

Now consider the M5 branes. Integrating the flux through the torus we have

$$N_{M5} = -\tfrac{1}{(2\pi)^3 l_p^3} \alpha \tfrac{2r_1^2}{R^3} (2\pi)^4 v l_s^4 = -4\pi N_1 \tfrac{l_s^6}{l_p^3} \tfrac{\alpha}{R^3} \,. \tag{A.4.5}$$

Notice that $N_{M2} = j|N_{M5}^{(\mathrm{T}^4)}|$.

Next integrating the flux through the four-cycle in (r, ρ, S^3) directions as described in section 3.4.2 gives, if $r_1 = 0$

$$N_{M5'} = \tfrac{1}{(2\pi)^3 l_p^3} 2\pi^2 \alpha R \bar{\rho}^2 = \tfrac{\alpha R}{4\pi l_p^3} \bar{\rho}^2 \tag{A.4.6}$$

where $\bar{\rho}$ corresponds to the limit of the ρ integration (starting at $\rho = 0$). Then charge quantisation requires

$$\bar{\rho}^2 = N \tfrac{4\pi l_p^3}{\alpha R} \,, \quad N \in \mathbb{N} \,. \tag{A.4.7}$$

The above results work remarkably well with the matching to the AdS solutions of [92]. Restoring the Planck length appropriately in the solution (3.4.14) such that ρ has units of length and ϱ is dimensionless, and carefully working through the identification with the AdS limit $r^2 f_1 = r_1^2$ of (A.4.1), the matching condition (3.4.16) and (3.4.18) become

$$\rho^2 = \frac{2l_p^3}{R\alpha} \varrho \,, \quad u = \alpha \frac{2r_1^2 \varrho}{l_p R} \,, \quad \hat{h}_4 = \alpha \frac{2r_1^2 l_p \varrho}{R^3} \,. \tag{A.4.8}$$

In [92] we have a sequence of intervals $\varrho \in [2\pi j, 2\pi(j+1)]$. Viewing our solution as lying in the first interval, $\varrho \in [0, 2\pi]$ we have $\bar{\rho}^2 = \frac{4\pi l_p^3}{\alpha R}$ giving one unit of charge. Meanwhile the relationship between the M2 and M5 charges matches that following from equations (3.6) to (3.8) of [92].

Finally we can try to fix the relationship between the 11-dimensional Planck length and the

10-dimensional string length appearing in the original solutions in type IIA on S^3. A crude way to do this is to reduce the Newton's constant prefactor of 11- and 10-dimensional supergravity to the 7-dimensional theory, via

$$\frac{1}{2\kappa_{11}^2}\int \mathsf{d}\rho\,\rho^3\mathsf{d}\Omega_3 = \frac{1}{2\kappa_{10}^2}\int R^3\mathsf{d}\Omega_3 \Rightarrow \frac{2\pi^2\frac{1}{4}\bar{\rho}^4}{(2\pi)^8 l_p^9} = \frac{2\pi^2 R^3}{(2\pi)^7 l_s^8} \Rightarrow \frac{l_s^3}{l_p^3} = \frac{\alpha^2 N_5^{5/2}}{2\pi}\,, \tag{A.4.9}$$

which implies

$$N_{M5} = 2N_5\alpha^3 N_1\,. \tag{A.4.10}$$

It seems most natural to take $\alpha = (2N_5)^{-1/3}$, as the field strength component giving rise to this flux comes directly from the three-form flux due to the F1 in the original brane solution.

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