On self-force for particles coupled to higher-order electrodynamics and scalar fields

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We address the question of point particle motion coupled to classical fields, in the context of scalar fields derived from higher-order Lagrangians and BLTP electrodynamics.

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1. Introduction

Finding a consistent and well-posed dynamical equations of motion for a system of point particles interacting with classical fields is an important problem of relativistic physics. The field could be scalar, gravitational or electromagnetic. For the gravitational field (Einstein-Infeld-Hoffmann problem), the subject's history is reviewed in [7], so we will restrict ourselves to brief remarks concerning electromagnetic interaction. The problem of point particle motion famously started with attempts to model the electron as point particles in classical electrodynamics. The field has been plagued ever since by the appearance of *infinities* for which renormalization procedures are required (see e.g. [6,16]). An approach free of infinities started in 1933 with a paper by Born [4], which was continued by Born and Infeld in [5]. This time, the field equations of electrodynamics were altered to the nonlinear Maxwell–Born–Infeld field equations. The MBI field equations are nonlinear and hence difficult to handle.

Bopp [2,3], Landé–Thomas [10,11], and Podolsky [13,14] went into a different direction by proposing linear, but *higher-order derivative* field equations to remove the infinite field energy problems. Recently the self-force problem in BLTP electrodynamics was studied in quite some generality by J. Gratus, W. Tucker and V. Perlick [9], after more limited earlier studies in [11] and [18]. Recently M. Kiessling and A.S. Tahvildar-Zadeh considered a system of N charged point particles together with their associated BLTP self-fields. Their paper [8] contains the first well-posedness result for the combined Cauchy problem of particles and electrodynamic fields.

In this paper, we report on recent results [1,12] to provide a rigorous framework for point particles interacting via *retarded fields*. As we explain in section 2, the usual Lagrangian framework does not lead to viable particle equations of motion. An additional principle is needed, which we take to be the conservation of energy and momentum (see also[8]). It turns out that for the higher-order field equations we describe here, a rigorous derivation of the particle equations of motion can be given, which is free of ad-hoc renormalization procedures.

2. Scalar Fields

The fundamental issues can be most readily illustrated in the context of scalar fields. We therefore first consider a scalar field $\phi(x^\alpha)$ defined on Minkowski space-time (with Minkowski metric $g_{\alpha\beta}$), interacting with a particle described by its world-line $\tau \mapsto q^\alpha(\tau)$. The particle world-line is associated with a scalar charge density

$$\rho(x^\alpha) = \kappa \int \frac{\delta^{(4)}(x - q(\tau))}{\sqrt{-g}} d\tau \tag{1}$$

where κ is coupling constant between particle and field. Note that the scalar current $\rho\dot{q}^\alpha$ is conserved, i.e. $\partial_\alpha(\rho\dot{q}^\alpha) = 0$. The action S of the system is *formally* given by

$$S[\phi, q] = S_{\text{field}} + S_{\text{int}} + S_{\text{particle}}$$

where

$$S_{\text{field}} = -\frac{1}{2}\int d^4x\sqrt{-g}\; g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \qquad S_{\text{int}} = -\int d^4x\sqrt{-g}\;\phi\rho$$
$$S_{\text{particle}} = \int m(\tau)d\tau$$

Here, $m(\tau)$ is the bare mass of the particle, which for scalar particles has to be taken as time-dependent (see[16]). Formal variation of the action leads to the equations

$$\frac{d}{d\tau}\left(m\dot{q}^\alpha\right) = -\kappa\partial^\alpha\phi\big|_{x^\beta = q^\beta(\tau)} \tag{2}$$
$$\Box\,\phi = -\rho. \tag{3}$$

Now the retarded solution of the wave equation (3), for all x^α not on the world-line, is given by

$$\phi(x^\beta) = \left.\frac{\kappa}{4\pi\dot{q}_\gamma(x^\gamma - q^\gamma)}\right|_{\text{ret}} \tag{4}$$

where "ret" indicates evaluation at the retarded time on the particle world-line, i.e. $\dot{q}_\gamma(x^\gamma - q^\gamma)$ is a light-like Minkowski vector. But combining the equations (2), (3) and (4) makes no apparent sense, since (2) requires evaluation of $\phi, \partial^\alpha\phi$ at the particle's position and (4) is not well-defined there. To remedy this situation, authors have proposed renormalization procedures, avereraging axioms and the use of a combination of retarded and advanced solutions of the field equation (3) (see e.g.[16]). Clearly the Lagrangian formulation does not determine the particle equations of motion uniquely and an additional principle is needed. Preferably, the additional principle should be as fundamental as possible. Here, we consider the possibility of using *conservation of energy and momentum* to derive the equations of motion. The most convenient, relativistic covariant way to implement this is to define the energy-momentum tensor of the particle by

$$T^{\mu\nu}_{\text{particle}}(x^\alpha) = m_0 \int \frac{u^\mu u^\nu \delta^{(4)}(x - q(\tau))}{\sqrt{-g}} \, d\tau \tag{5}$$

The conservation of energy and momentum is then simply

$$\partial_\mu \left(T^{\mu\nu}_{\text{particle}} + T^{\mu\nu}_{\text{field}} \right) = 0 \tag{6}$$

where $T^{\mu\nu}_{\text{field}}$ is the canonical energy-momentum tensor of the field. In view of (5), we need to be precise on how to mathematically interpret (6). The most straightforward way is to interpret (6) in a distributional sense. So as the definition of the action of $T^{\mu\nu}_{\text{field}}$ on any smooth tensor field $\varphi_{\mu\nu}$ we might take

$$T^{\mu\nu}_{\text{field}}(\varphi_{\mu\nu}) = \int \left[-\frac{1}{2} g^{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi + \partial^\mu \phi \partial^\nu \phi \right] \varphi_{\mu\nu} \; d^4x \tag{7}$$

where $\phi \in C_0^\infty(\mathbb{R}^4)$ is a smooth test function. The problem, of course, is that the integrand is *not locally integrable* if we plug in the retarded field (4). This is most easily seen for a charge at rest, for which $T^{00} \sim r^{-4}$ with r being the spatial distance between the charge and the point (x^α).

3. Higher-Order Scalar Fields

A possible way out without resorting to renormalization is to modify the field equations. It should be kept in mind that other choices of the action are conceivable and a modification of the Lagrangian is a direction worth pursuing. As an example, we would like to mention that there is a large body of literature on modifications of Einstein's general relativity (see e.g.[17]). For now, we consider the modified action

$$S_{\text{field}} = \int d^4x \sqrt{-g} \;\; \left[-g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + a (\Box\, \phi)^2 \right], \tag{8}$$

by the higher-order term $(\Box\, \phi)^2 = \left[\frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta \phi \right) \right]^2$. This modification keeps the field equations linear, but changes the *short-distance physics* in a way explained below. Observe also that (7) is the unique action functional involving second derivatives of ϕ that leads to linear field equations. The only other covariant quadratic contribution involving second order derivatives would be

$$g^{\alpha\gamma} g^{\beta\rho} (\partial_\alpha \partial_\beta \phi)(\partial_\gamma \partial_\rho \phi)$$

but this would reduce to $(\Box\, \phi)^2$ after an integration by parts. The resulting field equation for ϕ is

$$\Box\, (\phi + a \Box\, \phi) = -\rho. \tag{9}$$

where ρ is given by (1). It turns out that the higher-order equation (9) has a regularizing effect on fields with point sources. The retarded solution ϕ of (9) can be determined by Green's functions techniques and has the property that both ϕ and $\partial_\beta \phi$ are now bounded in the vicinity of the particle, although the value of $\partial_\beta \phi$ on the particle world-line is still ill-defined.

Recall that the canonical energy-momentum tensor is given by the variation of S_{field} with respect to $g^{\mu\nu}$:

$$\delta S_{\text{field}} = -\int d^4x\sqrt{-g}\ (T_{\text{field}})_{\mu\nu} g^{\mu\nu} \tag{10}$$

A lengthy calculation gives (for $g_{\alpha\beta}$ being the Minkowski metric):

$$\begin{aligned} T^{\alpha\beta}_{\text{field}} &= -\frac{1}{2} g^{\alpha\beta}\partial_\mu\phi\partial^\mu\phi + \partial^\alpha\phi\partial^\beta\phi \\ &+ 2a\partial^\alpha\Box\phi\partial^\beta\phi - \frac{a}{2}(\Box\phi)^2 - a\partial_\mu(\Box\phi)(\partial^\mu\phi)g^{\alpha\beta} \end{aligned} \tag{11}$$

The energy-momentum tensor of the field can now be defined as a distribution, since it contains singularities of order at most r^{-2}. A thorough analysis of (6) now yields a well-defined equation of motion (see[1]), from which the "self-force" contribution can be read off. The form of the self-force can depend on the type of problem considered (e.g. a scattering problem or initial-value problem for particles and fields).

4. BLTP Electrodynamics

Higher-order field linear equations for electrodynamics were proposed by Bopp[2,3], Landé–Thomas[10,11], and Podolsky[13,14]. Starting point is the usual Maxwell action with an additional term containing second-order derivatives of the field tensor $F_{\mu\nu}$:

$$S_{\text{BLTP}} = \int d^4x\sqrt{-g}\ \left[-\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{8\pi\kappa^2}\partial_\gamma F^{\gamma\alpha}\partial^\lambda F_{\lambda\alpha}\right] \tag{12}$$

This is the unique action giving linear field equations of order less or equal to four. $\kappa > 0$ is a fixed parameter throughout. Variation of the action with respect to the vector potential A_α gives

$$\begin{aligned} (I - \kappa^{-2}\Box)\partial^\alpha F_{\alpha\beta} &= -4\pi j_\beta \\ \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} &= 0 \end{aligned} \tag{13}$$

where j_α is usual particle 4-current vector. We will refer to a theory based on (13) as BLTP electrodynamics. As before, the Lagrangian formulation does not determine the particle equations of motion and we additionally impose (6) in a distributional sense. The canonical energy-momentum tensor of the field is given by

$$\begin{aligned} -4\pi T^{\alpha\gamma}_{\text{BLTP}} :=& g^{\alpha\rho}F_{\rho\mu}F^{\mu\gamma} + \frac{1}{4}g^{\alpha\gamma}F_{\rho\theta}F^{\rho\theta} - \kappa^{-2}(g^{\alpha\rho}F_{\rho\mu}\Box F^{\mu\gamma} + g^{\alpha\rho}F^{\gamma\mu}\Box F_{\mu\rho} \\ &+ g^{\alpha\rho}\partial_\mu F^{\mu\gamma}\partial^\beta F_{\rho\beta}) - \frac{1}{2}\kappa^{-2}(F_{\rho\theta}\Box F^{\rho\theta} + \partial_\rho F^{\rho\theta}\partial^\beta F_{\beta\theta})g^{\alpha\gamma} \end{aligned} \tag{14}$$

and it can be shown that

$$\varphi_{\mu\nu} \mapsto \int T^{\mu\nu}_{\mathrm{BLTP}} \varphi_{\mu\nu}\ d^4x$$

defines a well-defined distribution. In[12], we prove the following Theorem:

Theorem 4.1. *Suppose* $\{q_n^\alpha(\tau_n)\}_{n=1}^N$ *are a collection of non-intersecting world-lines satisfying certain conditions and let the field associated to the* n*-th particle be defined by*

$$(F_n)_{\alpha\beta}(x^\alpha) = e_n\kappa^2 \left.\frac{R_{[\alpha}u_{\beta]}}{2u^\gamma R_\gamma}\right|_{ret} + e_n\kappa^2 \int_{-\infty}^{\tau_{ret}(x^\alpha)} \frac{J_2(\kappa D) R_{[\alpha}u_{\beta]}}{D^2} d\tau \tag{15}$$

where e_n *is the charge of the* n*-th particle and the integral extends over the world-line* $q_n(\tau)$*.* R *is defined by* $R^\alpha(x^\beta,\tau) = x^\alpha - q^\alpha(\tau)$ *and* $D = (q_n^\gamma(\tau) - q_n^\gamma(\tau'))((q_n)_\gamma(\tau) - (q_n)_\gamma(\tau'))$*. Suppose that* (6) *holds. Then*

$$m_n \frac{du_n^\alpha}{d\tau_n} = e_n \left(\mathcal{F}_n^{\alpha\beta} + \sum_{m\neq n} F_m^{\alpha\beta} \right) u_\beta \tag{16}$$

holds. Note in particular that the force on the n*-th particle splits into a self-force* $\mathcal{F}^{\alpha\beta}u_\beta$ *and the Lorentz-force exerted by all other particles. The self-field* $\mathcal{F}^{\alpha\beta}$ *is given by the formula*

$$(\mathcal{F}_n)_{\alpha\beta}(\tau) = e_n\kappa^2 \int_{-\infty}^{\tau} \frac{J_2(\kappa D) R_{[\alpha}u_{\beta]}}{D^2} d\tau'. \tag{17}$$

More details can be found in[12]. The class of world-lines covered by the Theorem is extremely broad, comprising essentially arbitrary subluminal world-lines that are twice differentiable. Hence in the context of BLTP electrodynamics, the equations of motion rigorously follow from (6).

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