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ORIGINAL RESEARCH

Epistemic horizons from deterministic laws: lessons from a nomic toy theory

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Abstract

Quantum theory has an epistemic horizon, i.e. exact values cannot be assigned simultaneously to incompatible physical quantities. As shown by Spekkens' toy theory, positing an epistemic horizon akin to Heisenberg's uncertainty principle in a classical mechanical setting also leads to a plethora of quantum phenomena. We introduce a deterministic theory—nomic toy theory—in which information gathering agents are explicitly modelled as physical systems. Our main result shows the presence of an epistemic horizon for such agents. They can only simultaneously learn the values of observables whose Poisson bracket vanishes. Therefore, nomic toy theory has incompatible measurements and the complete state of a physical system cannot be known. The best description of a system by an agent is via an epistemic state of Spekkens' toy theory. Our result reconciles us to measurement uncertainty as an aspect of the inseparability of subjects and objects. Significantly, the claims follow even though nomic toy theory is essentially classical. This work invites further investigations of epistemic horizons, such as the one of (full) quantum theory.

Keywords Epistemic horizons · Predictability · Quantum uncertainty principles · Deterministic laws · Subject-object inseparability · Spekkens' toy theory

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1 Introduction

Whenever an agent cannot obtain a complete account of a physical phenomenon, we shall speak of an *epistemic horizon*. A standard example is the Heisenberg uncertainty principle. Its qualitative consequence is that, given two incompatible measurements[1] of a quantum system, an agent can only be certain about the outcome of at most one of the measurements. There is a multitude of ways in which uncertainty about a physical system, and thus an epistemic horizon, can emerge.

Sources of epistemic horizons. One potential source of uncertainty arises in *chaotic* systems, which exhibit high sensitivity to initial conditions. Unpredictability of such systems follows due to the unavoidable inaccuracy of any specification of boundary conditions (see, for instance, Batterman (1993)).

Learning about a non-chaotic system may still be intractable because of technological limits to measurement precision. Moreover, its behaviour may be unpredictable as a result of its astronomical computational complexity. In both cases, the lack of knowledge an agent has about the system is connected to *practical* considerations contingent on technological advances.

Logical paradoxes present another source of epistemic horizons. Self-referential reasoning has been employed to establish links with undecidability, uncomputability, and randomness (Svozil, 2019; Szangolies, 2018; Dalla Chiara, 1977). For example, the work of Bendersky et al. suggests that quantum randomness must be uncomputable (Bendersky et al., 2017). A similar conclusion was drawn in Del Santo and Gisin (2019), based on the idea of finite representability.

In the context of the theory of general relativity, it has been claimed that there is an upper bound on *information density*. See, for instance, Bekenstein's result expressing the maximum amount of information in a bounded system (Bekenstein, 1981). Thus, an epistemic horizon can arise from the nature of spacetime itself for agents of bounded size.

There are also more exotic possibilities. In Everettian Quantum Mechanics and Many Worlds interpretations of quantum theory, all possible outcomes of a given measurement actually happen and are experienced independently in parallel worlds. Nevertheless, our single-world experience carries a *self-locating* uncertainty, which leads to uncertainty about the outcome that can be described probabilistically (Barrett et al., 2010).

In a *causally indeterministic* world there is a fundamental epistemic horizon. This means that events need not be pre-determined by preceding conditions together with the laws of nature (Hoefer, 2024).

Yet another source of uncertainty is the nature of *dynamical* laws. For instance, in an extreme scenario of a physical theory with two types of systems without coupling, a system of one type cannot learn about the behaviour of systems of the other type when learning is mediated by interactions. A remote yet far-reaching example is the part of the Universe we will never interact with, which includes all systems beyond the horizon from which no information can reach us.

[1] Two measurements of a quantum system described by a Hermitian operator are incompatible if the operators do not commute.

Still, even in the presence of non-trivial interactions, learning faces limitations. In this work we study an epistemic horizon in the context of a specific physical theory introduced below as nomic toy theory. In particular, we prove that in this theory, one physical system can only obtain constrained information about another. Similar to quantum theory, measurements in nomic toy theory can exhibit incompatibility. Their outcomes cannot be known simultaneously by agents modelled as systems within the theory. The nature of interactions of nomic toy theory thus impacts the information gathering activities of agents and entails fundamental limits to what can be known about the world.

Dynamical epistemic horizons. In classical mechanics the values of the positions and momenta of all particles at a certain time, together with the physical laws, are purported to fully determine their entire future (and past) values. Moreover, so the story goes, the values at a given time can be precisely measured. A principal articulation of such causal determinism is the omniscient intellect of Laplace's demon (Laplace, 1814; Hoefer, 2024).

In Newtonian physics one often ignores an explicit account of measurement interactions and that they necessarily disturb the system being measured (Barad 2007, Chapter 3). This is traditionally justified by stipulating that the disturbance is determinable and thus can be accounted for. Adjusting one's measurement record based on known disturbance—if indeed possible—allows an agent to acquire arbitrary information about a system. Particularly in the context of quantum theory, measurements are said to introduce disturbance. Heisenberg's uncertainty relations were interpreted by himself as originating from an inevitable measurement disturbance: Whatever pre-determines the outcome of a measurement of a particle is inadvertently disturbed by its interaction with the apparatus (Heisenberg, 1925).[2]

Based on complementarity, i.e. the existence of mutually exclusive experimental arrangements, Niels Bohr argued that the measurement disturbance in quantum theory cannot be accounted for. According to him, this is due to discontinuous quantum jumps (Bohr, 1937). Thus, the discrete nature of measurement interactions spoils determinism and predictability.[3]

One perspective on our work is that it provides an account of uncertainty in Spekkens' toy theory (which reproduces stabiliser states in quantum theory (Pusey, 2012; Catani & Browne, 2017)) in terms of dynamical measurement disturbance. In a nutshell, there is a classical theory—the ontological model of Spekkens' toy theory—whose deterministic laws entail an epistemic horizon. In this sense, Heisenberg's original interpretation of uncertainty can be said to apply in the case of stabiliser quantum theory.

Spekkens' toy theory. In 2004 Robert Spekkens conceived of a toy theory based on the so-called knowledge-balance principle: "If one has maximal knowledge, then

[2] However, this early account of Heisenberg's uncertainty is but one possible interpretation. The properties that determine an individual measurement result need not exist in a quantum world. In particular, it is unclear whether a single particle can be said to possess properties of position and momentum prior to measurement (see, for example, Fankhauser (2022, Section 6.1)). Thus, one cannot straightforwardly argue that such properties (since they do not exist) would be disturbed in a measurement.

[3] Later, Heisenberg in part conceded to Bohr's views and acknowledged complementarity as the source of uncertainty (cf. Wheeler and Zurek (1983) on Heisenberg's postscript to his uncertainty article).

for every system, at every time, the amount of knowledge one possesses about the ontic state of the system at that time must equal the amount of knowledge one lacks" (Spekkens, 2007) (cf. similar in-principle restrictions on the detectable amount of information by Brukner and Zeilinger (2003)). The idea was to construct a theory in which (at least some) quantum states can be viewed as epistemic as opposed to ontic. That is, they would represent states of incomplete knowledge about a physical system instead of different states of physical reality. The theory is essentially classical, because it admits a deterministic non-contextual ontological model.[4] Specifically, its kinematics is given by phase spaces of classical particles and its dynamics preserves the phase space structure. Epistemic states of the toy theory arise from the ontic states via an epistemic restriction called classical complementarity: *Two linear observables on the phase space can be jointly known only if their Poisson bracket vanishes*. The toy theory qualitatively reproduces a large part of the operational predictions of quantum theory (Spekkens 2016, Table 2). For instance, it can recover the complete behaviour of states and measurements in the stabiliser subtheory of quantum theory, whose states are eigenstates of products of Pauli operators. With respect to the epistemic restriction of Spekkens' theory we ask the following question: *Can uncertainty in a physical theory arise without imposing an a priori restriction on the acquisition of knowledge?*

We give an affirmative answer. Namely, inspired by Hausmann et al. (2023), we provide a deterministic physical theory—nomic toy theory—and show that agents are limited in the amount of information they can gather. The limitation derives from the dynamics of nomic toy theory and a definition of information gathering agents modelled within the theory. Furthermore, the epistemic horizon we *derive* precisely matches the *postulated* epistemic restriction of Spekkens' toy theory. This is interesting since Spekkens' toy theory includes no formal account of how agents acquire knowledge and what is the source of the limitation. To our knowledge, our work constitutes the first account of an *a posteriori* epistemic horizon arising from dynamical laws.

Paper overview. We proceed as follows. First, in Sect. 2, we define nomic toy theory, its ontic state space, the notion of a toy system, the characterisation of agents, as well as the dynamics and the notion of a measurement. In Sect. 3 we present the main result, which represents a fundamental epistemic horizon in nomic toy theory. There, we also relate our work to Spekkens' toy theory, which is shown to arise as the epistemic counterpart of our nomic toy theory (Sect. 3.3). We furthermore comment on the possibility of self-measurement in Sect. 3.2. The findings are summarised in Sect. 4, where we also comment on the relationship to quantum and classical uncertainty more generally, and provide an outlook on related issues such as multi-agent scenarios and the participatory nature of the observer. Appendix A contains the details of a position and momentum measurement in nomic toy theory. In Appendix B we provide supplementary material on Spekkens' toy theory, including several new proofs. For additional details on this toy theory, closely related to our nomic toy theory, we refer the reader to Spekkens (2016), Hausmann et al. (2021).

[4] cf. also Catani et al. (2023) on classicality in quantum theory.

2 Nomic toy theory

To formulate our result on an epistemic horizon emerging from deterministic physical laws, we introduce nomic toy theory in which the subject-object relationship can be studied.[5] The key feature of nomic toy theory is that it explicitly models the agent performing the measurement as a physical system in the theory. Given the ontic state space (a classical phase space), deterministic dynamics (via symplectic maps), as well as a definition of the agent, the theory contains restrictions on what can be known about physical systems.

We first introduce the state space and dynamics of toy systems (Sect. 2.1) and elaborate on their properties in Sect. 2.2, to then define toy subjects within the theory (Sect. 2.3). In Sect. 2.4 we define measurements between subjects and objects as a physical interaction. Finally, Sect. 2.5 discusses what kind of information can be learned by a toy subject about a toy object via such interactions. An arbitrary learnable property is provided by the notion of a fixed variable (Definition 2.7). However, as we show in the crucial Proposition 2.10, the same information is carried by the smaller set of measurable variables (Definition 2.9).

2.1 Toy systems

The formalism of physical states in nomic toy theory closely follows that of ontic states in Spekkens' toy theory (cf. Appendix B.1, Hausmann et al. (2021) and Hausmann et al. (2023, Appendix A)). We begin with a description of the *kinematics* of nomic toy theory and the definition of a physical system.

Definition 2.1 A physical system V in nomic toy theory (a **toy system**) is specified by a symplectic vector space V.

We can also think of it as the phase space of a classical particle. Namely, V is a $2n$-dimensional $\mathcal{F}$-vector space[6] with an orthonormal basis $\{q_1, \ldots, q_n, p_1, \ldots, p_n\}$. It is furthermore equipped with a symplectic form $\omega : \mathrm{V} \times \mathrm{V} \to \mathcal{F}$ given by

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{1}$$

in matrix form in the above basis, where $\mathbb{1}_n$ is the $n \times n$ identity matrix. In particular, we have

$$\omega(x, y) = x^T \Omega y = \langle x, \Omega y \rangle, \tag{2}$$

[5] The use of the word nomic is motivated by the theory's emphasis on law-like interactions between an agent and another physical system.

[6] For a continuous toy system, $\mathcal{F}$ is $\mathbb{R}$, while for a discrete d-level system, it is $\mathbb{Z}_d$, in which case it is a field only if d is a prime. For degrees of freedom with other finite cardinalities, one can instead consider V to be a $\mathbb{Z}_d$-module.

where vectors are represented as columns, x^T is the transpose of x, and $\langle \, , \, \rangle$ is the canonical inner product.
A physical state of the toy system (an **ontic state**) is then specified by an element of V.

The choice of *dynamics* of the theory is inspired by the Hamiltonian formulation of classical mechanics. In particular, its time evolution via Hamiltonian flow is always a symplectomorphism—a map between symplectic manifolds that preserves the symplectic structure. For the manifolds considered here, i.e. symplectic affine spaces, there are two basic types of such transformations. One can be represented by a linear map $\mathrm{V} \to \mathrm{W}$ which preserves the symplectic form. The other corresponds to an affine map $\mathrm{V} \to \mathrm{V}$ that translates each state by a chosen vector in V. These are exactly the allowed transformations of ontic states in Spekkens' toy theory. The choice of dynamics of nomic toy theory is thus compatible with the epistemic restriction of Spekkens' toy theory (see Lemma B.3 and Proposition B.2 for a proof).

For a symplectic vector space V, the symplectic maps $\mathrm{V} \to \mathrm{V}$ form the symplectic group, whose matrix representation is

$$\mathrm{Sp(V)} := \{ M \in \mathrm{GL(V)} \mid M^T \Omega M = \Omega \}, \tag{3}$$

where $\mathrm{GL(V)}$ is the set of the invertible linear maps of type $\mathrm{V} \to \mathrm{V}$.
We thus define the group of **reversible transformations** of a given toy system in nomic toy theory to be the affine symplectic group: Its elements are pairs (t, v) of a symplectic map $t \in \mathrm{Sp(V)}$ and a vector $v \in \mathrm{V}$, which compose via

$$(s, u) \circ (t, v) = \big(s \circ t, u + s(v)\big). \tag{4}$$

As we can see, the dynamical evolution of ontic states in nomic toy theory is deterministic. That is, a given reversible transformation (t, v) acts uniquely on ontic states via $x \mapsto t(x) + v$.

2.2 Properties of toy systems

To facilitate our formal derivation of the epistemic horizon in nomic toy theory, we discuss several properties of toy systems in this section. Our main theorem (Theorem 3.1) later establishes which of these properties can be acquired by a toy subject through a measurement interaction (see Sect. 2.4). In particular, there are properties that cannot be learned in this way and thus lie beyond the epistemic horizon.

Our notion of a *variable* is intended to model an arbitrary property of a toy system (at a particular point in time[7]). On the other hand, a *Poisson variable* is a special property which, as we prove later in Theorem 3.1, is measurable by toy subjects. Table 1 provides an overview of the different kinds of properties of toy systems.

[7] Note that the notion of time is implicit but of no particular relevance for the results. It only matters that a transformation connects a pre-measurement state to a post-measurement state.

Table 1 Summary of the three different types of properties of a toy system V. Note that every functional is a Poisson variable

Property	Type	Values	Extra conditions
Variable Z	Function $\mathrm{V} \to \mathrm{Z}$	Set Z	–
Poisson variable Z	Linear map $\mathrm{V} \to \mathrm{Z}$	Vector space Z	$Z\Omega Z^T = 0$
Functional Z	Linear map $\mathrm{V} \to \mathcal{F}$	Scalar $\mathcal{F}$	–

Definition 2.2 Let V be a toy system with symplectic vector space V. A function $Z : \mathrm{V} \to \mathrm{Z}$ is termed a **variable** of V, where Z is the set of values of the variable. A variable is termed **Poisson** if Z is an $\mathcal{F}$-vector space and Z is a linear map that satisfies

$$Z\Omega Z^T = 0, \tag{5}$$

where Ω is the matrix representation of the symplectic form.
Every variable induces a partition

$$\{Z^{-1}(x) \mid x \in \mathrm{Z}\} \tag{6}$$

of the set V of ontic states. Variables that induce the same partition are considered to be equivalent. Note that Poisson variables are valued in a vector space, whose dimension tells us about the potential number of independent scalar properties it can describe. An important special case is when the dimension is 1, in which case we speak of a **functional** $\mathrm{V} \to \mathcal{F}$. Such a linear map automatically satisfies Eq. (5).
For any basis $\{z_i\}_{i=1}^{\dim(\mathrm{Z})}$ of a vector space Z, we can think of an arbitrary linear map $Z : \mathrm{V} \to \mathrm{Z}$ as a set of functionals $\{Z_i\}$, where Z_i is given by $z_i^T Z$ in matrix form. In this representation, Eq. (5) says that every pair of these functionals must have vanishing Poisson bracket, i.e.

$$\omega(Z_i^T, Z_j^T) = Z_i \Omega Z_j^T = 0 \tag{7}$$

holds for all i and all j. Therefore, Poisson variables precisely correspond to properties which, in Spekkens' toy theory, are assumed to be knowable about the toy system. While this epistemic horizon is traditionally postulated in Spekkens' toy theory, we derive it in nomic toy theory.

Remark 2.3 To see the connection to epistemic states of Spekkens' toy theory (Appendix B), note that the set of vectors $\{Z_i^T\}$ spans an isotropic subspace of V. Together with a value of Z, it thus specifies an epistemic state. The support of this epistemic state is an element of the partition from (6).

A canonical example of a Poisson variable is the projection of V onto the n-dimensional subspace spanned by the q_i basis vectors. It satisfies Eq. (5) because we have $\omega(q_i, q_j) = 0$ for all i and all j. In other words, the symplectic form vanishes on this subspace. The highest dimension of a subspace with this property is n. The following standard concept generalises such a *maximal* Poisson variable.

Definition 2.4 An n-dimensional subspace Q of a symplectic vector space V, on which the symplectic form ω vanishes, is called a **Lagrangian** subspace.

For any Lagrangian subspace, the associated projection $Q : \mathrm{V} \to \mathrm{Q}$ is a Poisson variable. Moreover, by Darboux's Theorem, there is a basis of its orthogonal complement $\mathrm{P} := \mathrm{Q}^{\perp}$, in which the symplectic form has the canonical form of Eq. (1) with respect to the decomposition $\mathrm{V} = \mathrm{Q} \oplus \mathrm{P}$.

2.3 Toy subjects

Physical theorising is often done from an omniscient point of view *external* to the world. That is, one introduces a theoretical domain of discourse—the physical world together with some law-like behaviour—to explain the phenomena that are directly observable through empirical data. For instance, according to an omniscient being like Laplace's demon the future and past of the world is completely fixed if the laws are deterministic.

However, observations of phenomena necessarily occur *within* the world. Therefore, every physical theory requires in addition an epistemology that stipulates what can be known, e.g. about the physical world. That is, intuitively, we need to specify what the empirical data can and cannot signify about the physical world.

And so it may happen that the two perspectives disagree. Even if the omniscient viewpoint contains no fundamental uncertainty about all details of the world, an internal agent could be bound to epistemic limitations. Whether the omniscient view is or is not conceivable, it may be unreachable for any agent as a result of the dynamical constraints of the world in which the agent operates. To study this tension, let us introduce the notion of agents in nomic toy theory. Note that we do not place any anthropocentric constraints on these, our agents are part of nature in the same way that their objects of study are. Since our agents are decidedly minimal and may not fulfil elaborate requirements for agency (van Lier, 2023; McGregor et al., 2024), we also call them toy subjects.

We only have two basic desiderata. Firstly, a toy subject is a physical systems of the same kind as any object to be observed and interacted with.[8] That is, an information gathering subject is an arbitrary toy system as introduced in Sect. 2.1.

Secondly, a toy subject ought to include a specification of its 'knowledge' variables. These are manifest properties of the subject that represent the directly accessible empirical data on which the subject's knowledge supervenes. The dynamics of nomic toy theory, in turn, dictates what the manifest variables of the subject can and cannot signify about the ontic properties of an object with which the subject interacts.

Definition 2.5 A **toy subject** is a toy system A equipped with a Lagrangian subspace Q of the symplectic vector space A. The associated Poisson variable $Q : \mathrm{A} \to \mathrm{Q}$ is called the **manifest** variable of the subject.

[8] See also Hausmann et al.'s more operational approach to modelling the memory register of an agent as a toy bit (Hausmann et al., 2023).

An example of a toy subject is a simple pointer apparatus. The manifest variable would be the value on a scale or the angle of a pointer needle. Inspired by such example, we call the manifest variable Q of A the **position** of A and its complementary variable P the **momentum** of A. Even though we do not assume what type of degrees of freedom the manifest ones are, we label them as 'positions' for the sake of simplicity.

Throughout this work we refer to subjects also as 'agents'. However, we do not intend the above definition of toy subjects to fully capture the complexities of agency, which may include qualities of awareness, intention, or free will. Indeed, both passive measurement devices and active agents with awareness can satisfy our purposefully minimal definition, which allows for broad applicability of our results.

In either case, when we speak of the subject's knowledge we merely refer to the information stored in its manifest variable. The measurement of another toy system by the subject (as discussed later in Sect. 2.4) is conceptualized as an interaction between the two systems and thus does not require any 'agency' on the part of the subject. Finally, for a toy subject to learn about another system means that it 'obtains information' by virtue of changes in its manifest variable. The epistemic horizon we derive (Theorem 3.1) is therefore a limitation on the possible acquisition of information via interactions.

The crucial property of a toy subject defined above is that its knowledge supervenes on its manifest variable. Importantly, it is a variable associated to a *Lagrangian* subspace of its own ontic state space. Therefore, the toy subject does not have direct access to the value of its own momentum variable P. This has implications for the feasibility of measurements that the toy subject can implement. In particular, given a specific value q of the position variable Q, the toy subject *can* perform a measurement of another toy system conditionally, i.e. so that its own position prior to the measurement has value q. On the other hand, we *cannot* grant it the power to fix its own momentum value before the measurement interaction since there is no a priori way for the toy subject to know its own momentum. We discuss measurements in more detail in Sect. 2.4.

One may be tempted to view the restriction on a toy subject's access to its own ontic state as a kind of epistemic horizon (on self-knowledge rather than on knowledge of the world). However, this is not fully justified. Even if an agent has no direct access to some of its own degrees of freedom, it could still learn about them indirectly. Whether this is possible or not depends on the dynamical laws of the world in which the agent operates. We discuss toy subjects measuring their own momentum in the context of nomic toy theory in Sect. 3.2.

Nevertheless, the fact that the knowledge of a toy subject supervenes on a Poisson variable (Definition 2.2) rather than its ontic state is a key ingredient in our derivation of the epistemic horizon in Sect. 3.1. Other agents, such as ones with direct access to their own ontic state, would be able to break the epistemic horizon of nomic toy theory.

2.4 Measurement interactions

Let us now turn to the discussion of how a toy subject A may learn about a toy system S by virtue of interacting with it. To distinguish S from A, we call such S the **toy object**.

We model this potential acquisition of knowledge as a process in nomic toy theory (Fig. 1), which transforms the joint system of S and A denoted by $S \oplus A$. The joint ontic state space is given by the direct sum $\mathrm{S} \oplus \mathrm{A}$, which carries a canonical symplectic structure induced by those of S and A. For more details on joint systems as well as joint and marginal states in Spekkens' toy theory, see Appendix B.2.

We also assume that the toy subject A is in a 'ready state' prior to the process, i.e. its position variable Q has a definite value. Since the value of Q is already assumed to be directly accessible to A (see Sect. 2.3), this presents no additional assumption.

Definition 2.6 Given a toy system S and a toy subject A (with manifest variable Q), a **measurement** of S by A is a pair of a ready state, specified by a value of Q, and a reversible transformation $m : \mathrm{S} \oplus \mathrm{A} \to \mathrm{S} \oplus \mathrm{A}$ of nomic toy theory.

That is, m is given by an affine symplectic map

$$x \mapsto Mx + v \tag{8}$$

where $x, v \in \mathrm{S} \oplus \mathrm{A}$ and M is a symplectic matrix.
The assumption that the measurement process is governed by reversible transformations does not pose any loss of generality if we assume that all irreversible transformations can be dilated to a reversible one with larger output (cf. Appendix B.3). Any information obtained by the irreversible process could then also be learned via its reversible dilation.

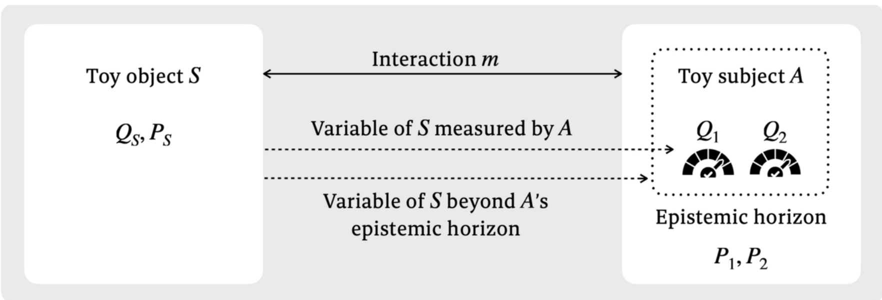

Fig. 1 A toy subject A with two manifest variables Q_1 and Q_2 gathers information about a toy object S via a measurement interaction m. Due to the physical nature of the measurement process, the subject can only acquire information about compatible variables. That is, its *internal* perspective (indicated in the white box) has an epistemic horizon, by which the toy subject can learn some properties but not others. In contrast, the *external* omniscient perspective (gray box) features the *complete* joint description of both the toy object and subject. See also Fig. 2

2.5 Measurable properties of toy systems

Regarding measurement interactions in nomic toy theory, we are concerned with the following question: Which variables $Z : \mathrm{S} \to \mathrm{Z}$ can be measured by the toy subject A via a measurement as in Definition 2.6? Our model of learning presumes that the toy subject A only has direct access to its own manifest variable Q. That is, there should be a way to extract the value of Z prior to the measurement from the value of Q after the measurement. The following definition formalises this notion.

Definition 2.7 Given a measurement m of a toy object S by a toy subject A and a variable Z of S, we say that Z is **fixed** by m if there exists a function $f : \mathrm{Q} \to \mathrm{Z}$ satisfying

$$Z(s) = f \circ Q \circ m(s + p) \tag{9}$$

for all $s \in \mathrm{S}$ and all $p \in \mathrm{P}$.
Here, $Z(s)$ is the value of the Z variable before the measurement took place, while $Q \circ m(s + p)$ is the subject's position after the measurement. Note that $\circ$ denotes, as usual, the composition of functions.

The fact that Eq. (9) is required to hold for every p expresses the assumption that the subject cannot use any direct information about its own initial momentum to learn about Z.

Note that the initial value of Q, which has a definite value because the subject enters the interaction in a ready state, is hidden in the choice of f. Specifically, let q_0 be the initial poistion of the toy subject A. Given a function f' satisfying

$$Z(s) = f' \circ Q \circ m(s + q_0 + p) \tag{10}$$

for all s and all p, one can define a new function

$$f(q) := f'\big(q + Q \circ m(q_0)\big), \tag{11}$$

which, by linearity of m and Q, satisfies Eq. (9). Thus, there is no loss of generality in setting $q_0 = 0$ in Definition 2.7. However, the fact that the suitable f depends non-trivially on q_0 means that assuming the toy subject to enter the interaction in a ready state is necessary.
Let us decompose the measurement interaction's matrix form into blocks with respect to $\mathrm{V} = \mathrm{S} \oplus \mathrm{Q} \oplus \mathrm{P}$ via

$$M = \begin{pmatrix} M_{\mathrm{SS}} & M_{\mathrm{SQ}} & M_{\mathrm{SP}} \\ M_{\mathrm{QS}} & M_{\mathrm{QQ}} & M_{\mathrm{QP}} \\ M_{\mathrm{PS}} & M_{\mathrm{PQ}} & M_{\mathrm{PP}} \end{pmatrix}, \tag{12}$$

where, for example, M_{QP} is the block that acts as the linear map $\mathrm{P} \to \mathrm{Q}$.

Definition 2.8 Given the notation from Eq. (12), the subspace $\mathrm{im}\,(M_{\mathrm{QP}})$ of Q provides the **contingent** manifest variable given by the orthogonal projection onto this subspace and denoted by $C : \mathrm{Q} \to \mathrm{C}$. Its orthogonal complement in Q specifies the **free** manifest variable denoted by $F : \mathrm{Q} \to \mathrm{F}$.

Here, $\mathrm{im}\,(M_{\mathrm{QP}})$ denotes the image of the map M_{QP}. The value of the contingent manifest variable after the transformation m depends on the initial momentum of A, which motivates its name. On the other hand, the value of the free manifest variable after the measurement m is independent of the initial momentum of A.

Thus, by definition we have $\mathrm{Q} = \mathrm{F} \oplus \mathrm{C}$. Furthermore, if we write the symplectic matrix M of the transformation in a block form with respect to the decomposition

$$\mathrm{S} \oplus \mathrm{A} = \mathrm{S} \oplus \mathrm{F} \oplus \mathrm{C} \oplus \mathrm{P}, \tag{13}$$

then the block M_{FP} vanishes by definition.
Among all the variables fixed by a given measurement, there is an essentially unique most discerning (i.e. most informative) one, as we show in Proposition 2.10 below. It is the variable $s \mapsto F \circ m(s)$, which is a linear map $\mathrm{S} \to \mathrm{F}$ that is given by M_{FS} in matrix form.

Definition 2.9 The variable **measured** by a measurement m is the linear map $M_{\mathrm{FS}} : \mathrm{S} \to \mathrm{F}$, where F is the free manifest variable. A variable $Z : \mathrm{S} \to \mathrm{Z}$ is called **measurable** if it is measured by some transformation in nomic toy theory.

The next proposition shows that any variable fixed by a measurement can be extracted from the variable measured by it. Therefore, considering variables that are fixed by some measurement does not give the agent any more information about the system than merely restricting attention to variables of the form M_{FS}. This result justifies our identification of the set of *measurable* variables as representing all properties of a toy object that a toy subject can acquire through a measurement interaction.

Proposition 2.10 *If Z is a variable fixed by a measurement m, then there is a function $f : \mathrm{F} \to \mathrm{Z}$ such that for each $s \in \mathrm{S}$ we have*

$$Z(s) = f\left(M_{\mathrm{FS}}\, s\right). \tag{14}$$

Proof Without loss of generality, we can assume that m is a linear map, so that $m(x) = Mx$ for any vector $x \in \mathrm{S} \oplus \mathrm{A}$. This is because affine shifts do not affect whether a variable is fixed by a measurement.

The fact that Z is fixed by m means that there is a function $f : \mathrm{F} \to \mathrm{Z}$ satisfying Eq. (9). Using the notation from Eq. (12) and that M_{FP} vanishes by the definition of the free manifest variable F, we thus have

$$Z(s) = f\big(M_{\mathrm{FS}}\, s + M_{\mathrm{CS}}\, s + M_{\mathrm{CP}}\, p\big) \tag{15}$$

for all $s \in \mathrm{S}$ and all $p \in \mathrm{P}$.
Since M_{CP} is surjective by the definition of the contingent manifest variable *C*, there is a $p_s \in \mathrm{P}$ that satisfies

$$M_{\mathrm{CP}}\, p_s = -M_{\mathrm{CS}}\, s \tag{16}$$

for a given value of s. Choosing p to be p_s in Eq. (15) thus completes the proof. □

3 Epistemic horizons from deterministic laws

We are now ready to present our main result (Theorem 3.1), which derives a limitation on the toy subject's abilities to learn about toy objects. Specifically, we show that a variable is measurable (Definition 2.9) if and only if it is a Poisson variable (Definition 2.2) in nomic toy theory (Sect. 3.1). In Sect. 3.2, we comment on why our agents know nothing about the object prior to learning and how this assumption can be justified with measurement disturbance. We also discuss a model of a toy subject measuring its own momentum and show that it does not break the epistemic horizon—unlike an agent that would have direct access to its own ontic state. Since Poisson variables in nomic toy theory are exactly those that can be known in Spekkens' toy theory, we conclude in Sect. 3.3 that Spekkens' toy theory is the epistemic counterpart of nomic toy theory.

3.1 Constraints on information acquisition

Recall that Poisson variables can be thought of as a collection of functionals with mutually vanishing Poisson brackets. Since the Poisson bracket of generic functionals does not vanish, this implies that not all properties of a toy system can be known simultaneously by an agent in the theory.

With all the definitions introduced in Sect. 2, we can now state our main theorem.

Theorem 3.1 *A variable is measurable in nomic toy theory if and only if it is a Poisson variable.*

Moreover, by Proposition 2.10, the only variables fixed by some measurement in nomic toy theory are those that can be written as a function of some Poisson variable. We illustrate this phenomenon in Fig. 2.

Proof We split the proof into two parts.

Part I: Poisson variables are measurable. In the first part, given any collection of *compatible* components of a Poisson variable, we construct a transformation that implements their joint measurement. That is, we consider an arbitrary Poisson variable $Z : \mathrm{S} \to \mathrm{Z}$ of the toy system S (see Definition 2.2). Recall that Z can be decom-

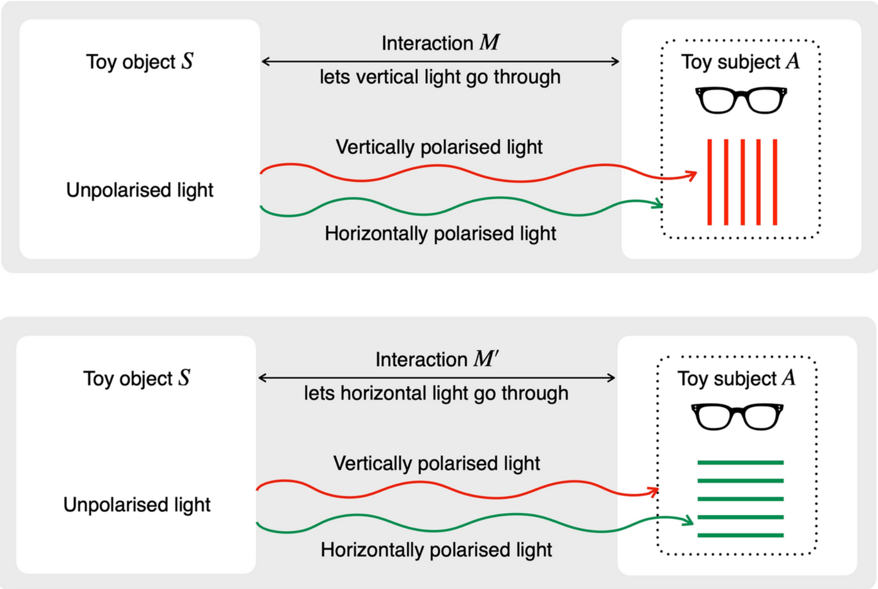

Fig. 2 The epistemic horizon ‘experienced’ by the toy subject is akin to the situation in which the subject would wear glasses that only let vertically polarised light or horizontally polarised light go through. The positioning of the glasses determines whether interaction m or m' takes place. The properties of the toy object are analogous to unpolarised light, which can be decomposed into vertical and horizontal components. The toy subject can choose the orientation of glasses, but cannot observe the toy object without the glasses

posed as a set of components (functionals) as $\{Z_i\}_{i=1}^{\dim(\mathrm{Z})}$. By definition, it satisfies the compatibility equation

$$Z\Omega_{\mathrm{S}}Z^T = 0, \tag{17}$$

which can be interpreted as saying that its components have mutually vanishing Poisson brackets, i.e. they satisfy $\omega(Z_i^T, Z_j^T) = 0$ for all i and j.

We then specify the phase space of the toy subject to be $\mathrm{A} = \mathrm{Q} \oplus \mathrm{P}$, where Q is defined to be Z—the vector space of possible values of Z. Here, we demand Q to be a Lagrangian subspace of A, which thus uniquely fixes P and the symplectic structure on A. The manifest variable of the toy subject A is chosen to be the projection map $\mathrm{A} \to \mathrm{Q}$.

We now construct a transformation $m : \mathrm{S} \oplus \mathrm{A} \to \mathrm{S} \oplus \mathrm{A}$ that measures the Poisson variable Z. It is the linear transformation given as a matrix by

$$M = \begin{pmatrix} \mathbb{1} & 0 & \Omega_{\mathrm{S}}Z^T \\ Z & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \tag{18}$$

in the block form relative to the decomposition. A specific example of the above matrix for the case of a position measurement can be found below.
Note that we have $M_{\mathrm{QP}} = 0$, which implies $\mathrm{F} = \mathrm{Q}$, and $M_{\mathrm{QS}} = Z$. Thus, by Definition 2.9, the transformation m measures Z if it is indeed a valid transformation in nomic toy theory. To show that it is, we have to prove that it is a symplectic matrix, i.e. that $M^T \Omega M = \Omega$ holds. The left-hand side of this equation gives

$$\begin{pmatrix} \Omega_{\mathrm{S}} & 0 & 0 \\ 0 & 0 & \mathbb{1} \\ 0 & -\mathbb{1} & Z\Omega_{\mathrm{S}} Z^T \end{pmatrix}, \tag{19}$$

which is indeed equal to Ω, provided that Z is a Poisson variable satisfying Eq. (17).
Part II: Measurable variables are Poisson variables. In the second part of the proof, we show that no other variables can be measured by valid transformations in nomic toy theory.

Our task is to show that if Z is measurable, then it must be Poisson, which means proving

$$Z\Omega_{\mathrm{S}} Z^T = 0, \tag{20}$$

since the fact that Z is a linear map follows from the definition of measurable variables.
Consider now a measurement m where the toy subject A is given by the symplectic vector space $\mathrm{Q} \oplus \mathrm{P}$ where Q is its manifest variable. Moreover, the linear part of m is denoted by M with blocks denoted with respect to the decomposition from (13). The fact that m is a transformation in nomic toy theory means that M is a symplectic matrix. Moreover, the transpose of every symplectic matrix is also symplectic, i.e. we have $M\Omega M^T = \Omega$. Extracting the FF block out of this set of 16 equations, we find

$$M_{\mathrm{FS}}\Omega_{\mathrm{S}} M_{\mathrm{FS}}^T - M_{\mathrm{FP}} \left(M_{\mathrm{FF}} + M_{\mathrm{FC}}\right)^T + \left(M_{\mathrm{FF}} + M_{\mathrm{FC}}\right) M_{\mathrm{FP}}^T = 0. \tag{21}$$

Since M_{FP} is the zero matrix by Definition 2.8, this implies

$$M_{\mathrm{FS}}\Omega_{\mathrm{S}} M_{\mathrm{FS}}^T = 0, \tag{22}$$

which is what we wanted to show, concluding the proof of Theorem 3.1. □

Note that every functional is a Poisson variable. Theorem 3.1 thus implies that every functional is measurable. Furthermore, by the construction in the first part of the proof, a 2-dimensional subject suffices to measure it.

3.1.1 An example of a position measurement

Let us illustrate the construction of the measurement of a generic Poisson variable Z with a concrete example. To this end, consider both the toy object S and the toy subject A to be 2-dimensional, i.e. each one comes with a single position and a single momentum degree of freedom. Moreover, we choose Z to be the position variable of S, which is a functional in this case. In matrix form, Z is given by

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \tag{23}$$

in the $\{q_{\mathrm{S}}, p_{\mathrm{S}}\}$ basis of S.
Before the measurement, the initial joint state of $S \oplus A$ is denoted by

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{24}$$

in the $\{q_{\mathrm{S}}, p_{\mathrm{S}}, q_{\mathrm{A}}, p_{\mathrm{A}}\}$ basis of $\mathrm{S} \oplus \mathrm{A}$. On the other hand, the measurement interaction from the proof of Theorem 3.1 is in general given by the matrix M from Eq. (18). Substituting the position variable from (23) for Z in this expression gives

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{25}$$

Thus, the post-measurement ontic state of the joint system is

$$Mv = \begin{pmatrix} v_1 \\ v_2 - v_4 \\ v_1 + v_3 \\ v_4 \end{pmatrix}. \tag{26}$$

We notice two crucial features. First, if the toy subject A is initially in a ready state, i.e. if v_3 has a definite value, then the manifest variable after the measurement encodes the initial position of S given by v_1. This illustrates one role of our assumption that the agent's manifest variable be fixed prior to the measurement.
Secondly, there is a back-reaction on the object's momentum—the conjugate variable to the measured position of S. In particular, its value after the measurement is disturbed by a value that equals the initial momentum of the toy subject A. This disturbance highlights the role of our assumption that the toy subject cannot directly know its own momentum. If it did, the measurement disturbance could be accounted for.

Let us discuss both of these points in more detail now.

3.2 A couple of caveats

Our claim that the learning of an agent in nomic toy theory is limited by an epistemic horizon hinges on the following caveat.

3.2.1 The relevance of (no) a priori knowledge

We assume that, prior to any measurement, the agent possesses no knowledge about the state of the toy object S. Indeed, imagine that, on the contrary, the following is true: The agent A is composed of two subsystems, i.e. we have $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$ where each A_i is a toy subject with an associated manifest variable Q_i. At time t_1 (labelling that the measurement process is yet to occur), the value of the manifest variable Q_1 encodes the momentum of S and, importantly, the agent A knows that this is the case. Furthermore, A_2 is in a ready state (see Definition 2.6).

Then, applying the transformation $m_q : \mathrm{S} \oplus \mathrm{A}_2 \to \mathrm{S} \oplus \mathrm{A}_2$ given by Eq. (18), where Z is the position variable of S, enables A_2 to learn the position of S. Since the state of A_1 is unchanged by this transformation, at time t_2 (labelling that the transformation has occurred) we have the following situation: The manifest variable Q_1 at time t_2 encodes the momentum of S at time t_1 and the manifest variable Q_2 at time t_2 encodes the position of S at time t_1. In conjunction, at time t_2, the agent A has a complete specification of the ontic state of S at time t_1 and thus breaks the purported epistemic horizon.

However, it is natural to assume that the agent has no knowledge of the toy object's state initially. After all, we are interested in deriving fundamental bounds on the learning capabilities of the agent. Any pre-existing knowledge should be accounted for by an explicit process that allows A to obtain information. Hence, we can circumvent the above caveat on grounds that the acquisition of information invariably involves an interaction with the physical world. This justifies our assumption that toy subjects possess no a priori knowledge of the state of the toy object.

3.2.2 Measurement disturbance

But how can we be certain that the knowledge of the object's momentum by A_1 could not be accounted for by an explicit process in nomic toy theory? Abstractly, this follows from Theorem 3.1.

More concretely, consider a measurement $m_p : \mathrm{S} \oplus \mathrm{A}_1 \to \mathrm{S} \oplus \mathrm{A}_1$ that is applied before m_q and given by Eq. (18), where Z is now the momentum of S (see Fig. 3). That is, m_p encodes the momentum of S at time t_0 (labelling that the measurement m_p is yet to occur) into the manifest variable Q_1 at time t_1. One can check that the momentum of S is unaffected by m_p and thus the manifest variable Q_1 at time t_1 also coincides with the momentum of S at time t_0. See Appendix A for the explicit computations.

The issue is that m_p disturbs the position of the toy object S by a shift that depends on the (unknown) momentum of A_1 at time t_0. As a result, the subsequent transformation m_q encodes the position of S at time t_1, rather than the one at time t_0, into the manifest variable Q_2. The composed process $m_q \circ m_p$ is therefore no counterexample to Theorem 3.1. Nevertheless, our discussion shows that the toy subject A can, at time t_2, have perfect knowledge of the ontic state of the toy object S at time t_1! This is in line with the fact that also the ontic state in Spekkens' toy theory can be perfectly known given both pre- and post-selection (Hausmann et al., 2023). It is worth mentioning that the same is true in quantum mechanics.

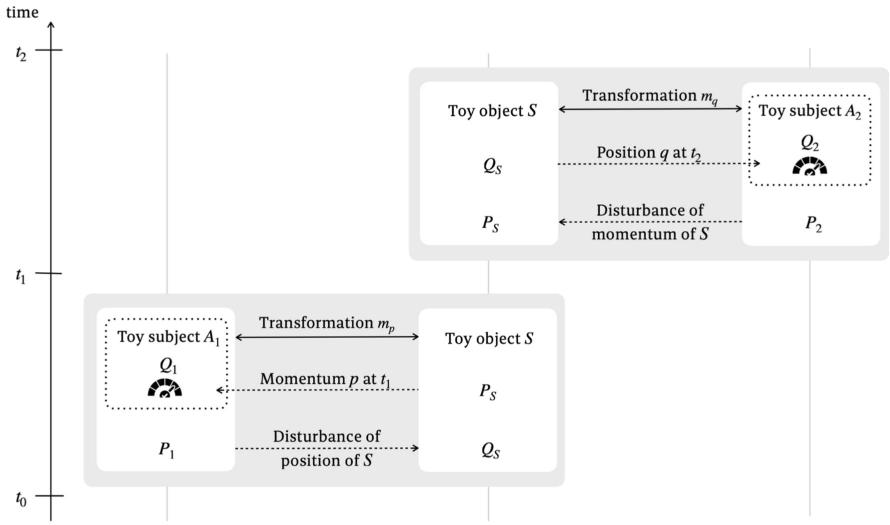

Fig. 3 At time t_0, both subjects are in a ready state with $Q_1 = Q_2 = 0$. Then, the toy subject A_1 measures the toy object S, so that the initial value of the object's momentum gets encoded into the manifest variable Q_1. At the same time, the toy object's position gets disturbed by an amount that depends on the initial momentum P_1 of the toy subject A_1. Subsequently, the other toy subject A_2 interacts with S, so that the object's position at time t_1 gets encoded into the manifest variable Q_2. During this interaction, the object's momentum is disturbed by an amount that depends on the initial momentum of A_2

Nevertheless, neither the ontic state of S at time t_0 nor the one at time t_2 is completely known to A. That is, it is still true that the subject A cannot at any time encode a previously unknown ontic state of the object S that it possesses at that very same time. The transformation m_p applied between times t_0 and t_1 disturbs the position of S, while the transformation m_q applied between times t_1 and t_2 disturbs the momentum of S. The results we derive in this paper rule out the possibility that after a measurement (time t_2 above), A would know the ontic state of S before the measurement (time t_0 above), which is what we call learning. The fact that A may have sufficient information to determine the value of incompatible variables at an intermediate time during the measurement is not ruled out by the epistemic horizon we derive.

3.2.3 Self-measurement of toy subjects

The reason why A cannot access the initial position of S in the above example is its disturbance by the initial momentum of A_1, which is unknown. But could a toy subject measure its own momentum and with this information correct for the disturbance?

As it turns out, such a self-measurement is implicitly accounted for by Theorem 3.1. Our main result therefore gives an abstract argument why measuring one's own momentum cannot break the epistemic horizon.

This stems from measurement disturbance again. Specifically, imagine a further toy subject A_3 that measures the momentum of A_1 before time t_0. This measurement disturbs the position of A_1 by a shift that depends on the (unknown) initial momentum of A_3. As a result, the momentum of S is no longer fixed by m_p. Self-measurement thus does not resolve the issue, the source of the uncertainty has been merely shifted from the initial momentum of A_1 to that of A_3. One could imagine introducing further pointers to measure initial momenta, but this inevitable leads to an infinite regress that does not stabilise to a reliable knowledge of the relevant parameters.

Our analysis of nomic toy theory implies that an epistemic horizon exists also in classical mechanics, given the assumption that agents modelled as mechanical systems can only directly access their own manifest variable (Definition 2.5). In contrast, traditional accounts claim that in classical mechanics arbitrary measurement precision can be achieved and that both position and momentum can be recorded simultaneously (see, for instance, Solé et al. (2016)). However, a closer look at these arguments reveals that this holds only under the assumption that the initial momentum of the measurement apparatus is known—in line with our discussion above.

3.3 Spekkens' toy theory as the epistemic counterpart of the nomic toy theory

In this section we briefly discuss the connection between measurable variables in nomic toy theory and epistemic states in Spekkens' toy theory (Spekkens, 2016).

While agents are not explicitly modelled as physical systems in Spekkens' toy theory, its epistemic restriction is introduced to specify what a hypothetical agent could learn about a physical system.

Ontic states and the associated reversible transformations in Spekkens' toy theory match those of nomic toy theory. While the latter posits no a priori notion of epistemic states, these are explicitly specified in Spekkens' toy theory (see our description in Appendix B for more details). Specifically, each epistemic state corresponds to the value of a variable that can be known according to the epistemic restriction in Spekkens' toy theory (and vice versa). Among these 'knowable' variables, the scalar-valued ones are called *quadrature functionals*. A generic one, an affine map $f : \mathrm{V} \to \mathcal{F}$, can be written as

$$f := a_1 q_1^T + b_1 p_1^T + \cdots + a_n q_n^T + b_n p_n^T + c, \tag{27}$$

where $\{q_1, \ldots, q_n, p_1, \ldots, p_n\}$ is the chosen orthonormal basis of the phase space V and a_j, b_j, c are all scalars in the field $\mathcal{F}$. As far as the resulting epistemic state is concerned, we can assume $c = 0$ without loss of generality (cf. our notion of equivalence of variables introduced in Sect. 2.2).

Generic (vector-valued) linear variable can be identified as a collection of quadrature functionals. The epistemic restriction of Spekkens' toy model says that such a collection is jointly knowable if and only if the Poisson bracket of each pair of them vanishes, i.e. $\{f_1, f_2\} = 0$ in Spekkens' standard notation for quadrature functionals. The theory postulates that variables whose value can be known are precisely Poisson variables in nomic toy theory as introduced in Definition 2.2. Furthermore, as we show in Proposition 2.10 and Theorem 3.1, Poisson variables in nomic toy theory coincide with those properties of toy systems that can be learned by a toy subject within the world. In this way, the epistemic restriction of Spekkens' toy theory arises from two ingredients:

(1) the allowed transformations of ontic states introduced in Sect. 2.1 (which coincide for nomic and Spekkens' toy theories), and
(2) the specification of information gathering agents and identification of their directly accessible information in the form of manifest variables (Definition 2.5).While nomic toy theory does not come with a pre-specified epistemology, the second ingredient allows us to derive an epistemic horizon for the model of toy subjects used in this article. Doing so, we find that the derived epistemic aspects of subjects in nomic toy theory coincide—at least as far as epistemic states are concerned—with the posited epistemic horizon in Spekkens' toy theory.

4 Conclusions

Let us now discuss implications of our results and related questions in the foundations of physics so as to put things into a broader perspective. We discuss the significance of our work for the relationship of internal and external observers, representationalism, the subject-object split and the reality of unobserved properties (Sect. 4.1). We also comment on the relationship of nomic toy theory to quantum theory, and a possible view of physical phenomena that supersedes the subject-object separability. We then conclude with an outlook on future directions of study in Sect. 4.2.

4.1 Discussion

4.1.1 Internal versus external perspective

Theorem 3.1 can be interpreted as describing a relationship of two distinct perspectives. One is the omniscient view that specifies the precise ontic state of every system in toy world, akin to the meticulous vision of the entire state of the toy universe by Laplace's demon. This view is by definition from 'outside', i.e. external to the world. Conversely, there is an internal perspective as experienced by an embedded toy subject. This view is shown to be limited relative to the omniscient one. As we prove, a subject in toy theory cannot learn the precise ontic state of another toy system by interacting with it. The best description it can have is an epistemic state, which necessarily retains uncertainty about the precise ontic state (see Appendix B.1 for details).

4.1.2 Subject-object inseparability

The derived epistemic horizon emphasises the participatory nature of the subject in the theory. It shows that the physically allowed information gathering activities of an agent affect the knowledge it can have about an object. This challenges the old divide between the subject and object. That is, our approach highlights that the standard notions of measurement, representability, and epistemology are intimately bound up. Relatedly, in Russo (2000) Russo builds on the ontoepistemology of Barad (2007) and the constructionist approach of Floridi (2011) to argue for an epistemology of techno-scientific practice.[9]

Moreover, the construction of a toy subject measuring itself (Sect. 3.2) introduces the possibility of self-reference, which in turn makes the knowledge of a toy subject liable to logical paradoxes. It is conceivable that our results could be linked to a logical argument about the impossibility for an observer to describe itself from within the world. In particular, recall that the crucial Definition 2.5 of toy subjects specifies what a toy subject knows about its own ontic state as well as how its knowledge is manifested in its ontic state. Relatedly, Ismael presents an argument for the instability in an embedded agent's ability to know the future due to self-reference (Ismael, 2023).

So it could be argued, perhaps, that what is 'real' to one subject is not 'real' to another. Furthermore, does it make sense for the subject to speak of a world as being separate from itself? What would a measurement outcome signify if we take the participatory nature of the subject seriously and abandon an observation-independent reality? What is the new referent of measurement? In other words, what supersedes the subject-object split?

4.1.3 Epistemic horizons and their implications for ontology

The idea that a physical theory may operate under the premise of an observer-dependent description is not new. Several interpretations of quantum theory take a similar stance, such as the non-realist (Fuchs and Schack, 2013; Rovelli, 1996; Faye, 2019), pragmatist (Healey, 2012), or Everett (1957) approaches.

Nomic toy theory gives an explicit account of the interdependency of subjects and objects. It invites us to study whether subjects are justified to posit the existence of ontic states that are only 'visible' from an omniscient perspective. Even though subjects in nomic toy theory are faced with an epistemic horizon, this limitation is compatible with a deterministic and classical description. Can the same be said for other kinds of subjective experiences featuring an epistemic horizon, such as the one of quantum theory? Are there operational theories whose predictions *rule out* the possibility that their epistemic horizon stems from the dynamical laws of a classical ontic theory?

Making these questions precise requires a careful construction of a more general framework than our investigation of nomic toy theory and its symplectic dynamics. With it, one may hope to classify the kinds of epistemic horizons that could arise based on the allowed subject-object interactions just like the one we derive in this paper. Similar efforts have been successful in the framework of ontological models (a.k.a. hidden variable models), in which one can formally derive the operational consequences of metaphysical assumptions such as Bell locality (Bell and Aspect, 2004) and non-contextuality (Kochen and Specker, 1967).

Importantly, the fact that the operational consequences of both are violated by behaviours of quantum systems constrains the possible underlying physical reality.

[9] We thank an anonymous reviewer for drawing our attention to Russo's work.

Answering the questions from previous paragraphs would likely constitute an analogous step in understanding quantum theory and its viable interpretations.

4.1.4 Relation to interpretations of quantum theory

Although we do not provide answers to the questions posed in the previous paragraphs, it is worthwhile to mention that a version of an observer-dependent realism aligns with the spirit of relational and QBist approaches to quantum mechanics (Rovelli, 1996; Fuchs and Schack, 2013). See also Barad's agential realism (Barad, 2007) and the quantum holism of Ismael and Schaffer (2020).

For instance, relational quantum mechanics purports that the notion of a subject has no metaphysical significance—any physical system could be one. Moreover, it emphasises "the way in which one part of nature manifests itself to any other single part of nature" (Rovelli et al., 2021, p. 67). In this view, properties of an object are relative to another system which interacts (and thus measures) the object. This resonates with the notion of the observer-dependent epistemic state in nomic toy theory.

Relational quantum mechanics, as well as many other interpretations, effectively posit that quantum properties *do not exist* prior to measurement or that there is no way to consistently describe them (see, for instance, Wheeler's participatory nature (Wheeler & Zurek, 1983, pp. 182–213). This is in contrast with the ontic status of unobservable variables in Spekkens' toy theory (and thus also nomic toy theory). There, we have an epistemic horizon featuring unpredictability, uncertainty, and complementarity, even though all properties of systems exist and have definite values at all times (at least from the omniscient perspective featuring the full ontic state description). From a toy subject's perspective, however, the view is very similar to one invoking participatory 'realism'. The subject-object divide can be therefore called into question even given a deterministic physical theory.

Furthermore, recall the intuition that the epistemic horizon of nomic toy theory is connected to the uncontrollable initial momentum of toy subjects, which introduces an unpredictable disturbance of the toy object (Sect. 3.2). This implies that a toy subject measuring position after a measurement of momentum (Fig. 3) may find a different position value than a toy subject measuring position prior to the measurement of momentum. More generally, Theorem 3.1 implies that there is no simultaneous measurement of both position and momentum—they are incompatible. In quantum theory, the incompatibility structure of observables leads to contextuality Kochen and Specker (1967). In contrast, Spekkens' toy theory is non-contextual. In this case, the incompatibility can be seen merely as an expression of measurement disturbance (see also the more general arguments in Erba et al. (2024)).

It is noteworthy that even full quantum theory can be given an ontological model in which measurement interactions entail disturbance—e.g. the de Broglie-Bohm theory (dBBT) (Bohm and Hiley, 1993). Part of the ontology of dBBT consists of particles like in nomic toy theory. However, dBBT also incorporates the quantum wave-function which can be seen as the analogue of the epistemic state from Spekkens' toy theory. In dBBT it is used to determine particles' trajectories as well as the initial distribution of their positions. This is in contrast to nomic toy theory where both the initial configurations and dynamics of particles are independent of epistemic states.

Studying epistemic horizons, i.e. defining measurable variables and characterising a subject's information-gathering capabilities within dBBT, in parallel with our investigations here, would be a promising direction for future research.

4.1.5 Limitations on predictability

We suspect that our result also implies that a toy subject cannot *prepare* the toy object in a fixed ontic state. Intuitively, this would follow from the fact that in order to prepare an exact ontic state, a subject would need to perform a measurement that signifies the preparation of this state. But as we have shown such a measurement process does not exist (a similar claim was proven by Hausmann et al. (2023) in the context of Spekkens' toy theory).

4.2 Summary and outlook

We have used nomic toy theory—an essentially *classical* theory—to propose an explicit account of the source of the epistemic horizon in Spekkens' toy theory. Subjects in nomic toy theory can only ever ascertain a coarse-grained description of objects in the world, namely one in terms of the epistemic states of Spekkens' toy theory. We attribute the source of the fundamental uncertainty to the nature of interactions between subjects and objects. Specifically, the learning process governed by such an interaction is invariably connected to a disturbance of the object, which prevents the subject from learning the complete state of the object.

At first glance, our result may be surprising in light of the claims that Newtonian mechanics should in principle allow for arbitrarily precise measurements of the properties of a classical particle. Bear in mind that Liouville's theorem in Hamiltonian mechanics implies preservation of phase space volume, but does not rule out arbitrary stretching and squeezing of a phase space volume such that conjugate variables become simultaneously sharply defined. However, we suspect that our result could be related to the claims of de Gosson on the relationship of symplectic geometry and quantum uncertainty principles (de Gosson, 2009). Basically, de Gosson derived an analogue of the quantum Robertson–Schrödinger inequality from the symplectic properties of the phase space alone. This essentially implies that Heisenberg's uncertainty relations already hold in Hamiltonian mechanics for all pairs of conjugate position and momentum variables.

Why does it seem that some aspects of quantum uncertainty can be explained in terms of Hamiltonian mechanics? Do uncertainty relations really have such an analogue in classical physics? Can the epistemic horizon in nomic toy theory be restated as a classical uncertainty relation akin to Heisenberg's uncertainty principle in quantum theory? We hope that our analysis will serve as a toy example to facilitate explorations of those pertinent questions.

For instance, one may study the role of hidden variables in quantum theory. Our result derives the consequences of positing a specific classical ontology for the learning capabilities of internal agents. Not all underlying ontic models may lead to the same information gathering capabilities of agents. Thus, the empirically observed epistemic horizon could potentially be used to rule out the ontological models that do not reproduce it. More on this is found in the "implications for ontology" part of the Discussion (Sect. 4.1).

A related question concerns the development of ontological models motivated and evaluated from within nomic toy theory. That is, one may investigate what kind of ontologies are consistent with the experience of an epistemically restricted toy subject. Is there a way to differentiate among them based on desiderata such as parsimony or naturalness? In a nutshell, what would such a subject conclude about the ontology underlying the phenomena observed? See also a potential link to problems about bootstrapping and reliabilist epistemology (Goldman and Beddor, 2021).

As one possibility, one could look at nomic toy theory in an Everettian setting where pointer states are not single valued. Could a many-worlds ontology lead to a single-world experience of the toy subject (Barrett et al., 2010, Chapter 2)? It would be interesting to study the problem of Everettian probabilities in this context (Barrett et al., 2010, Chapter 3). There may also exist connections to more elaborate models of agents such as those in Shrapnel et al. (2023).

In the future we also wish to shed light on multi-agent scenarios. A recent attempt to try to view quantum theory as an integration of perspectives of agents subject to the epistemic horizon of Spekkens' toy theory has been explored in Braasch Jr and Wootters (2022). It is particularly interesting to look at what different subjects can communicate intersubjectively (see also related ideas in the context of Spekkens' toy model Hausmann et al. (2023)). This may perhaps allow novel insights into the intricacies of many recently studied Wigner's friend type scenarios as well as no-go claims on 'observer-independent facts' (Wigner, 1961; Bong et al., 2020; Frauchiger and Renner, 2018; Lawrence et al., 2023; Brukner, 2018; Ormrod and Barrett, 2022). See also the reviews in Adlam (2024), Schmid et al. (2023), Brukner (2022) and more general results on quantum epistemic boundaries (Fankhauser, 2023).

We also leave open the question whether the participatory nature of the agent in our toy theory entails a more parsimonious account of the physical world. Could there exist a new *relational* physical state of the world relative to the internal observers of the theory describing the subject and object jointly? Such an account would go beyond the traditional subject-object split and take inseparability seriously.

Appendix A: Composing position and momentum measurements

Here, we give additional details on the attempted construction of a joint measurement of position and momentum from Sect. 3.2. Specifically, we consider three toy systems—S, A_1, and A_2—each of which has one position and one momentum degree of freedom. Moreover, the latter two are toy subjects with their positions acting as manifest variables (see Fig. 3).

The joint system $A_1 \oplus S \oplus A_2$ starts out at time t_0 in the ontic state denoted by

$$u(t_0) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{\mathrm{A}_1} \oplus \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}_{\mathrm{S}} \oplus \begin{pmatrix} u_5 \\ u_6 \end{pmatrix}_{\mathrm{A}_2} \tag{28}$$

in the $\{q_1, p_1, q_\mathrm{S}, p_\mathrm{S}, q_2, p_2\}$ basis of $\mathrm{A}_1 \oplus \mathrm{S} \oplus \mathrm{A}_2$.

The first interaction m_p is a measurement of the momentum of S by the toy subject A_1. Just as at the end of Sect. 3.1, we substitute the matrix form

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \tag{29}$$

of the momentum variable into Eq. (18) to obtain the matrix form of m_p:

$$M_p = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{30}$$

where one ought to be careful that the subject and object are now in reverse order compared to Eq. (18). Here, we merely write its action on $\mathrm{A}_1 \oplus \mathrm{S}$. The action on the full joint state space is then via $M_p \oplus \mathbb{1}_{\mathrm{A}_2}$.

At time t_1, i.e. once the interaction m_p has taken place, the joint state of all three toy systems is thus

$$u(t_1) = \begin{pmatrix} u_1 + u_4 \\ u_2 \end{pmatrix}_{\mathrm{A}_1} \oplus \begin{pmatrix} u_2 + u_3 \\ u_4 \end{pmatrix}_{\mathrm{S}} \oplus \begin{pmatrix} u_5 \\ u_6 \end{pmatrix}_{\mathrm{A}_2}. \tag{31}$$

As we can see, the manifest variable of A_1 now encodes the initial momentum of S, provided that A_1 started out in a ready state. Furthermore, the position of S has been disturbed by the initial momentum of A_1.

The second step of the composite transformation depicted in Fig. 3 is a measurement m_q of the position of S by the toy subject A_2. Its matrix form is as in Eq. (25):

$$M_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

After this interaction, at time t_2, the full ontic state is

$$u(t_2) = \begin{pmatrix} u_1 + u_4 \\ u_2 \end{pmatrix}_{\mathrm{A}_1} \oplus \begin{pmatrix} u_2 + u_3 \\ u_4 - u_6 \end{pmatrix}_{\mathrm{S}} \oplus \begin{pmatrix} u_2 + u_3 + u_5 \\ u_6 \end{pmatrix}_{\mathrm{A}_2}. \tag{33}$$

If we assume the ready states of the toy subjects have vanishing manifest variables, this reduces to

$$\begin{pmatrix} u_4 \\ u_2 \end{pmatrix}_{\mathrm{A}_1} \oplus \begin{pmatrix} u_2 + u_3 \\ u_4 - u_6 \end{pmatrix}_{\mathrm{S}} \oplus \begin{pmatrix} u_2 + u_3 \\ u_6 \end{pmatrix}_{\mathrm{A}_2}. \tag{34}$$

The values of the manifest variables at time t_2 are thus u_4 and $u_2 + u_3$ respectively. The former encodes the correct momentum of S at times t_0 and t_1, while the latter encodes the correct position of S at times t_1 and t_2.

Appendix B: Supplementary material on Spekkens' toy theory

As we mention throughout the text, nomic toy theory shares both the kinematics and dynamics with Spekkens' toy theory (Spekkens, 2007). This is not an accident. We are specifically interested in the latter because it features both an epistemic restriction as well as deterministic dynamics at the ontic level. As we discuss in Sect. 3.3, our results show that the epistemic restriction of Spekkens' toy theory coincides with the epistemic horizon of nomic toy theory that we derive. To make this precise, we provide a description of the epistemic level of Spekkens' toy theory here including several auxilliary results. Our presentations closely follows that of Hausmann et al. (2021). For additional details on Spekkens' toy theory, see (Spekkens, 2016; Catani and Browne, 2017).

B.1 Systems in Spekkens' toy theory

The ontic state space of a system V is a symplectic vector space V, just as we discuss in Sect. 2.1.

Remark B.1 If the underlying field of V is that of real numbers, we obtain continuous toy systems. Basic finite systems are associated with an integer *d*. Their ontic state space is a (symplectic) $\mathbb{Z}_d$-module, which is a vector space if *d* is a prime power. Other finite systems can be obtained as composites of the basic ones (see Appendix B.2.1).

For a linear subspace W of V, we can define the **symplectic complement**

$$\mathrm{W}^{\omega} := \{v \in \mathrm{V} \mid \omega(\mathrm{W}, v) = 0\} \tag{35}$$

where

$$\omega(\mathrm{W}, v) = 0 \quad :\Longleftrightarrow \quad \omega(x, v) = 0 \;\; \forall x \in \mathrm{W}. \tag{36}$$

Such a subspace W is

- a **symplectic subspace** if $\mathrm{W}^{\omega} \cap \mathrm{W} = \{0\}$,
- **isotropic** if $\mathrm{W} \subseteq \mathrm{W}^{\omega}$, i.e. if the symplectic form vanishes on W, and
- Lagrangian if $\mathrm{W} = \mathrm{W}^{\omega}$, i.e. if it is a maximal isotropic subspace (cf. Definition 2.4). An **epistemic state** of Spekkens' toy theory (U, a) is specified by an isotropic subspace U of V and a vector $a \in \mathrm{V}$. Via an isomorphism of V and its

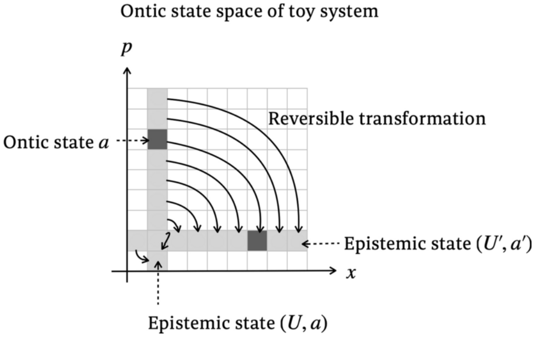

Fig. 4 While the ontic state a is an element in the ontic state space of the toy system, the support of an epistemic state (U, a) is a collection of such elements, namely those given by Eq. (38). After a reversible transformation, these are transformed to another collection (U', a') given by Eq. (40), where a' is the image of a under the transformation. In the figure, U is the position variable and U' is the momentum variable

dual V^*, the subspace U is interpreted as consisting of those functionals whose values are known. Alternatively, we can think of U as the set of values of the orthogonal projection $U : \mathrm{V} \to \mathrm{U}$. This is an isotropic variable if and only if U is isotropic.

The vector a is interpreted as one of the ontic states that is deemed possible by this epistemic state.

It fixes the value of any functional $u \in \mathrm{U}$ to be

$$\langle u, a \rangle \tag{37}$$

where $\langle \_, \_ \rangle$ is the canonical inner product on V. Thus, the set of all ontic states that are possible according to the epistemic state (U, a) is

$$\mathrm{U}_a := \left\{ v \in \mathrm{V} \,\middle|\, \langle u, a \rangle = \langle u, v \rangle \;\; \forall\, u \in \mathrm{U} \right\} = \mathrm{U}^\perp + a, \tag{38}$$

where $\mathrm{U}^\perp$ is the orthogonal complement of U. In other words, the possible ontic states must share the value of the variable U. We call U_a the **support** of the epistemic state (U, a). Note that it is an affine subspace of V. We do not distinguish between epistemic states that have the same support. An epistemic state (U, a) is called **pure** if U is Lagrangian.

The reversible transformations of Spekkens' toy theory form the affine symplectic group (Sect. 2.1) and act on ontic states via the canonical action. That is, its elements are pairs (t, v) of a symplectic map $t \in \mathrm{Sp}(\mathrm{V})$ and a vector $v \in \mathrm{V}$, which compose via

$$(s, u) \circ (t, v) = \big(s \circ t, u + s(v)\big). \tag{39}$$

A given reversible transformation (t, v) then acts on ontic states via $x \mapsto t(x) + v$.

The following proposition shows that epistemic states are mapped to epistemic states under affine symplectic transformations and derives Equation (A.3) from Hausmann et al. (2023).

Proposition B.2 *Let (f, v) be an affine symplectic map on a symplectic vector space* V *and let* U_a *be the support of an epistemic state* (U, a). *The affine subspace*

$f(\mathrm{U}_a)+v$, *which is the image of* U_a *under* (f, v), *coincides with the support of the epistemic state*

$$\left(f^{-1^T}(\mathrm{U}), f(a)+v\right). \tag{40}$$

Proof First, let us show that (40) is indeed an epistemic state. To this end, note that the inverse of any symplectic matrix $M \in \mathrm{Sp}(\mathrm{V})$ is given by

$$M^{-1} = \Omega^T M^T \Omega. \tag{41}$$

Therefore, f^{-1^T} is given by

$$f^{-1^T}(v) = -\Omega F \Omega v \tag{42}$$

where F is the matrix representation of f. In particular, it is also a symplectic map.

By Lemma B.3 proven below, the image of U under f^{-1^T} is an isotropic subspace and (40) is thus an epistemic state.

The rest of the proof establishes that $f(\mathrm{U}_a)+v$ *is the support of this epistemic state. We have*

$$f(\mathrm{U}_a)+v = \left\{ f(x)+v \;\middle|\; \langle u, a\rangle = \langle u, x\rangle \;\; \forall\, u \in \mathrm{U} \right\} \tag{43}$$

by definition. Let us denote $f(x)+v$ by w, so that we have $x = f^{-1}(w-v)$. Then, the right-hand side of Eq. (43) is the set of all $w \in \mathrm{V}$ satisfying

$$\langle u, a\rangle = \left\langle u, f^{-1}(w-v)\right\rangle \;\; \forall\, u \in \mathrm{U}. \tag{44}$$

Since we have $f^T f^{-1^T} = \mathbb{I}$, the left-hand side of Eq. (44) is equal to either side of

$$\left\langle f^T f^{-1^T}(u), a\right\rangle = \left\langle f^{-1^T}(u), f(a)\right\rangle, \tag{45}$$

while the right-hand side of Eq. (44) is

$$\left\langle f^{-1^T}(u), w-v\right\rangle. \tag{46}$$

Thus we obtain

$$f(\mathrm{U}_a)+v = \left\{ w \;\middle|\; \left\langle f^{-1^T}(u), f(a)+v\right\rangle = \left\langle f^{-1^T}(u), w\right\rangle \;\; \forall\, u \in \mathrm{U} \right\}, \tag{47}$$

which is the support of the epistemic state in (40). □

Therefore, reversible transformations preserve the set of epistemic states. In other words, if a function $f : \mathrm{V} \to \mathrm{V}$ maps (the support of) some epistemic state to a subset of V that is not (the support of) an epistemic state, then f is not a valid reversible transformation. For example, this directly implies Corollary 1 (Restrictions on conditional transformations: example) in Hausmann et al. (2023).

Lemma B.3 *Symplectic maps preserve the set of isotropic subspaces. That is, if* $f : \mathrm{V} \to \mathrm{V}$ *is a symplectic map and* W *is an isotropic subspace of* V, *then* $f(\mathrm{W})$ *is also an isotropic subspace.*

This is a standard result, we give the proof for completeness.

Proof Note that a subspace W is isotropic if and only if the implication

$$v \in \mathrm{W} \quad \Longrightarrow \quad \omega(x, v) = 0 \quad \forall x \in \mathrm{W} \tag{48}$$

holds. Moreover, since f is bijective, we have $y \in f(\mathrm{W})$ if and only if $y = f(v)$ for some $v \in \mathrm{W}$. Thus we have

$$y \in f(\mathrm{W}) \quad \Longrightarrow \quad \omega\left(x, f^{-1}(y)\right) = 0 \quad \forall x \in \mathrm{W} \tag{49}$$

$$\Longleftrightarrow \quad \omega\left(f^{-1}(z), f^{-1}(y)\right) = 0 \quad \forall z \in f(\mathrm{W}) \tag{50}$$

$$\Longleftrightarrow \quad \omega\left(z, y\right) = 0 \quad \forall z \in f(\mathrm{W}) \tag{51}$$

where the last equivalence holds because f^{-1} is itself symplectic. In conclusion, $f(\mathrm{W})$ is isotropic. □

Note, furthermore, that symplectic maps in $\mathrm{Sp}(V)$ act transitively on the Lagrangian Grassmanian (Calegari (2022), Lemma 1.12).

B.2 Description of multiple systems in Spekkens' toy theory

B.2.1 Joint states

Each ontic state of the joint (bipartite) system is given by a pair of ontic states from each of the components respectively. Its underlying vector space is thus the direct product of the individual ones, which is isomorphic to their direct sum.

Definition B.4 Given two toy systems (V_1, ω_1) and (V_2, ω_2), the **joint system** describing their composite is given by $(\mathrm{V}_1 \oplus \mathrm{V}_2, \omega_1 \oplus \omega_2)$.

Every joint ontic state $v \in \mathrm{V} := \mathrm{V}_1 \oplus \mathrm{V}_2$ has a unique decomposition $v = v_1 + v_2$ for $v_i \in \mathrm{V}_i$. Moreover, there are linear projections $V_i : \mathrm{V} \to \mathrm{V}_i$, such that V_i maps v

to v_i. Similarly, for any choice of epistemic states (U_1, a_1), and (U_2, a_2) of V_1 and V_2 respectively, the joint state of $\mathrm{V}_1 \oplus \mathrm{V}_2$ is the epistemic state $(\mathrm{U}_1 \oplus \mathrm{U}_2, a_1 + a_2)$ with support[10]

$$(\mathrm{U}_1 \oplus \mathrm{U}_2)^{\perp} + a_1 + a_2 = \left(\mathrm{U}_1^{\perp} + a_1\right) \oplus \left(\mathrm{U}_2^{\perp} + a_2\right). \tag{52}$$

These constitute the so-called **product states** of the joint system.

Besides product states, there are also correlated joint states. As an example, consider the joint system of two toy bits with its epistemic state given by

$$\left(\mathrm{span}\left\{\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right\}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right). \tag{53}$$

It is a state for which both the positions and momenta of the two systems are perfectly correlated. Its support is the subset of $(\mathbb{Z}_2)^4$ given by

$$\left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}. \tag{54}$$

B.2.2 Reduced states

Appendix B.2.1 describes global states of multiple systems. For any such global state, we can marginalize any of its subsystems to obtain the local description of the remaining subsystems. This notion also appears in Definition 2.6 of pointer-preserving measurements.

Definition B.5 Given a possibilistic state[11] ρ of a composite system $\mathrm{V}_1 \oplus \mathrm{V}_2$, its $\mathrm{V}_{\boldsymbol{i}}$ **-marginal** (also referred to as the reduced state to V_i) is the image of ρ under the projection V_i.

Whenever ρ is the support of an epistemic state (U, a), we can find its marginal by projecting *a* and restricting the set of known functionals in U to the local ones.

Proposition B.6 (Marginals of epistemic states) *Consider an epistemic state* (U, a) *of the composite* $\mathrm{V} = \mathrm{V}_1 \oplus \mathrm{V}_2$. *Then the* V_1*-marginal of its support is the support of the epistemic state of* V_1 *given by*

[10] Note that on the right-hand side, $\mathrm{U}_i^{\perp}$ refers to the orthogonal complement of U_i *within* V_i, as opposed to the left-hand side, where it denotes the orthogonal complement in V.

[11] A possibilistic state of a toy system is a subset of its underlying vector space of ontic states. Key examples of possibilistic states are supports of epistemic states.

$$\left(\mathrm{U} \cap \mathrm{V}_1, V_1(a)\right). \tag{55}$$

Proof The V_1-marginal of (the support of) (U, a) is

$$V_1\left(\mathrm{U}^\perp\right) + V_1(a), \tag{56}$$

while the support of the epistemic state in (55) is

$$(\mathrm{U} \cap \mathrm{V}_1)^\perp + V_1(a), \tag{57}$$

where the orthogonal complement is within V_1 here. The task is to show that these two affine subspaces of V_1 coincide. Writing expression (57) instead in terms of the orthogonal complement within V, we thus have to show

$$V_1\left(\mathrm{U}^\perp\right) = (\mathrm{U} \cap \mathrm{V}_1)^\perp \cap \mathrm{V}_1. \tag{58}$$

It is an elementary fact that $(\mathrm{U} \cap \mathrm{V}_1)^\perp = \mathrm{U}^\perp + \mathrm{V}_1^\perp$ holds, see for example (Hausmann et al. 2023, Lemma B.3). Therefore, we can rewrite the right-hand side of Eq. (57) as

$$\left(\mathrm{U}^\perp + \mathrm{V}_1^\perp\right) \cap \mathrm{V}_1, \tag{59}$$

which can be further simplified as follows

$$\left(\mathrm{U}^\perp + \mathrm{V}_1^\perp\right) \cap \mathrm{V}_1 = \left(V_1\left(\mathrm{U}^\perp\right) \oplus \mathrm{V}_1^\perp\right) \cap \mathrm{V}_1 \tag{60}$$

$$= V_1\left(\mathrm{U}^\perp\right), \tag{61}$$

because $V_1\left(\mathrm{U}^\perp\right)$ is a subspace of V_1 and $\mathrm{V}_1^\perp$ is orthogonal to V_1. Thus, we get the desired equality. □

B.3 General physical transformations

Section 2.1 introduces the reversible transformations of nomic toy theory (which are identical to those of Spekkens' toy theory). A generic physical transformation may also involve discarding of subsytems, and as a result become irreversible.

Definition B.7 A **physical transformation** between two toy systems given by symplectic vector spaces V and W respectively is an affine symplectic map $\mathrm{V} \to \mathrm{W}$.

Proposition B.8 *An affine map* $f : \mathrm{V} \to \mathrm{W}$ *is a physical transformation if and only if there is a decomposition* $\mathrm{V} \cong \mathrm{W} \oplus \mathrm{W}^{\perp}$, *a reversible transformation* $\tilde{f} \in \mathrm{Sp}(\mathrm{V})$, *and an* $w \in \mathrm{W}$ *satisfying*

$$f(v) = W \circ \tilde{f}(v) + w, \tag{62}$$

where $W : \mathrm{V} \to \mathrm{W}$ *is the symplectic, orthogonal projection of* $\mathrm{W} \oplus \mathrm{W}^{\perp}$ *onto* W.

Proof The "if" direction is immediate. For the "only if" part, note that since the symplectic form is non-degenerate and the symplectic part of *f* preserves it, the image of *f* must coincide with W. Thus, by the first isomorphism theorem, we have

$$\mathrm{W} \cong {}^{\mathrm{V}}\!/_{\ker(f)}, \tag{63}$$

which implies $\mathrm{V} = \mathrm{W} \oplus \mathrm{W}^{\perp}$ as symplectic vector spaces, since $\ker(f) = \mathrm{W}^{\perp}$ is necessarily a symplectic subspace.

Now we can let $\tilde{f} := f|_{\mathrm{W}} \oplus \mathbb{1}_{\mathrm{W}^{\perp}}$, which satisfies Eq. (62). □

One of the consequences of Proposition B.8 is that the dimension of V cannot be smaller than the dimension of W. Another is that every physical transformation has a reversible dilation given by $v \mapsto \tilde{f}(v) + w$.

B.4 Measurable variables in nomic toy theory are copyable

Definition B.9 A variable Z is an **information variable** if there is a reversible transformation $f : \mathrm{V} \oplus \mathrm{V} \to \mathrm{V} \oplus \mathrm{V}$ and an epistemic state (U, a) of V, satisfying

$$Z(v) \oplus Z(v) = (Z \oplus Z) \circ f(v + x) \tag{64}$$

for every ontic state $v \in \mathrm{V}$ and every ontic state *x* in the support of (U, a).
In other words, information variables carry information that can be copied.

The following result says that a variable is copyable if and only if it is a collection of functionals whose Poisson brackets vanish. Together with Theorem 3.1, it entails that a variable in nomic toy theory is measurable if and only if it is an information variable.

Proposition B.10 *A variable is an information variable if and only if it is a Poisson variable.*

Proof First of all, let us show that if $Z : \mathrm{V} \to \mathrm{Z}$ is a Poisson variable, then it is an information variable, i.e. that it is copyable.

To this end, we denote the vector space $\left(\ker(Z)\right)^{\perp} \cong \mathrm{S}/\ker(Z)$ by D and the isomorphism arising from the first isomorphism theorem by $K : \mathrm{D} \to \mathrm{im}(Z)$. Since Z is a Poisson variable, D must be an isotropic subspace of V.

The copying of Z is then achieved by the transformation $M : \mathrm{V} \oplus \mathrm{V} \to \mathrm{V} \oplus \mathrm{V}$ introduced in Eq. (18) and using $K^{-1}Z : \mathrm{V} \to \mathrm{D}$ instead of its M_{QP} component. It is symplectic for the same reason as in the proof of Theorem 3.1, i.e. by virtue of Z being a Poisson variable. Applying this M to an arbitrary ready state input, written

$$\begin{pmatrix} v \\ 0 \\ x \end{pmatrix} \tag{65}$$

in the $\mathrm{V} \oplus \mathrm{D} \oplus \ker(Z)$ decomposition, gives

$$\begin{pmatrix} s + \Omega_{\mathrm{V}} Z^T \left(K^{-1}\right)^T x \\ s_{\mathrm{D}} \\ x \end{pmatrix}, \tag{66}$$

Here, s_{D} denotes $K^{-1}Zs$, which is the orthogonal projection of s onto D. Applying the variable $Z \oplus Z$ to the output state yields

$$Z\left(s + \Omega_{\mathrm{V}} Z^T \left(K^{-1}\right)^T x\right) = Zs \tag{67}$$

in the first instance of V and

$$Z(s_{\mathrm{D}} + x) = Zs \tag{68}$$

in the second instance. Equation (67) follows because of the property $Z\Omega_{\mathrm{V}}Z^T = 0$ satisfied by every Poisson variable. Equation (68) is a consequence of $Zx = 0$ (since x is in the kernel of Z) and $Zs_{\mathrm{D}} = Zs$ (since s_{D} is the orthogonal projection of s onto D). As a result, we have shown that Z is an information variable.

Let us now prove the converse. Namely, we assume that Z is an information variable. Then the copying transformation, composed with applying Z to the second instance of V, is a transformation that measures Z. Therefore, by Theorem 3.1, Z is a Poisson variable. □

B.5 Pointer-preserving measurements

For any value a of the position Q of a toy subject A, we can associate an epistemic state (Q, a) of Spekkens' toy theory, called a **pointer state** of A. The name comes from the fact that we think of the manifest variable of the toy subject also as a pointer of a measurement apparatus.

A specific class of measurements in nomic toy theory are those that preserve pointer states. As we show in Remark B.13 below, they have the special property that the contingent manifest variable is trivial. In particular, for a pointer-preserving measurement, the post-measurement manifest variable of A is independent of its initial momentum, as a result of the property $M_{\mathrm{QP}} = 0$. For this reason, they are transformations that measure the variable M_{QS} (see Definition 2.9). The characterisation of pointer-preserving measurements from Remark B.13 says that they satisfy an additional property, namely that M_{PP} is non-degenerate. This suggests that there may be other transformations besides pointer-preserving measurements that also measure the variable M_{QS}.

Definition B.11 A **pointer-preserving measurement** consists of an affine symplectic transformation $m : \mathrm{S} \oplus \mathrm{A} \to \mathrm{S} \oplus \mathrm{A}$ and a Lagrangian subspace Q of A, such that for every $s \in \mathrm{S}$, the associated map $\mathrm{A} \to \mathrm{A}$ given by the composite (here, $A : \mathrm{S} \oplus \mathrm{A} \to \mathrm{A}$ denotes the projection map)

$$\begin{array}{c} \mathrm{A} \xrightarrow{\;s \oplus \mathbb{1}\;} \mathrm{S} \oplus \mathrm{A} \xrightarrow{\;\;m\;\;} \mathrm{S} \oplus \mathrm{A} \xrightarrow{\;\;A\;\;} \mathrm{A} \\ a \longmapsto s + a \longmapsto m(s + a) \longmapsto A \circ m(s + a) \end{array} \tag{69}$$

maps (the support of) each pointer state (Q, a) to that of another pointer state (Q, a').

Proposition B.12 (Characterisation of pointer-preserving measurements) *Let M be the matrix representation of the linear part of the transformationm above. Then the following are equivalent*:

(i) *m and Q make up a pointer-preserving measurement.*
(ii) *Given $q \in \mathrm{A}$, we have $M^T q \in \mathrm{S} \oplus \mathrm{Q}$ if and only if $q \in \mathrm{Q}$.*

Proof Let us analyse how a pointer state (Q, a) is transformed by the respective maps in the composite (69). After the first step, adjoining the ontic state s to the possibilistic state $\mathrm{P} + a$ (note that we have $\mathrm{P} = \mathrm{Q}^T$ by definition of the momentum variable) leads to the possibilistic state[12]

$$\big(\mathrm{S} + \mathrm{Q}, s + a\big) \;\text{ with support }\; \big(\mathrm{P} \cap \mathrm{A}\big) + a + s. \tag{70}$$

By Proposition B.2 (extended from epistemic states to all possibilistic state that are affine subspaces), the possibilistic state after the measurement interaction *m* is

$$\Big(\big(M^T\big)^{-1}(\mathrm{S} + \mathrm{Q}),\, m(s + a)\Big), \tag{71}$$

[12] Note that here we use the same notation for possibilistic states as for epistemic states. That is, for any subspace U (which need not be isotropic), the possibilistic state associated to (U, a) is $\mathrm{U}^T + a$. This notation works for any possibilistic state that is an affine subspace.

which becomes

$$\left(\left[\left(M^T\right)^{-1}(\mathrm{S}+\mathrm{Q})\right] \cap \mathrm{A},\ A \circ m(s+a)\right) \tag{72}$$

after the marginalization to A via the projection $A : \mathrm{S} \oplus \mathrm{A} \to \mathrm{A}$. Thus, the condition that this is another pointer state amounts to

$$\mathrm{Q} = \left[\left(M^T\right)^{-1}(\mathrm{S}+\mathrm{Q})\right] \cap \mathrm{A}. \tag{73}$$

The inclusion of Q within the right hand side is equivalent to

$$q \in \mathrm{Q} \quad \Longrightarrow \quad M^T q \in \mathrm{S} \oplus \mathrm{Q}. \tag{74}$$

The reverse inclusion, on the other hand, is equivalent to

$$q \in \mathrm{A} \text{ and } M^T q \in \mathrm{S} \oplus \mathrm{Q} \quad \Longrightarrow \quad q \in \mathrm{Q}, \tag{75}$$

so that the result follows. □

Remark B.13 Note that if we write M in block form as in Eq. (12), then condition (ii) says that

- $M_{\mathrm{QP}} : \mathrm{P} \to \mathrm{Q}$ is equal to 0, and
- $M_{\mathrm{PP}} : \mathrm{P} \to \mathrm{P}$ is non-degenerate.

Proof First, consider an arbitrary $q \in \mathrm{Q}$, so that in this block form we have

$$M^T q = M^T \begin{pmatrix} 0 \\ q \\ 0 \end{pmatrix} = \begin{pmatrix} M^T_{\mathrm{QS}} q \\ M^T_{\mathrm{QQ}} q \\ M_{\mathrm{QP}} q \end{pmatrix}. \tag{76}$$

Thus $M^T q$ is an element of $\mathrm{S} \oplus \mathrm{Q}$ if and only if M_{QP} vanishes.

On the other hand, consider an arbitrary $a \in \mathrm{A}$ with components a_{Q} and a_{P} in Q and P respectively. Then, using $M_{\mathrm{QP}} = 0$, we have

$$M^T a = M^T \begin{pmatrix} 0 \\ a_{\mathrm{Q}} \\ a_{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} M^T_{\mathrm{QS}}\, a_{\mathrm{Q}} + M^T_{\mathrm{PS}}\, a_{\mathrm{P}} \\ M^T_{\mathrm{QQ}}\, a_{\mathrm{Q}} + M^T_{\mathrm{PQ}}\, a_{\mathrm{P}} \\ M^T_{\mathrm{PP}}\, a_{\mathrm{P}} \end{pmatrix}, \tag{77}$$

so that the implication

$$M^T a \in \mathrm{S} \oplus \mathrm{Q} \quad \Longrightarrow \quad a \in \mathrm{Q} \tag{78}$$

becomes

$$M_{\mathrm{PP}}^{T}\, a_{\mathrm{P}} = 0 \quad \Longrightarrow \quad a_{\mathrm{P}} = 0. \tag{79}$$

This implication is satisfied if and only if M_{PP}^{T} is non-degenerate, which is equivalent to M_{PP} itself being non-degenerate. □

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Declarations

Conflict of interest Not applicable.

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