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Topic Collection
Gravitational Waves as a New Probe for Astronomy and Fundamental Physics

Edited by
Dr. Houri Ziaeepour

MDPI

https://doi.org/10.3390/universe10110426 *universe*

Article

Probing the Dark Universe with Gravitational Waves

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Abstract: Gravitational waves (GWs) are expected to interact with dark energy and dark matter, affecting their propagation on cosmological scales. To model this interaction, we derive a gauge-invariant effective equation and action valid for all GW polarizations. This is achieved by encoding the effects of GW interactions at different orders of perturbation into a polarization-, frequency-, and time-dependent effective speed. The invariance of perturbations under time-dependent conformal transformations and the gauge invariance of GWs allow us to derive the unitary gauge effective action in any conformally related frame, thereby clarifying the relationship between the Einstein and Jordan frames. Tests of the polarization and frequency dependencies in the propagation time and luminosity distance of different GW polarizations allow us to probe the dark Universe, which acts as an effective medium, modeled by the GW effective speed.

Keywords: gravitational waves; effective speed; gravitational luminosity distance

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Citation: Romano, A.E. Probing the Dark Universe with Gravitational Waves. *Universe* **2024**, *10*, 426. https://doi.org/10.3390/universe10110426

Academic Editor: Houri Ziaeepour

Received: 24 October 2024
Revised: 12 November 2024
Accepted: 14 November 2024
Published: 15 November 2024

1. Introduction

The direct observation of gravitational waves (GWs) by the Laser Interferometer Gravitational-Wave Observatory (LIGO) and Virgo has initiated the era of GW astronomy [1]. These observations are consistent with general relativity (GR) predictions, but even in GR, tensor modes are expected to acquire a frequency- and polarization-dependent effective speed [2] due to interactions with other fields—an effect that can be tested via multimessenger observations across different bands and by measuring the luminosity distance of different polarizations modes.

To study the effects of GW interactions, we generalize the effective speed approach [2–4]—originally derived for tensor modes—to include all possible GW polarizations, including gauge-invariant scalar and vector modes. We derive a general effective propagation equation and action, which can be applied to model-independent analyses of the effects of interactions between different GW polarizations, or between GWs and other fields [5], such as dark matter or dark energy. In this effective approach, other fields and GW polarizations act as an effective medium for each GW polarization, which consequently propagates with a frequency- and time-dependent effective speed. The effective speed encodes interaction effects that are conceptually analogous to those described by the modified dispersion relation for electromagnetic waves (EMW) propagating through a plasma [6,7]. This effective approach is particularly suitable for model-independent analysis of observational data, predicting that the GW speed c_{ij} and the gravitational luminosity distance may depend on the frequency and polarization of GWs.

2. Effective Speed of Arbitrary GW Polarization

GWs h_{ij} propagating in the z-direction [8], can be decomposed into different polarizations as

$$h_{ij} = \begin{pmatrix} h_S + h_+ & h_\times & h_{V1} \\ h_\times & h_S - h_+ & h_{V2} \\ h_{V1} & h_{V2} & h_L \end{pmatrix}, \quad (1)$$

where h_+ and $h_\times$ are the tensor modes, $h_{V1,2}$ the vector polarizations, and h_S and h_L the transverse and longitudinal scalar modes. Note that the gauge has not been fully fixed in the above equation, and this can lead to GW modes which depend on the observer frame [9]. We will resolve this ambiguity in the next section, in which we will obtain the effective action for gauge-invariant GWs.

In order to show the general applicability of the effective speed approach [2], let us consider for example this ansatz for the modified GW propagation equation [10]

$$h''_{ij} + (2+\nu)\mathcal{H}h'_{ij} + (c_g^2 k^2 + m^2 a^2)h_{ij} = \Pi_{ij}(h_{ij}, h_{pq}, \phi_m)\,, \tag{2}$$

$$L_m(\phi_m, h_{ij}) = 0\,, \tag{3}$$

where ν accounts for a modification of the friction term, c_g for an modified propagation speed, m is an effective mass, Π_{ij} is the source term associated with the self-interaction, interaction with other fields or other GWs polarizations h_{pq}, ϕ_m denotes abstractly other fields interacting with GWs, and L_m is the differential operator corresponding to the equation of motion (EOM) of each ϕ_m. We can rewrite Equation (2) as

$$h''_{ij} + 2\mathcal{H}h'_{ij} + k^2 h_{ij} = \Pi_{ij} - \nu\mathcal{H}h'_{ij} - \Big[(c_g^2-1)k^2 + m^2a^2\Big]h_{ij} = a^2\Pi_{ij}^{eff}\,, \tag{4}$$

where we have defined the effective anisotropic stress tensor Π_{ij}^{eff} as an effective source term on the right-hand side of the GW propagation equation in the theory of general relativity, and we have used units in which $8\pi G = c = 1$. The above equation shows that any solution of the modified GW propagation equation can be obtained as the solution of the GR equation with an appropriately defined source term.

Let us denote with a hat a solution $\hat{h}_{ij}$ of Equation (4), and with $\hat{\Pi}_{ij}^{eff}$ the quantity obtained from substituting in Π_{ij}^{eff} the solutions of the equations of motions, including $\hat{h}_{ij}$, implying

$$\hat{h}''_{ij} + 2\mathcal{H}\hat{h}'_{ij} + k^2\hat{h}_{ij} - a^2\hat{\Pi}_{ij}^{eff} = 0\,. \tag{5}$$

The above equation can be manipulated as following

$$\hat{h}''_{ij} + 2\mathcal{H}\hat{h}''_{ij} - a^2\hat{\Pi}_{ij}^{eff} + k^2\hat{h}_{ij} = \frac{(\hat{h}'_{ij}a^2)'}{a^2} - \frac{\left(\int^{\eta} a^4(\eta')\hat{\Pi}_{ij}(\eta')^{eff}d\eta'\right)'}{a^2} + k^2\hat{h}_{ij} =$$
$$\frac{1}{a^2}\left[a^2\hat{h}'_{ij}\left(1-\frac{\hat{g}_{ij}}{\hat{h}'_{ij}}\right)\right]' + k^2\hat{h}_{ij} = \frac{1}{a^2}\left(\frac{a^2\hat{h}'_{ij}}{c_{ij}^2}\right)' + k^2\hat{h}_{ij} \tag{6}$$

where we have defined [2] the effective polarization, momentum and time dependent speed c_{ij} as

$$c_{ij}^2(\eta,k) = \left(1-\frac{\hat{g}_{ij}}{\hat{h}'_{ij}}\right)^{-1} \quad , \quad \hat{g}_{ij} = \frac{1}{a^2}\int^{\eta} a^4(\eta')\hat{\Pi}_{ij}^{eff}(\eta')\,d\eta'\,, \tag{7}$$

and η' denotes a dummy variable of integration. In the above manipulation we used the first fundamental theorem of calculus to write the source term as the derivative of its indefinite integral

$$a^4(\eta)\hat{\Pi}_{ij}^{eff}(\eta) = \left(\int^{\eta} a^4(\eta')\hat{\Pi}_{ij}^{eff}(\eta')d\eta'\right)'\,, \tag{8}$$

and this implies that the lower limit of the integral in the definition of $\hat{g}_{ij}$ is not physically relevant, i.e., this is not a non-local quantity involving the integration over an infinite time range, but it only involves the indefinite integral of the source term.

Equation (6) can finally be written as

$$\hat{h}''_{ij} + 2\left(\frac{a'}{a} - \frac{c'_{ij}}{c_{ij}}\right)\hat{h}'_{ij} + k^2c_{ij}^2\hat{h}_{ij} = 0\,, \tag{9}$$

showing that c_{ij}^2 is indeed the quantity playing the role of effective speed. In the above equations a hat denotes quantities obtained by substituting the solutions $\hat{h}_{ij}, \hat{\phi}_m$ of the system of coupled differential equations given in Equations (2) and (3), i.e., $\hat{\Pi}_{ij}^{eff}$ is just a function of space-time coordinates after the substitution, which accounts for the integrated modified propagation effects. Equation (7) shows that each polarization mode h_{ij} can have a different frequency and time dependent effective speed, depending on the effective anisotropic stress tensor $\hat{\Pi}_{ij}^{eff}$, evaluated along the propagation path.

For a set of initial conditions consistent with those used to obtain $\hat{h}_{ij}$, Equation (9) gives by construction the same solution $\hat{h}_{ij}$. Solutions of Equation (9) corresponding to different initial conditions are not of physical relevance, such as for example $h_{ij} = 0$. Any solution of Equation (2), or any other ansatz which can be manipulated to put it in the form given in Equation (4), can always be obtained by solving the effective Equation (9), with the effective speed defined in Equation (7). This effective approach is convenient to relate the modified propagation effects to observations, because the GW-EMW luminosity distance ratio is related to the effective speed ratio [2], making the effective equation useful for model independent observational data analysis.

3. Einstein Frame Definition

In the previous section, we did not explictly specify the frame in which the ansatz (2) is given, since our goal was to give an example of the general applicability of the effective speed approach. In order to derive a general effective action, without assuming any ad hoc ansatz, it is important to clarify the relation between the Einstein and Jordan frames. We defined the Jordan frame Lagrangian of a modified gravity theory with Jordan frame matter-coupling (JMC) as

$$\mathcal{L}_{\rm JMC} = \sqrt{g_{\rm J}}\Big[\Omega^2 R_{\rm J} + L_{\rm J}^{\rm MG} + L_{\rm J}^{\rm matter}(g_{\rm J})\Big] \,, \tag{10}$$

where $L^{\rm MG}$ and $L^{\rm matter}$ are, respectively, the modified gravity and matter Lagrangians. After performing a conformal transformation $g_{\rm E} = \Omega^2 g_{\rm J}$, in the Einstein frame we have

$$\mathcal{L}_{\rm JMC} = \sqrt{g_{\rm E}}\Big[R_{\rm E} + L_{\rm E}^{\rm MG} + L_{\rm E}^{\rm matter}(\Omega^{-2} g_{\rm E})\Big] \,. \tag{11}$$

Note that while the tensor modes speed c_T is invariant under conformal transformations, it is not invariant under disformal transformations, allowing to define a combination of disformal, conformal and coordinate transformations taking to a frame [11] in which $c_T = 1$. Note nevertheless that the ratio c_T/c between the speed of gravitational and electromagnetic waves is disformal invariant, implying that, if c_T/c is time dependent, in the $c_T = 1$ frame the speed of light is time dependent. For this reason, while the $c_T = 1$ frame is useful to study the resilience of inflationary predictions [11], we define the Einstein frame as that in which the coefficient of the Ricci scalar and the speed of light are constant, and not the $c_T = 1$ frame, in agreement with the definition adopted in [12] for example.

4. Gauge-Invariant Gravitational Waves

For the purpose of understanding the role played by gauge transformation it is convenient to interpret the GW polarization tensor defined in Equation (1) in terms of cosmological perturbations theory [13–15]. Since tensor perturbations are gauge invariant at first order, we will just consider scalar and vector perturbations. Using the scalar-vector-tensor (SVT) decomposition [13], the scalar and vector perturbations of the flat FRW line element can be written as

$$ds^2 = (1+2\Phi)dt^2 - 2a(t)\Big(B_{,i} + S_i\Big)dx^i dt - a^2(t)\Big[(1-2\Psi)\delta_{ij} + 2(E_{,ij} + F_{(i,j)})\Big]dx^i dx^j \,, \tag{12}$$

where S_i and F_i are the vector perturbations, satisfying $S_{i,i} = F_{i,i} = 0$, and Φ, B, Ψ, E are the scalar perturbations. Applying the SVT decomposition to an infinitesimal time and space translation, we obtain

$$\begin{aligned} t &\to t + T\,, && (13)\\ x^i &\to x^i + \beta^i + \delta^{ij}\beta_{,j}\,. && (14) \end{aligned}$$

where β_i and β are, respectively, the vector and scalar part of the infinitesimal space translation, satisfying $\beta_{i,i} = 0$, and T is the infinitesimal time translation.

Under the above infinitesimal coordinate transformations the metric tensor transformations implies the gauge transformations for the perturbations [15]

$$\begin{aligned} S_i &\to S_i + a\dot{\beta}_i\,, && (15)\\ F_i &\to F_i - \beta_i\,, && (16)\\ \Phi &\to \Phi - \dot{T}\,, && (17)\\ B &\to B + a^{-1}T - a\dot{\beta}\,, && (18)\\ E &\to E - \beta\,, && (19)\\ \Psi &\to \Psi + HT\,. && (20) \end{aligned}$$

while scalar fields transform as

$$\delta\phi \to \delta\phi - \dot{\phi}\,T\,. \qquad (21)$$

Note that the above gauge transformations have a purely geometrical origin, i.e., they are independent of the specific theory, since they are simply a direct consequence of the fact that the metric and the energy-momentum tensor transform as tensors under a coordinate transformation. This allows us to obtain a gauge-invariant definition of GW polarizations valid in any gravity theory, since the SVT decomposition of a tensor is also purely geometrical, and is not based on assuming any symmetry of the action defining the theory. In other words, gauge transformations are just a manifestation of general covariance, which any theory should satisfy, and correspond to the Lagrangian not containing any free tonsorial index, i.e., being a coordinate invariant. By comparing Equation (1) with the spatial part of the perturbed metric, we obtain:

$$\begin{aligned} h_S &= 1 - 2\Psi \quad , \quad h_L = 1 - 2\Psi + E_{,zz}\,, && (22)\\ h_{V1} &= -F_{x,z} \quad , \quad h_{V2} = -F_{y,z}\,, && (23) \end{aligned}$$

from which we obtain the gauge transformations of the GW polarizations

$$\begin{aligned} h_S &\to h_S - 2HT\,, && (24)\\ h_L &\to h_L - 2HT - \beta_{,zz}\,, && (25)\\ h_{V1} &\to h_{V1} + \beta_{x,z}\,, && (26)\\ h_{V2} &\to h_{V2} + \beta_{y,z}\,. && (27) \end{aligned}$$

We can now use the above gauge transformations to define gauge-invariant GW polarizations. It is convenient to fix the Einstein gauge (EG), defined by the condition

$$\nabla^\mu\left(h_{\mu\nu} - \frac{1}{2}hg^0_{\mu\nu}\right) = 0 \qquad (28)$$

where $g^0_{\mu\nu}$ denotes the background metric, $h = h^{ab}g^0_{ab}$ is the trace of h_{ab}, and the covariant derivative is with respect to the background metric. The above gauge fixing condition gives a set of four differential equations which can be solved to obtain the infinitesimal translations necessary to switch to the EG. Denoting the solutions of the gauge fixing equations as $\{T^E, \beta^E, \beta_i^E\}$, we can define the gauge invariant GWs polarizations $\bar{h}_{ij}$

$$\bar{h}_S = h_S - 2HT^E\,, \tag{29}$$

$$\bar{h}_L = h_L - 2HT^E - \beta^E_{,zz}\,, \tag{30}$$

$$\bar{h}_{V1} = h_{V1} + \beta^E_{x,z}\,, \tag{31}$$

$$\bar{h}_{V2} = h_{V2} + \beta^E_{y,z}\,, \tag{32}$$

which are gauge invariant by construction, and we will call EGWs. We are denoting with a bar the gauge invariant quantities EGWs defined above, while tensor modes are gauge invariant a first order in perturbations, i.e., $h_\times = \bar{h}_\times, h_+ = \bar{h}_+$. Some residual gauge freedom is present even after fixing the EG [16], but we will not use it, since our main purpose is to define gauge invariant variables satisfying a wave equation, and imposing the EG is enough to achieve this, as we will show in the next section.

5. Gauge-Invariant Gravitational Waves Equation

The field equations of a generic JMC modified gravity theory defined by the action (10) can be written in the Einstein frame as

$$G_{\mathrm{E},\mu\nu} = T^{tot}_{\mathrm{E},\mu\nu}\,, \tag{33}$$

where E denotes the Einstein frame, $T^{tot}_{\mathrm{E},\mu\nu}$ is the sum of terms associated with the matter fields, the modified gravity fields, and their interaction with matter due to the non minimal coupling of matter with the Einstein frame metric. The linearized perturbed Einstein equations with respect to a curved background [8] in the Einstein gauge read

$$\Box\psi_{\mu\nu} + 2R^{\mathrm{B}}_{\mu\alpha\nu\beta}\psi^{\alpha\beta} = 2\delta T_{\mu\nu} + 2R^{\mathrm{B}}{}_{(\mu}{}^{\alpha}\psi_{\nu)\alpha} - R^{\mathrm{B}}\psi_{\mu\nu} + g^0_{\mu\nu}R^{\alpha\beta}_{\mathrm{B}}\psi_{\alpha\beta}\,, \tag{34}$$

where $\psi_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}g^0_{\mu\nu}$. The general form of the equation without fixing the Einstein gauge can be found for example in [17,18]. From the above equation we obtain

$$\Box\bar{h}_{\mu\nu} = \delta T^{eff}_{\mu\nu}\,, \tag{35}$$

where the d'Alambert operator is defined with respect to the background metric and $\delta T^{eff}_{\mu\nu}$ is an effective energy-stress (ES) tensor given by the sum of the matter ES tensor, and other terms involving GWs, the Ricci tensor, and possible additional fields related to the gravity modification. Equation (35) is quite general, since it is valid for any theory which admits an Einstein frame formulation, for a general background metric, and as mentioned earlier, the definitions of the gauge invariant variables is even more general, since it does not assume any form of the action, and the EG can be fixed even for a background different from the FRW solution.

Note that the transverse traceless (TT) gauge cannot be imposed in a generic curved space [16], and the scalar and vector modes cannot be gauged away in a generic modified gravity theory, since this is only possible in vacuo, but the extra fields associated with the gravity modification act as an effective source in the r.h.s. of Equation (35), which in general is not zero even in absence of matter fields. In GR, assuming a flat background, and in vacuo, the effective source term is zero, because the Ricci and Riemann tensors are zero. This allows use the residual gauge freedom to set $\bar{h} = 0$, and fix the TT gauge, in which only the two tensor modes survive.

Considering a FRW background, following a method similar to the one used in the previous section, an effective equation for the gauge invariant GWs can be derived from Equation (35)

$$\bar{h}''_{ij} + 2\left(\frac{a'}{a} - \frac{c'_{ij}}{c_{ij}}\right)\bar{h}'_{ij} + k^2c^2_{ij}\bar{h}_{ij} = 0\,, \tag{36}$$

where

$$c_{ij}^2(\eta,k) = \left(1 - \frac{\hat{g}_{ij}}{\hat{\bar{h}}'_{ij}}\right)^{-1} \quad , \quad \hat{g}_{ij} = \frac{1}{a^2}\int a^4 \delta T_{ij}^{eff}\, d\eta \,, \tag{37}$$

Note that, contrary to the previous section derivation, which was based on assuming a non gauge invariant ansatz for the GWs propagation equation, the effective equation above is gauge invariant by construction and is quite general, since it can be obtained for any theory which admits an Einstein frame formulation (Appendix A).

6. Generalization to Higher Order in Perturbations

Note that the gauge invariant quantities EGWs and Equation (35) are defined at linear order in perturbations, and at higher order new gauge invariant variables $\bar{h}^{(i)}_{\mu\nu}$ can be defined [19–21]. The expansion of the Einstein Equation (33) will give new equations which can always be put in the canonical form

$$\Box \bar{h}^{(i)}_{\mu\nu} = \delta T^{(i)eff}_{\mu\nu} \,, \tag{38}$$

by appropriately defining the effective perturbed ES tensor $\delta T^{(i)eff}_{\mu\nu}$, even if the d'Alambert operator does not appear explicitly, by adding it on both sides of the expanded equations. We have fixed the gauge by imposing the condition

$$\nabla^\mu \left(\bar{h}^{(i)}_{\mu\nu} - \frac{1}{2}\bar{h}^{(i)} g^0_{\mu\nu}\right) = 0 \,, \tag{39}$$

where $g^0_{\mu\nu}$ denotes the background metric, $\bar{h}^{(i)} = \bar{h}^{(i)}_{\mu\nu} g^{0,\mu\nu}$ is the trace of $\bar{h}^{(i)}_{\mu\nu}$, and the covariant derivative represents the background metric. This effective equation allows to interpret GWs at any order in perturbations as the solutions of a wave equation with an appropriately defined source term. Since the d'Alambert operator is defined with respect to the same background metric in all these equations, the equations can be summed to give

$$\Box \bar{h}^{(N)}_{\mu\nu} = \delta T^{(N)eff}_{\mu\nu} \,. \tag{40}$$

where we have defined the summed GWs and ES tensor perturbations

$$\bar{h}^{(N)}_{\mu\nu} = \sum_{i=1}^{N} \bar{h}^{(i)}_{\mu\nu} \quad , \quad T^{(N)eff}_{\mu\nu} = \sum_{i=1}^{N} \delta T^{(i)eff}_{\mu\nu} \,. \tag{41}$$

Note that $\bar{h}^{(N)}_{\mu\nu}$ are the physically observable GWs, given by the sum of the contributions from different orders in perturbations. From Equation (40), similarly to what shown in the previous section for linear perturbations, it is possible to define an effective speed, equation, and action for the summed GWs $\bar{h}^{(N)}_{\mu\nu}$. Note that this effective speed is not simply the sum of the effective speeds corresponding to perturbations equations at different orders, due to the coupling between perturbations at different orders. After solving the system of coupled differential equations for the perturbations equations at all relevant orders, the solutions $\hat{\bar{h}}^{(i)}_{\mu\nu}$ can be substituted to obtain $\hat{T}^{(N)eff}_{\mu\nu}$, from which the effective speed and equation can be obtained

$$\bar{h}^{(N)''}_{ij} + 2\left(\frac{a'}{a} - \frac{c'_{ij}}{c_{ij}}\right)\bar{h}^{(N)'}_{ij} + k^2 c^2_{ij} \bar{h}^{(N)}_{ij} = 0 \,, \tag{42}$$

$$c_{ij}^2(\eta,k) = \left[1 - \frac{\hat{g}_{ij}}{\hat{\bar{h}}^{(N)'}_{ij}}\right]^{-1} \quad , \quad \hat{g}_{ij} = \frac{1}{a^2}\int a^4 \delta \hat{T}^{(N)eff}_{ij}\, d\eta \,. \tag{43}$$

The effective speed is encoding in a single quantity all the interaction effects between different GWs polarizations, and between GWs and other fields, up to order N in perturbations.

7. Effective Lagrangian and Metric

The Lagrangian corresponding to Equation (42) is

$$\mathcal{L}_h^{eff} = \frac{a^2}{c_{ij}^2}\left[\left(\bar{h}_{ij}^{(N)'}\right)^2 + k^2 c_{ij}^2 \left(\bar{h}_{ij}^{(N)}\right)^2\right], \tag{44}$$

generalizing to scalar and vector modes the tensor perturbations effective action [2]. The effective method can also be applied in physical space, and it can be shown that [2] the effective Lagrangian can be obtained from the GR Lagrangian density

$$\mathcal{L}_h^{GR} = a^2\left[h_{ij}'^2 - c^2(\nabla h_{ij})^2\right] = \sqrt{-g}(\partial_\mu h_{ij}\partial^\mu h_{ij})\,, \tag{45}$$

via the transformation

$$a \to \alpha_{ij} = \frac{a}{\hat{c}_{ij}} \quad , \quad c \to \hat{c}_{ij} \quad , \quad h_{ij} \to \bar{h}_{ij}^{(N)}\,, \tag{46}$$

where we have denoted with c the speed of light, to avoid ambiguity, and the space effective sound speed $\hat{c}_{ij}(\eta, x^i)$ [2], is defined in terms of the physical space effective ES tensor, not of its Fourier transform. Note that $c_{ij}(\eta, k)$ is not the Fourier transform of $\hat{c}_{ij}(\eta, x^i)$ [2]. The physical space effective action is

$$\mathcal{L}_h^{eff} = \sqrt{-g^{\text{eff}}}(\partial_\mu h_{ij}\partial^\mu h_{ij})\,, \tag{47}$$

where the effective metric is

$$ds_{eff}^2 = g_{\mu\nu}^{\text{eff}} dx^\mu dx^\nu = a^2\left[\hat{c}_{ij} d\eta^2 - \frac{\delta_{mm}}{\hat{c}_{ij}} dx^m dx^m\right], \tag{48}$$

from which the EOM are given in terms of the d'Alembert operator

$$\Box h_{ij} = \frac{1}{\sqrt{-g^{\text{eff}}}}\partial_\mu\left(\sqrt{-g^{\text{eff}}}\partial^\mu h_{ij}\right) = 0\,. \tag{49}$$

In this effective geometrical description, the effects of the interactions of GWs are encoded in the effective metric, and in the eikonal approximation, the solutions of the EOM are geodesics in the effective curved space corresponding to the effective metric.

8. Jordan Frame Effective Action

Cosmological perturbations with respect to a flat FRW background are invariant under time dependent conformal transformations, since they correspond to a scale factor redefinition $a = \Omega\,\tilde{a}$, which has no effects on the perturbations

$$ds^2 = a^2\left[(\eta_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu\right] = \Omega^2(\eta)\tilde{a}^2\left[(\eta_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu\right]. \tag{50}$$

The gauge invariant GWs do not depend on the coordinate choice by definition, and can in particular be evaluated in the unitary gauge, defined by $\delta\phi = 0$, in which the conformal factor Ω becomes a function of time only. Using the conformal and gauge invariance of the GWs $\bar{h}_{ij}$, we can obtain the Jordan frame effective action in the unitary gauge

$$\mathcal{L}_{\bar{h}}^{eff} = \frac{\Omega^2(\eta)\tilde{a}^2}{\hat{c}_{ij}^2}\left[\bar{h}_{ij}'^2 - \hat{c}_{ij}^2(\nabla\bar{h}_{ij})^2\right] = \hat{M}_{ij}^2\tilde{a}^2\left[\bar{h}_{ij}'^2 - \hat{c}_{ij}^2(\nabla\bar{h}_{ij})^2\right]. \tag{51}$$

where we have defined $\hat{M}_{ij} = \Omega/\hat{c}_{ij}$. Note that the coefficient of the Ricci scalar in the action is playing the role of effective Planck mass, but in the literature $\hat{M}_{ij}$ is sometime called effective Planck mass, which is correct only for luminal modified gravity theories, when $\hat{c}_{ij} = 1$ and $\hat{M} = \Omega$ does not depend on polarization.

Formally the effective approach we derived assuming an Einstein frame formulation, can also be applied to a generic modified gravity theory [22], by appropriately defining and effective ES tensor, allowing to recast the fields equation in an Einstein-like form.

9. Polarization and Frequency Dependency of the Luminosity Distance

After Defining the Effective Scale Factor

$$\alpha_{ij} = \frac{a}{c_{ij}} \, , \tag{52}$$

Equation (42) can be re-written as

$$\bar{h}_{ij}^{(N)''} + 2\frac{\alpha'_{ij}}{\alpha_{ij}}\bar{h}_{ij}^{(N)'} + k^2 c_{ij}^2 \bar{h}_{ij}^{(N)} = 0 \, , \tag{53}$$

from which we obtain

$$\chi''_{ij} + \left(c_{ij}k^2 - \frac{\alpha''_{ij}}{\alpha_{ij}}\right)\chi_{ij} = 0 \, , \tag{54}$$

where we have defined $\bar{h}_{ij}^{(N)} = \chi_{ij}/\alpha_{ij}$. In the sub-horizon limit $\alpha''_{ij}/\alpha_{ij}$ is negligible, and the amplitude of $\bar{h}_{ij}^{(N)}$ is proportional to $1/\alpha_{ij}$, giving [2]

$$\frac{d_{ij}^{GW}}{d_L^{EM}}(z) = \frac{a(z)}{\alpha_{ij}(z)}\frac{\alpha_{ij}(0)}{a(0)} = \frac{c_{ij}(z,k)}{c_{ij}(0,k)} \, , \tag{55}$$

where we have used $d_{ij}^{GW} = r\,\alpha_{ij}(0)/\alpha_{ij}(z)$, $d_L^{EM} = r\,a(0)/a(z)$, assuming $(1+z) = a(0)/a(z)$, i.e., that matter is minimally coupled to the Einstein frame metric.

If matter is minimally coupled to the Jordan frame metric, Equation (53) is still valid, due to the conformal invariance of GWs, but the relation between the scale factor and α is modified

$$\alpha_{ij} = \frac{a}{c_{ij}} = \frac{\Omega\tilde{a}}{c_{ij}} \, , \tag{56}$$

implying that

$$\frac{d_{ij}^{GW}}{d_L^{EM}}(z) = \frac{\tilde{a}(z)}{\alpha_{ij}(z)}\frac{\alpha_{ij}(0)}{\tilde{a}(0)} = \frac{c_{ij}(z,k)}{c_{ij}(0,k)}\frac{\Omega(0)}{\Omega(z)} = \frac{M_{ij}(0,k)}{M_{ij}(z,k)} \, , \tag{57}$$

where we have used $d_{ij}^{GW} = r\,\alpha_{ij}(0)/\alpha_{ij}(z)$, $d_L^{EM} = r\,\tilde{a}(0)/\tilde{a}(z)$, assuming $(1+z) = \tilde{a}(0)/\tilde{a}(z)$, i.e., that matter is minimally coupled to the Jordan frame metric.

Note that independently of the type of matter-gravity coupling, the GWs luminosity distance is predicted to be frequency and polarization dependent.

10. Observational Implications

The frequency and polarization dependency of the effective speed implies that different polarizations of GWs emitted by the same source at different frequencies can spend different times to reach the observer. This effect could be detected by comparing the time delay between the detection of different GWs polarizations with different observatories, operating at different frequencies, and can be observed even in absence of an electromagnetic counterpart.

Another observable effect is the polarization and frequency dependency of the GW luminosity distance. For theories with matter minimally coupled to the Jordan frame metric the GW luminosity distance is related to the M_{ij} ratio

$$d^{GW}_{ij}(z) = \frac{M_{ij}(0,k)}{M_{ij}(z,k)} d^{EM}_L(z)\,, \tag{58}$$

while for theories minimally coupled to the Einstein frame is related to the effective speed ratio

$$d^{GW}_{ij}(z) = \frac{c_{ij}(0,k)}{c_{ij}(z,k)} d^{EM}_L(z)\,. \tag{59}$$

Time delay observations allow to constrain at different frequencies the ratio between the speed of different GWs polarizations

$$r^c_{ijpq}(k,z) = \frac{c_{ij}(k,z)}{c_{pq}(k,z)}\,, \tag{60}$$

while the GWs waveform, which are inversely proportional to the GW luminosity distance, allow us to constrain the corresponding distance ratio

$$r^d_{ijpq}(k,z) = \frac{d^{GW}_{ij}(z)}{d^{GW}_{pq}(z)}. \tag{61}$$

For multimessenger events, additional constraints can be set on $c_{ij}(k,z)/c$ and $d^{GW}_{ij}(z)/d^{EM}_L(z)$.

For GWs propagating in vacuum according to GR, only tensor modes are expected, and there should not be any redshift, frequency or polarization dependency, i.e., $c_\times(k,z) = c_+(k,z) = c$ and $d^{GW}_\times(z) = d^{GW}_+(z) = d^{EM}_L(z)$ at any frequency. If the effective ES tensor defined in Equation (41) is not negligible along the GW propagation, either because of GWs interaction with matter fields, or due to the effects of gravity modification, other polarizations modes could be detected and the corresponding speed and luminosity distance could be constrained observationally using time delay and GWs waveform observations. These observations can be used to probe the dark Universe by its interaction with GWs, modeled by the effective GWs speed.

11. Possible Applications

The effective speed approach is based on the theory of cosmological perturbations and the SVT decomposition, without assuming any specific physical scenario. The general applicability of this method stems from the broad assumptions made to derive the effective speed. Specifically, by appropriately defining the effective anisotropic stress tensor, we should always be able to express the GW propagation equation in the form given by Equation (4). As an example, we can consider the interaction between GWs and free-streaming neutrinos, which is known to induce a damping effect [23] due to the anisotropic stress tensor. At linear order, this can be written in a simple form as the integral of the product of a kernel function and h'_{ij}. For GW–neutrino interactions, second-order perturbations of the anisotropic stress tensor have also been computed [24], which could be used to calculate higher-order contributions to the effective speed using Equation (43). The method can also be applied to modified gravity theories. For example, in scenarios such as those presented in [6,7], a damping effect occurs due to the coupling between different GW polarizations and other fields, extending the results obtained in [2,22] to include other polarizations and higher-order perturbations. Given its generality, the effective speed can also be used for model-independent analysis of observational data, such as searching for possible deviations from GR or, more generally, modeling the effects of interactions between GWs and other fields, or among different GW polarizations. In this context, some general phenomenological ansatz for $c_{ij}(k,\eta)$ can be made, whose parameters can then be constrained through observational data analysis. We leave this for future work.

12. Conclusions

We have derived a gauge-invariant effective equation and action for GWs, encoding the effects of interaction at different orders of perturbations theory into a polarization-, frequency-, and time-dependent effective speed. The invariance of perturbations under time-dependent conformal transformations and the gauge invariance of GWs allows us to obtain the unitary gauge effective action in any conformally related frame. The propagation time and luminosity distance of different GW polarizations, emitted at different frequencies and redshifts by dark or bright sirens, provide an opportunity to probe interactions with other fields, particularly those involving the dark Universe. The interaction of GWs with other fields can induce polarization- and frequency-dependent propagation time delays, as well as differences in the luminosity distance ratio between gravitational and electromagnetic sources. By combining observations from GW detectors operating at different frequencies, it will be possible to test this polarization and frequency dependency. To connect the effective approach with theoretical models, it will be important to compute the effective GW speed for different dark matter and dark energy theories, allowing us to constrain these theories using luminosity distance and time delay observations. Moreover, the effective speed approach could be used for model-independent data analysis by making some general phenomenological ansatz for $c_{ij}(k,\eta)$, whose parameters can then be constrained through data analysis.

Funding: AER was supported by the UDEA projects 2021-44670, 2019-28270, 2023-63330.

Data Availability Statement: No new data were created or anlyzed in this study. Date sharing is not applicable to this article.

Acknowledgments: I thank Antonaldo Diaferio, Francesco Pace, Piero Rettegno, Luca Baiotti, and Flippo Vernizzi for insightful discussions regarding GW polarizations. I am also grateful to YITP and the Osaka University Theoretical Astrophysics group for their kind hospitality during the preparation of this paper.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Application to Generic Theories

The effective approach we derived assuming an Einstein frame formulation, can also be applied to a generic theory by appropriately defining an effective energy-stress tensor [22], according to

$$F[\hat{g}_{\mu\nu}] = T_{\mu\nu}[\hat{g}_{\mu\nu},\hat{\phi}_i] \quad , \quad M_i[\hat{\phi}_i] = 0\,, \tag{A1}$$

$$G_{\mu\nu}[\hat{g}_{\mu\nu}] = \hat{T}^{eff}_{\mu\nu}\,, \tag{A2}$$

$$\hat{T}^{eff}_{\mu\nu}(x^\rho) = T_{\mu\nu}[\hat{g}_{\mu\nu},\hat{\phi}_i] - F[\hat{g}_{\mu\nu}] + G_{\mu\nu}[\hat{g}_{\mu\nu}]\,. \tag{A3}$$

In the above equations, F and M_i correspond to the gravity and matter field equations, $\{\hat{g}_{\mu\nu},\hat{\phi}_i\}$ denote the solutions of Equation (A1), and $\hat{T}^{eff}_{\mu\nu}$ is obtained by substituting the solutions into $T^{eff}_{\mu\nu}$. The solutions of Equation (A2) are by construction also solutions of Equation (A1), and the effective approach can be applied to the perturbations of Equation (A2).

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