Conformal Field Theory and Functions of Hypergeometric Type

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Zusammenfassung

Die konforme Feldtheorie (CFT) bietet eine universelle Beschreibung verschiedener Phänomene in den Naturwissenschaften. Ihre schnelle und erfolgreiche Entwicklung gehört zu den wichtigsten Höhepunkten der theoretischen Physik des späten 20. Jahrhunderts. Demgegenüber ging der Fortschritt der hypergeometrischen Funktionen durch die Jahrhunderte langsamer vonstatten. Funktionale Identitäten, die von dieser mathematsichen Disziplin untersucht werden, sind faszinierend sowohl in ihrer Komplexität, als auch ihrer Schönheit. Diese Arbeit untersucht das Zusammenspiel dieser beiden Themen anhand der direkten Analyse dreier CFT-Problemen: Zweipunktfunktionen der zweidimensionalen 'strange metal CFT', Dreipunktfunktionen von primären Feldern der nichtrationalen Toda CFT und kinematischen Teilen von Mellin-Amplituden für skalare Vierpunktfunktionen in beliebigen Dimensionen. Wir heben verschiedene Verallgemeinerungen der hypergeometrischen Funktionen als eine natürliche mathematische Sprache für zwei dieser Probleme hervor. Einige neue Methoden, die durch klassische Resultate über hypergeometrische Funktionen inspiriert wurden, werden vorgestellt. Diese Arbeit basiert sowohl auf unseren Publikationen [1–3], als auch auf einem Papier, dass sich in seiner abschliessenden Vorbereitung befindet [4].

Abstract

Conformal field theory provides a universal description of various phenomena in natural sciences. Its development, swift and successful, belongs to the major highlights of theoretical physics of the late XX century. In contrast, advances of the theory of hypergeometric functions always assumed a slower pace throughout the centuries of its existence. Functional identities studied by this mathematical discipline are fascinating both in their complexity and beauty. This thesis investigates the interrelation of two subjects through a direct analysis of three CFT problems: two-point functions of the 2d strange metal CFT, three-point functions of primaries of the non-rational Toda CFT and kinematical parts of Mellin amplitudes for scalar four-point functions in general dimensions. We flash out various generalizations of hypergeometric functions as a natural mathematical language for two of these problems. Several new methods inspired by extensions of classical results on hypergeometric functions, are presented. This work is based on our publications [1–3] as well as on one paper at the final stage of preparation [4].

This thesis is based on following publications:

- M. Isachenkov, I. Kirsch and V. Schomerus, *Chiral Primaries in Strange Metals,* Nucl. Phys. B **885**, 679 (2014) [arXiv:1403.6857 [hep-th]].

- M. Isachenkov, I. Kirsch and V. Schomerus, *Chiral Ring of Strange Metals: The Multicolor Limit,* Nucl. Phys. B **897**, 660 (2015) [arXiv:1410.4594 [hep-th]].

- M. Isachenkov, V. Mitev and E. Pomoni, *Toda 3-point Functions From Topological Strings II,* [arXiv:1412.3395 [hep-th]].

Contents

Chapter 1

Introduction

Modern *quantum field theory* (QFT) is a beautiful and complex framework. Having unified the principles of quantum mechanics and special relativity, it evolved into a powerful instrument which ever since has been helping us to understand how the world behaves on microscopic scales and at large energies. The Standard Model of particle physics, based on principles of QFT and fully justified after the discovery of a Higgs particle in 2012, works with unprecedented precision for the observed high energy phenomena and will certainly open new technological horizons in the future. The present thesis studies several examples of QFTs with additional, conformal symmetry which come from different corners of the huge space of physically interesting theories. There is however a red thread that underlies our considerations: universal generalizations of special functions appearing as solutions of these models. To make it visible to the reader, we zoom out for a while and pursue a suggestive historical parallel. It will tune our mind into the right station and set the flow of subsequent exposition.

Put into a perspective, the present state of quantum field theory can be compared to the one of classical mechanics in the early XVIII century. After the mathematical substrate of calculus was prepared by Newton and Leibnitz, and Newton formulated his laws of motion, the classical mechanics hit its stride in explaining various physical phenomena. Kepler's problem of orbital motion was solved and consistency of Newton's gravitational theory with Kepler's laws was shown. Problems of motion in more elaborate, continuous systems, such as stretched string or flow of a fluid, were also addressed. However, there was still a long way to go, until clear physical principles were recast into mathematically invincible form, allowing for a wide range of exactly solvable physical models. It required another century and such great minds as Euler, d'Alembert, Laplace, Lagrange, Hamilton, Jacobi for classical mechanics to become what we know today, a measure of clarity and mathematical beauty among physical theories.

What was happening throughout this century? Without dwelling on details, one can certainly tell that it was an extremely fruitful age when scientists analysed many concrete examples of classical systems and refined their mathematical tools. A discipline of mathematical physics has begun to grow, in close connection to various techniques of solving equations of motion, a number of tricks and observations which later evolved to mathematical theories of differential equations and of special functions.

Word by word, all said above can be seen to characterize today's quantum field theory. Having clear physical principles laid in its foundation and being extremely successful in describing various phenomena, it is still quite below the level of clarity exemplified by classical mechanics. The strict mathematical formulation of QFT has many caveats and is often very obscure. Other side of the coin, as in the above historical example, is the fact that our abilities for analytical computations in quantum field theory are still quite restricted, although most of advances of the last 40 years were related to accessing non-perturbative physics, going beyond evaluations of Feynman diagrams. Alongside, more elaborate physical theories, such as string (M-) theory were developed and found to be intimately connected to QFT, much as the interest to continuous systems followed the studies of point masses. Continuing the parallel, it is very likely that some useful progress can be made on the way of extending and developing modern analogues of the theory of special functions. An opinion shared by many physicists is that such *'special functions of the XXI century'* are conformal field theories.

Conformal field theory (CFT) is one of the highlights of theoretical physics in the XX century. The power of conformal symmetry in QFT, first realized by Polyakov and Migdal [5, 6][1], led to a breakthrough in understanding exact dynamics of two-dimensional quantum field theories after the seminal paper of Belavin, Polyakov and Zamolodchikov [8]. Ever since, conformal field theory has found many applications, from describing a variety of critical phenomena in condensed matter [9], to the analysis of turbulence [10–12] and even of bond markets [13]. It is also a crucial building block of string theory, as the string worldsheet itself is governed by a two-dimensional CFT [14]. In a generic quantum field theory, conformal field theories arise as fixed points of the renormalization group flows, describing for instance the infrared dynamics of the model. In a way, the CFTs can be regarded as 'boundary conditions' for QFTs, their core.[2]

What about more traditional special functions, the ones everybody knows? In this thesis we will support a claim that those are naturally at the root of the framework: the observables in CFTs are often given by extensions of classical special functions. To focus the attention, we will look at the particular class of those: *hypergeometric functions*.[3]

The theory of hypergeometric series[4] is of respectable age. It debuted in mathematical physics just at the historical period described above: the term generalizing a notion of geometric series was coined by Wallis in 1695. Series of this type were then studied by Euler, Gauss, Kummer, Riemann and many others. Throughout XIX and XX century, a lot of further generalizations were found. Patterns of hypergeometric type, closely associated to various discrete symmetries, are universal, emerging in many different areas of mathematics and physics [18].

To see the actual relation between classical special functions and CFTs, let us stick to our parallel and recall that the special functions often appeared as solutions of equations of motion for various classical systems, continuous and discrete. Translated to quantum language, these solutions are replaced by correlation functions of local operators. These describe how various fields interfere with each other and in principle fully determine a generic QFT. Then we can ask a question: what are the types of functions representing correlators if the theory also possesses a conformal symmetry? In this thesis we will show that, for particular CFTs, the correlators as well as some related quantities are indeed given by extensions

[1]The first to consider conformal quantum mechanics was Dirac in [7].

[2]E.g. for $\mathcal{N} = 2$ supersymmetric four-dimensional theories of class $\mathcal{S}$, the way to justify this is to notice that the partition function, information essentially on their vacua, can be translated into information on full dynamics (correlators) of some CFT via 2d/4d duality. [15] In other words, this CFT indeed forms kind of a skeleton for such theories, which then becomes dressed by dynamics of non-supersymmetric excitations. See also Chapter 5.

[3]Actually, we will reduce the class to consider even further: functions we will work with in what follows are mostly generalizations of ${}_{p+1}F_p$ hypergeometric functions.

[4]In this thesis, we will mostly use series representations of hypergeometric functions and say very little about their integral forms, although sometimes the latter are the only way to proceed, e.g. in the case of non-terminating elliptic hypergeometric functions. [16, 17]

of well-known hypergeometric functions. So, the meaning of motto 'CFTs are new special functions' as well as correspondence between 'old' and 'new' become somewhat more precise.

Let us notice that there is no news by itself in the fact that hypergeometric functions and conformal field theories are related. Indeed, just at the very birth of CFT [8], a Gauss hypergeometric function ${}_2F_1$ was shown to express the four-point functions of so-called minimal models, exceptionally simple examples of conformal theories. More generally, the Knizhnik-Zamolodchikov equations constraining correlators in 2d CFT were shown to have rich mathematical structure and a web of connections with generalized hypergeometric functions [19]. Another, somewhat related, fusion point of conformal field theory and hypergeometric functions is the mathematics describing symmetry algebras of 2d CFTs and various deformations thereof. Hypergeometric functions then enter the story through the representation theory of these algebras, e.g. expressing their intertwining properties in the form of $3nj$-symbols [20].

The present work's valuable addition to the above is describing important analogues of hypergeometric functions which were not looked at before in this context, letting some more species from the zoo of possible hypergeometric extensions to the game. We believe that our observations are not of local use and can bring more interesting results in the future. The overall message we aim to convey through this thesis is that it proves very fruitful to think about complicated sums and integrals encountered in quantum field theory calculations as (deformations of) hypergeometric functions. This brings structure into somewhat messy expressions and allows to use the 200 years of cumulative mathematical experience in summing up and simplifying. Such a correspondence is also bound to work the other way around: as we know, physical intuition is often far ahead of rigorous mathematical constructions. Summarizing, there are good reasons to expect that the connection between generalizations of special functions and CFTs has profound meaning and, understood well enough, will become a fountainhead of new exact analytical calculations in QFT and string theory.

Having set up the stage and explained the objectives of our research, we now move to a more detailed exposition. As mentioned in the beginning, the physical problems we address are all related to calculations in conformal field theory: namely, we deal with two-point functions of the strange metal coset model in Chapter 4, three-point functions of primaries of the Toda CFT in Chapter 5 and Mellin amplitudes for conformal blocks of scalars in Chapter 6. Whereas in the two latter problems we will use extensively the properties of (generalized) hypergeometric functions to arrive at our results, hypergeometric functions do not play an important role in the former analysis.

Zooming in, this thesis is organized as follows. After short introductions to conformal field theory in Chapter 2 and to the theory of hypergeometric functions in Chapter 3, Chapter 4 will deal with a particular model for $1+1$-dimensional strange metals. Strange metals are media that display a breakdown of a standard Fermi liquid behaviour [21]. Although most interesting strange metals are experimentally obtained as two-dimensional layers, one can restrict to one dimension in a hope that the strongly interacting model will prove fully solvable and give some insights into $2+1$ dynamics. Also, in $1+1$ dimensions one has to be a bit more careful with the above definition and make sure to also exclude the case of Luttinger liquid – the one-dimensional analogue of Fermi liquid which nevertheless has some properties, unusual for its higher-dimensional cousins, e.g. the absence of well-defined quasiparticles [22]. Luttinger liquid can be conveniently described by a simple free-boson CFT. A so-called Luttinger relation, common both for Fermi and Luttinger liquids and saying essentially that the interacting liquid can be adiabatically connected to a bunch of free fermions, is violated for strange metals. Moreover, the Fermi surface of strange metals is very often hidden, which therefore suggests an emergent gauge theory description, interpreting this fact as inobservability of gauge non-invariant one-point function.

In [23] it was shown that adjoint QCD with with fermions of large density flows to a specific CFT in the infrared, which possesses some features of a strange-metallic state. This $\mathcal{N} = (2,2)$ supersymmetric CFT has a coset WZNW description and was named *'strange metal CFT'*. In Chapter 4, we describe its large symmetry algebra $\mathcal{W}_N$, find the operator spectrum for small numbers of colors $N < 6$ and show that for general N the spectrum is naturally labeled by necklaces, certain combinatorial objects. Finally, we find the spectrum of operators protected by supersymmetry in the multicolor limit $N \to \infty$. One of surprising interrelations between QFT and string theory mentioned above, is the so-called gauge/string duality [24], viewing an asymptotically free gauge theory as a hologram of some dual string theory living in a warped space, the emergent bulk dimensions representing a collective effect of quantum fluctuations [25]. If the gauge theory possesses conformal symmetry, the habitat of the dual strings is bound to contain a factor of constant curvature, the Anti-de-Sitter space, so that the duality refines its name to *AdS/CFT correspondence* [26]. Thinking this way, the BPS operator content we identify in Chapter 4 is a unique footprint of the strange metal CFT detectable from the Anti-de-Sitter side, which imposes powerful constraints on its possible string description. The exposition of the results of this Chapter is based on our papers [1, 2].

The next Chapter 5 contains a very non-trivial check of a novel conjecture for three-point functions of generic primaries in *Toda field theory*. To set up some background, let us recall that the Toda conformal field theory is an important example of non-rational two-dimensional CFT which possesses another large symmetry algebra. This algebra called $\mathbf{W}_N$ further extends the usual 2d enhancement from conformal to Virasoro symmetry. Toda CFT is a multifield generalization of the renowned Liouville field theory, crucial ingredient of a formulation of non-critical string theory. Toda theory has a number of applications, but most importantly is the simplest example of a non-rational conformal theory, possessing rich $\mathbf{W}_N$ symmetry. Surprisingly enough, after the expressions for three-point functions of Liouville theory[5] were proposed and derived in the middle of 1990's, the three-point functions of generic Toda primaries remained elusive for the subsequent 20 years. The standard CFT methods proved quite hard to develop and worked only for some subset of generic three-point functions of primary fields, so the solution circumventing the difficulties of the CFT approach finally came from gauge theory.

In [27] an explicit answer was obtained for the above three-point functions, a rather complicated expression. The precise way to derive it is quite involved and will be discussed in detail in Chapter 5. It combines different technical tools, including *AGT(-W) correspondence* between two-dimensional CFT correlators and partition functions of certain four-dimensional gauge theories, machinery of topological string theory allowing geometrical engineering of corresponding isolated gauge theories and some working assumptions, regarding the counting of BPS strings in five dimensions. In particular, the answer for generic three-point functions contains a piece analogous to a Nekrasov partition function[6] which is a counting function for instantons in 4d $\mathcal{N} = 2$ (5d $\mathcal{N} = 1$) gauge theories, generically a convoluted sum over many number partitions. The highlight of Chapter 5 is that the Nekrasov functions are often useful to think about as (extensions of) *hypergeometric functions of Kaneko-Macdonald type*. Based on this observation, we check the above formula against an important known special case when one of the

[5]One of the nice features of quantum field theories with conformal symmetry is that it is enough to know only two-point functions (spectrum) and three-point functions (structure constants) to reconstruct the full dynamics of the theory. This lays the foundation for the so-called conformal bootstrap program, a way to solve CFTs from the constraints imposed by symmetry. See Chapters 2, 6 for a more detailed discussion.

[6]We write 'analogous' here because the corresponding four-dimensional T_N gauge theory doesn't have a known Lagrangian description, so that there is no usual notions of 'perturbative' and 'instanton' in this case. See Chapter 5 for more details.

primary fields contains a null-vector at level one, a Fateev-Litvinov formula [28]. Actually, this very non-trivial calculation turns out to be quite simple if one thinks in the paradigm 'Nekrasov functions are hypergeometric functions'. The sums which correspond to the residues of an integrand giving the structure constants are simplified to products of standard Υ-functions via using a binomial identity for the Kaneko-Macdonald(-Warnaar) sl(N) basic hypergeometric functions. The resuts of Chapter 5 were obtained in our paper [3].

The considerations of Chapter 6 go beyond 2d, discussing properties of conformal blocks of a CFT living in a generic spacetime dimension. The higher-dimensional ($d > 2$) *conformal bootstrap*, a way to constrain conformal field theories using consistency conditions, has been experiencing a resurgence in the last few years, mostly due to advances in numerical solving of crossing symmetry equations. The basis for this was laid in the previous decade by exact computations of higher-dimensional conformal blocks for scalar fields due to Dolan and Osborn [29, 30] and by Mack [31] who also observed the analogy between Mellin representations for conformal blocks and scattering amplitudes. It resulted in obtaining impressive numerical bounds for lowest operator dimensions of 3d Ising model [32] and, more generally, of the class of 3d models possessing O(N) symmetry [33, 34]. These numerical considerations are being extended to supersymmetric models and to fields of higher spin [35–37], the ultimate goal of interest here being bootstrapping a spin 2 energy-momentum tensor. However, going beyond numerics still seems to be quite technically demanding even in the case of scalar fields, due to complicated outfit of corresponding exact expressions.

Based on our forthcoming paper [4], in Chapter 6 we find out that specific generalizations of hypergeometric functions, called *Srivastava-Daoust multiple hypergeometric*, naturally capture the behaviour of Mellin amplitudes for CFT correlators. Using identities for them we study a Mellin amplitude of the conformal block with four external scalars, also known as *Mack polynomial*. There are two different expressions for this quantity. One was derived by Dolan and Osborn who solved a recursion relation following from an eigenvalue problem for the conformal Casimir. The second was obtained by Mack via direct calculation using modified Symanzik formula. Both expressions are known to agree numerically. Our computational method provides an analytical bridge between the two. As an example, we prove an identity for a triple sum conjectured in [38], confirming the proposed formula for a first residue of the factorized Mellin amplitude in the simple case of four-point function. The applicability range for bunch of tricks which allow us to simplify the expressions typically encountered when dealing with Mellin amplitudes, doesn't seem to be restricted particularly to the case of scalars, which should render them as quite useful for analytical études on higher-dimensional bootstrap.

Chapter 7 is a conclusion. It gives an outlook of the results and discusses some interesting future prospects.

Chapter 2

Conformal field theory

This Chapter is an introduction to CFT, a QFT possessing conformal symmetry. The presentation is rather meant to give reader a flavour of the thing than to go into detailed explanations. Since the subject has been actively developing for more than 30 years, there is a huge amount of literature and many comprehensive introductory courses. One may want to consult [39], lecture notes [40–45] and, of course, the 'big yellow book' [46] for the thorough treatment of two-dimensional CFTs.

The plan for this Chapter is as follows. First, we shortly recall the physical origins of conformal symmetry and its mathematical description. Then, we discuss how the conformal symmetry is imposed on a quantum field theory and list some properties we expect from a sensible CFT. In particular, we dwell on the bootstrap program of solving conformal theories. Finally, we describe the subtleties of two-dimensional CFTs due to enhancement of conformal symmetry and discuss some particular examples of those which will be useful for next Chapters.

2.1 Conformal symmetry

Conformal transformation is the one locally preserving the angles. Imposing a symmetry under it on a physical theory is quite restrictive and one may start to wonder if it describes anything in the real world. Indeed, the reader who likes sometimes to pore over old maps can immediately see non-triviality of the assertion. Let us choose two different angle-preserving cartographic projections – e.g. Mercator and stereographic – of Europe, and measure the distance, say from Moscow to Hamburg. Clearly, it will strongly depend on the type of projection. Now, imposing conformal invariance would mean that we declare all such projections physically equivalent, and to ensure that there should be no characteristic scale at all! Amazing though it would be in terms of travel time, one empirically knows it's not the actual case. So, what makes us think that a relatively small class of theories with restrictive symmetry of spacetime has something in common with physics?

There are at least two good answers to this question. Historically first type of relevant physical situations is connected to critical points in condensed matter physics, i.e. endpoints of phase equilibrium curves where the system undergoes a continuous phase transition. The vicinity of such a critical point on the phase diagram was noticed to exhibit quite peculiar properties. For instance, for a liquid/gas phase transtition fluctuations of any wavelength become allowed, as one can vividly see by observing critical opalescence, so at least the scale invariance seems to be at work here. In a short while, we will make a small argument why the conformal invariance doesn't overimpose much and a microscopic description for a generic example of this type is actually a CFT. A very important feature to mention is the universality of such continuous phase transitions: microscopically very different models demonstrate same correlation properties on the large scales.

The second set of physical examples is relevant for quantum field theory. Inspired by analogy with condensed matter physics, Wilson [47] was first to think about QFT as a renormalization group flow from the ultraviolet (UV), high-energetic dynamics to the large-distance infrared (IR) description. If the model we look at has a Lagrangian, this flow can be viewed as generated by beta-functions of various couplings, describing the response of the theory to changing them, if it doesn't – as generated by relevant operators which push one out of the UV. At the endpoints of the flow, there sit quantum theories for which all the beta-functions vanish.[1] It is often the case, that the fixed point theory is actually non-trivial (interacting) and scale invariant. For instance, if a QFT model enjoys such a fixed point in IR, the way to think about it is that microscopic quantum effects sum up to yield a macroscopic theory with emergent scale-independence. The RG language will now help us to sketch the above mentioned argument, showing that generically this fixed point theory is also conformal. Namely, we want to draw a parallel to lattice models of condensed matter. With an additional knowledge that the liquid/gas critical point is in the same universality class as the 3d Ising lattice model and a QFT can be often viewed as continuous limit of some lattice theory, the subsequent reasoning provides intuition for both described cases.

So, let us use a specific realization of the RG, called exact renormalization group (ERG). We look at a lattice theory, where at each node a spin is placed, transforming according to representation of some symmetry algebra, and the overall energy is described by a Hamiltonian, function of these spins. If one now considers a regular tesselation of the lattice and changes to new variables where all the spins belonging to one block are described by one effective spin variable, one gets a system with new, more strongly coupled Hamiltonian. Iteration of this process can be approximated with a flow in the space of theories. These flows are the ones forming exact renormalization group. Provided that the particular flow we study has a fixed point, we can take an effective Hamiltonian corresponding to it and stir by small perturbations. It is easy to see that local operators of the fixed point theory should correspond to solutions of an eigenvalue problem for a ERG flow linearized around it. With a little bit of additional thought, the assertion that scale invariance implies conformal can be reformulated as follows: if the fixed point theory is invariant under the ERG flow, it is also invariant under a 'non-uniform' ERG tranformation, by which the lattice on each step gets also conformally distorted. This additional requirement of 'smoothness' of the RG flow in the vicinity of fixed point can be believed to hold for a sufficiently nice physical theory, just as in the theory of differential equations we usually agree that nice solutions are differentiable at the boundary for sufficiently many times. And indeed, there are theorems saying that under some mild assumptions scale invariance enhances to conformal: e.g. if the fixed point Hamiltonian is not very non-local [49]. For a recent progress in this direction, see [50].

Having said enough about physical lineage of conformal symmetry, we now briefly describe its mathematical implementation. Continuous symmetries in physics are often captured by Lie algebras. As CFTs are special cases of QFTs, the algebra of conformal transformations we look for should be some extension

[1]The mathematics describing RG flows is essentially theory of differential equations and dynamical systems. As we know, the phase space of those can in principle contain various types of special points: for instance, a given RG flow can be attracted to a fixed point, but never reach it [48]. We leave the discussion of physicality of such special points out and consider only generic, non-pathological flows.

of a usual Poincaré algebra. Actually, let us do a Wick rotation right now and keep the spacetime Euclidean until the end of this work.[2] Then, the sought-for extension of the d-dimensional Euclidean algebra with generators given by P_μ and $L_{\mu\nu}$ where $\mu, \nu \in \{1, \dots d\}$, is generated by additional elements labeled D and K_μ and satisfying following Lie-algebraic commutation relations:

$$\begin{aligned}
&[D, P_\mu] = iP_\mu, \ [D, K_\mu] = -iK_\mu, \\
&[L_{\mu\nu}, P_\varrho] = i(\eta_{\nu\varrho}P_\mu - \eta_{\mu\varrho}P_\nu), \\
&[L_{\mu\nu}, K_\varrho] = -i(\eta_{\nu\varrho}K_\mu - \eta_{\mu\varrho}K_\nu), \\
&[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}).
\end{aligned} \tag{2.1.1}$$

Here D is a dilatation generator responsible for scale transformations, and K_μ are so-called special conformal generators, describing the 'non-uniform' part of conformal symmetry. By a linear change of basis, this Lie algebra can be seen as isomorphic to $SO(d+1, 1)$, a Lorentz algebra in $d+2$ dimensions. To make us confident that this is the algebra we want, let us notice that its action on $\mathbb{R}^d$ can be represented through following differential operators:

$$\begin{aligned}
&p_\mu = -i\partial_\mu && \text{(infinitesimal translations)}, \\
&l_{\mu\nu} = ix_\mu\partial_\nu - ix_\nu\partial_\mu && \text{(infinitesimal rotations)}, \\
&d = -ix^\mu\partial_\mu && \text{(infinitesimal dilations)}, \\
&k_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) && \text{(infinitesimal special conformal transformations)}.
\end{aligned} \tag{2.1.2}$$

Looking at these expressions for a while, one can indeed see that they generate a most general symmetry of a d-dimensional Euclidean space compatible with preserving an angle between two vectors.[3] The corresponding global transformations on $\mathbb{R}^d$ read as

$$\begin{aligned}
&x_\mu \to x'_\mu = x_\mu + \alpha_\mu && \text{(translations)}, \\
&x_\mu \to x'_\mu = \omega_\mu^\nu x_\nu && \text{(rotations)}, \\
&x_\mu \to x'_\mu = \gamma \cdot x_\mu && \text{(dilations)}, \\
&x_\mu \to x'_\mu = \frac{x_\mu - \beta_\mu \cdot x^2}{1 - 2\beta_\mu x^\mu + \beta^2 \cdot x^2} && \text{(special conformal transformations)}.
\end{aligned} \tag{2.1.3}$$

Since physical states of our quantum theory will be expected to organize into multiplets of conformal algebra, one needs to study its representations in order to learn about the kinematics of CFT. A thorough treatment of the representation theory of conformal algebra can be found in [53]. Importantly for us, the unitary representations of it(s covering) with positive energy are labeled by a finite-dimensional irreducible

[2] We are allowed to do it provided all the observables of our theory can be analytically continued to Euclidean time. This is, of course, non-trivial and is usually postulated or derived from the adopted set of axioms. To yield a local unitary Minkowski theory, the Euclidean formulation should also satisfy a very important reflection-positivity property, see further.

[3] An elegant way to show it is an embedding space formalism. [51, 52]

representation of the d-dimensional rotation algebra (i.e. a set of matrices $\Sigma_{\mu\nu} = \left(\Sigma_{\mu\nu}\right)_A^B$ with indices A, B, satisfying appropriate commutation relations) and a real number $\Delta \geq 0$ called conformal weight which relates to a non-compact part of Euclidean conformal algebra.

To construct representations, we now need to define a field of lowest weight. It can be shown that the following definition works for the conformal algebra. A field $\Phi = (\phi_A)$ is said to be a quasi-primary field[4] of spin Σ and conformal weight Δ if

$$\begin{aligned} D\Phi(x) &= (-x^\nu\partial_\nu - i\Delta)\Phi(x), \\ L_{\mu\nu}\Phi(x) &= \left(ix_\mu\partial_\nu - ix_\nu\partial_\mu + \Sigma_{\mu\nu}\right)\Phi(x), \\ K_\mu\Phi(x) &= \left(ix^2\partial_\mu - 2ix_\mu x^\nu\partial_\nu - 2ix_\mu\Delta - 2x^\nu\Sigma_{\mu\nu}\right)\Phi(x). \end{aligned} \tag{2.1.4}$$

Before concluding this short section, we make a last important remark. Representing the conformal symmetry in a generic spacetime dimension d, the described algebra actually undercounts when $d = 2$: there is an occasional enhancement of the local symmetry such that the augmented Lie algebra is actually infinite-dimensional. We will discuss it in more detail further, in a section about two-dimensional CFT.

2.2 Quantization

The next step we want to take is the actual imposing the conformal symmetry on a QFT. As mentioned in the introduction, precise mathematical definition of a quantum field theory is quite hard. We will now give the reader a flavor of this, before describing a more manageable 'phenomenological' approach.

One of the mathematical descriptions QFT is a framework called functorial quantum field theory (FQFT) which aims to generalize for QFT a Schrödinger approach to quantum mechanics. We remind that the central object in Schrödinger picture is a wave function/ket-vector satisfying a linear first-order differential equation. To each moment of time one assigns a Hilbert space in which the quantum states live whereas to each pair of such moments one assigns a (unitary) operator encoding a local time evolution. The evolution operator is given by a parallel transport of Hamiltonian and, as soon as inner product is defined, allows us to calculate probability amplitudes.

Lifting this up to field theory, one can say that the quantum field theory is given by a (monoidal) functor

$$Z : \mathrm{Bord}_d^S \longrightarrow \mathrm{Vect} \tag{2.2.5}$$

acting from a (monoidal) category of cobordisms to a category of vector spaces which is also monoidal [54]. The cobordisms are, in leading approximation, just smooth interpolations between different field configurations, and the monoidality imposed on a category of them would mean that we impose locality on our QFT. The monoidality of category of vector spaces is crucial and means, as in quantum-mechanical case, that we can 'add' quantum systems and form mixed states out of pure ones.

To believe that this definition contains everything we usually want from a QFT, let us unpack it a bit. Take, for example, codimension one and zero slices of spacetime. The above functor then implies:

[4]Due to historical reasons, there is often a minor clash of notation here: the defined quasi-primaries are usually called primaries in dimensions $d > 2$, whereas in two dimensions the word 'primary' refers to a Virasoro primary, see further.

- To a codimension one slice of spacetime M_{d-1} it assigns a vector space $Z(M_{d-1})$ which is, tautologically, a space of quantum states at a given time. So, our spacetime is foliated by space-like leaves, and each of them is endowed with its own Hilbert space.

- To a manifold M of spacetime dimensionality having boundary ∂M it associates a linear 'propagator map' $Z(M) : Z(\partial_{\text{in}} M) \longrightarrow Z(\partial_{\text{out}} M)$ which takes 'incoming' states to 'outcoming' ones. This looks precisely like an S-matrix for moving from $\partial_{\text{in}} M$ to $\partial_{\text{out}} M$! So, we can connect Hilbert spaces belonging to different leaves by an S-matrix. It describes the evolution of our system.

Let us take one step further and see what this definition implies. Provided we can go to a dual vector space belonging to dual category Vect*, the propagator map is rewritten as

$$\mathbb{C} \longleftarrow Z(\partial_{\text{out}} M) \otimes Z(\partial_{\text{in}} M)^* = Z(\partial M). \tag{2.2.6}$$

To a given cobordism it assigns a complex number. If we look at a class of cobordisms obtained by point-like insertions of local operators to a vacuum state, this assignment is nothing else as a correlation function of these operators. So, we can specify the theory by a full set on correlation functions instead of an S-matrix.

In that way, one can seemingly always go between amplitudes and correlators in a QFT.[5] However, the requirement that a dual vector space to our Hilbert space exists puts quite strong bounds on such a relation. E.g., the simplest sufficient condition would be a finite dimensionality of the Hilbert spaces, as the category of finite-dimensional vector spaces is compact. This essentially restricts a QFT to a topological theory (TQFT) where observables can depend only on the topological invariants of spacetime manifold. It shows why most results in the framework of FQFT are actually on TQFTs. Not without physical relevance [55], topological theories are, of course, a bit boring in terms of dynamics.

A physically interesting conformal field theory is also believed to be described solely by specifying all the correlation functions of local operators. To try and justify it from FQFT perspective one would need to formulate a conformal theory in this language, endowing the category Bord_d^S with additional conformal structure.For a CFT of general dimension, it still seems pretty hard to implement in all details, although not impossible [54]. On the contrary, for a very important class of rational two-dimensional field theories (for definition, see section on two-dimensional CFTs), there is a beautiful explicit contruction in this spirit due to Fröhlich, Fuchs, Runkel and Schweigert [56], providing full classification of sufficiently nice[6] 2d RCFTs.

With cobordisms as a multi-dimensional analogue of quantum-mechanical time, just by looking at variability range of possible manifolds one sees how much more complicated it becomes to describe a generic quantum theory via FQFT when one grows in dimension. One may like to try and save the situation by changing the point of view. We remember that quantum mechanics can be equally well treated in a Heisenberg picture focusing on an algebra of observables. This was historically the first formulation of QM. There is a corresponding framework called algebraic quantum field theory (AQFT) aiming to extend the Heisenberg approach to QFT. It is focused on deformation quantization of algebras formed by observables via so-called local nets, see [57]. Analogously, to pass to conformal theories one needs to

[5]This is a topic of so-called Wightman/Osterwalder-Schrader reconstruction theorems, stating that one can reconstruct fields of the theory from the correlation functions under certain assumptions, see further.

[6]Here: existing on any compact 2d manifold.

introduce more structure to this toolkit. What perhaps shouldn't surprise us is that this approach also runs into numerous difficulties as soon as one goes up in dimension. So, since from approaches trying to be as mathematically rigorous as quantum mechanics one doesn't get a clean and immediate view of the peculiar properties of CFTs, let us move towards more handy definition.

A renowned set of axioms for QFT is due to Wightman [58]. It represents quantum fields as operator-valued distributions and manifests an old-school approach to quantum field theory, the one everybody has studied at university. Its Euclidean version bears names of Osterwalder-Schrader [59] and for a theory with conformal symmetry reads as follows. Fundamental objects are the Euclidean N-point functions (Schwinger functions) denoted as $\langle \Phi_1(x_1) \dots \Phi_N(x_N) \rangle$. They are assumed being analytical functions with reasonable growth at infinity in some subdomain[7] of $\left(\mathbb{R}^d\right)^N$ and should satisfy the following three axioms:

- Locality: ordering of fields in the correlation function is indifferent (modulo sign). Translated to field language, it amounts to requiring (anti-)commutativity and associativity for local operators.

- The correlators respect the symmetries of the theory (2.1.3), in particular d-dimensional rotations and dilations.

- Define an involution $\Theta : \mathbb{R}^d \to \mathbb{R}^d$ reversing the Euclidean time:

  $$\Theta(x_1, x_2, \dots, x_d) = (-x_1, x_2, \dots, x_d) \, .$$

  Then

  $$\langle \Phi_N(\Theta x_N) \dots \Phi_1(\Theta x_1) \Phi_1(x_1) \dots \Phi_N(x_N) \rangle \geq 0 \, . \tag{2.2.7}$$

  This condition, called reflection positivity, is an analogue of unitarity[8] for Euclidean theories. In the precise formulation, the fields should be smeared by test functions.

Provided these three axioms are satisfied, there are theorems (see [60]) allowing one to reconstruct the Hilbert space of the theory, i.e. all its operators, and analytically continue to Minkowski space.

However, that's not the end. The above assertions giving a well-defined QFT (CFT) should be supplemented by requiring physical properties we usually observe in a majority of interesting (conformal) theories:

- First assumption is related to existence of a special spin 2 field $T^{\mu\nu}(x)$ called stress tensor, a Euclidean version of energy-momentum tensor. This condition is quite restrictive. Indeed, up to now we essentially demanded only existence of globally conserved energy and symmetry charges, but didn't say anything about local balance of currents. Our additional condition now implies covariance of correlators with respect to local transformations (2.1.2).

  As an example of a physically interesting model without local stress tensor, one can consider a 'non-local Ising model' [61] in $2 \leq d < 4$, continuum limit of the following lattice Hamiltonian:

  $$H = \frac{1}{T} \sum_{\text{all pairs of sites } i,j} \frac{1 - s_i s_j}{|i-j|^\gamma},$$

[7] This subdomain depends on locality requirements in a particular spacetime signature. For a Euclidean theory, it is enough to make sure that the hyperplanes $x_i = x_j$, $i < j = 1, \dots N$ are excluded. In a Minkowski formulation, every pair of x_i should be space-like separated.

[8] Let us mention that unitarity assumption being very important for a generic QFT, sometimes is too restrictive for physical models related to condensed matter, such as Lee-Yang singularity. [46]

where the value of γ needs to be appropriately fine-tuned to yield an interacting fixed point different from usual Ising model. This is however rather on exotic side, so generically we assume a symmetric conserved stress tensor $T^{\mu\nu}(x)$, $\partial_\mu T^{\mu\nu}(x) = 0$ to exist.

It is easy to understand that local conservation of dilation current $x_\mu T^{\mu\nu}(x)$ then actually amounts to requiring $T^\mu_\mu(x) = 0$. To see it without writing equations, we notice that $T^\mu_\mu(x)$ splits in two parts transforming irreducibly under SO(d): symmetric traceless one and the trace itself. A priori, different SO(d) representations are exprected to have different scaling dimensions. This cannot be the case, because trace of the stress tensor has the same dimension as its symmetric traceless part. So, the tracelessness of $T^{\mu\nu}(x)$ is unavoidable in a generic non-trivial conformal theory.

It turns out that with the conservation of the above two currents, the special conformal current $2x^\mu x_\rho T^{\rho\nu}(x) - x^2 T^{\mu\nu}(x)$ is automatically conserved. This is consistent with our previously acquired intuition about special conformal transformations being just a non-uniform enhancement of the rescaling symmetry.

Again, there are some rare examples of scale invariant theories with $T^\mu_\mu \neq 0$, such as Euclidean theory of elasticity [62] in $d = 2, 3$ dimensions:

$$\mathcal{L} = a(\partial_\mu u_\nu)^2 + b(\partial_\mu u_\nu), \quad a + b \neq 0.$$

The trace of the stress tensor in this case is a total derivative which modifies the dilation current. This modification kills the special conformal part of the symmetry. The theory, however, does not have a reflection-positivity property. It is not a coincidence. In two and four dimensions, there are theorems that a unitary scale invariant theory under some additional assumptions is bound to have its stress tensor traceless.

- The second assumption is very peculiar for CFT. It demands existence of the following associative operator product expansion (OPE) of local operators:

$$\Phi_{\sigma_1}(x_1)\Phi_{\sigma_2}(x_2) = \sum_{\sigma \in \text{spectrum}} C^\sigma_{\sigma_1,\sigma_2}(x_1, x_2)\Phi_\sigma(x_2)$$

  where $C^\sigma_{\sigma_1,\sigma_2}(x_1, x_2)$ is a c-function.

  This assumption is also quite restrictive, as generically there is much more room for operators which can appear in the right-hand side than just those belonging to the spectrum. So, as this is an expansion in a complete Hilbert space, it is not just asymptotic as in generic QFT, but actually convergent with a non-zero radius. This fact gives to the conformal theory powerful properties which we describe in the next section.

Although the listed set of axioms in some respects is rather bottom-up, it provides a convenient basis to explore the space of quantum theories having physical applications. As we already saw, some of the assumptions can be relaxed to yield more general classes of CFTs, but the amount we know about those rapidly shrinks to zero, as soon as we start. One, however, should be careful in regarding these axioms as fundamental[9], since, for example, no well-defined 4d interacting theory satisfying Wightman axioms

[9]Two-dimensions again represent a more advanced case: the axioms for (important classes of) those are quite well understood. There are actually many different sets of axioms for rational theories, see [63, 64] and [65] for a comprehensive account of developments.

is known up to now [60]. So, without treating the above as a full description of what any quantum field theory is, we take it as a very successful working receipt, capturing the main features of an interesting QFT (CFT).

2.3 Properties

In this section we list some implications of the above requirements for a generic quantum theory with conformal symmetry.

First, we discuss a very special feature of CFT involving its space of operators. This space actually turns out to be isomorphic to the space of states of CFT. This is, of course, a striking simplification with respect to a generic QFT, as generically the former is expected to be 'exponentially' bigger that the latter. The reasons for this phenomenon called *state-field correspondence* are conformal symmetry and reflection-positivity.

Let us go back to the picture of a spacetime foliated by codimension one slices. Each of them is endowed with its own Hilbert space, and there is a distinct symmetry generator which moves states between different Hilbert spaces, an analogue of a usual time-evolution operator $U = e^{iP^0\Delta t}$ in QFT. In our case, it is natural to choose a foliation consistent with conformal symmetry. The choice most useful for us is given by a family of concentric $d-1$ dimensional spheres S^{d-1} and is called radial quantization. The unitary operator which moves between different leaves is an exponentiated dilation generator $U = e^{iD\Delta\tau}$, where $\tau := \ln r$. The states living on the spheres are then characterized by eigenvalues of generators that commute with dilations: D itself and angular momentum $L_{\mu\nu}$, which are precisely the quantum numbers characterizing unitary infinite-dimensional representations of conformal algebra. It is not hard to show that the reflection-positivity condition (2.2.7) in this coordinate system amounts to

$$\langle [\Phi_N(x_N)]^\dagger \dots [\Phi_1(x_1)]^\dagger \Phi_1(x_1) \dots \Phi_N(x_N) \rangle \geq 0 \tag{2.3.8}$$

where $[\Phi(x)]^\dagger$ is defined for scalar quasi-primaries through

$$[\Phi(x)]^\dagger = x^{-2\Delta}\Phi\left(\frac{x}{x^2}\right) . \tag{2.3.9}$$

This gives us a positive-definite scalar product on each slice, making the construction of Hilbert spaces consistent.

In such a way, we can think about states as functions on the space of field configurations inserted along a surface τ =const of constant time τ. Since we are free to use any such surface we will choose $\tau = -\infty$ which, according to $\tau = \ln r$, suggests we define states as

$$|\Phi\rangle \equiv \lim_{x\to 0} |\Phi(x)\rangle. \tag{2.3.10}$$

Thus, we can obtain all states by inserting fields at the point $x = 0$, which means, there is an isomorphism between linear spaces of states and of fields. As we said before, this very peculiar feature of the conformal field theory is called state-field correspondence. For the state that is associated with the identity field $\mathbf{1}$ we use the special symbol $|0\rangle = \lim_{x\to 0} |\mathbf{1}\rangle$ and refer to it as the vacuum state. Since the identity field $\mathbf{1}$ does not depend on an insertion point x, the limiting process is trivial in this case. Our definition (2.3.10) also suggests writing

$$|\Phi\rangle = \lim_{x\to 0} |\Phi(x)\rangle = \lim_{x\to 0} |\Phi(x)\mathbf{1}\rangle = \lim_{x\to 0} \Phi(x)|0\rangle = \Phi(0)|0\rangle.$$

Let us notice, however, that the existence of a normalizable vacuum state in CFT is not a neccessary condition: for example, this is not the case for non-rational theories in two dimensions, see next section.

A small consistency check can be done. One of the defining features of conformal field theory was that the space of fields carries an action of the conformal algebra. With respect to the scalar product introduced above, the adjoint $X^\dagger$ of the conformal symmetry operators X on this space is given by

$$\begin{array}{ll} K_\rho^\dagger = -P_\rho & P_\rho^\dagger = -K_\rho \\ D^\dagger = -D & L_{\mu\nu}^\dagger = L_{\mu\nu} \end{array} .$$

One can notice that operation † introduced in that way is indeed consistent with the commutation relations (2.1.1).

According to our general definition (2.1.4) of quasi-primary states of conformal weight Δ and spin Σ, the action of conformal symmetry generators on the associated states (2.3.10) is given by

$$L_{\mu\nu}|\Phi\rangle = \Sigma_{\mu\nu}|\Phi\rangle(0) \quad , \quad D|\Phi\rangle = -i\Delta|\Phi\rangle \quad , \quad K_\mu|\Phi\rangle = 0 \, .$$

The last relation confirms our initial way of thinking about quasi-primaries as lowest-weight states. Namely, notice that, while the momentum operators (derivatives), increase the scaling weight of a field, each application of a special conformal generator K_μ decreases the scaling weight by one unit. Put differently, the generators P_μ and K_μ may be thought of as creation and annihilation operators, respectively. Quasi-primary fields are those that sit at a bottom of the annihilation process, i.e. they are exactly those fields that cannot be written as derivatives of others. This insight will soon help us to rearrange the OPE in a convenient way.

A short remark about properties of vacuum state is in order. Since the identity field **1** is a quasi-primary field with scaling weight $\Delta = 0$ and spin $\Sigma = 0$, we obtain

$$L_{\mu\nu}|0\rangle = 0 \quad , \quad D|0\rangle = 0 \quad , \quad K_\mu|0\rangle = 0 \quad , \quad P_\mu|0\rangle = 0 \, ,$$

i.e. the vacuum state is invariant under all conformal transformations. Invariance under translations follows from the fact that the identity field does not depend on an insertion point x. Using the notation we introduced above, we can now write

$$L_{\mu\nu}|\Phi\rangle = (L_{\mu\nu}\Phi)(0)|0\rangle = [L_{\mu\nu}, \Phi(0)]|0\rangle$$

and similarly for the other generators D, P_μ and K_μ of the conformal Lie algebra.

Another nice feature worth mentioning here is the existence of *unitarity bounds*. They represent 'convexity' of the space of unitary conformal theories: scaling dimensions of fields in an interacting theory are required to be equal or bigger than those in a free theory. For symmetric traceless fields of spin l in d dimensions these bounds read:

$$\begin{aligned} \Delta_{\min}(l) &= l + d - 2, \text{ if } \ l = 1, 2, \ldots \\ \Delta_{\min}(l) &= \frac{d}{2} - 1, \text{ if } \ l = 0. \end{aligned} \tag{2.3.11}$$

There are similar results for spinor fields, either. [66] The unitarity bounds are very useful additional constraints for the (numerical) bootstrap method, see further. Notice that in case of two dimensions there are scalars of conformal dimension zero saturating the unitarity bound which can be present in a theory. This is actually one of the reasons for existence of non-rational conformal field theories in two dimensions, having non-normalizable vacuum and continuous spectrum.

Next thing to look at are the correlation functions. As we mentioned, these are expected to be covariant under conformal transformations:

$$\delta_X \langle \Phi_1(x_1) \dots \Phi_N(x_N) \rangle := \sum_{i=1}^{N} \langle \Phi_1(x_1) \dots X\Phi_i(x_i) \dots , \Phi_N(x_N) \rangle = 0$$

for all $X \in SO(1, d+1)$. To illustrate the implications, let us take the simplest case of scalar quasi-primary fields and combine the above conditions for different conformal generators. For an average of quantum fluctuations of a field itself one gets:

$$\langle \varphi(x) \rangle = 0 \quad \text{if} \quad \Delta_\varphi \neq 0\ . \tag{2.3.12}$$

Further, the expression for two-point correlator invariant under conformal symmetry reads:

$$\langle \varphi_1(x_1)\varphi_2(x_2) \rangle = \frac{C_{12}\, \delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}}\ . \tag{2.3.13}$$

The constants C_{12} here may be absorbed into the normalization. Hence, the two-point functions are determined solely by the conformal weights of the fields. Three-point functions of scalar quasi-primaries take the following form[10]:

$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3) \rangle = \frac{C_{123}}{x_{12}^{\Delta_{12}} x_{23}^{\Delta_{23}} x_{13}^{\Delta_{13}}}. \tag{2.3.14}$$

Here we used the notation $\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$ etc. and $x_{ij} = |x_i - x_j|$. The result shows that 3-point functions or scalar quasi-primaries are determined by conformal symmetry up to the constants C_{123}, called structure constants. They are impossible to be absorbed in the definitions of three-point functions, as this would spoil the normalization of two-point functions. In that way, C_{123} encode independent dynamical information and are not constrained by the symmetry. The stress in finding exact solution of CFT very often falls upon identifying these structure constants.

As we see, the correlators simplify drastically: they are not some complicated analytic functions, but just power functions, with all the non-trivial information about them zipped to a bunch of numbers independent on spacetime coordinates. Conformal symmetry is crucial in obtaining such simple expressions – the invariance just under scale transformations would produce much less restriction. When one starts to look at correlators with fields having non-zero spin, the above equations acquire additional appropriate tensor/spinor structures. Conformal symmetry imposes equally powerful restrictions here, see [42]. The correlation functions of descendants can be then obtained from those of primaries by using conformal Ward identities.

For N-point functions with $N > 3$, the nice properties of operator product expansion that we assumed become crucial. Because of the one-to-one correspondence between states and fields, we can write $\Phi(0)|0\rangle$

[10] In two dimensions, pathologies can occur leading to a similar formula decorated with logarithms. This is a benchmark of a so-called logarithmic CFT, see further.

for the state $|\Phi\rangle$, reminding us that $|\Phi\rangle$ is obtained by acting with the field $\Phi(0)$ on the vacuum state. When a field $\Phi_2(x)$ acts on the state $|\Phi_1\rangle$ of the theory, we obtain a new state that can be expanded as follows

$$\Phi_2(x)\Phi_1(0)|0\rangle = \sum_{\phi \in \text{quasi-primaries}} \lambda_\phi C_\phi(x, \partial_y)\phi(y)|0\rangle\Big|_{y=0} \tag{2.3.15}$$

where, as argued before, the radius of convergence is finite and actually equals $|x|$, if there are no other operators inserted in this domain. Notice that we rearranged the sum to run only over quasi-primary (i.e. non-derivative) fields ϕ of the theory and all the descendant fields entering the OPE are encoded in a differential operator C_ϕ whose coefficients are functions of the argument x of the field Φ_2. The object C_ϕ must contain arbitrary numbers of derivatives. We also have extracted a constant factor λ_ϕ from C_ϕ such that C_ϕ satisfies the normalization condition $\lim_{x\to 0}\left(|x|^c C_\phi(x, \partial_y)\right) = 1$ for an appropriate choice of the exponents c. The explicit calculation shows that λ_ϕ is precisely the structure constant $C_{12\phi}$ whereas the operators C_ϕ are completely fixed by conformal symmetry.

Now, we see that knowing the OPE and the two- and three-point functions actually makes it possible to recontruct any correlation function in the theory. Having an arbitrary N-point function to compute, one inductively uses the OPE inside it in the above manner, until the calculation boils down just to three-point functions. Of course, there can be still very cumbersome expressions as a result of this process which require additional understanding, but there are no principal obstacles for that. So, one usually says that a conformal field theory is solved by determining its two- and three-point functions.

Still, if that would be the end of the story, one wouldn't praise a CFT so much as one should. The spectrum and three-point functions are extremely hard to calculate for almost any theory having physical relevance, so there is seemingly no reason to bother building all the complicated machinery to face a wall in the end. However, something much more superb happens for the CFT: its consistency conditions often turn out to be powerful enough to determine the full theory just knowing the spectrum, and even (quite ambitiously) to think about classifying possible conformal theories that way. It is convergence and associativity of the OPE that play a crucial role here. The corresponding physical program is called *conformal bootstrap*, referring to an image of such theory solving itself with minimal external input. We now briefly describe ingredients for the conformal bootstrap.

For simplicity, let us consider correlation functions of four scalar quasi-primary fields $\varphi_i(x_i)$ of scaling weight Δ_i for which one needs some additional notations. Firstly, it can be noticed that from any four points x_i in the d-dimensional Euclidean space $\mathbb{R}^d$ one can build two independent quntities invariant under conformal transformations, so-called conformal cross ratios. Let us, for example, pick

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad , \quad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \,. \tag{2.3.16}$$

Secondly, let us introduce the following function Ω of the insertion points x_i,

$$\Omega_{(12)(34)}(x_1, x_2, x_3, x_4) = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}}\left(\frac{x_{41}^2}{x_{31}^2}\right)^{\frac{\Delta_{34}}{2}} \,. \tag{2.3.17}$$

By application of conformal generators, much like above for two- and three-point functions, one can then observe that a four-point function of scalar quasi-primaries can be written as

$$G(x_i) = \langle \prod_{i=1}^{4} \varphi_i(x_i)\rangle = \Omega_{(12)(34)}(x_1, x_2, x_3, x_4)\; g_{(12)(34)}(u, v) \tag{2.3.18}$$

where $g_{(12)(34)}$ is a function of the cross ratios u, v only.

Let us note that the representation (2.3.18) of the scalar four-point function proposed, i.e. the definition of the function $g_{(12)(34)}$, depends on a choice of a pairing between the insertion points. Above we paired the insertion point of φ_1 with that of φ_2. Alternatively, one could have paired e.g. the insertion point of φ_1 with that of φ_4 to obtain the following representation of G,

$$G(x_i) = \Omega_{(14)(23)}(x_1, x_4, x_3, x_2) g_{(14)(23)}(v, u) \tag{2.3.19}$$

These two representations of the scalar 4-point function are related by an exchange of the fields φ_2 and φ_4 along with their insertion points x_2 and x_4. Note that under the exchange of the insertion points, the cross ratios u and v are mapped to each other, i.e.

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \leftrightarrow \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = v\ . \tag{2.3.20}$$

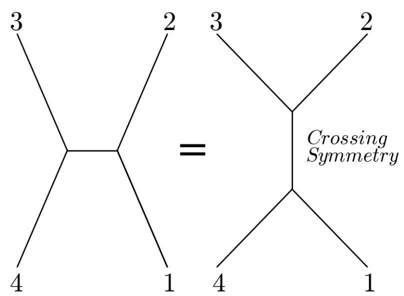
Let us also introduce here the folowing variables $z, \bar{z}$ traditionally used along with the previously defined cross-ratios:

$$u = z\bar{z}, \qquad v = (1-z)(1-\bar{z}). \tag{2.3.21}$$

Due to associativity of OPE, the way of pairing shouldn't affect the value of correlator, so we have:

$$g_{(12)(34)}(u, v) v^{\frac{1}{2}(\Delta_2+\Delta_3)} = g_{(14)(23)}(v, u) u^{\frac{1}{2}(\Delta_3+\Delta_4)}. \tag{2.3.22}$$

This is the crossing symmetry property of the functions g which can be represented graphically as follows



In order to obtain the crossing symmetry constraint on the unknown 3-point couplings λ of a d-dimensional conformal field theory we decompose the functions g into so-called conformal partial waves

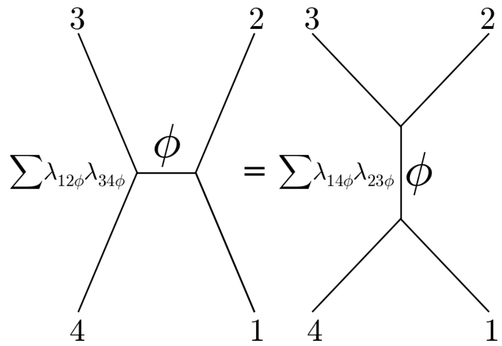
$$g_{(12)(34)}(u, v) = \sum_{\phi \in \text{quasi-primaries}} \lambda_{12\phi} \lambda_{34\phi} g^{\phi}_{(12)(34)}(u, v). \tag{2.3.23}$$

Here, the summation runs over all quasi-primary fields of the model. The conformal partial waves which should be normalized appropriately are completely determined by the conformal symmetry, being purely kinematical information. A more detailed discussion of corresponding exact expressions is a subject of Chapter 6.

Once these normalized conformal partial waves are introduced, we can decompose the crossing symmetry relation (2.3.22) to finally obtain the central equation of conformal bootstrap:

$$\sum_{\phi \in \text{quasi-primaries}} \lambda_{12\phi}\lambda_{34\phi} g^{\phi}_{(12)(34)}(u,v) v^{\frac{1}{2}(\Delta_2+\Delta_3)} = \sum_{\phi \in \text{quasi-primaries}} \lambda_{14\phi}\lambda_{23\phi} g^{\phi}_{(14)(23)}(v,u) u^{\frac{1}{2}(\Delta_3+\Delta_1)}. \tag{2.3.24}$$

This relation can be represented graphically as



The bootstrap program worked brilliantly for several important models of two-dimensional CFT, including so-called minimal models [8] and Liouville theory [67]. However, enormous technical complications related to achieving any progress in this direction render it as rather an art than estabilished technology: generically, one needs to solve an infinite system of crossing-symmetry equations. Nevertheless, a numerical higher-dimensional conformal bootstrap is gaining a considerable momentum in the last years [32, 33], and one might hope that analytical advances are to be expected in the near future.

2.4 Two dimensions

As we mentioned before, two-dimensional conformal field theories generically are much easier to describe and solve. This is, of course, related to the small number of spacetime dimensions: restrictions that we usually impose to build 'good' theories are ahead of the game in 1 + 1 and in some sense do not let a theory to run far away from free, one-body reducible dynamics. For instance, it can very often happen that the dynamics of a full quantum theory in 1 + 1 (2 Euclidean) dimensions is two-body reducible and, correspondingly, integrable. This is precisely the case for a generic 2d local CFT [68]: (the universal enveloping algebra of) its symmetry algebra has a huge center, providing an infinite family of motion integrals. Not surprisingly, the theory then allows for a much finer description.[11] The naturalness of assumptions leading to such an enhancement depends on the physical context in which a CFT is used: it is most natural for string theory where the worldsheet metric is unphysical and in 2d gravity where the independence on background is expected, whereas it can be regarded as more artificial for condensed

[11]Examples of 2d models with conformal symmetry should be mentioned here, where such enhancement does not happen, such as already described non-local Ising model in two dimensions or light asymptotic limit of the Liouville theory [41]. In general, one should expect a huge uncharted space of models with only global conformal invariance, but we will not discuss them further.

matter applications in which flat space is usually considered. Further we briefly describe the hallmarks of a symmetry enhancement in generic 2d CFTs.

Let us take a two-dimensional Euclidean plane, compactify it by adding one point at infinity and introduce complex coordinates:

$$z = x^0 + ix^1$$
$$\bar{z} = x^0 - ix^1.$$

One can check that the global conformal transformations (2.1.3) now can be repackaged into Möbius transformation

$$z \to z' = \frac{az+b}{cz+d} \qquad + \text{ complex conjugate,} \tag{2.4.25}$$

where matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of the group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$. These transformations describe one-to-one isometries of our compactified complex plane. Then, one notices that the six generators of corresponding Lie algebra can be written as holomorphic and anti-holomorphic copy[12] of

$$l_n = -z^{n+1}\frac{\partial}{\partial z}, \qquad n = 0, \pm 1. \tag{2.4.26}$$

Now, the crucial step comes: assume one lowers his expectations and doesn't care about global existence of corresponding finite conformal transformations. Precisely in two dimensions, one can then employ the complex structure and find more transformations locally preserving angles on a complex plane: the Cauchy-Riemann conditions make sure that any analytic function of z will do the job. The resulting algebra is much bigger: actually, it is generated by vector fields (2.4.26) where now $n \in \mathbb{Z}$. Their commutation relations read:

$$[l_n, l_m] = (n-m)\, l_{n+m}.$$

This infinite-dimensional algebra is called Witt algebra, it is the Lie algebra of meromorphic vector fields defined on the Riemann sphere that are holomorphic except at two fixed points.

So, now we have two (holomorphic and antiholomorphic) copies of Witt algebra extending the usual conformal algebra in two dimensions. Going along the lines of previous sections, we would like to impose this as a symmetry of the quantum space of states. There is one subtlety which we haven't spelled out yet. It is well-known that physical quantum states are represented not just by vectors in Hilbert space, but by unit rays therein: the additional complex phase they can acquire is not observable. This mild non-linearity of the quantum description never causes problems in the basic QFT setting, since there is a theorem due to Bargmann showing that for representations of finite-dimensional semisimple algebras one can always remove the additional phase [69]. As usual, infinite-dimensional Lie algebras have much less convenient properties, so one could get really alarmed when quantizing a theory with infinite-dimensional symmetry. But the price to pay here is just trading unitary representations of the Lie algebra acting on states by projective ones, representations 'up to phase factor'. Usually it is convenient to treat this technicality in

[12]Starting from here, we regard z and $\bar{z}$ as independent variables, assuming that actual physical space is the two-dimensional submanifold $\{(z,\bar{z})|z^* = \bar{z}\}$ called real surface.

a different, more explicit manner. Well-known fact about Lie algebras is that a projective representation can be pulled back to a linear one, but of the centrally extended algebra where possible central extensions of algebra are classified by its two-cocycles, elements of the second cohomology. It turns out, that in this case there is a unique (up to equivalence) non-trivial extension of the Witt algebra: $\mathrm{H}^2(\mathrm{Witt}, \mathbb{C}) \cong \mathbb{C}$. This extension is called *Virasoro algebra*. Its commutation relations read:

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12} n\, (n^2-1)\, \delta_{n+m,0}, \qquad n, m \in \mathbb{Z}. \tag{2.4.27}$$

Notice that this algebra still contains the global conformal algebra (2.4.26) as its subalgebra, so that any theory possessing Virasoro symmetry is also globally conformal. This is, of course, consistent with our expectations.

The value of central charge c of Virasoro algebra is an important quantity. It is determined by short-distance behaviour of the theory and can be thought of as an extensive number counting microscopical degrees of freedom. The limit $c \to \infty$ is often called a quasi-classical regime. The central charge also describes response of the theory to the 'soft' breaking of conformal symmetry via putting it on a curved manifold, a so-called conformal anomaly.

Before we go to implications of the enhanced symmetry, let us dwell for a while on the representation theory of our new beast.[13] Not surprisingly, the notion of a quasi-primary field now gets refined with respect to global conformal case. As the reader may remember, from a quasi-primary field we required only to be annihilated by special conformal generators and give a certain eigenvalue under rescalings. It can be seen that in our two-dimensional language special conformal transformations translate to the generator L_1 of Virasoro algebra and dilations – to L_0. The primary fields with respect to Virasoro algebra $V_h(z)$ of holomorphic conformal dimension h are then defined as the following infinite-dimensional analogue of above:

$$\begin{aligned} L_{n>0} V_h(z) &= 0, \\ L_0 V_h(z) &= h\, V_h(z) \end{aligned} \tag{2.4.28}$$

This definition of Virasoro primaries was exactly the reason we termed global primary fields as quasi-primary in the very beginning of this Chapter. It trivially extends to a physical field $V_{h,\bar{h}}(z,\bar{z})$, transforming in the representations of $\mathrm{Vir} \times \overline{\mathrm{Vir}}$.

To build a representation of Virasoro algebra, we now should act on the primary fields with 'raising' generators. There are two types of such highest-weight Virasoro representations. The first is a Verma module $\mathcal{V}_h$ whose basis consists of the states[14] $\{L_{-n_1} \dots L_{-n_p}|h\rangle\}_{1\leq n_1 \cdots \leq n_p}$, where an integer $n = \sum_{i=1}^{p} n_i$ is called the level of the correponding state. It is the largest possible representation with the lowest holomorphic conformal dimension h. However, Verma modules sometimes can contain vectors $|\chi\rangle$ of zero norm, called null-vectors. Correspondingly, there is the second type of Virasoro representations called degenerate and represented as a quotient $\mathcal{V}_h / U(\mathrm{Vir}^+)|\chi\rangle$ of one Verma module by another Verma module, where the latter is generated by a null-vector contained in the former. Here $U(\mathrm{Vir}^+)|\chi\rangle$ is a module generated by negative

[13]For simplicity, we now discuss representations of a single Virasoro algebra. The full quantum symmetry is $\mathrm{Vir} \times \overline{\mathrm{Vir}}$. Combining left- and right-moving fields is discussed further.

[14]Strictly speaking, it is a basis for a module of the universal enveloping algebra of Virasoro. There is a natural isomorphism between the two, though.

’creation’ operators $\{L_n\}_{n<0}$ belonging to the universal enveloping algebra of Virasoro. Representations one encounters in a physical theory, of course, shouldn’t contain states of zero norm. From analyzing unitarity requirements for the Virasoro modules, one can get powerful constraints on possible representations when the central charge is small enough: these are much more powerful analogues of unitarity bounds from the previous section.

The existence of the null-vectors in Virasoro representations is actually one of the things giving the 2d CFT such computational power. By looking at correlators containing degenerate null-fields and employing Ward identities for Virasoro algebra, one can obtain relatively simple differential equations for many exact correlation functions. This is precisely the way Virasoro minimal models were solved in [8]. As a more advanced example, Fateev-Litvinov formula for three-point function of the Toda CFT was obtained by inserting a degenerate field into a four-point function and solving the corresponding differential equation [28], see also further.

By Noether’s theorem, there is a conserved quantity corresponding to any continuous symmetry generator. We saw in the discussion of global conformal symmetry that the corresponding global currents can be written in terms of stress tensor. If a similar analysis is performed for Virasoro algebra, one can show that the quantities corresponding to Virasoro generators are given just by coefficients in the Laurent expansion of the holomorphic stress tensor field. Namely, by introducing notations

$$T := -2\pi T_{zz}, \quad \overline{T} := -2\pi T_{\bar{z}\bar{z}}$$

for components of the stress tensor (mixed $z\bar{z}$ components are zero due to tracelessness), one can write

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{(z-z_0)^{n+2}}, \qquad L_n = \frac{1}{2\pi i}\oint_{z_0} dz (z-z_0)^{n+1} T(z). \tag{2.4.29}$$

Using such mode expansions, one can move between OPEs of the fields and commutation relations of their modes. For example, the translated commutation relations of Virasoro algebra take the form:

$$T(y)T(z) = \frac{c/2}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z} + \text{ reg.}, \tag{2.4.30}$$

where ’reg.’ denotes terms regular in $y-z$.

So, we see that quantum symmetry of our locally conformal field theory consists of two mutually commuting copies of Virasoro algebra[15] spanned by left- and right-moving generators $(L_n)_{n\in\mathbb{Z}}$ and $(\bar{L}_n)_{n\in\mathbb{Z}}$ (remember that we regard z and $\bar{z}$ as independent variables). The spacetime in this description is actually complexified, so that we need to impose appropriate reality conditions on observable quantities. This clearly shows us that, although the complexified spacetime and the symmetry algebra are naturally factorized into holomorphic and antiholomorphic parts, the theory itself cannot be a mere juxtaposition of two independent theories. For generic models, a spectrum actually decomposes into sum of factorized representations, not neccessarily irreducible, of two copies of Virasoro algebras $R\otimes\bar{R}'$ with some multiplicities. The neccessary condition for them to be single-valued on the Riemann sphere reads $h-\bar{h}\in\mathbb{Z}$ for bosonic fields and $h-\bar{h}\in\mathbb{Z}+\frac{1}{2}$ for fermionic ones, where h and $\bar{h}$ are left- and right-moving conformal dimensions, correspondingly. Models where only isomorphic right and left representations $R=R'$ are combined in the spectrum, are called diagonal.

[15]For physically interesting models, left central charge is usually assumed equal to the right one.

For conformal field theories describing, for example, string worldsheet, the spectrum should actually satisfy further conditions. Namely, in order to be able doing string perturbation theory, a corresponding worldsheet CFT should be well-defined on a Riemann surface of any genus. Neccessarily, it should be consistent on an orientable surface of genus one, the torus. We can easily see what it implies for the spectrum. Let us take a part of a Riemann sphere of annulus shape which one can further conformally map to a cylinder. Given the boundary conditions, the dynamics on this annulus is fully determined by dynamics on the sphere. In particular, the states which can propagate in the annulus are precisely the states of CFT living on the Riemann sphere. When gluing together two endpoints of the cylinder to obtain a torus, we take a trace over the Hilbert space of states and can in principle pick up an additional twist q. This means that the vacuum correlator of the theory on a torus (zero-point function) is given by a function of this complex twist parameter $q := e^{2\pi i\tau}$. Since the tori described by τ and $A\tau$, where $A \in \mathrm{SL}(2, \mathbb{Z})$ are equivalent, in order to be a well-defined function the partition function should be $\mathrm{SL}(2, \mathbb{Z})$-invariant, which imposes a restriction on possible spectra. So, the modular-invariant partition function is a convenient way to encode the spectrum, in addition possessing rich underlying mathematical structure related to modular forms, which in some cases allows for classification of the former [46]. Further discussion regarding modular invariants of certain CFTs see in Chapter 4.

All remarks of the previous section regarding state-field correspondence essentially stay almost the same, just enhancing themselves in a similar way from global to local conformal symmetry. For example, the conjugation rules for conformal algebra generators defining positive scalar product on the Hilbert space now get extended to

$$L_n^\dagger = L_{-n}.$$

One subtlety arises for non-rational theories: there is no normalizable vacuum state there, as the identity field is not part of the Hilbert space anymore.

The expressions for one-, two- and three-point functions on the sphere stay literally the same as in equations (2.3.12), (2.3.13), (2.3.14) upon replacing x's by z, $\bar{z}$ and 'quasi-primaries' by 'primaries'[16]. Two subtleties should be mentioned. First, right and left conformal dimensions of corresponding fields can be different. Second, in case of two-point functions of non-RCFT's a certain care should be taken: due to continuity of spectrum, the two-point functions are non-normalizable. So, one should replace Kronecker delta by a delta-function to normalize them in a distributional sense. This is, of course, related to the above comment on absence of a vacuum state in these theories.

The N-point functions of a CFT with Virasoro symmetry can be in principle also reconstructed out of two-and three-point functions of primary fields, much like in global case. However, expressions for Virasoro conformal blocks are not surprisingly more complex than those for global ones [70], because of refinement of algebraic structure. In a way similar to previous section, one can build correlators of Virasoro descendants out of those of primaries by using corresponding Ward identities.

The bootstrap program in 2d also gets a boost and, as we already mentioned, becomes much more powerful than in a generic dimension. Before, we noticed that the spectrum of a global CFT has infinitely many quasi-primary states, highest-weight states of the global conformal algebra. Descending to two dimensions, we now get a strong organizing principle for these: they assemble themselves into modules

[16]Note that there are no additional tensor structures appearing for correlators of primary fields of any integer spin $l \equiv h - \bar{h}$ here. This is, of course, because they trivialize in two dimensions.

of Virasoro algebra. Now, the counting in terms of such new blocks gives two alternatives of how the spectrum of the theory is structured:

- There is a finite number of irreducible representations of Virasoro algebra (or its finite extensions) making up the spectrum. These theories are called *rational CFTs* (RCFTs). A quite advanced understanding of them is reached as today [56]. Nice basic examples of such theories are so-called minimal models [8]. Their solution via bootstrap approach was a tremendous success and the first major impulse for developing CFT technology. More advanced examples of rational theories include Wess-Zumino-Novikov-Witten (WZNW) theories with affine Lie-algebraic symmetry and coset models (see the further discussion and Chapter 4).

- The spectrum contains uncountably many irreducible Virasoro representations. Such theories are called *non-rational CFTs*. They are much less studied and less convenient to handle. An example of such a theory is the Liouville theory describing scattering on a simple exponential potential. Liouville theory proved much harder to solve, but eventually, after exact expressions for its three-point functions were proposed by Dorn, Otto [71], A. Zamolodchikov and Al. Zamolodchikov [72], the bootstrap program was implemented by Teschner and Ponsot [73,74]. Another example of a non-RCFT is the Toda CFT. The bootstrap program for it is not realized yet (see the further discussion and Chapter 5).

In principle, there is also a third option: the spectrum of a theory can involve reducible, but indecomposable representations of Virasoro algebra which is translated to non-diagonalizability of the operator L_0. One can then show from Ward identities that correlators can include mild logarithmic corrections to a usual power-law dependence on coordinates. Such theories are called *logarithmic CFTs* (LCFTs) and represent, in a sense, a class of theories inbetween rational and non-rational CFTs. We will not discuss them in this work.

Now it is time to describe two extensions of Virasoro algebra relevant for theories we will study in Chapter 4. Both these extended algebras are not symmetries of a physical theory by themselves: as in Virasoro case, they need to be supplemented by a second, anti-holomorphic copy to yield an observable spectrum.

The first type of extension we look at, is called an *affine Lie algebra*, or Kac-Moody algebra [75]. It is generated by currents of holomorphic dimension one satisfying following commutation relations:

$$[J_n^a, J_m^b] = \sum_c i\, f_{abc} J_{n+m}^c + k\, n\, \delta_{a,b}\delta_{n+m,0}, \qquad n, m \in \mathbb{Z}, \quad a, b, c = 1, \ldots, \dim g, \tag{2.4.31}$$

where f_{abc} are structure constants of a corresponding Lie algebra, $\dim g$ denotes its dimension and k is an integer called level. As one can see, the affine Lie algebras are the central extensions of corresponding Lie algebras, hence the name. A so-called Sugawara construction makes sure that an appropriate Virasoro field

$$T(z) = \frac{1}{2(k+g)} \sum_a (J_a J_a)(z), \qquad \text{where} \quad J_a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J_n^a \tag{2.4.32}$$

can be constructed out of these currents, which has the central charge $c = \frac{k \dim g}{k+g}$. Here g is the dual Coxeter number of a corresponding Lie algebra. Affine Lie algebras often arise in CFTs as symmetries of WZNW models. [76–79] By taking quotients of affine Lie algebras, many more interesting conformal field theories can be constructed, see Chapter 4. The exposition of that Chapter will use current algebras $SU(N)_k$ and $SO(2M)_1$ as building blocks.

The other extension, called $\mathcal{N} = 2$ *superconformal algebra* (SCA) [80], is generated just by adding to a Virasoro field additional symmetry currents, namely two fermionic currents $G^{\pm}_r$ of spin 3/2 and a bosonic current J_m of spin 1. The R-symmetry rotating between two supersymmetric generators here is U(1). Commutation relations of Virasoro are extended as follows:

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0},\\
&[L_n, J_m] = -m\, J_{n+m},\\
&[J_n, J_m] = \frac{c}{3}n\delta_{n+m,0},\\
&\{G^+_r, G^-_s\} = L_{r+s} + \frac{1}{2}(r-s)J_{r+s} + \frac{c}{6}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},\\
&\{G^+_r, G^+_s\} = \{G^-_r, G^-_s\} = 0,\\
&[L_n, G^{\pm}_r] = \left(\frac{n}{2} - r\right)G^{\pm}_{n+r},\\
&[J_n, G^{\pm}_r] = \pm G^{\pm}_{n+r}.
\end{aligned} \tag{2.4.33}$$

Here $n, m \in \mathbb{Z}$, as for pure Virasoro case. For spinorial modes r, s there are however two distinct possibilities related to different choices of boundary conditions: if $r, s \in \mathbb{Z}$, this yields the so-called $\mathcal{N} = 2$ Ramond algebra, if $r, s \in \mathbb{Z} + \frac{1}{2}$, it defines the $\mathcal{N} = 2$ Neveu-Schwarz algebra.

The Ramond and Neveu-Schwarz algebras are not disconnected: actually, they go into each other via deformation of the SCA which preserves the $\mathcal{N} = 2$ superconformal structure. Such smooth automorphism called spectral flow is a very peculiar feature of this algebra. A zero-mode of the U(1)-current J_0 is actually part of the SCA Cartan torus, so that representations of the algebra are labeled by two quantum numbers: conformal dimension h and charge Q.

Another very peculiar feature of the $\mathcal{N} = 2$ SCA can be observed in its spectrum: there is a special subset of so-called *chiral primary states* saturating the BPS bound $h \leq |Q|$. Representation theory then implies that OPE of two chiral primary fields does not include any singular terms. Together with usual assumption on OPE associativity, it means that the space of chiral primaries admits an algebraic ring structure. This space is called *chiral ring* [81] and is a rigid structure of superconformal theory protected by SUSY. For example, if a theory has Lagrangian description, the operators corresponding to chiral primaries do not get any loop corrections whatsoever. In a full $\mathcal{N} = (2, 2)$ superconformal theory there are four (anti-)chiral rings: holomorphic and anti-holomorphic rings can pair between themselves in obvious ways yielding a physical spectrum of chiral primaries. The counting of states in a chiral ring is usually convenient to encode by a corresponding supersymmetric index called Poincaré polynomial (series). We will derive such polynomials for a $\mathcal{N} = (2, 2)$ superconformal model under consideration in Chapter 4. To conclude this rather short characteristic, let us mention that the $\mathcal{N} = (2, 2)$ SCA has very important applications in superstring theory, describing the worldsheet of so-called Gepner models [82].

Compared to RCFTs, the non-rational CFTs which are relevant for discussion in Chapter 5, have much more complicated structure, consisting of uncountably many Virasoro representations, and no normalizable vacuum state. Since properties of non-RCFTs are less studied and not so well formalized, also due to the small number of explicitly solved examples, we find it a better idea to look at a particular model of this type, the *Toda CFT*, and restrict from making general statements. We closely follow [28, 83, 84].

The Lagrangian of the A_{N-1} Toda CFT is given by

$$L = \frac{1}{8\pi}(\partial_\nu\varphi, \partial^\nu\varphi) + \mu\sum_{k=1}^{N-1} e^{b(e_k,\varphi)}, \tag{2.4.34}$$

where $\varphi := \sum_{i=1}^{N-1}\varphi_i\omega_i$, with e_k, ω_k being the simple roots and the fundamental weights of sl(N) respectively. The definition of the inner product $(\cdot,\cdot)$ along with other useful Lie-algebraic definitions and notations we will need in Chapter 5 are collected in appendix 5.A.The parameter μ is called the *cosmological constant*, in analogy to the Liouville case ($N = 2$) where it determines the constant curvature of a surface described by the classical equation of motion. The normalization of the Lagrangian is chosen in such a way that

$$\varphi_i(z,\bar{z})\,\varphi_j(0,0) = -\delta_{ij}\log|z|^2 + \cdots \quad \text{at } z \to 0. \tag{2.4.35}$$

Following [83, 84], we consider the correlators on a two-sphere, which prescribes putting a background charge at the north pole in order to render the Toda action finite:

$$\varphi(z,\bar{z}) = -\boldsymbol{Q}\log|z| + \cdots \quad \text{at } z \to \infty, \tag{2.4.36}$$

where $\boldsymbol{Q} := Q\rho = (b + b^{-1})\rho$ with the Weyl vector ρ defined in (5.A.3).

Analyzing the path integral of the theory (2.4.34), one can argue that the Toda CFT must have an exchange symmetry $b \leftrightarrow b^{-1}$ on a quantum level which simultaneously sends the cosmological constant to its dual $\tilde{\mu}$, defined as

$$\left(\pi\tilde{\mu}\gamma(b^{-2})\right)^{b} \overset{!}{=} \left(\pi\mu\gamma(b^{2})\right)^{\frac{1}{b}} \Longrightarrow \tilde{\mu} = \frac{\left(\pi\mu\gamma(b^2)\right)^{1/b^2}}{\pi\gamma(1/b^2)}, \tag{2.4.37}$$

where $\gamma(x) := \frac{\Gamma(x)}{\Gamma(1-x)}$. As we mentioned in the introduction, the Toda CFT also has a $\mathbf{W}_N$ higher spin chiral symmetry [85] generated by the fields $W_2 \equiv T, W_3, \ldots, W_N$ of spins $2, \ldots, N$. The primaries under the full symmetry algebra $\mathbf{W}_N \times \overline{\mathbf{W}}_N$ are the exponential fields of spin zero labeled by a weight of sl(N):

$$V_{\boldsymbol{\alpha}} := e^{(\boldsymbol{\alpha},\varphi)}. \tag{2.4.38}$$

In what follows, we will parametrize the fundamental weight decomposition of a weight $\boldsymbol{\alpha}_i$ as

$$\boldsymbol{\alpha}_i = N\sum_{j=1}^{N-1}\alpha_i^j\omega_j. \tag{2.4.39}$$

By looking at the corresponding OPEs, one reads off the central charge c of the Toda CFT and the conformal dimensions $\Delta(\alpha)$ of its primary fields:

$$c = N - 1 + 12\,(\boldsymbol{Q},\boldsymbol{Q}) = (N-1)\left(1 + N(N+1)Q^2\right), \qquad \Delta(\alpha) = \frac{(2\boldsymbol{Q} - \boldsymbol{\alpha}, \boldsymbol{\alpha})}{2}, \tag{2.4.40}$$

with the anti-holomorphic conformal dimensions of the primary fields being equal to the holomorphic ones.

The conformal dimension, as well as the eigenvalues of all the other higher spin currents W_k are invariant under the affine[17] Weyl transformations (5.A.8) of the weights $\boldsymbol{\alpha}_i$, which roughly means that several exponential fields correspond to the same 'physical' field. The primary fields of Toda CFT transform under affine Weyl transformations $\boldsymbol{\alpha} \to \mathsf{w} \circ \boldsymbol{\alpha}$ given in (5.A.8) as

$$V_{\mathsf{w}\circ\alpha} = \mathsf{R}^{\mathsf{w}}(\alpha) V_\alpha \tag{2.4.41}$$

with the reflection amplitude R given by the expression

$$\mathsf{R}^{\mathsf{w}}(\boldsymbol{\alpha}) := \frac{\mathsf{A}(\boldsymbol{\alpha})}{\mathsf{A}(\mathsf{w} \circ \boldsymbol{\alpha})} \tag{2.4.42}$$

in terms of the function

$$\mathsf{A}(\boldsymbol{\alpha}) := \left(\pi\mu\gamma(b^2)\right)^{\frac{(\alpha-Q,\rho)}{b}} \prod_{e>0} \Gamma\left(1 - b\,(\boldsymbol{\alpha} - \boldsymbol{Q}, e)\right) \Gamma\left(-b^{-1}\,(\boldsymbol{\alpha} - \boldsymbol{Q}, e)\right), \tag{2.4.43}$$

where Euler gamma function Γ is defined in Chapter 3.

The two-point correlation functions of primary fields are fixed by conformal invariance and by the normalization (2.4.38). They read

$$\left\langle V_{\alpha_1}(z_1,\bar{z}_1) V_{\alpha_2}(z_2,\bar{z}_2) \right\rangle = \frac{(2\pi)^{N-1}\delta(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 - 2\boldsymbol{Q}) + \text{Weyl-reflections}}{|z_1 - z_2|^{4\Delta(\alpha_1)}}, \tag{2.4.44}$$

where 'Weyl-reflections' stands for additional δ-contributions that come from the field identifications (2.4.41). A Dirac delta-function is present in the right-hand side in agreement with the earlier discussion.

As usual, the coordinate dependence of 3-point functions of primary fields (2.4.38) is fixed by conformal symmetry up to an overall coefficient $C(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)$ called the 3-point structure constant:

$$\left\langle V_{\alpha_1}(z_1,\bar{z}_1) V_{\alpha_2}(z_2,\bar{z}_2) V_{\alpha_3}(z_3,\bar{z}_3) \right\rangle = \frac{C(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}}, \tag{2.4.45}$$

where $z_{ij} := z_i - z_j$ and Δ_i is the (holomorphic) conformal dimension of the primary V_{α_i}. As the spectrum of Toda theory is diagonal, the left and right dimensions are equal here.

Up to now, the CFT machinery has produced expressions only for a restricted subset of three-point functions, as well as for some interesting physical limits of those, see [28, 83, 84] for the state of the art. The formula of Fateev and Litvinov [28] which we will quote in a moment gives the Toda structure constants for a particular *semi-degenerate case* when one of the fields contains a null-vector at level one, implying that the corresponding weight becomes proportional to the first ω_1 or to the last ω_{N-1} fundamental weight of sl(N). Specifically, if one sets[18] $\boldsymbol{\alpha}_1 = N\varkappa\omega_{N-1}$, the structure constants read

$$\begin{aligned} C(N\varkappa\omega_{N-1}, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = &\left(\pi\mu\gamma(b^2) b^{2-2b^2}\right)^{\frac{\left(2Q-\sum_{i=1}^{3}\alpha_i,\rho\right)}{b}} \times \\ &\times \frac{\Upsilon'(0)^{N-1}\Upsilon(N\varkappa)\prod_{e>0}\Upsilon((\boldsymbol{Q}-\boldsymbol{\alpha}_2,e))\Upsilon((\boldsymbol{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^{N}\Upsilon(\varkappa + (\boldsymbol{\alpha}_2 - \boldsymbol{Q}, h_i) + (\boldsymbol{\alpha}_3 - \boldsymbol{Q}, h_j))}, \end{aligned} \tag{2.4.46}$$

[17]One should not confuse the affine Weyl transformation, *i.e.* Weyl reflections accompanied by two translations, with Weyl 'reflections' belonging to the Weyl group of the affine Lie algebra.

[18]We use a slightly different convention than [28]. One has to rescale $\varkappa \to \frac{\varkappa}{N}$ to match the expressions.

where the function Υ is an entire function defined in Chapter 3.

To be able to present the general formula formula for the 3-point functions of Toda primaries in Chapter 5, we will actually need to introduce a deformation of Toda CFT, known as *q-deformed Toda theory*. Albeit no Lagrangian description of the q-deformed version of Toda field theory has been found yet, many quantities of this conjectural deformation are algebraically well-defined, in full analogy to the Toda CFT (see [86] and references therein). While the q-deformed Toda CFTs are vastly unexplored, for the q-deformed Liouville case a bit more is known [87–101]. The building blocks of our proposal are q-deformed functions which reproduce the known limit as $q := e^{-\beta} \to 1$, keep the symmetries and transformation properties as well as the poles and zeros[19] of the undeformed ones. In Toda CFT, dependence on the cosmological constant μ is fully fixed by a Ward identity coming from the path integral formulation. The absence of a path integral formulation for the q-deformed Toda implies that such quantities as structure constants of the theory are ambiguous up to a function of μ, b and q. Due to this, we define the q-deformed structure constants here up to the $\pi\mu\gamma(b^2)$ term, q-deforming only the part respecting the symmetry $b \leftrightarrow b^{-1}$:

$$\begin{aligned} C_q(N\varkappa\omega_{N-1}, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \cong &\left(\frac{(1-q^b)^2(1-q^{b^{-1}})^{2b^2}}{(1-q)^{2(1+b^2)}}\right)^{\frac{\left(2Q-\sum_{i=1}^3 \alpha_i, \rho\right)}{b}} \\ &\times \frac{\Upsilon_q'(0)^{N-1}\Upsilon_q(N\varkappa)\prod_{e>0}\Upsilon_q((Q-\boldsymbol{\alpha}_2, e))\Upsilon_q((Q-\boldsymbol{\alpha}_3, e))}{\prod_{i,j=1}^N \Upsilon_q(\varkappa + (\boldsymbol{\alpha}_2 - Q, h_i) + \left(\boldsymbol{\alpha}_3 - Q, h_j\right))}, \end{aligned} \tag{2.4.47}$$

where the function Υ_q is a q-deformation of Υ, also defined in Chapter 3. To match with the undeformed Toda structure constants in the limit $q \to 1$, one has to set, respectively:

$$C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \xrightarrow{q\to 1} \left(\pi\mu\gamma(b^2)\right)^{-\frac{\left(2Q-\sum_{i=1}^3 \alpha_i, \rho\right)}{b}} C(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)\,. \tag{2.4.48}$$

In Chapter 5, we will reproduce the q-deformed Fateev-Litvinov formula (2.4.47) which then gives the undeformed one (2.4.46) upon taking the limit $q \to 1$ and reintroducing the μ-dependence as in (2.4.48).

This concludes our general discussion of CFT. Summarizing the rather long exposition, by now we have introduced all the relevant entities of conformal field theory and some of its models that we will need further. Having met Dr. Jekyll, let us now look at Mr. Hyde and discuss basics of hypergeometric functions.

[19] To be more precise, the q-deformed functions get a whole tower of zeros/poles from each zero/pole of the undeformed function. The tower is generated by beginning with the undeformed zero/pole and translating it by $r\frac{2\pi i}{\log q} = -r\frac{2\pi i}{\beta}$, where r is a positive integer.

Chapter 3

Functions of hypergeometric type

As we already noted in the Introduction, a history of hypergeometric functions extends back over centuries. A combinatorial and analytical genius of many generations was involved in unraveling new identities and developing the corresponding theory. In the last decades, a unification of mathematical language as well as enormously increased computational resources led to an outburst of new discoveries and insights. The task of giving a more or less concise survey of advances in this area seems therefore hopeless from the start. Instead of trying, we will adopt a different strategy. After a brief motivational part, we will dwell for quite a while on properties of the archetypical Gauss ${}_2F_1$ function and then, opening out, take a short walk through the garden of possible extensions. Two particular species will keep our eye for a slightly longer time: these are Kaneko-Macdonald(-Warnaar) sl(N) basic hypergeometric and Srivastava-Daoust multiple hypergeometric functions. We will list several identities for those and present some formulae/methods which will prove useful in the next Chapters.

3.1 A working definition

Let us start by asking: what is, broadly speaking, a hypergeometric series? As we know, these series, more precisely analytic functions defined by them, generalize elementary functions and introduce many more new ones, the useful special functions of mathematical physics. This answer, being very correct, is nevertheless a bit dull. Indeed, there are many different physical models which are solved by the same functions. Various hierarchies of these models are known, and corresponding solutions can be observed to go to each other in different limits. The above form of an answer doesn't provide the optics fine enough to catch a bigger picture. So, we can challenge ourselves with finding an image having more intrinsic logic, some uniform way to characterize a variety of generalizations and identities from voluminous books on hypergeometric functions.

There are at least two possible routes to follow. To illustrate one of them, let us, for instance, consider Newton's binomial series

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n, \quad |x|<1 \tag{3.1.1}$$

which yields the geometric series in the simple limit $\alpha = -1$. This binomial series can be regarded as a generating function for (generalized) binomial coefficients $\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$ which, as we know, count the number of ways to choose from a set of identical objects. One can directly observe that the interrelations among binomial coefficients as well as information on their growth are encoded in identities and analytical properties of the above function. Let us now see how far we can push this correspondence. To start with, assume that one wants to study the behaviour of more general combinatorial quantities constructed out of binomial coefficients and factorials. These numbers will characterize some more complicated choosing patterns. Then, in the same way one builds generating functions for them. Obviously enough, by these newly constructed functions one sweeps out precisely the domain of classical one-variable hypergeometric functions.[1]

However, it is not the most general combinatorics we can study via generating functions. One can start and go with this toolbox to the four winds. First, one can extend to more than just one counting parameter and consider generating series summing over multiple integer indices. This is what will give rise to so-called multiple hypergeometric functions. [102] One may also need not just to label the objects to count on by positive integers, but by partitions, i.e. Young diagrams, or even by many of them. This is a way of extension pioneered by Macdonald and Kaneko. [103, 104] One shouldn't neccessarily stop here and can keep the pace proceeding to multidimensional analogues of partitions, such as plane partitions, or going on with vectors of such objects... In the end, the discrete sets we take to count on, are in principle restricted only by our interest, motivation and time required to write down complicated expressions.

But there is more than that: the actual combinatorial numbers one counts with a generating function can be refined as well. One can promote them to various rational (and more complicated) functions by including additional information on interlacing and braiding of different object combinations between themselves. One can as well insist on coloring those in different ways, with colors corresponding, for instance, to Cartan generators of some Lie algebra. And again, the number of parameters we refine and the particular ways in which we play with them are, in principle, all up to our inner child. The whole bunch of possibly complicated properties and interrelations of these combinatorial numbers/functions is then again repackaged into behaviour and analytic temper of corresponding hypergeometric functions.

Of course, the details of actual implementing the above prescription in a well-defined way can be very cumbersome. Also, not for all functions we will discuss in the following such a combinatorial interpretation is completely clear and clean. However, there seems to be no obvious reason prohibiting one to proceed further and further to a realm of such extensions.

The second route for motivating existence and use of various hypergeometric generalizations is very much connected to the first. Overdrawing a bit, one can tell that mathematics is very much about a deep duality between numbers and geometry. So, applying this intuition to the present case, we can often regard the above described numbers as counting geometric invariants, thereby 'categorifying' corresponding combinatorial generating functions to, say, characteristic classes of some manifolds. A basic example of such thinking is to observe that periods of elliptic functions actually satisfy a ${}_2F_1$ hypergeometric differential equation. A more advanced example is represented by a Nekrasov instanton partition function which counts free sheaves on blow-ups of certain orbifolds.

Both of the above pictures can give us a feeling for why hypergeometric functions are so much connected to physics. Let us recall that for a CFT, there is a lot of geometrical data encoded via quantum description of such a system (remember the functorial quantization described above), but there is also a lot of combinatorial information, related, for instance, to counting of states on different 'energy' levels of the theory. To get a grip on a few cases where such a connection reveals itself, we will now state some

[1] We do not discuss here subtleties related to convergence of the corresponding generating functions.

essentials of hypergeometric functions.

3.2 Gauss ${}_2F_1$ hypergeometric function

We mostly list in what follows the results we need, without deriving them. Some of them are pretty standard and should be familiar to the reader from the undergraduate mathematics courses. Others are more involved and require certail skill for proving them. Wherever possible, we will always mark the analogy of seemingly more complicated expressions to elementary ones.

Let us define Euler gamma function as a solution to the following functional equation:

$$f(x+1) = xf(x), \qquad\qquad f(1) = 1. \tag{3.2.2}$$

It can be shown that it is given by the following integral ($\operatorname{Re} x > 0$):

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{3.2.3}$$

Sometimes it is also convenient to use a closely related function

$$\gamma(x) := \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{3.2.4}$$

The function $\Gamma(x)$ can actually be analytically continued to $\mathbb{C}\backslash\{\mathbb{Z}_{\leq 0}\}$, the full complex plane excluding non-positive integers. The following well-known properties are often useful:

$$\begin{aligned} \Gamma(x)\Gamma(1-x) &= \frac{\pi}{\sin \pi x} \\ \Gamma(2x) &= \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+\frac{1}{2}). \end{aligned} \tag{3.2.5}$$

The most important formula to remember for the subsequent exposition is the following infinite product:

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)e^{-\frac{x}{k}}, \tag{3.2.6}$$

where γ is an Euler-Mascheroni constant. This representation explicitly shows the poles of gamma function. Mild exponential smearing is introduced in order to regularize the infinite product.

The Euler beta function which we will occasionally use later on is then defined as

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \tag{3.2.7}$$

for $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$ and can be shown to satisfy

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3.2.8}$$

whenever this equation makes sense.

Having defined the Euler functions, let us introduce the (rising) Pochhammer symbol:

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} \equiv a(a+1)\dots(a+n-1), \tag{3.2.9}$$

so that now we have all ingredients ready to introduce the hypergeometric functions.

Despite the title of this Chapter, the ${}_2F_1$ function was introduced by Euler, as a power series expansion of the form[2]

$${}_2F_1\!\left(\begin{matrix} a, & b \\ & c \end{matrix}; x\right) := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}, \quad c \neq 0, -1, -2, \dots \tag{3.2.10}$$

on the unit disc $\{x \in \mathbb{C}, |x| < 1\}$. It is called terminating whenever number of terms is finite and non-terminating otherwise. Gauss was the first to research an analytical continuation of the above series, as well as summarizing and deriving numerous identities for it. In particular, the following integral representation is useful:

$${}_2F_1\!\left(\begin{matrix} a, & b \\ & c \end{matrix}; x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, dt, \tag{3.2.11}$$

which converges for $\operatorname{Re} c > \operatorname{Re} b > 0$, whereas the x plane is assumed to have a cut along the real axis from 1 to ∞, such that $\arg t = \arg(1-t) = 0$ and $(1-xt)^{-a}$ takes its principal value. The branch $|\arg\,(1-x)| < \pi$ is called a principal branch of ${}_2F_1$. It will be assumed in the subsequent discussion. Importantly, the principal branch of

$$\frac{1}{\Gamma(c)}\, {}_2F_1\!\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)$$

is an entire function of a, b and c. The convergence condition can be relaxed by considering Pochhammer double-loop integral.

One of the many misterious facts about hypergeometric functions is that there is a plethora of formulae relating various very different-looking hypergeometric series and integrals. An important subclass of those comprises the cases of summability, when the corresponding hypergeometric function can be reduced just to a ratio of products of gamma functions. Our basic example here is ${}_2F_1(1)$: there is a Gauss formula stating that

$${}_2F_1\!\left(\begin{matrix} a, & b \\ & c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{3.2.12}$$

for $\operatorname{Re}(c-a-b) > 0$. In particular case of terminating series, it reduces to a so-called Chu-Vandermode summation

$${}_2F_1\!\left(\begin{matrix} -n, & b \\ & c \end{matrix}; 1\right) = \frac{(c-b)_n}{(c)_n}, \qquad n \in \mathbb{Z}_{\geq 0}, \;\; c \neq 0, -1, \dots, -n. \tag{3.2.13}$$

We will see in what follows that the higher hypergeometric functions at the point $x = 1$ are generically not summable: only for values of parameters adjusted in a very special way they can be reduced to a simple summed-up form.

[2]Euler assumed a, b and c to be rational numbers.

Many special values are also known for ${}_2F_1$ at points different from $x = 1$. However, this function has enough interesting properties just as an analytic function of a general argument x. As an example, we list here Pfaff and Euler transformation formulae:

$$ {}_2F_1\left(\begin{matrix} a, \ b \\ c \end{matrix}; x\right) = (1-x)^{-a} \, {}_2F_1\left(\begin{matrix} a, \ c-b \\ c \end{matrix}; \frac{x}{x-1}\right) \tag{3.2.14} $$

$$ {}_2F_1\left(\begin{matrix} a, \ b \\ c \end{matrix}; x\right) = (1-x)^{c-a-b} \, {}_2F_1\left(\begin{matrix} c-a, \ c-b \\ c \end{matrix}; x\right) \tag{3.2.15} $$

for $|\arg(1-x)| < \pi$.

Let us mention another very useful integral representation for gamma function called Barnes representation:

$$ {}_2F_1\left(\begin{matrix} a, \ b \\ c \end{matrix}; x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{+i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s \, \frac{ds}{2\pi i} \tag{3.2.16} $$

for $a, b \notin \mathbb{Z}_{\leq 0}$ and $x \neq 0$, where the integration contour mostly goes along the imaginary line, but should be sometimes indented to separate the poles going to real $-\infty$ from those going to real $+\infty$. Such prescription for the contour ensures the correct analytical continuation of ${}_2F_1$ into different regions and is archetypical for so-called *Mellin-Barnes integrals*.

Riemann was the first to undertake a comprehensive study of the hypergeometric function as a solution to the second-order differential equation with three regular singularities. To state the latter in generalizable form, let us introduce a differential operator

$$ \Theta_x := x \frac{\partial}{\partial x}. \tag{3.2.17} $$

The hypergeometric differential equation then reads as

$$ x(\Theta_x + a)(\Theta_x + b) f = \Theta_x(\Theta_x + c - 1) f \tag{3.2.18} $$

or

$$ x(x-1) f'' + ((a+b+1)z - c) f' + ab f = 0, \tag{3.2.19} $$

being more explicit.

Obviously, there are two linearly independent solutions to the hypergeometric equation near each of the branch points $x = 0, 1, \infty$. In a generic situation when none of c, $c-a-b$ and $a-b$ is an integer, they are given simply by Gauss hypergeometric functions multiplied by some powers. For instance, two fundamental solutions around $x = 0$ are given by

$$ f_1(x) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right), \qquad f_2(x) = x^{1-c} {}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix}; x\right). \tag{3.2.20} $$

Whenever any of c, $c-a-b$ and $a-b$ is an integer, such two fundamental solutions become linearly dependent, so the second solution is then given by a certain logarithmically corrected expression. By studying hypergeometric equation, Kummer obtained his famous list of total 24 generic solutions, all expressible through the Gauss hypergeometric function. The connection coefficients allowing one to go between bases of solutions with different monodromy properties are given by particular ratios of gamma functions. This way of characterizing hypergeometric functions through differential equations they satisfy is what we will not use in the following. However, it is very important to keep in mind, especially in connection to classification issues, see also the end of the next section. Many more interesting properties of ${}_2F_1$ function are known which we will not list here. Instead, let us move further along the way described in the previous, introductory section.

A higher one-variable generalization of ${}_2F_1$ is actually straightforward to define as

$$ {}_rF_s\binom{a_1,\ \dots a_r}{b_1,\ \dots b_s}; x\Big) := \sum_{m=0}^{\infty} \frac{(a_1)_m \dots (a_r)_m}{(b_1)_m \dots (b_s)_m} \frac{x^m}{m!}, \tag{3.2.21} $$

which converges absolutely for $r \leq s$ or for $r = s + 1$ when[3] $|x| < 1$ and diverges otherwise, unless $x = 0$. We see that these functions can be characterized by a property that the ratio $P(m) = A(m)/A(m-1)$ of coefficients in their series expansion $\sum_{m=0}^{\infty} A(m)x^m$ is a rational function of m. This observation will help us to generalize hypergeometric setting to a multivariable case. In what follows, we will mostly look at (generalizations of) the above hypergeometric functions with $r = s + 1$.

In a full analogy, it can be noticed that the above functions satisfy general hypergeometric equation

$$ x \prod_{i=1}^{r} (\Theta_x + a_i) f = \Theta_x \prod_{i=1}^{s} (\Theta_x + b_i - 1) f. \tag{3.2.22} $$

The analysis of solutions similar to the one described above can be done.

Similarly, many identities are known for higher one-variable hypergeometric functions [105]. We have no intention to survey them here, in order to avoid overloading of the text with cumbersome expressions. Some of these formulae we will encounter in the exposition below.

We will now trace in more detail two particular footpaths of extending the one-variable hypergeometric functions described above. They will lead us to two interesting kinds of functions which we will then exploit in the following Chapters.

3.3 Srivastava-Daoust hypergeometric functions

To arrive at the Srivastava-Daoust hypergeometric series, let us pursue the way of generalization, first unveiled in the end of XIX century by Appell, Pochhammer, Horn, Goursat, Picard, Lauricella and Kampé de Fériet. We will extend the hypergeometric to include more than one summation, in a straightforward way. Let us start with a double power series $\sum_{m,n=0}^{\infty} A(m,n)x^m y^n$. Recalling one of the above characteristics of a single-variable hypergeometric function, we can call a double series hypergeometric if the ratios $P(m,n) = A(m,n)/A(m,n-1)$ and $Q(m,n) = A(m,n)/A(m-1,n)$ are rational functions in m, n and if the compatibility conditions $P(m,n)\,Q(m-1,n) = P(m,n-1)\,Q(m,n)$ are satisfied. The largest among the degrees of the polynomials making up the rational functions $P(m,n)$ and $Q(m,n)$ (provided the corresponding ratios are irreducible) is called an *order* of a double hypergeometric series.

[3]For $|x| = 1$ some further conditions should be imposed for hypergeometric function ${}_{s+1}F_s$ to be convergent.

Proceeding in this manner and analyzing possible cases, one can, e.g., obtain the renowned *Appell series*[4]

$$F_1\left(\begin{matrix} a;\ b, b' \\ c \end{matrix};\ x, y\right) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad \max(|x|, |y|) < 1,$$

$$F_2\left(\begin{matrix} a;\ b, b' \\ c, c' \end{matrix};\ x, y\right) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad |x| + |y| < 1,$$

$$F_3\left(\begin{matrix} a, a';\ b, b' \\ c \end{matrix};\ x, y\right) := \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad \max(|x|, |y|) < 1,$$

$$F_4\left(\begin{matrix} a;\ b \\ c, c' \end{matrix};\ x, y\right) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad \sqrt{|x|} + \sqrt{|y|} < 1.$$

where neither of parameters c, c' is a non-positive integer, and 10 *Horn series*, with more fancy combinations of summation indices, such as

$$G_3\left(\begin{matrix} a, b \\ - \end{matrix};\ x, y\right) := \sum_{m,n=0}^{\infty} (a)_{2m-n}(b)_{2n-m} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{3.3.23}$$

where the region of convergence becomes somehow difficult to state in a compact form, see [102] for details. It turns out that these 14 series exhaust possible double hypergeometric series of order two which generalize Gauss ${}_2F_1$ function. In a similar manner, one can go on and generalize the whole zoo of higher one-variable hypergeometric functions.[5] The most 'regular' members of the large resulting family, resembling the Appell functions, are captured by a so-called *Kampé de Fériet function*:

$$F^{p:\,q;\,k}_{l:\,m;\,n}\left(\begin{matrix} (a_p) : (b_q);\ (c_k) \\ (\alpha_l) : (\beta_m);\ (\gamma_n) \end{matrix};\ x, y\right) = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r}{r!}\frac{y^s}{s!} \tag{3.3.24}$$

which is convergent when

$$p + q < l + m + 1, \qquad p + k < l + n + 1. \tag{3.3.25}$$

If

$$p + q = l + m + 1, \qquad p + k = l + n + 1, \tag{3.3.26}$$

the convergence additionally requires

$$\begin{cases} |x|^{\frac{1}{p-l}} + |y|^{\frac{1}{p-l}} < 1, & \text{if } p > l \\ \max\ (|x|, |y|) < 1, & \text{if } p \le l. \end{cases} \tag{3.3.27}$$

[4]In a full analogy, the corresponding series may be found to satisfy certain partial differential equations resembling one-variable hypergeometric equation. However, the explicit form of these equations is usually not very enlightening, so that we restrain from writing them out. We will briefly comment on a problem of finding such a description of multiple hypergeometric functions at the end of this section, in relation to GKZ hypergeometric functions.

[5]To drop some names here, there are, for instance, generalizations of ${}_3F_2$ to two variables known as Clausenian double hypergeometric functions [106].

Notice, however, that already the above mentioned Horn functions are generically not from this class. Instead of incorporating them into a more general definition right now, let us first see how far one can proceed by further increasing number of variables.

Perfectly mimicking the above discussion, one can go to three-variable hypergeometric functions, the nicest examples here being so-called three-variable *Lauricella hypergeometric functions* F_A, F_B, F_C and F_D. They are complete analogues of Appell functions for a higher-variable case. For example,

$$F_D\left(\begin{matrix} a;\ b, b', b'' \\ c \end{matrix};\ x, y, z\right) := \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b)_m(b')_n(b'')_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \qquad \max(|x|,|y|,|z|) < 1$$

Again, alike the two-variable case, there are many more Gaussian three-variable hypergeometric functions than just four Lauricella ones. For a full list of 205 functions, including so-called Lauricella-Saran and Srivastava triple hypergeometric functions, see extensive tables in [102].

From the above, it is quite easy to see how one would proceed to higher-dimensional analogues of hypergeometric functions. One of the interesting general definitions yielding most of the mentioned series as special cases is due to Srivastava and Daoust [102]. It reads as:

$$\begin{aligned} &\mathcal{S}^{A:\,B';\dots;\,B^{(n)}}_{C:\,D';\dots;\,D^{(n)}}\left(\begin{matrix} [(a) : \theta', \ \dots, \ \theta^{(n)}\ ] : [(b') : \varphi'] \,; \dots ; \,[(b^{(n)}) : \varphi^{(n)}] \\ [(c) : \psi', \ \dots, \ \psi^{(n)}] : [(d') : \delta'] \,; \dots ; \,[(d^{(n)}) : \delta^{(n)}] \end{matrix};\ x_1, \dots, x_n\right) \\ &= \sum_{m_1,\dots,m_n \geq 0} \frac{\prod_{j=1}^{A} (a_j)_{m_1\theta'_j+\dots+m_n\theta^{(n)}_j} \prod_{j=1}^{B'} (b'_j)_{m_1\varphi'_j} \cdots \prod_{j=1}^{B^{(n)}} (b^{(n)}_j)_{m_n\varphi^{(n)}_j}}{\prod_{j=1}^{C} (c_j)_{m_1\psi'_j+\dots+m_n\psi^{(n)}_j} \prod_{j=1}^{D'} (d'_j)_{m_1\delta'_j} \cdots \prod_{j=1}^{D^{(n)}} (d^{(n)}_j)_{m_n\delta^{(n)}_j}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \end{aligned} \tag{3.3.28}$$

where all the parameters satisfy

$$\theta'_1, \dots, \theta'_A, \dots, \delta^{(n)}_1, \dots, \delta^{(n)}_{D^{(n)}} > 0. \tag{3.3.29}$$

The convergence conditions for this series are also known, but will not be put here. The main new thing with respect to functions defined above is that now any linear combination of summation variables with positive coefficients is allowed to appear in the indices of corresponding Pochhammer symbols. This, of course, significantly increases the number of possible functions. Notice that even in certain cases when some of the parameters in (3.3.29) are negative, a multiple hypergeometric function still can be brought to a Srivastava-Daoust form via linear change of the summation variables. A particular case of the Srivastava-Daoust function we will need is the Srivastava triple hypergeometric function:

$$\begin{aligned} &F^{(3)}\left(\begin{matrix} (a_{k_a}) :: (b_{k_b}); (c_{k_c}); (d_{k_d}) : (e_{k_e}); (f_{k_f}); (g_{k_g}) \\ (a'_{l_{a'}}) :: (b'_{l_{b'}}); (c'_{l_{c'}}); (d'_{l_{d'}}) : (e'_{l_{e'}}); (f'_{l_{f'}}); (g'_{l_{g'}}) \end{matrix};\ x, y, z\right) := \sum_{m,n,p\geq 0} \\ &\times \frac{\prod_{j=1}^{k_a} (a_j)_{m+n+p} \prod_{j=1}^{k_b} (b_j)_{m+n} \prod_{j=1}^{k_c} (c_j)_{n+p} \prod_{j=1}^{k_d} (d_j)_{m+p} \prod_{j=1}^{k_e} (e_j)_m \prod_{j=1}^{k_f} (f_j)_n \prod_{j=1}^{k_g} (g_j)_p}{\prod_{j=1}^{l_{a'}} (a'_j)_{m+n+p} \prod_{j=1}^{l_{b'}} (b'_j)_{m+n} \prod_{j=1}^{l_{c'}} (c'_j)_{n+p} \prod_{j=1}^{l_{d'}} (d'_j)_{m+p} \prod_{j=1}^{l_{e'}} (e'_j)_m \prod_{j=1}^{l_{f'}} (f'_j)_n \prod_{j=1}^{l_{g'}} (g'_j)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \end{aligned} \tag{3.3.30}$$

Plenty of useful methods and tricks exists allowing one to deal with the above functions and to find various identities. To our knowledge however, they unfortunately do not admit a perfect systematics and are to large extent based on a case-by-case study of combinatorial properties. For a thorough survey, see the book [102] as well as numerous subsequent papers citing it.

We will mark here just one particular method to derive identities for multiple hypergeometric functions, called *Burchnall-Chaundy method* [107]. The identities derived by applying it, will be of use in Chapter 6. Let us introduce the following mutually inverse formal operators depending on a parameter h:

$$\nabla_{xy}(h) := \frac{\Gamma(h)\Gamma(\Theta_x + \Theta_y + h)}{\Gamma(\Theta_x + h)\Gamma(\Theta_y + h)} \qquad \Delta_{xy}(h) := \frac{\Gamma(\Theta_x + h)\Gamma(\Theta_y + h)}{\Gamma(h)\Gamma(\Theta_x + \Theta_y + h)} \tag{3.3.31}$$

where Θ_x is defined in (3.2.17). It is easy to see how these operators act on monomials. E.g., for the first one we have:

$$\nabla_{xy}(h)_m(h)_n\, x^m y^n = (h)_{m+n}\, x^m y^n. \tag{3.3.32}$$

One observes that the operator $\nabla_{xy}(h)$ can be regarded as 'gluing' for corresponding Pochhammer symbols in the numerator and 'separating' for corresponding Pochhammer symbols in the denominator of a hypergeometric sum. Vice versa, the operator $\Delta_{xy}(h)$ is a 'separating' one for Pochhammer symbols in the numerator and 'gluing' for Pochhammer symbols in the denominator.

The idea of Burchnall and Chaundy is then to relate two multiple hypergeometric functions by consecutive action of the above operators and then to use the following lemmas, which are instances of well-known hypergeometric formulae:

$$\begin{aligned}
\frac{\Gamma(h)\Gamma(m+n+h)}{\Gamma(m+h)\Gamma(n+h)} &= \sum_{r=0}^{\infty} \frac{(-m)_r(-n)_r}{r!(h)_r} && \text{(Gauss)},\\
\frac{\Gamma(m+h)\Gamma(n+h)}{\Gamma(h)\Gamma(m+n+h)} &= \sum_{r=0}^{\infty} \frac{(-m)_r(-n)_r}{r!(-h-m-n+1)_r} && \text{(Gauss)},\\
&= \sum_{r=0}^{\infty}(-1)^r \frac{(h)_{2r}(-m)_r(-n)_r}{r!(h+r-1)_r(m+h)_r(n+h)_r} && \text{(Dougall)},\\
\frac{\Gamma(h)\Gamma(m+n+h)\Gamma(m+k)\Gamma(n+k)}{\Gamma(m+h)\Gamma(n+h)\Gamma(k)\Gamma(m+n+k)} &= \sum_{r=0}^{\infty} \frac{(k-h)_r(k)_{2r}(-m)_r(-n)_r}{r!(k+r-1)_r(m+k)_r(n+k)_r(h)_r} && \text{(Dougall)},\\
&= \sum_{r=0}^{\infty} \frac{(h-k)_r(-m)_r(-n)_r}{r!(h)_r(-k-m-n+1)_r} && \text{(Saalschütz)}.
\end{aligned} \tag{3.3.33}$$

Notice that for functions of more than two variables additional lemmas might be required [108] which we will not write down here.

This way to act then yields many non-trivial identities. Without spelling out straightforward details, we notice that in particular the following useful decompositions[6] can be obtained:

$$F^{1:\,r;\,u}_{1:\,s;\,v}\left(\begin{matrix}\alpha\,:\,(a_r);\,(c_u)\\ \gamma\,:\,(b_s);\,(d_v)\end{matrix};\,x,y\right)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma-\alpha)_n\prod_{j=1}^{r}(a_j)_n\prod_{j=1}^{u}(c_j)_n}{(\gamma+n-1)_n(\gamma)_{2n}\prod_{j=1}^{s}(b_j)_n\prod_{j=1}^{v}(d_j)_n}\frac{(xy)^n}{n!}$$
$$\times\,{}_{r+1}F_{s+1}\left(\begin{matrix}\alpha+n,(a_r)+n\\ \gamma+2n,(b_s)+n\end{matrix};\,x\right){}_{u+1}F_{v+1}\left(\begin{matrix}\alpha+n,(c_u)+n\\ \gamma+2n,(d_v)+n\end{matrix};\,y\right) \tag{3.3.34}$$

$$F^{1:\,r;\,u}_{0:\,s;\,v}\left(\begin{matrix}\alpha\,:\,(a_r);\,(c_u)\\ -\,:\,(b_s);\,(d_v)\end{matrix};\,x,y\right)=\sum_{n=0}^{\infty}\frac{(\alpha)_n\prod_{j=1}^{r}(a_j)_n\prod_{j=1}^{u}(c_j)_n}{\prod_{j=1}^{s}(b_j)_n\prod_{j=1}^{v}(d_j)_n}\frac{(xy)^n}{n!}$$
$$\times\,{}_{r+1}F_{s+1}\left(\begin{matrix}\alpha+n,(a_r)+n\\ (b_s)+n\end{matrix};\,x\right){}_{u+1}F_{v+1}\left(\begin{matrix}\alpha+n,(c_u)+n\\ (d_v)+n\end{matrix};\,y\right) \tag{3.3.35}$$

$$F^{0:\,r;\,u}_{1:\,s;\,v}\left(\begin{matrix}-\,:\,(a_r);\,(c_u)\\ \gamma\,:\,(b_s);\,(d_v)\end{matrix};\,x,y\right)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{r}(a_j)_n\prod_{j=1}^{u}(c_j)_n}{(\gamma+n-1)_n(\gamma)_{2n}\prod_{j=1}^{s}(b_j)_n\prod_{j=1}^{v}(d_j)_n}\frac{(-xy)^n}{n!}$$
$$\times\,{}_{r+1}F_{s+1}\left(\begin{matrix}(a_r)+n\\ \gamma+2n,(b_s)+n\end{matrix};\,x\right){}_{u+1}F_{v+1}\left(\begin{matrix}(c_u)+n\\ \gamma+2n,(d_v)+n\end{matrix};\,y\right) \tag{3.3.36}$$

provided both sides are well-defined.

In the concluding part of this section we will briefly discuss a classification issue. Actually, it is not so straightforward for the above series, as there are generically many equivalent Srivastava-Daoust representations of one multiple hypergeometric function, related by linear changes of summation variables. So, naturally, one can ask if there exists a description factoring out such redundancy. It turns out that a characterization via corresponding systems of partial differential equations in the most satisfactory in this respect. This is known as Gelfand-Kapranov-Zelevinsky (GKZ) approach [110], or A-hypergeometric functions [111, 112]. In this setting, the multiple hypergeometric functions are characterized through certain matrices which encode the systems of differential equations they solve. To each such matrix one then associates a specific convex polytope (e.g., a square for Gauss ${}_2F_1$), whose symmetries encode symmetries of corresponding function. Many nice properties of GKZ hypergeometric functions are known. For example, around any non-singular point the analytic solution space of A-hypergeometric systems is of finite dimension and for a very important subclass of functions ('non-resonant') the number of solutions is just proportional to Euclidean volume of the polytope. We will, however, not need this nice systematic approach in the next Chapters.

3.4 Kaneko-Macdonald hypergeometric functions

To hit the definition of Kaneko-Macdonald (basic) hypergeometric functions, we should take a different route of generalization, starting again from one-variable hypergeometric functions. This one dates back to Heine, Cauchy, Jackson, Bailey and many others who explored so-called basic, q-deformed hypergeometric functions. We will now notice an interesting pattern: as soon as one defines a proper generalization/deformation of gamma function, yielding a corresponding Pochhammer symbol, and specifies the appropriate monomials to sum on, there is an immediate generalization of a hypergeometric function which can be written down and in many cases can be shown to satisfy familiar one-variable hypergeometric identities.

[6] These decompositions were first obtained in [109] by using the inverse Laplace transform method.

So, let us briefly describe a setting of basic hypergeometric functions, following closely [17]. A receipt is simple: we introduce a dummy variable q which is usually taken to satisfy[7] $|q| < 1$ and then replace every number n by a q-number (also called trigonometric number [113], in different parametrization) $[n]_q := \frac{1-q^n}{1-q}$. In such a way, we, for instance, get for the (rising) q-Pochhammer symbol

$$(a;q)_n = \prod_{k=1}^{n}(1-aq^k). \tag{3.4.37}$$

It can be then continued to negative values of n via

$$(a;q)_n = \frac{1}{(aq^n;q)_{-n}}. \tag{3.4.38}$$

In particular for $n \to \infty$, and for arbitrary number of q's, we have (requiring for convergence that $|q_i| < 1$ for all i)

$$(a;q_1,\ldots,q_r)_\infty := \prod_{i_1=0,\ldots,i_r=0}^{\infty}(1-aq_1^{i_1}\cdots q_r^{i_r}). \tag{3.4.39}$$

We can extend thi definition of the shifted factorial for all values of q_i by imposing the relations

$$(a;q_1,\ldots,q_i^{-1},\ldots,q_r)_\infty = \frac{1}{(aq_i;q_1,\ldots,q_r)_\infty}. \tag{3.4.40}$$

The above is consistent with the following definition for q-deformed gamma function:

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}, \qquad 0 < q < 1, \tag{3.4.41}$$

which can be then analytically continued to other values of q. In analogy to the undeformed case, the q-gamma function can be also defined as satisfying the functional equation

$$f(x+1) = \frac{1-q^x}{1-q}f(x), \qquad\qquad f(1) = 1. \tag{3.4.42}$$

With a little effort, one can see that this definition indeed implies $\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x)$, as it should. The q-analogue of function $\gamma(x)$ is straightforwardly

$$\gamma_q(x) := \frac{\Gamma_q(x)}{\Gamma_q(1-x)} = (1-q)^{1-2x}\frac{(q^{1-x};q)_\infty}{(q^x;q)_\infty}, \tag{3.4.43}$$

valid in this form for $|q| < 1$. The analogue of the infinite product (3.2.6) for $\frac{1}{\Gamma(x)}$ now reads as follows:

$$\frac{1}{\Gamma_q(x)} = (1-q)^{x-1}\prod_{k=1}^{\infty}\frac{1-q^{k-1+x}}{1-q^k}. \tag{3.4.44}$$

[7]Depending on the series regarded, others choices can be made. [17]

One can see here that something interesting happens: unlike the undeformed case, it doesn't require an exponential regularization anymore. The q-deformation provides a natural 'UV-cutoff' by itself. The poles of $\Gamma(x)$ give rise to towers of poles of $\Gamma_q(x)$.

For completeness, let us also add here the analogues of (3.2.5):

$$\Gamma_q(x)\Gamma_q(1-x) = \frac{2iq^{\frac{x}{2}}}{1-q}\frac{\theta_1(i\ln q^{\frac{x}{2}}; q^{\frac{1}{2}})}{\theta_1'(0; q^{\frac{1}{2}})} \tag{3.4.45}$$

$$\Gamma_q(2x) = \frac{(1+q)^{2x-1}}{\Gamma_{q^2}(\frac{1}{2})}\Gamma_{q^2}(x)\Gamma_{q^2}(x+\frac{1}{2}) \tag{3.4.46}$$

where the theta-function θ_1 is defined in (4.B.1).

Now we are ready to define so-called q-deformed (basic) hypergeometric functions. Essentially, by copying and pasting the non-deformed definition[8], we can write:

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1, \ \dots a_{r+1} \\ b_1, \ \dots b_r \end{matrix}; q, x\right) := \sum_{m=0}^{\infty} \frac{(a_1;q)_m \dots (a_{r+1};q)_m}{(b_1;q)_m \dots (b_r;q)_m} \frac{x^m}{(q;q)_m}, \qquad |x| < 1. \tag{3.4.47}$$

It is really stunning when first encountering it how much the theory of q-deformed hypergeometric functions resembles the theory of ordinary ones. The corresponding q-hypergeometric equation can be written down which is now a finite difference equation, as opposed to a differential one. Numerous identities for ordinary hypergeometric functions prove quite easy to lift up to identities for basic ones. For example, an analogue of the binomial theorem[9] (3.1.1) is now a formula due to Cauchy and Heine:

$$ {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, x\right) \equiv \sum_{m=0}^{\infty} \frac{(a;q)_m}{(q;q)_m} x^m = \frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{3.4.48}$$

The notion of Jackson q-integral can be introduced to mimic all the integral representations and interrelations we demonstrated for non-deformed hypergeometric functions. For example, let us state one[10] of q-analogues of Mellin-Barnes representations for ${}_3F_2$

$$ {}_3\phi_2\left(\begin{matrix} ac, bc, ad \\ abcg, acdh \end{matrix}; q, x\right) = \frac{(q, ac, bc, ag, bg, ch; q)_\infty}{(f, {}^{q}/_{f}, {}^{cf}/_{g}, {}^{qg}/_{cf}, abcg, acdh; q)_\infty} \frac{1}{2\pi i} \oint_K \frac{dz}{z} \frac{({}^{fz}/_{g}, {}^{qz}/_{cf}, dhz, {}^{qg}/_{fz}, {}^{cf}/_{z}; q)_\infty}{(az, bz, hz, {}^{c}/_{z}, {}^{g}/_{z}; q)_\infty}, \tag{3.4.49}$$

where the contour K now goes along the unit circle, indented so as to separate the poles going to 0 from those going to ∞. Such prescription for the contour ensures the correct analytical continuation of ${}_3\phi_2$ into different regions and is archetypical for so-called *q-deformed Mellin-Barnes integrals*. We will encounter this prescription when discussing integration contour in the general formula for three-point functions of Toda CFT in Chapter 5.

Instead of q-reformulating now every equation listed in previous sections, let us, before moving further, say a few more words about symmetries of hypergeometric functions, both ordinary and basic. There is a well-known result due to Hardy that the function ${}_3F_2(1)$ multiplied by appropriate gamma-factors

[8]We will not use more general basic hypergeometric functions ${}_r\phi_s$ with $r \neq s+1$, so they are not defined here.

[9]To match the corresponding variables, one needs to change $\alpha \to -a$ and $x \to -x$ in (3.1.1).

[10]There are actually several possible q-deformations of Mellin-Barnes integral, see [17].

actually has an S_5 symmetry, where S_n denotes a group of n-permutations. This is quite an enhancement comparing to an a priori expected trivial $S_3 \times S_2$ symmetry. The corresponding transformations permuting inside the coset $\frac{S_5}{S_3 \times S_2}$ constitute so-called Thomae relations. Let us notice that $S_5 = W(A_4)$, where the latter expression denotes Weyl group of root system A_4. Moreover, by moving away from the point $x = 1$ it can be seen that the corresponding function (to be more precise, the differential equation describing it) actually has a full A_4 Lie symmetry.

It turns out that this result is not a coincidence, being a particular case of something resembling an ADE-classification of hypergeometric functions. The functions participating in it assemble together to a so-called Bailey hierarchy. Important entries of this hierarchy (we list only ordinary and basic hypergeometric functions here) are [114]:

- ${}_9F_8$ and ${}_{10}\phi_9$ (very-well poised), both terminating and non-terminating: possess E_6 symmetry
- ${}_7F_6$ and ${}_8\phi_7$ (very-well poised), non-terminating: possess D_5 symmetry
- ${}_4F_3$ and ${}_4\phi_3$ (balanced), terminating: possess A_5 symmetry

where the terms in round brackets denote various linear restrictions on parameters of corresponding functions. For non-deformed hypergeometric functions these considerations are, of course, related to GKZ hypergeometric functions, mentioned at the end of previous section. The corresponding dual polytopes here are the ones having appropriate discrete symmetry. Hierarchies of hypergeometric functions different from Bailey hierarchy are known [115].

To introduce the further generalization of hypergeometric functions due to Macdonald [103] and Kaneko [104], the one we actually aim for in this section, we need first to take a closer look on a corresponding analogue of gamma function. Let us meditate on the equation (3.2.2). It is very simple. One may wonder what will happen if one regards a sequence of equations obtained via iteratively replacing the multiplier in the right-hand side by the solution of a previous equation. This is what leads to a notion of so-called Barnes multiple gamma functions. For our further purposes, we will just need a second member of this family, called *Barnes double gamma function* $G(x)$.

According to what we just said, this function $G(x)$ is defined as a solution to:

$$f(x+1) = \Gamma(x) f(x), \qquad f(1) = 1. \tag{3.4.50}$$

One of the reasons to call it double is that the functions of this class, roughly speaking, possess a 2d lattice of simple poles, as opposed to a line of simple poles for ordinary $\Gamma(x)$. To see it clearly, let us define a rescaled function via

$$\frac{\Gamma_2(s+\omega_1|\omega_1,\omega_2)}{\Gamma_2(s|\omega_1,\omega_2)} = \frac{\sqrt{2\pi}}{\omega_2^{\frac{s}{\omega_2}-\frac{1}{2}}\Gamma\left(\frac{s}{\omega_2}\right)}, \qquad \frac{\Gamma_2(s+\omega_2|\omega_1,\omega_2)}{\Gamma_2(s|\omega_1,\omega_2)} = \frac{\sqrt{2\pi}}{\omega_1^{\frac{s}{\omega_1}-\frac{1}{2}}\Gamma\left(\frac{s}{\omega_1}\right)}, \tag{3.4.51}$$

where the two complex numbers ω_1 and ω_2 are taken as parameters. In what follows we will often specialize them as $\omega_1 = b$ and $\omega_1 = b^{-1}$, denoting[11] $b + b^{-1} \equiv Q$, or use corresponding Omega-background

[11] In Chapter 4 the letter Q is reserved for charges.

parameters $\epsilon_1 \equiv \omega_1$ and $\epsilon_2 \equiv \omega_2$. It can be shown that the function solving these difference equations is given by a folowing formula (the sum is only well-defined if $\mathrm{Re}\,(t) > 2$):

$$\ln \Gamma_2(s|\omega_1, \omega_2) = \left[\frac{\partial}{\partial t} \sum_{n_1, n_2=0}^{\infty} (s + n_1\omega_1 + n_2\omega_2)^{-t} \right]_{t=0}. \tag{3.4.52}$$

The related function

$$\Gamma_b(x) := \frac{\Gamma_2(x|b, b^{-1})}{\Gamma_2(\frac{Q}{2}|b, b^{-1})}, \tag{3.4.53}$$

normalized by $\Gamma_b(\frac{Q}{2}) = 1$, turns out to be represented by an integral:

$$\ln \Gamma_b(x) = \int_0^{\infty} \frac{dt}{t} \left(\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1 - e^{-tb})(1 - e^{-tb^{-1}})} - \frac{\left(\frac{Q}{2} - x\right)^2}{2} e^{-t} - \frac{\frac{Q}{2} - x}{t} \right). \tag{3.4.54}$$

However, the very function we will need for the following is rather

$$\Upsilon(x) := \frac{1}{\Gamma_b(x)\Gamma_b(Q - x)}. \tag{3.4.55}$$

Letters used in this equation are not coincidental: actually, $\Upsilon(x)$ is a complete ’2d’ analogue of $\frac{1}{\Gamma(x)}$. For instance, an infinite product can be written down which is a two-dimensional lattice counterpart of (3.2.6), also with an appropriate regularization. It shows that $\Upsilon(x)$ is an entire function on the complex plane with zeros at

$$x = -n_1 b - n_2 b^{-1}, \qquad \text{or} \qquad x = (n_1 + 1)b + (n_2 + 1)b^{-1}, \tag{3.4.56}$$

where $n_1, n_2 \in \mathbb{Z}_{\geq 0}$.

Let us dwell a bit more on properties of $\Upsilon(x)$, to persuade ourselves it is a decent analogue of gamma function and to list some identities which will be useful in Chapter 5. It is easy to see that

$$\Upsilon(x) = \Upsilon(Q - x), \qquad \Upsilon\left(\frac{Q}{2}\right) = 1. \tag{3.4.57}$$

Moreover,

$$\Upsilon(x + b) = \gamma(xb)\, b^{1-2bx}\, \Upsilon(x), \qquad \Upsilon(x + b^{-1}) = \gamma(xb^{-1})\, b^{2xb^{-1}-1}\, \Upsilon(x), \tag{3.4.58}$$

from which we see that $\Upsilon(x)$ solves the analogue of Barnes difference equation, with $\Gamma(x)$ replaced by its ’uniformized’ cousine $\gamma(x)$. Using these shift properties, we can obtain:

$$\Upsilon(x + Q) = b^{2(b^{-1}-b)x} \frac{\Gamma(1 + bx)\Gamma(b^{-1}x)}{\Gamma(1 - bx)\Gamma(-b^{-1}x)} \Upsilon(x). \tag{3.4.59}$$

An integral representation

$$\ln \Upsilon(x) = \int_0^{\infty} \frac{dt}{t} \left[\left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\sinh^2 \left[\left(\frac{Q}{2} - x\right)\frac{t}{2}\right]}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right], \qquad 0 < \mathrm{Re}\,(x) < Q \tag{3.4.60}$$

is sometimes useful.

Very much like previously described, the double gamma function, as well as $\Upsilon(x)$, can be q-deformed. For instance, the function $\Upsilon_q(x)$ can be straightforwardly defined via an appropriate deformation of the difference equation. Let us, however, for the sake of variety introduce it here as an infinite product:

$$\Upsilon_q(x|\epsilon_1,\epsilon_2) := (1-q)^{-\frac{1}{\epsilon_1\epsilon_2}\left(x-\frac{\epsilon_+}{2}\right)^2} \prod_{n_1,n_2=0}^{\infty} \frac{(1-q^{x+n_1\epsilon_1+n_2\epsilon_2})(1-q^{\epsilon_+-x+n_1\epsilon_1+n_2\epsilon_2})}{(1-q^{\epsilon_+/2+n_1\epsilon_1+n_2\epsilon_2})^2}. \tag{3.4.61}$$

Like the case of ordinary gamma function, the q-deformation itself provides a sufficient regularization for an infinite product here, and each zero of $\Upsilon_q(x)$ gives rise to a tower of zeros of $\Upsilon(x)$.

If follows from the definition (3.4.67) that $\Upsilon_q(\epsilon_+/2|\epsilon_1,\epsilon_2) = 1$, $\Upsilon_q(x|\epsilon_1,\epsilon_2) = \Upsilon_q(\epsilon_+ - x|\epsilon_1,\epsilon_2)$ and $\Upsilon_q(x|\epsilon_1,\epsilon_2) = \Upsilon_q(x|\epsilon_2,\epsilon_1)$. Furthermore, we can see:

$$\Upsilon_q(x+\epsilon_1|\epsilon_1,\epsilon_2) = \left(\frac{1-q}{1-q^{\epsilon_2}}\right)^{1-2\epsilon_2^{-1}x} \gamma_{q^{\epsilon_2}}(x\epsilon_2^{-1})\Upsilon_q(x|\epsilon_1,\epsilon_2)\,, \tag{3.4.62}$$

together with a similar equation for the shift with ϵ_2. It can be also shown that

$$\Upsilon_q(x|\epsilon_1,\epsilon_2) \xrightarrow{q\to 1} \Upsilon(x|\epsilon_1,\epsilon_2)\,, \tag{3.4.63}$$

as expected. We shall often just write $\Upsilon_q(x)$ instead of $\Upsilon_q(x|\epsilon_1,\epsilon_2)$ and indicate in the text whether the ϵ_i parameters are arbitrary or whether $b = \epsilon_1 = \epsilon_2^{-1}$. One last thing to mention about this function is its derivative at $x = 0$ which is equal to

$$\Upsilon_q'(0) = \frac{\beta}{1-q}\Upsilon_q(b)\,. \tag{3.4.64}$$

Despite all the nice properties, the (deformed) double gamma function/Υ-function is still not exactly something we would use to formulate an appropriate generalization of hypergeometric functions. Indeed, the function $\frac{1}{\Upsilon(x)}$, for instance, has a lattice of poles, whereas the ordinary gamma function has just a line of those. So, in a sense, to be close to the source we would need to 'factorize' the lattice of the poles and take, roughly, a 'square root' of it as a corresponding analogue of gamma function. Let us introduce the following notations

$$\mathfrak{q} = q^{\epsilon_1}\,, \qquad \mathfrak{t} = q^{-\epsilon_2} \tag{3.4.65}$$

and the function

$$\mathcal{M}(U;\mathfrak{t},\mathfrak{q}) := (U\mathfrak{q};\mathfrak{t},\mathfrak{q})_\infty^{-1} = \begin{cases} \prod_{i,j=1}^{\infty}(1-U\mathfrak{t}^{i-1}\mathfrak{q}^j)^{-1} & \text{for } |\mathfrak{t}|<1, |\mathfrak{q}|<1 \\ \prod_{i,j=1}^{\infty}(1-U\mathfrak{t}^{i-1}\mathfrak{q}^{1-j}) & \text{for } |\mathfrak{t}|<1, |\mathfrak{q}|>1 \\ \prod_{i,j=1}^{\infty}(1-U\mathfrak{t}^{-i}\mathfrak{q}^j) & \text{for } |\mathfrak{t}|>1, |\mathfrak{q}|<1 \\ \prod_{i,j=1}^{\infty}(1-U\mathfrak{t}^{-i}\mathfrak{q}^{1-j})^{-1} & \text{for } |\mathfrak{t}|>1, |\mathfrak{q}|>1 \end{cases}, \tag{3.4.66}$$

converging for all U. One can immediately see that this function can be regarded as a 'square root' of Υ_q:

$$\Upsilon_q(x|\epsilon_1,\epsilon_2) = (1-q)^{-\frac{1}{\epsilon_1\epsilon_2}\left(x-\frac{\epsilon_+}{2}\right)^2} \left\| \frac{\mathcal{M}(q^{-x};\mathfrak{t},\mathfrak{q})}{\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}};\mathfrak{t},\mathfrak{q})} \right\|^2, \tag{3.4.67}$$

where the following definition is used for the norm-squared:

$$||f(U_1,\dots,U_r;\mathfrak{t},\mathfrak{q})||^2 := f(U_1,\dots,U_r;\mathfrak{t},\mathfrak{q})f(U_1^{-1},\dots,U_r^{-1};\mathfrak{t}^{-1},\mathfrak{q}^{-1})\,. \tag{3.4.68}$$

A few words on properties of $\mathcal{M}(U;\mathfrak{t},\mathfrak{q})$ are in order. First, it can be written as a plethystic exponential

$$\mathcal{M}(U;\mathfrak{t},\mathfrak{q}) = \exp\left[\sum_{m=1}^{\infty}\frac{U^m}{m}\frac{\mathfrak{q}^m}{(1-\mathfrak{t}^m)(1-\mathfrak{q}^m)}\right], \tag{3.4.69}$$

which converges for all $\mathfrak{t}$ and all $\mathfrak{q}$, provided that $|U| < \mathfrak{q}^{-1+\theta(|\mathfrak{q}|-1)}\mathfrak{t}^{\theta(|\mathfrak{t}|-1)}$, where $\theta(x)$ is the Heaviside step-function. The following identity is obvious from the definition:

$$\mathcal{M}(U;\mathfrak{q},\mathfrak{t}) = \mathcal{M}(U\mathfrak{t}/\mathfrak{q};\mathfrak{t},\mathfrak{q})\,. \tag{3.4.70}$$

From the analytic properties of the shifted factorials (3.4.40), we read the identities:

$$\mathcal{M}(U;\mathfrak{t}^{-1},\mathfrak{q}) = \frac{1}{\mathcal{M}(U\mathfrak{t};\mathfrak{t},\mathfrak{q})}, \qquad \mathcal{M}(U;\mathfrak{t},\mathfrak{q}^{-1}) = \frac{1}{\mathcal{M}(U\mathfrak{q}^{-1};\mathfrak{t},\mathfrak{q})}, \tag{3.4.71}$$

as well as shifting identities:

$$\mathcal{M}(U\mathfrak{t};\mathfrak{t},\mathfrak{q}) = (U\mathfrak{q};\mathfrak{q})_\infty\mathcal{M}(U;\mathfrak{t},\mathfrak{q}), \qquad \mathcal{M}(U\mathfrak{q};\mathfrak{t},\mathfrak{q}) = (U\mathfrak{q};\mathfrak{t})_\infty\mathcal{M}(U;\mathfrak{t},\mathfrak{q})\,. \tag{3.4.72}$$

In Chapter 5 we will use the short-hand notation

$$\Lambda := \left\|\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}};\mathfrak{t},\mathfrak{q}\right)\right\|^2\,. \tag{3.4.73}$$

There is one more identity for a function

$$||M(\mathfrak{t},\mathfrak{q})||^2 := \lim_{U\to 1}\frac{||\mathcal{M}(U;\mathfrak{t},\mathfrak{q})||^2}{1-U^{-1}} = ||\mathcal{M}(\mathfrak{q}^{-1};\mathfrak{t},\mathfrak{q})||^2 = (1-q)^{\frac{(\epsilon_1-\epsilon_2)^2}{4\epsilon_1\epsilon_2}}\Lambda\Upsilon_q(\epsilon_1) \tag{3.4.74}$$

which we will use in Chapter 5. Namely, for $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$:

$$||M(\mathfrak{t},\mathfrak{q})||^2 = \frac{1}{\beta}(1-q)^{\left(\frac{Q}{2}\right)^2}\Lambda\Upsilon_q'(0)\,. \tag{3.4.75}$$

Now, having introduced a more appropriate analogue of gamma function, we are one step closer to defining corresponding hypergeometric function. Not surprisingly, the 'square root' which we took above, is quite a change: it also modifies the actual definition of a corresponding Pochhammer symbol which is the next in our program. Speaking more intuitively, to yield a nice analogue of hypergeometric function, the 'summation pattern' now needs to have more 'degrees of freedom' which remember the old, 'unfactorized' lattice of poles. It turns out that in this case solution is quite simple: the integers we sum over should be replaced via their 'two-dimensional analogues', number partitions or Young diagrams.

Let us list some notations for partitions which we shall use in Chapter 5:

$$|\lambda| := \sum_{i=1}^{\ell(\lambda)}\lambda_i, \qquad ||\lambda||^2 := \sum_{i=1}^{\ell(\lambda)}\lambda_i^2, \qquad n(\lambda) := \sum_{i=1}^{\ell(\lambda)}(i-1)\lambda_i = \frac{||\lambda^t||^2-|\lambda|}{2}, \tag{3.4.76}$$

where $\ell(\lambda)$ is the number of rows of the partition λ. We also define the relative arm-length $a_\mu(s)$, arm-colength $a'_\mu(s)$, leg-length $l_\mu(s)$ and leg-colength $l'_\mu(s)$ of a given box s of the partition λ with respect to another partition μ as:

$$a_\mu(s) := \mu_i - j\,, \qquad a'_\mu(s) := j-1\,, \qquad l_\mu(s) := \mu^t_j - i\,, \quad l'_\mu(s) := i-1\,. \tag{3.4.77}$$

It is, of course, also possible to have $\lambda = \mu$.

The $(\mathfrak{q},\mathfrak{t})$-deformed Pochhammer symbol of U depending on a partition λ is then given as a following product over its boxes:

$$(U;\mathfrak{q},\mathfrak{t})_\lambda := \prod_{i=1}^{\ell(\lambda)} (U\mathfrak{t}^{1-i};\mathfrak{q})_{\lambda_i} = \prod_{s\in\lambda}(1-U\mathfrak{q}^{a'(s)}\mathfrak{t}^{-l'(s)})\,. \tag{3.4.78}$$

The next piece of notation that we need are the $(\mathfrak{q},\mathfrak{t})$-deformations of the hook product of a Young diagram λ, which are deformed analogues of factorials. There are two inequivalent ways for a hook product to be deformed to a two-variable polynomial, and we will use them both. Namely:

$$h_\lambda(\mathfrak{q},\mathfrak{t}) := \prod_{s\in\lambda}(1-\mathfrak{q}^{a(s)}\mathfrak{t}^{l(s)+1})\,, \qquad h'_\lambda(\mathfrak{q},\mathfrak{t}) := \prod_{s\in\lambda}(1-\mathfrak{q}^{a(s)+1}\mathfrak{t}^{l(s)})\,. \tag{3.4.79}$$

The last ingredient to specify are monomials we sum over. Before, it wasn't an issue for us: the basis of monomials x^m indexed by integers (or direct product of those bases) turned out to be always appropriate, even when we defined multiple hypergeometric functions. Clearly, this is not the case now, when we want to sum over partitions. So, what is the replacement? In other words, are there any polynomials, which are labeled by number partitions and orthogonal with respect to some measure? It is well-known that the symmetric polynomials, such as Schur polynomials, give the positive answer: they form the basis in the ring of symmetric functions indexed exactly by number partitions. Even better, there are many different symmetric polynomials, their appearance depending on the numbers/functions we allow as their coefficients. It turns out that, for our present purposes, so-called *Macdonald polynomials* are of the finest design.

The Macdonald polynomials $P_\lambda(\mathsf{x};\mathfrak{q},\mathfrak{t})$, which are referred to in the case of infinite alphabet x as the Macdonald symmetric functions, are labeled by a number partition $\lambda = (\lambda_1,\ldots,\lambda_{\ell(\lambda)})$ and form an especially convenient basis in the ring of symmetric functions of $\mathsf{x} = (x_1, x_2, \ldots)$ over the field $\mathbb{F} = \mathbb{Q}(\mathfrak{q},\mathfrak{t})$ of rational functions in two variables $\mathfrak{q}$ and $\mathfrak{t}$ [116]. These polynomials are defined as unique symmetric polynomials[12] having the expansion

$$P_\lambda(\mathsf{x}) = m_\lambda(\mathsf{x}) + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu(\mathsf{x}). \tag{3.4.80}$$

Here '<' denotes the dominance ordering on partitions and $m_\mu(\mathsf{x})$ denotes the monomial symmetric function, i.e. a sum of all monomials x^β, such that β ranges over all distinct permutations of rows of Young diagram μ.

Macdonald polynomials form an orthogonal basis of symmetric functions with respect to the following inner product

$$\langle f, g\rangle := \frac{1}{n!}\,\mathrm{C.T.}\left(f(\mathsf{x})\,g(\mathsf{x}^{-1})\Delta_{\mathfrak{q}}(\mathsf{x})\right), \tag{3.4.81}$$

[12]In what follows, we only regard Macdonald polynomials of A type.

where

$$\Delta_{\mathfrak{q}}(\mathsf{x}) := \prod_{1\le i<j\le n} \left(\frac{x_i}{x_j};\mathfrak{q}\right)_k \left(\frac{x_j}{x_i};\mathfrak{q}\right)_k \tag{3.4.82}$$

and 'C.T.' stands for constant term[13] of the corresponding expression. The normalization of polynomials with respect to above inner product reads as

$$\langle P_\lambda, P_\lambda\rangle = \frac{h_\lambda(\mathfrak{q},\mathfrak{t})}{h'_\lambda(\mathfrak{q},\mathfrak{t})}\frac{(\mathfrak{t}^n)_\lambda}{(\mathfrak{q}\mathfrak{t}^{n-1})_\lambda}\langle 1,1\rangle. \tag{3.4.83}$$

In the following, we will need a so-called *principal specialization* of a Macdonald polynomial, for which the string of arguments x is set to $\tilde{\mathsf{x}} := z(1,t,\ldots,t^{k-1})$ with $k=\ell(\lambda)$:

$$P_\lambda(\tilde{\mathsf{x}};\mathfrak{q},\mathfrak{t}) = z^{|\lambda|}\mathfrak{t}^{n(\lambda)}\frac{(\mathfrak{t}^k;\mathfrak{q},\mathfrak{t})_\lambda}{h_\lambda(\mathfrak{q},\mathfrak{t})}. \tag{3.4.84}$$

More general evaluation maps exist which we will not use below.

Having introduced all needed ingredients promised above, we now can define the generalization of basic hypergeometric functions due to Kaneko-Macdonald as

$${}_{r+1}\Phi_r\left(\begin{matrix}A_1,\ldots,A_{r+1}\\ B_1,\ldots,B_r\end{matrix};\mathfrak{q},\mathfrak{t};\mathsf{x}\right) := \sum_\lambda \mathfrak{t}^{n(\lambda^{(s)})}\frac{(A_1,\ldots,A_{r+1};\mathfrak{q},\mathfrak{t})_\lambda}{(B_1,\ldots,B_r;\mathfrak{q},\mathfrak{t})_\lambda}\frac{P_\lambda(\mathsf{x};\mathfrak{q},\mathfrak{t})}{h'_\lambda(\mathfrak{q},\mathfrak{t})}. \tag{3.4.85}$$

The functions defined in this way satisfy many identities which are remarkably similar to those for usual one-variable (basic) hypergeometric functions. For simplicity, let us stick to the binomial identity which started this Chapter. It turns out that there is a straightforward analogue of a q-binomial identity[14] (3.4.48) for the Kaneko-Macdonald function ${}_1\Phi_0$ [104]:

$${}_1\Phi_0\left(\begin{matrix}A\\ -\end{matrix};\mathfrak{q},\mathfrak{t};\mathsf{x}\right) = \prod_{s=1}^n \frac{(ax_i;\mathfrak{q})_\infty}{(x_i;\mathfrak{q})_\infty}. \tag{3.4.86}$$

Now we are finally reaching the goal of this section: the *sl*(N) *Kaneko-Macdonald-Warnaar basic hypergeometric functions*. To our knowledge, they were first considered by Warnaar in [117]. It will turn out that certain cases of five-dimensional Nekrasov partition functions in Chapter 5 are naturally described by these hypergeometric generalizations. Specifically, the following sl(N) analogue of above Kaneko-Macdonald basic hypergeometric functions is well-defined:

$$\begin{aligned}
&{}_{r+1}F_r\left(\begin{matrix}A_1,\ldots,A_{r+1}\\ B_1,\ldots,B_r\end{matrix};\mathfrak{q},\mathfrak{t};\mathsf{x}^{(1)},\ldots,\mathsf{x}^{(N-1)}\right) := \\
&\sum_{\lambda^{(1)},\ldots,\lambda^{(N-1)}}{}' \frac{(A_1,\ldots,A_{r+1};\mathfrak{q},\mathfrak{t})_{\lambda^{(N-1)}}}{(\mathfrak{q}\mathfrak{t}^{k_{N-1}-1},B_1,\ldots,B_r;\mathfrak{q},\mathfrak{t})_{\lambda^{(N-1)}}}\prod_{s=1}^{N-1}\left[\mathfrak{t}^{n(\lambda^{(s)})}\frac{(\mathfrak{q}\mathfrak{t}^{k_s-1};\mathfrak{q},\mathfrak{t})_{\lambda^{(s)}}}{h'_{\lambda^{(s)}}(\mathfrak{q},\mathfrak{t})}P_{\lambda^{(s)}}(\mathsf{x}^{(s)};\mathfrak{q},\mathfrak{t})\right] \\
&\times\prod_{s=1}^{N-2}\prod_{i=1}^{k_s}\prod_{j=1}^{k_{s+1}}\frac{(\mathfrak{q}\mathfrak{t}^{j-i-1+k_s-k_{s+1}};\mathfrak{q})_{\lambda_i^{(s)}-\lambda_j^{(s+1)}}}{(\mathfrak{q}\mathfrak{t}^{j-i+k_s-k_{s+1}};\mathfrak{q})_{\lambda_i^{(s)}-\lambda_j^{(s+1)}}},
\end{aligned} \tag{3.4.87}$$

[13]More precisely, it stands for 'constant term' of corresponding expression when specialized as $\mathfrak{t}=\mathfrak{q}^k$. Otherwise, this should be understood as an integral.

[14]All x_i with $i>n$ are assumed to be zero here. The reduction to principally specialized x is straightforward.

where the integer parameters k_s are such that $0 \equiv k_0 < k_1 < k_2 < \cdots < k_{N-1}$ and the summations are performed over partitions $\lambda^{(s)}$, $1 \leq s \leq N-1$ satisfying $k_s \geq \ell(\lambda^{(s)})$. The prime symbol above marks the fact that entries of the partitions giving a non-zero contribution to the sum all satisfy an additional condition $\lambda_i^{(s)} \geq \lambda_{i-k_s+k_{s+1}}^{(s+1)}$ for $1 \leq i \leq k_s$. It provides a convenient visualization of the multiple sum as running over single skew plane partitions of shape $\eta - \nu$, where $\eta = (k_{N-1}^{N-1})$ is a rectangle and $\nu = (k_{N-1} - k_1, \ldots, k_{N-1} - k_{N-2})$.

The corresponding (specialization of the) sl(N) *q-binomial theorem* [[117], Cor. 3.1] is then written as:

$$
{}_1\Phi_0\left(\begin{matrix} A \\ - \end{matrix}; \mathfrak{q}, \mathfrak{t}; \tilde{\mathsf{x}}^{(1)}, \ldots, \tilde{\mathsf{x}}^{(N-1)}\right) = \prod_{s=1}^{N-1} \prod_{i=1}^{k_s - k_{s-1}} \frac{(A z_s \cdots z_{N-1} \mathfrak{t}^{i+s+k_{s-1}+\cdots+k_{N-2}-N}; \mathfrak{q})_\infty}{(z_s \cdots z_{N-1} \mathfrak{t}^{i+s+k_{s-1}+\cdots+k_{N-2}-N}; \mathfrak{q})_\infty}
$$

$$
\times \prod_{1 \leq s \leq r \leq N-2} \prod_{i=1}^{k_s - k_{s-1}} \frac{(\mathfrak{q} z_s \cdots z_r \mathfrak{t}^{i+s-r+k_{s-1}+\cdots+k_r-k_{r+1}-2}; \mathfrak{q})_\infty}{(z_s \cdots z_r \mathfrak{t}^{i+s-r+k_{s-1}+\cdots+k_{r-1}-1}; \mathfrak{q})_\infty}, \tag{3.4.88}
$$

where $\tilde{\mathsf{x}}^{(s)} := z_s(1, \mathfrak{t}, \ldots, \mathfrak{t}^{k_s-1})$ for $1 \leq s \leq N-1$. It is this identity which we will need in Chapter 5. The right-hand side of this formula can be also written using the $\mathcal{M}$ functions introduced above which then constitutes a full analogy of this formula with previously described deformations. Using this identity to prove the one involving five-dimensional Nekrasov partition functions with certain restictions on parameters, is a topic of appendix 5.C.

One last thing to mention in present section is that by putting $\mathfrak{t} = \mathfrak{q}^\alpha$ and going to the limit $\mathfrak{q} \to 1^-$, Macdonald polynomials yield so-called Jack ones [116], another class of symmetric polynomials. Carefully taking the corresponding limit in all λ-factorials defining the above hypergeometric functions, one will obtain the non-deformed Kaneko-Macdonald hypergeometric functions. The relation of these functions to $\Upsilon(x)$ is exactly the same as the relation of basic Kaneko-Macdonald functions to $\Upsilon_q(x)$.

This remark concludes our very incomplete survey of hypergeometric functions. Other special functions we will require are introduced throughout the text and in corresponding appendices. From now on, the exposition of this thesis will go to analysis of concrete physical examples. The discussion will be based on our papers [1–4].

Chapter 4

Spectrum of strange metal CFT

Recently it was suggested in [23] that the study of 1-dimensional QCD with fermions in the adjoint representation could lead to an interesting toy model for strange metals and their holographic formulation. In the high density regime, the infrared physics of this theory is described by a constrained free fermion theory with an emergent $\mathcal{N} = (2,2)$ superconformal symmetry. In this Chapter we systematically study the spectrum of such models, focusing on their chiral ring. We argue that the bosonic part of the superconformal algebra can be extended to a coset chiral algebra of the form $\mathcal{W}_N = \mathrm{SO}(2N^2-2)_1/\mathrm{SU}(N)_{2N}$. In terms of this algebra the spectrum of the low energy theory decomposes into a finite number of sectors which are parametrized by special necklaces. We explicitly construct chiral primaries for a small number $N < 6$ of colors and then analyze the multicolor limit where N is sent to infinity. We shall find that chiral primaries are labeled by partitions in this limit and identify the ring they generate as the ring of Schur polynomials. Our finding imposes strong constraints on the possible dual description through string theory in an AdS_3 compactification. The presentation is based on papers [1,2], joint with I. Kirsch and V. Schomerus.

4.1 Interlude

Low dimensional examples of dualities between conformal field theories and gravitational models in Anti-deSitter (AdS) space provide an area of active research. There are several reasons why such developments are interesting. On the one hand, many low dimensional critical theories can actually be realized in condensed matter systems. As they are often strongly coupled, the AdS/CFT correspondence might provide intriguing new analytic tools to compute relevant physical observables. On the other hand, low dimensional incarnations of the AdS/CFT correspondence might also offer new views on the very working of dualities between conformal field theories and gravitational models in AdS backgrounds. This applies in particular to the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence since there exist many techniques to solve 2-dimensional models directly, without the use of a dual gravitational theory. Recent examples in this direction include the correspondence between certain 2-dimensional coset conformal field theories and higher spin gauge theories [118, 119], see also [120–123] for examples involving supersymmetric conformal field theories and [124, 125] for a more extensive list of the vast literature on the subject. It would clearly be of significant interest to construct new examples of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence which involve full string theories in AdS_3.

In 2012, Gopakumar, Hashimoto, Klebanov, Sachdev and Schoutens [23] studied a two-dimensional adjoint QCD in which massive Dirac fermions Ψ are coupled to an SU(N) gauge field. The fermions were assumed to transform in the adjoint rather than the fundamental representation of the gauge group. In the strongly coupled high density region of the phase space, the corresponding infrared fixed point, called also *strange metal CFT*, is known to develop an $\mathcal{N} = (2,2)$ superconformal symmetry. For gauge groups SU(2) and SU(3) the fixed points possess Virasoro central charge $c_2 = 1$ and $c_3 = 8/3$, respectively. These central charges are smaller than the critical value of $c = 3$ below which one can only have a discrete set of $\mathcal{N} = (2,2)$ superconformal minimal models. Such theories are very well studied. But in order to compare with tree level string theory, one needs to explore the multicolor limit in which N goes to infinity. This regime is much less understood. Note that the central charge $c_N = (N^2 - 1)/3$ of these models grows quadratically with the rank $N - 1$ of the gauge group. While this is very suggestive of a string theory dual, there existed very little further clues on the appropriate choice of the 7-dimensional compactification manifold M^7 of the relevant AdS background.

The most interesting structure inside any $\mathcal{N} = (2,2)$ superconformal field theory is its chiral ring. Recall that the $\mathcal{N} = (2,2)$ superconformal algebra contains a U(1) R-charge Q. The latter provides a lower bound on the conformal weights h in the theory, i.e. physical states ϕ in a unitary superconformal field theory obey the condition $h(\phi) \geq Q(\phi)$. States in the Neveu-Schwarz sector that saturate this bound, i.e. for which $h(\phi) = Q(\phi)$, are called *chiral primaries*. Since chiral primaries are protected by supersymmetry, they are expected to play a key role in discriminating between potential gravitational duals for the infrared fixed point of adjoint QCD. More concretely, the space of chiral primaries in the limit of large N should carry essential information on the compactification manifold M^7 of the dual AdS_3 background.

The objective of this Chapter is a systematic study of the chiral ring for the models proposed by Gopakumar et al. In [23] the partition function of the infrared fixed point was studied for $N = 2, 3$. In these two cases the chiral ring is well understood through the relation with $\mathcal{N} = (2,2)$ minimal models, as we mentioned above. The chiral primaries that are found in these two simple models are special representatives of a larger class of *regular* chiral primaries that can be constructed for all N. But once we leave the territory of minimal models, these do not exhaust the set of chiral primary operators. First, we will push the study of chiral primaries to $N > 3$ and construct all such operators for $N = 4$ and $N = 5$. In both cases, we will find new chiral primary operators that we dubbed *exceptional*. The total number of such exceptional chiral primaries can be shown to grow very rapidly with N.[1] We will then proceed to the analysis of multicolor case to show that the large N limit of the chiral ring receives contributions only from regular chiral primaries. The latter can be counted quite easily: namely, by partitions or Young diagrams. Moreover, their operator product expansions may be argued to agree with the product of Schur polynomials. This provides a complete description of the chiral ring in the large N limit.

Let us briefly discuss the plan of this Chapter. The next section will discuss the model and its symmetries. Namely, we shall describe the low energy theory, identify its chiral algebra, discuss the emergent $\mathcal{N} = (2,2)$ superconformal symmetry, construct the relevant modular invariant partition function and finally look at several particular low-lying examples of such models, explicitly listing all the representations of chiral algebra up to $N = 5$. In section 3 we turn to the main theme of our research in this Chapter, the set of chiral primaries. After explaining some general bounds on their conformal weights we describe the set of regular chiral primaries and study some of their properties. Finally, we construct all additional

[1] In particular, we observed that the number of exceptionals grows faster than the number of regular chiral primaries whose number grows as 2^N. For example, while there is a single exceptional at $N = 4$ along with 7 regulars, the $N = 8$ theory possesses 153 exceptional chiral primaries which outnumber the 125 regulars.

exceptional chiral primaries for $N = 4$ and $N = 5$, which were not known previously. Section 4 contains the second portion of important results. There we shall show that chiral primaries can only contribute in the limit $N \to \infty$ if they are regular. The operator products of regular chiral primaries are discussed in the concluding section along with a few open problems that should be addressed in future studies of the model.

4.2 The model and its symmetries

In this section, we will first review the setup described in [23]. Starting from 2-dimensional adjoint QCD we remind how the low energy description emerges in the limit of large density and strong coupling. Special attention is paid to the chiral symmetries of the theory which are identified at the end of the first subsection. The algebra we construct there is a bit larger than the one that was considered in [23]. In the second subsection we then describe how the state space of the low energy theory decomposes into representations of left- and right-moving chiral algebra. After some comments on an emergent $\mathcal{N} = (2, 2)$ superconformal symmetry and the role of chiral primaries for future studies of AdS duals, we will construct the state space of our model and provide several different ways to think about the pairs (A, a) that label non-trivial branching functions of the chiral algebra. Next, in the third and fourth subsections we will have a closer look on construction of the characters of chiral algebra and identification of the correct modular invariant partition function combining those. The remaining, fifth subsection discusses explicit examples of strange metal cosets up to $N < 6$, to the extent which will be relevant in the following.

4.2.1 Review of the model

The model we start with is a 2-dimensional version of QCD with fermions in the adjoint representations, i.e.

$$\mathcal{L}(\Psi, A) = \text{Tr}\left[\overline{\Psi}(i\gamma^\mu D_\mu - m - \mu\gamma^0)\Psi\right] - \frac{1}{2g_{\text{YM}}^2}\text{Tr}\,F_{\mu\nu}F^{\mu\nu}\ . \tag{4.2.1}$$

Here, A denotes an SU(N) gauge field with field strength F and gauge coupling g_{YM}. The complex Dirac fermions Ψ transform in the adjoint of the gauge group and D_μ denote the associated covariant derivatives. The two real parameters m and μ describe the mass and chemical potential of the fermions, respectively.

We are interested in the strongly coupled high density regime of the theory, i.e. in the regime of very large chemical potential $\mu \gg m$ and g_{YM}. As is well known, we can approximate the excitations near the zero-dimensional Fermi surface by two sets of relativistic fermions, one from each component of the Fermi surface. These are described by the left- and right-moving components of massless Dirac fermions. At strong gauge theory coupling, the resulting (Euclidean) Lagrangian reads

$$\mathcal{L}_{\text{eff}}(\psi, \overline{\psi}, A) = \text{Tr}\left(\overline{\psi}^*\partial\,\overline{\psi} + \psi^*\overline{\partial}\,\psi + A_z[\,\psi^*, \psi\,] + A_{\bar{z}}[\,\overline{\psi}^*, \overline{\psi}\,]\right)\ . \tag{4.2.2}$$

Here we have dropped the term involving the field strength F, using that $g_{\text{YM}} \to \infty$. Upon integrating out the two components A_z and $A_{\bar{z}}$ of the gauge field we obtain the constraints

$$J(z) := [\,\psi^*, \psi\,] \sim 0 \quad , \quad \bar{J}(\bar{z}) := [\,\overline{\psi}^*, \overline{\psi}\,] \sim 0\,. \tag{4.2.3}$$

These constraints are to be implemented on the state space of the N^2-1 components of the complex fermion ψ such that all the modes $J_n, n>0$, of $J(z)=\sum J_n z^{-1-n}$ vanish on physical states, as is familiar from the standard Goddard-Kent-Olive coset construction [126].

In order to describe the chiral symmetry algebra of the resulting conformal field theory we shall start with the unconstrained model, which we refer to as the *numerator* theory. It is based on $M=N^2-1$ complex fermions $\psi_\nu, \nu=1,\dots,M$. These give rise to a Virasoro algebra with central charge $c_\mathrm{N}=N^2-1$, where the subscript N stands for numerator. We can decompose each complex fermion into two real components $\psi^n_\nu, n=1,2$, such that $\psi_\nu=\psi^1_\nu+i\psi^2_\nu$. From time to time we shall combine ν and n into a single index $\alpha=(\nu,n)$. Let us recall that the $2M$ real fermions ψ_α can be used to build $\widehat{\mathrm{SO}}(2M)$ currents $K_{\alpha\beta}$ at level $k=1$. The central charge of the associated Virasoro field coincides with the central charge c_N of the original fermions. The $\mathrm{SO}(2M)_1$ current algebra generated by the modes of $K_{\alpha\beta}$ forms the numerator in the coset construction.

In order to describe the *denominator*, i.e. the algebra generated by the constraints (4.2.3), we need to recall a second way in which our fermions ψ^n_ν give rise to currents. According to the usual constructions, we can employ the representation matrices of the adjoint representation to build two sets of $\mathrm{SU}(N)$ currents at level $k=N$. These currents will be denoted by j^n_ν with $\nu=1,\dots,M$ and $n=1,2$. The currents J that were introduced in eq. (4.2.3) are obtained as $J_\nu=j^1_\nu+j^2_\nu$. The chiral $\mathrm{SU}(N)$ currents J_ν form an affine algebra at level $k=2N$. Through the Sugawara construction we obtain a Virasoro algebra with central charge $c_\mathrm{D}=2(N^2-1)/3$, where the subscript D stands for denominator. Now we have assembled all the elements that are needed in defining the coset chiral algebra

$$\mathcal{W}_N := \mathrm{SO}(2N^2-2)_1/\mathrm{SU}(N)_{2N}\ . \tag{4.2.4}$$

The parameter N keeps track of the gauge group $\mathrm{SU}(N)$. The algebra $\mathcal{W}_N$ is a key element in our subsequent analysis. It is larger than the chiral symmetry considered in [23] which uses the subalgebra $\mathrm{SU}(N)_N\times \mathrm{SU}(N)_N\subset \mathrm{SO}(2N^2-2)_1$ to encode symmetries of the numerator theory.

According to the usual Goddard-Kent-Olive (GKO) construction [126], the chiral algebra $\mathcal{W}_N$ contains a Virasoro field whose central charge is given by the difference of the central charges in the numerator and the denominator,

$$c=c(\mathcal{W}_N)=c_\mathrm{N}-c_\mathrm{D}=N^2-1-\frac{2}{3}(N^2-1)=\frac{1}{3}(N^2-1)\ . \tag{4.2.5}$$

Of course, the coset chiral algebra contains many more fields. To be precise, any element of the numerator algebra that has trivial operator product with respect to the denominator currents makes it into our algebra $\mathcal{W}_N$. In the case at hand, the condition is also satisfied by the U(1) current

$$J(z)=\tfrac{1}{3}\textstyle\sum_{\nu,\mu}\psi^1_\nu(z)\psi^2_\mu(z)\kappa^{\nu\mu} \tag{4.2.6}$$

where $\kappa^{\nu\mu}$ denotes the Killing form of $\mathrm{SU}(N)$.

It was observed in [127] that the conformal symmetry is actually enhanced to a $\mathcal{N}=(2,2)$ superconformal one. This means that the state space admits the action of fermionic generators $G^\pm$ and an additional U(1) current J. While $G^\pm$ are not contained in our chiral algebra $\mathcal{W}_N$, the U(1) current is. In fact, it is precisely the current we found in the previous paragraph. The zero mode of this current turns out to measure the R-charge of fields in $\mathcal{N}=(2,2)$ superconformal low energy limit of 2D adjoint QCD. It will therefore play a very important role in the subsequent analysis.

Let us recall from Chapter 2 that in models with $\mathcal{N}=2$ supersymmetry there is an important subset of fields, namely the (anti-)chiral primaries. By definition, these correspond to states in the Neveu-Schwarz sector of the theory, i.e. with A =*id,v*, such that $h = |Q|$ where Q denotes the U(1) charge and h the conformal weight. As mentioned, chiral primaries have many interesting properties. In particular, they give rise to the so-called chiral ring. In addition the space of chiral primaries is protected under deformations preserving the $\mathcal{N}=(2,2)$ superconformal symmetry. Therefore, it can serve as a 'fingerprint' of our model.

Namely, as we discussed in the prologue, chiral primaries should play an important role when it comes to identifying the AdS dual of the superconformal field theory we are dealing with. AdS duals of 2-dimensional (super-)conformal field theories have recently attracted quite some attention. In the existing examples, the central charge is linear in N and the dual model is a higher spin theory in AdS_3. The case we are dealing with here is different: The central charge (4.2.5) is quadratic in N and hence standard arguments would suggest a richer dual model which is described by a full string theory in AdS_3 rather than a higher spin theory. The identification of this string theory would be significant progress. Clearly, the chiral primaries could play a central role in identifying the string dual.

4.2.2 The state space

Our aim in this subsection is to discuss the state space of the coset model. We shall start by discussing the sectors of the chiral algebra $\mathcal{W}_N$ and describe two relevant labeling systems before concluding with some remarks on counting of the representations. Since the Ramond (R) and Neveu-Schwarz (NS) sector of an $\mathcal{N}=(2,2)$ superconformal field theory are related by spectral flow [81], our discussion will mostly focus on the NS sector.

Let us denote the state space that is created with chiral fields of the numerator theory in the NS sector by $\mathcal{H}^{\text{NS}}$. It consists of two sectors[2], *id* and *v*, and can be constructed from the modes of free fermions. Under the action of the denominator chiral algebra $SU(N)_{2N}$ the space $\mathcal{H}^{\text{NS}}$ decomposes as

$$\mathcal{H}^{\text{NS}} \cong \bigoplus_{a\in\mathcal{J}_N} \mathcal{H}^{\text{C}}_{\{a\}} \otimes \mathcal{H}^{\text{D}}_a \,. \tag{4.2.7}$$

Here, $\mathcal{H}^{\text{D}}_a$ denotes the sectors of the denominator algebra $SU(N)_{2N}$ and $a \in \mathcal{J}_N$ is the corresponding weight.[3] We shall consider $\mathcal{J}_N$ as the set of $N-1$ tuples

$$a = [\lambda_1, \dots, \lambda_{N-1}] \quad \text{with} \quad \sum_s^{N-1} \lambda_s \le 2N \,. \tag{4.2.8}$$

Alternatively, the elements of $\mathcal{J}_N$ may be thought of as SU(N) Young diagrams $Y = Y_a$. Given $a = [\lambda_1, \dots, \lambda_{N-1}]$ the length of the ith row is

$$Y_a = (l_1, \dots, l_{N-1}) \quad \text{is} \quad l_i = \sum_{s=i}^{N-1} \lambda_s \,. \tag{4.2.9}$$

[2]Considering Ramond boundary conditions adds two more, such that possible representations of the numerator algebra belong to the set $A = \{id, v, sp, c\}$.

[3]Due to conventions usually adopted in the corresponding fields, this Chapter will have a minor clash of notations with Chapter 5. Namely, the SU(N) Dynkin labels we denote here by λ_i will be denoted as l_i in the next Chapter. Some further conventions which we will not need discussing strange metals are collected in appendix 5.A.

Of course it is just as easy to reconstruct $a = a(Y)$ from a Young diagram Y. The factor $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ has been introduced to denote sectors of the coset chiral algebra $\mathcal{W}_N$. It will become clear momentarily why we placed the index a in brackets $\{\cdot\}$.

As usual in the coset construction, for $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ not to be empty, the label a must satisfy certain selection rules. In addition, some of the spaces $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ carry equivalent representations of $\mathcal{W}_N$. In order to describe the relevant selection rules and field identifications, we need to introduce the following map γ

$$\gamma([\lambda_1, \dots, \lambda_{N-1}]) = [2N - \sum_{s=1}^{N-1} \lambda_s, \lambda_1, \dots, \lambda_{N-2}] \ . \tag{4.2.10}$$

Obviously, γ maps elements $a \in \mathcal{J}_N$ back into $\mathcal{J}_N$ and it obeys $\gamma^N = id$. One can show that two sectors $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ and $\mathcal{H}^{\mathrm{C}}_{\{b\}}$ of the coset chiral algebra are isomorphic provided that the weights a and b are related to each other by repeated application of γ or, equivalently,

$$\mathcal{H}^{\mathrm{C}}_{\{a\}} \cong \mathcal{H}^{\mathrm{C}}_{\{\gamma(a)\}} \quad \text{for} \quad a \in \mathcal{J}_N \ . \tag{4.2.11}$$

The isomorphism respects the action of the coset chiral algebra $\mathcal{W}_N$ on the sectors $\mathcal{H}^{\mathrm{C}}_{\{a\}}$. In order to state the selection rules we recall that the conformal weight $h^{\mathrm{D}} : \mathcal{J}_N \to \mathbb{R}$ is given by

$$h^{\mathrm{D}}(a) = \frac{C_2(a)}{3N} \ , \tag{4.2.12}$$

where the quadratic Casimir of an SU(N) representation $a = [\lambda_1, \dots, \lambda_{N-1}]$ takes the form

$$C_2(a) = \frac{1}{2}\left[-\frac{n^2}{N} + nN + \sum_{i=1}^{r}(l_i^2 + l_i - 2il_i)\right] .$$

Here, we have used the row length parameters l_i introduced in eq. (4.2.9) and n denotes the total number of boxes $|Y| = n = \sum_i l_i$ in the Young diagram $Y = Y_a$. With these notations let us introduce the so-called monodromy charge

$$Q_\gamma(a) \equiv h^{\mathrm{D}}(\gamma(a)) - h^{\mathrm{D}}(a) \bmod 1 \ . \tag{4.2.13}$$

For the carrier space $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ of the coset algebra to be non-vanishing, the label a should be taken from the set $\mathcal{J}^0_N$ of $\mathrm{SU}(N)_{2N}$ labels a with vanishing monodromy charge $Q_\gamma(a) = 0$,

$$\mathcal{H}^{\mathrm{C}}_{\{a\}} \cong \emptyset \quad \text{if} \quad Q_\gamma(a) \neq 0 \ . \tag{4.2.14}$$

Since representations of the coset chiral algebra $\mathcal{W}_N$ are invariant under the action (4.2.10) of the identification group $\mathbb{Z}_N$, isomorphism classes of representations of the coset chiral algebra are labeled by orbits $\{a\} \in O_N = \mathcal{J}^0_N/\mathbb{Z}_N$.

A very useful way to parametrize elements of $\mathcal{J}^0_N$, i.e. $\mathrm{SU}(N)_{2N}$ weights a with vanishing monodromy charge, is through a pair of Young diagrams Y' and Y''. Namely, we introduce two SU(N) Young diagrams Y' and Y'' with equal number $n' = |Y'| = |Y''|$ of boxes, subject to the additional conditions

$$r' + c'' \leq N \quad , \quad r'' + c' \leq 2N \tag{4.2.15}$$

where r', r'' and c', c'' denote the numbers of rows and columns of Y', Y'', respectively. Let us denote the row lengthes of the Young diagrams Y' and Y'' by

$$Y' = (l'_1, \ldots, l'_{r'}) \quad , \quad Y'' = (l''_1, \ldots, l''_{r''}) .$$

As before, we arrange the l'_i and l''_i in decreasing order, i.e. $l'_i \geq l'_{i+1}$ etc. so that the largest entries are $l'_1 = c'$ and $l''_1 = c''$. From these two Young diagrams we can build a new diagram $Y = Y(Y', Y'') = (l_1, \ldots, l_{N-1})$ through

$$l_i = \begin{cases} r'' + l'_i & \text{for} \quad i = 1, ..., r' \\ r'' & \text{for} \quad i = r' + 1, ..., N - l''_1 \\ r'' - k & \text{for} \quad i = N - l''_k + 1, ..., N - l''_{k+1} , \quad k = 1, ..., r'' - 1 \\ 0 & \text{for} \quad i = N - l''_{r''} + 1, ..., N - 1 . \end{cases} \tag{4.2.16}$$

This prescription extends a construction in [128] and it gives a special family of so-called composite representations $Y = \bar{Y}''Y'$ in the sense of [129]. The latter have been defined without the additional condition $|Y'| = |Y''|$. It is not too difficult to show that all diagrams $Y = Y(Y', Y'')$ obtained in this way correspond to an $\mathrm{SU}(N)_{2N}$ weight $a = a\,(Y) = a\,(Y', Y'')$ with vanishing monodromy charge. Conversely, any such weight arises from a suitably chosen pair (Y', Y'').

The inverse procedure of obtaining diagrams Y' and Y'' from a given diagram $Y = Y_a = (l_1, \ldots, l_{N-1})$ satisfying the zero monodromy charge condition

$$\sum_i l_i \equiv 0 \bmod N, \tag{4.2.17}$$

goes as follows. One defines $r'' := \frac{1}{N} \sum_i l_i$, $c' := l_1 - r''$. Then the entries of the small Young diagrams Y' and Y'' can be written as

$$\begin{cases} Y' := (l_1 - r'', l_2 - r'', \ldots, l_{N-1} - r''), \\ Y''^{\mathrm{T}} := (r'', r'' - l_{N-1}, \ldots, r'' - l_1). \end{cases} \tag{4.2.18}$$

In both expressions the order of entries is non-decreasing and all non-positive entries are to be skipped from the end of these strings.

The prescriptions (4.2.16) and (4.2.18) might appear somewhat heavy at first, but they possess a very simple pictorial representation, see figure 1. Suppose we are given the two Young diagrams Y' and Y''. Then we need to flip Y'' and place it on the bottom line of the image which is N boxes below the top line. This Young diagram has to start in the first column and hence will extend over r'' columns. We now fill all the boxes above the flipped diagram before we attach the second Young diagram Y' on the right-hand side. Conversely, if we are given Y, we must first construct the flipped Y''. It is made from all the boxes that are needed to fill the space below the Young diagram Y, including the N-th row. On the right-hand side, we include as many columns r'' as are needed for the flipped Y'' to possess as many boxes as the Young diagram Y' that appears to the right of the r''th column. This can be done by increasing the number of columns one by one until the appropriate r'' is found. If no appropriate choice of r'' exists, the original Young diagram Y does not correspond to a sector with vanishing monodromy charge.

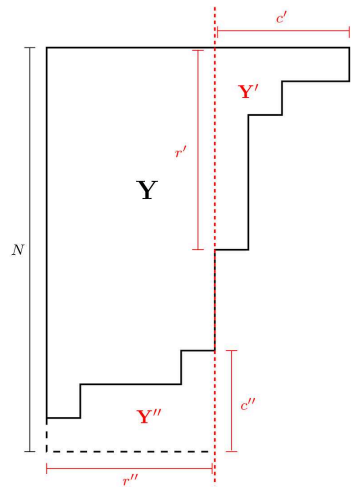


Figure 1: Dissecting the Young diagram Y by the vertical dashed line identifies Young diagrams Y' and Y'' as soon as their numbers of boxes match. This can happen exactly once.

From the above construction, it is easy to see that the number of elements in $\mathcal{J}_N^0$ is given by

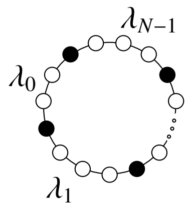
$$|\mathcal{J}_N^0| = \frac{1}{3N} \sum_{n|N} \varphi(N/n) \binom{3n}{n}, \tag{4.2.19}$$

where $\varphi(n)$ is Euler phi function. When $N = 7$, for example, we obtain $|\mathcal{J}_7^0| = 5538$ representations of the affine algebra $\mathrm{SU}(7)_{14}$ with vanishing monodromy charge. Let us now mention two properties of $Y(Y', Y'')$ that become relevant later on. To begin with, it is evident from the geometric construction we described in the previous paragraph that the diagram $Y = Y(Y', Y'')$ possesses $|Y| = n = r''N$ boxes. Furthermore, as we show in the appendix 4.A, the value of the quadratic Casimir in the corresponding representation of $\mathrm{SU}(N)$ is given by

$$C_2(a(Y', Y'')) = C_2(Y(Y', Y'')) = n'N + C_2(Y') - C_2(Y''), \tag{4.2.20}$$

where $C_2(Y')$ and $C_2(Y'')$ are the quadratic Casimirs of the $\mathrm{SU}(N)$ representations associated with Y' and Y'', respectively. Concluding the discussion of the formula (4.2.7), let us remark that even though we wrote that the summation index a is taken out of $\mathcal{J}_N$, we should keep in mind that the summands are trivial unless $a \in \mathcal{J}_N^0$ simply because the corresponding spaces $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ vanish. Hence, we can think of the summation as running over pairs (Y', Y'') of Young diagrams subject to the conditions (4.2.15).

Notice that, while the representation spaces $\mathcal{H}^{\mathrm{D}}_a$ of the denominator current algebra depend on the weight $a = a(Y', Y'')$, the sectors $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ are invariant under the action (4.2.10) of γ and hence only depend on the $\mathbb{Z}_N$ orbit $\{a\} = \{a(Y', Y'')\}$ of a. There is actually another labeling system taking this invariance into account. Namely, the orbits of the identification current possess a nice representation in terms of *necklaces* with N black and $2N$ white beads. A necklace is constructed from the affine Dynkin labels $[\lambda_0, \lambda_1, ..., \lambda_{N-1}]$ of any representation within a given orbit. The additional entry λ_0 of the affine Dynkin label is simply given by $\lambda_0 = 2N - \sum_{i=1}^{N-1} \lambda_i$. Necklaces are direct graphical representations of the $\mathrm{SU}(N)_{2N}$ affine Dynkin labels. The entries of the affine Dynkin label determine the number of white beads which are separated by the black beads, i.e. the structure of a necklace is: λ_0 white beads, black bead, λ_1 white beads, black bead, etc.



A necklace represents the whole orbit $\{a\}$, since the action of the identification current γ corresponds to a rotation of the necklace but not a modification of the necklace itself. The identification of orbits with necklaces enables us to find a simple formula for the number

$$|O_N| = \sum_{N_a|N} \frac{1}{3N_a^2} \sum_{n|N_a} \mu(N_a/n) \binom{3n}{n} \tag{4.2.21}$$

of orbits $O_N = \mathcal{J}_N^0/\mathbb{Z}_N$. Here, μ denotes the classical Möbius function.

Let us stress that the number $4|O_N|$ counts the number of inequivalent branching functions and not the number of representations of our chiral algebra. If we assume that a branching function associated to an orbit of length N_a can be decomposed into characters of N/N_a inequivalent representations, then the number of $\mathcal{W}_N$ sectors is given by

$$|\mathcal{R}_N^{\mathcal{W}}| = 4 \sum_{N_a|N} \frac{N}{N_a} \frac{1}{3N_a^2} \sum_{n|N_a} \mu(N_a/n) \binom{3n}{n} .$$

The factor of 4 in front of the sum stems from the summation over A, running through four sectors *id,v,sp,c*. This formula produces the correct results at least when N is prime.

4.2.3 Characters of the chiral algebra

Having parametrized and counted the orbits of (A, a) we will discuss the associated branching functions and the closely related characters of the chiral algebra $\mathcal{W}_N$ in more detail. By definition, the $\mathcal{N} = 2$ character of a $\mathcal{W}_N$ representation R is obtained through

$$\chi_R^{\mathcal{W}}(q, x) = \mathrm{tr}_R \left(q^{L_0^{\mathcal{W}} - \frac{c}{24}} x^{2J_0} \right) \tag{4.2.22}$$

where $L_0^{\mathcal{W}}$ denotes the zero mode of the coset Virasoro algebra and J_0 is the zero mode of the current (4.2.6). The subscript R refers to the choice of a representation of the chiral algebra $\mathcal{W}_N$.

As usual in coset conformal field theory we can obtain branching functions by decomposing the characters $\chi^{\mathrm{N}}(q, x, z_i)$ of the numerator $\mathrm{N} \equiv \mathrm{SO}(2M)_1$ into characters $\chi^{\mathrm{D}}(q, z_i)$ of the denominator $\mathrm{D} \equiv \mathrm{SU}(N)_{2N}$,

$$\chi_A^{\mathrm{N}}(q, x, z_i) = \mathrm{tr}_A \left(q^{L_0^{\mathrm{N}} - \frac{c}{24}} x^{2J_0} \prod_{i=1}^{N-1} z_i^{H_{i0}} \right) = \sum_{A,a} \mathcal{X}_{(A,a)}^{\mathcal{W}}(q, x) \chi_a^{\mathrm{D}}(q, z_i) . \tag{4.2.23}$$

Here we have twisted the characters of the numerator free fermion model with the zero modes H_{i0} of the Cartan currents of D in Chevalley basis (constructed from fermions). By the very definition of the coset chiral algebra $\mathcal{W}_N$ this implies that states of the coset algebra possess vanishing H_{i0} charge. In other words, all the dependence on the variables z_i on the right-hand side of the previous equation is contained in the characters χ^{D} of the denominator algebra $\mathrm{SU}(N)_{2N}$. The summation in eq. (4.2.23) runs over representations a of the denominator algebra, i.e. over weights $a = [\lambda_1, \ldots, \lambda_{N-1}]$ of SU(N) subject to the condition $\sum_{s=1}^{N-1} \lambda_s \leq 2N$. Note that the generator of the latter carry no charge with respect to the U(1) current $J(z)$ so that the corresponding characters are independent of x. The label A runs through the four sectors $A = id, v, sp, c$ of the SO($2M$) current algebra at level $k = 1$.

As we already noted when discussing the coset state space, the branching functions of eq. (4.2.23),

$$\mathcal{X}^{\mathcal{W}}_{(A,a)}(q, x) = 0 \qquad \text{if } Q_\gamma(a) \neq 0 \,, \tag{4.2.24}$$

$$\mathcal{X}^{\mathcal{W}}_{(A,a)}(q, x) = \mathcal{X}^{\mathcal{W}}_{(B,b)}(q, x) \qquad \text{if } B = A \,,\ b = \gamma^n(a) \tag{4.2.25}$$

for some choice of n. Using these two properties, we can rewrite eq. (4.2.23) in the form

$$\chi^{\mathrm{N}}_A(q, x, z_i) = \sum_{\{a\}, Q_\gamma(a)=0} \mathcal{X}^{\mathcal{W}}_{(A,a)}(q, x) S_{\{a\}}(q, z_i) \tag{4.2.26}$$

where the sum extends over orbits $\{a\}$ of denominator labels under the identification current γ whose monodromy charge vanishes and we defined

$$S_{\{a\}}(q, z_i) = \sum_{b \in \{a\}} \chi^{\mathrm{D}}_b(q, z_i) \,. \tag{4.2.27}$$

In order to progress, we must now insert explicit formulae for the various characters. The functions on the left-hand side of eq. (4.2.26) are actually very easy to construct from the free fermion representation which gives

$$\begin{aligned}
&\chi^{\mathrm{N}}_{id}(q, x, z_i) \pm \chi^{\mathrm{N}}_{v}(q, x, z_i) \\
&\quad = \prod_{X \in \mathrm{SU}(N)} \left(q^{-1/24} \prod_{n=1}^{\infty} (1 \pm x^{1/3} z^{\alpha(X)} q^{n-1/2})(1 \pm x^{-1/3} z^{\alpha(X)} q^{n-1/2}) \right) ,
\end{aligned} \tag{4.2.28}$$

$$\begin{aligned}
&\chi^{\mathrm{N}}_{sp}(q, x, z_i) \pm \chi^{\mathrm{N}}_{c}(q, x, z_i) \\
&\quad = \prod_{X \in \mathrm{SU}(N)} \left(q^{1/12} x^{1/6} \prod_{n=1}^{\infty} (1 \pm x^{1/3} z^{\alpha(X)} q^{n})(1 \pm x^{-1/3} z^{\alpha(X)} q^{n-1}) \right) ,
\end{aligned} \tag{4.2.29}$$

where

$$z^{\alpha(X)} \equiv z_1^{\alpha_1(X)} z_2^{\alpha_2(X)} \cdots z_{N-1}^{\alpha_{N-1}(X)} \tag{4.2.30}$$

and $\alpha(X) = (\alpha_1(X), ..., \alpha_{N-1}(X))$ is a root vector of SU(N). Of course, we can obtain explicit formulae for the characters $\chi^{\mathrm{N}}_A, A = id, v$ by taking the sum and difference of the expressions in the first line.

Characters of the denominator algebra $\mathrm{SU}(N)_{2N}$ are a little bit more complicated but of course also well known. In terms of the string functions $c^b_\lambda(q)$ of the denominator theory, the characters can be written as

$$\chi_b^{\mathrm{D}}(q,z_i) = \sum_{\lambda\in P/kM} c_\lambda^b(q)\Theta_\lambda(q,z_i)\,, \tag{4.2.31}$$

$$\Theta_\lambda(q,z_i) = \sum_{\beta^\vee\in M^\vee} q^{\frac{k}{2}|\beta^\vee+\lambda/k|^2}\prod_{i=1}^{N-1} z_i^{k(\beta^\vee+\lambda/k,\alpha_i^\vee)}\,, \tag{4.2.32}$$

where P and M ($M^\vee$) denote the weight and (co)root lattice of SU(N), respectively, and $\alpha_i^\vee$ are the simple coroots of SU(N). By comparing the q-expansion of the right-hand side of equation (4.2.26) with that of expressions (4.2.28, 4.2.29), we find the x-dependence of the branching functions order by order in x and q.

We have worked out the first few terms in the expansion of the $\mathcal{W}_N$ characters for all representations with $N \leq 5$. The results will be sketched in a while, at least to the extent to which they are needed later on.[4] Before coming to this, let us briefly discuss the modular invariant partition function of our model.

4.2.4 Modular invariant partition function

The decomposition (4.2.7) is just used to build the representations $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ of the coset chiral algebra $\mathcal{W}_N$ but it does not tell us yet how these sectors are combined with those of the right moving chiral algebra in order to build a fully consistent conformal field theory. The aim of this subsection is to explain how the characters we introduced are put together in order to construct the partition function of the coset model.

We shall begin with a few simple comments on the numerator theory. As we reviewed above, its state space carries the action of a chiral SO($2M$) algebra at level $k = 1$. This current algebra possesses four sectors which are denoted by *id,v,sp* and c, respectively. When decomposed into the associated characters, the partition function takes the form

$$Z^{\mathrm{N}}(q,\bar q) = |\chi^{\mathrm{N}}_{id}|^2 + |\chi^{\mathrm{N}}_{v}|^2 + |\chi^{\mathrm{N}}_{sp}|^2 + |\chi^{\mathrm{N}}_{c}|^2 \;=\; M^{\mathrm{N}}_{AB}\chi^{\mathrm{N}}_A(q)\bar\chi^{\mathrm{N}}_B(\bar q)\,. \tag{4.2.33}$$

The labels A, B on the right-hand side run through $A, B = id,v,sp$ and c and M^{N}_{AB} are integers which are defined through the expression on the left-hand side. Explicitly, these integers are given by $M^{\mathrm{N}}_{AB} = \mathrm{diag}(1,1,1,1)$.

Now we need to describe a similar set of integers M^{D}_{ab} for the denominator theory. This is obtained from the D-type modular invariant partition function for the SU(N)$_{2N}$ Wess-Zumino-Witten model, reading as

$$Z^{\mathrm{D}}(q,\bar q) \;=\; \sum_{\{a\};Q_\gamma(a)\equiv 0}\frac{N}{N_a}|\sum_{b\in\{a\}}\chi_b|^2 \;=\; \sum_{ab} M^{\mathrm{D}}_{ab}\chi_a^{\mathrm{SU}(N)_{2N}}(q)\bar\chi_b^{\mathrm{SU}(N)_{2N}}(\bar q)\,. \tag{4.2.34}$$

The first summation is over orbits $\{a\}$ of weights for the affine SU(N)$_{2N}$ under the action (4.2.10) of the identification current γ. The length of a generic orbit agrees with the size N of the gauge group SU(N). Some orbits $\{a\}$, however, possess fixed points so that their length $N_a = N_{\{a\}}$ can be a nontrivial divisor of N. For more details on simple current modular invariants see [130].

[4] More details can be found in appendix 4.C.

From eq. (4.2.34) we can read off the integer coefficients M^{D} of the decomposition [131],

$$M^{\mathrm{D}}_{ab} = \begin{cases} \sum_{p=1}^{N} \delta_{a,\sigma^p(b)} & \text{if } t(a) = 0 \mod N \\ 0 & \text{otherwise} \end{cases}, \tag{4.2.35}$$

where $t(a) = \sum_{s=1}^{N-1} s\,\lambda_s$ is the N-ality of an $\mathrm{SU}(N)_{2N}$ representation $a = [\lambda_1, ..., \lambda_{N-1}]$ and $\sigma^r(a) = 2N\omega_r + c^r(a), r = 1, ..., N$, are associated with group automorphisms of $\mathrm{SU}(N)_{2N}$. They are defined in terms of the fundamental weights ω_r and the Coxeter rotations $c^r(\omega_i) = \omega_{i+r} - \omega_r$, see [131, 132] for more details and [133, 134] for M^{D} at $N = 2, 3$. Note that the N-ality constraint coincides with the condition of vanishing monodromy charge that was built into eq. (4.2.34).

We are now prepared to construct a modular invariant that is associated with our coset model. In fact, following the standard procedures in coset conformal field theory we are led to consider

$$\tilde{Z}_N(q, \bar{q}) = \frac{1}{N^2} \sum_{ABab} M^{\mathrm{D}}_{ab} M^{\mathrm{N}}_{AB} \mathcal{X}^{\mathcal{W}}_{(A,a)}(q) \bar{\mathcal{X}}^{\mathcal{W}}_{(B,b)}(\bar{q}) \,. \tag{4.2.36}$$

The summation runs over the same range as in eqs. (4.2.33) and (4.2.34), where $\mathcal{X}$ are branching functions we introduced before. For most values of the label (A, a), the branching function $\mathcal{X}$ is a character χ of an irreducible representation of $\mathcal{W}_N$. More precisely, one finds that

$$\mathcal{X}^{\mathcal{W}}_{(A,a)}(q) = \chi^{\mathcal{W}}_{(A,a)}(q) \quad \text{when} \quad N_a = N \,, \tag{4.2.37}$$

i.e. when the orbit $\{a\}$ of a under the action of $\gamma \in \mathbb{Z}_N$ consists of N elements. The orbit $\{0\}$ of the vacuum representation is always such a long one. Consequently, in order for the vacuum to contribute with unit multiplicity, we had to divide the sum in eq. (4.2.36) by N^2. But this is a dangerous division. In order to see the problem, let us insert M^{N}_{AB} and M^{D}_{ab} as in eqs. (4.2.33) and (4.2.34), respectively. Then our modular invariant (4.2.36) reads

$$\tilde{Z}_N(q, \bar{q}) = \sum_{A} \sum_{\{a\}, Q_\gamma(a)=0} \frac{N_a}{N} |\mathcal{X}^{\mathcal{W}}_{(A,a)}(q)|^2 \,. \tag{4.2.38}$$

Here, we sum over orbits $\{a\}$ instead of $\mathrm{SU}(N)$ representations a with vanishing monodromy charge. For short orbits we have $N_a < N$ so that the corresponding branching functions are divided by a non-trivial integer. Typically, one finds that these fractions are not compensated by corresponding multiplicities in the branching functions so that the modular invariant (4.2.36) possesses non-integer coefficients. This problem is of course well known and may be overcome by a process known as fixed point resolution, see [130,135]. In the case of short orbits, i.e. when $N_a \neq N$, the branching function $\mathcal{X}^{\mathcal{W}}_{(A,a)}$ turns out to decompose into a sum of $\mathcal{W}_N$ characters $\chi^{\mathcal{W}}_{(A,a,m)}$ for irreducibles labeled by m. General formulae for such decompositions exist only for some coset chiral algebras, see e.g. [135]. Experience shows that the characters $\chi^{\mathcal{W}}_{(A,a,m)}$ can be used as building blocks for modular invariants Z^{res}_N such that

$$Z_N(q) = \tilde{Z}_N(q) + Z^{\mathrm{res}}_N(q) \tag{4.2.39}$$

has integer coefficients only. Z_N can therefore be interpreted as the partition function of the system.

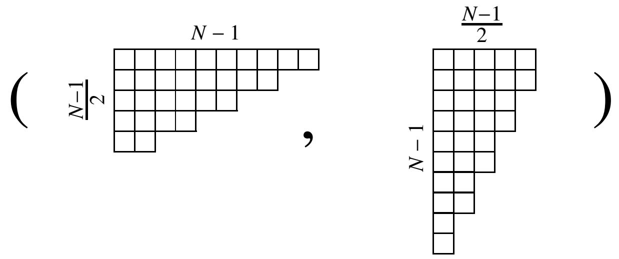


Figure 2: The pair $(Y'(a_*), Y''(a_*))$ corresponding to $a_* = [2, 2, \ldots, 2]$, N assumed odd.

To spell out details, we shall mostly assume that N is a prime number. With this assumption, all but one of the sectors $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ carry irreducible representations of the chiral algebra. Only the sector $\mathcal{H}^{\mathrm{C}}_{\{a_*\}}$ for $a_* = [2, 2, \ldots, 2]$ can be decomposed into several irreducibles $\mathcal{H}^{\mathrm{C}}_{\{a_*\};m}$, where $m = 1, \ldots, N$. The full state space of the conformal field theory then takes the form

$$\mathscr{H}^{\mathrm{C}} = \frac{1}{N} {\bigoplus_{Y',Y''}}' \mathcal{H}^{\mathrm{C}}_{\{a(Y',Y'')\}} \otimes \overline{\mathcal{H}}^{\mathrm{C}}_{\{a(Y',Y'')\}} \;\oplus\; \mathscr{H}^{\mathrm{C}}_{\mathrm{fix}} \,. \tag{4.2.40}$$

As should be clear from above discussion, we sum over all pairs (Y', Y'') of Young diagrams that obey the conditions (4.2.15) with the exception of the unique pair that gives the Young diagram $Y = Y_{a_*}$, see figure 2. The term $\mathscr{H}^{\mathrm{C}}_{\mathrm{fix}}$ is built out of the sectors $\mathcal{H}^{\mathrm{C}}_{\{a_*\};m}$ and their right moving counterparts.

To write down explicitly characters of the $\mathcal{W}_N$ representations belonging to the fixed point sector, we follow a recipe first described in [130] and define

$$\chi^{\mathcal{W}}_{(A,a_*,m)}(q) = \frac{1}{N}\left(\chi^{\mathcal{W}}_{(A,a_*)}(q) + d_{(A,m)}\right) , \tag{4.2.41}$$

$$\text{where} \quad d_{(A,m)} \in \mathbb{Z} \quad \text{with} \quad \sum_{m=1}^{N} d_{(A,m)} = 0 \,, \tag{4.2.42}$$

where A runs through its four possible values, as usual. Note that the proposed characters indeed sum up to the branching functions. For $N > 2$ we propose the following values for $d_{(A,m)}$:[5]

$$\begin{aligned}
d_{(id,1)} &= N-1\,, & d_{(id,p)} &= -1\,,\\
d_{(v,1)} &= 0\,, & d_{(v,p)} &= 0\,,\\
d_{(sp,1)} &= -\frac{(N-1)^2}{2}\,, & d_{(sp,p)} &= \frac{N-1}{2}\,,\\
d_{(c,1)} &= \frac{N^2-1}{2}\,, & d_{(c,p)} &= -\frac{N+1}{2}\,,
\end{aligned} \tag{4.2.43}$$

$p = 2, ..., N$. Given these characters we can now construct Z^{res} through

$$Z^{\mathrm{res}}_N(q) \;=\; \frac{1}{N^2}\sum_A \sum_{m=1}^{N} d^2_{(A,m)} = \frac{N^2+3}{2}\frac{N-1}{N} \tag{4.2.44}$$

[5]For $N = 2$, we have $d_{(sp,1)} = -1$, $d_{(sp,2)} = 1$ and $d_{(c,1)} = 1$, $d_{(c,2)} = -1$, all others are zero.

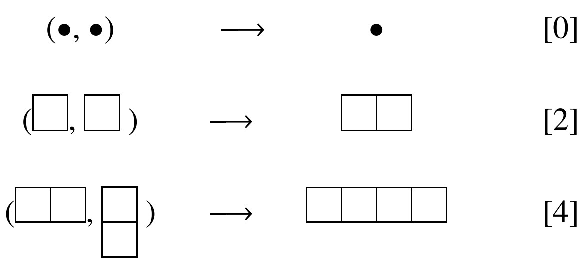


Figure 3: Pairs of Young diagrams (Y', Y'') inducing $Y \in \mathcal{J}_N^0$ for $N = 2$.

for N prime. Since Z^{res} is a constant, it is obviously modular invariant. In addition, if we add this term to the modular invariant $\tilde{Z}_N$ we obtain an expression in which squares of characters are summed with integer coefficients,

$$Z_N(q) = \sum_A \sum_{\substack{\{a\}, a \neq a_* \\ Q(a)=0}} |\chi^{\mathcal{W}}_{(A,a)}(q)|^2 + \sum_A \sum_{m=1}^{N} |\chi^{\mathcal{W}}_{(A,a_*,m)}(q)|^2 \,. \tag{4.2.45}$$

Here, the first summation is over all orbits of length N with vanishing monodromy charge. Of course our assumption that N be prime is crucial for the validity of the expression (4.2.45) for the partition function of our model.

4.2.5 Examples with small N

In order to illustrate the constructions we outlined above and to prepare for our search of chiral primaries, we want to work out some explicit results with $N \leq 5$. Let us recall that the central charge of the models with $N = 2$ and $N = 3$ satisfies $c_N < 3$ so that these two models are part of the minimal series of $\mathcal{N} = (2, 2)$ superconformal theories. The other two cases, $N = 4$ and $N = 5$, however, are outside this range and hence our results here are new.

N = 2

For $N = 2$, there are $|\mathcal{J}_2^0| = 3$ representations of $SU(2)_4$ with vanishing monodromy charge. Such representations can be constructed from pairs of Young diagrams (Y', Y'') by eq. (4.2.16), as shown in figure 3. Under the action of $\gamma \in \mathbb{Z}_2$ these representations form two orbits. The first one is long, i.e. $N_{\{0\}} = N = 2$ and it consists of $\{[0], [4]\}$. There is a second orbit of length $N_{\{2\}} = 1$ which is given by $\{[2]\}$.

In the case at hand, it is actually possible to derive explicit expressions for the branching functions $\mathcal{X}^{W}$

from the general decomposition formula (4.2.26), see appendix 4.B,

$$
\begin{aligned}
\chi^{\mathcal{W}}_{(id\pm v,[0])}(q,x) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(\pm 1)^n q^{\frac{3}{2}n^2} x^n\,,\\
\chi^{\mathcal{W}}_{(id\pm v,[2])}(q,x) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(\pm 1)^{n+1} q^{\frac{3}{2}\left(n+\frac{1}{3}\right)^2}\left(x^{n+\frac{1}{3}}+x^{-(n+\frac{1}{3})}\right),\\
\chi^{\mathcal{W}}_{(sp\pm c,[0])}(q,x) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(\pm 1)^n q^{\frac{3}{2}\left(n+\frac{1}{2}\right)^2}\,,\\
\chi^{\mathcal{W}}_{(sp\pm c,[2])}(q,x) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(\pm 1)^{n+1} q^{\frac{3}{2}\left(n+\frac{1}{6}\right)^2}\left(x^{n+\frac{1}{6}}\pm x^{-(n+\frac{1}{6})}\right).
\end{aligned}
\tag{4.2.46}
$$

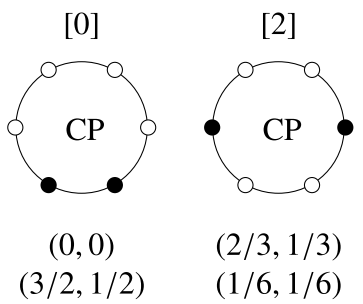
From these expressions we can read off the conformal weights of the ground states in all 8 sectors. Similarly, we can also determine the maximal value Q the U(1) charge can assume among the ground states of these sectors. In particular there are two sectors with A=id. The sector $(id,[0])$ is the vacuum sector with $h=0$ and $Q=0$.

As discussed at the end of subsection 4.2.2, the branching functions associated with the fixed points $(A,a_*)=(A,[2])$ can be decomposed into two characters of our algebra $\mathcal{W}_2$. For $A=id$ and v, for example, these characters read

$$
\begin{aligned}
\chi^{\mathcal{W}}_{(id,[2],1)} &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{6\left(n+\frac{1}{3}\right)^2} x^{2(n+\frac{1}{3})}\,,\\
\chi^{\mathcal{W}}_{(id,[2],2)} &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{6\left(n+\frac{1}{3}\right)^2} x^{-2(n+\frac{1}{3})}\,,\\
\chi^{\mathcal{W}}_{(v,[2],1)} &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{6\left(n+\frac{1}{6}\right)^2} x^{2(n+\frac{1}{6})}\,,\\
\chi^{\mathcal{W}}_{(v,[2],2)} &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{6\left(n+\frac{1}{6}\right)^2} x^{-2(n+\frac{1}{6})}\,.
\end{aligned}
\tag{4.2.47}
$$

It is easy to check that these formulae agree with the expressions (4.2.41) and (4.2.42) when $x=1$.

Let us also display the necklace patterns for the two orbits $\{[0],[4]\}$ and $\{[2]\}$. The affine Dynkin labels for these orbits are $[4,0]$ (or $[0,4]$) and $[2,2]$. These correspond to the following two necklaces,



In the two lines below the necklace we use tuples (h,Q) to display the ground state energy h and maximal U(1) charge Q among the ground states of the sectors with A=id (first line) and A=v (second line). In principle, there are also two sectors with A=*sp, c* which we do not show here. The label 'CP' we placed in the center of the two necklaces will be explained in the next section.

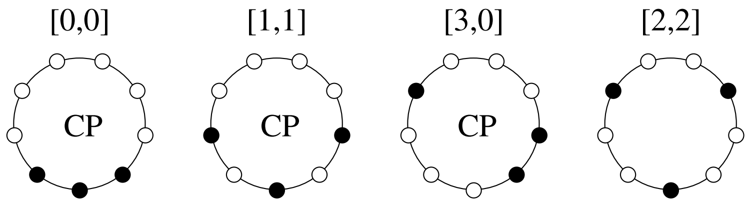
Let us finally also spell out the full partition function of the model. In the case at hand, our general expression (4.2.45) reads as follows

$$\begin{aligned} Z_2 &= \sum_A \left(|\chi^{\mathcal{W}}_{(A,[0])}|^2 + |\chi^{\mathcal{W}}_{(A,[2],1)}|^2 + |\chi^{\mathcal{W}}_{(A,[2],2)}|^2 \right) \\ &= \frac{1}{|\eta(q)|^2} \sum_{n,w\in\mathbb{Z}} q^{\frac{k_L^2}{2}} \bar{q}^{\frac{k_R^2}{2}} x^{rk_L} \bar{x}^{rk_R} \quad \text{with} \quad k_{L,R} = \frac{n}{r} \pm \frac{wr}{2}\,, \quad r = 2\sqrt{3} \end{aligned} \tag{4.2.48}$$

see appendix 4.B.3 for a few more details. The resummation leading to the second line is in principle straightforward. The final result coincides with the usual partition function of a free boson compactified on a circle of radius $2\sqrt{3}$.

N = 3

After having gone through the example of $N = 2$ quite carefully, we can now be a bit more sketchy with $N = 3$. In this case we obtain $|\mathcal{J}^0_3| = 10$ representations of $SU(3)_6$ with vanishing monodromy charge. They can be grouped into four orbits, three of which have length $N = 3$ while the last one has length $N_{[2,2]} = 1$. More explicitly, the orbits are given by {[0,0], [6,0], [0,6]},{[1,1], [4,1], [1,4]}, {[3,0], [0,3], [3,3]} and {[2,2]}. The reader is invited to recover this list from pairs of Young diagrams Y' and Y'', as explained in subsection 4.2.2 above. The four orbits are associated with the following four necklaces



(0, 0)	(2/3, 1/3)	(1/3, 1/3)	(1/9, 0)
(3/2, 1/2)	(1/6, 1/6)	(5/6, 1/2)	(11/18, 1/2)

The lines below these diagrams display again some information about the associated branching functions $\mathcal{X}^{\mathcal{W}}_{(A,a)}$ for A=*id* and A=*v*, namely the ground state energy h and the maximum Q of the U(1) charge. These results can be read off from the branching functions which we computed numerically, see appendix 4.C.1 for the first few terms.

According to the general formula (4.2.38), the function $\tilde{Z}_N$ takes the form

$$\tilde{Z}_3(q) = \sum_A \left(|\mathcal{X}^{\mathcal{W}}_{(A,[0,0])}|^2 + |\mathcal{X}^{\mathcal{W}}_{(A,[1,1])}|^2 + |\mathcal{X}^{\mathcal{W}}_{(A,[3,0])}|^2 + \tfrac{1}{3}|\mathcal{X}^{\mathcal{W}}_{(A,[2,2])}|^2 \right) . \tag{4.2.49}$$

Once again, the x-dependent branching functions for the short orbit can be decomposed into a sum of $\mathcal{W}_N$ characters. For instance, the branching function[6]

$$\mathcal{X}^{\mathcal{W}}_{(id,[2,2])}(q,x) = 1 + (2x^{4/3} + 3x^{2/3} + 5 + 3x^{-2/3} + 2x^{-4/3})\, q + O(q^2)\,, \tag{4.2.50}$$

[6] In comparison with appendix 4.C.1, we have reintroduced the factor $q^{-c_3/24}$ in $\mathcal{X}^{\mathcal{W}}$ ($c_3 = 8/3$).

can be written as the sum of three characters,

$$
\begin{aligned}
\chi^{\mathcal{W}}_{(id,[2,2],1)}(q,x) &= \tfrac{1}{2}\Big(\mathrm{ch}^{\mathrm{NS,ext}}_{1/9} + \widetilde{\mathrm{ch}}^{\mathrm{NS,ext}}_{1/9}\Big) = 1 + (x^{2/3} + 3 + x^{-2/3})\, q + O(q^2)\,, \\
\chi^{\mathcal{W}}_{(id,[2,2],p)}(q,x) &= \tfrac{1}{2}\Big(\mathrm{ch}^{\mathrm{NS,ext}}_{11/18} + \widetilde{\mathrm{ch}}^{\mathrm{NS,ext}}_{11/18}\Big) = (x^{4/3} + x^{2/3} + 1 + x^{-2/3} + x^{-4/3})\, q + O(q^2)
\end{aligned}
\tag{4.2.51}
$$

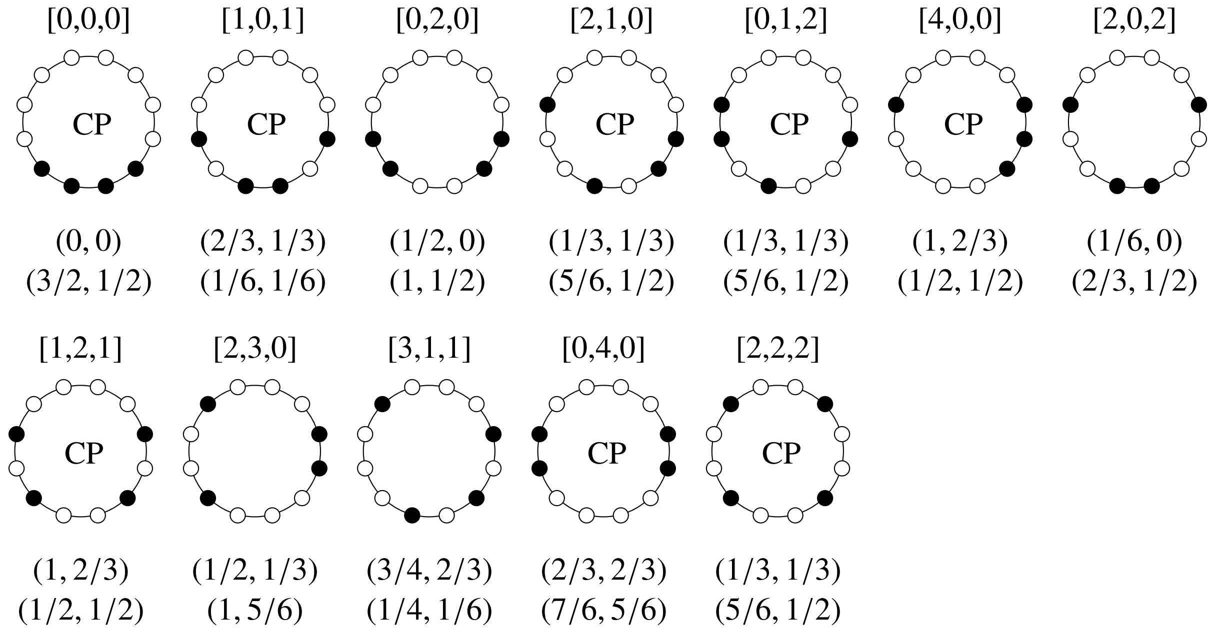
for $p = 2, 3$. Here, the $\mathrm{ch}^{\mathrm{NS,ext}}$ are extended $\mathcal{N} = 2$ characters, as defined in [23]. After the resolution of the fixed point in the sectors $(A, [2, 2])$, we obtain the partition function

$$
Z_3(q,x) = \sum_A \Big(|\chi^{\mathcal{W}}_{(A,[0,0])}|^2 + |\chi^{\mathcal{W}}_{(A,[1,1])}|^2 + |\chi^{\mathcal{W}}_{(A,[3,0])}|^2 + |\chi^{\mathcal{W}}_{(A,[2,2],1)}|^2 + 2|\chi^{\mathcal{W}}_{(A,[2,2],2)}|^2\Big)\,, \tag{4.2.52}
$$

where all summands are considered as functions of both q and x. Of course, for $x = 1$ we recover the expression (4.2.45) for the modular invariant partition function we described above.

N = 4

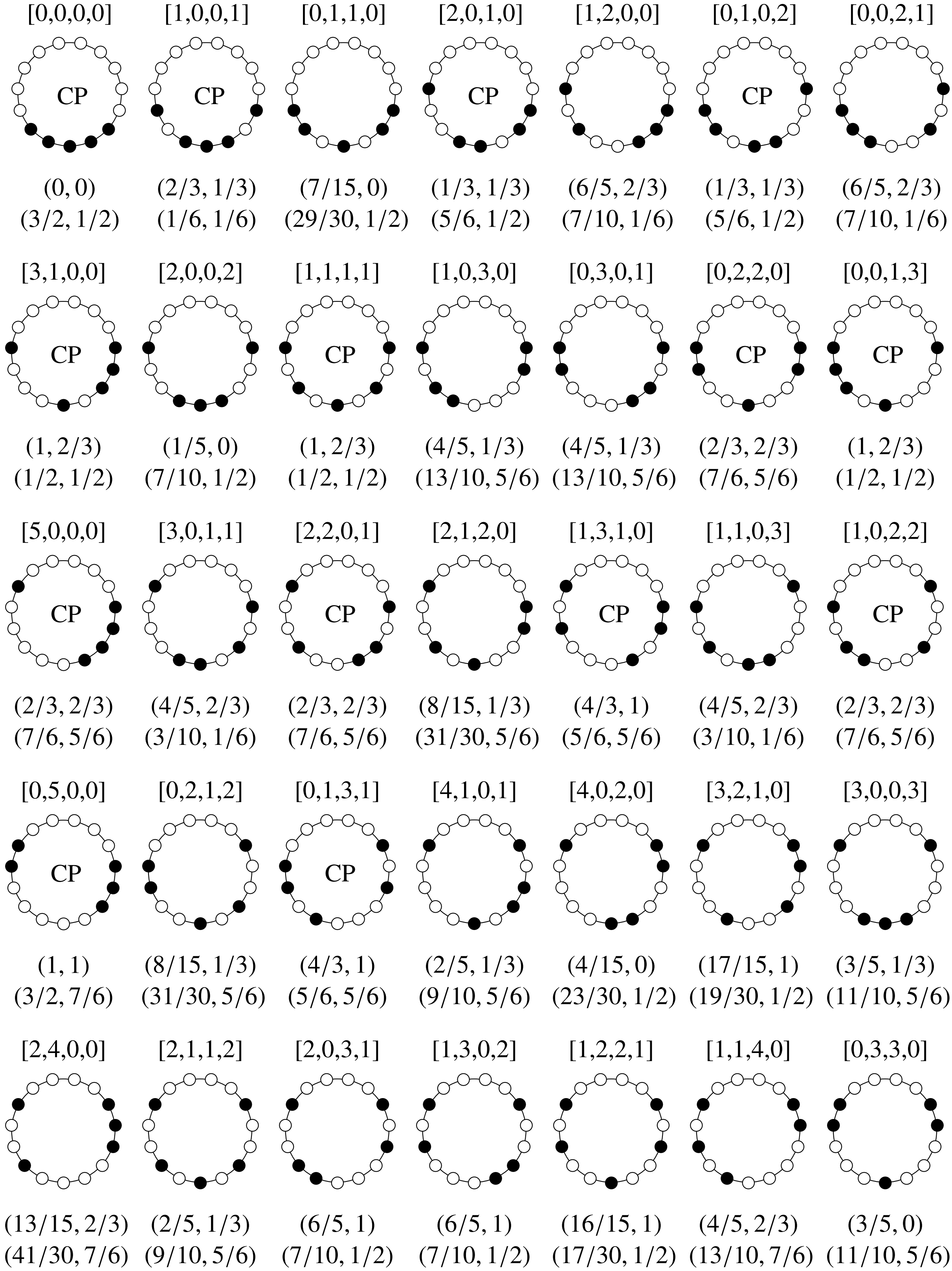
For $N = 4$ there exist twelve different orbits which are labeled by the following necklaces



Note that in this case there are two short orbits. While the orbit $\{[0,4,0], [4,0,4]\}$ has length $N_{[0,4,0]} = 2$, the element $a_* = \{[2,2,2]\}$ is fixed under the action of γ and hence gives an orbit of length $N_{[2,2,2]} = 1$. The branching functions of all orbits are displayed in appendix 4.C.2. Note that 4×2 of these branching functions should be decomposed into the sum of $\mathcal{W}_N$ characters since they are associated to short orbits. For the remaining ones, the branching functions coincide with the characters. We shall not discuss the resolution of fixed points and the partition function of the system in any more detail.

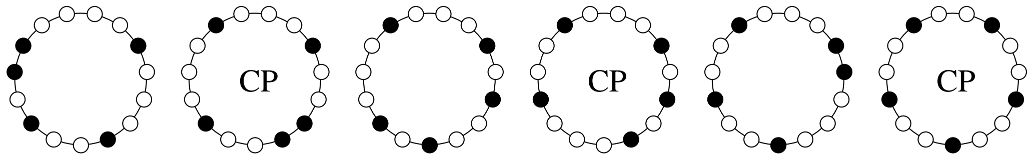
N = 5

Since $N = 5$ is the first prime number beyond the minimal model bound, the final case in our discussion is the most important one. For $N = 5$ there are 41 different orbits which are labeled by the following necklaces:



[0,0,0,0]
CP
(0, 0)
(3/2, 1/2)
[1,0,0,1]
CP
(2/3, 1/3)
(1/6, 1/6)
[0,1,1,0]
(7/15, 0)
(29/30, 1/2)
[2,0,1,0]
CP
(1/3, 1/3)
(5/6, 1/2)
[1,2,0,0]
(6/5, 2/3)
(7/10, 1/6)
[0,1,0,2]
CP
(1/3, 1/3)
(5/6, 1/2)
[0,0,2,1]
(6/5, 2/3)
(7/10, 1/6)
[3,1,0,0]
CP
(1, 2/3)
(1/2, 1/2)
[2,0,0,2]
(1/5, 0)
(7/10, 1/2)
[1,1,1,1]
CP
(1, 2/3)
(1/2, 1/2)
[1,0,3,0]
(4/5, 1/3)
(13/10, 5/6)
[0,3,0,1]
(4/5, 1/3)
(13/10, 5/6)
[0,2,2,0]
CP
(2/3, 2/3)
(7/6, 5/6)
[0,0,1,3]
CP
(1, 2/3)
(1/2, 1/2)
[5,0,0,0]
CP
(2/3, 2/3)
(7/6, 5/6)
[3,0,1,1]
(4/5, 2/3)
(3/10, 1/6)
[2,2,0,1]
CP
(2/3, 2/3)
(7/6, 5/6)
[2,1,2,0]
(8/15, 1/3)
(31/30, 5/6)
[1,3,1,0]
CP
(4/3, 1)
(5/6, 5/6)
[1,1,0,3]
(4/5, 2/3)
(3/10, 1/6)
[1,0,2,2]
CP
(2/3, 2/3)
(7/6, 5/6)
[0,5,0,0]
CP
(1, 1)
(3/2, 7/6)
[0,2,1,2]
(8/15, 1/3)
(31/30, 5/6)
[0,1,3,1]
CP
(4/3, 1)
(5/6, 5/6)
[4,1,0,1]
(2/5, 1/3)
(9/10, 5/6)
[4,0,2,0]
(4/15, 0)
(23/30, 1/2)
[3,2,1,0]
(17/15, 1)
(19/30, 1/2)
[3,0,0,3]
(3/5, 1/3)
(11/10, 5/6)
[2,4,0,0]
(13/15, 2/3)
(41/30, 7/6)
[2,1,1,2]
(2/5, 1/3)
(9/10, 5/6)
[2,0,3,1]
(6/5, 1)
(7/10, 1/2)
[1,3,0,2]
(6/5, 1)
(7/10, 1/2)
[1,2,2,1]
(16/15, 1)
(17/30, 1/2)
[1,1,4,0]
(4/5, 2/3)
(13/10, 7/6)
[0,3,3,0]
(3/5, 0)
(11/10, 5/6)

[0,1,2,3] [3,2,0,2] [3,1,2,1] [2,3,1,1] [2,2,3,0] [2,2,2,2]



(17/15, 1) (1, 2/3) (13/15, 2/3) (2/3, 2/3) (7/15, 1/3) (1/3, 0)
(19/30, 1/2) (1/2, 1/2) (11/30, 1/6) (7/6, 5/6) (29/30, 5/6) (5/6, 5/6)

As usual, the length of the orbit $a_* = \{[2,2,2,2]\}$ is $N_{a_*} = 1$. All other orbits are of maximal length N. The first few terms of the branching functions are displayed in appendix 4.C.3.

Let us briefly describe how to resolve the fixed point when we work with x-dependent branching functions and characters. One may find the following expression for the branching function

$$\mathcal{X}^{\mathcal{W}}_{(id,[2,2,2,2])} = 1 + (4y^2 + 19y^{4/3} + 36y^{2/3} + 47)\, q + O(q^2)\,, \tag{4.2.53}$$

in appendix 4.C.3. It can be written as a sum of five functions,

$$\begin{aligned} \chi^{\mathcal{W}}_{(id,[2,2,2,2],1)} &= 1 + (3y^{4/3} + 8y^{2/3} + 11)\, q + O(q^2)\,, \\ \chi^{\mathcal{W}}_{(id,[2,2,2,2],p)} &= (y^2 + 4y^{4/3} + 7y^{2/3} + 9)\, q + O(q^2) \end{aligned} \tag{4.2.54}$$

for $p = 2, ..., 5$, which we propose for the characters. Here we have introduced the shorthand $y^n \equiv x^n + x^{-n}$. Note that for $x = 1$ the coefficients of q in the characters must equal $165/5 = 33$. Then, after resolution of the fixed point, we get

$$Z_5 = \sum_A \Big(\sum_{\{a\}\in O_5/\{[2,2,2,2]\}} |\chi^{\mathcal{W}}_{(A,a)}|^2 + |\chi^{\mathcal{W}}_{(A,[2,2,2,2],1)}|^2 + 4|\chi^{\mathcal{W}}_{(A,[2,2,2,2],2)}|^2 \Big)\,. \tag{4.2.55}$$

This concludes our brief discussion of branching functions, characters and partition functions for the examples with $N \le 5$.

4.3 Chiral primary fields

In this section we will start to describe the main results of the present Chapter. We have described the chiral symmetry $\mathcal{W}_N$ and the complete modular invariant partition function Z_N for a family of field theories with $\mathcal{N} = (2,2)$ superconformal symmetry. Our goal now is to determine the chiral primaries of these models. Since we know how the spectrum of the model is built from the various representations of the chiral algebra $\mathcal{W}_N$ all that is left to do is to find (anti-)chiral primaries in the individual sectors. In principle this is straightforward once the characters of the chiral algebra are known. Indeed, for $N \le 5$, the chiral primaries can be read off from the q-expanded branching functions listed in appendix 4.C. In section 4.3.1, we show that there exists an upper bound on the conformal weight of a chiral primary. In order to organize the chiral primaries, we will then define and discuss in section 4.3.2 the class of *regular* chiral primaries. In section 4.3.3, we discuss a few examples and show that for $N \le 3$ there are no other chiral primaries besides the regular ones. This will change for theories with $N \ge 4$, as we shall show in section 4.3.4.

4.3.1 Bound on the dimension of chiral primaries

There are a few general results on the dimension of chiral primaries that are useful to discuss before we get into concrete examples. In any $\mathcal{N} = 2$ superconformal field theory, the conformal weight of chiral primaries is bounded from above by

$$h(\phi_{\mathrm{cp}}) \leq \frac{c}{6} \tag{4.3.1}$$

where c is the central charge of the Virasoro algebra [81]. This bound is independent of the sector in which the chiral primary resides.

In order to derive stronger sector-dependent bounds, we recall that the fields in the numerator theory satisfy $h^{\mathrm{N}} \geq 3|Q^{\mathrm{N}}|$. States that make it into the coset sector $(A, \{a\})$ contain the highest weight vector of a $\mathrm{SU}(N)_{2N}$ representation $b \in \{a\}$ in the orbit of a. The latter has weight h_b^{D} and charge $Q_b^{\mathrm{D}} = 0$. For the dimension h and charge Q of the coset fields we obtain the constraint $h + h_b^{\mathrm{D}} = h^{\mathrm{N}} \geq 3|Q^{\mathrm{N}}| = 3|Q|$ and consequently for coset states ϕ in the sector $(A, \{a\})$,

$$h(\phi) \geq 3|Q(\phi)| - \min_{b\in\{a\}}(h_b^{\mathrm{D}}) \, .$$

For (anti-)chiral primaries ϕ_{cp} with $h(\phi_{\mathrm{cp}}) = |Q(\phi_{\mathrm{cp}})|$ this inequality implies that

$$2h(\phi_{\mathrm{cp}}) \leq \min_{b\in\{a\}}(h_b^{\mathrm{D}}) \quad \text{or} \quad h(\phi_{\mathrm{cp}}) \leq \min_{b\in\{a\}}\left(\frac{C_2(b)}{6N}\right) . \tag{4.3.2}$$

In addition to this constraint, the U(1) charges must also satisfy $|Q| = k/6$ ($k \in \mathbb{N}_0$). It is easy to see that this implies

$$\min_{b\in\{a\}}\left(\frac{C_2(b)}{N}\right) \in \mathbb{Z}$$

if the sector $(A, \{a\})$ is to contain a chiral primary and N is odd. For even N a similar condition holds with N replaced by $N/2$.

As we shall see below there exist some important sectors for which this bound is so strong that it does not permit chiral primaries above the ground states. In other sectors, however, our bound (4.3.2) is much less powerful. This applies in particular to those that are associated with the fixed point a_*. In fact, in the representation a_*, the quadratic Casimir assumes its largest eigenvalue

$$C_2(a_*) = \frac{1}{3}N(N^2 - 1) \quad \text{or} \quad \frac{C_2(a_*)}{6N} = \frac{c_N}{6} \, .$$

Hence, in the fixed point sectors our bound (4.3.2) coincides with the universal bound (4.3.1). This appears to leave a lot of room for chiral primaries.

4.3.2 Regular chiral primaries

As we stressed before, there exists a large set of chiral primaries that may be constructed very explicitly for any value of N. Their description is particularly simple when we use our parametrization of orbits in terms of two Young diagrams Y' and Y'', see section 4.2.2. We will determine $\mathcal{W}$ sectors containing regular chiral primaries in the first half of this subsection and then count regular chiral primaries of the full (non-chiral) conformal field theory in the second.

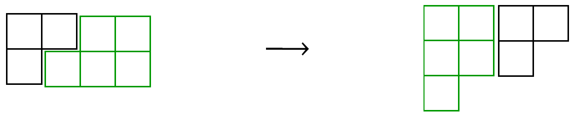


Figure 4: Generation of a representation $Y \in \mathcal{S}_4$. The Young diagram Y' with lengths $(l'_1, l'_2) = (2, 1)$ (black) induces the SU(4) representation Y with lengths $(l_1, l_2, l_3) = (4, 3, 1)$.

Parametrization and properties

In section 4.2.2 we constructed all solutions $a \in \mathcal{J}_N$ of the vanishing monodromy condition $Q_\gamma(a) = 0$ in terms of a pair of Young diagrams $Y' = Y'(a)$ and $Y'' = Y''(a)$. As indicated in our notation we now think of these Young diagrams as functions of the sector label a. This map is obtained by reversing our formula (4.2.16) for the construction of $Y(a)$ from Y' and Y''. As we shall show below, for elements of the following subset

$$\mathcal{S}_N = \{a = (l_1, \ldots, l_r) | Y'(a) = Y''(a); Q_\gamma(a) = 0\} \subset \mathcal{J}_N^0 \tag{4.3.3}$$

we can find a regular chiral primary in the coset sectors $(A, \{a\})$ with $A = id$ if $|Y'|$ is even and $A = v$ otherwise.

There is a simple geometrical construction of the Young diagrams $Y(a)$ that can be obtained with $Y' = Y''$, simplifying the general prescription from previous section. In fact, it follows from eq. (4.2.16) that a Young diagram $Y(a), a \in \mathcal{S}_N$ is obtained from Y' by first completing Y' to an $r' \times N$ rectangular Young diagram and then attaching the (rotated) 'complementary' diagram $(N - l'_{r'}, ..., N - l'_1)$ from the left to the original Young diagram Y'. An example is shown in figure 4.

Let us add a few comments that will later allow us to enumerate regular chiral primaries, i.e. the orbits of diagonal sectors $a(Y', Y')$. We should stress that most elements in such an orbit $\{a(Y', Y'' = Y')\}$ are not obtained from diagonal pairs $(Y', Y'') = (Y', Y')$. So, if we would like to decide whether the sector $\mathcal{H}^{\mathrm{C}}_{\{b\}}$ contains a chiral primary field, we need to construct the pair $(Y'(a), Y''(a))$ for each element a in the orbit $\{b\}$ of the element $b \in \mathcal{J}_N^0$ and check whether at least one of these pairs satisfies the condition $Y'(a) = Y''(a)$. Let us note that the orbit $\{a_*\}$ of the weight $a_* = [2, 2, \ldots, 2]$ consists of a single element a_* and $Y'(a_*) \neq Y''(a_*)$ for any prime $N > 2$. Hence the distinction between coset sectors and labels $(A, \{a\})$, as well as all our discussion of fixed point resolutions, is not relevant for the discussion of regular chiral primaries when N is a prime number.

More importantly, one can show that most orbits $\{b\}$ contain at most one representative $a \in \{b\}$ such that $Y'(a) = Y''(a)$. This follows from the following expression for the action of γ^k in the weights of (Y', Y''),

$$\begin{aligned}
\gamma^k(Y') \quad &:= (2(N-k) + l_{N-k+1} - r'', \ldots, 2(N-k) + l_{N-1} - r'', \\
&\qquad 2(N-k) - r'', l_1 - 2k - r'', \ldots, l_{N-k-1} - 2k - r'') \\
\gamma^k(Y''^{\mathrm{T}}) \quad &:= (r'' + 2k - l_{N-k}, \ldots, r'' + 2k - l_1, r'' - 2(N-k), \\
&\qquad r'' - 2(N-k) - l_{N-1}, \ldots, r'' - 2(N-k) - l_{N-k+2}).
\end{aligned}$$

In both expressions the order of entries is non-decreasing and all non-positive entries are to be skipped from the end of these strings. The Y^{T} is used to denote the transpose of a Young diagram Y. If we now require $Y' = Y''$ and $\gamma^k(Y') = \gamma^k(Y'')$ it is easy to infer that the only solutions satisfying these two constraints are the Young diagrams of rectangular shape. These correspond to the orbits $\{a_v\}$ of the weight

$a_\nu = [0, ..., 0, N, 0, ..., 0]$ for $\nu = 1, \ldots, N-1$ where the only non-zero entry N can appear in any position ν, i.e. $\lambda_\nu = N$. As we have just demonstrated, field identifications can map a_ν to $a_{N-\nu}$, so that we have now found $\lfloor (N-1)/2 \rfloor$ orbits that contain two elements of $\mathcal{S}_N$. These weights are associated with diagonal pairs $(Y', Y'' = Y')$ of Young diagrams.

For coset sectors $(A, \{a\})$ with $a \in \mathcal{S}_N$ there exists a simple formula to compute their exact conformal weight. By eq. (4.2.20) the quadratic Casimir of a representation $a \in \mathcal{S}_N$ is simply $C_2(a) = n'N$. This implies that the conformal weight of the ground states is

$$h(\psi_{(A,\{a\})}) = \frac{C_2(a)}{6N} = \frac{n'}{6}, \tag{4.3.4}$$

for $a \in \mathcal{S}_N$. Let us add that the number $n' = |Y'| = |Y''|$ of boxes in Y' is given in terms of the representation labels $(l_1, ..., l_r)$ of a by

$$n' = \sum_{i=1}^{r'} l_i' = \sum_{i=1}^{r'} (l_i - r') \tag{4.3.5}$$

with $r' = n/N$ and where $n = \sum_{i=1}^{r} l_i$ is the number of boxes of $Y = Y(a)$. With the help or sector dependent bound from the previous subsection, see eq. (4.3.2), we can now show that in all the sectors associated with $a \in \mathcal{S}_N$, chiral primaries must be ground states of the $\mathcal{W}$ algebra. In fact, by combining the conformal weight (4.3.4) with the bound (4.3.2), we find

$$\frac{n'}{6} = h(\psi_{(A,\{a\})}) \leq h(\phi_{\text{cp}}) \leq \min_{b \in \{a\}} \left(\frac{C_2(b)}{6N} \right) = \frac{n'}{6}.$$

Hence, these sectors cannot contain any chiral primaries in addition to the ones we will find among their ground states.

Counting of regular chiral primaries

Before we look into examples let us count the regular chiral primaries along with their conformal weight. This will proceed in several steps. First we shall count the number of elements in $\mathcal{S}_N$, then we employ the result to count the number representations of our chiral algebra $\mathcal{W}$ that contain a regular chiral primary and finally we determine the counting function for regular chiral primaries from the full partition function of the model, at least for N prime.

Our description of the set $\mathcal{S}_N$ in terms of Young diagrams $Y' = Y''$ makes it an easy task to determine $|\mathcal{S}_N|$. The conditions for the choice of Y' and Y'' we spelled out before eq. (4.2.16). They imply that diagrams Y' corresponding to elements in $\mathcal{S}_N$ must fit into a rectangle of size $r' \times c'$ with $r' + c' = N$. Such Young diagrams are counted through the series

$$\tilde{T}_N(\mathfrak{q}) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_{\mathfrak{q}} - \sum_{k=0}^{N-1} \begin{bmatrix} N-1 \\ k \end{bmatrix}_{\mathfrak{q}} = \sum_{k=0}^{N-1} \mathfrak{q}^k \begin{bmatrix} N-1 \\ k \end{bmatrix}_{\mathfrak{q}}, \tag{4.3.6}$$

which is denoted by A161161 in [136]. The $\mathfrak{q}$-binomial coefficient[7] that multiplies $\mathfrak{q}^k$ counts all Young diagrams Y' that fit into a rectangle with $(N-1-k) \times k$ boxes. The factor $\mathfrak{q}^k$ corresponds to attaching to

[7] It is a q-analogue of the usual binomial coefficient, see Chapter 3.

each of these Young diagrams from the left a single column of k boxes. As a consequence, the individual summands in eq. (4.3.6) count the number of Young diagrams fitting into a rectangle with $(N-k)\times k$ boxes. By the binomial theorem we find

$$|\mathcal{S}_N| = \tilde{T}_N(1) = \sum_{k=0}^{N-1}\binom{N-1}{k} = 2^{N-1} \,.$$

Let us list also the coefficients $\tilde{t}_n^N$ of the function $\tilde{T}_N(\mathfrak{q}) = \sum_n \tilde{t}_n^N \mathfrak{q}^n$ for all values with $N \leq 7$,

$$\begin{aligned}
N=2: &\quad 1,1\\
N=3: &\quad 1,1,2\\
N=4: &\quad 1,1,2,3,1\\
N=5: &\quad 1,1,2,3,5,2,2\\
N=6: &\quad 1,1,2,3,5,7,5,4,3,1\\
N=7: &\quad 1,1,2,3,5,7,11,8,9,7,6,2,2\,.
\end{aligned}$$

Let us note in passing that, at large N, the coefficients $\tilde{t}_n^N$ of $\tilde{T}_N(\mathfrak{q})$ coincide with the number $p(n)$ of partitions of n, i.e.

$$\lim_{N\to\infty}\tilde{T}_N(\mathfrak{q}) = \prod_{k=1}^{\infty}\frac{1}{1-\mathfrak{q}^k} = \sum_{n=0}^{\infty} p(n)\mathfrak{q}^n \,. \tag{4.3.7}$$

In order to count the number of $\mathcal{W}$ representations that contain a regular chiral primary we recall two facts discussed above. The first one concerns the fixed point resolution. When $N > 2$ is prime, there is only one short orbit a_* and since a_* is not a element of $\mathcal{S}_N$ the counting of *regular* chiral primaries is not affected by the fixed point resolution. On the other hand, there are a few orbits that contain two elements of $\mathcal{S}_N$. These need to be subtracted from the counting function $\tilde{T}_N(\mathfrak{q})$ in order to obtain a counting function $T_N(\mathfrak{q})$ for $\mathcal{W}$ sectors containing regular chiral primaries

$$T_N(\mathfrak{q}) := \tilde{T}_N(\mathfrak{q}) - \sum_{n=1}^{\lfloor\frac{N-1}{2}\rfloor}\mathfrak{q}^{n(N-n)/6} \,. \tag{4.3.8}$$

As explained above, the over-counting we are trying to make up for is associated with the Dynkin labels $a_\nu \in \mathcal{S}_N$. Since the sector (A, a_ν) containing the associated regular chiral primary has conformal weight $h_{(A,[0\ldots N\ldots 0])} = n(N-n)/6$, we have included the appropriate power of $\mathfrak{q}$ in our subtraction.

After this preparation we can now turn to the counting of (regular) chiral primaries. By their very definition, (anti-)chiral primaries are fields in the Neveu-Schwarz sector of the theory for which the conformal weight h and the U(1) charge Q satisfy $h = \pm Q$. Here, the upper sign applies to chiral primaries while the lower one is relevant for anti-chiral primaries. It is then obvious that chiral primaries are counted by

$$Z_N^{\mathrm{cp}}(q,\bar{q}) = \frac{1}{(2\pi i)^2}\oint\frac{dx}{x}\oint\frac{d\bar{x}}{\bar{x}}\, Z_N^{\mathrm{NS}}(qx^{-2},\bar{q}\bar{x}^{-2},x,\bar{x}) \tag{4.3.9}$$

where Z_N^{NS} denotes the contribution from the NS sector of the model, i.e. the summands $A{=}id$ and $A = \nu$, to the full (resolved) partition function. For anti-chiral primaries, the first two arguments of the partition function in the integrand must be replaced by qx^2 and $\bar{q}\bar{x}^2$, respectively. We know that this counting function for chiral primaries receives contributions from the regular ones. The latter have been determined above so that

$$Z_N^{\rm cp}(q,\bar{q}) = Z_N^{\rm cp,reg}(q,\bar{q}) + Z_N^{\rm cp,exc}(q,\bar{q}) = T_N\left((q\bar{q})^{1/6}\right) + Z_N^{\rm cp,exc}(q,\bar{q})\ . \tag{4.3.10}$$

The counting function T_N for regular chiral primaries has been constructed in eqs. (4.3.8) and (4.3.6) above. If all chiral primaries were regular, there would be no additional contributions. But we shall see below that this is not the case. Starting from $N = 4$ not all chiral primaries are regular. The additional *exceptional* chiral primaries are counted by $Z_N^{\rm cp,exc}$.

4.3.3 Examples with $N \leq 3$: Minimal models

The aim of this subsection is to illustrate our general constructions through the first two examples, namely $N = 2$ and $N = 3$. These possess central charge $c_N < 3$ and hence they belong to the minimal series of $\mathcal{N} = (2,2)$ superconformal minimal models. For models from this series the chiral primaries are well known. Our only task is therefore to show that the general constructions of regular chiral primaries outlined in the previous subsection allow us to recover all known chiral primaries.

N = 2

Let us start by reviewing briefly the case of $N = 2$ which gives a CFT with Virasoro central charge $c_2 = 1 \leq 3$. In section 4.2.5 we have listed all the sectors of this model along with the conformal weight and maximal R-charge of their ground states. From the results we can easily deduce that there are only two sectors containing chiral primaries, namely the sectors $(id,[0])$ and $(v,[2],1)$. Recall that the label $(v,[2])$ labels a branching function that can be decomposed into a sum of two characters. These characters, which were displayed in eq. (4.2.47), show that only one of the corresponding sectors contains a chiral primary. Moreover, since the conformal weight of all chiral primaries is bounded by $c_2/6 = 1/6$ there can be no chiral primaries among the excited states of the model. Hence, we conclude that model contains two chiral primaries. One is the identity field, the other one a chiral primary of weight $h = 1/6$.

Let us reproduce this simple conclusion from the construction of regular chiral primaries. The construction we sketched above instructs us to list all Young diagrams Y' that can fit into a rectangle of size $r' \times c'$ where $r' + c' = N = 2$. Obviously, there are only two such Young diagrams, namely the trivial one and the single box. These are depicted in the leftmost column of figure 5. Applying the general prescription (4.2.16) (with $Y'' = Y'$) we obtain two Young diagrams Y in the second column. From the two columns we can read off the labels of the corresponding coset sectors $(A,\{a\})$. These are displayed in the third column. As we explained above, the first label A is determined by the number of boxes n' of the Young diagram Y' in the first column. It is $A = id$ if n' is even and $A = v$ otherwise. The second entry contains the orbit $\{a\}$ of the $SU(2)_4$ representation a that is associated with the Young diagram Y in the second column. According to eq. (4.3.4), the conformal weights of the corresponding chiral primaries are given by $h(\psi_{(A,a)}) = |Y'|/6$. In this case, we recovered all chiral primaries through the construction of the regular ones. The counting function for chiral primaries is given by

$$Z_2^{\rm cp}(q,\bar{q}) = 1 + (q\bar{q})^{1/6} \ = T_2\left((q\bar{q})^{1/6}\right) \tag{4.3.11}$$

and it obviously coincides with the counting function for regular chiral primaries we stated in the previous subsection.

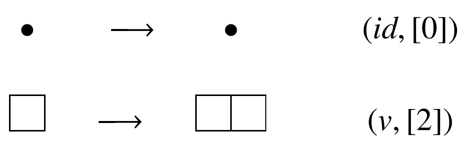


Figure 5: The sets of Young diagrams Y' and Y for $N = 2$.

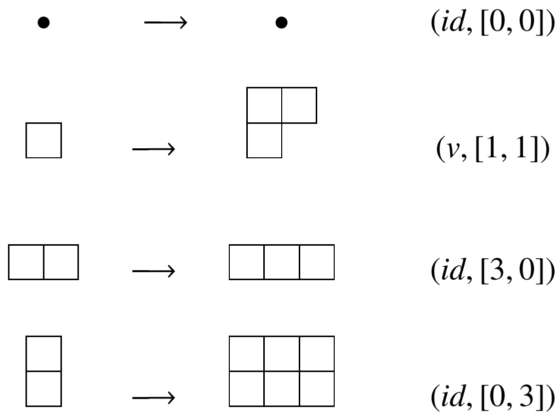


Figure 6: The sets of Young diagrams Y' and Y for $N = 3$.

N = 3

For $N = 3$ we can proceed similarly. In this case, the model has central charge $c_3 = 8/3$, still below the critical value $c = 3$. It is well known to possess 3 chiral primaries of conformal weights $h = 0, 1/6, 1/3$ [23] and one can verify this statement through a quick glance at the data we provided with the list of necklaces in section 4.2.5. Note that the three necklaces that are associated with chiral primaries have been marked with the letters 'CP' in the center.

Let us now apply our general construction of regular chiral primaries to the case $N = 3$. To begin with, we must list all the Young diagrams Y' that fit into a rectangle of size 1×2 or 2×1. There are four such diagrams which are depicted in the leftmost column of figure 6. Application of the construction (4.2.16) (with $Y'' = Y'$) gives four representations of SU(3), as shown in the second column of figure 6. The corresponding coset sectors are listed in the right column. In this case two of the obtained sectors coincide since the $\mathrm{SU}(3)_6$ sectors $[0, 3]$ and $[3, 0]$ are related by the simple current automorphism. Hence we end up with three inequivalent coset representations whose ground states can provide a regular chiral primary. Their labels are displayed in the third column of figure 6.

We can easily scan the partition function Z_3 given in eq. (4.2.52) for chiral primaries from the list in the third column of figure 6 to obtain

$$Z_3^{\mathrm{cp}}(q, \bar{q}) = 1 + (q\bar{q})^{1/6} + (q\bar{q})^{1/3} \ . \tag{4.3.12}$$

The answer agrees with the counting function for regular chiral primaries we proposed in the previous subsection, i.e.

$$Z_3^{\mathrm{cp}}(q, \bar{q}) = Z_3^{\mathrm{cp,reg}}(q, \bar{q}) = \tilde{T}_3((q\bar{q})^{1/6}) - (q\bar{q})^{1/3} \ .$$

The subtraction of $(q\bar{q})^{1/3}$ is explained by the field identification $(id, [3, 0]) = (id, [0, 3])$.

4.3.4 Exceptional chiral primaries for $N \geq 4$

For $N = 2, 3$ the complete set of chiral primaries is given by the class of regular chiral primaries. While this class still plays a role for higher N we shall find additional chiral primaries when $N \geq 4$. We refer to them as *exceptional* chiral primaries.

N = 4

For $N = 4$ the central charge $c_4 = 5$ exceeds the bound $c = 3$ that can be reached with supersymmetric minimal models. Therefore we can no longer rely on known results on the set of chiral primaries. Let us therefore first apply our general constructions of regular chiral primaries and then check whether they provide the complete set of chiral primaries.

The analysis is summarized in figure 7. In the first column we list all the Young diagrams Y' which can fit into rectangles of size 1×3 or 3×1 or 2×2. From these we build Young diagrams for representations of SU(4) with the help of eq. (4.2.16). The results are shown in the second column. Taking the first two columns together we determine the list of coset sectors shown in the third column. Note that $(v, [4, 0, 0])$ and $(v, [0, 0, 4])$ refer to the same sector of the model since $[4, 0, 0]$ may be obtained from $[0, 0, 4]$ by applying the simple current automorphism. Hence, our construction gives seven different coset sectors whose ground states are chiral primary.

In order to check whether we are missing any chiral primaries of the model, we must scan the space of states with conformal weight $h \leq c_4/6 = 5/6$, or a little less if we used the sector dependent bound (4.3.2). Since $5/6 < 1$, all chiral primaries must be ground states. Hence we can perform the scan by looking through the pairs (h, Q) we displayed when we listed the necklaces for $N = 4$ in section 4.2.5. Those necklaces that give rise to chiral primaries have already been marked by a ‘CP’ in the center. Not surprisingly we find all the seven regular chiral primaries from the third column of figure 7.

On the other hand, the scan we just performed gives one more chiral primary that does not appear in the right column of figure 7, namely a ground state of the coset sector $(id, [2, 2, 2])$. This new chiral primary has conformal weight $h = Q = 1/3$ and it is our first example of an exceptional chiral primary. Our findings may be summarized in the following expression

$$Z_4^{\mathrm{cp}}(q, \bar{q}) = 1 + (q\bar{q})^{1/6} + 3(q\bar{q})^{1/3} + 2(q\bar{q})^{1/2} + (q\bar{q})^{2/3} \tag{4.3.13}$$

which is equal to

$$Z_4^{\mathrm{cp}}(q, \bar{q}) = Z_4^{\mathrm{cp,reg}}(q, \bar{q}) + (q\bar{q})^{1/3} .$$

The additional term $(q\bar{q})^{1/3}$ counts the exceptional chiral primary. A word of caution is in order. In general, chiral primaries can appear in coset sectors which are fixed points of the theory. As we discussed before, such fixed points must be resolved, and it is not *a priori* clear whether this changes the multiplicity of the chiral primaries or not. For $N = 4$ both the sector $(id, [0, 4, 0])$ and $(id, [2, 2, 2])$ are fixed points and their ground states are chiral primary. The chiral primaries appear with multiplicity one in both $\mathcal{X}^{\mathcal{W}}_{(id+v,[0,4,0])}$ and $\mathcal{X}^{\mathcal{W}}_{(id+v,[2,2,2])}$, as can be seen from the their q-expansions in appendix 4.C.2. The result (4.3.13) for Z_4^{cp} holds true provided that the fixed point resolution does *not* change the multiplicities. Otherwise the counting of chiral primaries would need to be modified accordingly.

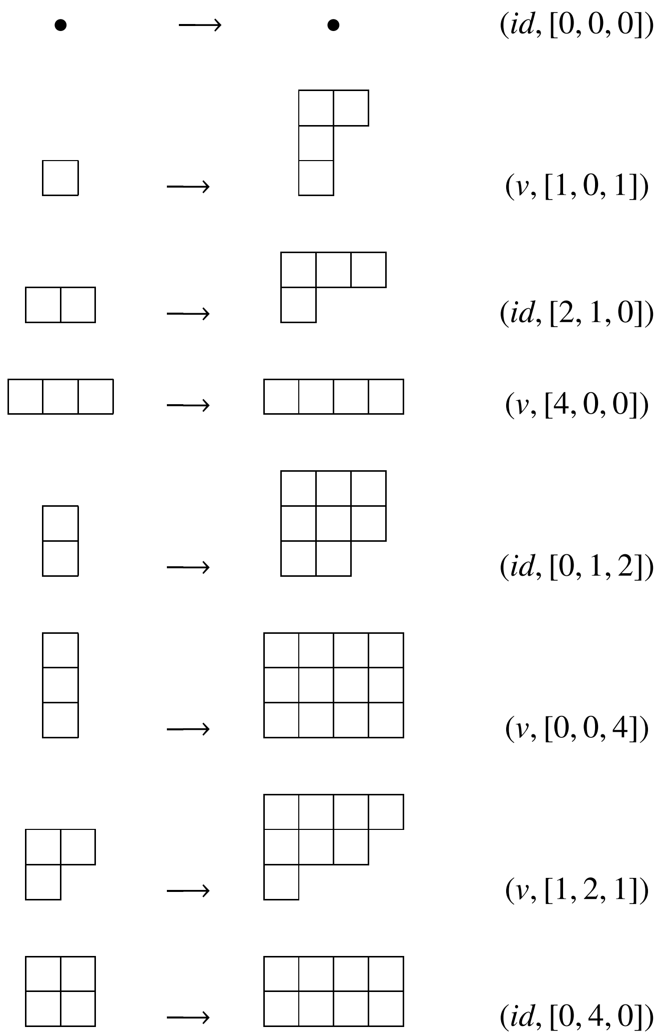


Figure 7: The sets of Young diagrams Y' and Y for $N = 4$.

N = 5

As in the previous discussion we shall begin by listing all the regular chiral primaries for $N = 5$. After constructing all Young diagrams Y' which fit into rectangles of size 2×3, 3×2, 1×4 or 4×1 we apply eq. (4.2.16) to obtain the 16 Young diagrams Y. It would take quite a bit of space to display all of them. So, let us simply produce a list of the corresponding Dynkin labels,

$$\begin{aligned}\mathcal{S}_5 = \{&(id, [0,0,0,0]), (v, [1,0,0,1]), (id, [2,0,1,0]), (id, [0,1,0,2]), (v, [0,0,1,3]),\\ &(v, [1,1,1,1]), (v, [3,1,0,0]), (id, [5,0,0,0]), (id, [2,2,0,1]), (id, [0,2,2,0]),\\ &(id, [1,0,2,2]), (id, [0,0,0,5]), (v, [0,1,3,1]), (v, [1,3,1,0]), (id, [0,5,0,0]),\\ &(id, [0,0,5,0])\}\,.\end{aligned}$$

Note that both $(id, [5,0,0,0])$ and $(id, [0,0,0,5])$ as well as $(id, [0,5,0,0])$ and $(id, [0,0,5,0])$ are identified by the simple current automorphism. Hence we would expect 14 coset sectors whose ground states are (regular) chiral primary.

Let us now look for the complete set of chiral primaries. In this case, the sector independent bound (4.3.1) restricts the conformal weight of chiral primaries to satisfy $h \leq c_5/6 = 4/3$. Here we inserted the central charge $c_5 = 8$. One can again do a little better using the sector dependent bound (4.3.2), but in the case at hand we also listed all contributions to branching functions up to weight $h = 4/3$, see appendix 4.C.3. The results show that once more all chiral primaries are $\mathcal{W}_N$ ground states so that we can detect chiral primaries from the data that were provided in section 4.2.5 where we listed the necklaces for $N = 5$. We see 17 sectors of our $\mathcal{W}_N$ algebra contain a chiral primary among its ground states. This is three more that the 14 regular chiral primaries we described in the previous paragraph. The exceptional chiral primaries correspond to ground states of the sectors $(v, [3, 2, 0, 2])$, $(id, [2, 3, 1, 1])$ and $(v, [2, 2, 2, 2])$ and they possess conformal weights $h = 1/2$, $h = 2/3$ and $h = 5/6$, respectively.

From the resolved partition function (4.2.55), we thus find

$$Z_5^{\mathrm{cp}} = 1 + (q\bar{q})^{1/6} + 2(q\bar{q})^{1/3} + 4(q\bar{q})^{1/2} + 5(q\bar{q})^{2/3} + 3(q\bar{q})^{5/6} + q\bar{q} \tag{4.3.14}$$

$$= T_5\left((q\bar{q})^{1/6}\right) + (q\bar{q})^{1/2} + (q\bar{q})^{2/3} + (q\bar{q})^{5/6} \,. \tag{4.3.15}$$

The last three terms give the counting function $Z_5^{\mathrm{cp,exc}}$ for exceptional chiral primaries. Let us point out that one of the new chiral primaries is sitting inside the fixed point sector (v, a_*), something that could not happen with the regular chiral primaries for N prime.

4.4 Chiral primaries at large N

We now address the second important goal of this Chapter, namely to construct the chiral ring in the limit of large N. As we are about to vary N, many of the objects we encountered in the previous sections will carry an additional label N. This applies in particular to the quadratic Casimir $C_2^{(N)}$, the sectors $\mathcal{H}^{\mathrm{C},(N)}$ of the coset chiral algebra as well as the maps $a_N = a_N(Y', Y'')$ and $Y_N = Y_N(Y', Y'')$ that associate a weight a or a Young diagram Y to a pair of Young diagrams Y' and Y''.

Since the coset model is built from representations of the coset chiral algebra, we should first explain how to take the large N limit of the sectors $\mathcal{H}^{\mathrm{C},(N)}_{\{a\}}$. In the subsection 4.2.2 we learned how to parametrize the allowed values of a in terms of two Young diagrams Y' and Y''. In taking the limit, we keep these Young diagrams fixed, i.e. we define

$$\mathcal{H}_{\{Y',Y''\}} \equiv \lim_{N\to\infty} \mathcal{H}^{\mathrm{C},(N)}_{\{a_N(Y',Y'')\}} \,.$$

Let us stress that the Young diagram $Y = Y_a$ that we construct from Y' and Y'' depends on the value of N. This is why it was so important to place a subscript $_N$ on the corresponding SU(N) weight $a = a_N$. One can show that the sectors $\mathcal{H}_{\{Y',Y''\}}$ are well defined. In particular, the dimension of the subspaces with fixed conformal weight h stabilizes as we send N to infinity. We are now trying to find those pairs (Y', Y'') for which the space $\mathcal{H}_{\{Y',Y''\}}$ contains chiral primaries. Our claim is that this happens if and only if $Y' = Y''$. As we explained in the previous section, such diagonal pairs of Young diagrams are associated with regular chiral primaries.

In order to establish these claims let us consider any of the summands

$$\mathcal{H}_a^{\mathrm{NS}} = \mathcal{H}^{\mathrm{C},(N)}_{\{a\}} \otimes \mathcal{H}_a^{\mathrm{D}}$$

in the decomposition (4.2.7). The space $\mathcal{H}_a^{\rm NS}$ comes equipped with the action of several commuting operators. To begin with, we mention the zero modes of the coset Virasoro field and the U(1) currents, i.e. $L_0 = L_0^G - L_0^H$ and Q. In addition, we can also introduce the fermion number operator K_0 which is defined by

$$K_0 = \sum_{r\geq 1/2} \psi^1_{\mu,-r}\psi^1_{\nu,r}\kappa^{\mu\nu} + \psi^2_{\mu,-r}\psi^2_{\nu,r}\kappa^{\mu\nu} \ .$$

K_0 commutes with Q and L_0 and hence can be measured simultaneously on $\mathcal{H}_a^{\rm NS}$.

Proposition 4.4.1. *The conformal weight h_ϕ of states ϕ in the subspace $\mathcal{H}_a^{NS}$ of the NS-sector is bounded from below by*

$$h_\phi \geq \frac{K_\phi}{2} - \frac{C_2^{(N)}(a)}{3N} \ . \tag{4.4.16}$$

Similarly, the U(1) charge Q_ϕ of the state ϕ is bounded from above by

$$|Q| \leq \frac{K_\phi}{6} \ . \tag{4.4.17}$$

In both inequalities, the number K_ϕ denotes the fermion number, i.e. the eigenvalue of the fermion number operator K_0 on the state ϕ.

The two inequalities follow straightforwardly from the fact that the complex fermion multiplets Ψ and Ψ^* have conformal weight $h_\Psi = 1/2$ and that their real and imaginary part ψ^1 and ψ^2 possess U(1) charge $|Q_{\psi^j}| = 1/6$. In the first relation, the two sides are equal in case the construction of ϕ does not involve any derivatives of the fermionic fields. The second relation becomes an equality for states ϕ that are built from ψ^1 or ψ^2 and its derivatives only.

There is another simple proposition we need to discuss. Before we state it, let us recall from the previous section that a sector $\mathcal{H}^{\rm C,(N)}_{\{a\}}$ of the coset model can only contain a chiral primary if

$$\min_{b\in\{a\}}\left(C_2^{(N)}(b)\right) \equiv 0 \bmod N \ ,$$

i.e. the minimum $C_2(b)$ assumed in the orbit $\{a\}$ of a must be divisible by N, at least when N is odd. Under the action of the identification current, the value of the quadratic Casimir can only shift by an integer multiple[8] of N so that a sector $\mathcal{H}^{(N)}_{\{a\}}$ can only contain a chiral primary if

$$C_2^{(N)}(a) \equiv 0 \bmod N \ .$$

As we explained before, when we vary N we are instructed to keep Y' and Y'' fixed. Let us assume that N_0 is the minimal number for which the two inequalities

$$r' + c'' \leq N_0 \quad , \quad r'' + c' \leq 2N_0$$

are satisfied. Then Y' and Y'' define a sector of the coset theory for all $N \geq N_0$. We can use the rules stated above to construct a diagram $Y_N(Y', Y'')$ for all $N \geq N_0$. The associated representation is denoted by $a_N = a_N(Y', Y'')$, as before.

[8]The precise amount of this shift is $C_2(\gamma^i(a)) - C_2(a) = 3N\left(\sum_{j=1}^{i-1} l_{N-j} + i\,(N - r'' - i)\right)$.

Proposition 4.4.2. *The family of sectors $\mathcal{H}^{C,(N)}_{\{a_N(Y',Y'')\}}$ can only contain a chiral primary if the two Young diagrams Y' and Y'' of $SU(N_0)$ belong to representations with the same value of the quadratic Casimir element,*

$$C_2^{(N_0)}(Y') = C_2^{(N_0)}(Y'') \ .$$

To prove this statement we recall from eq. (4.2.20) that the value of the quadratic Casimir element in the representation a_N of SU(N) is given by

$$C_2^{(N)}(a_N(Y',Y'')) = n'N + C_2^{(N)}(Y') - C_2^{(N)}(Y'') \ .$$

For the sector a_N to contain a chiral primary, the right-hand side must be divisible by N. Since the first term is, we need to determine the conditions under which the $C_2^{(N)}(Y') - C_2^{(N)}(Y'')$ is a multiple of N. The difference of the Casimir reads

$$C_2^{(N)}(Y') - C_2^{(N)}(Y'') = \frac{1}{2}\left(\sum_i (l_i'^2 + l_i' - 2il_i') - (l_i''^2 + l_i'' - 2il_i'')\right) = C_2^{(N_0)}(Y') - C_2^{(N_0)}(Y'') \ , \qquad (4.4.18)$$

and thus does not depend on N. Hence it clearly cannot be divisible by (a sufficiently large)[9] N unless the difference vanishes. This is what we had to prove.

Our third proposition is a little more difficult to prove, but it is absolutely crucial for what we are about to establish.

Proposition 4.4.3. *For the states ϕ in the sector $\mathcal{H}^{NS}_a$, the fermion number satisfies the inequality*

$$K_\phi \geq n' \ .$$

The number n' is determined by the choice of a. It is computed from the associated Young diagram $Y = Y_a$ by, see eq. (4.2.18),

$$n' = \sum_{i=1}^{N-1} \theta\left(l_i - \frac{1}{N}\sum_{i=1}^{N-1} l_i\right),$$

where $\theta(x)$ denotes the Heaviside step-function. If the representation a is diagonal, i.e. $Y'(a) = Y''(a)$, the above formula simplifies to

$$n' = \sum_{i=1}^{n/N}\left(l_i - \frac{n}{N}\right) , \quad \textit{where} \quad n = \sum_{i=1}^{N-1} l_i \ .$$

We will first give a somewhat heuristic graphical argument using Young diagrams before we outline a formal proof of this proposition. Let us recall that all our fermions transform in the tensor product of the fundamental and the dual fundamental representations. These correspond to Young diagrams that consist of a single box and a single column of maximal length $N-1$, respectively.

We need to show that it takes at least n' fermionic fields in order to build a state in the representation a, i.e. the first time the representation a appears in the tensor power $\mathbf{adj}^{\otimes K}$ is for $K = n'$. For the SU(N) Lie algebra, the adjoint representation decomposes as $\mathbf{adj} = \square \otimes \overline{\square}$. Here $\overline{\square}$ denotes the (Young diagram of) the dual fundamental representation, i.e. a column of $N-1$ boxes.

[9] A rough estimate for what 'sufficiently large' means is given by $N > N_0(N_0^2 - 1)/12$.

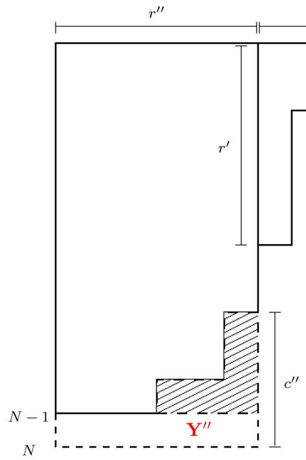


Figure 8: The Young diagram Y is obtained from a rectangle of size $N \times r''$ by attaching the Young diagram Y' and removing (a reflected version of) the Young diagram Y''. Note that $r' + c'' \leq N$ is needed for the resulting diagram to be a Young diagram of SU(N). The shaded region shows those boxes from the tensor power of the dual fundamental that must be moved to the left.

The graphical proof goes as follows. In order to build the Young diagram Y we start with the K columns $\bar{\square}$ of size $N-1$. If we put them all side by side, we would obtain a rectangular Young diagram of size $(N-1) \times K$. As we increase the number K, the diagram Y'' starts to cut into the rectangle, see figure 8. This means that we have to remove a few boxes from those columns and move them to one of the previous columns. But since all these have maximal length, every time we take out one of the boxes and move it to a full column to the left, we lose an entire column with N boxes. It is easy to see that in total we need to move $n' - r''$ boxes which make us lose $N(n' - r'')$ boxes altogether. Hence we need $K = r'' + n' - r'' = n'$ fermions to begin with. Of course, this number is also sufficient since we can build Y' from the n' fundamentals $\square$. This concludes the graphical proof.

Let us now go through a somewhat more formal argument. There is an explicit decomposition [137] found by studying so-called walled Brauer algebras [138, 139] which provides us with the irreducible content of the K-th tensor power of the SU(N) adjoint representation, at least up to $2K \leq N$,

$$\mathbf{adj}^{\otimes K} = \sum_{n'=0}^{K} b_{n'}^{(K)} \sum_{Y',Y'' \vdash n'} \frac{n'!}{\prod_{(l',m') \in Y'} h(l',m')} \frac{n'!}{\prod_{(l'',m'') \in Y''} h(l'',m'')} \cdot a(Y',Y'') \,. \tag{4.4.19}$$

Here we use the notation $Y', Y'' \vdash n'$ to express that both Y' and Y'' are partitions of n', the products run over all boxes in $Y'$and Y'' and the integers $h(l,m)$ denote the length of a hook that is associated to the box (l,m), definition of it can be found in appendix 4.D. The multiplicities $b_{n'}^{(K)}$ are of combinatorial nature and explicitly given by

$$b_{n'}^{(K)} := \sum_{i=0}^{K-n'} (-1)^{i+K+n'} \; i! \binom{K}{n'} \binom{K-n'}{i} \binom{i+n'}{i} \,. \tag{4.4.20}$$

The derivation of formula (4.4.19) is discussed in more detail in the appendix 4.D. What is most important for us right now is that a representation a composed out of two small Young diagrams of n' boxes can appear on the right-hand side of eq. (4.4.19) only when we start with the product of $K = n'$ adjoints on the left-hand side of eq. (4.4.19). This concludes the proof of proposition 4.4.3.

Now let us combine the previous three statements. According to proposition 4.4.2, the sectors that can contribute a chiral primary which has finite weight in the large N limit have $C_2(a) \sim n'N$. Hence, the two inequalities in proposition 4.4.1 become

$$h_\phi \geq \frac{K_\phi}{2} - \frac{n'}{3} = \frac{n'}{6} + \frac{K_\phi - n'}{2}\,, \tag{4.4.21}$$

$$|Q_\phi| \leq \frac{K_\phi}{6} = \frac{n'}{6} + \frac{K_\phi - n'}{6}\,. \tag{4.4.22}$$

In the second step we have sightly rewritten the bounds. From proposition 4.4.3 we know that the quantity $K_\phi - n'$ is non-negative. Hence, the equality $h_{\phi_{\rm cp}} = Q_{\phi_{\rm cp}}$ between the conformal weight and U(1) charge of a chiral primary $\phi_{\rm cp}$ can only be satisfied for $K_{\phi_{\rm cp}} = n'$. This implies that both the weight and the U(1) charge of such chiral primaries,

$$h_{\phi_{\rm cp}} = \frac{n'}{6} \quad \text{and} \quad Q_{\phi_{\rm cp}} = \frac{n'}{6}\,, \tag{4.4.23}$$

saturate the bounds given in proposition 4.4.1. As we explained in the text below proposition 4.4.1, this implies that the state $\phi_{\rm cp}$ is constructed from the fermionic fields ψ^1_ν only without any derivatives and components of ψ^2. States with these features must transform in the anti-symmetric tensor power of the adjoint representation. It is actually possible to work out the precise content of the anti-symmetrized part of the k-th power of adjoint representation for values of $k \leq N-1$,

$$\left\{\mathbf{adj}^{\otimes k}\right\}_{\rm antisymm} = \sum_{n'=1}^{k} d^{(k)}_{n'} \sum_{Y' \vdash n'} a\,(Y', Y')\,. \tag{4.4.24}$$

A more detailed discussion and the precise values of the coefficients $d^{(n')}_k$ can be found in appendix 4.E. What is most important about formula (4.4.24), at least in our present context, is that all representations that appear in the decomposition are of the form $a(Y', Y'') = a(Y', Y')$. Now we only need to recall from section 4.3.2 that such diagonal sectors are associated with regular chiral primaries to establish our central claim: The chiral primaries of the large N limit are regular. Let us stress once again that for any given finite value of N, chiral primaries can be constructed that do not satisfy eqs. (4.4.23) and hence are not regular.

4.5 Conclusions

In this Chapter we described the chiral symmetry $\mathcal{W}_N$ and the complete modular invariant partition function Z_N for a family of field theories with $\mathcal{N} = (2,2)$ superconformal symmetry that arise in the low energy limit of 1-dimensional adjoint QCD. We developed techniques to study these theories for $N \geq 4$, where the theory does not correspond to a supersymmetric minimal model. Special attention was payed to the set of chiral primaries which are counted by a function $Z^{\rm cp}_N(q)$ which we introduced in eq. (4.3.10). One of our main results is the discovery of exceptional chiral primaries for $N = 4, 5$, which lie outside the set of regular chiral primaries. In fact, we found one such chiral primary for $N = 4$ and three of them for

$N = 5$. Regular chiral primaries were described in some detail in section 4.3.2. These fields are counted by a function $T_N(\mathfrak{q})$ which we defined in eq. (4.3.8).

The last part of our analysis was dwelling on a large N behaviour of the chiral ring. Namely, we proved that chiral primary fields in the low energy limit of multi-color adjoint QCD are regular in the sense we defined in section 4.3.2. We have seen before that such regular chiral primaries are in one-to-one correspondence with Young diagrams Y', at least if we approach the multicolor limit through a sequence of prime numbers N. In case N is prime, the only contribution to the state space (4.2.40) that is not simply a diagonal product of left- and right-movers is the term $\mathscr{H}_{\rm fix}$ which does not contain any regular chiral primaries, see section 4.3.2. Furthermore, as we approach the large N theory, the only orbits $\{a\}$ that are associated with two different diagonal pairs $(Y', Y'' = Y')$, namely the orbits $\{a_\nu\}$, give rise to regular chiral primaries of weight $h = \nu(N-\nu)/6$. Hence, they are not part of the spectrum of chiral primaries as N tends to infinity. For all remaining regular chiral primaries, the orbit is associated with a unique Young diagram Y'. Combining all these facts, we introduce the symbol $\phi_{\rm cp}(Y')$ to denote the unique chiral primary

$$\phi_{\rm cp}(Y') \in \mathcal{H}_{Y',Y''} \otimes \overline{\mathcal{H}}_{Y',Y''} \quad \text{with} \quad h(\phi_{\rm cp}(Y')) = n' = |Y'|\,.$$

As usual in $\mathcal{N} = (2,2)$ supersymmetric theories, the chiral fields form a chiral ring which closes under operator product expansions. It is not difficult to argue that the chiral ring at large N must be isomorphic to a standard graded ring of symmetric functions $\Lambda_R = \oplus_{i\in\mathbb{N}} \Lambda_R^{(i)}$, which is a ring of formal infinite sums of monomials. Its Hilbert-Poincaré series

$$\sum_{i\in\mathbb{N}} \dim(\Lambda_R^{(i)})\, t^i := \prod_{i=1}^{\infty} \frac{1}{1-t^i}\,, \tag{4.5.1}$$

i.e. the function which generates dimensions of subspaces of grade i, is the generating function of integer partitions. The operator product of two chiral primaries at large N thus takes the form

$$\phi_{\rm cp}(Y_1') \cdot \phi_{\rm cp}(Y_2') = \sum_{Y_3'} C_{Y_1',Y_2'}^{Y_3'}\, \phi_{\rm cp}(Y_3'), \tag{4.5.2}$$

where $C_{Y_1',Y_2'}^{Y_3'}$ are the Littlewood-Richardson coefficients [140, 141]. The ring is freely generated, e.g. by the elementary symmetric polynomials e_k, $k = 1, 2, \ldots$ corresponding to those chiral primaries whose Young diagrams Y' consist of only one column. Obviously, there is exactly one such generator at each grade. The construction of generators representing chiral primaries $\phi_{\rm cp}(Y')$ corresponding to other Young diagrams $Y' \neq e_k$ is then performed with the help of the second Jacobi-Trudi identity.

In the special case of fusion with a chiral primary corresponding to the partition f_n of one row with n boxes, the Pieri's formula implies

$$\phi_{\rm cp}(Y_1') \cdot \phi_{\rm cp}(f_n) = \sum_{Y_3'} \phi_{\rm cp}(Y_3'), \tag{4.5.3}$$

where the summation goes only over Young diagrams obtained from Y_1' by adding n boxes, no two in the same column. From this formula one can see that an iterative fusion of the vacuum with the lowest non-trivial chiral primary $C_\square$ precisely generates the Young lattice (the lattice of Young diagrams ordered by inclusion). This is a nice way to picture a subring of the chiral ring generated by the grade 1 generator alone (see figure 9).

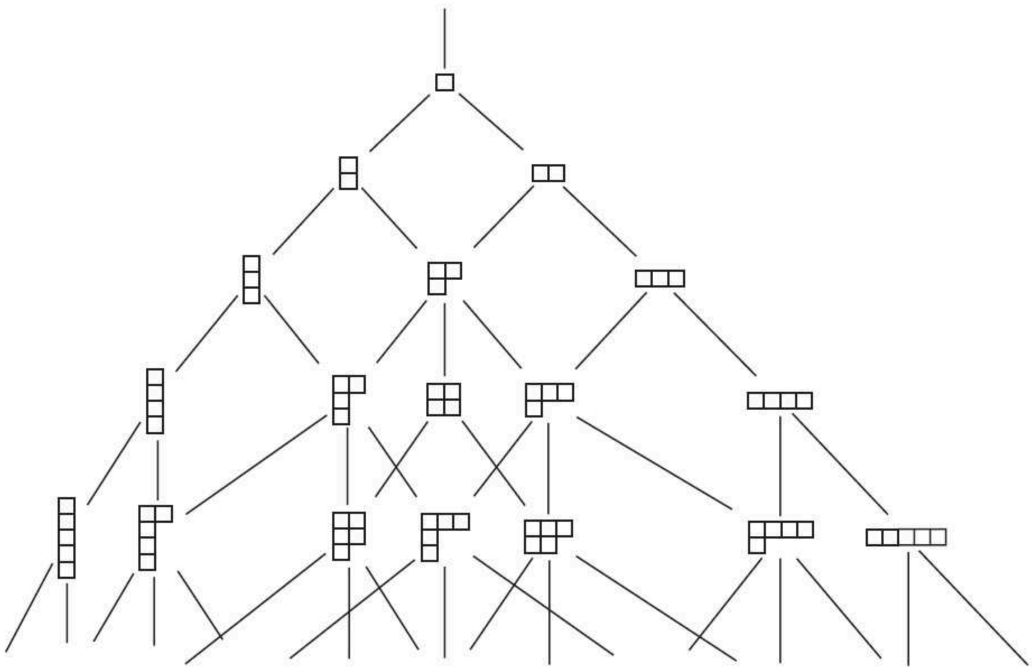


Figure 9: Hasse diagram of the Young lattice.

This concludes our discussion of the results in this Chapter. Let us recall that the main motivation of our work stems from the desire to constrain the dual higher spin or string theory. Let us recall that the family of 'strange metal' coset models we analysed here has a matrix-like structure in which the central charge behaves as $c \sim N^2$. While theories with a vector-like dependence $c \sim N$ have been argued to be dual to Vasiliev higher spin theories in AdS, the dual of strange metal coset theories with chiral algebra $\mathcal{W}_N$ is believed to possess a much larger symmetry than Vasiliev theory and could well be a string theory. The background geometry of such a potential dual string theory is severely constrained by the result we reported above. Since the spectrum of chiral primaries does not depend on the string length, we have argued that the background geometry should give rise to chiral primaries which are in one-to-one correspondence with partitions or Young diagrams. While we do not have any concrete proposal for now, we want to point out that infinite families of AdS_3 backgrounds with at least $\mathcal{N} = (2, 0)$ supersymmetry have been constructed in [142]. It would be interesting to scan those solutions or apply the methods of Donos et al. in order to find a geometry that gives rise to the desired chiral ring. Let us note in passing that one chiral (left-moving) half of the strange metal coset theory was recently argued to arise in string theory on near horizon geometries of certain fast rotating black holes in an AdS space [143]. The constructions of Berkooz et al. provide the entire state space of the chiral strange metal coset, obviously including all the chiral primaries we described above. In case the aforementioned results or methods do not suffice to identify a dual string background, one might obtain valuable additional constraints on the dual theory by decomposing the spectrum of the strange metal coset theory into representations of higher spin symmetries, much along the lines of [144]. We plan to come back to these issues in future research.

Appendix

4.A $\mathrm{SU}(N)_{2N}$ representations with zero monodromy charge

In this appendix we will prove the formula (4.2.20) for the quadratic Casimir $C_2(Y)$ of a representation $a \in \mathcal{J}_N^0$ associated with a Young diagram Y. As described in subsection 4.2.2 we pick up a pair of SU(N) Young diagrams Y' and Y'' satisfying the conditions listed there. From these two Young diagrams we build a new diagram $Y = (l_1, \ldots, l_{N-1})$ through our prescription (4.2.16).

We now claim that the resulting Young diagram Y possesses

$$|Y| = r''N \tag{4.A.1}$$

boxes and that the eigenvalue of the SU(N) quadratic Casimir on Y takes the value

$$C_2(Y) = n'N + C_2(Y') - C_2(Y'') . \tag{4.A.2}$$

In order to prove these two statements, we use eq. (4.2.16) to obtain

$$|Y| = \sum_i l_i = \sum_{i=1}^{r'} (r'' + l'_i) + r''(N - r' - l''_1) + \sum_{i=1}^{r''-1} (r'' - i)(l''_i - l''_{i+1}) . \tag{4.A.3}$$

Since

$$\sum_{i=1}^{r''-1} i(l''_i - l''_{i+1}) = \sum_{i=1}^{r''-1} i\, l''_i - \sum_{i=2}^{r''} (i-1) l''_i = n' - r'' l''_{r''} , \tag{4.A.4}$$

we arrive at

$$|Y| = r'r'' + n' + r''(N - r' - l''_1) + r''(l''_1 - l''_{r''}) - (n' - r'' l''_{r''}) = r''N , \tag{4.A.5}$$

which proves eq. (4.A.1).

The quadratic Casimir on Y is therefore given by

$$C_2(Y) = \frac{1}{2}\left[N r''(N + 1 - r'') + \sum_i l_i(l_i - 2i) \right] .$$

Let us compute the last term in the brackets,

$$\sum_i l_i(l_i - 2i)$$

$$
\begin{aligned}
&= \sum_{i=1}^{r'} (r'' + l'_i)(r'' + l'_i - 2i) + \sum_{i=1}^{N-l''_1-r'} r''(r'' - 2(r' + i)) \\
&\quad + \sum_{i=1}^{l''_1-l''_2} (r'' - 1)(r'' - 1 - 2(N - l''_1 + i) + \cdots + \sum_{i=1}^{l''_{r''-1}-l''_{r''}} 1 \cdot (1 - 2(N - l''_{r''-1} + i)) \\
&= \sum_{i=1}^{r'} (r'' + l'_i)(r'' + l'_i - 2i) - \sum_{i=1}^{N-l''_1-r'} r''(r'' + 2i) + \sum_{i=1}^{r''-1} (r'' - i)(r'' - i - 2N)(l''_i - l''_{i+1}) \\
&\quad + 2\sum_{i=1}^{r''-1} (r'' - i) l''_i (l''_i - l''_{i+1}) - \sum_{i=1}^{r''-1} (r'' - i)(l''_i - l''_{i+1} + 1)(l''_i - l''_{i+1}) \\
&=: \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 \,. \qquad (4.A.6)
\end{aligned}
$$

Using the identities analogous to eq. (4.A.4),

$$
\begin{aligned}
\sum_{i=1}^{r''-1} i(l''^2_i - l''^2_{i+1}) &= \sum_{i=1}^{r''} l''^2_i - r'' l''^2_{r''} \,, \\
\sum_{i=1}^{r''-1} i^2(l''_i - l''_{i+1}) &= 2\sum_{i=1}^{r''} i l''_i - n' - (r''^2 - r'') l''_{r''} \,,
\end{aligned}
\qquad (4.A.7)
$$

we can conclude

$$
\begin{aligned}
\Sigma_1 + \Sigma_2 &= \sum_{i=1}^{r'} l'_i (l'_i - 2i) + 2r''n' - r''(N - l''_1)(N - l''_1 - r'' + 1) \,, \\
\Sigma_3 &= 2\sum_{i=1}^{r''} i l''_i + (2N - 2r'' - 1)n' + (r''^2 - 2r''N) l''_1 \,, \\
\Sigma_4 + \Sigma_5 &= \sum_{i=1}^{r''-1} (r'' - i)((l''^2_i - l''^2_{i+1}) - (l''_i - l''_{i+1})) \\
&= -\sum_{i=1}^{r''} l''^2_i + n' + r''(l''^2_1 - l''_1) \,.
\end{aligned}
\qquad (4.A.8)
$$

When summed up, this contributions give

$$
\sum_i l_i(l_i - 2i) = -Nr''(N + 1 - r'') + 2n'N + \sum_{i=1}^{r'} l'_i(l'_i - 2i) - \sum_{i=1}^{r''} l''_i(l''_i - 2i) \qquad (4.A.9)
$$

and thus

$$
C_2(Y) = Nn' + \frac{1}{2}\left(\sum_{i=1}^{r'} l'_i(l'_i - 2i) - \sum_{i=1}^{r''} l''_i(l''_i - 2i) \right) . \qquad (4.A.10)
$$

Since $|Y'| = |Y''| = n'$ holds by construction, this expression is equivalent to eq. (4.A.2).

4.B Branching functions and fixed-point resolution for $N = 2$

In this appendix we explain how to compute the branching functions for $N = 2$, how to resolve the fixed point and how to recover the partition function of a compactified free boson. The first subsection contains a list of relevant functions and identities. Branching functions of the model are computed in the second subsection before we discuss the partition function in the final part.

4.B.1 Notations

We use the following notation for theta functions

$$\begin{aligned}\Psi_k(a|b)_x &:= \sum_{n\in\mathbb{Z}} q^{\frac{a}{2}(n+\frac{1}{k})^2} x^{b(n+\frac{1}{k})}\\ \tilde{\Psi}_k(a|b)_x &:= \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{a}{2}(n+\frac{1}{k})^2} x^{b(n+\frac{1}{k})}\,.\end{aligned}$$

For $k = 1, 2$, these reduce to ordinary Jacobi theta functions,

$$\begin{aligned}\theta_3(a|b)_x &:= \Psi_1(a|b)_x\,, \quad \theta_4(a|b)_x := \tilde{\Psi}_1(a|b)_x\,,\\ \theta_2(a|b)_x &:= \Psi_2(a|b)_x\,, \quad \theta_1(a|b)_x := -i\tilde{\Psi}_2(a|b)_x.\end{aligned} \tag{4.B.1}$$

Whenever we set $x = 1$, we omit the second parameter in the brackets and the small subscript.

Next, let us introduce Ramanujan's theta function,

$$f(a,b) := \sum_{n\in\mathbb{Z}} (ab)^{\frac{n^2}{2}} \left(\frac{a}{b}\right)^{\frac{n}{2}} = \prod_{n\in\mathbb{N}} \left(1 + \frac{1}{a}(ab)^n\right)\left(1 + \frac{1}{b}(ab)^n\right)\left(1 - (ab)^n\right).$$

It is related to theta functions through

$$\begin{aligned}f(a,b) &= q^{-\frac{\beta^2}{8\alpha}} u^{-\frac{\beta}{2\alpha}} \Psi_{\frac{2\alpha}{\beta}}(\alpha|1)_u\\ f(-a,-b) &= q^{-\frac{\beta^2}{8\alpha}} u^{-\frac{\beta}{2\alpha}} \tilde{\Psi}_{\frac{2\alpha}{\beta}}(\alpha|1)_u\end{aligned}$$

where the variables u and q on the right-hand side are related to a and b through $ab = q^\alpha$ and $a/b = q^\beta u^2$. Ramanujan's theta function obeys Weierstrass' three-term relation [145],

$$f(a,b)f(c,d) = f(ad,bc)f(ac,bd) + a\,f(c/a,a^2bd)f(d/a,a^2bc)$$

whenever $ab = cd$, and Hirschhorn's generalized quintuple product identity [146],

$$\begin{aligned}f(a,b)f(c,d) &= f(ac,bd)f(ac\cdot b^3, bd\cdot a^3)\\ &+ a\,f\left(\frac{d}{a},\frac{a}{d}(abcd)\right) f\left(\frac{bc}{a},\frac{a}{bc}(abcd)^2\right) + b\,f\left(\frac{c}{b},\frac{b}{c}(abcd)\right) f\left(\frac{ad}{b},\frac{b}{ad}(abcd)^2\right)\end{aligned}$$

for $(ab)^2 = cd$. This concludes our brief list of mathematical functions and identities.

4.B.2 Branching functions

In order to illustrate how branching functions are computed, let us focus on the decomposition of the $id+v$ sector of the $SO(6)_1$ WZW model. According to our general prescription (4.2.28), the corresponding character reads

$$
\begin{aligned}
&\chi^{SO(6)_1}_{id+v}(q,x,z) \\
&= q^{-1/8}\prod_{n\in\mathbb{N}}(1+x^{1/3}q^{n-1/2})(1+x^{-1/3}q^{n-1/2})(1+x^{1/3}z^2q^{n-1/2}) \\
&\times(1+x^{-1/3}z^{-2}q^{n-1/2})(1+x^{1/3}z^{-2}q^{n-1/2})(1+x^{-1/3}z^2q^{n-1/2}) \\
&= \eta^{-3}f(q^{1/2}x^{1/3},q^{1/2}x^{-1/3})f(q^{1/2}x^{1/3}z^2,q^{1/2}x^{-1/3}z^{-2})f(q^{1/2}x^{1/3}z^{-2},q^{1/2}x^{-1/3}z^2) \\
&= \eta^{-3}f(q^{1/2}x^{1/3},q^{1/2}x^{-1/3})\left(f(qx^{2/3},qx^{-2/3})f(qz^4,qz^{-4})+q^{1/2}x^{1/3}z^2f(q^2x^{2/3},x^{-2/3})f(q^2z^4,z^{-4})\right)
\end{aligned}
\tag{4.B.2}
$$

where $\eta\equiv\eta(q)$ is the Dedekind eta function. In the final step we inserted Weierstrass' three-term relation. Using Hirschhorn's quintuple product, one can then write

$$
\begin{aligned}
&f(q^{1/2}x^{1/3},q^{1/2}x^{-1/3})f(qx^{2/3},qx^{-2/3}) \\
&\quad = f(q^{3/2}x,q^{3/2}x^{-1})f(q^3,q^3)+q^{1/2}f(q,q^5)\left(x^{1/3}f(q^{5/2}x,q^{1/2}x^{-1})+x^{-1/3}f(q^{5/2}x^{-1},q^{1/2}x)\right)
\end{aligned}
$$

for $a=q^{1/2}x^{1/3}$, $b=q^{1/2}x^{-1/3}$, $c=qx^{2/3}$, $d=qx^{-2/3}$ and

$$
\begin{aligned}
&x^{1/3}f(q^{1/2}x^{1/3},q^{1/2}x^{-1/3})f(q^2x^{2/3},x^{-2/3}) \\
&\quad = q^{1/2}f(q^{3/2}x,q^{3/2}x^{-1})f(1,q^6)+f(q^2,q^4)\left(x^{1/3}f(q^{5/2}x,q^{1/2}x^{-1})+x^{-1/3}f(q^{5/2}x^{-1},q^{1/2}x)\right)
\end{aligned}
$$

for $a=q^{1/2}x^{1/3}$, $b=q^{1/2}x^{-1/3}$, $c=q^2x^{2/3}$, $d=x^{-2/3}$. In the second case we employed the following obvious symmetry property of f,

$$\frac{1}{\sqrt{u}}f\left(u,\frac{A}{u}\right)=\sqrt{u}f\left(Au,\frac{1}{u}\right).$$

Substituting these products back and simplifying, one arrives at

$$
\begin{aligned}
&\chi^{SO(6)_1}_{id+v}(q,x,z) \\
&= \eta^{-1}\Psi_1(3|1)_x\left(\frac{\theta_3(6)}{\eta^2}\theta_3(2|4)_z+\frac{\theta_2(6)}{\eta^2}\theta_2(2|4)_z\right) \\
&+\eta^{-1}\left(\Psi_3(3|1)_x+\Psi_3(3|-1)_x\right)\left(\frac{\Psi_3(6)}{\eta^2}\theta_3(2|4)_z+\frac{\Psi_6(6)}{\eta^2}\theta_2(2|4)_z\right).
\end{aligned}
\tag{4.B.3}
$$

In this formula $\theta_3(6)/\eta^2$, $\theta_2(6)/\eta^2$, $\Psi_3(6)/\eta^2$ and $\Psi_6(6)/\eta^2$ are combinations of $SU(2)_4$ string functions $c^0_0+c^0_4$, $2c^0_2$, c^2_0 and c^2_2, respectively (see e.g. [147] for more information). Expressions in round brackets can be then recognized as $S_{\{0\}}$ and $S_{\{2\}}$ which were defined in eq. (4.2.27). Indeed, taking into account the symmetry properties of the $SU(2)_k$ string functions,

$$c^{\Lambda}_{\lambda}=c^{J(\Lambda)}_{J(\lambda)}=c^{\Lambda}_{\lambda+2k}=c^{\Lambda}_{-\lambda}$$

where J is the $SU(2)_k$ simple current, one readily sees from

$$S^{N=2}_{\{a\}} = \sum_{\Lambda\in\{a\}} \sum_{\lambda\in\{-2,0,2,4\}} c^{\Lambda}_{\lambda} \sum_{n\in\mathbb{Z}} q^{4\left(n+\frac{\lambda}{8}\right)^2} z^{8\left(n+\frac{\lambda}{8}\right)}$$

that

$$S_{\{0\}} = (c^0_0 + c^0_4)\,\theta_3(2|4)_z + 2c^0_2\,\theta_2(2|4)_z$$
$$S_{\{2\}} = c^2_0\,\theta_3(2|4)_z + c^2_2\,\theta_2(2|4)_z.$$

The decomposition (4.B.3) of the $SO(6)_1$ character (4.B.2) thus leads to the following branching functions

$$\mathcal{X}^{\mathcal{W}}_{(id+v,[0])}(q,x) = \eta^{-1}\Psi_1(3|1)_x\,, \qquad \mathcal{X}^{\mathcal{W}}_{(id+v,[2])}(q,x) = \eta^{-1}\left(\Psi_3(3|1)_x + \Psi_3(3|-1)_x\right)\,.$$

Going along the same lines, one may compute the six remaining branching functions. These are given by

$$\begin{aligned}
&\mathcal{X}^{\mathcal{W}}_{(id-v,[0])}(q,x) = \eta^{-1}\tilde{\Psi}_1(3|1)_x\,, && \mathcal{X}^{\mathcal{W}}_{(id-v,[2])}(q,x) = -\eta^{-1}\left(\tilde{\Psi}_3(3|1)_x + \tilde{\Psi}_3(3|-1)_x\right)\,,\\
&\mathcal{X}^{\mathcal{W}}_{(sp+c,[0])}(q,x) = \eta^{-1}\Psi_2(3|1)_x\,, && \mathcal{X}^{\mathcal{W}}_{(sp+c,[2])}(q,x) = \eta^{-1}\left(\Psi_6(3|1)_x + \Psi_6(3|-1)_x\right)\,,\\
&\mathcal{X}^{\mathcal{W}}_{(sp-c,[0])}(q,x) = \eta^{-1}\tilde{\Psi}_2(3|1)_x\,, && \mathcal{X}^{\mathcal{W}}_{(sp-c,[2])}(q,x) = -\eta^{-1}\left(\tilde{\Psi}_6(3|1)_x - \tilde{\Psi}_6(3|-1)_x\right)\,.
\end{aligned} \tag{4.B.4}$$

4.B.3 The partition function

The 'unresolved' partition function $\tilde{Z}_2$ of the $N = 2$ model is constructed from the branching functions (4.B.4) according to the general prescription (4.2.38),

$$\begin{aligned}
\tilde{Z}_2 &= \sum_{A=id,v,sp,c} \left(|\mathcal{X}^{\mathcal{W}}_{(A,[0])}|^2 + \tfrac{1}{2}|\mathcal{X}^{\mathcal{W}}_{(A,[2])}|^2\right) = \frac{1}{2}\sum_{B=id+v,id-v,sp+c,sp-c} \left(|\mathcal{X}^{\mathcal{W}}_{(B,[0])}|^2 + \tfrac{1}{2}|\mathcal{X}^{\mathcal{W}}_{(B,[2])}|^2\right)\\
&= \frac{1}{2|\eta|^2}\Big\{|\Psi_1(3|1)_x|^2 + |\tilde{\Psi}_1(3|1)_x|^2 + |\Psi_2(3|1)_x|^2 + |\tilde{\Psi}_2(3|1)_x|^2 + \frac{1}{2}[|\Psi_3(3|1)_x + \Psi_3(3|-1)_x|^2\\
&\qquad + |\tilde{\Psi}_3(3|1)_x + \tilde{\Psi}_3(3|-1)_x|^2 + |\Psi_6(3|1)_x + \Psi_6(3|-1)_x|^2 + |\tilde{\Psi}_6(3|1)_x - \tilde{\Psi}_6(3|-1)_x|^2]\Big\}\,.
\end{aligned}$$

As we explained before, the model suffers from a fixed point in the sectors $(A, [2])$ so that $\tilde{Z}_2$ does not describe the partition function of a well-defined CFT: The multiplicities of some states inside the square brackets are non-integer. In order to cure the issue, let us add the following modular-invariant contribution

$$\begin{aligned}
Z^{\text{res}}_2 := \frac{1}{4|\eta|^2}[&|\Psi_3(3|1)_x - \Psi_3(3|-1)_x|^2 + |\tilde{\Psi}_3(3|1)_x - \tilde{\Psi}_3(3|-1)_x|^2\\
&+ |\Psi_6(3|1)_x - \Psi_6(3|-1)_x|^2 + |\tilde{\Psi}_6(3|1)_x + \tilde{\Psi}_6(3|-1)_x|^2]\,.
\end{aligned} \tag{4.B.5}$$

Note that this expression reduces to $Z^{res}_2(x = 1) = 1$ due to Euler's pentagonal number theorem. Regrouping terms, we end up with

$$\begin{aligned}
Z_2 &:= \tilde{Z}_2 + Z^{\text{res}}_2 \qquad\qquad (4.B.6)\\
&= \frac{1}{2|\eta|^2}\Big\{|\Psi_1(3|1)_x|^2 + |\tilde{\Psi}_1(3|1)_x|^2 + |\Psi_2(3|1)_x|^2 + |\tilde{\Psi}_2(3|1)_x|^2 + |\Psi_3(3|1)_x|^2 + |\Psi_3(3|-1)_x|^2\\
&\quad + |\tilde{\Psi}_3(3|1)_x|^2 + |\tilde{\Psi}_3(3|-1)_x|^2 + |\Psi_6(3|1)_x|^2 + |\Psi_6(3|-1)_x|^2 + |\tilde{\Psi}_6(3|1)_x|^2 + |\tilde{\Psi}_6(3|-1)_x|^2\Big\}\,.
\end{aligned}$$

With a little bit of additional effort, this expression may be resummed into a more compact form

$$Z_2 = \frac{1}{|\eta|^2} \sum_{n,w\in\mathbb{Z}} q^{\frac{k_L^2}{2}} \bar{q}^{\frac{k_R^2}{2}} x^{rk_L} \bar{x}^{rk_R} \qquad \text{with} \qquad k_{L,R} = \frac{n}{r} \pm \frac{wr}{2}, \quad r = 2\sqrt{3} \tag{4.B.7}$$

which is the well known partition function of a free boson that has been compactified on a circle of radius $r = 2\sqrt{3}$.

4.C Branching functions for $N = 3, 4, 5$

In this appendix we give the q-expansions of the branching functions $\mathcal{X}^{\mathcal{W}}$ up to order $O(q^{c_N/6})$. In order to better read off the conformal weights h, we omit the overall factor $q^{-c_N/24}$ in the $\mathcal{X}^{\mathcal{W}}$'s. As shown in section 4.3.1, there are no chiral primaries with conformal weight larger than h=$c_N/6$. Chiral primary fields (with $h = Q$) are marked by CP_h. We restrict to the NS sector, i.e. we only display the branching functions $\mathcal{X}^{\mathcal{W}}_{(id+v,a)}$ ($a \in \mathcal{J}^0_N$). Similar expansions exist for all the branching functions $\mathcal{X}^{\mathcal{W}}_{(sp+c,a)}$ in the R sector. The calculations of this appendix were performed using Mathematica package affine.m [148].

4.C.1 $N = 3$

The central charge is $c_3 = 8/3$. Chiral primaries exist only for $h \leq 4/9$. The expansion of the branching functions $\mathcal{X}^{\mathcal{W}}_{(id+v,a)}$ ($a \in \mathcal{J}^0_3$) is given by

$$\begin{aligned}
\mathcal{X}^{\mathcal{W}}_{(id+v,[0,0])} &= 1 + O(q^1) && \boxed{\mathrm{CP}_0}\\
\mathcal{X}^{\mathcal{W}}_{(id+v,[1,1])} &= \left(x^{-1/3} + x^{1/3}\right) q^{1/6} + O(q^{2/3}) && \boxed{\mathrm{CP}_{1/6}}\\
\mathcal{X}^{\mathcal{W}}_{(id+v,[3,0])} &= \left(x^{-2/3} + 1 + x^{2/3}\right) q^{1/3} + O(q^{5/6}) && \boxed{\mathrm{CP}_{1/3}}\\
\mathcal{X}^{\mathcal{W}}_{(id+v,[2,2])} &= q^{1/9} + O(q^{11/18}) \,.
\end{aligned}$$

4.C.2 $N = 4$

The central charge is $c_4 = 5$. The sector independent bound on the conformal weight of a chiral primary state is therefore $h \leq 5/6$.

The expansion of the branching functions $\mathcal{X}^{\mathcal{W}}_{(id+v,a)}$ ($a \in \mathcal{J}^0_4$) is given by

$$\begin{aligned}
\mathcal{X}^{\mathcal{W}}_{(id+v,[0,0,0])} &= 1 + O(q^1) && \boxed{\mathrm{CP}_0}\\
\mathcal{X}^{\mathcal{W}}_{(id+v,[1,0,1])} &= \left(x^{1/3} + x^{-1/3}\right) q^{1/6} + \left(x^{2/3} + 1 + x^{-2/3}\right) q^{2/3} + O(q^{7/6}) && \boxed{\mathrm{CP}_{1/6}}\\
\mathcal{X}^{\mathcal{W}}_{(id+v,[0,2,0])} &= q^{1/2} + O(q^1)\\
\mathcal{X}^{\mathcal{W}}_{(id+v,[2,1,0])} = \mathcal{X}^{\mathcal{W}}_{(id+v,[0,1,2])} &= \left(x^{2/3} + 1 + x^{-2/3}\right) q^{1/3}\\
&\quad + \left(x + 2x^{1/3} + 2x^{-1/3} + x^{-1}\right) q^{5/6} + O(q^{4/3}) && \boxed{\mathrm{CP}_{1/3}}
\end{aligned}$$

$$\mathcal{X}^{W}_{(id+v,[4,0,0])} = \left(x + x^{1/3} + x^{-1/3} + x^{-1}\right) q^{1/2} + O(q) \qquad \boxed{\mathrm{CP}_{1/2}}$$

$$\mathcal{X}^{W}_{(id+v,[2,0,2])} = q^{1/6} + \left(x + 2x^{1/3} + 2x^{-1/3} + x^{-1}\right) q^{2/3} + O(q^{7/6})$$

$$\mathcal{X}^{W}_{(id+v,[1,2,1])} = \left(x + 2x^{1/3} + 2x^{-1/3} + x^{-1}\right) q^{1/2} + O(q) \qquad \boxed{\mathrm{CP}_{1/2}}$$

$$\mathcal{X}^{W}_{(id+v,[2,3,0])} = \left(x^{2/3} + 1 + x^{-2/3}\right) q^{1/2} + O(q)$$

$$\mathcal{X}^{W}_{(id+v,[3,1,1])} = \left(x^{1/3} + x^{-1/3}\right) q^{1/4} + O(q^{5/4})$$

$$\mathcal{X}^{W}_{(id+v,[0,4,0])} = \left(x^{4/3} + x^{2/3} + 2 + x^{-2/3} + x^{-4/3}\right) q^{2/3} + O(q^{7/6}) \qquad \boxed{\mathrm{CP}_{2/3}}$$

$$\begin{aligned}\mathcal{X}^{W}_{(id+v,[2,2,2])} &= \left(x^{2/3} + 2 + x^{-2/3}\right) q^{1/3} \\ &\quad + \left(3x + 5x^{1/3} + 5x^{-1/3} + 3x^{-1}\right) q^{5/6} + O(q^{4/3}) \qquad \boxed{\mathrm{CP}_{1/3}}\ .\end{aligned}$$

4.C.3 $N = 5$

The central charge is $c_5 = 8$, and we expand up to order $O(q^{4/3})$ in order to capture all contributions from chiral primaries with conformal weight $h \leq c_5/6$. In order to write the expansion of branching functions $\mathcal{X}^{W}_{(id+v,a)}$ ($a \in \mathcal{J}^0_5$) we shall introduce the shorthand $y^n \equiv x^n + x^{-n}$.

$$\mathcal{X}^{W}_{(id+v,[0,0,0,0])} = 1 + q + O(q^{3/2}) \qquad \boxed{\mathrm{CP}_{0}}$$

$$\mathcal{X}^{W}_{(id+v,[1,0,0,1])} = y^{1/3} q^{1/6} + (y^{2/3} + 1)\, q^{2/3} + (y + 3y^{1/3})\, q^{7/6} + O(q^{5/3}) \qquad \boxed{\mathrm{CP}_{1/6}}$$

$$\mathcal{X}^{W}_{(id+v,[0,1,1,0])} = q^{7/15} + (y + 2y^{1/3})\, q^{29/30} + O(q^{22/15})$$

$$\begin{aligned}\mathcal{X}^{W}_{(id+v,[2,0,1,0])} &= (y^{2/3} + 1)\, q^{1/3} + (y + 2y^{1/3})\, q^{5/6} \\ &\quad + (y^{4/3} + 5y^{2/3} + 6)\, q^{4/3} + O(q^{11/6}) \qquad \boxed{\mathrm{CP}_{1/3}}\end{aligned}$$

$$\mathcal{X}^{W}_{(id+v,[1,2,0,0])} = y^{1/3} q^{7/10} + (y^{4/3} + 3y^{2/3} + 4)\, q^{6/5} + O(q^{17/10})$$

$$\mathcal{X}^{W}_{(id+v,[0,1,0,2])} = \mathcal{X}^{W}_{(id+v,[2,0,1,0])} \qquad \boxed{\mathrm{CP}_{1/3}}$$

$$\mathcal{X}^{W}_{(id+v,[0,0,2,1])} = \mathcal{X}^{W}_{(id+v,[1,2,0,0])}$$

$$\mathcal{X}^{W}_{(id+v,[3,1,0,0])} = (y + y^{1/3})\, q^{1/2} + (y^{4/3} + 2y^{2/3} + 3)\, q + O(q^{3/2}) \qquad \boxed{\mathrm{CP}_{1/2}}$$

$$\mathcal{X}^{W}_{(id+v,[2,0,0,2])} = q^{1/5} + (y + 2y^{1/3})\, q^{7/10} + (2y^{4/3} + 4y^{2/3} + 7)\, q^{6/5} + O(q^{17/10})$$

$$\mathcal{X}^{W}_{(id+v,[1,1,1,1])} = (y + 2y^{1/3})\, q^{1/2} + (2y^{4/3} + 6y^{2/3} + 8)\, q + O(q^{3/2}) \qquad \boxed{\mathrm{CP}_{1/2}}$$

$$\mathcal{X}^{W}_{(id+v,[1,0,3,0])} = \mathcal{X}^{W}_{(id+v,[0,3,0,1])}$$

$$\mathcal{X}^{W}_{(id+v,[0,3,0,1])} = (y^{2/3} + 1)\, q^{4/5} + (y^{5/3} + 4y + 6y^{1/3})\, q^{13/10} + O(q^{9/5})$$

$$\mathcal{X}^{W}_{(id+v,[0,2,2,0])} = (y^{4/3} + y^{2/3} + 2)\, q^{2/3} + (y^{5/3} + 3y + 5y^{1/3})\, q^{7/6} + O(q^{5/3}) \qquad \boxed{\mathrm{CP}_{2/3}}$$

$$\mathcal{X}^{W}_{(id+v,[0,0,1,3])} = \mathcal{X}^{W}_{(id+v,[3,1,0,0])} \qquad \boxed{\mathrm{CP}_{1/2}}$$

$$\mathcal{X}^{W}_{(id+v,[5,0,0,0])} = (y^{4/3} + y^{2/3} + 1)\, q^{2/3} + (y^{5/3} + y + 2y^{1/3})\, q^{7/6} + O(q^{5/3}) \qquad \boxed{\mathrm{CP}_{2/3}}$$

$$\begin{aligned}\mathcal{X}^{W}_{(id+v,[3,0,1,1])} &= y^{1/3} q^{3/10} + (y^{4/3} + 3y^{2/3} + 4)\, q^{4/5} \\ &\quad + (2y^{5/3} + 7y + 13y^{1/3})\, q^{13/10} + O(q^{41/30})\end{aligned}$$

$$\mathcal{X}^{W}_{(id+v,[2,2,0,1])} = (y^{4/3} + 2y^{2/3} + 3)\, q^{2/3} + (2y^{5/3} + 6y + 10y^{1/3})\, q^{7/6} + O(q^{5/3}) \qquad \boxed{\mathrm{CP}_{2/3}}$$

$$\mathcal{X}^{W}_{(id+v,[2,1,2,0])} = (y^{2/3} + 1)\, q^{8/15} + (y^{5/3} + 4y + 7y^{1/3})\, q^{31/30} + O(q^{23/15})$$

$$\begin{aligned}\mathcal{X}^{W}_{(id+v,[1,3,1,0])} &= (y^{5/3} + 2y + 3y^{1/3})\, q^{5/6} \\ &\quad + (2y^{2} + 6y^{4/3} + 11y^{2/3} + 13)\, q^{4/3} + O(q^{11/6}) \qquad \boxed{\mathrm{CP}_{5/6}}\end{aligned}$$

$$\mathcal{X}^{W}_{(id+v,[1,1,0,3])} = \mathcal{X}^{W}_{(id+v,[3,0,1,1])}$$

$$\mathcal{X}^{W}_{(id+v,[1,0,2,2])} = \mathcal{X}^{W}_{(id+v,[2,2,0,1])} \qquad \boxed{\mathrm{CP}_{2/3}}$$

$$\mathcal{X}^{W}_{(id+v,[0,5,0,0])} = (y^{2} + y^{4/3} + 2y^{2/3} + 2)\, q + O(q^{3/2}) \qquad \boxed{\mathrm{CP}_{1}}$$

$$\mathcal{X}^{W}_{(id+v,[0,2,1,2])} = (y^{2/3} + 1)\, q^{8/15} + (y^{5/3} + 4y + 7y^{1/3})\, q^{31/30} + O(q^{23/15})$$

$$\mathcal{X}^{W}_{(id+v,[0,1,3,1])} = \mathcal{X}^{W}_{(id+v,[1,3,1,0])} \qquad \boxed{\mathrm{CP}_{5/6}}$$

$$\mathcal{X}^{W}_{(id+v,[4,1,0,1])} = (y^{2/3} + 1)\, q^{2/5} + (y^{5/3} + 3y + 5y^{1/3})\, q^{9/10} + O(q^{7/5})$$

$$\mathcal{X}^{W}_{(id+v,[4,0,2,0])} = q^{4/15} + (y + 2y^{1/3})\, q^{23/30} + (3y^{4/3} + 6y^{2/3} + 10)\, q^{19/15} + O(q^{7/5})$$

$$\mathcal{X}^{W}_{(id+v,[3,2,1,0])} = (y + 2y^{1/3})\, q^{19/30} + (y^{2} + 4y^{4/3} + 9y^{2/3} + 11)\, q^{17/15} + O(q^{49/30})$$

$$\mathcal{X}^{W}_{(id+v,[3,0,0,3])} = (y^{2/3} + 1)\, q^{3/5} + (y^{5/3} + 3y + 5y^{1/3})\, q^{11/10} + O(q^{8/5})$$

$$\mathcal{X}^{W}_{(id+v,[2,4,0,0])} = (y^{4/3} + y^{2/3} + 2)\, q^{13/15} + O(q^{41/30})$$

$$\mathcal{X}^{W}_{(id+v,[2,1,1,2])} = (y^{2/3} + 2)\, q^{2/5} + (y^{5/3} + 5y + 9y^{1/3})\, q^{9/10} + O(q^{7/5})$$

$$\mathcal{X}^{W}_{(id+v,[2,0,3,1])} = (y + 2y^{1/3})\, q^{7/10} + (y^{2} + 5y^{4/3} + 10y^{2/3} + 13)\, q^{6/5} + O(q^{17/10})$$

$$\mathcal{X}^{W}_{(id+v,[1,3,0,2])} = \mathcal{X}^{W}_{(id+v,[2,0,3,1])}$$

$$\mathcal{X}^{W}_{(id+v,[1,2,2,1])} = (y + 2y^{1/3})\, q^{17/30} + (y^{2} + 5y^{4/3} + 11y^{2/3} + 14)\, q^{16/15} + O(q^{47/30})$$

$$\begin{aligned}\mathcal{X}^{W}_{(id+v,[1,1,4,0])} &= (y^{4/3} + 2y^{2/3} + 2)\, q^{4/5} \\ &\quad + (y^{7/3} + 4y^{5/3} + 9y + 13y^{1/3})\, q^{13/10} + O(q^{9/5})\end{aligned}$$

$$
\begin{aligned}
\mathcal{X}^{W}_{(id+v,[0,3,3,0])} &= q^{3/5} + (y^{5/3} + 3y + 5y^{1/3})\, q^{11/10} + O(q^{8/5}) \\
\mathcal{X}^{W}_{(id+v,[0,1,2,3])} &= \mathcal{X}^{W}_{(id+v,[3,2,1,0])} \\
\mathcal{X}^{W}_{(id+v,[3,2,0,2])} &= (y + 2y^{1/3})\, q^{1/2} + (3y^{4/3} + 7y^{2/3} + 9)\, q + O(q^{3/2}) && \boxed{\mathrm{CP}_{1/2}} \\
\mathcal{X}^{W}_{(id+v,[3,1,2,1])} &= y^{1/3} q^{11/30} + (2y^{4/3} + 6y^{2/3} + 8)\, q^{13/15} + O(q^{41/30}) \\
\mathcal{X}^{W}_{(id+v,[2,3,1,1])} &= (y^{4/3} + 3y^{2/3} + 4)\, q^{2/3} \\
&\quad + (4y^{5/3} + 12y + 19y^{1/3})\, q^{7/6} + O(q^{5/3}) && \boxed{\mathrm{CP}_{2/3}} \\
\mathcal{X}^{W}_{(id+v,[2,2,3,0])} &= (y^{2/3} + 1)\, q^{7/15} + (y^{5/3} + 4y + 7y^{1/3})\, q^{29/30} + O(q^{22/15}) \\
\mathcal{X}^{W}_{(id+v,[2,2,2,2])} &= q^{1/3} + (y^{5/3} + 5y + 9y^{1/3})\, q^{5/6} \\
&\quad + (4y^{2} + 19y^{4/3} + 36y^{2/3} + 47)\, q^{4/3} + O(q^{11/16}) && \boxed{\mathrm{CP}_{5/6}}\ .
\end{aligned}
$$

4.D Tensor powers of the SU(N) adjoint representation

In this appendix we discuss the phylogeny of the formula (4.4.19) and we briefly outline its derivation.

Let us start with the following very well known formula for decomposing the k-th tensor power of the SU(N) fundamental representation into irreducible SU(N) representations

$$
\square^{\otimes k} = \sum_{Y \vdash k} \frac{k!}{\prod_{(i,j)\in Y} h(i,j)}\, a(Y)\,. \tag{4.D.1}
$$

Here $Y \vdash k$ denotes a Young diagram $Y = (l_1, \ldots l_r)$ which has k boxes, i.e. is a partition of k, we use $a(Y)$ to label the corresponding SU(N) representation and

$$
h(i,j) := l^{\mathrm{T}}_j - i + l_i - j + 1
$$

is defined as the length of a hook (i, j) belonging to the given partition $Y = (l_1, \ldots l_r)$. The product runs over the boxes of the Young diagram Y.

The raison d'être of formula (4.D.1) is the renowned Schur-Weyl duality [149]: The image of the action of the symmetric group S_k on the k-th tensor power of the $\mathrm{GL}_N(\mathbb{C})$ fundamental representation space can be identified with the centralizer algebra of $\mathrm{GL}_N(\mathbb{C})$ and vice versa. It means that under the joint action of S_k and $\mathrm{GL}_N(\mathbb{C})$, the tensor power decomposes into a direct sum of tensor products of irreducible modules for these two groups thus yielding formula (4.D.1). The coefficient

$$
\frac{k!}{\prod_{i,j)\in Y} h(i,j)}
$$

is just the dimension of a corresponding representation of the symmetric group S_k.

It turns out that for a tensor power of the adjoint representation $\mathbf{adj}^k = \square^k \otimes \overline{\square}^k$ a similar correspondence holds, only that now the symmetric group algebra gets replaced by a more sophisticated structure known as the *walled Brauer algebra* [138, 139]. The associated decomposition reads

$$\mathbf{adj}^{\otimes k} = \sum_{m=0}^{k} b_m^{(k)} \sum_{Y', Y'' \vdash m} \frac{m!}{\prod_{(i',j') \in Y'} h(i', j')} \frac{m!}{\prod_{(i'',j'') \in Y''} h(i'', j'')} \cdot a(Y', Y'') \,. \tag{4.D.2}$$

Here $a(Y', Y'')$ denotes an SU(N) representation generated from two Young diagrams Y' and Y'' according to (4.2.16), $Y', Y'' \vdash m$ means that Y' and Y'' are Young diagrams corresponding to partitions of m. The products in (4.D.2) run over boxes of the Young diagrams Y' and Y''. The range of validity here is $k \le \left\lfloor \frac{N}{2} \right\rfloor$, otherwise not all of the listed representations $a(Y', Y'')$ are allowed to appear on the right-hand side which results in a reshuffling of the remaining multiplicities. The multiplicities

$$b_m^{(k)} := \sum_{i=0}^{k-m} (-1)^{i+k+m} \; i! \binom{k}{m} \binom{k-m}{i} \binom{i+m}{i} \equiv \frac{k!}{m!} \sum_{i=0}^{k-m} \frac{(-1)^i}{i!} \binom{k-i}{m} \tag{4.D.3}$$

are actually the most interesting feature of formula (4.D.2). They reflect the fact that the new algebra replacing the symmetric group algebra in this case is not just a direct product of two copies of the latter. We refer the reader to [137] for background on walled Brauer algebras as well as the representation-theoretic discussion of decomposition formulae, such as the one displayed above.

There is a simple way to argue that the coefficients $b_m^{(k)}$ should have the form (4.D.3). Indeed, let us notice that they can be rewritten as

$$b_m^{(k)} = (-1)^{k+m} \binom{k}{m} \, {}_2F_0\!\left(\begin{matrix} m+1, -(k-m) \\ - \end{matrix} ; 1 \right) \tag{4.D.4}$$

where ${}_2F_0$ is the hypergeometric function of type $(2,0)$. It is now straightforward to see that the coefficients $b_m^{(k)}$ actually satisfy the recursion relation

$$b_m^{(k)} = \frac{k(k-1)}{k-m} (b_m^{(k-2)} + b_m^{(k-1)}) \tag{4.D.5}$$

with the initial conditions $b_m^{(m-1)} = 0$, $b_m^{(m)} = 1$. This nice recursion readily suggests a way to proceed in proving the decomposition (4.D.2) with coefficients (4.D.3). Acting by induction, the inductive step is just to apply the Littlewood-Richardson rule [140, 141] for multiplying all the Young diagrams present in the decomposition of the $(k-2)$-nd adjoint power by another two adjoint representations, carefully factoring out two hook multipliers which describe adding boxes to 'small' Young diagrams Y' and Y''. The relation (4.D.5) then allows to disentangle the obtained expression bringing it to the needed k-th step's outfit. For the original combinatorial proof involving a generalization of the Schensted insertion algorithm, see [150].

4.E Antisymmetric part of the powers of adjoints

Before we begin with our discussion of eq. (4.4.24) let us give the precise statement and introduce a bit of additional notation. According to eq. (4.4.24), the part of the decomposition of the SU(N) adjoint power which transforms in the totally antisymmetric representation of the permutation group S_k is given by

$$\left\{\mathbf{adj}^{\otimes k}\right\}_{\text{antisymm}} = \sum_{m=1}^{k} d_m^{(k)} \sum_{Y' \vdash m} a\,(Y', Y')\,. \tag{4.E.1}$$

Here $a\,(Y', Y')$ denotes an SU(N) representation generated from two Young diagrams $Y' = Y''$ according to (4.2.16), $Y' \vdash m$ means that Y' is a Young diagram satisfying $|Y'| = m$, i.e. is a partition of m.

The coefficients $d_m^{(k)}$ read

$$d_m^{(k)} := \sum_{1 \le m_1 < \cdots < m_{k-1} \le m-1} r_{m_1} r_{m_2 - m_1} \ldots r_{m_{k-1} - m_{k-2}} r_{m - m_{k-1}} \tag{4.E.2}$$

where r_m are expressed as

$$r_m := \frac{1}{m!} \left(\frac{d}{dq}\right)^{m-1} \left[\left(\frac{q}{\frac{\phi^3(q^2)}{\phi(q)\phi(q^4)} - 1}\right)^m\right]_{q=0} \equiv \frac{1}{m!} \left(\frac{d}{dq}\right)^{m-1} \left[\left(\frac{q}{\phi(-q) - 1}\right)^m\right]_{q=0} \tag{4.E.3}$$

and ϕ is the Euler function

$$\phi(q) := \prod_{i=1}^{\infty} (1 - q^i). \tag{4.E.4}$$

The range of validity of the formula (4.E.1) is restricted by $k \le N - 1$, otherwise not all of the listed representations $a\,(Y', Y')$ are allowed to appear on the right-hand side which results in a reshuffling of the remaining multiplicities.

We checked this formula by direct computation up to $k = 9$. Unfortunately, we were not able to find it in the literature. One immediate aspect to notice is that upon applying the Lagrange inversion formula, the coefficients $d_m^{(k)}$ turn out to be just coefficients of the series expansion of $q^k(\phi)$, where $q(\phi)$ denotes the function inverse to $\phi(-q) - 1$ around $q = 0$.

The parts of the SU(N) adjoint powers' decomposition transforming in other representations of the symmetric group will, of course, involve the 'non-diagonal' representations $a\,(Y', Y'')$, with $Y' \neq Y''$, to yield the full decomposition (4.D.1) when summed up over all representations of the symmetric group S_k. It is tempting to speculate that the multiplicities in those partial decompositions may be characterized by other modular forms replacing the Dedekind eta $\eta(q) = q^{\frac{1}{24}}\phi(q)$. The exact formulae of this type will be discussed elsewhere.

Chapter 5

Three-point functions of Toda CFT

In this Chapter we study a formula for the 3-point structure constants of generic primary fields in Toda field theory, conjectured in [27]. Its motivation comes, via AGT-W correspondence, from expressions for partition functions of certain non-Lagrangian gauge theories, computable using topological string techniques. In order to check the validity of the proposal for previously known special cases, we will obtain here a renowned formula by Fateev and Litvinov and show that the degeneration on a first level of one of the three primary fields on the Toda side corresponds to a particular Higgsing of the T_N theories. The exposition is based on our paper [3], joint with V. Mitev and E. Pomoni.

5.1 Interlude

The solution of the Liouville field theory is a success story in the two-dimensional CFT. This theory describes scattering on a simple exponential potential. It has been long ago realized to govern also world-sheet reparametrizations of a bosonic string in non-critical dimensions [25], thus representing an ideal playground for studying two-dimensional quantum gravity. Its various symmetry properties have played a key role in obtaining a DOZZ formula for the three-point functions [71, 72] which was then derived in [67, 73]. Together with explicit expressions for Virasoro conformal blocks, first conjectured in [15] and then obtained in [151], this constituted a full solution of this CFT, the higher-point correlation functions being constructed via conformal bootstrap.

Computation of correlation functions of a multifield generalization of the Liouville CFT, the A_{N-1} Toda conformal field theory, is naturally a next challenge to undertake. Alike Liouville, Toda CFT nourishes an intuition for a variety of topics. At its core this [1] has to do with two key features of the theory: first, with the structure of its symmetry algebra whose chiral half, denoted $\mathbf{W}_N$, is a particular non-linear extension of Virasoro algebra by currents of spins $3, \ldots, N$ [153], second, with the fact that the set of $\mathbf{W}_N \times \overline{\mathbf{W}}_N$ representations making up the Hilbert space of the Toda CFT is continuous, i.e. that the theory is non-compact. Both these properties are mirrored in the fact that the sl(N) Toda CFT is actually a particular Hamiltonian reduction of the $\widehat{\mathrm{sl}}(N)$ WZNW model. [152]

The reasons to be interested in solving a theory with these two features are manifold. The $\mathbf{W}_N$ symmetry is known to govern the critical behaviour of numerous statistical systems, including various general-

[1] In this Chapter we discuss only the A_{N-1} Toda CFT. Toda theories related to a wider class of W-algebras can be obtained by more general Hamiltonian reductions of the $\widehat{\mathrm{sl}}(N)$ WZNW model, see [152].

izations of Ising model ($\mathbb{Z}_N$, tricritical), Potts model ($\mathbb{Z}_3$, Ashkin-Teller) and many others [154]. Recently it was also realized that the $\mathbf{W}_N$ minimal models are the key players in the novel higher-spin versions of AdS/CFT duality [155]. To cover links to somewhat more formal material, let us mention that $\mathbf{W}_N$ can be also promoted to a local symmetry of a space-time, giving rise to the theories termed correspondingly as W-gravity and W-strings.[2] A geometrical approach to Toda CFT [158, 159], relevant for these constructions, is also intrinsically valuable. More specifically, the classical Toda equations of motion can be written as a flatness condition. Quantization then relates this problem to the quantization of moduli spaces of flat SL(N) connections and higher quantum Teichmüller theory (see [160] for the discussion of $N = 2$ case and [161]).

In a broader physical setting, Toda CFT is also a reference point in studying a class of massive deformations of CFTs [162] called affine Toda theories. A massive deformation of a conformal field theory is specified by the CFT data and the choice of a particular relevant operator perturbing the conformal dynamics. For an important class of deformations when the resulting QFT is integrable, Toda CFT provides short-distance asymptotics of correlation functions and vacuum expectation values of exponential fields [163, 164].

Surprisingly, despite the diverse connections of the Toda theory to various physical and mathematical topics and a longstanding interest in it, even the three-point functions of its primary fields remained elusive until very recently, for about twenty years since discovering those of Liouville. The state of art, due to works of Fateev and Litvinov [28, 83, 84] comprised either particular limits (semi-classical, mini-superspace) or special cases constraining symmetry algebra representations to which one of the primary fields belongs.

Recently, a new formula was conjectured for a *three-point function of generic primary fields* [27]. A keystone of this proposal is the AGT-W correspondence [15, 165], which is a relation between 4d $\mathcal{N} = 2$ SU(N) quiver gauge theories and the 2d $\mathbf{W}_N$ Toda CFT. Specifically, upon an appropriate identification of the parameters, the correlation functions of the 2d Toda CFT are equal to the partition functions of the corresponding 4d $\mathcal{N} = 2$ gauge theories. The conformal blocks of the 2d CFTs are given by the instanton partition functions of Nekrasov [15,165], while the 3-point structure constants are obtained by the partition functions of the T_N superconformal theories [94, 166]. The T_N theories have no Lagrangian description and thus their partition functions were unknown until recently [27, 94, 167]. The sole exception, already mentioned above, was the $\mathbf{W}_2 \equiv$ Vir case, *i.e.* the Liouville theory, whose 3-point structure constants given by the DOZZ formula equal to a partition function of four free hypermultiplets [94, 168].

The fact that the T_N theories have no known Lagrangian description was by-passed by using a *generalized* version of AGT-W: a relation between 5d gauge theories compactified on S^1 and 2d q-deformed Liouville/Toda CFT [87–101], where the circumference β of the S^1 corresponds to the deformation parameter $q = e^{-\beta}$ of the CFT. In 5d, the partition functions can be computed not only using localization, which requires a Lagrangian, but also by using the powerful tool of topological strings [169]. Employing this technology, the partition functions of the 5d T_N theories were calculated in [94] (see also [167]), and it was suggested that they should be interpreted as the 3-point structure constants of the q-deformed Toda. In [27] it was shown how to take the 4d limit, corresponding to $\beta \to 0$ or equivalently to $q \to 1$, thus obtaining the partition function (5.2.6) of the 4d T_N theories. Let us stress that taking this limit is a tricky business, as the expression includes non-trivial multiple sums and integrals. This is the reason why we will

[2]The physical states in such a string theory, for example, lie in the cohomology of a BRST operator involving the full $\mathbf{W}_N$ algebra, as opposed to Virasoro algebra for ordinary strings, see [156, 157] for more details.

always work with the q-deformed formulae and take the limit only at the end. Without going into details here, let us state that the techniques of [27] will provide the solution not only for the 3-points functions of $\mathbf{W}_N$ primaries, but also for those involving descendant fields. This is left for a future work. Let us now come to an objective of the present Chapter.

If one has a general formula, the main interest, of course, would be to *reproduce all previously known Toda results* from this finding. Here lies the essential difficulty we start to address in our work: the corresponding expression is quite involved, including non-trivial multiple summations over partitions and multiple integrations. The cumbersomeness of this expression as opposed to the elegant DOZZ formula probably unveils one of the reasons it has not been found for such a long time as well as suggests to look for a possibly hidden simplification. The purpose of this Chapter is to initiate the program of making precise and checking the general proposal against known special cases. Namely, we show here how to reproduce an important formula by Fateev and Litvinov (2.4.46), giving a three-point function in which one of the primary fields contains a null-vector at level one. In the rest of the Chapter we will refer to them as semi-degenerate[3], as opposed to completely degenerate ones, containing $N-1$ linearly independent null-vectors.

Having advertized the general formula for the 3-point function of generic primary fields of Toda CFT quite a lot already, we finish this interlude by spelling it out:

$$C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \text{const}\times\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{\left(2Q-\sum_{i=1}^3\alpha_i,\rho\right)}{b}} \times\lim_{\beta\to 0}\beta^{-2Q\sum_{i=1}^3(\alpha_i,\rho)}\oint\prod_{i=1}^{N-2}\prod_{j=1}^{N-1-i}\left[\frac{d\tilde{A}_i^{(j)}}{2\pi i\tilde{A}_i^{(j)}}\,||M(\mathfrak{t},\mathfrak{q})||^2\right]\left\|\mathcal{Z}_N^{\text{top}}\right\|^2 \quad (5.1.1)$$

where by 'const' we mean an overall function of only b that is independent of the weights of the CFT primaries. To grasp various details of the right-hand side, in particular the topological string amplitude $\mathcal{Z}_N^{\text{top}}$, we require some notions and notations which will come in the next section. The impatient reader may skip the explanations and proceed straight to the formulae (5.2.8), (3.4.68), (5.2.16), (5.2.17), (5.2.18), (5.B.1) consulting also Chapter 3 for definitions of the encountered special functions.

The rest of this Chapter is organized as follows. First, we will need some basics of the AGT dictionary collected in section 5.2. In the subsequent section 5.3, the discussion temporarily deviates from the CFT matters focusing rather on the interplay between the moduli spaces of the corresponding gauge theories and 5-brane web physics. We argue that the semi-degeneration of a primary field on the (q-deformed) CFT side mirrors a Higgsing of the T_N theory on the 4d (5d) side. A more CFT-oriented reader can skip this section. The AGT genesis of Fateev-Litvinov formula for $\mathbf{W}_3$ Toda 3-point function, via pinching an integration contour by residues of the corresponding integrand and applying non-trivial summation theorems, is what section 5.4 focuses upon. Discussion of a general $\mathbf{W}_N$ case is given in section 5.5. The conclusion and outlook follow, whereas the appendices are devoted to an overview of notations and, most importantly, to describing an interplay of the Kaneko-Macdonald-Warnaar sl(N) hypergeometric functions with Nekrasov partition functions which is of major importance for our calculations.

[3] A representation of $\mathbf{W}_N$ can contain a null vector at some level higher than one. Such representations are called semi-degenerate as well. The three-point functions containing one primary belonging to such representation will not be considered in the present note.

5.2 AGT dictionary

According to the AGT-W correspondence [15, 165], the correlation functions of the 2d Toda CFT are obtained from the partition functions of the corresponding 4d $\mathcal{N} = 2$ gauge theories as

$$\mathcal{Z}^{S^4} = \int [da] \left| \mathcal{Z}^{\text{4D}}_{\text{Nek}}(a, m, \tau, \epsilon_{1,2}) \right|^2 \propto \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_n}(z_n) \rangle_{\text{Toda}} , \tag{5.2.2}$$

where the Omega-deformation parameters are related to the Toda coupling constant[4] via $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$. Moreover, a stands for the set of Coulomb moduli of the theory, m for the masses of the hypermultiplets and τ for the coupling constants. The correspondence relates the masses m to the weights $\boldsymbol{\alpha}_i$ and the couplings constants τ to the insertion points z_i of the primary fields. In particular, the conformal blocks of the 2d CFTs are given by the appropriate Nekrasov instanton partition functions [15,165] and the 3-point structure constants by the partition functions of the T_N superconformal theories on S^4 [94, 166].

A similar relation between 5d gauge theories and 2d q-CFT exists [87–101], which relates the 5d Nekrasov partition functions on $S^4 \times S^1$ to correlation functions the of q-deformed Liouville/Toda field theory:

$$\mathcal{Z}^{S^4\times S^1} = \int [da] \left| \mathcal{Z}^{\text{5D}}_{\text{Nek}}(a, m, \tau, \beta, \epsilon_{1,2}) \right|^2 \propto \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_n}(z_n) \rangle_{q\text{-Toda}} , \tag{5.2.3}$$

where $\beta = -\log q$ is the circumference of the S^1. The exponentiated Omega-background parameters

$$\mathfrak{q} = e^{-\beta\epsilon_1} , \qquad \mathfrak{t} = e^{\beta\epsilon_2} , \tag{5.2.4}$$

are used in this case. The partition function on $S^4 \times S^1$ is the 5d superconformal index, which as discussed in [169] can also be computed using topological string theory techniques

$$\mathcal{Z}^{S^4\times S^1} = \int [da] \, |\mathcal{Z}^{\text{5D}}_{\text{Nek}}(a)|^2 \propto \int [da] \, |\mathcal{Z}_{\text{top}}(a)|^2 . \tag{5.2.5}$$

The authors of [94] computed the partition functions of the 5d T_N theories on $S^4 \times S^1$ (see also [167]) and suggested that they should be interpreted as the 3-point structure constants of q-deformed Toda. Those are read off the toric-web diagrams of the T_N junctions of [170] by employing the refined topological vertex formalism of [171, 172]. In a subsequent paper [27] the authors showed how the 4d limit, corresponding to $\beta \to 0$ or $q \to 1$, is to be taken. The partition function of the 4d T_N theories on S^4 was thus shown to be

$$\mathcal{Z}^{S^4}_N = \text{const} \times \lim_{\beta\to 0} \beta^{-\frac{\chi_N}{\epsilon_1\epsilon_2}} \mathcal{Z}^{S^4\times S^1}_N , \tag{5.2.6}$$

where by 'const' we mean a function of ϵ_1, ϵ_2 that is independent of the mass parameters of the theory. The degree of divergence was determined as proportional to the quadratic Casimir of SU(N)[3]

$$\chi_N = - \sum_{1\le i<j\le N} \left[(m_i - m_j)^2 + (n_j - n_i)^2 + (l_i - l_j)^2 \right] = -N \sum_{i=1}^{3} (\boldsymbol{\alpha}_i - \boldsymbol{Q}, \boldsymbol{\alpha}_i - \boldsymbol{Q}) , \tag{5.2.7}$$

where $\boldsymbol{Q} := Q\rho = (b + b^{-1})\rho$ with the SU(N) Weyl vector ρ defined in (5.A.3). After the first equality of (5.2.7), we have introduced the mass parameters m_i, n_i and l_i of the T_N theory, which, as shown in figure 10, are connected to the Toda theory parameters [27]

[4] We also use the notation $\epsilon_+ = \epsilon_1 + \epsilon_2$. When we specialize $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$ in order to connect the topological string expressions to the Toda expressions, we have $\epsilon_+ = b + b^{-1} = Q$.

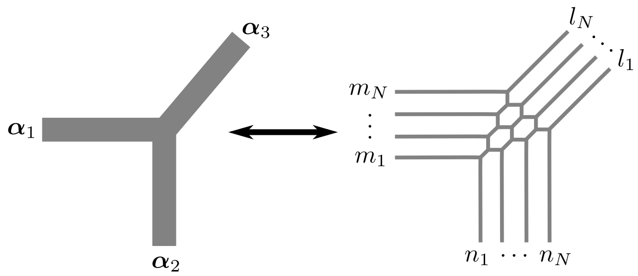


Figure 10: *This figure depicts the identification of the α weights appearing on the Toda CFT side with the position of the flavor branes on the T_N side, here drawn for the case $N = 5$.*

$$\begin{aligned} m_i &= (\boldsymbol{\alpha}_1 - \boldsymbol{Q}, h_i) = N \sum_{j=i}^{N-1} \alpha_1^j - \sum_{j=1}^{N-1} j\alpha_1^j - \frac{N+1-2i}{2} Q\,, \\ n_i &= -(\boldsymbol{\alpha}_2 - \boldsymbol{Q}, h_i) = -N \sum_{j=i}^{N-1} \alpha_2^j + \sum_{j=1}^{N-1} j\alpha_2^j + \frac{N+1-2i}{2} Q\,, \\ l_i &= -(\boldsymbol{\alpha}_3 - \boldsymbol{Q}, h_{N+1-i}) = -N \sum_{j=N+1-i}^{N-1} \alpha_3^j + \sum_{j=1}^{N-1} j\alpha_3^j - \frac{N+1-2i}{2} Q\,. \end{aligned} \tag{5.2.8}$$

It is important to note, that the mass parameters are not all independent, but obey

$$\sum_{i=1}^{N} m_i = \sum_{i=1}^{N} n_i = \sum_{i=1}^{N} l_i = 0\,, \tag{5.2.9}$$

which is reflected in the fact that the sum of the weights h_i of the fundamental SU(N) representation is zero. Then the structure constants of three primary operators in the q-Toda theory are given by the T_N partition functions on $S^4 \times S^1$ as

$$C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \text{const} \times \left[\prod_{j=1}^{3} Y_q(\boldsymbol{\alpha}_j)\right] (1-q)^{-\chi_N} \mathcal{Z}_N^{S^4\times S^1}\,, \tag{5.2.10}$$

where by 'const' we mean a function of ϵ_1, ϵ_2 and β that is independent of the mass parameters of the theory. We stress that the superconformal index $\mathcal{Z}_N^{S^4\times S^1}$ is invariant under the affine Weyl transformations (5.A.7) and that all the non-trivial Weyl transformation properties of the structure constants are captured by the following special functions:

$$Y_q(\boldsymbol{\alpha}) := \left[\frac{(1-q^b)^{2b^{-1}}(1-q^{b^{-1}})^{2b}}{(1-q)^{2Q}}\right]^{-(\boldsymbol{\alpha},\rho)} \prod_{e>0} \Upsilon_q\left((\boldsymbol{Q}-\boldsymbol{\alpha}, e)\right)\,, \tag{5.2.11}$$

with the functions Υ_q defined in Chapter 3 and the product taken over all positive roots e of SU(N). The partition function on $S^4 \times S^1$, or the superconformal index, for the T_N theory is given by an integral over the refined topological string amplitude with an integration measure containing the refined MacMahon

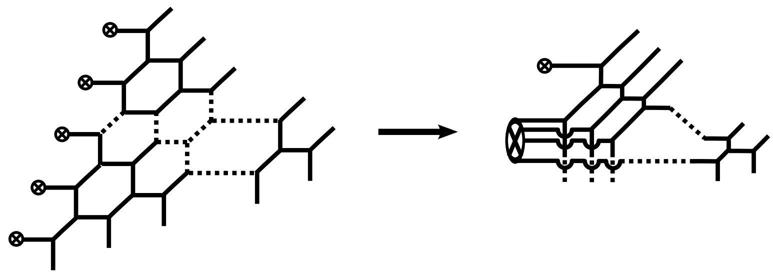


Figure 11: *The figure illustrates the desired Higgsing procedure for the general T_N diagram. We denote 7-branes by crossed circles. The left part of the figure shows the original T_N 5-brane web diagram, while the right one depicts the web diagram obtained by letting $N-1$ of the left 5-branes terminate on the same 7-brane.*

function[5] $M(\mathfrak{t},\mathfrak{q})$ [169]

$$\mathcal{Z}_N^{S^4\times S^1} := \oint \prod_{i=1}^{N-2}\prod_{j=1}^{N-1-i}\left[\frac{d\tilde{A}_i^{(j)}}{2\pi i \tilde{A}_i^{(j)}}\,\|M(\mathfrak{t},\mathfrak{q})\|^2\right]\left\|\frac{\mathcal{Z}_N^{\text{top}}}{\mathcal{Z}_N^{\text{dec}}}\right\|^2 . \tag{5.2.12}$$

Here, we have removed the decoupled degrees of freedom, referred to as 'non-full spin content' in [94],

$$\begin{aligned}\left\|\mathcal{Z}_N^{\text{dec}}\right\|^2 &:= \prod_{1\le i<j\le N}\left\|\mathcal{M}(\tilde{M}_i\tilde{M}_j^{-1})\mathcal{M}({}^{\mathfrak{t}}/_{\mathfrak{q}}\tilde{N}_i\tilde{N}_j^{-1})\mathcal{M}(\tilde{L}_i\tilde{L}_j^{-1})\right\|^2\\ &= \text{const}\times\prod_{k=1}^{3}(1-q)^{N(\alpha_k,\alpha_k-2Q)}\left((1-q^b)^{2b^{-1}}(1-q^{b^{-1}})^{2b}\right)^{(\alpha_k,\rho)}Y_q(\alpha_k)\,,\end{aligned} \tag{5.2.13}$$

where the function $\mathcal{M}$ is defined in (3.4.66). Interestingly enough, as noted in [27], these degrees of freedom are responsible for the Weyl covariance of the Toda structure constants. We use the shorthand notation

$$\|f(U_1,\ldots,U_r;\mathfrak{t},\mathfrak{q})\|^2 := f(U_1,\ldots,U_r;\mathfrak{t},\mathfrak{q})f(U_1^{-1},\ldots,U_r^{-1};\mathfrak{t}^{-1},\mathfrak{q}^{-1})\,, \tag{5.2.14}$$

see also Chapter 3. Inserting (5.2.12) into (5.2.10), we find the nice expression

$$C_q(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \text{const}\times\oint\prod_{i=1}^{N-2}\prod_{j=1}^{N-1-i}\left[\frac{d\tilde{A}_i^{(j)}}{2\pi i\tilde{A}_i^{(j)}}\,\|M(\mathfrak{t},\mathfrak{q})\|^2\right]\left\|\mathcal{Z}_N^{\text{top}}\right\|^2 . \tag{5.2.15}$$

The topological string amplitude is $\mathcal{Z}_N^{\text{top}}$ obtained from the T_N web-diagram by using the refined topological vertex formalism and reads

$$\mathcal{Z}_N^{\text{top}} = \mathcal{Z}_N^{\text{pert}}\mathcal{Z}_N^{\text{inst}}\,, \tag{5.2.16}$$

where the 'perturbative' partition function[6] is

$$\mathcal{Z}_N^{\text{pert}} := \prod_{r=1}^{N-1}\prod_{1\le i\le j\le N-r}\frac{\mathcal{M}\Big(\frac{\tilde{A}_i^{(r-1)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_{j+1}^{(r-1)}}\Big)}{\mathcal{M}\Big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r-1)}\tilde{A}_{j-1}^{(r)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_j^{(r)}}\Big)\mathcal{M}\Big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r-1)}}\Big)}\prod_{1\le i\le j\le N-r-1}\mathcal{M}\Big(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r)}}\Big), \tag{5.2.17}$$

[5]See (3.4.74) for the definition of the norm of refined MacMahon function $M(\mathfrak{t},\mathfrak{q})$.

[6]We put the words 'perturbative' and 'instanton' inside quotation marks because for the T_N there is no notion of instanton expansion, since there is no coupling constant.

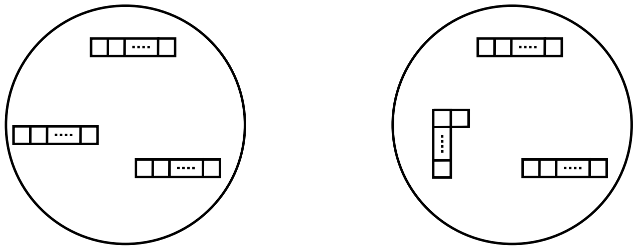


Figure 12: On the left we depict the sphere with three full punctures that corresponds to the un-Higgsed T_N with $\mathrm{SU}(N)^3$ global symmetry. On the right we show the sphere with two full punctures and one L-shaped $\{N-1,1\}$ puncture. This particular Higgsing of T_N leads to a theory with with $\mathrm{SU}(N)\times\mathrm{SU}(N)\times\mathrm{U}(1)$ global symmetry. The partition function of this theory will lead to the Toda 3-point function with one semi-degenerate primary insertion.

and the 'instanton' one is

$$\begin{aligned}
\mathcal{Z}_N^{\text{inst}} :=\; & \sum_{\nu}\prod_{r=1}^{N-1}\prod_{i=1}^{N-r}\left(\frac{\tilde{N}_r\tilde{L}_{N-r}}{\tilde{N}_{r+1}\tilde{L}_{N-r+1}}\right)^{\frac{|\nu_i^{(r)}|}{2}}\prod_{r=1}^{N-1}\prod_{1\le i\le j\le N-r}\left[\frac{\mathsf{N}^{\beta}_{\nu_i^{(r-1)}\nu_j^{(r)}}\left(a_i^{(r-1)}+a_{j-1}^{(r)}-a_{i-1}^{(r-1)}-a_j^{(r)}-\epsilon_+/2\right)}{\mathsf{N}^{\beta}_{\nu_i^{(r-1)}\nu_{j+1}^{(r-1)}}\left(a_i^{(r-1)}+a_j^{(r-1)}-a_{i-1}^{(r-1)}-a_{j+1}^{(r-1)}\right)}\right.\\
&\left.\times\frac{\mathsf{N}^{\beta}_{\nu_i^{(r)}\nu_{j+1}^{(r-1)}}\left(a_i^{(r)}+a_j^{(r-1)}-a_{i-1}^{(r)}-a_{j+1}^{(r-1)}-\epsilon_+/2\right)}{\mathsf{N}^{\beta}_{\nu_i^{(r)}\nu_j^{(r)}}\left(a_i^{(r)}+a_{j-1}^{(r)}-a_{i-1}^{(r)}-a_j^{(r)}-\epsilon_+\right)}\right],
\end{aligned}\tag{5.2.18}$$

with $a_i^{(j)}$ defined via $\tilde{A}_i^{(j)} = e^{-\beta a_i^{(j)}}$, while $\mathsf{N}^{\beta}_{\lambda\mu}$ being given in (5.C.1). The summation goes over $\frac{N(N-1)}{2}$ partitions $\nu_i^{(r)}$, $r=1,\ldots,N-1$, $i=1,\ldots,N-r$. The 'interior' Coulomb moduli $\tilde{A}_j^{(i)} = e^{-\beta a_i^{(j)}}$ are independent, while the 'border' ones are given by

$$\tilde{A}_i^{(0)}=\prod_{k=1}^{i}\tilde{M}_k\,,\qquad \tilde{A}_0^{(i)}=\prod_{k=1}^{i}\tilde{N}_k\,,\qquad \tilde{A}_i^{(N-i)}=\prod_{k=1}^{i}\tilde{L}_k\,,\tag{5.2.19}$$

where $\tilde{M}_k := e^{-\beta m_k}$ and similarly for $\tilde{N}_k$ and $\tilde{L}_k$. See appendix 4.B for more details on the parametrization of the T_N junction. The contours of integration in (5.2.15) are taken according to the q-deformed Mellin-Barnes prescription[7], for each of the variables $\tilde{A}_i^{(j)}$.

The formula (5.1.1) (correspondingly, (5.2.10)) for the structure constants of three primary fields of (q-deformed) Toda CFT, has the correct symmetry properties, the zeros that it should and, for $N=2$, gives the known answer for the Liouville CFT [27]. However, it is very implicit, requiring to perform $\frac{N(N-1)}{2}$ sums over the partitions $\nu_i^{(j)}$, followed by a $\frac{(N-1)(N-2)}{2}$-dimensional[8] integral over the Coulomb moduli $\tilde{A}_i^{(j)}$ and finally to take the 4d ($q\to 1$) limit (5.2.6). In the subsequent parts of this Chapter we will show how to derive the special case (2.4.46), known due to Fateev and Litvinov [28,83,84], from the formula (5.1.1). This provides a strong check of this general proposal.

[7] See the discussion around formula (3.4.49) in Chapter 3.

[8] It is the number of faces of the left diagram in figure 11.

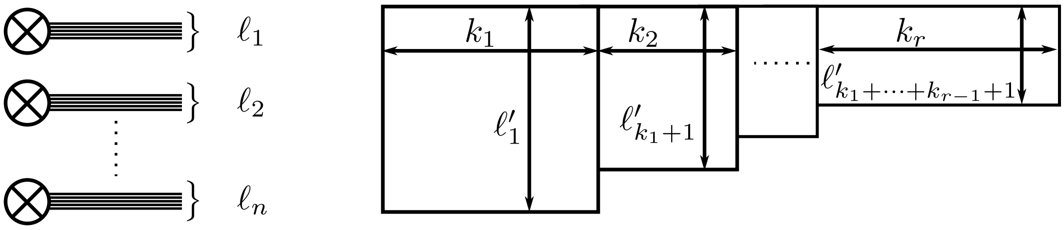


Figure 13: On the left part of this figure, we see N 5-branes ending on n 7-branes in bunches of $\ell_1, \ldots, \ell_n$ 5-branes each. On the right side of the figure, we depict the Young diagram $\{\ell'_1, \ell'_2, \ldots, \ell'_n\}$ that gives the flavor symmetry of the corresponding puncture. Having n bunches of 5-branes, each ending of a 7-brane leads to a puncture in the Gaiotto curve with flavor symmetry $S\,(\mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r))$, where the widths k_i of the boxes are equal to the numbers of stacks with the same number of branes per stack.

5.3 Semi-degeneration from Higgsing the T_N theories

In this section we argue that a particular way of Higgsing the T_N theories, as depicted in figure 11, corresponds to the degeneration with one simple and two full punctures. On the Toda side, this is equivalent to the semi-degeneration of Fateev and Litvinov. On the gauge theory side, the partition function of the theory with one simple and two full punctures is the partition function of N^2 free hypermultiplets. Our discussion is based on the physics of (p, q) 5-brane webs and their symmetries. In particular, we identify which Higgsing mechanism corresponds to the Fateev and Litvinov semi-degeneration by introducing 7-branes on the 5-brane web. In the next sections we will use the intuition acquired here to explicitly substitute the values dictated by the web diagram, (5.3.29) and (5.3.25), in (5.2.10) so as to obtain the formula (2.4.46) by Fateev and Litvinov.

5.3.1 Higgsing the T_N: review

The physics of the (p, q) 5-brane webs that we will need in the context of this section is studied in [167, 170, 173, 174]. First, we give a short review of the relevant results. A very useful way of realizing 4d $\mathcal{N} = 2$ quiver gauge theories in string theory is by using type IIA string theory and the Hanany-Witten construction [175] of D4 branes suspended between NS5 branes [176]. This configuration can be lifted to M-theory, where both the D4 and the NS5 branes become a single M5 brane with non-trivial topology, physically realizing the Seiberg-Witten curve in which all the low energy data are encoded [176]. Similarly, 5d $\mathcal{N} = 1$ gauge theories can be realized using type IIB string theory with D5 branes suspended between NS5 branes forming (p, q) 5-brane webs [177, 178]. A large class of $\mathcal{N} = 2$ SCFTs, called class $\mathcal{S}$, can be reformulated (from the realization in [176] with a single M5 brane with non-trivial topology) as a compactification of N M5 branes on a sphere [179]. This point of view is very useful since intersections of these N M5 branes with other M5 branes can be thought of as insertions of defect operators on the world volume of the M5 branes and thus punctures on the sphere. The name *simple puncture* is used for defects that are obtained from the intersection of the original N M5 branes with a single M5 brane (originating from D4's ending on an NS5 in the Hanany-Witten construction), while *full or maximal punctures* stem from defects corresponding to intersections with N semi-infinite M5 branes (external flavor semi-infinite D4's in [176]).

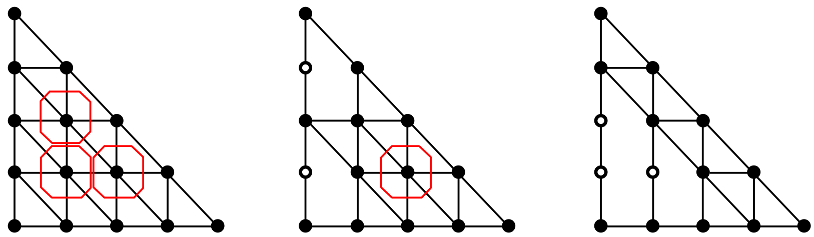


Figure 14: *In this figure we present the dot diagrams of T_4 with three different Higgsings. On the left we have the un-Higgsed dot diagram with three full punctures, $SU(4)^3$ global symmetry and three Coulomb moduli. In the middle, the four D5 branes end on two D7 branes with two D5 branes on each, which corresponds to the Young diagram $\{2,2\}$. This theory has apparent global symmetry $SU(4)^2 \times SU(2)$ and one closed polygon corresponding to one leftover Coulomb modulus. Finally, on the right we have the fully Higgsed theory with three D5 branes on the first D7 brane and one D5 brane on the second D7. This theory has no Coulomb moduli left.*

More general punctures, naturally labeled by Young diagrams consisting of N boxes, are also possible [179, 180]. In the (p,q) 5-brane web language, they can be described when additional 7-branes are introduced [170]. Semi-infinite (p,q) 5-branes are equivalent to (p,q) 5-branes ending on (p,q) 7-branes [181]. Consider N 5-branes and let them end on n 7-branes, as shown on the left of figure 13. The j^{th} 7-brane carries ℓ_j 5-branes. We define the numbers ℓ'_j as a permutation of the ℓ_j such that they are ordered

$$\ell'_1 \geq \ell'_2 \geq \cdots \geq \ell'_n \,, \tag{5.3.20}$$

and arrange them as the columns of a Young diagram.As we started with N 5-branes, the ℓ'_js must obey the condition $\sum_{j=1}^{n} \ell'_j = N$. The integers k_a are defined recursively

$$k_a = \{\#\, \ell'_j : \ell'_j = \ell'_{k_1+\cdots k_{a-1}+1}\} \,, \tag{5.3.21}$$

and are equal to the number of columns of equal height. Since the diagonal U(1) of the whole set of the N 5-branes is not realized on the low energy theory [181], the flavor symmetry of the corresponding puncture in the Gaiotto curve is $\mathrm{S}(\mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r))$ [179].

The Coulomb branch of the T_N theories, corresponding to normalizable deformations of the web which do not change its shape at infinity, has dimension equal to the number of faces in the T_N web diagram, see the left part of figure 11, and has dimension $\frac{(N-1)(N-2)}{2}$, as it should [180]. Moreover, the dimension of the Higgs branch of the T_N theories, known to be $\frac{3N^2-N-2}{2}$ [180], was obtained by terminating all the external semi-infinite 5-branes on 7-branes and counting the independent degrees of freedom for moving them around on the web-plane [170]. Finally, the global symmetry $\mathrm{SU}(N)^3$ of the T_N theories is realized on the 7-branes.

Higgsed T_N theories can also be understood in this way [170]. Beginning with the T_N 5-brane webs which correspond to the sphere with three full punctures (labeled by the Young diagrams $\{1^N\}$) and grouping the N parallel 5 branes of the punctures into smaller bunches (labeled by the Young diagrams $\{\ell'_1, \ell'_2, \ldots, \ell'_n\}$), 5-brane configurations which realize 5d theories with $E_{6,7,8}$ flavor symmetry were obtained. These theories have Coulomb and Higgs branches of smaller dimension than the original T_N which can be counted using a generalization of the s-rule [182–184] from the so-called dot diagrams[9], see

[9]The dot diagrams are the dual graphs of the web diagrams with the additional information about the 7-branes encoded in

also [167, 173, 174]. For us, the important result from [170] is that the dimension of the Higgs moduli space of a puncture corresponding to the Young diagram depicted in figure 13 is

$$\dim_{\mathbb{H}} \mathcal{M}_H^p = \sum_{j=1}^{n} (j-1)\, \ell'_j\,, \tag{5.3.22}$$

and that the Coulomb branch is the number of closed dual polygons in the dot diagram.

5.3.2 The Fateev-Litvinov degeneration from Higgsing

We need to decide which puncture (Young diagram $\{\ell'_1, \ell'_2, \ldots, \ell'_n\}$) corresponds to the Fateev-Litvinov semi-degenerate primary operator. This puncture should have only U(1) symmetry (for $N > 2$). Thus, it can be obtained by grouping the N 5-branes in two bunches of unequal number of 5-branes, $N-1$ and 1 respectively, forming the L-shaped Young diagram $\{N-1, 1\}$ shown in figure 12. For $N = 2$, the puncture has an SU(2) flavor symmetry, while for $N \geq 3$ the flavor symmetry gets reduced to U(1), as required for the semi-degenerate field. This Young diagram $\{N-1, 1\}$ corresponds to the simple punctures discussed before. The Higgs moduli space of this configuration has $\dim_{\mathbb{H}} \mathcal{M}_H^{\text{semi-deg}} = 1$ which is consistent with the fact that we have only one parameter $\varkappa$ on the CFT side. Finally, the dot diagrams tell us that the dimension of the Coulomb branch in this case is zero, which, as we will see later, is consistent with what one gets by just substituting (5.3.26) in (5.2.10).

Now, let us discuss what happens with the Kähler moduli that parametrize the T_N partition functions as we bring together $N-1$ parallel horizontal external D5 branes on a single D7 brane. These we will then translate to the language of mass parameters m_i, n_i, l_i ($i = 1, \ldots, N$) and Coulomb moduli a_r ($r = 1, \ldots, {}^{(N-1)(N-2)}/_2$) using the dictionary of appendix 5.B and in particular equation (5.B.4) and, finally, to the Toda weights $\boldsymbol{\alpha}_{1,2,3}$ using (5.2.8). We follow closely the discussion in [173]. For simplicity, we begin with two parallel D5 branes that originally end on different D7 branes. This process is depicted in figure 15. First we need to shrink u_2 of $U_2 = e^{-\beta u_2}$ to zero while still having two 7-branes. In the process of sending the u_1 of $U_1 = e^{-\beta u_1}$ to zero, one of the two D7 branes will meet a D5 brane and the two parallel D5 branes will fractionate on the D7 branes. After moving the cut piece to infinity, it effectively decouples from the rest of the web.

For the unrefined topological strings, *i.e.* for $\epsilon_2 = -\epsilon_1$, shrinking the length of a 5-brane parametrized[10] by $U = e^{-\beta u}$ corresponds to setting $U = 1$. This is not true any more in the case of the refined topological string where zero size will correspond either to $U = \sqrt{{}^{\mathfrak{t}}/_{\mathfrak{q}}}$ or $U = \sqrt{{}^{\mathfrak{q}}/_{\mathfrak{t}}}$ [185–188]. It turns out that both choices are equivalent as is extensively discussed in [173]. In this Chapter we wish to consider only the parameter space that corresponds to Toda CFT with $Q = \epsilon_1 + \epsilon_2 > 0$, *i.e.* ${}^{\mathfrak{t}}/_{\mathfrak{q}} > 1$, and thus we have to pick $U = \sqrt{{}^{\mathfrak{t}}/_{\mathfrak{q}}}$.

For the T_3 case the situation is exactly the same as in the simple example depicted on figure 15. The following two Kähler parameters

$$Q^{(1)}_{m;1} = \mathbf{A}^{-1} \tilde{M}_1 \tilde{N}_1 \quad \text{and} \quad Q^{(1)}_{l;1} = \mathbf{A} \tilde{M}_2^{-1} \tilde{N}_1^{-1} \tag{5.3.23}$$

white and black dots.

[10] The parameter u in the exponent is the length of the 5-brane segment.

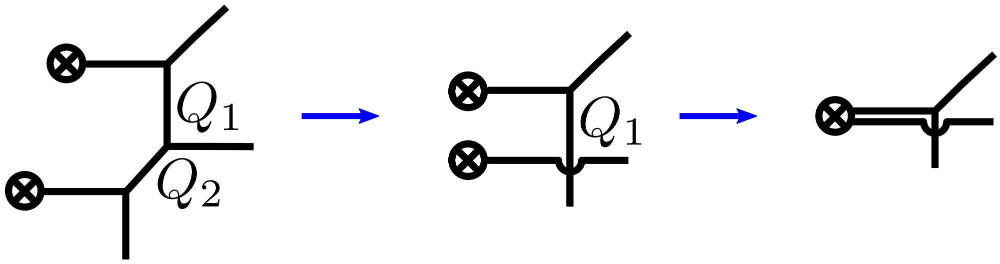


Figure 15: This figure shows the way two 5-branes are brought on the same 7-brane [167].

are the ones we have to shrink, where $\mathbf{A} \equiv \tilde{A}_1^{(1)}$ is the Coulomb modulus of T_3.[11] Thus, we have to set

$$Q_{m;1}^{(1)} = Q_{l;1}^{(1)} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\,. \tag{5.3.24}$$

For the general T_N case, as depicted in figure 20, we must tune

$$Q_{m;i}^{(j)} = Q_{l;i}^{(j)} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}} \qquad \text{with} \qquad i = 1, \ldots, N-2, \quad j = 1, \ldots, N-1-i\,. \tag{5.3.25}$$

Going back to the Toda side, we wish to semi-degenerate the weight $\boldsymbol{\alpha}_1$, *i.e.* set it to

$$\boldsymbol{\alpha}_1 = N\varkappa\omega_{N-1} \qquad \Longleftrightarrow \qquad m_i = \begin{cases} \varkappa - \frac{N+1-2i}{2}Q & i = 1, \ldots, N-1\,, \\ -(N-1)\varkappa + \frac{N-1}{2}Q & i = N\,, \end{cases} \tag{5.3.26}$$

where we used (5.2.8). For the T_3 case that implies for the exponentiated mass parameters that

$$\tilde{M}_1 = \frac{\mathfrak{t}}{\mathfrak{q}}\tilde{K} = e^{-\beta(\varkappa - Q)} \qquad \text{and} \qquad \tilde{M}_2 = \tilde{K} \tag{5.3.27}$$

which is consistent with (5.3.23) and (5.3.24) when the Coulomb moduli is tuned to the value

$$\mathsf{A} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1\,. \tag{5.3.28}$$

This is compatible with the statement that after Higgsing, the T_3 the dimension of the Coulomb branch is zero, and with another fact we will discuss in next section: the contour integral gets pinched once one substitutes (5.3.26) in (5.2.10). In the general T_N case, Higgsing forces the Coulomb parameters to become

$$\tilde{A}_i^{(j)} = \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{i(N-i-j)}{2}} \tilde{K}^i \prod_{k=1}^{j} \tilde{N}_k\,, \tag{5.3.29}$$

where $i, j = 1, \ldots, N-2$, $i + j \leq N-1$ and $\tilde{K} = e^{-\beta\varkappa}$. This implies that the Kähler parameters obey (5.3.25).

At the level of partition functions, the Fateev-Litvinov formula for the special 3-point functions can be identified with the partition function of N^2 free hypermultiplets, after removal of the decoupled degrees

[11] See appendix 5.B for notations.

of freedom (5.2.13). We know from [94, 168], that the partition function of a single free hypermultiplet is given by

$$\mathcal{Z}^{S^4}_{\text{free hyper}} = \frac{1}{\Upsilon(m - \frac{\epsilon_+}{2})}\,, \tag{5.3.30}$$

$$\mathcal{Z}^{S^4\times S^1}_{\text{free hyper}} = \frac{1}{||\mathcal{M}(e^{-\beta m}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}})||^2} = \frac{(1-q)^{-\frac{m^2}{\epsilon_1\epsilon_2}}}{\left\|\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}};\mathfrak{t},\mathfrak{q}\right)\right\|^2}\frac{1}{\Upsilon_q(m-\frac{\epsilon_+}{2})}\,. \tag{5.3.31}$$

Thus, the 5d superconformal index of N^2 free hypermultiplets is the product of N^2 such partition functions

$$\mathcal{Z}^{S^4\times S^1}_{N^2\text{ free hypers}} = \frac{1}{\prod_{i,j=1}^N\left\|\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}e^{-\beta m_{ij}}\right)\right\|^2}\,. \tag{5.3.32}$$

Up to some dropped factors, by using (5.2.13) we can identify

$$\frac{C_q(N\varkappa\omega_{N-1},\alpha_2,\alpha_3)}{\left\|\mathcal{Z}^{\text{dec}}_N\right\|^2} \sim \frac{1}{\prod_{i,j=1}^N\Upsilon_q(\varkappa+(\alpha_2-Q,h_i)+(\alpha_3-Q,h_j))} \sim \mathcal{Z}^{S^4\times S^1}_{N^2\text{ free hypers}}\,. \tag{5.3.33}$$

From this knowledge, one could in principle guess some of the complicated summation formulae like (5.4.50), as was done by [173] for the T_3 case.

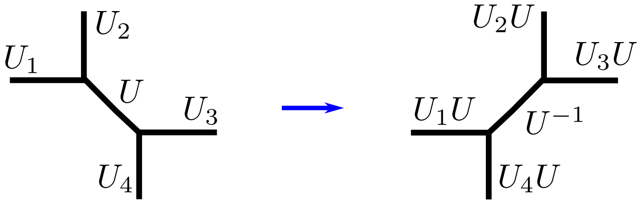


Figure 16: The figure illustrates the change of the Kähler parameters upon flopping.

We wish to conclude this section by stressing that the formulae we are dealing with have different outfit depending on the values of masses, just like in (3.4.66). In the topological string language this corresponds to different geometries that are related by flopping. For each Kähler parameter U, we distinguish between the region $|U| > 1$ and the one with $|U| < 1$; to each one associates a different (p, q) 5-brane web diagram. Going from one region to the other involves 'flopping' which transforms the Kähler parameters as depicted in figure 16, see [189] for a recent discussion of the topic. In the next section, we explain how the contour in (5.2.10) is to be chosen, such that the number of pinching points is equal to the one of relevant flopping frames.

5.4 The semi-degenerate W_3 3-point functions

In the present section we explicitly derive the Fateev-Litvinov formula for special 3-point functions of sl(3) Toda theory from the general formula (5.1.1). Along the way, we also highlight the relation between semi-degeneration of external momenta m_i and Higgsing of the corresponding four-dimensional T_3 gauge theory described in the previous section. For this, we show how semi-degeneration of the masses m_i 'pinches' the contour integral, so that the result is given by the small number of residues.

To illustrate a general phenomenon let us regard a simple situation. Assume g is a meromorphic function in domain $D \subset \mathbb{C}$ that has only simple poles at the points a, b and p_i, meaning that it can be written as

$$g(z) = \frac{f(z)}{(z-a)(z-b)\prod_i (z-p_i)}, \tag{5.4.34}$$

where f is a holomorphic function in D. Let C be a closed contour in D that encircles a as well as the p_i but not b. We write $a = p + \delta$ and $b = p - \delta$ and take the limit $\delta \to 0$, thus letting the two points a and b collide on the contour C on both sides, as depicted in figure 17. If we now compute the contour integral of

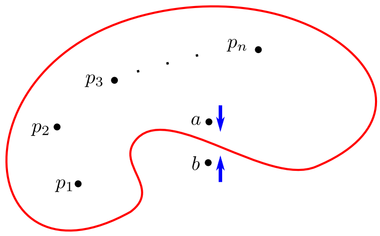


Figure 17: *The figure shows an example of contour pinching. As the poles at a and at b collide, the contour integral diverges. Scaling it with $a-b$, we see that in the limit $a \to b$, the integral is given by a single residue.*

g around C and multiply by $a-b$, we obtain

$$\begin{aligned}(a-b)\oint_C \frac{dz}{2\pi i} g(z) &= \frac{f(a)}{\prod_i (a-p_i)} + \sum_i \frac{(a-b)f(p_i)}{(p_i-a)(p_i-b)\prod_{j\neq i}(p_i-p_j)} \\ &\overset{\delta\to 0}{\longrightarrow} \frac{f(p)}{\prod_i (p-p_i)} = \lim_{a\to b}\left[(a-b)\mathrm{Res}(g(z),a)\right] .\end{aligned} \tag{5.4.35}$$

Thus, in the limit $a \to b$, the contour gets pinched at the point $a = b = p$ and the integral is given by a single residue. Essentially, this is just an integral version of the identity $\lim_{\varepsilon\to 0} \frac{\varepsilon}{(x+i\varepsilon)(x-i\varepsilon)} = \pi\delta(x)$.

Let us see how this simple example applies to our integral formulae for the correlation functions of T_3. The formula (5.2.15) for the structure constants now reads

$$C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \text{const} \times \oint \frac{d\mathbf{A}}{2\pi i \mathbf{A}} \, ||M(\mathfrak{t}, \mathfrak{q})||^2 \, ||\mathcal{Z}_3^{\text{top}}||^2 \,, \tag{5.4.36}$$

where $A_1^{(1)} \equiv \mathbf{A} = e^{-\beta \mathbf{a}}$.

Using (5.2.16), we get the following expression for topological string amplitude:

$$
\begin{aligned}
\|\mathcal{Z}_3^{\text{top}}\|^2 = \|\mathcal{Z}_3^{\text{pert}}\|^2\|\mathcal{Z}_3^{\text{inst}}\|^2 = &\left\| \frac{\prod_{1\le i<j\le 3}\mathcal{M}\left(\frac{\tilde{M}_i}{\tilde{M}_j}\right)}{\prod_{k=1}^3\left[\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_k\tilde{L}_3\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}}{\tilde{M}_k\tilde{N}_1}\right)\right]} \frac{\mathcal{M}\left(\frac{\mathbf{A}^2\tilde{L}_3}{\tilde{N}_1}\right)\mathcal{M}\left(\frac{\tilde{N}_1}{\mathbf{A}^2\tilde{L}_3}\right)}{\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}\tilde{N}_2}{\tilde{L}_1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}\tilde{N}_3}{\tilde{L}_2}\right)} \right\|^2 \\
&\times \left\| \sum_{\boldsymbol{\nu}} \left(\frac{\tilde{N}_1\tilde{L}_2}{\tilde{N}_2\tilde{L}_3}\right)^{\frac{|\nu_1^{(1)}|+|\nu_2^{(1)}|}{2}} \left(\frac{\tilde{N}_2\tilde{L}_1}{\tilde{N}_3\tilde{L}_2}\right)^{\frac{|\nu_1^{(2)}|}{2}} \frac{\prod_{k=1}^3\left[\mathsf{N}^{\beta}_{\nu_1^{(1)}\emptyset}(\mathbf{a}-m_k-n_1-Q/2)\,\mathsf{N}^{\beta}_{\emptyset\nu_2^{(1)}}(\mathbf{a}+m_k+l_3-Q/2)\right]}{\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_1^{(1)}}(0)\,\mathsf{N}^{\beta}_{\nu_2^{(1)}\nu_2^{(1)}}(0)\,\mathsf{N}^{\beta}_{\nu_1^{(2)}\nu_1^{(2)}}(0)} \right. \\
&\times \left. \frac{\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_1^{(2)}}(\mathbf{a}+n_2-l_1-Q/2)\,\mathsf{N}^{\beta}_{\nu_1^{(2)}\nu_2^{(1)}}(\mathbf{a}+n_3-l_2-Q/2)}{\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_2^{(1)}}(2\mathbf{a}-n_1+l_3)\,\mathsf{N}^{\beta}_{\nu_2^{(1)}\nu_1^{(1)}}(-2\mathbf{a}+n_1-l_3)} \right\|^2,
\end{aligned}
\tag{5.4.37}
$$

with the sum going over partitions $\boldsymbol{\nu} = \{\nu_1^{(1)}, \nu_2^{(1)}, \nu_1^{(2)}\}$. Since we wish to evaluate the contour integral (5.4.36) in the semi-degenerate limit $\boldsymbol{\alpha}_1 = 3\varkappa\omega_2$, let us introduce a regulator δ and parametrize the three 'masses' labeling positions of the branes as

$$
m_1 = \varkappa + \delta - Q\,, \qquad m_2 = \varkappa - \delta\,, \qquad m_3 = -2\varkappa + Q\,, \tag{5.4.38}
$$

which implies that the exponentiated masses $\tilde{M}_i = e^{-\beta m_i}$ are

$$
\tilde{M}_1 = \frac{\mathfrak{t}}{\mathfrak{q}}\tilde{K}e^{-\beta\delta}\,, \qquad \tilde{M}_2 = \tilde{K}e^{\beta\delta}\,, \qquad \tilde{M}_3 = \frac{\mathfrak{q}}{\mathfrak{t}}\tilde{K}^{-2}\,, \tag{5.4.39}
$$

with $\tilde{K} = e^{-\beta\varkappa}$. The semi-degenerate limit then corresponds to $\delta \to 0$. For these values of the masses, the numerator of $\|\mathcal{Z}_3^{\text{top}}\|^2$ in (5.4.37) goes to zero, just like the term $a - b$ in (5.4.35), since

$$
\|\mathcal{M}(\tilde{M}_1\tilde{M}_2^{-1})\|^2 = (1 - e^{-2\beta\delta}) \times \text{reg.} \approx \delta \times \text{reg.}\,, \tag{5.4.40}
$$

where 'reg' are terms that don't vanish for $\delta \to 0$.

To specify the integration contour, let us now look at poles of the integrand (5.4.37). One can argue that they can only come from perturbative part of the string amplitude. Indeed, as we already mentioned discussing the contour, the integral in our formula (5.1.1) for Toda three-point function should be regarded as complicated deformation of a conventional Mellin-Barnes contour integral (of ratio of gamma functions multiplying a hypergeometric function). 'Perturbative' part of the integrand corresponds to deformed gamma functions, whereas 'instanton' part is an analogue of a hypergeometric function. As (a principal branch of) the usual/basic hypergeometric function is an entire function of its parameters, it cannot yield additional poles. The same property is natural to expect from the deformation.

Poles of the integrand come from the zeros of the functions $\|\mathcal{M}(U)\|^2$ in denominator of first line of (5.4.37). Since, in order to obtain the Toda theory from topological strings, we wish to have $b > 0$, so that $|\mathfrak{q}| < 1$ and $|\mathfrak{t}| > 1$, we get from (3.4.66) the expression

$$
\|\mathcal{M}(U;\mathfrak{t},\mathfrak{q})\|^2 = \mathcal{M}(U;\mathfrak{t},\mathfrak{q})\mathcal{M}(U^{-1};\mathfrak{t}^{-1},\mathfrak{q}^{-1}) = \prod_{i,j=1}^{\infty}(1 - U\mathfrak{t}^{-i}\mathfrak{q}^{j})(1 - U^{-1}\mathfrak{t}^{1-i}\mathfrak{q}^{j-1})\,. \tag{5.4.41}
$$

Thus, the zeros of $||\mathcal{M}(U)||^2$ are to be found at

$$U = \mathfrak{t}^{-m}\mathfrak{q}^{n}\,, \qquad U = \mathfrak{t}^{m+1}\mathfrak{q}^{-n-1}\,, \tag{5.4.42}$$

for $m, n \in \mathbb{Z}_{\geq 0}$. We see that there are two classes of poles of $||\mathcal{Z}^{\text{top}}||^2$: those that condense around zero in the $\mathbf{A}$ complex plane and those that condense around infinity. The Mellin-Barnes contour should separate exactly these two families for all functions $\mathcal{M}$.

When we then take the limit $\delta \to 0$, some poles from the exterior of the contour integral with coincide with some from the interior, leading to a divergence that will cancel the zero of (5.4.41), just like in the equation (5.4.35). We easily see that relevant terms in the denominator of the first line of (5.4.37) are

$$\left|\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_1^{-1}\tilde{N}_1^{-1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_2^{-1}\tilde{N}_1^{-1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_1\tilde{L}_3\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_2\tilde{L}_3\right)\right|^2 . \tag{5.4.43}$$

The other zeros in the denominator will not pinch the integral once the regulator δ is set to zero and can be ignored, just like the p_i terms in (5.4.35). Numbering the functions $\mathcal{M}$ as 1 to 4 in (5.4.43) from left to right, using (5.4.42) and the parametrization (5.4.38), we know that we have first order poles in the integrand if

$$\begin{aligned}
&(1) \qquad \mathbf{A} = \tilde{K}\tilde{N}_1 e^{-\beta\delta}\mathfrak{t}^{-m+\frac{1}{2}}\mathfrak{q}^{n-\frac{1}{2}}\,, && (\bar{1}) \qquad \mathbf{A} = \tilde{K}\tilde{N}_1 e^{-\beta\delta}\mathfrak{t}^{m+\frac{3}{2}}\mathfrak{q}^{-n-\frac{3}{2}}\,,\\
&(2) \qquad \mathbf{A} = \tilde{K}\tilde{N}_1 e^{\beta\delta}\mathfrak{t}^{-m-\frac{1}{2}}\mathfrak{q}^{n+\frac{1}{2}}\,, && (\bar{2}) \qquad \mathbf{A} = \tilde{K}\tilde{N}_1 e^{\beta\delta}\mathfrak{t}^{m+\frac{1}{2}}\mathfrak{q}^{-n-\frac{1}{2}}\,,\\
&(3) \qquad \mathbf{A} = \tilde{K}^{-1}\tilde{L}_3^{-1} e^{\beta\delta}\mathfrak{t}^{-m-\frac{3}{2}}\mathfrak{q}^{n+\frac{3}{2}}\,, && (\bar{3}) \qquad \mathbf{A} = \tilde{K}^{-1}\tilde{L}_3^{-1} e^{\beta\delta}\mathfrak{t}^{m-\frac{1}{2}}\mathfrak{q}^{-n+\frac{1}{2}}\,,\\
&(4) \qquad \mathbf{A} = \tilde{K}^{-1}\tilde{L}_3^{-1} e^{-\beta\delta}\mathfrak{t}^{-m-\frac{1}{2}}\mathfrak{q}^{n+\frac{1}{2}}\,, && (\bar{4}) \qquad \mathbf{A} = \tilde{K}^{-1}\tilde{L}_3^{-1} e^{-\beta\delta}\mathfrak{t}^{m+\frac{1}{2}}\mathfrak{q}^{-n-\frac{1}{2}}\,,
\end{aligned} \tag{5.4.44}$$

for $m, n \in \mathbb{Z}_{\geq 0}$. We have labeled with $\bar{\cdot}$ those sets of poles that coalesce around $\mathbf{A} = \infty$.

We can see that, when one of the poles from 1 collides with one of the poles from $\bar{2}$ for $m = n = 0$, the integral gets pinched as $\delta \to 0$. The result is given by the residue at

$$\mathbf{A} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1 e^{-\beta\delta}\,. \tag{5.4.45}$$

The flopping geometry corresponding to the residue at (5.4.45) is depicted in figure 18.

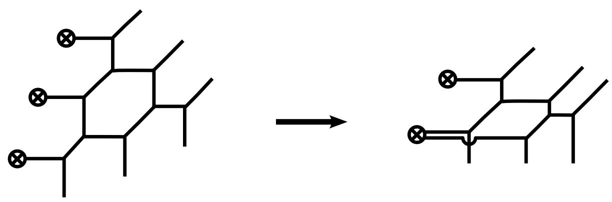


Figure 18: *The figure shows the Higgsed geometry corresponding to the residue* (5.4.45). *For this residue, the Kähler parameters take the values* (5.3.24).

For the corresponding residue we then have

$$\begin{aligned}
\lim_{\delta\to 0}\oint \frac{d\mathbf{A}}{2\pi i\mathbf{A}}\,||M(\mathfrak{t},\mathfrak{q})||^2\,||\mathcal{Z}_3^{\text{top}}||^2 &= ||M(\mathfrak{t},\mathfrak{q})||^2\,\text{Res}\left(||\mathcal{Z}_3^{\text{top}}||^2, \mathbf{A} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1 e^{-\beta\delta}\right) \\
&= \frac{\left\|\mathcal{M}(\tilde{K}^{-3})\right\|^2}{\left\|\prod_{k=1}^{3}\mathcal{M}(\frac{\tilde{N}_k\tilde{L}_{4-k}}{\tilde{K}})\right\|^2}\left\|Z_3^{\text{inst}}\right\|^2_{\mathbf{A}=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1}\,.
\end{aligned} \tag{5.4.46}$$

One can observe that due to (5.C.3), the sum over $\nu_1^{(1)}$ in $\left\|\mathcal{Z}_3^{\text{inst}}\right\|^2_{\big|\mathbf{A}=\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}\tilde{K}\tilde{N}_1}$ drops out and we obtain the result

$$\left(\mathcal{Z}_3^{\text{inst}}\right)_{\big|\mathbf{A}=\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}\tilde{K}\tilde{N}_1} = \sum_{\nu_1,\nu_2}\left(\frac{\tilde{N}_2\tilde{L}_1}{\tilde{N}_3\tilde{L}_2}\right)^{\frac{|\nu_1|}{2}}\left(\frac{\tilde{N}_1\tilde{L}_2}{\tilde{N}_2\tilde{L}_3}\right)^{\frac{|\nu_2|}{2}} \times \frac{\mathsf{N}^{\beta}_{\nu_1\emptyset}(n_3+l_1-\varkappa)\mathsf{N}^{\beta}_{\nu_2\nu_1}(n_2+l_2-\varkappa)\mathsf{N}^{\beta}_{\emptyset\nu_2}(n_1+l_3-\varkappa)}{\mathsf{N}^{\beta}_{\nu_1\nu_1}(0)\mathsf{N}^{\beta}_{\nu_2\nu_2}(0)}, \quad (5.4.47)$$

where we denoted $\nu_1^{(2)} \equiv \nu_1$, $\nu_2^{(1)} \equiv \nu_2$.

The second contribution to our integral comes from the collision of poles belonging to families $\bar{3}$ and 4, which results in pinching at

$$\mathbf{A} = \sqrt{\frac{\mathrm{q}}{\mathrm{t}}}\tilde{K}^{-1}\tilde{L}_3^{-1}e^{-\beta\delta}. \quad (5.4.48)$$

The corresponding flopping frame is shown in figure 19. It turns out, however, that the relevant residue is

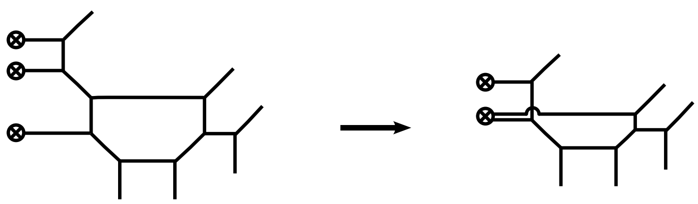


Figure 19: *The figure shows the Higgsed geometry corresponding to the residue* (5.4.48).

given by same contribution as for the previous pole. This is almost immediate to see for the perturbative part. Alike, after relabeling $\nu_2^{(1)} \leftrightarrow \nu_1^{(1)}$ and using (5.C.4), we find that the 'instanton' contribution in (5.4.47) is unchanged as well, *i.e.*

$$\left(\mathcal{Z}_3^{\text{inst}}\right)_{\big|\mathbf{A}=\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}\tilde{K}\tilde{N}_1} = \left(\mathcal{Z}_3^{\text{inst}}\right)_{\big|\mathbf{A}=\sqrt{\frac{\mathrm{q}}{\mathrm{t}}}\tilde{K}^{-1}\tilde{L}_3^{-1}}. \quad (5.4.49)$$

This fact is a manifestation of a general phenomenon: symmetries of the general formula (5.1.1) are sufficient to equate all the residues pinching the contour.

Now is the important moment when an inflow from the theory of hypergeometric functions occurs. Namely, in order to complete the computation, we need to calculate the sum in (5.4.47) over two remaining partitions. For this purpose, the following identity will be used which we state in full generality in section 5.5 and prove in appendix 4.C:

$$\sum_{\nu_1,\nu_2}\left(V_1\sqrt{U_1U_2}\right)^{|\nu_1|}\left(V_2\sqrt{U_2U_3}\right)^{|\nu_2|}\frac{\mathsf{N}^{\beta}_{\nu_1\emptyset}(u_1-\mathcal{Q}/2)\,\mathsf{N}^{\beta}_{\nu_2\nu_1}(u_2-\mathcal{Q}/2)\,\mathsf{N}^{\beta}_{\emptyset\nu_2}(u_3-\mathcal{Q}/2)}{\mathsf{N}^{\beta}_{\nu_1\nu_1}(0)\,\mathsf{N}^{\beta}_{\nu_2\nu_2}(0)} \quad (5.4.50)$$

$$= \frac{\mathcal{M}(U_1V_1)\mathcal{M}(\frac{\mathrm{t}}{\mathrm{q}}V_1U_2)\mathcal{M}(U_2V_2)\mathcal{M}(\frac{\mathrm{t}}{\mathrm{q}}V_2U_3)\mathcal{M}(U_1V_1U_2V_2)\mathcal{M}(\frac{\mathrm{t}}{\mathrm{q}}V_1U_2V_2U_3)}{\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}V_1)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}V_2)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}U_1V_1U_2)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}V_1U_2V_2)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}U_2V_2U_3)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}U_1V_1U_2V_2U_3)},$$

where $U_i := e^{-\beta u_i}$. Essentially, it is a q-binomial identity for Kaneko-Macdonald(-Warnaar) sl(3) basic hypergeometric functions rewritten in terms of Nekrasov partition functions. Upon making the following substitutions in (5.4.50)

$$U_k = \sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{\tilde{N}_{4-k}\tilde{L}_k}{\tilde{K}}, \qquad V_1 = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{K}}{\tilde{N}_3\tilde{L}_2}, \qquad V_2 = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{K}}{\tilde{N}_2\tilde{L}_3}, \tag{5.4.51}$$

where $k = 1, 2, 3$, we arrive at

$$\left(\mathcal{Z}_3^{\text{inst}}\right)\Big|_{\mathbf{A}=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1} = \frac{\mathcal{M}(\frac{\tilde{L}_1}{\tilde{L}_2})\mathcal{M}(\frac{\tilde{L}_2}{\tilde{L}_3})\mathcal{M}(\frac{\tilde{L}_1}{\tilde{L}_3})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{N}_1}{\tilde{N}_2})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{N}_2}{\tilde{N}_3})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{N}_1}{\tilde{N}_3})}{\mathcal{M}(\frac{\tilde{N}_1\tilde{L}_1}{\tilde{K}})\mathcal{M}(\frac{\tilde{N}_1\tilde{L}_2}{\tilde{K}})\mathcal{M}(\frac{\tilde{N}_2\tilde{L}_1}{\tilde{K}})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{K}}{\tilde{N}_2\tilde{L}_3})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{K}}{\tilde{N}_3\tilde{L}_2})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{K}}{\tilde{N}_3\tilde{L}_3})}. \tag{5.4.52}$$

Inserting the above into (5.4.46), we get

$$\begin{aligned}\lim_{\delta\to 0}\oint \frac{d\mathbf{A}}{2\pi i\mathbf{A}}\,||M(\mathfrak{t},\mathfrak{q})||^2\,||\mathcal{Z}_3^{\text{top}}||^2 &= 2\,\frac{\left\|\mathcal{M}(\tilde{K}^{-3})\prod_{1\le i<j\le 3}\mathcal{M}(\tilde{N}_j/\tilde{N}_i)\,\mathcal{M}(\tilde{L}_i/\tilde{L}_j)\right\|^2}{\left\|\prod_{i,j=1}^3\mathcal{M}(\tilde{N}_i\tilde{L}_j\tilde{K}^{-1})\right\|^2}\\ &= 2\,\frac{(1-q)^{\varphi_3}}{\Lambda^2}\frac{\Upsilon_q(3\varkappa)\prod_{1\le i<j\le 3}\Upsilon_q(n_i-n_j)\Upsilon_q(l_{4-i}-l_{4-j})}{\prod_{i,j=1}^3\Upsilon_q(\varkappa-n_i-l_{4-j})}.\end{aligned} \tag{5.4.53}$$

Here we have used (3.4.67), (3.4.73) and defined the exponent

$$\begin{aligned}\varphi_3 &= \left(\frac{Q}{2}-3\varkappa\right)^2 + \sum_{1\le i<j\le 3}\left[\left(\frac{Q}{2}+n_j-n_i\right)^2+\left(\frac{Q}{2}+l_{4-j}-l_{4-i}\right)^2\right] - \sum_{i,j=1}^3\left(\frac{Q}{2}+n_i+l_{4-j}-\varkappa\right)^2\\ &= 2Q\left(3\varkappa+\sum_{i=1}^3 i(n_i+l_{4-i})\right)-\frac{Q^2}{2} = -2Q\left(2\boldsymbol{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)-\frac{Q^2}{2},\end{aligned} \tag{5.4.54}$$

where in the last line we have used our sl(3) conventions, see appendix 5.A and equation (5.2.8). Now we employ (3.4.75) and rearrange the prefactors of (5.4.53) to obtain the q-deformed $\mathbf{W}_3$ Fateev-Litvinov structure constants (2.4.47) in the form[12] conjectured by [27]:

$$\begin{aligned}&C_q(3\varkappa\omega_2,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) =\\ &= \frac{1}{2}\left(\beta\,||M(\mathfrak{t},\mathfrak{q})||^2\right)^2\left((1-q^b)^{2b^{-1}}(1-q^{b^{-1}})^{2b}\right)^{\left(2\boldsymbol{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}\lim_{\delta\to 0}\oint\frac{d\mathbf{A}}{2\pi i\mathbf{A}}\,||M(\mathfrak{t},\mathfrak{q})||^2\,||\mathcal{Z}_3^{\text{top}}||^2 \qquad (5.4.55)\\ &= \frac{1}{2}\left(\frac{(1-q^b)^{2b^{-1}}(1-q^{b^{-1}})^{2b}}{(1-q)^{2Q}}\right)^{\left(2\boldsymbol{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}\frac{\Upsilon_q'(0)^2\Upsilon_q(3\varkappa)\prod_{e>0}\Upsilon_q((\boldsymbol{Q}-\boldsymbol{\alpha}_2,e))\Upsilon_q((\boldsymbol{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^3\Upsilon_q(\varkappa+(\boldsymbol{\alpha}_2-\boldsymbol{Q},h_i)+(\boldsymbol{\alpha}_3-\boldsymbol{Q},h_j))}.\end{aligned}$$

Taking here the 4d limit $q\to 1$ and reintroducing the cosmological constant dependence according to (2.4.48) leads to the Fateev-Litvinov formula (2.4.46) for $N = 3$.

5.5 The general $\mathbf{W}_N$ case

Having computed the structure constants for the $\mathbf{W}_3$ case in the previous section, let us proceed to a general case. Like the previous section, we parametrize the masses as follows

$$\tilde{M}_i = \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{N+1-2i}{2}}\tilde{K}d_i \qquad \text{for } i = 1,\dots,N-1, \qquad \tilde{M}_N = \left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{N-1}{2}}\frac{1}{\tilde{K}^{N-1}}, \tag{5.5.56}$$

[12] The factor 1/2 introduced here is accounted for by 'const' in the corresponding general formula.

where $d_i = e^{-\beta\delta_i}$ are regulators satisfying $\prod_{i=1}^{N-1} d_i = 1$ and $\tilde{K} = e^{-\beta\varkappa}$. The numerator of $\left\|\mathcal{Z}^{\text{top}}\right\|^2$ has a zero of order $\frac{(N-2)(N-1)}{2}$ in the limit $\delta_i \to 0$ since

$$\prod_{1\le i<j\le N} \left\|\mathcal{M}\Big(\frac{\tilde{M}_i}{\tilde{M}_j}\Big)\right\|^2 = \text{reg} \times \prod_{1\le i<j\le N-1} \left\|\mathcal{M}\left(\Big(\frac{\mathfrak{t}}{\mathfrak{q}}\Big)^{j-i}\frac{d_i}{d_j}\right)\right\|^2, \tag{5.5.57}$$

and $\left\|\mathcal{M}\left((\mathfrak{t}/\mathfrak{q})^n\right)\right\|^2 = 0$ for $n \ge 0$. These zeros are cancelled by the divergences coming from pinching of the $\frac{(N-2)(N-1)}{2}$ Mellin-Barnes integrals.

Again, we will find that the relevant residues are in one-to-one correspondence with the possible flopping frames. Thus, the final answer is obtained by taking one residue in the integration variables $\tilde{A}_i^{(j)}$, e.g. at

$$\tilde{A}_i^{(j)} = \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{i(N-i-j)}{2}} \tilde{K}^i \prod_{k=1}^{j} \tilde{N}_k\,, \tag{5.5.58}$$

and multiplying it by the number of possible flopping frames. For the corresponding residue, we get, much like before:

$$\begin{aligned}
&\lim_{\delta_a\to 0} \oint \prod_{i=1}^{N-2}\prod_{j=1}^{N-1-i}\Big[\frac{d\tilde{A}_i^{(j)}}{2\pi i \tilde{A}_i^{(j)}}\|M(\mathfrak{t},\mathfrak{q})\|^2\Big]\|\mathcal{Z}_N^{\text{top}}\|^2 = \\
&= \frac{\left\|\mathcal{M}(\tilde{K}^{-N})\right\|^2}{\left\|\prod_{k=1}^{N}\mathcal{M}\big(\frac{\tilde{N}_k\tilde{L}_{N+1-k}}{\tilde{K}}\big)\right\|^2} \times \Bigg\| \sum_{\nu_1,\ldots,\nu_{N-1}} \Bigg[\prod_{i=1}^{N-1}\left(\frac{\tilde{N}_{N-i}\tilde{L}_i}{\tilde{N}_{N-i+1}\tilde{L}_{i+1}}\right)^{\frac{|\nu_i|}{2}}\Bigg] \\
&\times \frac{\mathsf{N}^{\beta}_{\nu_1\varnothing}(n_N + l_1 - \varkappa)\left[\prod_{i=1}^{N-2}\mathsf{N}^{\beta}_{\nu_{i+1}\nu_i}(n_{N-i} + l_{i+1} - \varkappa)\right]\mathsf{N}^{\beta}_{\varnothing\nu_{N-1}}(n_1 + l_N - \varkappa)}{\prod_{i=1}^{N-1}\mathsf{N}^{\beta}_{\nu_i\nu_i}(0)}\Bigg\|^2.
\end{aligned} \tag{5.5.59}$$

Here ν_i for $i = 1,\ldots,N-1$ denote partitions corresponding to the $N-1$ brane junctions not affected by Higgsing at a given pole. For our particular choice of flopping frame depicted on the figure 11, these partitions are readily identified as $\nu_i := \nu_i^{(N-i)}$, $i = 1,\ldots,N-1$.

The remaining sums in (5.5.59) will be now performed by using the summation identity (5.C.21) proven in appendix 5.C, which we reproduce here for convenience.

Theorem

$$\begin{aligned}
&\sum_{\nu_1,\ldots,\nu_{N-1}}\left[\prod_{i=1}^{N-1}\frac{\left(V_i\sqrt{U_iU_{i+1}}\right)^{|\nu_i|}}{\mathsf{N}^{\beta}_{\nu_i\nu_i}(0)}\right]\mathsf{N}^{\beta}_{\nu_1\varnothing}(u_1 - \epsilon_+/2)\left[\prod_{i=1}^{N-2}\mathsf{N}^{\beta}_{\nu_{i+1}\nu_i}(u_{i+1} - \epsilon_+/2)\right]\mathsf{N}^{\beta}_{\varnothing\nu_{N-1}}(u_N - \epsilon_+/2) = \\
&= \prod_{i=1}^{N-1}\prod_{j=1}^{N-i}\frac{\mathcal{M}\big(\prod_{s=j}^{i+j-1}U_sV_s\big)\mathcal{M}\big(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{U_{i+j}}{U_j}\cdot\prod_{s=j}^{i+j-1}U_sV_s\big)}{\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_{i+j}\prod_{s=j}^{i+j-1}U_sV_s\big)\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{1}{U_j}\prod_{s=j}^{i+j-1}U_sV_s\big)}.
\end{aligned} \tag{5.5.60}$$

This identity is a version of q-binomial identity for the Kaneko-Macdonald(-Warnaar) sl(N) basic hypergeometric functions. Setting the parameters here to

$$U_i = \sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{\tilde{N}_{N-i+1}\tilde{L}_i}{\tilde{K}}\,, \qquad V_j = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{K}}{\tilde{N}_{N-j+1}\tilde{L}_{j+1}}\,, \tag{5.5.61}$$

for $i = 1, \cdots N$ and $j = 1, \cdots N-1$, one straightforwardly obtains:

$$\sum_{\nu_1,\ldots,\nu_{N-1}} \Big[\prod_{i=1}^{N-1}\Big(\frac{\tilde{N}_{N-i}\tilde{L}_i}{\tilde{N}_{N-i+1}\tilde{L}_{i+1}}\Big)^{\frac{|\nu_i|}{2}}\Big]$$

$$\times \frac{\mathsf{N}^{\beta}_{\nu_1\emptyset}(n_N + l_1 - \varkappa)\Big[\prod_{i=1}^{N-2}\mathsf{N}^{\beta}_{\nu_{i+1}\nu_i}(n_{N-i} + l_{i+1} - \varkappa)\Big]\mathsf{N}^{\beta}_{\emptyset\nu_{N-1}}(n_1 + l_N - \varkappa)}{\prod_{i=1}^{N-1}\mathsf{N}^{\beta}_{\nu_i\nu_i}(0)}$$

$$= \prod_{1\le i<j\le N} \frac{\mathcal{M}(\frac{\tilde{L}_i}{\tilde{L}_j})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{N}_{N-j+1}}{\tilde{N}_{N-i+1}})}{\mathcal{M}(\frac{\tilde{N}_{N-j+1}\tilde{L}_i}{\tilde{K}})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{K}}{\tilde{N}_{N-i+1}\tilde{L}_j})}\,. \tag{5.5.62}$$

Now, substituting (5.5.62) in (5.5.59) and expressing everything in terms of the Υ_q functions via formula (3.4.67), one obtains

$$\lim_{\delta_a\to 0}\oint\prod_{i=1}^{N-2}\prod_{j=1}^{N-1-i}\Big[\frac{d\tilde{A}_i^{(j)}}{2\pi i\tilde{A}_i^{(j)}}||M(\mathfrak{t},\mathfrak{q})||^2\Big]||\mathcal{Z}_N^{\text{top}}||^2 =$$

$$= \text{const}\,\frac{(1-q)^{\varphi_N}}{\Lambda^{N-1}}\frac{\Upsilon_q(N\varkappa)\prod_{1\le i<j\le N}[\Upsilon_q(n_i-n_j)\Upsilon_q(l_{N+1-i}-l_{N+1-j})]}{\prod_{i,j=1}^{N}\Upsilon_q(\varkappa - n_i - l_{N+1-j})} \tag{5.5.63}$$

where the exponent

$$\begin{aligned}\varphi_N =& \Big(\frac{Q}{2} - N\varkappa\Big)^2 + \sum_{1\le i<j\le N}\Big[\Big(\frac{Q}{2} + n_j - n_i\Big)^2 + \Big(\frac{Q}{2} + l_{N+1-j} - l_{N+1-i}\Big)^2\Big]\\ &- \sum_{i,j=1}^{N}\Big(\frac{Q}{2} + n_i + l_{N+1-j} - \varkappa\Big)^2\end{aligned} \tag{5.5.64}$$

after a little algebra simplifies into

$$\varphi_N = 2Q\left(\frac{N(N-1)}{2}\varkappa + \sum_{i=1}^{N} i(n_i + l_{N+1-i})\right) - \frac{N-1}{4}Q^2 = -2Q\left(2\boldsymbol{Q} - \sum_{i=1}^{3}\boldsymbol{\alpha}_i, \rho\right) - \frac{N-1}{4}Q^2\,. \tag{5.5.65}$$

Let us now employ the sl(N) conventions, listed in appendix 5.A, equation (5.2.8) as well as equations (3.4.73), (3.4.75) and rearrange the prefactors. In a full analogy to the T_3 case, we will obtain[13] a q-deformed Fateev-Litvinov 3-point function in the form conjectured by [27]:

$$\begin{aligned}C_q(N\varkappa\omega_{N-1},\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) =& \text{ const}\,\Big(\beta\,||M(\mathfrak{t},\mathfrak{q})||^2\Big)^{N-1}\Big((1-q^b)^{2b^{-1}}(1-q^{b^{-1}})^{2b}\Big)^{(2\boldsymbol{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho)}\\ &\times\lim_{\delta_a\to 0}\oint\prod_{i=1}^{N-2}\prod_{j=1}^{N-1-i}\Big[\frac{d\tilde{A}_i^{(j)}}{2\pi i\tilde{A}_i^{(j)}}||M(\mathfrak{t},\mathfrak{q})||^2\Big]||\mathcal{Z}_N^{\text{top}}||^2\\ =& \left(\frac{(1-q^b)^{2b^{-1}}(1-q^{b^{-1}})^{2b}}{(1-q)^{2Q}}\right)^{(2\boldsymbol{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho)}\frac{\Upsilon_q'(0)^{N-1}\Upsilon_q(N\varkappa)\prod_{e>0}\Upsilon_q((\boldsymbol{Q}-\boldsymbol{\alpha}_2,e))\Upsilon_q((\boldsymbol{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^{N}\Upsilon_q(\varkappa + (\boldsymbol{\alpha}_2-\boldsymbol{Q},h_i) + (\boldsymbol{\alpha}_3-\boldsymbol{Q},h_j))}\,.\end{aligned} \tag{5.5.66}$$

Taking the 4d limit $q\to 1$ and reintroducing the cosmological constant dependence according to (2.4.48) leads to the Fateev-Litvinov formula (2.4.46) for general N.

[13]The constant here equals one over the number of relevant flopping frames.

5.6 Conclusions and outlook

In this Chapter we provided a very convincing check of a general proposal for primary three-point functions of Toda CFT (5.1.1). Namely, we reproduced an important known special case when one of the primaries has a null-vector at level one, a result due to Fateev and Litvinov [28]. Before giving an outlook of interesting problems which should be addressed next, let us briefly summarize some crucial points of our exposition.

After introducing a required background material, we discussed how the degeneration of the primary fields on the Toda side corresponds to Higgsing on the (p, q) 5-brane web diagram side. Upon analyzing possible residues, we used a summation formula derived from q-binomial identities (5.5.60) for Kaneko-Macdonald-Warnaar sl(n) hypergeometric functions to sum the corresponding expressions. Eventually, we found that the result (5.5.66) indeed gives an expression of Fateev and Litvinov (2.4.46) after one takes the $q \to 1$ limit and reintroduces (2.4.48) the dependence on the cosmological constant μ that is fixed from a corresponding Ward identity.

Reproducing the Fateev-Litvinov formula is a powerful test in support of the proposal for 3-point functions of generic Toda exponential fields. It is, of course, interesting to obtain further checks of (5.2.10) which is currently the work in progress. There are two natural steps to take here. The first one involves placing a more general semi-degenerate field to the 3-point function. Specifically for $\mathbf{W}_3$, if a semi-degenerate condition reads $\boldsymbol{\alpha}_1 = N\varkappa\omega_2 - mb\omega_1$, where m is a positive integer, it corresponds to a primary field having a null-vector on a level $m + 1 > 1$. The Toda 3-point functions containing such a field are also known from [84]. In fact, these are the best of the CFT knowledge for 3-point functions of generic primaries. The corresponding formula (see (3.11) and appendix B of [84]) involves two very different pieces: a straightforward generalization of (2.4.46) and a $4m$-dimensional Coulomb integral. This intriguing factorization indeed looks like to be reproducible from our general perspective.

The second natural step is matching the known semi-classical asymptotics [83]. We observe that in such a limit the combinatorial functions $\mathsf{N}^{\beta}_{\lambda\mu}$ factorize as

$$\mathsf{N}^{\beta}_{\lambda\mu}(m; b, b^{-1}) \xrightarrow{b\to\infty} \mathsf{N}^{\beta}_{\lambda\emptyset}(m; b, b^{-1})\mathsf{N}^{\beta}_{\emptyset\mu}(m; b, b^{-1})\,. \tag{5.6.67}$$

The sums over partitions thus disentangle, and proper generalizations of hypergeometric identities for the case of sl(2) KM hypergeometric functions can be found to perform them. In fact, this step could then serve as a launch pad for a more ambitious goal of guessing a still unknown 'Lagrangian' for the q-deformed Toda theory. One would have to begin here by looking for the Lagrangian description of the q-deformed Liouville theory, returning to the work of [93, 95]. It could well be that the 2d space has to be made non-commutative [190–192].

Having checked the known cases, it is very interesting to go beyond them, the ultimate goal being to compute the contour integral in (5.2.15) exactly for generic values of the parameters. This will mean a considerable simplification of the general formula for the 3-point functions of Toda primaries. Doing so requires finding a closed form expression for the 'instanton' sum of (5.2.18), meaning that a suitable generalization of the KMW sl(N) hypergeometric functions, as well as corresponding summation identities for them, have to be found. As an exercise to do before going for this serious problem, one could like to compute the corresponding sums for the cases with $E_{6,7,8}$ flavor symmetry studied in [167, 170, 173, 174] which are obtained from the general T_N by a less severe Higgsing than the one we perform here.

Putting the above into the perspective of a full solution of the Toda theory, let us mention the remaining ingredients of it. First, a well-known fact is that, unlike the Virasoro case, the $\mathbf{W}_N$ symmetry is not restrictive enough to constrain the 3-point functions of descendant fields from those of primaries [193]. The number of corresponding Ward identities is simply too small to find from them the descendant structure constants. This means that in order to find all the 3-point correlators, one needs to calculate independently the 3-point structure constants of two primaries and one descendant. It is however rather straightforward from the topological strings point of view.

The second ingredient of a complete solution of Toda CFT are the $\mathbf{W}_N$ conformal blocks. The paper [194] describes the particular family of blocks which can be obtained by gluing the Fateev-Litvinov 3-point functions (2.4.46). Gluing the general (q-deformed) Toda 3-point functions in the same way would give the general conformal blocks of the (q-deformed) Toda CFT. Addressing this problem for q-Liouville would be a starting point in such an investigation. Due to many uncertainties in properly defining a q-deformed Liouville (Toda) theory, such a finding would then as well work in opposite direction, allowing to know more about the q-deformed AGT-W correspondence and its relation to topological strings (see [195]). The novel identities for Kaneko-Macdonald-Warnaar sl(N) hypergeometric functions could probably be as helpful here as they were in the present note, to sum up known and new expressions for conformal blocks.

We finish with two remarks on the gauge theory side. The degeneration we study in this Chapter, and in general Higgsing, should also be understood on the 4d/5d gauge theory side using a generalization of the AGT correspondence with additional co-dimension two half-BPS surface defects [196] as in [185, 197–199]. See also [200, 201]. The partition functions with half-BPS surface operators can be obtained form certain 2d partition functions [202]. This 2d/4d relation has its q-deformation to a 3d/5d relation that was initiated by [93] and further studied by [95–97]. See [203] for the recent advances on the subject.

Lastly, by observing that the Higgsed geometry corresponding to the degeneration, see the right side of figure 11, is related to the strip geometry $\tilde{T}_N$, see figure 7 in [166], by the Hanany-Witten effect. We refer the interested reader to [166, 204] for a nice discussion on the subject. The invariance of the topological string amplitude under the Hanany-Witten transition is non-trivial. It would be important to see how one can relate formula (2.4.47) for the q-deformed structure constants to the topological string amplitude for the strip, see equation (4.66) of [94].

Appendix

5.A Conventions for SU(N) and sl(N)

We summarize here our conventions for the SU(N), as well as sl(N), weights, roots and Weyl group used in Chapter 5 and at the end of Chapter 2. The weights of the fundamental representation of these Lie algebras are h_i with $\sum_{i=1}^{N} h_i = 0$. We remind that the scalar product is defined via $\left(h_i, h_j\right) = \delta_{ij} - \frac{1}{N}$. The simple roots are

$$e_k := h_k - h_{k+1}\,, \qquad k = 1, \ldots, N-1\,, \tag{5.A.1}$$

and the positive roots $e > 0$ are contained in the set

$$\Delta^+ := \{h_i - h_j\}_{i<j=1}^{N} = \{e_i\}_{i=1}^{N-1} \cup \{e_i + e_{i+1}\}_{i=1}^{N-2} \cup \cdots \cup \{e_1 + \cdots + e_{N-1}\}\,. \tag{5.A.2}$$

The Weyl vector ρ for SU(N) and sl(N) is given by

$$\rho := \frac{1}{2}\sum_{e>0} e = \frac{1}{2}\sum_{i<j=1}^{N}(h_i - h_j) = \sum_{i=1}^{N}\frac{N+1-2i}{2}h_i = \omega_1 + \cdots + \omega_{N-1}, \tag{5.A.3}$$

and obeys $(\rho, e_i) = 1$ for all i. The $N-1$ fundamental weights ω_i are given correspondingly by

$$\omega_i = \sum_{k=1}^{i} h_k\,, \qquad i = 1, \ldots, N-1 \tag{5.A.4}$$

and the corresponding finite dimensional representations are the i-fold antisymmetric tensor product of the fundamental representation. They obey the scalar products $\left(e_i, \omega_j\right) = \delta_{ij}$, *i.e.* form a dual basis. Furthermore, we find the following scalar products useful

$$\left(\rho, h_j\right) = \frac{N+1}{2} - j, \qquad (\rho, \omega_i) = \frac{i(N-i)}{2}, \qquad (h_j, \omega_i) = \begin{cases} 1 - \frac{i}{N} & j \leq i \\ -\frac{i}{N} & j > i \end{cases}\,, \tag{5.A.5}$$

as well as

$$\left(\omega_i, \omega_j\right) = \frac{\min(i,j)\,(N - \max(i,j))}{N}, \qquad (\rho, \rho) = \frac{N(N^2-1)}{12}\,. \tag{5.A.6}$$

The Weyl group of these Lie algebras is isomorphic to S_N and is generated by the $N-1$ Weyl reflections associated to the simple roots. If $\boldsymbol{\alpha}$ is a weight, we define a Weyl reflections with respect to the simple root e_i as

$$\mathsf{w}_i \cdot \boldsymbol{\alpha} := \boldsymbol{\alpha} - 2\frac{(e_i, \boldsymbol{\alpha})}{(e_i, e_i)}e_i = \boldsymbol{\alpha} - (e_i, \boldsymbol{\alpha})\, e_i\,. \tag{5.A.7}$$

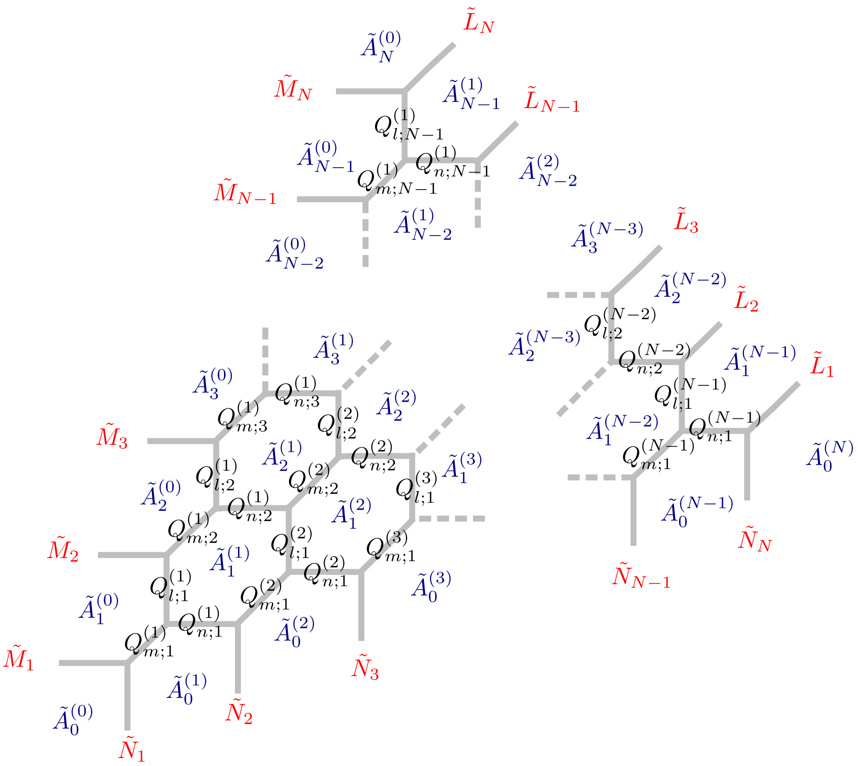


Figure 20: Parametrization for T_N. We denote the Kähler moduli parameters corresponding to the horizontal lines as $Q_{n;i}^{(j)}$, to the vertical lines as $Q_{l;i}^{(j)}$, and to tilted lines as $Q_{m;i}^{(j)}$. We denote the breathing modes as $\tilde{A}_i^{(j)}$. The index j labels the strips in which the diagram can be decomposed.

Furthermore, we define the affine Weyl reflections with respect to e_i as follows:

$$\mathsf{w}_i \circ \boldsymbol{\alpha} := \mathcal{Q} + \mathsf{w}_i \cdot (\boldsymbol{\alpha} - \mathcal{Q}) = \mathsf{w}_i \cdot \boldsymbol{\alpha} + Q e_i = \boldsymbol{\alpha} - (\boldsymbol{\alpha} - \mathcal{Q}, e_i)\, e_i \,, \tag{5.A.8}$$

where $\mathcal{Q} := Q\rho = (b + b^{-1})\rho$.

5.B Parametrization of the T_N junction

This appendix encompasses essential formulae for the parametrizations of the Kähler moduli of the T_N. First, the 'interior' Coulomb moduli $\tilde{A}_j^{(i)} = e^{-\beta a_i^{(j)}}$ are independent, while the 'border' ones are given by

$$\tilde{A}_i^{(0)} = \prod_{k=1}^{i} \tilde{M}_k, \qquad \tilde{A}_0^{(j)} = \prod_{k=1}^{j} \tilde{N}_k, \qquad \tilde{A}_i^{(N-i)} = \prod_{k=1}^{i} \tilde{L}_k. \tag{5.B.1}$$

The parameters labeling the positions of the flavor branes obey the relations

$$\prod_{k=1}^{N} \tilde{M}_k = \prod_{k=1}^{N} \tilde{N}_k = \prod_{k=1}^{N} \tilde{L}_k = 1 \Longleftrightarrow \sum_{k=1}^{N} m_k = \sum_{k=1}^{N} n_k = \sum_{k=1}^{N} l_k = 0. \tag{5.B.2}$$

Therefore, $\tilde{A}_0^{(0)} = \tilde{A}_N^{(0)} = \tilde{A}_0^{(N)} = 1$ and we can invert relation (5.B.1) as

$$\tilde{M}_i = \frac{\tilde{A}_i^{(0)}}{\tilde{A}_{i-1}^{(0)}}, \qquad \tilde{N}_i = \frac{\tilde{A}_0^{(i)}}{\tilde{A}_0^{(i-1)}}, \qquad \tilde{L}_i = \frac{\tilde{A}_i^{(N-i)}}{\tilde{A}_{i-1}^{(N-i+1)}}. \tag{5.B.3}$$

All placements are illustrated in figure 20. The Kähler parameters associated to the edges of the T_N junction are related to the $\tilde{A}_i^{(j)}$ as follows

$$Q_{n;i}^{(j)} = \frac{\tilde{A}_i^{(j)}\tilde{A}_{i-1}^{(j)}}{\tilde{A}_i^{(j-1)}\tilde{A}_{i-1}^{(j+1)}}, \qquad Q_{l;i}^{(j)} = \frac{\tilde{A}_i^{(j)}\tilde{A}_i^{(j-1)}}{\tilde{A}_{i-1}^{(j)}\tilde{A}_{i+1}^{(j-1)}}, \qquad Q_{m;i}^{(j)} = \frac{\tilde{A}_i^{(j-1)}\tilde{A}_{i-1}^{(j)}}{\tilde{A}_i^{(j)}\tilde{A}_{i-1}^{(j-1)}}. \tag{5.B.4}$$

For each inner hexagon of (20), the following two constraints are satisfied

$$Q_{l;i}^{(j)}Q_{m;i+1}^{(j)} = Q_{m;i}^{(j+1)}Q_{l;i}^{(j+1)}, \qquad Q_{n;i}^{(j)}Q_{m;i}^{(j+1)} = Q_{m;i+1}^{(j)}Q_{n;i+1}^{(j)}. \tag{5.B.5}$$

5.C The summation formula for Nekrasov functions

This appendix contains derivation of the important summation formula (5.5.60) used in the main text of this Chapter. It exploits a binomial identity for the Kaneko-Macdonald(-Warnaar) extension of basic hypergeometric functions [117] introduced in Chapter 3.

Let us first define the 5d uplift of Nekrasov functions, which we write as

$$\begin{aligned}
\mathsf{N}_{\lambda\mu}^{\beta}(u;\epsilon_1,\epsilon_2) :=& \prod_{(i,j)\in\lambda} 2\sinh\frac{\beta}{2}\left[u + \epsilon_1(\lambda_i - j + 1) + \epsilon_2(i - \mu_j^t)\right] \\
&\times \prod_{(i,j)\in\mu} 2\sinh\frac{\beta}{2}\left[u + \epsilon_1(j - \mu_i) + \epsilon_2(\lambda_j^t - i + 1)\right] \\
=& \prod_{s\in\lambda} 2\sinh\frac{\beta}{2}\left[u + \epsilon_1\left(a_\lambda(s) + 1\right) - \epsilon_2 l_\mu(s)\right] \prod_{s\in\mu} 2\sinh\frac{\beta}{2}\left[u - \epsilon_1 a_\mu(s) + \epsilon_2\left(l_\lambda(s) + 1\right)\right]
\end{aligned} \tag{5.C.1}$$

where the products are taken over boxes of partitions λ and μ, respectively. By pulling some factors out of the products, the definition can also be rewritten as

$$\begin{aligned}
\mathsf{N}_{\lambda\mu}^{\beta}(u;\epsilon_1,\epsilon_2) :=& \left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}\frac{1}{U}}\right)^{\frac{|\lambda|+|\mu|}{2}} \mathfrak{t}^{\frac{\|\lambda^t\|^2 - \|\mu^t\|^2}{4}} \mathfrak{q}^{\frac{\|\mu\|^2 - \|\lambda\|^2}{4}} \prod_{(i,j)\in\lambda}\left(1 - U\mathfrak{t}^{\mu_j^t - i}\mathfrak{q}^{\lambda_i - j + 1}\right) \\
&\times \prod_{(i,j)\in\mu}\left(1 - U\mathfrak{t}^{-\lambda_j^t + i - 1}\mathfrak{q}^{-\mu_i + j}\right),
\end{aligned} \tag{5.C.2}$$

where $U = e^{-\beta u}$. For particular values of the parameter u, the introduced functions behave like Kronecker$-\delta$ functions, namely

$$\mathsf{N}_{\lambda\emptyset}^{\beta}(-\epsilon_+) = \mathsf{N}_{\emptyset\lambda}^{\beta}(0) = \delta_{\lambda\emptyset}, \tag{5.C.3}$$

where $\epsilon_+ = \epsilon_1 + \epsilon_2$. Furthermore, they obey the exchange identities

$$\begin{aligned}
\mathsf{N}_{\lambda\mu}^{\beta}(u;-\epsilon_2,-\epsilon_1) &= \mathsf{N}_{\mu^t\lambda^t}^{\beta}(u - \epsilon_+;\epsilon_1,\epsilon_2), \\
\mathsf{N}_{\lambda\mu}^{\beta}(-u;\epsilon_1,\epsilon_2) &= (-1)^{|\lambda|+|\mu|}\mathsf{N}_{\mu\lambda}^{\beta}(u - \epsilon_+;\epsilon_1,\epsilon_2), \\
\mathsf{N}_{\lambda\mu}^{\beta}(u;\epsilon_2,\epsilon_1) &= \mathsf{N}_{\lambda^t\mu^t}^{\beta}(u;\epsilon_1,\epsilon_2).
\end{aligned} \tag{5.C.4}$$

There are two relations connecting the functions we just defined to our definitions from Chapter 3.

$$\frac{1}{h_\lambda(\mathfrak{q},\mathfrak{t})h'_\lambda(\mathfrak{q},\mathfrak{t})} = \frac{(-1)^{|\lambda|}\mathfrak{t}^{-\frac{\|\lambda^t\|^2}{2}}\mathfrak{q}^{-\frac{\|\lambda\|^2}{2}}}{\mathsf{N}^\beta_{\lambda\lambda}(0)} \tag{5.C.5}$$

as well as

$$(U)_\lambda \equiv (U;\mathfrak{q},\mathfrak{t})_\lambda = \left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U\right)^{\frac{|\lambda|}{2}} \mathfrak{t}^{-\frac{\|\lambda^t\|^2}{4}}\mathfrak{q}^{\frac{\|\lambda\|^2}{4}}\mathsf{N}^\beta_{\lambda\emptyset}(u-\epsilon_+), \tag{5.C.6}$$

where $U = e^{-\beta u}$. These formulae will be important in what follows.

It will be crucial for the subsequent argument to rewrite the q-binomial identity for the sl(N) Kaneko-Macdonald(-Warnaar) basic hypergeometric functions (3.4.88) in the topological string conventions. This turns out to be possible due to identities (3.4.66), (5.C.5), (5.C.6) and the following lemma:

Lemma

$$\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}\frac{(A\mathfrak{t}^{j-i})_{\lambda_{1,i}-\lambda_{2,j}}}{(A\mathfrak{t}^{j-i+1})_{\lambda_{1,i}-\lambda_{2,j}}} = \mathfrak{t}^{\frac{k_1|\lambda_2|-k_2|\lambda_1|}{2}}\frac{\mathsf{N}^\beta_{\lambda_2\lambda_1}(-a)}{\mathsf{N}^\beta_{\lambda_2\emptyset}(-a-k_1\epsilon_2)\,\mathsf{N}^\beta_{\emptyset\lambda_1}(-a+k_2\epsilon_2)}, \tag{5.C.7}$$

where $\ell(\lambda_1) \le k_1$, $\ell(\lambda_2) \le k_2$ and $A := e^{-\beta a}$.

Proof. Let us first notice that by using definition (5.C.2) as well as exchange identities (5.C.4), the right-hand side of the above formula can be written as a following product:

$$\mathfrak{t}^{\frac{k_1|\lambda_2|-k_2|\lambda_1|}{2}}\frac{\mathsf{N}^\beta_{\lambda_2\lambda_1}(-a)}{\mathsf{N}^\beta_{\lambda_2\emptyset}(-a-k_1\epsilon_2)\,\mathsf{N}^\beta_{\emptyset\lambda_1}(-a+k_2\epsilon_2)}$$

$$= \prod_{(i,j)\in\lambda_1}\frac{1-A\frac{\mathfrak{t}}{\mathfrak{q}}\mathfrak{t}^{\lambda^t_{2,j}-i}\mathfrak{q}^{\lambda_{1,i}-j+1}}{1-A\frac{\mathfrak{t}}{\mathfrak{q}}\mathfrak{t}^{k_2-i}\mathfrak{q}^{\lambda_{1,i}-j+1}}\prod_{(i,j)\in\lambda_2}\frac{1-A\frac{\mathfrak{t}}{\mathfrak{q}}\mathfrak{t}^{-\lambda^t_{1,j}+i-1}\mathfrak{q}^{-\lambda_{2,i}+j}}{1-A\frac{\mathfrak{t}}{\mathfrak{q}}\mathfrak{t}^{-k_1+i-1}\mathfrak{q}^{-\lambda_{2,i}+j}}. \tag{5.C.8}$$

In proving the lemma, we will deal with formal power series in variables $\mathfrak{t}$ and $\mathfrak{q}$, so that we will not be concerned with issues of convergence of the intermediate expressions, requiring only that $\mathfrak{t},\mathfrak{q} \neq 1$. We also extend the entries of partitions λ_1 and λ_2, such that

$$\lambda_{1,i} := 0,\ i > \ell(\lambda_1), \qquad \lambda_{2,i} := 0,\ i > \ell(\lambda_2) \tag{5.C.9}$$

and for now assume $\ell(\lambda_1) = k_1$, $\ell(\lambda_2) = k_2$.

So, let us start with the following obvious identity:

$$\sum_{i,j=1}^{\infty}\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right) = \left(\sum_{i=1}^{k_1}\sum_{j=1}^{k_2} + \sum_{i=k_1+1}^{\infty}\sum_{j=1}^{k_2} + \sum_{i=1}^{k_1}\sum_{j=k_2+1}^{\infty}\right)\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right). \tag{5.C.10}$$

Taking the last two sums of the right-hand side, shifting their summation indices and using convention (5.C.9), one gets:

$$\left(\sum_{i=k_1+1}^{\infty}\sum_{j=1}^{k_2} + \sum_{i=1}^{k_1}\sum_{j=k_2+1}^{\infty}\right)\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right) = \sum_{i=1}^{\infty}\sum_{j=1}^{k_2}\mathfrak{t}^{j-i-k_1}\left(1-\mathfrak{q}^{-\lambda_{2,j}}\right) + \sum_{i=1}^{k_1}\sum_{j=1}^{\infty}\mathfrak{t}^{j-i+k_2}\left(1-\mathfrak{q}^{\lambda_{1,i}}\right)$$

$$= \frac{1}{\mathfrak{t}^{-1}-1}\left(-\sum_{j=1}^{k_2}\mathfrak{t}^{j-1-k_1}\left(1-\mathfrak{q}^{-\lambda_{2,j}}\right) + \sum_{i=1}^{k_1}\mathfrak{t}^{-i+k_2}\left(1-\mathfrak{q}^{\lambda_{1,i}}\right)\right), \tag{5.C.11}$$

where in the last step we used the sum of an infinite geometric progression. Substituting this back and multiplying the whole expression by $\mathfrak{t}^{-1}-1$, we obtain:

$$(\mathfrak{t}^{-1}-1)\sum_{i,j=1}^{\infty}\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right)=(\mathfrak{t}^{-1}-1)\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right)$$
$$-\sum_{j=1}^{k_2}\mathfrak{t}^{j-1-k_1}\left(1-\mathfrak{q}^{-\lambda_{2,j}}\right)+\sum_{i=1}^{k_1}\mathfrak{t}^{-i+k_2}\left(1-\mathfrak{q}^{\lambda_{1,i}}\right). \tag{5.C.12}$$

Now we will use the following identity which the reader can find for instance in [171]:

$$-(\mathfrak{t}^{-1}-1)\sum_{i=1}^{\infty}\mathfrak{q}^{\lambda_{1,i}}\mathfrak{t}^{1-i}=(\mathfrak{q}^{-1}-1)\sum_{i=1}^{\infty}\mathfrak{t}^{-\lambda^t_{1,i}}\mathfrak{q}^{i}. \tag{5.C.13}$$

Multiplying it by $\sum_{j=1}^{\infty}\mathfrak{t}^{j-1}\mathfrak{q}^{-\lambda_{2,j}}$ and subtracting from the result the same with λ_1, λ_2 set to zero, we find:

$$(\mathfrak{t}^{-1}-1)\sum_{i,j=1}^{\infty}\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right)=(\mathfrak{q}^{-1}-1)\sum_{i,j=1}^{\infty}\mathfrak{t}^{j-1}\mathfrak{q}^{i}\left(\mathfrak{t}^{-\lambda^t_{1,i}}\mathfrak{q}^{-\lambda_{2,j}}-1\right). \tag{5.C.14}$$

Substituting this back as a left-hand side of (5.C.12) and dividing everything by $\mathfrak{q}^{-1}-1$, we obtain the following:

$$\sum_{i,j=1}^{\infty}\mathfrak{t}^{j-1}\mathfrak{q}^{i}\left(\mathfrak{t}^{-\lambda^t_{1,i}}\mathfrak{q}^{-\lambda_{2,j}}-1\right)=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\mathfrak{q}\left(\mathfrak{t}^{j-i-1}-\mathfrak{t}^{j-i}\right)\frac{1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}}{1-\mathfrak{q}}$$
$$+\sum_{j=1}^{k_2}\mathfrak{q}^{1-\lambda_{2,j}}\mathfrak{t}^{j-1-k_1}\frac{1-\mathfrak{q}^{\lambda_{2,j}}}{1-\mathfrak{q}}+\sum_{i=1}^{k_1}\mathfrak{q}\mathfrak{t}^{-i+k_2}\frac{1-\mathfrak{q}^{\lambda_{1,i}}}{1-\mathfrak{q}}, \tag{5.C.15}$$

where one can now use the formula for finite geometric progression to get rid of the fractions in the right-hand side:

$$\sum_{i,j=1}^{\infty}\left(\mathfrak{t}^{j-1-\lambda^t_{1,i}}\mathfrak{q}^{i-\lambda_{2,j}}-\mathfrak{t}^{j-1}\mathfrak{q}^{i}\right)=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{l=1}^{\lambda_{1,i}-\lambda_{2,j}}\left(\mathfrak{t}^{j-i-1}-\mathfrak{t}^{j-i}\right)\mathfrak{q}^{l}$$
$$+\sum_{j=1}^{k_2}\sum_{i=1}^{\lambda_{2,j}}\mathfrak{t}^{j-1-k_1}\mathfrak{q}^{i-\lambda_{2,j}}+\sum_{i=1}^{k_1}\sum_{j=1}^{\lambda_{1,i}}\mathfrak{t}^{-i+k_2}\mathfrak{q}^{j}. \tag{5.C.16}$$

For clarity, the upper bound of the first summation on the right is written schematically, implying that for terms having $\lambda_{1,i}-\lambda_{2,j}<0$ the sum should be replaced by an equivalent corresponding to a negative Pochhammer symbol.

For the left-hand side one now should employ an identity from [205] (our $\mathfrak{t}$ and $\mathfrak{q}$ are interchanged with respect to the formula there):

$$\sum_{i,j=1}^{\infty}\left(\mathfrak{t}^{j-1-\lambda^t_{1,i}}\mathfrak{q}^{i-\lambda_{2,j}}-\mathfrak{t}^{j-1}\mathfrak{q}^{i}\right)=\sum_{s\in\lambda_1}\mathfrak{t}^{l_{\lambda_2}(s)}\mathfrak{q}^{a_{\lambda_1}(s)+1}+\sum_{s\in\lambda_2}\mathfrak{t}^{-l_{\lambda_1}(s)-1}\mathfrak{q}^{-a_{\lambda_2}(s)}$$
$$\equiv\sum_{(i,j)\in\lambda_1}\mathfrak{t}^{\lambda^t_{2,j}-i}\mathfrak{q}^{\lambda_{1,i}-j+1}+\sum_{(i,j)\in\lambda_2}\mathfrak{t}^{i-\lambda^t_{1,j}-1}\mathfrak{q}^{j-\lambda_{2,i}}. \tag{5.C.17}$$

Interchanging the indices in the second summand of the right-hand side of (5.C.16), changing the summation order in the third summand and moving them to the left, one finally obtains:

$$\sum_{(i,j)\in\lambda_1}\left(\mathrm{t}^{\lambda^t_{2,j}-i}-\mathrm{t}^{k_2-i}\right)\mathrm{q}^{\lambda_{1,i}-j+1}+\sum_{(i,j)\in\lambda_2}\left(\mathrm{t}^{-\lambda^t_{1,j}+i-1}-\mathrm{t}^{-k_1+i-1}\right)\mathrm{q}^{-\lambda_{2,i}+j}$$
$$=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{l=1}^{\lambda_{1,i}-\lambda_{2,j}}\left(\mathrm{t}^{j-i-1}-\mathrm{t}^{j-i}\right)\mathrm{q}^{l}. \tag{5.C.18}$$

Substituting here $\mathrm{t},\mathrm{q}\longrightarrow\mathrm{t}^r,\mathrm{q}^r$, multiplying by $\left(A\frac{\mathrm{t}}{\mathrm{q}}\right)^r/r$ and using a series expansion of the logarithm, we get

$$\sum_{(i,j)\in\lambda_1}\ln\left(\frac{1-A\frac{\mathrm{t}}{\mathrm{q}}\mathrm{t}^{\lambda^t_{2,j}-i}\mathrm{q}^{\lambda_{1,i}-j+1}}{1-A\frac{\mathrm{t}}{\mathrm{q}}\mathrm{t}^{k_2-i}\mathrm{q}^{\lambda_{1,i}-j+1}}\right)+\sum_{(i,j)\in\lambda_2}\ln\left(\frac{1-A\frac{\mathrm{t}}{\mathrm{q}}\mathrm{t}^{-\lambda^t_{1,j}+i-1}\mathrm{q}^{-\lambda_{2,i}+j}}{1-A\frac{\mathrm{t}}{\mathrm{q}}\mathrm{t}^{-k_1+i-1}\mathrm{q}^{-\lambda_{2,i}+j}}\right)$$
$$=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\ln\left(\prod_{l=1}^{\lambda_{1,i}-\lambda_{2,j}}\frac{1-A\mathrm{t}^{j-i}\mathrm{q}^{l-1}}{1-A\mathrm{t}^{j-i+1}\mathrm{q}^{l-1}}\right). \tag{5.C.19}$$

Exponentiation concludes the proof.

Remark. Tracing the above argument, one can see that it can be literally extended to the case $\ell(\lambda_1)\leq k_1$, $\ell(\lambda_2)\leq k_2$. This will be crucial for what follows. □

Having the lemma, we now can show that (3.4.88) is equivalent to:

$$\sum_{\lambda^{(1)},\ldots,\lambda^{(N-1)}}{}^{\prime}\left[\prod_{i=1}^{N-2}\left(\frac{z_i}{\mathrm{t}}\mathrm{t}^{\frac{k_{i-1}}{2}+k_i-\frac{k_{i+1}}{2}}\right)^{|\lambda^{(i)}|}\right]\cdot\left(\sqrt{A\frac{\mathrm{t}}{\mathrm{q}}}\frac{z_{N-1}}{\mathrm{t}}\mathrm{t}^{\frac{k_{N-2}+k_{N-1}}{2}}\right)^{|\lambda^{(N-1)}|}$$
$$\times\left[\prod_{i=1}^{N-1}\frac{\mathsf{N}^{\beta}_{\lambda^{(i)}\lambda^{(i-1)}}((k_{i-1}-k_i)\epsilon_2-\epsilon_+)}{\mathsf{N}^{\beta}_{\lambda^{(i)}\lambda^{(i)}}(0)}\right]\cdot\mathsf{N}^{\beta}_{\emptyset\lambda^{(N-1)}}(-a) \tag{5.C.20}$$
$$=\prod_{1\leq i\leq j\leq N-2}\frac{\mathcal{M}(\mathrm{t}^{i-(j+1)+k_i-k_{j+1}}\cdot\prod_{s=i}^{j}(z_s\mathrm{t}^{k_s}))\mathcal{M}(\frac{\mathrm{t}}{\mathrm{q}}\cdot\mathrm{t}^{(i-1)-j+k_{i-1}-k_j}\cdot\prod_{s=i}^{j}(z_s\mathrm{t}^{k_s}))}{\mathcal{M}(\mathrm{t}\cdot\mathrm{t}^{(i-1)-(j+1)+k_{i-1}-k_{j+1}}\cdot\prod_{s=i}^{j}(z_s\mathrm{t}^{k_s}))\mathcal{M}(\frac{1}{\mathrm{q}}\cdot\mathrm{t}^{i-j+k_i-k_j}\cdot\prod_{s=i}^{j}(z_s\mathrm{t}^{k_s}))}$$
$$\times\prod_{i=1}^{N-1}\frac{\mathcal{M}(\frac{A}{\mathrm{q}}\cdot\mathrm{t}^{i-(N-1)+k_i-k_{N-1}}\cdot\prod_{s=i}^{N-1}(z_s\mathrm{t}^{k_s}))\mathcal{M}(\frac{\mathrm{t}}{\mathrm{q}}\cdot\mathrm{t}^{(i-1)-(N-1)+k_{i-1}-k_{N-1}}\cdot\prod_{s=i}^{N-1}(z_s\mathrm{t}^{k_s}))}{\mathcal{M}(\frac{A\mathrm{t}}{\mathrm{q}}\cdot\mathrm{t}^{(i-1)-(N-1)+k_{i-1}-k_{N-1}}\cdot\prod_{s=i}^{N-1}(z_s\mathrm{t}^{k_s}))\mathcal{M}(\frac{1}{\mathrm{q}}\cdot\mathrm{t}^{i-(N-1)+k_i-k_{N-1}}\cdot\prod_{s=i}^{N-1}(z_s\mathrm{t}^{k_s}))}.$$

Finally, we are in position to prove the required summation formula:

Theorem

$$\sum_{\lambda^{(1)},\ldots,\lambda^{(N-1)}}\left[\prod_{i=1}^{N-1}\frac{\left(V_i\sqrt{U_iU_{i+1}}\right)^{|\lambda^{(i)}|}}{\mathsf{N}^{\beta}_{\lambda^{(i)}\lambda^{(i)}}(0)}\right]\mathsf{N}^{\beta}_{\lambda^{(1)}\emptyset}(u_1-\epsilon_+/2)$$
$$\times\left[\prod_{i=1}^{N-2}\mathsf{N}^{\beta}_{\lambda^{(i+1)}\lambda^{(i)}}(u_{i+1}-\epsilon_+/2)\right]\mathsf{N}^{\beta}_{\emptyset\lambda^{(N-1)}}(u_N-\epsilon_+/2) \tag{5.C.21}$$
$$=\prod_{i=1}^{N-1}\prod_{j=1}^{N-i}\frac{\mathcal{M}(\prod_{s=j}^{i+j-1}(V_sU_s))\mathcal{M}(\frac{\mathrm{t}}{\mathrm{q}}\frac{U_{i+j}}{U_j}\cdot\prod_{s=j}^{i+j-1}(V_sU_s))}{\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}U_{i+j}\cdot\prod_{s=j}^{i+j-1}(V_sU_s))\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}\frac{1}{U_j}\cdot\prod_{s=j}^{i+j-1}(V_sU_s))},$$

with N site parameters $U_i = e^{-\beta u_i}$ and $N-1$ link parameters V_j. One can visualize the right-hand side of this formula by noticing that the arguments of numerator are precisely all the simply-connected combinations of even number of site and link parameters (multiplied by $\frac{\mathfrak{t}}{\mathfrak{q}}$ when starting with a link parameter), whereas the arguments of denominator represent all the simply-connected combinations of odd number of site and link parameters (multiplied by $\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}$, single site parameters are excluded).

Proof. We use a so-called *specialization technique* [116]. Let us group all terms on the left having the same powers of V_i, $i = 1, \ldots, N-1$, *i.e.* grade our infinite sum with respect to a number of boxes of partitions we sum over. The coefficient of each combination of $V_1^{i_1} \cdots V_{N-1}^{i_{N-1}}$ is a polynomial in variables U_i, $i = 1, \ldots, N$ of degree $2(i_1 + \cdots + i_{N-1})$, having its coefficients in $\mathbb{F}$. Similarly, expanding the right-hand side as a series in V_i and re-summing geometric progressions in $\mathfrak{q}, \mathfrak{t}$ into rational functions, we learn that the corresponding coefficients are as well polynomial in variables U_i with coefficients in $\mathbb{F}$.

Let us now take any ordered combination of positive integers k_i, $k_1 < \cdots < k_{N-1}$, such that

$$k_{i+1} - k_i \geq \ell(\lambda^{(i+1)}). \tag{5.C.22}$$

One can see that the condition $\lambda_s^{(i)} \geq \lambda_{s-k_i+k_{i+1}}^{(i+1)}$ is trivially satisfied in this way, turning the corresponding skew plane partition into a horizontal strip plane partition. Making the following specialization of U_i (remember that $k_0 \equiv 0$):

$$U_i = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\, \mathfrak{t}^{k_i - k_{i-1}}, \qquad i = 1, \ldots, N-1 \tag{5.C.23}$$

and reparametrizing the remaining variables as

$$V_j = \sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\, \frac{z_j}{\mathfrak{t}}\, \mathfrak{t}^{k_{j-1}+k_j-k_{j+1}}, \qquad j = 1, \ldots, N-2 \tag{5.C.24}$$

as well as

$$U_N = \sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\, \frac{1}{A}, \qquad V_{N-1} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\, A\, \frac{z_{N-1}}{\mathfrak{t}}\, \mathfrak{t}^{k_{N-2}} \tag{5.C.25}$$

one can readily check that formula (5.C.21) then degenerates to the established sl(N) $\mathfrak{q}$-binomial identity (5.C.20). Correspondingly, the above statement on equality of two polynomial coefficients translates into a statement on equality of corresponding polynomial coefficients of $z_1^{i_1} \cdots z_{N-1}^{i_{N-1}}$, which holds true.

We see that two polynomials in $N-1$ variables[14] coincide on an $(N-1)$-dimensional semilattice, meaning they just coincide. Term by term, this proves the theorem.

□

Finally, let us remark that the summation formula (5.C.21) for $N = 2$

$$\sum_{\lambda^{(1)}} \left(V_1 \sqrt{U_1 U_2}\right)^{|\lambda^{(1)}|} \frac{\mathsf{N}^{\beta}_{\lambda^{(1)}\emptyset}(u_1 - \epsilon_+/2)\, \mathsf{N}^{\beta}_{\emptyset\lambda^{(1)}}(u_2 - \epsilon_+/2)}{\mathsf{N}^{\beta}_{\lambda^{(1)}\lambda^{(1)}}(0)} = \frac{\mathcal{M}(U_1V_1)\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}V_1U_2)}{\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}V_1)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_1V_1U_2)} \tag{5.C.26}$$

[14] According to the above specialization, U_N can be kept generic.

reproduces the non-trivial part of (5.3) of [166], whereas, taken for $N = 3$

$$\sum_{\lambda^{(1)},\lambda^{(2)}} \left(V_1 \sqrt{U_1U_2}\right)^{|\lambda^{(1)}|} \left(V_2 \sqrt{U_2U_3}\right)^{|\lambda^{(2)}|} \frac{\mathsf{N}^{\beta}_{\lambda^{(1)}\emptyset}(u_1 - \epsilon_+/2)\, \mathsf{N}^{\beta}_{\lambda^{(2)}\lambda^{(1)}}(u_2 - \epsilon_+/2)\, \mathsf{N}^{\beta}_{\emptyset\lambda^{(2)}}(u_3 - \epsilon_+/2)}{\mathsf{N}^{\beta}_{\lambda^{(1)}\lambda^{(1)}}(0)\, \mathsf{N}^{\beta}_{\lambda^{(2)}\lambda^{(2)}}(0)}$$
$$= \frac{\mathcal{M}(U_1V_1)\mathcal{M}(\frac{\mathrm{t}}{\mathrm{q}}V_1U_2)\mathcal{M}(U_2V_2)\mathcal{M}(\frac{\mathrm{t}}{\mathrm{q}}V_2U_3)\mathcal{M}(U_1V_1U_2V_2)\mathcal{M}(\frac{\mathrm{t}}{\mathrm{q}}V_1U_2V_2U_3)}{\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}V_1)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}V_2)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}U_1V_1U_2)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}V_1U_2V_2)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}U_2V_2U_3)\mathcal{M}(\sqrt{\frac{\mathrm{t}}{\mathrm{q}}}U_1V_1U_2V_2U_3)}, \tag{5.C.27}$$

it is equivalent to the formula (6.7) conjectured in [167].

Chapter 6

Mellin amplitudes of conformal partial waves

This Chapter discusses a new technique for dealing with cumbersome expressions one encounters analyzing Mellin amplitudes of CFT correlators in d dimensions. Our method describes a way to move between different forms of such expressions and, in some cases, to simplify them. It draws inspiration from classical summation theorems for hypergeometric functions. The Chapter is based on our forthcoming paper [4], joint with V. Schomerus.

6.1 Interlude

As discussed in Chapter 2, crossing symmetry imposes a strong constraint on the coefficients of the operator product expansion. Unfortunately, this constraint is very difficult to analyze and so far no non-trivial analytic solutions are known beyond $d = 2$. One of the reasons is the complicated form of all the ingredients of conformal bootstrap for $d > 2$. Therefore, a good understanding of the corresponding expressions is very crucial for possible future advances in this field. In the present Chapter, we discuss Mellin representation for correlators, focusing on Mellin amplitudes for conformal partial waves. The Mellin representation for conformal correlators was first proposed by Mack in [31]. It is useful in many respects.

The main physical motivation of using Mellin amplitudes is very simple. Namely, it is known that from the decomposition of the four-point functions into conformal partial waves (CPW) it is difficult to keep track of crossing symmetry since the conformal partial waves transform non-trivially under the exchange of the cross-ratios u and v. In the Mellin representation, on the other hand, we decompose the correlation function with respect to the set of functions $u^s v^t$ which have a trivial behavior under the exchange of u and v. In other words, the Mellin representation makes crossing symmetry manifest.

A very surprising feature of the Mellin amplitudes for conformal correlators is their striking resemblance to scattering amplitudes [31]. This property of Mellin amplitudes which is only starting to be explored and exploited, will certainly bring many non-trivial insights. For instance, the study of factorization properties of Mellin amplitudes initiated in [31, 38] should ultimately bring to more efficient ways of computing Mellin amplitudes, much like the BCFW relations do the job for scattering amplitudes [206].

The main purpose of this Chapter is to present a new method of dealing with complicated expressions encountered in studies of Mellin amplitudes. We will see that the machinery of multiple hypergeometric functions is quite appropriately designed for this range of problems. From the analogy to scattering amplitudes, one can easily guess the root of such usefulness. It is, at least partially, in the fact that multiple hypergeometric functions are very natural expressions for Feynman integrals yielding corresponding amplitudes. We will demonstrate the working profile of our technique on a simple example of a residue in the four-point function of scalars which controls contribution of quasi-primaries and descendants of the minimal twist. We expect that ultimately, our method should provide a combinatorial bridge between expressions obtained via solving eigenvalue problems for conformal Casimirs [30] and Mack's approach of calculating conformal integrals [31].

The plan of this Chapter is as follows. We begin with a formula for the Mellin representation of four-point functions and outline its analogues for higher-point functions. After this, we turn to a brief discussion of the Mellin transform of conformal partial waves and sketch the original derivation of Mack. Explicitly nailing down from CPWs the residues controlling operator product expansion of two scalars, we show how the first residue, controlling contribution of a quasi-primary operator and all its descendants with minimal twist, can be drastically simplified using our new technique. Namely, we prove the conjecture of [38], asserting that this residue is given by a simple single-variable hypergeometric function.

6.2 The Mellin representations of 4-point functions

The Mellin amplitude M_4 of a 4-point function $G(x_i)$ of scalar fields with conformal weight Δ_i is defined by

$$G(x_1, x_2, x_3, x_4) = \int [d\delta] M_4(\delta_{ij}) \prod_{i<j} x_{ij}^{-2\delta_{ij}} \Gamma(\delta_{ij}) \ . \tag{6.2.1}$$

Here, $i, j = 1, \ldots 4$ and $\delta_{ij} = \delta_{ji}$ are solutions to the constraints

$$\Delta_{ii} = 0 \quad , \quad \sum_j \delta_{ij} = \Delta_i \ .$$

Note that the definition of the Mellin amplitude and the variables δ_{ij} admits for an obvious generalization to more than four scalar fields. In the case of a 4-point function, the space of solutions δ_{ij} to the constraints we imposed is 2-dimensional, regardless of the space-time dimension d our theory is defined on. Actually, the the 2-dimensional space of solutions can be spelled out very explicitly,

$$\begin{aligned}
\delta_{12} &= \frac{1}{6}(2\Delta_1 + 2\Delta_2 - \Delta_3 - \Delta_4) - \tilde{s} \quad , \quad & \delta_{23} &= \frac{1}{6}(2\Delta_2 + 2\Delta_3 - \Delta_1 - \Delta_4) - \tilde{t} \\
\delta_{13} &= \frac{1}{6}(2\Delta_1 + 2\Delta_3 - \Delta_2 - \Delta_4) + \tilde{t} + \tilde{s} \quad , \quad & \delta_{24} &= \frac{1}{6}(2\Delta_2 + 2\Delta_4 - \Delta_1 - \Delta_3) + \tilde{s} + \tilde{t} \\
\delta_{14} &= \frac{1}{6}(2\Delta_1 + 2\Delta_4 - \Delta_2 - \Delta_3) - \tilde{t} \quad , \quad & \delta_{34} &= \frac{1}{6}(2\Delta_3 + 2\Delta_4 - \Delta_1 - \Delta_2) - \tilde{s}.
\end{aligned} \tag{6.2.2}$$

When we perform the integration in eq. (6.2.1) we are instructed to pick any two independent δ_{ij} and integrate along a contour following the usual Mellin-Barnes prescription. If all four scalar fields possess equal weight $\Delta_i = \Delta$, the Mellin representation takes the form

$$G(x_i) = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} \int_{-i\infty}^{+i\infty} \frac{d\tilde{s}}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{d\tilde{t}}{2\pi i} u^{\tilde{s}+\frac{2\Delta}{3}} v^{\tilde{t}-\frac{\Delta}{3}} M_4(\tilde{s}, \tilde{t}) \Gamma^2\left(\frac{\Delta}{3} - \tilde{s}\right) \Gamma^2\left(\frac{\Delta}{3} - \tilde{t}\right) \Gamma^2\left(\frac{\Delta}{3} + \tilde{t} + \tilde{s}\right) \ . \tag{6.2.3}$$

Explicitly working this representation out for simple models of conformal field theory, e.g. for 2d minimal models, one can see that the Mellin amplitude resembles a bit the typical structure of scattering amplitudes in string theory, like the famous Veneziano or Virasoro-Shapiro amplitudes [80], in that it is constructed from a product of gamma functions.

This intriguing resemblance between the Mellin amplitudes of correlation functions and scattering amplitudes goes further. Note that we can solve the constraints we imposed on the parameters δ_{ij} if we write $\delta_{ij} = p_i \cdot p_j$ where p_i are d-vectors satisfying

$$p_j^2 = -\Delta_j \quad , \quad \sum_j p_j = 0 \, .$$

With this parametrization of δ_{ij}, the Mellin amplitude becomes a function which depends on the momenta p_i of massive particles through the usual kinematic invariants just as a scattering amplitude. Like scattering amplitudes, the Mellin amplitude may be shown to possess poles when the sum of any two of the four external momenta satisfy on-shell conditions. Note that

$$-(p_1+p_2)^2 = \Delta_1 + \Delta_2 - 2\delta_{12} = \frac{4}{3}\overline{\Delta} + 2\tilde{s} \quad , \quad -(p_1+p_4)^2 = \Delta_1 + \Delta_4 - 2\delta_{14} = \frac{4}{3}\overline{\Delta} + 2\tilde{t} \tag{6.2.4}$$

where we introduced the shorthand $4\overline{\Delta} = \sum_i \Delta_i$ for the average of the conformal weight of our external fields. Hence, the amplitude takes the general form

$$M_4(s,t) \sim \sum_\phi \lambda_{12\phi}\lambda_{34\phi}\left(\frac{1}{2}\sum_{K=0}^{\infty}\frac{Q_K(\Delta,J)[\tilde{t}]}{\tilde{s}-\varpi_K(\Delta,J)}\right) . \tag{6.2.5}$$

Here, the first sum runs over all quasi-primary fields ϕ that appear in operator product of φ_1 and φ_2. It turns out that each such field of spin J and weight $\Delta \equiv h - i\nu$, where $h \equiv d/2$, determines a whole series of poles at

$$\varpi_K(\Delta,J) = \frac{\tau_\phi}{2} + K - \frac{2}{3}\overline{\Delta} \tag{6.2.6}$$

where $K = 0, 1, 2, 3, \ldots$, see [31,207] for details. We point out that the weight Δ and the spin J only enter through the so-called twist $\tau_\phi = \Delta - J$.

More remarkably, there also exist relatively simple formulae for the residues Q_K. Before we state some of them we point out that $Q_K(\Delta, J)$ may be shown to be a polynomial of order J in the variable $\tilde{t}$. Once again, such a feature is well known for simple string theory amplitudes. A concrete, though somewhat complicated, formula for the residue Q_J was derived by Mack in [31]. Later, Gonçalves et al. provided the following simpler expression, at least for the first residue Q_0 [38]

$$Q_0(\Delta,J) = -\frac{\lambda_{12\phi}\lambda_{34\phi}\Gamma(\Delta+J)(\delta_1)_J(\delta_3)_J}{2^J\Gamma(\delta_{12})\Gamma(\delta_{34})\prod_{\nu=1}^4\Gamma(\delta_\nu+J)}\, {}_3F_2\!\left(\begin{matrix}-J,\delta_{13},\delta_{24}\\ 1-J-\delta_{14},1-J-\delta_{23}\end{matrix};1\right), \tag{6.2.7}$$

where we need to set $\tilde{s} \equiv \tau_\phi/2 - \overline{\Delta}/6$ so that all the $\delta_{ij} = \delta_{ij}(\tilde{s},\tilde{t})$ are functions of $\tilde{t}$ only. Furthermore, the δ_i are given by

$$2\delta_1 = \Delta_1 - \Delta_2 + \tau_\phi \, , \; 2\delta_2 = \Delta_2 - \Delta_1 + \tau_\phi \, , \; 2\delta_3 = \Delta_3 - \Delta_4 + \tau_\phi \, , \; 2\delta_4 = \Delta_4 - \Delta_3 + \tau_\phi \, .$$

Expressions for the higher residues Q_K, which may be obtained through some recursion relation, will not be relevant in the following discussion.

Before we conclude this section, we want to come back to the issue of positivity, which is somewhat obscure in the Mellin representation. However, the results we reviewed in the last few paragraphs allow to check for positivity, at least in principle. In order to do so, we need to extract the coefficients λ of the operator product expansion from the Mellin amplitude M_4. Subsequently we can locate first poles from each series that appears in M_4 and read off the corresponding residue Q_0. These residues contain the products $\lambda_{12\phi}\lambda_{34\phi}$ along with a factor that depends on the weight Δ and spin J, but is known explicitly. Thus, we can remove this numerical factor and check for positivity of the remaining λ_ϕ^2.

6.3 Mack polynomials

As we already mentioned, conformal partial waves g^ϕ are crucial for the conformal bootstrap program, separating kinematical information on the four-point function from the dynamical one given by three-point couplings C_ϕ. In this section, we briefly recall how the expression for a scalar conformal partial wave was derived in [31], constructed from an integral over a product of two normalized 3-point vertex functions V. For this, we first introduce some notations.

First thing to discuss are the fields transforming in more general representations of the conformal algebra than just scalars, namely the symmetric traceless tensors. The corresponding fields $\phi^J_{\mu_1,\dots,\mu_J}$ are completely symmetric under exchange of the indices and satisfy

$$\sum_\mu \phi^J_{\mu_1,\dots,\mu,\mu} = 0\ .$$

Since it is a bit awkward to deal a large number of indices, we shall convert the field ϕ^J into a function of d complex variables $\zeta = (\zeta_1, \dots, \zeta_d)$,

$$\phi^J(\zeta) \equiv \phi^J_{\mu_1,\dots,\mu_J}\zeta^{\mu_1}\cdots\zeta^{\mu_J}\ .$$

In order to make the number of components of a symmetric traceless tensor of rank J match the number of monomials of order J in the d complex variables ζ_μ, we impose the additional condition

$$\zeta\cdot\zeta = \sum_\mu \zeta_\mu^2 = 0\ .$$

Let us point out that conformal field theories certainly contain quasi-primary fields that transform in representations other than the symmetric traceless ones. The reason we restrict here to the latter is that these are the only ones that appear in the operator product expansion of scalar fields.

The three-point functions we will need to construct our expressions are from the special class involving two scalar fields φ_1 and φ_2 of scaling weight Δ_1 and Δ_2 along with a single quasi-primary ϕ_3^J of weight Δ_3 and spin J. Very much like the pure scalar case (2.3.14) the 3-point function of these fields is determined by conformal symmetry, except for an overall normalization, to take the form

$$\langle\varphi_1(x_1)\varphi_2(x_2)\phi_3^J(\zeta;x)\rangle = \lambda_{12\phi_3}V(x_1,x_2;\zeta,x) \qquad (6.3.1)$$

where

$$V^{(J)}_{12;3}(x_1,x_2;\zeta,x_3) = \frac{N_J(\Delta_1,\Delta_2;\Delta_3)}{(2\pi)^{d/2}}\left(\frac{2}{x_{12}^2}\right)^{\frac{\Delta_{12,3}+J}{2}}\left(\frac{2}{x_{13}^2}\right)^{\frac{\Delta_{13,2}-J}{2}}\left(\frac{2}{x_{23}^2}\right)^{\frac{\Delta_{23,1}-J}{2}}(\mu_{12;3}\cdot\zeta)^J\ . \qquad (6.3.2)$$

Here, we introduced $\Delta_{ij,k} = \Delta_i + \Delta_j - \Delta_k$ and

$$\mu_{12;3} = 2\left(\frac{x_{13}}{x_{13}^2} - \frac{x_{23}}{x_{23}^2}\right) .$$

Note that for $J = 0$ we recover the formula (2.3.13) from the Chapter 2.

As should be clear from the discussion in Chapter 2, one can characterize the conformal partial waves

$$g^{\phi}_{(12)(34)}(z,\bar{z}) = g_{\Delta,J}(z,\bar{z}) \tag{6.3.3}$$

as solutions to the following second order differential equations

$$D^{(2)}_{z,\bar{z}}\, g_{\Delta,J}(z,\bar{z}) = C_{\Delta,J}\, g_{\Delta,J}(z,\bar{z}) \tag{6.3.4}$$

satisfying the boundary condition

$$g_{\Delta,J}(z,\bar{z}) \sim (z\bar{z})^{\frac{1}{2}(\Delta-J)}(z+\bar{z})^J + \dots\ . \tag{6.3.5}$$

Here

$$C_{\Delta,J} = \Delta(\Delta-d) + J(J+d-2) \tag{6.3.6}$$

denotes the eigenvalue of the quadratic Casimir[1] $C_2 = \frac{1}{2}M_{AB}M^{AB} = -\frac{1}{2}M_{AB}M^{AB}$, $A, B = 0, 1, \dots d+1$ of conformal algebra and $D^{(2)}_{z,\bar{z}}$ is a differential operator representing it, which reads as

$$\begin{aligned} \frac{1}{2}D^{(2)}_{z,\bar{z}} &= \frac{1}{4}\Omega_{(12)(34)}(x_i)\left(M^{(1)}_{AB} + M^{(2)}_{AB}\right)\left(M^{(1)AB} + M^{(2)AB}\right)\Omega^{-1}_{(12)(34)}(x_i) \\ &= \bar{z}^2(1-\bar{z})\bar{\partial}^2 + z^2(1-z)\partial^2 + (d-2)\frac{z\bar{z}}{\bar{z}-z}\left(\bar{\partial}-\partial\right) \\ &+ (d-2)(z^2\partial - \bar{z}^2\bar{\partial}) - ab(z+\bar{z}) - (a+b+1)(\bar{z}^2\bar{\partial} + z^2\partial), \end{aligned} \tag{6.3.7}$$

where we denoted

$$a = -\frac{1}{2}\Delta_{12}, \qquad b = \frac{1}{2}\Delta_{34}. \tag{6.3.8}$$

It can be shown that such particular solution is extracted from an integral

$$\Omega_{(12)(34)}(x_i)\, \mathrm{g}_{\Delta,J}(z,\bar{z}) := \frac{(2\pi)^d}{N_J^2}\int d^d x_0 \langle V^{(J)}_{12;0}(x_1,x_2;\xi,x_0), V^{(J)}_{34;\tilde{0}}(x_3,x_4;\xi',x_0)\rangle \tag{6.3.9}$$

where $\Omega_{(12)(34)}(x_i)$ was defined in (2.3.17). The brackets here $\langle\cdot,\cdot\rangle$ denote the scalar product in the space of traceless polynomials. More precisely, we have the following 'multipole' expansion

$$\langle(\mu\cdot\xi)^J, (\mu'\cdot\xi')^J\rangle = \frac{J!}{(d/2-1)_J}2^{-J}(|\mu||\mu'|)^J C_J^{d/2-1}\left(\frac{\mu\cdot\mu'}{|\mu||\mu'|}\right) = (|\mu||\mu'|)^J \hat{C}_J^{d/2-1}\left(\frac{\mu\cdot\mu'}{|\mu||\mu'|}\right)$$

[1]The generators M_{AB} of conformal algebra used here are linear combinations of those introduced in Chapter 2.

where the (normalized) Gegenbauer polynomials are defined as

$$\hat{C}_n^\lambda(y) := \frac{n!}{2^n(\lambda)_n} C_n^\lambda(y) := \frac{(2\lambda)_n}{2^n(\lambda)_n} \, {}_2F_1\left(\begin{matrix} -n, n+2\lambda \\ \lambda+\frac{1}{2} \end{matrix}; \frac{1-y}{2}\right). \tag{6.3.10}$$

Note that we have used the symbol $\mathfrak{g}$ instead of g we used before. Equation (6.3.9) should be considered as a definition of $\mathfrak{g}$ which mimics the construction of conformal partial waves. By this very definition, the function $\mathfrak{g}$ satisfies the same second order differential equation as the conformal partial wave g. On the other hand, it does not obey the same boundary conditions: in addition to the term that is proportional to the conformal partial wave it contains a second, so-called shadow contribution [208]. In order to construct the conformal partial wave g from the integral $\mathfrak{g}$, one has to remove the shadow term. This will later translate to a requirement of picking only half of the relevant poles when calculating corresponding Mellin-Barnes integral, see also further.

Let us compute the conformal integral (6.3.9). Integrals of this type were first discussed in 1970's, long before the era of CFT, in studies of scattering amplitudes. Particularly, for parameters δ_i with $\operatorname{Re}\,\delta_i > 0 \;\; \forall i$ such that $\delta_i = d$, the following formula was known due to Symanzik [209][2]:

$$\frac{1}{\pi^h}\int d^d u \prod_{i=1}^{n} \frac{1}{\left((u-x_i)^2\right)^{\delta_i}} = \frac{1}{(2\pi i)^{n(n-3)/2}} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} d\tilde{s}_i \ldots d\tilde{s}_{n(n-3)/2} \tag{6.3.11}$$

$$\times \prod_{1\le i<j\le n} \frac{\Gamma\left(\delta_{ij}^0 + \sum_k c_{ij,k}\tilde{s}_k\right)}{\Gamma(\delta_i)\,(x_{ij}^2)^{\delta_{ij}^0+\sum_k c_{ij,k}\tilde{s}_k}},$$

where δ_{ij}^0 is a particular solution of

$$\sum_{j\neq i} \delta_{ij}^0 = \delta_i \tag{6.3.12}$$

with positive real parts and the real $c_{ij,k} = c_{ji,k}$ obey

$$c_{ii,k} = 0, \qquad \sum_{j\neq i} c_{ij,k} = 0. \tag{6.3.13}$$

Here k runs over $1, \ldots, n(n-3)/2$ and the $(n(n-3)/2 \times n(n-3)/2)$ matrix $c_{ij,k}$, with $2 \le i < j \le n$ (excluding the pair $(i,j) = (2,3)$, such that corresponding $c_{23,k}$ may be taken as independent parameters), is required to have determinant ± 1. For $n = 4$, fixing e.g. $c_{23,1} = c_{23,2} = -1$ and choosing parameters $\delta_{ij} = \delta_{ij}^0 + \sum_k c_{ij,k}\tilde{s}_k$ as in (6.2.2), one gets a so-called 4-star formula:

$$\frac{1}{\pi^h}\int d^d u \prod_{i=1}^{4} \frac{\Gamma(\delta_i)}{\left((u-x_i)^2\right)^{\delta_i}} = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} d\tilde{s}\, d\tilde{t} \prod_{1\le i<j\le 4} \frac{\Gamma\left(\delta_{ij}\right)}{(x_{ij}^2)^{\delta_{ij}}}. \tag{6.3.14}$$

In [31], Mack generalized this formula to integrands whose numerator additionally contains polynomials of type

$$T_l^{(h)}(y) = \sum_{r=0}^{\lfloor \frac{J}{2} \rfloor} a_{J,r}\, y^{J-2r}. \tag{6.3.15}$$

[2] This generalizes earlier result of D'Eramo, Peliti and Parisi [210].

Notice that Gegenbauer polynomials $\hat{C}_J^{h-1}(y)$ (6.3.10) are exactly of such type, with

$$a_{J,r} = \frac{J!}{2^{2r} r!(J-2r)!(2-h-J)_r}. \tag{6.3.16}$$

The idea of Mack is to expand the scalar product in (6.3.9), so that the integral reduces to a sum of known Symanzik integrals. Namely, we have

$$\mu = 2\left(\frac{x_{10}}{x_{10}^2} - \frac{x_{20}}{x_{20}^2}\right) \qquad \mu' = 2\left(\frac{x_{40}}{x_{40}^2} - \frac{x_{30}}{x_{30}^2}\right) \tag{6.3.17}$$

in (6.3), so that

$$\mu^2 = \frac{4x_{12}^2}{x_{10}^2 x_{20}^2}, \qquad \mu' = \frac{4x_{34}^2}{x_{30}^2 x_{40}^2}, \qquad \langle \mu, \mu' \rangle = 2\left(\frac{x_{13}^2}{x_{10}^2 x_{30}^2} + \frac{x_{24}^2}{x_{20}^2 x_{40}^2} - \frac{x_{14}^2}{x_{10}^2 x_{40}^2} - \frac{x_{23}^2}{x_{20}^2 x_{30}^2}\right).$$

Multinomial expansion then asserts

$$\langle \mu, \mu' \rangle^{J-2r} = 2^{J-2r}(J-2r)! \sum_{k_{ij}:\ \sum k_{ij} = J-2r}' (-1)^{k_{14}+k_{23}} \prod_{k_{ij}}' \frac{1}{k_{ij}!}\left(\frac{x_{ij}^2}{x_{i0}^2 x_{i0}^2}\right)^{k_{ij}}, \tag{6.3.18}$$

where the notations Σ', Π' instruct us to take a sum/product allowing just four combinations of indices (ij): (13), (14), (23) and (24). Substituting this to (6.3.9), one indeed observes that the latter boils down to integrals of type (6.3.11), with shifted δ_i now depending on summation indices.

Before we spell out a final answer for Mack polynomials, a few notational changes are in order. Namely, starting from this point we will adopt conventions of [211] which make the Mack polynomial look more compact. In particular, let us relabel the Mellin integration variables as:

$$\begin{aligned} \tilde{s} &= \frac{t}{2} - \frac{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4}{6} \\ \tilde{t} &= \frac{2\Delta_2 + 2\Delta_3 - \Delta_1 - \Delta_4}{6} - \frac{t+s}{2}. \end{aligned} \tag{6.3.19}$$

Then, upon appropriately choosing the normalization constant N_J in (6.3.9), result of the above integration (6.3.9) becomes:

$$\begin{aligned} \mathfrak{g}_{\nu,J}(u,v) = \int_{-i\infty}^{i\infty} \frac{dtds}{(4\pi i)^2} M_{\nu,J}(s,t)\, u^{t/2} v^{-(s+t)/2}\, \Gamma\!\left(\frac{\Delta_1+\Delta_2-t}{2}\right)\Gamma\!\left(\frac{\Delta_3+\Delta_4-t}{2}\right) \\ \Gamma\!\left(\frac{\Delta_{34}-s}{2}\right)\Gamma\!\left(\frac{-\Delta_{12}-s}{2}\right)\Gamma\!\left(\frac{t+s}{2}\right)\Gamma\!\left(\frac{t+s+\Delta_{12}-\Delta_{34}}{2}\right). \end{aligned} \tag{6.3.20}$$

As we mentioned above, $\mathfrak{g}_{\nu,J}(u,v)$ is given by the sum of two conformal blocks with dimensions $h+i\nu$ and $h-i\nu$. More precisely, one can write

$$\mathfrak{g}_{\nu,J}(u,v) = \kappa_{\nu,J}\, g_{h+i\nu,J}(u,v) + \kappa_{-\nu,J}\, g_{h-i\nu,J}(u,v)\,. \tag{6.3.21}$$

Here the normalization constant

$$\kappa_{\nu,J} = \frac{i\nu}{2\pi K_{h+i\nu,J}} \tag{6.3.22}$$

can be fixed by comparing the residues of $M_{\nu,J}(s,t)$ at $t = h \pm i\nu - J + 2K$ with the general expression (6.3.35), and

$$K_{\Delta,J} = \frac{\Gamma(\Delta+J)\,\Gamma(\Delta-h+1)\,(\Delta-1)_J}{4^{J-1}\Gamma\left(\frac{\Delta+J+\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta+J-\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta+J+\Delta_{34}}{2}\right)\Gamma\left(\frac{\Delta+J-\Delta_{34}}{2}\right)} \times \frac{1}{\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta+J}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-\Delta+J}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta+J-2h}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4+\Delta+J-2h}{2}\right)}\,. \tag{6.3.23}$$

The Mellin amplitude

$$M_{\nu,J}(s,t) = \omega_{\nu,J}(t)\,P_{\nu,J}(s,t)\,, \tag{6.3.24}$$

turns out to decompose into a generic part $\omega_{\nu,J}(t)$ which provides the poles expected from OPE, and a polynomial part $P_{\nu,J}(s,t)$ where

$$\omega_{\nu,J}(t) = \frac{\Gamma\left(\frac{\Delta_1+\Delta_2+J+i\nu-h}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4+J+i\nu-h}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2+J-i\nu-h}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4+J-i\nu-h}{2}\right)}{8\pi\Gamma(i\nu)\Gamma(-i\nu)} \times \frac{\Gamma\left(\frac{h+i\nu-J-t}{2}\right)\Gamma\left(\frac{h-i\nu-J-t}{2}\right)}{\Gamma\left(\frac{\Delta_1+\Delta_2-t}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-t}{2}\right)}\,. \tag{6.3.25}$$

The polynomials, first introduced in [31], can be written as

$$P_{\nu,J}(s,t) = \sum_{r=0}^{\lfloor J/2\rfloor} a_{J,r}\frac{2^{J+2r}\left(\frac{h+i\nu-J-t}{2}\right)_r\left(\frac{h-i\nu-J-t}{2}\right)_r(J-2r)!}{(h+i\nu-1)_J\,(h-i\nu-1)_J} \times \sum_{\sum k_{ij}=J-2r}^{\prime}(-1)^{k_{13}+k_{24}}\prod_{(ij)}^{\prime}\frac{(\gamma_{ij})_{k_{ij}}}{k_{ij}!}\prod_{n=1}^{4}(\alpha_n)_{J-r-\sum_j k_{jn}}\,. \tag{6.3.26}$$

The variables γ_{ij} are

$$\gamma_{13} = \frac{\Delta_{34}-s}{2}\,,\quad \gamma_{24} = -\frac{\Delta_{12}+s}{2}\,,\quad \gamma_{23} = \frac{t+s}{2}\,,\quad \gamma_{14} = \frac{t+s+\Delta_{12}-\Delta_{34}}{2} \tag{6.3.27}$$

whereas the variables α_n are given by

$$\alpha_1 = 1-\frac{h+i\nu+J+\Delta_{12}}{2}\,,\qquad \alpha_2 = 1-\frac{h+i\nu+J-\Delta_{12}}{2}\,,$$
$$\alpha_3 = 1-\frac{h-i\nu+J+\Delta_{34}}{2}\,,\qquad \alpha_4 = 1-\frac{h-i\nu+J-\Delta_{34}}{2}\,. \tag{6.3.28}$$

There are several immediate symmetry properties that follow from the formula (6.3.26) by relabelling the summation variables, namely

$$P_{-\nu,J}(s,t,\Delta_{12},\Delta_{34}) = P_{\nu,J}(s,t,-\Delta_{34},-\Delta_{12}) \tag{6.3.29}$$
$$P_{-\nu,J}(s,t,\Delta_{12},\Delta_{34}) = P_{\nu,J}(s+\Delta_{12}-\Delta_{34},t,\Delta_{34},\Delta_{12}) \tag{6.3.30}$$
$$P_{\nu,J}(s,t,\Delta_{12},-\Delta_{34}) = (-1)^J P_{\nu,J}(-s-t-\Delta_{12},t,\Delta_{12},\Delta_{34}) \tag{6.3.31}$$
$$P_{\nu,J}(s,t,-\Delta_{12},\Delta_{34}) = (-1)^J P_{\nu,J}(-s-t+\Delta_{34},t,\Delta_{12},\Delta_{34}). \tag{6.3.32}$$

The property which is of most importance for us is that the Mack polynomials, at specific values of t, reduce to the functions $Q_{J,K}(s)$ that control a scalar OPE

$$P_{i(\Delta-h),J}(s,\Delta-J+2K)=Q_{J,K}(s)\,. \tag{6.3.33}$$

More precisely, the expression

$$Q_{J,K}(s)=-\frac{2\Gamma(\Delta+J)(\Delta-1)_J}{4^J\Gamma\left(\frac{\Delta+J+\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta+J-\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta+J+\Delta_{34}}{2}\right)\Gamma\left(\frac{\Delta+J-\Delta_{34}}{2}\right)} \frac{Q_{J,K}(s)}{K!(\Delta-h+1)_K\,\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta+J}{2}-K\right)\Gamma\left(\frac{\Delta_3+\Delta_4-\Delta+J}{2}-K\right)} \tag{6.3.34}$$

represents a residue of Mellin amplitude $M(s,t)$, such that

$$M(s,t)\approx\frac{\lambda_{12\phi}\lambda_{34\phi}\,Q_{J,K}(s)}{t-\Delta+J-2K}\,,\qquad K=0,1,2,\dots\,, \tag{6.3.35}$$

where, as before, Δ and J are the dimension and spin of an operator ϕ^J that appears in both OPEs $\varphi_1\varphi_2$ and $\varphi_3\varphi_4$. This shows that the $K>0$ poles correspond to conformal descendant operators with twist greater than $\Delta-J$.

For further convenience, we will now write the Mack polynomials using the multiple hypergeometric functions defined in Chapter 3. To do so, we need to solve the condition $\sum k_{ij}=J-2r$ with respect to one of the indices k_{ij}. It will spoil the symmetric form of the sum, but using the above symmetry properties one can translate between different forms of rewriting the Mack polynomial this way. Let us take, for instance, k_{14} and express it through others. The $P_{\nu,J}$ then reads:

$$\begin{aligned}P_{\nu,J}(s,t)=&\frac{2^J(\alpha_2)_J(\alpha_3)_J(\gamma_{14})_J}{(h+i\nu-1)_J\,(h-i\nu-1)_J}\sum_{r\geq0}\frac{(-J)_{2r}\left(\frac{h+i\nu-J-t}{2}\right)_r\left(\frac{h-i\nu-J-t}{2}\right)_r(\alpha_1)_r(\alpha_4)_r}{(2-J-h)_r\,(1-J-\alpha_2)_r\,(1-J-\alpha_3)_r}\\&\times F^{(3)}\left(\begin{matrix}-J+2r::\alpha_1+r;\alpha_4+r;-:\gamma_{24};\gamma_{23};\gamma_{13}\\1-J-\gamma_{14}+2r::1-J-\alpha_2+r;1-J-\alpha_3+r;-:-;-;-\end{matrix};1,1,1\right).\end{aligned} \tag{6.3.36}$$

Obviously, this expression can be written as just a single quadruple Srivastava-Daoust hypergeometric function. Being in a sense more universal, such notation will not bring us any fruit in the present work, so we will restrain from presenting it.

6.4 First residue of the scalar OPE

According to (6.3.35), for taking the polynomial controlling K-th residue of the OPE, we set $t=\Delta-J+2K$, such that it reads as

$$Q_{J,K}(s)=\frac{2^J(A+C-x_1)_J\,(B+C)_J\,(D+K)_J}{(h+x_2-1)_J\,(h-x_2-1)_J} \tag{6.4.1}$$

$$\times\sum_{r\geq0}\frac{(-J)_{2r}\,(K)_r\,(-x_1-K)_r\,(1-J-B-D+x_2)_r(1-J-A-D)_r}{(1-J-K-D)_{2r}(2-J-h)_r\,(A+C-x_1)_r\,(B+C)_r} \tag{6.4.2}$$

$$\times F^{(3)}\left(\begin{matrix}-J+2r::1-J-B-D+x_2+r;1-J-A-D+r;-:A;C+K;B\\1-J-K-D+2r::A+C-x_1+r;B+C+r;-:-;-;-\end{matrix};1,1,1\right),$$

where we introduced further notations

$$A = \gamma_{24}\,, \quad B = \gamma_{13}\,, \quad C = \gamma_{23} - K\,, \quad D = \gamma_{14} - K\,, \quad x_1 = x_2 = \Delta - h \tag{6.4.3}$$

making the subsequent proof somewhat cleaner.

We have already quoted above a simplified expression (6.2.7) for the first residue, $K = 0$, which was described and motivated in [211] via using conformal Casimir operator. We will now present a new, direct combinatorial derivation of this finding[3], starting from (6.4.1). The tricks we will use in it seem actually to have a much more general range of applicability.

Substituting $K = 0$ to (6.4.1), we see that the sum over r vanishes, so that only one triple hypergeometric function survives. The conjecture we would like to prove now reads as:

$$\begin{aligned} F^{(3)} &:= F^{(3)}\left(\begin{matrix} -J & :: 1-J-B-D+x_2\,; 1-J-A-D\,; - : A; C; B \\ 1-J-D :: A+C-x_1 & ; B+C \qquad ; - : -; -; - \end{matrix} ; 1,1,1 \right) \\ &= (-1)^J \,\frac{(A)_J\,(B)_J\,(A+B+C+D-x_1-x_2+J-1)_J}{(B+C)_J\,(D)_J\,(A+C-x_1)_J}\, {}_3F_2\left(\begin{matrix} -J, C, D \\ 1-J-A, 1-J-B \end{matrix} ; 1 \right), \end{aligned} \tag{6.4.4}$$

which is, of course, can be marked as a drastic simplification with respect to general situation.

Before starting, let us state a useful identity due to Srivastava [212]:

$$\begin{aligned} &F^{p:\,2;\,1}_{q:\,1;\,0}\left(\begin{matrix} (a_p) :\ \alpha, \beta\,; \gamma-\alpha-\beta \\ (b_q) : \gamma+N; - \end{matrix} ; 1,1 \right) \\ &= \frac{(\gamma)_N(\gamma-\alpha-\beta)_N}{(\gamma-\alpha)_N(\gamma-\beta)_N} \sum_{k=0}^{\infty} \frac{(-N)_k(\alpha)_k(\beta)_k}{(\gamma)_k(1-N+\alpha+\beta-\gamma)_k\,k!}\; {}_{p+2}F_{q+1}\left(\begin{matrix} (a_p),\ \gamma-\alpha,\ \gamma-\beta \\ (b_q),\ \gamma+k \end{matrix} ; 1 \right), \end{aligned} \tag{6.4.5}$$

where $N \in \mathbb{Z}_{\geq 0}$. It can be justified by shifting summation variables of the left-hand side followed by employing a so-called contiguous Saalschütz identity and rearranging the sums.

We will prove the identity (6.4.4) by imposing a certain specialization on one of the parameters:

$$D = 1 - J - B + N, \qquad\qquad N \geq 0. \tag{6.4.6}$$

Due to a polynomial nature of the particular function we study, validity of this identity for all N will mean validity of the initial, 'complete' identity.[4]

Substituting (6.4.6) to (6.4.4) and using the property $(\lambda)_{p+i} = (\lambda)_p\,(\lambda+p)_i$ to separate the sum over p, we have

$$F^{(3)} = \sum_{p=0}^{\infty} \frac{(-J)_p(B-A-N)_p(B)_p}{(B-N)_p\,(B+C)_p\,p!}\, F^{2:\,2;\,1}_{2:\,1;\,0}\left(\begin{matrix} -J+p, x_2-N :\ B-A-N+p, C\,; A \\ B-N+p, A+C-x_1\ : B+C+p; - \end{matrix} ; 1,1 \right). \tag{6.4.7}$$

Noticing that parameters of the double hypergeometric function now allow the use of the identity (6.4.5), we employ it[5] and, after reshuffling the Pochhammer symbols to make a sum over k the outermost,

[3]The identity we prove here is precisely the conjecture of appendix B.4.2 in [38].

[4]There exists a so-called Carlson's theorem making this statement precise for any analytic function with modest growth at infinity, see [213] for more information.

[5]With parameters $\alpha = B - A - N + p, \beta = C$ and $\gamma = B + C - N + p$.

get:

$$F^{(3)} = \frac{(A)_N(1-B-C)_N}{(1-B)_N(A+C)_N} \sum_{k=0}^{\infty} \frac{(-N)_k(B-A-N)_k(C)_k}{(B+C-N)_k(1-N-A)_k\, k!}$$
$$\times F^{1:2;1}_{1:1;0}\left(\begin{matrix} -J : A+C, x_2-N ; B-A-N+k \\ B+C-N+k : A+C-x_1 ; - \end{matrix} ; 1, 1\right). \tag{6.4.8}$$

Let us now elaborate a bit more on the inner double hypergeometric function $F^{1:2;1}_{1:1;0}$. First, we take and cast on it one of the Burchnall-Chaundy identities (3.3.34) of Chapter 3, obtaining

$$F^{1:2;1}_{1:1;0} := \sum_{n=0}^{\infty} \frac{(-J)_n(B+C+J-N+k)_n(A+C)_n(x_2-N)_n(B-A-N+k)_n}{(B+C-N+k+n-1)_n(B+C-N+k)_{2n}(A+C-x_1)_n\, n!}$$
$$\times {}_3F_2\left(\begin{matrix} -J+n, A+C+n, x_2-N+n \\ B+C-N+k+2n, A+C-x_1+n \end{matrix} ; 1\right) {}_2F_1\left(\begin{matrix} -J+n, B-A-N+k+n \\ B+C-N+k+2n \end{matrix} ; 1\right). \tag{6.4.9}$$

The ${}_2F_1$ function here is summable according to Gauss formula (3.2.12):

$${}_2F_1\left(\begin{matrix} -J+n, B-A-N+k+n \\ B+C-N+k+2n \end{matrix} ; 1\right) = \frac{(A+C)_J}{(B+C-N+k)_J} \frac{(B+C-N+k)_{2n}}{(A+C)_n(B+C-J-N+k)_n},$$

where we again used properties of Pochhammer symbols to rewrite the result. Let us now plug this back to (6.4.9) and shift the summation indices of the resulting double sum as $m \mapsto m-n$, where m denotes a summation index of the ${}_3F_2$ function. After some massage, involving the easily proved formula

$$\frac{1}{(\lambda+n-1)_n(\lambda+2n)_m} = \frac{1}{(\lambda)_m} \frac{(\lambda-1)_n(\frac{\lambda+1}{2})_n}{(\lambda+m)_{2n}(\frac{\lambda-1}{2})_n} \tag{6.4.10}$$

with $\lambda = B+C-N+k$, one can bring function (6.4.9) to a following outfit:

$$F^{1:2;1}_{1:1;0} = \frac{(A+C)_J}{(B+C-N+k)_J} \sum_{m=0}^{\infty} \frac{(-J)_m(x_2-N)_m(A+C)_m}{(A+C-x_1)_m(B+C-N+k)_m\, m!} \tag{6.4.11}$$
$$\times \sum_{n=0}^{m} \frac{(B+C-1-N+k)_n(1+\frac{B+C-1-N+k}{2})_n(B-A-N+k)_n(-m)_n}{(\frac{B+C-1-N+k}{2})_n(A+C)_n(B+C-N+k+m)_n} \frac{(-1)^n}{n!}.$$

A bit heavy though this expression might seem at the first glance, it is actually quite nice: the inner sum can be simplified using the following limiting case of Dougall's theorem[6]

$${}_4F_3\left(\begin{matrix} a, \frac{a}{2}+1, c, -m \\ \frac{a}{2}, 1+a-c, 1+a+m \end{matrix} ; -1\right) = \frac{(1+a)_m}{(1+a-c)_m}, \tag{6.4.12}$$

with obvious identification of parameters. Substituting this back, we will find ourselves left just with a ${}_2F_1(1)$ which is again easily summed up by Gauss formula, finally yielding:

$$F^{1:2;1}_{1:1;0} = \frac{(A+C)_J}{(B+C-N+k)_J} \frac{(A+C+N-x_1-x_2)_J}{(A+C-x_1)_J}. \tag{6.4.13}$$

[6]This classical result for a very well-poised ${}_5F_4(1)$ can be looked up, for instance, in [214].

Let us substitute this back to (6.4.8) and use one of the well-known Thomae relations [214][7] [8]:

$$ {}_3F_2\left(\begin{matrix}-N, b, c\\ d, e\end{matrix}; 1\right) = \frac{(-1)^N (e-b)_N (e-c)_N}{(d)_N (e)_N} {}_3F_2\left(\begin{matrix}-N, 1-N-e, 1-N+b+c-d-e\\ 1-N+b-e, 1-N+c-e\end{matrix}; 1\right), \tag{6.4.14} $$

with $b = B - A - N$, $c = C$, $d = B + C + J - N$ and $e = 1 - N - A$ to get

$$ F^{(3)} = \frac{(A+C)_J (A+C+N-x_1-x_2)_J}{(B+C-N)_J (A+C-x_1)_J} \frac{(A)_N (1-B-C)_N}{(A)_N (1-J-B-C)_N} \tag{6.4.15} $$
$$ \times {}_3F_2\left(\begin{matrix}-N, 1-N-e, 1-N+b+c-d-e\\ 1-N+b-e, 1-N+c-e\end{matrix}; 1\right). $$

However, at this point we are not yet done: by matching the arguments of hypergeometric function and Pochhammer symbols with our assertion above, one sees they are not exactly what we expect. To finalize, we need one more ingredient: a nice formula due to Karlsson [215], which transforms a ${}_3F_2$ function with integer parameter difference

$$ {}_3F_2\left(\begin{matrix}a, b, f+N\\ c, f\end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_3F_2\left(\begin{matrix}-N, a, b\\ f, 1+a+b-c\end{matrix}; 1\right). \tag{6.4.16} $$

Notice that Thomae's relations on their own cannot connect the two such ${}_3F_2(1)$. This means that the Karlsson's formula, resembling them a lot at the first look, actually represents a completely different class of identities. Using it with obvious parameter substitutions, we arrive to a needed ${}_3F_2$ 'domain', where, upon reversing the summation order, one gets:

$$ F^{(3)} = (-1)^J \frac{(A)_J (B)_J (A+C+N-x_1-x_2)_J}{(B+C)_J (1-J-B+N)_J (A+C-x_1)_J} {}_3F_2\left(\begin{matrix}-J, C, 1-J-B+N\\ 1-J-A, 1-J-B\end{matrix}; 1\right), \tag{6.4.17} $$

which is exactly (6.4.4) specialized as $D = 1 - J - B + N$. Concluding the proof, we remark that one can clearly obtain several equivalent expressions from here by using the Thomae's relations and/or reversing the summation order.

6.5 Outlook

The main achievement of this Chapter is presenting a new way to think about expressions encountered when studying Mellin amplitudes. Namely, we look at them through the lens of multiple hypergeometric functions. To demonstrate our new approach, we derived, starting from the Mack polynomial, a simplified expression for the first residue of scalar OPE. This expression was obtained earlier by solving the eigenvalue problem for conformal Casimir [211], but, unlike the latter approach, our formalism can be easily extended to more general situations. There are two possible ways of such an extension which ideally should be combined. The first is to study the further residues in the scalar OPE, corresponding to $K > 0$ in (6.3.35). A recursion relation is known for those [211] allowing, in principle, to obtain polynomials corresponding to any $K > 0$ from $K = 0$. This recursion, however, has to our knowledge not

[7] This particular identity is a Thomae's transformation combined with reversal of the summation order.

[8] Thomae's relations are exactly those forming the S_5 group of transformations discussed in Chapter 3. To match the terminology, we notice that this type of formulae is the same as what is called Sheppard identities in [30].

yet been solved expicitly, so that up to now no general expressions are known for $Q_{J,K}(s)$ with $K > 0$ simpler than the ones provided by Mack. Judging by the $K = 0$ case though, one may expect a certain simplification also for higher residues. Our method provides a direct way to obtain such if it exists. The calculation which in many ways resembles the one presented here, is work in progress. The second way of extension assumes looking at residues appearing in factorization of higher-point functions of external scalars. In [38], a formula was conjectured for the first residue of an N-point Mellin amplitude for scalars which extends the four-point case. Our method should also prove useful for direct uncovering analogues of simplifications for those if there are any. Combining these two footpaths to analyze higher residues of multipoint amplitudes would be an interesting next problem to address.

Chapter 7

Conclusion

When our mind perceives things with sufficiently complex structure it often renders them as beautiful, perharps to create a shield from what otherwise would be too hard to process. Emblazonment of such a shield can then tell something about *la belle Dame*, if it was chosen with care. In the present thesis we took one truly beautiful subject and put into a spotlight the motto 'CFTs as special functions of the XXI century'. We tried to give some flesh to the latter, studying its impications while looking at several particular examples of conformal field theories. Although our choice of those might have seemed a bit scattered, upon giving it a thought one can see this is so due to the very essence of the subject: It is plausible that interpolating between two such different and vast fields should require somewhat bold strokes, before the finer picture reveals itself. We hope that this thesis will be a helpful addition to a long-term progress in understanding relations between CFT and hypergeometric functions. Clearly, the subject deserves a profound further study. Before listing some ways to proceed along the lines sketched in this work, we will briefly remind substantial results of it.

In Chapter 4, we studied a particular coset model describing strongly coupled one-dimensional strange metals of high density. We described its large bosonic symmetry algebra and studied representations of the latter into which excitations of these media fall. Due to supersymmetry of the coset model, a particular subset of fields exists in it which is closed under fusion. These chiral primaries saturating the BPS bound assemble themselves into an algebraic ring structure. We found the chiral ring of such one-dimensional strange metals to be isomorphic to a simple ring of symmetric polynomials. Nailing down this rigid structure is a strong result which should allow us to identify a dual string theory in Anti-de-Sitter space. Hypergeometric functions in this Chapter were used more on a case-by-case basis, popping up, for instance, as ${}_2F_0$ Charlier polynomials (4.D.4) counting interlacings of basis elements in walled Brauer algebras. So, in a sense, here we saw the hypergeometric functions in a role they traditionally play, being just one of many types of known special functions. In the two subsequent Chapters, however, properties of hypergeometric functions played a more active character and took a substantial part in our analysis of physical problems.

Thus, Chapter 5 was devoted to studies of Toda conformal field theory. Another duality, the AGT(-W) correspondence, is at work here helping to circumvent computational difficulties of more traditional CFT methods. It allowed to calculate the three-point functions of Toda primary fields by using a toolkit of refined topological string for geometrically engineering corresponding gauge theories. The interesting part of the resulting formulae are the (5d) Nekrasov instanton sums. In our work, we showed how fruitful it is to think about those as (deformed) Kaneko-Macdonald hypergeometric functions. Namely, we performed a very non-trivial calculation using binomial identity for them to sum up instanton contributions, yielding an important particular case known from pure CFT considerations of Fateev and Litvinov. This calculation provides a path connecting the new advances in understanding gauge theories and dualities to the well-established CFT results. It also justifies this proposed formula *per se*, as its topological string derivation used several non-trivial, though physically motivated conjectures.

Finally, Chapter 6 discussed Mellin amplitudes of scalar four-point functions. Those are expandable into integrals over universal kinematical quantities, the conformal partial waves. Understanding a structure of the latter is crucial for advances in higher-dimensional conformal bootstrap program. Expressions for the conformal partial waves are generically quite complicated. We presented a new method of dealing with them involving Srivastava-Daoust multiple hypergeometric functions and showed its application to a particular interesting problem. Namely, we proved a conjecture due to Gonçalves et al. yielding a simple expression for the leading twist residue in scalar OPE. The role of the theory of hypergeometric functions was obviously crucial in this case.

If the subject is interesting and deep enough, its discussion in some sense never ends, so that there is always a tiny fraction of artificiality in bringing that to a close. Assuming it inevitable, let us in the very last part of this thesis condense down to a mere listing some future prospects of our investigations, Chapter by Chapter.

As we mentioned before, an interesting next goal of considerations in Chapter 4 is to identify a dual stringy background. In [216], a pretty general family was constructed of AdS_3 backgrounds with $\mathcal{N} = (2,2)$ supersummetry. It is possible to calculate chiral rings they produce and then to match with our finding. Identifying the dual theory could shed new light not only onto the structure of excitations of one-dimensional strange metals, but also on the dynamics of corresponding tensionless string theory. Indeed, the conformal field theory in two dimensions provides plenty of methods to solve a theory exactly. To be able to apply those, it is of course desirable to calculate a full spectrum of the corresponding CFTs for general N and thoroughly investigate the new chiral algebra we found. This could make another direction of possible developments. For instance, our labeling of the spectrum via necklaces suggests there should be a way to read quantum numbers of symmetry representations directly from those. In terms of corresponding coset characters, this is closely connected to studying modular properties of the $SU(N)_{2N}$ string functions. Finally, an SU(2) symmetry[1] was observed for the space of BPS states, appropriately truncated, such that the corresponding partition function is the one of a certain Haldane-Shastry spin chain. [217]. It would be interesting to understand a Hamiltonian of the latter in terms of the chiral algebra generators, as well as to explore corresponding Haldane-Shastry motives. Possible relations to fractional quantum Hall effect deserve an exploration as well.

Further prospects of the research conducted in Chapter 5, first, include a thorough understanding of the four-dimensional limit of the general 5d (q-deformed Toda) formulae. On the level of corresponding hypergeometric functions, this amounts to taking the limit of basic Kaneko-Macdonald functions to ordinary ones. The next immediate thing to do is an analogous topological string derivation of the most general three-point functions of Toda theory: those having one descendant and two primary fields. Another line of research where the theory of hypergeometric functions can bring an interesting fruit is the following. From an outfit of the general formula for three-point functions of Toda primaries, one can clearly see it to be a Mellin-Barnes representation of something we may call a double sl(N) basic Kaneko-Macdonald(-Warnaar) hypergeometric function. A thorough definition of the corresponding class of functions and

[1]This SU(2) symmetry seems to build itself up to a full Yangian symmetry. [217]

study of their properties could make an interesting direction to follow. For instance, it may well be that these particular hypergeometric functions are (partially) summable. More generally, extending that to root systems other than A_{N-1}, such developments could help in understanding general conditions for simplification of the sums over partitions one always encounters in applying the AGT machinery.

Possible advances in considerations of Chapter 6, as we already mentioned, include applications of our newly found technique to simplifying further residues encountered by dissecting Mellin amplitudes for N-point functions, whenever a particular (conjectured) expression is available. This would constitute a minimal expected outcome of our method. Depending on the difficulty of corresponding computations, one may as well try to advance with the tricks we presented somewhat further. Namely, there is a considerable unexplored room for generalizing to calculations of higher spin conformal blocks, which are a subject of active research at present. Some particular results are known for corresponding polynomials, involving, e.g., those for one external vector field replacing a scalar. Whereas recursion relations are sufficient for numerical research on conformal bootstrap, the analytical approach certainly requires better control. In terms of finding new interesting hypergeometric patterns playing a role in CFT, there are at least two promising directions. First, it would be interesting to look at the corresponding hypergeometric expressions for $6j$-symbols of conformal algebra in general dimensions. These $6j$-symbols are of quite an importance, as they constitute a main ingredient of the higher-dimensional bootstrap. The second direction involves analysis of explicit expressions [70] for Virasoro blocks in two dimensions, recently obtained by solving the corresponding recursion relation. Virasoro bootstrap, although having worked spectacularly for some two-dimensional models, has its limitations. The absence of explicit decompositions for the conformal blocks was one of those. It would be interesting to analyze functions obtained in [70] in detail and bring them to a hypergeometric form. Finally, while the curtain is starting to fall, there is the last, ambitious goal to mention: making analytical calculations in higher-dimensional bootstrap really possible. The generalized hypergeometric functions could in principle be ample enough to provide a general mathematical language for analytically solving crossing symmetry constraints. Whether this would or would not be the case, is a question for future.

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Zusammenfassung

Die konforme Feldtheorie (CFT) bietet eine universelle Beschreibung verschiedener Phänomene in den Naturwissenschaften. Ihre schnelle und erfolgreiche Entwicklung gehört zu den wichtigsten Höhepunkten der theoretischen Physik des späten 20. Jahrhunderts. Demgegenüber ging der Fortschritt der hypergeometrischen Funktionen durch die Jahrhunderte langsamer vonstatten. Funktionale Identitäten, die von dieser mathematsichen Disziplin untersucht werden, sind faszinierend sowohl in ihrer Komplexität, als auch ihrer Schönheit. Diese Arbeit untersucht das Zusammenspiel dieser beiden Themen anhand der direkten Analyse dreier CFT-Problemen: Zweipunktfunktionen der zweidimensionalen 'strange metal CFT', Dreipunktfunktionen von primären Feldern der nichtrationalen Toda CFT und kinematischen Teilen von Mellin-Amplituden für skalare Vierpunktfunktionen in beliebigen Dimensionen. Wir heben verschiedene Verallgemeinerungen der hypergeometrischen Funktionen als eine natürliche mathematische Sprache für zwei dieser Probleme hervor. Einige neue Methoden, die durch klassische Resultate über hypergeometrische Funktionen inspiriert wurden, werden vorgestellt. Diese Arbeit basiert sowohl auf unseren Publikationen [1–3], als auch auf einem Papier, dass sich in seiner abschliessenden Vorbereitung befindet [4].

Abstract

Conformal field theory provides a universal description of various phenomena in natural sciences. Its development, swift and successful, belongs to the major highlights of theoretical physics of the late XX century. In contrast, advances of the theory of hypergeometric functions always assumed a slower pace throughout the centuries of its existence. Functional identities studied by this mathematical discipline are fascinating both in their complexity and beauty. This thesis investigates the interrelation of two subjects through a direct analysis of three CFT problems: two-point functions of the 2d strange metal CFT, three-point functions of primaries of the non-rational Toda CFT and kinematical parts of Mellin amplitudes for scalar four-point functions in general dimensions. We flash out various generalizations of hypergeometric functions as a natural mathematical language for two of these problems. Several new methods inspired by extensions of classical results on hypergeometric functions, are presented. This work is based on our publications [1–3] as well as on one paper at the final stage of preparation [4].

This thesis is based on following publications:

- M. Isachenkov, I. Kirsch and V. Schomerus, *Chiral Primaries in Strange Metals,* Nucl. Phys. B **885**, 679 (2014) [arXiv:1403.6857 [hep-th]].

- M. Isachenkov, I. Kirsch and V. Schomerus, *Chiral Ring of Strange Metals: The Multicolor Limit,* Nucl. Phys. B **897**, 660 (2015) [arXiv:1410.4594 [hep-th]].

- M. Isachenkov, V. Mitev and E. Pomoni, *Toda 3-point Functions From Topological Strings II,* [arXiv:1412.3395 [hep-th]].

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen benutzt habe.

Hamburg, den 11. September 2015