Superconformal indices, dualities and integrability

D I S S E R T A T I O N

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)
im Fach Physik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
Humboldt-Universität zu Berlin

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Tag der mündlichen Prüfung: 22. July 2016

Abstract

In this thesis we discuss exact, non-perturbative results achieved using superconformal index technique in supersymmetric gauge theories with four supercharges (which is $\mathcal{N} = 1$ supersymmetry in four dimensions and $\mathcal{N} = 2$ supersymmetry in three).

We use the superconformal index technique to test several duality conjectures for supersymmetric gauge theories. We perform tests of three-dimensional mirror symmetry and Seiberg-like dualities.

The purpose of this thesis is to present recent progress in non-perturbative supersymmetric gauge theories in relation to mathematical physics. In particular, we discuss some interesting integral identities satisfied by basic and elliptic hypergeometric functions and their relation to supersymmetric dualities in three and four dimensions.

Methods of exact computations in supersymmetric theories are also applicable to integrable statistical models, which we discuss in the last chapter of the thesis.

Zusammenfassung

In dieser Arbeit behandeln wir exakte, nicht-perturbative Ergebnisse, die mithilfe der superkonformen Index-Technik, in supersymmetrischen Eichtheorien mit vier Superladungen (d. h. N=1 Supersymmetrie in vier Dimensionen und N=2 in drei Dimensionen) gewonnen wurden.

Wir benutzen die superkonforme Index-Technik um mehrere Dualitäts Vermutungen in supersymmetrischen Eichtheorien zu testen. Wir führen Tests der dreidimensionalen Spiegelsymmetrie und Seiberg ähnlicher Dualitäten durch.

Das Ziel dieser Promotionsarbeit ist es moderne Fortschritte in nicht-perturbativen supersymmetrischen Eichtheorien und ihre Beziehung zu mathematischer Physik darzustellen. Im Speziellen diskutieren wir einige interessante Identitäten der Integrale, denen einfache und hypergeometrische Funktionen genügen und ihren Bezug zu supersymmetrischen Dualitäten in drei und vier Dimensionen.

Methoden der exakten Berechnungen in supersymmertischen Eichtheorien sind auch auf integrierbare statistische Modelle anwendbar. Dies wird im letzten Kapitel der vorliegenden Arbeit behandelt.

To my son Adnan

Acknowledgements

I wish to express my gratitude to my supervisor Prof. Jan Plefka for giving me the opportunity to do research at Humboldt University. I am grateful to him for supporting me during these years and for the possibility to write this thesis on my subject of choice.

I have the pleasant duty to thank my various collaborators on the projects presented in this thesis. I am grateful to Hjalmar Rosengren, Vyacheslav Spiridonov and Grigory Vartanov for very interesting discussions and collaborations in which a major part of the thesis was done. I have also benefited a lot from discussions with Edvard Musaev, Edoardo Vescovi, Wei Li and Stefan Theisen.

I would like to thank Grigory Vartanov for introducing me into the subject and for many stimulating discussions from which I have personally benefited.

Furthermore, I express my sincere gratitude to Edvard Musaev for his immense effort by careful reading of the thesis, many useful comments and suggestions, which were very helpful by eliminating several mistakes.

Many thanks are due to the people who supported me throughout my PhD studies, especially I would like to thank Emil Akhmedov and George Jorjadze.

I also thank Herman Nicolai, Masahito Yamazaki, Matthias Staudacher and Igor Sokolov for being members of my thesis defense committee.

Finally, let me thank my wife, Urnisa, for her patience, understanding and love, and my parents Balakhan and Reyhan, as well as my sister Svetlana for their support. They have shaped me into the person I am today.

Contents

Publications

The results presented in this thesis are based on the publications [1, 2, 3, 4] as well as [5] which is in preparation.

Chapter 3

- [1]: I. B. Gahramanov and G. S. Vartanov, "Superconformal indices and partition functions for supersymmetric field theories," XVIIth Intern. Cong. Math. Phys. 695-703 (2013)

Chapter 4

- [2]: I. Gahramanov and G. Vartanov, "Extended global symmetries for 4D $N = 1$ SQCD theories," J. Phys. A **46** (2013) 285403

Chapter 5

- [5]: I. Gahramanov and H. Rosengren, "Basic hypergeometry of susy dualities", to appear

- [3]: I. Gahramanov and H. Rosengren, "A new pentagon identity for the tetrahedron index," JHEP **1311**, 128 (2013)

Chapter 6

- [4]: I. Gahramanov and V. P. Spiridonov, "The star-triangle relation and 3d superconformal indices," JHEP **1508**, 040 (2015)

1 Introduction

Supersymmetry and dualities

Supersymmetry is a powerful idea in theoretical physics, which provides a non-trivial extension of the Poincare algebra. It transforms bosons into fermions and vice versa. Nowadays, supersymmetry is one of the key tools for high energy physics research beyond the Standard Model of particle physics.

Supersymmetry cannot be an exact symmetry of Nature. No one has ever seen a supersymmetric particle and so far no experimental evidence for supersymmetry has been discovered at the Large Hadron Collider and Tevatron. However the possibility that the world is supersymmetric at high energies attracts attention of scientists.

On the other hand the supersymmetric theories play role of a theoretical laboratory for studying non-perturbative effects in realistic theories, in particular they are an excellent technical playground for Quantum Chromodynamics. Supersymmetric gauge theories exhibit some of the same non-perturbative phenomena as Quantum Chromodynamics, such as confinement, chiral symmetry breaking, etc [6, 7, 8, 9].

In recent years, there have been extensive studies on exactly calculable quantities of supersymmetric gauge theories in diverse dimensions [10, 11, 12, 13, 14, 15, 16, 17, 18, 19] due to the use of the supersymmetric localization technique [20]. This technique enables us to compute exact quantities such as superconformal indices, partition functions on compact manifolds, Wilson loops, 't Hooft loops, surface operators and so on. These exact results gave a new and fresh look to the old and challenging problems. This thesis is mainly devoted to one of such exact quantities – the superconformal index.

In the case of a supersymmetric field theory one can generalize the Witten index [21] by including global symmetries of a theory commuting with a particular supercharge [22, 23, 24]. The superconformal index is a regularized index for the d-dimensional supersymmetric theory on $S^d \times S^1$ which counts short multiplets that cannot combine into long ones. For a d-dimensional supersymmetric theory the superconformal index is schematically defined as follows

$$\mathrm{I}(\{t_i\}) = \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}} \prod t_i^{F_i} \,, \tag{1.1}$$

where the trace is taken over the Hilbert space on S^{d-1}, F_i are generators for global symmetries that commute with Q and $Q^\dagger$, and t_i are additional regulators corresponding to the global symmetries.

The superconformal index has some properties that make it useful for studying supersymmetric theories. For insatance, since the superconformal index is invariant under renormalization group flow, it can be computed for weakly coupled theories and it must be the same in the strongly coupled regime. The main application of the superconformal index is checking supersymmetric dualities and providing non-trivial evidences for them [25, 26, 27, 28, 29, 30, 31].

In the 1990's Seiberg [32] and many others (e.g. [33, 34, 35, 36, 37, 38]) found a non-trivial quantum equivalence between different supersymmetric theories, called supersymmetric duality. To be more precise it was shown that two or more different theories may describe the same physics in the far infrared limit, i.e. an observer testing the low energy physics (or physics at long distances) cannot distinguish the dual theories[1].

The supersymmetric duality was first constructed [32] for four-dimensional $\mathcal{N} = 1$ gauge theory with matter in the fundamental representation. Later many examples of dualities have been found with complicated matter content, different gauge and flavor groups in different dimensions.

The basic example of supersymmetric duality [32] is an $SU(N_c)$ "electric" gauge theory with N_f flavors of quarks which possesses a dual description in terms of N_f "magnetic" flavors of quarks charged under $SU(N_f - N_c)$ gauge group[2] in the so-called conformal

[1] It is worth mentioning that supersymmetric dual theories are not identical, but they give rise to the same physics at long distances.

[2] In this case the gauge singlets of the dual theory interact with the flavors via the superpotential

window $\frac{3}{2}N_c < N_f < 3N_c$. These two theories flow to the same infrared fixed point.

It is worth mentioning that before the superconformal index technique the main consistency check for the conjectured supersymmetric dualities were the 't Hooft anomaly matching conditions [39, 40]. These conditions require that the values of the triangle anomalies corresponding to the global symmetries must coincide for the dual theories. Unfortunately, the anomaly matching is insufficient [41, 42] to check supersymmetric duality. There are cases when 't Hooft anomaly conditions of non-dual theories coincidentally match[3], interestingly that the superconformal indices of such theories do not coincide [43].

There is an interesting observation made in [44] that the 't Hooft anomaly matching conditions for dual theories are related to $SL(3, Z)$ modular transformation properties of the kernels of dual superconformal indices written as an integral over Coulomb branch moduli for a gauge group of the theory. There is also a recent observation that the central charges a and c [45], their difference $c - a$ [46, 47] for $\mathcal{N} = 1$ and $2a - c$ conformal anomaly [48] for $\mathcal{N} = 2$ theories can be obtained directly from the superconformal index.

Over the last ten years supersymmetric dualities for theories with different number of supersymmetric charges in different dimensions have been subjected to several new checks including the matching of sphere partition functions, superconformal indices, lens indices etc.

Supersymmetric duality has now become a key tool for studying strongly coupled effects and for this reason it is worthy learning this subject. It appears in many different gauge and superstring theories. The physical origin of supersymmetric duality is still unclear. Hopefully, study of dualities via superconformal indices may shed light on the dynamics of strongly coupled gauge theories and on the nature of supersymmetric duality itself. In the thesis supersymmetric duality plays a crucial role.

Integrability

Integrability is a beautiful phenomenon which plays a very important role in theoretical physics. One of the key structural elements leading to integrability is the Yang-Baxter

term.

[3]For instance, for the $\mathcal{N} = 1$ $SO(N)$ theory with a traceless symmetric tensor R–anomalies of the UV and IR theories match, however these theories are not dual.

equation

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{23}(v) = \mathbb{R}_{23}(v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{12}(u-v)$$

where the operators $\mathbb{R}_{ik}(u)$ act in the tensor product $\mathbb{V}\otimes\mathbb{V}\otimes\mathbb{V}$ of some vector space $\mathbb{V}$. Study of solutions of this equation has led to major breakthroughs in many areas of physics and mathematics including quantum field theory, knot theory, string theory, statistical physics, etc.

Recently, there has been observed connections of integrable statistical models to supersymmetric gauge theories [49, 50, 51, 52, 53] and special functions [4, 54]. One of such connections is a correspondence between quiver gauge theories and integrable lattice models such that the integrability emerges as a manifestation of supersymmetric dualities [49, 55, 50, 56]. Particularly, superconformal indices of $\mathcal{N}=1$ quiver gauge theories can be identified with partition functions of two-dimensional exactly solvable statistical mechanics models in the context of this correspondence.

This relationship has led to construction of new exactly solvable models of statistical mechanics, namely the Yang-Baxter equation was solved in terms of new special functions [4].

Mathematical results inspired by physics

There exist interesting relations between exact results in supersymmetric gauge theories and different branches of mathematics including knot theory [57, 28], integrability [49, 50, 56, 58, 4, 51, 52], quantum groups [59], cluster algebras [60, 61], invariants of 3-manifolds [62, 57] and so on. In particular, computations of partition functions for supersymmetric dual theories in different dimensions lead to many new results for special functions of hypergeometric type.

In the thesis we will mainly focus on basic and elliptic hypergeometric functions. The theory of elliptic hypergeometric functions is quite a new research area in mathematics. The first example of the elliptic analogues of hypergeometric series was discovered about 20 years ago by Frenkel and Turaev [63] in the context of elliptic 6j-symbol [64]. This family of functions is the top level of hypergeometric functions [65]. Recently they have attracted attention of physicists since they proved to be a useful tool in theoretical and mathematical physics.

The entry of elliptic hypergeometric integral identities into high energy physics occurred in 2008 when Dolan and Osborn observed [25] that the superconformal index can be expressed in terms of elliptic hypergeometric integral. Matching of superconformal indices of supersymmetric dual theories lead to various complicated integral identities for the elliptic hypergeometric integrals [26, 27, 66, 28, 67, 68, 69]. Some of them were known earlier, but most of them has not been proven yet.

There is a similar story for three-dimensional supersymmetric gauge theories. Namely three-dimensional superconformal index can be expressed in terms of basic hypergeometric integrals and three-dimensional sphere partition function has a form of hyperbolic hypergeometric integral [1, 3, 29, 70, 31].

The thesis is organized as follows:

- **Chapter 2** is devoted to several aspects of four-dimensional $\mathcal{N} = 1$ Super-Yang-Mills theory including supersymmetric algebra, supersymmetric Lagrangians, superconformal algebra and superconformal index technique. Note that the material in this chapter is collected from various sources.

- In **Chapter 3**we review the basic aspects of three-dimensional $\mathcal{N} = 2$ supersymmetric theories with focus on the necessary elements for the superconformal index computations. We also discuss relationship between the computation of the partition function on the three-sphere and the four-dimensional superconformal index.

- **Chapter 4** contains a review of the multiple duality for $\mathcal{N} = 1$ $SP(2)$ SQCD. We discuss the possibility of global symmetry enhancement of strongly coupled gauge theories, in particular for we show that for a four-dimensional $\mathcal{N} = 1$ SQCD with 3 flavors the explicit $SU(6)$ global symmetry is enhanced to an E_6 symmetry in the presence of $5d$ hypermultiplets. We also show connections between indices of different theories in three and four dimensions.

- **Chapter 5** contains mainly unpublished results. Using superconformal index technique we study three-dimensional Seiberg-like dualities and a particular kind of duality called mirror symmetry and present explicit expressions of superconformal indices for certain supersymmetric dual theories in terms of basic hypergeometric integrals.

- **Chapter 6** entirely dedicated to the relationship between supersymmetric dualities and quantum integrable models. The investigation is restricted to two-dimensional spin models from statistical physics side and to three-dimensional supersymmetric gauge theories from other side of the correspondence. We present a new solution of the star-triangle relation and other forms of Yang-Baxter equation in terms of the basic hypergeometric integral. The new solution corresponds to the generalized superconformal index of certain three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory.

- Our notations for the special functions we use are summarized in Appendix A.

2 $\mathcal{N} = 1$ SUSY in Four Dimensions

In this chapter, we summarize several aspects of four-dimensional $\mathcal{N} = 1$ Super-Yang-Mills theories including supersymmetric algebra, supersymmetric Lagrangians, superconformal algebra.

We especially review aspects of superconformal indices in four dimensions. The superconformal index was introduced [24, 23, 22] as a nontrivial generalization of the Witten index [21], which counts BPS states in superconformal field theories in curved spacetime [71]. The index is one of the useful tools in the study of non-perturbative characteristics of supersymmetric gauge theories. It provides a justification of the known supersymmetric dualities [23, 24, 25, 26, 27, 72, 68, 67, 28, 73] and holographic dualities [22, 74, 75, 76, 77, 78, 79]. Moreover one can use the index technique to discover new dualities [26], to study inclusion of surface and line operators [80, 81, 82, 83, 84], to get 't Hooft anomaly matching condition for dual theories [67, 44], to obtain new and interesting mathematical structures [27, 85, 28] etc.

This chapter is mostly for setting up basic terminology for the rest of the thesis.

2.1 The supersymmetry algebra

Let us recall below some basic notions of supersymmetry algebra. We use the supersymmetry conventions of [86]. The metric has the following signature $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

In four dimensions the Lorentz group has six generators: three generators J_i of the group of rotations in three dimensions and three boosts K_i along three spatial directions with the following commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \ , \tag{2.1}$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k \ , \tag{2.2}$$
$$[J_i, K_j] = i\epsilon_{ijk}K_j \ . \tag{2.3}$$

The Poincare group contains Lorentz transformations and translations:

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{2.4}$$

Translations do not commute with Lorentz transformations. If we denote the Lorentz generators as $M_{0i} = K_i$ and $M_{ij} = \epsilon_{ijk}J_k$, then the Poincare algebra becomes:

$$[P_\mu, P_\nu] = 0 \ , \tag{2.5}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\nu\rho}M_{\rho\sigma} - ig_{\mu\rho}M_{\nu\sigma} - ig_{\nu\sigma}M_{\mu\rho} + ig_{\mu\sigma}M_{\nu\rho} \ , \tag{2.6}$$
$$[M_{\mu\nu}, P_\rho] = -ig_{\rho\mu}P_\nu + ig_{\rho\nu}P_\mu \ . \tag{2.7}$$

The universal cover of the Lorentz group is $SL(2, C)$. The elements $M \in SL(2, C)$ are automorphisms of a spinor space. Let ψ_α be an arbitrary element (called spinor) of the spinor space. Consider an $SL(2, C)$–transformation of ψ_α:

$$\psi_\alpha \to \psi'_\alpha = M_\alpha^\beta \psi_\beta. \tag{2.8}$$

It is the fundamental representation of $SL(2, C)$. The conjugate representation is defined by

$$\bar{\psi}_\alpha \to \bar{\psi}'_{\dot{\alpha}} = M^{*\beta}_{\dot{\alpha}} \psi_\beta. \tag{2.9}$$

One can enlarge the Poincare algebra by generators that transform either as undotted spinors Q^N_α or as dotted spinors $\bar{Q}^N_{\dot{\alpha}}$ under the Lorentz group and that commute with translations:

$$
\begin{aligned}
&[P_\mu, Q^N_\alpha] = 0 \\
&[P_\mu, \bar{Q}^N_{\dot\alpha}] = 0 \\
&[M_{\mu\nu}, Q^N_\alpha] = i(\sigma_{\mu\nu})^\beta_\alpha Q^N_\beta \\
&[M_{\mu\nu}, \bar{Q}^N_{\dot\alpha}] = i(\bar\sigma_{\mu\nu})^{\dot\beta}_{\dot\alpha} \bar{Q}^N_{\dot\beta}
\end{aligned} \tag{2.10}
$$

The only possibility that the algebra does not require extra generators is found to be the algebra [87, 86]

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2(\sigma_\mu)_{\alpha,\dot\beta} P^\mu \delta^{IJ} \ , \tag{2.11}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \ , \tag{2.12}$$

$$\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} \bar{Z}^{IJ} \ . \tag{2.13}$$

here $Z^{IJ} = -Z^{JI}$ commute with all generators of supersymmetry algebra and called central charges.

There are no central charges in $\mathcal{N} = 1$ supersymmetry algebra , therefore we have [86]

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \tag{2.14}$$

$$\{Q_\alpha, Q_\beta\} = 0 \,, \tag{2.15}$$

$$\{\overline{Q}_{\dot\alpha} \,, \overline{Q}_{\dot\beta}\} = 0 \,. \tag{2.16}$$

The supersymmetry generators commute with the momentum operator P_μ and hence, with P^2. Therefore, all states in a given representation of the algebra have the same mass. For a theory to be supersymmetric, it is necessary that its particle content form a representation of the above algebra.

2.2 $4d$ $\mathcal{N} = 1$ supersymmetric theory

2.2.1 Superspace and superfields

In order to make a local realization of supersymmetry it is convenient to use the superspace formalism. Superspace is obtained by adding four spinor coordinates $\theta^\alpha, \overline{\theta}_{\dot{\alpha}}$ to the set of spacetime coordinates x^μ. The generator of supersymmetric transformations in superspace with transformation parameters ξ and $\overline{\xi}$ is then given by

$$\xi^\alpha Q_\alpha + \overline{\xi}_{\dot{\alpha}} \overline{Q}^{\dot{\alpha}} \tag{2.17}$$

To make consistency with the algebra of supersymmetry superspace transformations are chosen to be

$$x^\mu \to x'^\mu = x^\mu + i\theta\sigma^\mu\overline{\xi} - i\xi\sigma^\mu\overline{\theta}\,, \tag{2.18}$$
$$\theta \to \theta' = \theta + \xi\,, \tag{2.19}$$
$$\overline{\theta} \to \overline{\theta}' = \overline{\theta} + \overline{\xi}\,. \tag{2.20}$$

Action of the supercharges on (x, θ) can be written as follows:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\,\partial_\mu\,, \tag{2.21}$$
$$\overline{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\,. \tag{2.22}$$

These supercharges satisfy the anti-commutation relations

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\,. \tag{2.23}$$

It is easy to see that $\partial/\partial\theta$ and $\partial/\partial\overline{\theta}$ are not invariant under the transformations (2.18)-(2.20). Therefore, one needs to introduce the super-covariant derivatives. A standard choice of new derivatives is provided by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\,, \tag{2.24}$$
$$\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\sigma^\mu_{\alpha\dot{\alpha}}\theta^\alpha\,\partial_\mu\,. \tag{2.25}$$

They satisfy the following anti-commutation relations

$$\{D_\alpha, \overline{D}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu \tag{2.26}$$
$$\{D_\alpha, \bar{Q}_{\dot\alpha}\} = 0\ , \tag{2.27}$$
$$\{Q_\alpha, \bar{D}_{\dot\alpha}\} = 0\ . \tag{2.28}$$

Quantum fields transform as components of a superfield defined on superspace, $H(x, \theta, \overline{\theta})$. Since the θ coordinates are anti-commuting, the Taylor expansion of $H(x, \theta, \overline{\theta})$ in odd coordinates is finite, the most general superfield can always be expanded in the fermionic variables

$$\begin{aligned} H(x,\theta,\bar\theta) &= f(x) + \theta\psi(x) + \bar\theta\bar\xi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^\mu\bar\theta v_\mu(x) \\ &= +\theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\psi(x) + \theta\theta\bar\theta\bar\theta d(x)\ , \end{aligned} \tag{2.29}$$

where the coefficients of the expansion are the component fields.

To recover irreducible representations one must impose constraints on the superfields. There are two different $\mathcal{N} = 1$ irreducible multiplets in four dimensions: the chiral multiplet and the vector multiplet.

The chiral multiplet is represented by a superfield Φ, satisfying the following constraint

$$\overline{D}_{\dot\alpha}\Phi = 0\,. \tag{2.30}$$

Note that for

$$y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}\ , \tag{2.31}$$

we have

$$\overline{D}_{\dot\alpha}y^\mu = 0, \tag{2.32}$$
$$\overline{D}_{\dot\alpha}\theta^\beta = 0\,. \tag{2.33}$$

Therefore, any function of (y, θ) is a chiral superfield. The chiral superfield can be expanded in terms of components in the following way

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\,, \tag{2.34}$$

where ψ and ϕ are the fermionic and scalar components, respectively and F is an auxiliary field.

Similarly, an anti-chiral superfield satisfies the following condition

$$D_\alpha \Phi^\dagger = 0 \tag{2.35}$$

and it can be expanded as

$$\Phi^\dagger(y^\dagger, \overline{\theta}) = \phi^\dagger(y^\dagger) + \sqrt{2}\overline{\theta\psi}(y^\dagger) + \overline{\theta\theta}F^\dagger(y^\dagger)\,, \tag{2.36}$$

where, $y^{\mu\dagger} = x^\mu - i\theta\sigma^\mu\overline{\theta}$.

The vector multiplet is defined by a real scalar superfield

$$V = V^\dagger. \tag{2.37}$$

It can be expanded, in the Wess-Zumino gauge and gets the following form

$$V = -\theta\sigma^\mu\overline{\theta}A_\mu + i\theta^2\overline{\theta\lambda} - i\overline{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\overline{\theta}^2 D\,. \tag{2.38}$$

In this gauge

$$V^2 = \frac{1}{2}A_\mu A^\mu \theta^2\overline{\theta}^2 \;\text{ and }\; V^3 = 0\,. \tag{2.39}$$

The Wess-Zumino gauge breaks supersymmetry keeping the gauge symmetry of the Abelian gauge field A_μ. The abelian field strength is given by a combination

$$W_\alpha = -\frac{1}{4}\overline{D}^2 D_\alpha V\ , \tag{2.40}$$

$$\overline{W}_{\dot{\alpha}} = -\frac{1}{4}D^2\overline{D}_{\dot{\alpha}}V\,. \tag{2.41}$$

The non-Abelian gauge field strength is defined by the superfield

$$W_\alpha = \frac{1}{8}\overline{D}^2 e^{2V} D_\alpha e^{-2V} \tag{2.42}$$

and transforms as

$$W_\alpha \to W'_\alpha = e^{-i\Lambda}W_\alpha e^{i\Lambda}\ , \tag{2.43}$$

where $\Lambda = \Lambda^a T^a$ is a chiral superfield and T^a are chosen in the group representation carried by chiral superfields.

2.2.2 Supersymmetric Lagrangians

The most general $\mathcal{N}=1$ supersymmetric Lagrangian for the scalar multiplet is given by

$$\mathcal{L} = \int d^4\theta\, K(\Phi, \Phi^\dagger) + \int d^2\theta \mathcal{W}(\Phi) + \int d^2\overline{\theta}\overline{\mathcal{W}}(\Phi^\dagger)\,. \tag{2.44}$$

Recall that the θ-integrals pick up the highest component of the superfield as the integration formulas for Grassmann variables read $\int d^2\theta\; \theta^2 = 1$ and $\int d^2\overline{\theta}\; \overline{\theta}^2 = 1$.

In terms of the non-holomorphic function called Kähler potential $K(\Phi, \Phi^\dagger)$, the metric in field space is given by $g^{ij} = \partial^2 K / \partial\Phi_i \partial\Phi_j^\dagger$, therefore the target space for chiral superfields is always a Kähler space.

We can include the gauge coupling constant and the θ parameter in the Lagrangian in the following form

$$\begin{aligned}\mathcal{L} &= \frac{1}{4\pi}\mathrm{Im}\Big(\tau\,\mathrm{Tr}\int d^2\theta\, W^\alpha W_\alpha\Big)\\ &= -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} + \frac{1}{g^2}(\frac{1}{2}D^aD^a - i\lambda^a\sigma^\mu D_\mu\overline{\lambda}^a)\,,\end{aligned} \tag{2.45}$$

where, $\tau = \theta/2\pi + 4\pi i/g^2$.

Then the full $\mathcal{N}=1$ supersymmetric gauge invariant Lagrangian as

$$\begin{aligned}\mathcal{L} = &\frac{1}{8\pi}\,\mathrm{Im}\Big(\tau\mathrm{Tr}\int d^2\theta\, W^\alpha W_\alpha\Big)\\ &+\int d^2\theta d^2\overline{\theta}\,(\Phi^\dagger e^{-2V}\Phi) + \int d^2\theta\,\mathcal{W} + \int d^2\overline{\theta}\,\overline{\mathcal{W}}\,.\end{aligned} \tag{2.46}$$

In terms of the component fields, the Lagrangian (2.46) becomes

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} - \frac{i}{g^2}\,\lambda^a\sigma^\mu D_\mu\overline{\lambda}^a + \frac{1}{2g^2}D^aD^a \\ &+ (\partial_\mu\phi - iA^a_\mu T^a\phi)^\dagger(\partial^\mu\phi - iA^{a\mu}T^a\phi) - D^a\phi^\dagger T^a\phi \\ &- i\,\overline{\psi}\overline{\sigma}^\mu(\partial_\mu\psi - iA^a_\mu T^a\psi) + F^\dagger F \\ &+ \left(-i\sqrt{2}\,\phi^\dagger T^a\lambda^a\psi + \frac{\partial\mathcal{W}}{\partial\phi}\,F - \frac{1}{2}\,\frac{\partial^2\mathcal{W}}{\partial\phi\partial\phi}\,\psi\psi + h.c.\right) .\end{aligned} \tag{2.47}$$

Here, $\mathcal{W}$ stands for the scalar component of the superpotential. The auxiliary fields F and D^a can be eliminated by using their equations of motion:

$$F = \frac{\partial\mathcal{W}}{\partial\phi} , \tag{2.48}$$

$$D^a = g^2(\phi^\dagger T^a\phi) . \tag{2.49}$$

The terms involving these fields give rise to the scalar potential

$$V = |F|^2 + \frac{1}{2g^2}D^aD^a . \tag{2.50}$$

Using the supersymmetry algebra it is easy to show that the Hamiltonian is a positive semi-definite operator and that the ground state has zero energy if and only if it is supersymmetry invariant. The equation (2.50) means that the supersymmetric ground state configuration is such that

$$F = D^a = 0 . \tag{2.51}$$

2.2.3 $\mathcal{N} = 1$ superconformal algebra

In this section we outline the construction of the $\mathcal{N} = 1$ superconformal algebra in four dimensions. The section will mainly follow the exposition in [25, 27].

We consider an $\mathcal{N} = 1$ superconformal field theory on $S^3 \times R$. The $\mathcal{N} = 1$ superconformal group in four–dimensions is $SU(2,2|1)$ group, which has the following generators[1]:

$$J_i,\ \bar{J}_i\text{— Lorentz rotations}$$
$$P_\mu,\ Q_\alpha,\ \bar{Q}_{\dot\alpha}\text{— Supertranslations}$$

As in any conformal invariant field theory, one also has superconformal generators

$$K_\mu,\ S_\alpha,\ S_{\dot\alpha}\text{— Special superconformal transformation}$$
$$H\text{— Dilatations}$$

The action is invariant under

$$R\text{— } U(1)_R \text{ rotations.} \tag{2.52}$$

Supercharges satisfy the anticommutator relations (2.14)-(2.16). The superconformal charges obey the following relations

$$\{\overline{S}^{\dot\alpha}, S^\alpha\} = 2K^{\dot\alpha\alpha}\ , \tag{2.53}$$
$$\{\overline{S}^{\dot\alpha}, \overline{S}^{\dot\beta}\} = 0\ , \tag{2.54}$$
$$\{S^\alpha, S^\beta\} = 0. \tag{2.55}$$

The cross-anti-commutators of Q_α and S_α have the form

$$\{Q_\alpha, \overline{S}^{\dot\alpha}\} = 0,\ \{S^\alpha, \overline{Q}_{\dot\alpha}\} = 0, \tag{2.56}$$

while

$$\begin{aligned} \{Q_\alpha, S^\beta\} &= 4\left(M_\alpha^\beta + \frac{1}{2}\delta_\alpha^\beta H + \frac{3}{4}\delta_\alpha^\beta R\right), \\ \{\overline{S}^{\dot\alpha}, \overline{Q}_{\dot\beta}\} &= 4\left(\overline{M}_{\dot\beta}^{\dot\alpha} - \frac{1}{2}\delta_{\dot\beta}^{\dot\alpha} H + \frac{3}{4}\delta_{\dot\beta}^{\dot\alpha} R\right)\ . \end{aligned} \tag{2.57}$$

[1]For more details, see [27].

The bosonic and fermionic generators cross-commute as

$$\begin{aligned}
&[M_\alpha^\beta, Q_\gamma] = \delta_\gamma^\beta Q_\alpha - \frac{1}{2}\delta_\alpha^\beta Q_\gamma, [M_\alpha^\beta, \overline{Q}_{\dot\gamma}] = 0,\\
&[M_\alpha^\beta, S^\gamma] = -\delta_\alpha^\gamma S^\beta + \frac{1}{2}\delta_\alpha^\beta S^\gamma,\ [M_\alpha{}^\beta, \overline{S}^\gamma] = 0,\\
&[\overline{M}_\beta^{\dot\alpha}, Q_\gamma] = 0,\ [\overline{M}_{\dot\beta}^{\dot\alpha}, \overline{Q}_{\dot\gamma}] = -\delta_{\dot\gamma}^{\dot\alpha}\overline{Q}_{\dot\beta} + \frac{1}{2}\delta_{\dot\beta}^{\dot\alpha}\overline{Q}_{\dot\gamma},\\
&[\overline{M}_\beta^{\dot\alpha}, S^\gamma] = 0,\ [\overline{M}_{\dot\beta}^{\dot\alpha}, \overline{S}^{\dot\gamma}] = \delta_{\dot\beta}^{\dot\gamma}\overline{S}^{\dot\alpha} - \frac{1}{2}\delta_{\dot\beta}^{\dot\alpha}\overline{S}^{\dot\gamma},\\
&[P_{\alpha\dot\beta}, S^\gamma] = \delta_\alpha^\gamma \overline{Q}_{\dot\beta},\ [P_{\alpha\dot\beta}, \S^{\dot\gamma}] = \delta_{\dot\beta}^{\dot\gamma} Q_\alpha,\\
&[K^{\dot\alpha\beta}, Q_\gamma] = \delta_\gamma^\beta \overline{S}^{\dot\alpha},\ [K^{\dot\alpha\beta}, \overline{Q}_\gamma] = \delta_{\dot\gamma}^{\dot\alpha} S^\beta,\\
&[H, Q_\alpha] = \frac{1}{2}Q_\alpha,\ [H, \overline{Q}_{\dot\alpha}] = \frac{1}{2}\overline{Q}_{\dot\alpha},\\
&[H, S^\alpha] = -\frac{1}{2}S^\alpha,\ [H, \overline{S}^{\dot\alpha}] = -\frac{1}{2}\overline{S}^{\dot\alpha}.
\end{aligned} \tag{2.58}$$

The R-charge commutes with the bosonic generators, while it has non-trivial commutators with the supercharges and their superconformal partners

$$\begin{aligned}
&[R, Q_{\dot\alpha}] = -Q_\alpha,\ [R, \overline{Q}_{\dot\alpha}] = \overline{Q}_{\dot\alpha},\\
&[R, S^\alpha] = S^\alpha,\ [R, \overline{S}^{\dot\alpha}] = -\overline{S}^{\dot\alpha}.
\end{aligned} \tag{2.59}$$

2.3 Witten index

We start by giving a very brief introduction to the Witten index. More details on the subject can be found in the original paper of Witten [21] and in the review papers [88, 89].

For concreteness let us consider a supersymmetric quantum mechanics. Generators of the supersymmetry algebra satisfy following relations

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0\ , \tag{2.60}$$

and

$$\{Q, Q^\dagger\} = 2H\ . \tag{2.61}$$

Supersymmetry is unbroken if there is at least one state with vanishing energy, i.e. the vacuum state is annihilated by a supersymmetry generator. Indeed, just from the supersymmetry algebra one can see that the Hamiltonian is positive definite and if a state is annihilated by the Hamiltonian H, then it is also annihilated by the supercharge Q

$$Q|0\rangle = 0 \quad \Rightarrow \quad E_{vac} = 0. \tag{2.62}$$

In 1982 Witten suggested [21] an elegant and effective way of characterizing spontaneous supersymmetry breaking. He introduced a topological invariant of the theory which tells us whether supersymmetry is broken or not. This topological invariant, called the Witten index, is defined as follows

$$I_W = \mathrm{Tr}_{H=0}(-1)^F, \tag{2.63}$$

where F is the fermion number which takes value 0 on bosons and 1 on fermions[2] and $\{(-1)^F, Q\} = 0$. The trace is taken over all states in the Hilbert space of the theory. The index computes the difference between the numbers of bosonic and fermionic ground states. If $I_W \neq 0$ then supersymmetry is unbroken, since supersymmetry is unbroken if there is at least one state with vanishing energy.

The index can be defined also in the following way

$$I_W = \mathrm{Tr}(-1)^F e^{-\beta H} . \tag{2.64}$$

It is β – independent for the one–dimensional supersymmetric quantum mechanics, because of the discrete spectrum of the Hamiltonian. In fact, due to pairing of non–zero states, contributions of bosonic and fermionic states to the index cancel each other, since they have the opposite $(-1)^F$.

Note that the Witten index is an analogue of the Atiyah-Singer index [90].

[2]For instance, F can be taken to be twice the spin.

2.4 Superconformal index

We are now in a position to introduce the central object in the thesis – the superconformal index. The d–dimensional superconformal index is a generalization of the Witten index[3] defined on $S^{d-1} \times R$. It is a nontrivial function of flavor and superconformal fugacities [23, 22, 24]. One can define the superconformal index as [22, 78]

$$\mathrm{I}(\{t_i\}) = \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}} \prod t_i^{F_i} \ , \tag{2.65}$$

where Q, $Q^\dagger$ are the supercharges with $H = \{Q, Q^\dagger\}$, F_i are generators of global symmetries which commute with Q and $Q^\dagger$, and t_i are the corresponding fugacities (additional regulators). The trace in the definition of the index is over the Hilbert space of the theory on a $(d-1)$–dimensional sphere S^{d-1}, where d is dimension of the spacetime. The states with $H \neq 0$ come in pair and cancel out because of the factor $(-1)^F$, therefore the superconformal index is β–independent and counts states with $H = 0$. The index does not depend on coupling constants of the theory and it is invariant under marginal deformations of the theory.

Let us consider the $\mathcal{N} = 1$ superconformal theory in four dimensions. To construct the superconformal index we pick up one supercharge, for example, the supercharge $\bar{Q}_1$ and its conjugate $\bar{S}^1$. They satisfy the following relation

$$\{\bar{Q}_1, \bar{S}^1\} = -2(H - 2\bar{J}_3 - \frac{3}{2}R) \ . \tag{2.66}$$

Then one defines the superconformal index in the following way

$$\mathrm{I}(y, t, \{f_j\}) \ = \ \mathrm{Tr}(-1)^{\mathrm{F}} y^{2J_3} t^{\mathcal{R}} e^{\sum_{j=1}^{rank\,F} f_j F^j} \ . \tag{2.67}$$

Here $(-1)^F$ is the fermion number operator, $t^{\mathcal{R}}$ and y^{2J_3} are additional regulators with $|t| < 1$ and $|y| < 1$, f_j is the chemical potential for a group F, where F is a flavor group with maximal torus generators F_j, $j = 1, \dots,$ rank F and $\mathcal{R} = H - \frac{1}{2}R$.

According to the Romelsberger prescription [24] for $\mathcal{N} = 1$ theory with a weakly-coupled description one can write the full index via the so-called “plethystic” exponen-

[3] The original Witten index for supersymmetric gauge theories gives the dual Coxeter number for the gauge group.

tial [91] by integrating over the gauge group[4]

$$I(y,t,\{t_i\}) = \int_{G_c} d\mu(g)\, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \text{ind}\Big(y^n, t^n, \underline{z}^n, t_i^n\Big)\right), \tag{2.68}$$

where $d\mu(g)$ is the G–invariant Haar measure and $\text{ind}(y,t,z,t_i)$ is the index for single particle states.

Dolan and Osborn realized [25] that the exponential sum in (2.68) can be evaluated using elliptic Gamma function

$$\Gamma(z;p,q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1}p^{i+1}q^{j+1}}{1 - zp^i q^j}, \quad |p|,|q| < 1 \;, \tag{2.69}$$

and as a result the superconformal index can be expressed in terms of elliptic hypergeometric integrals.

The four-dimensional superconformal index is a powerful tool to test Seiberg–like dualities in $\mathcal{N}=1$ [24, 25, 26, 27, 28], S–dualities in $\mathcal{N}=2$ [72, 73] and $\mathcal{N}=4$ [72, 68] supersymmetric theories and has an elegant mathematical structure described by the theory of elliptic hypergeometric integrals [92].

We will work out the explicit expression of a single letter index and the full superconformal index for several cases.

2.4.1 Calculating the index

In [23, 24] Römelsberger introduced a simple procedure for explicit computation of the superconformal index. According to his prescription, to obtain the superconformal index one should first compute a single letter index $ind(\{f_i\})$ summing over all the fields contributing to the index.

Therefore let us first compute a single letter index. We start with the Lagrangian for the bosonic field in the free chiral multiplet of the R-charge q [23, 24]

$$L_\phi = (\partial_t - i\frac{3q-2}{2})\phi^\dagger(\partial_t + i\frac{3q-2}{2})\phi - 4\sigma_i^{(L)}\phi^\dagger\sigma_i^{(L)}\phi - \phi^\dagger\phi \;, \tag{2.70}$$

[4]Since we are interested in gauge invariant physical observables.

where the space derivatives $\sigma_i^{(L)}$ have the following form in terms of the Euler angles (ϕ_1, ϕ_2, ϕ_3)

$$\sigma_1^{(L)} = \cos\phi_2 \partial_{\phi_1} + \frac{\sin\phi_2}{\sin\phi_1}\partial_{\phi_3} - \sin\phi_2 \cot\phi_1 \partial_{\phi_2} \ , \tag{2.71}$$

$$\sigma_2^{(L)} = \sin\phi_2 \partial_{\phi_1} + \frac{\cos\phi_2}{\sin\phi_1}\partial_{\phi_3} - \cos\phi_2 \cot\phi_1 \partial_{\phi_2} \ , \tag{2.72}$$

$$\sigma_3^{(L)} = \partial_{\phi_2} \ . \tag{2.73}$$

Let us expand the sacalar field ϕ in spherical harmonics

$$\phi = \sum_{j,j_3,\tilde{j}_3} \phi_{j,j_3,\tilde{j}_3} Y_{j,j_3,\tilde{j}_3}. \tag{2.74}$$

where the sum is taken over the quantum numbers of $SO(4)$ angular momenta and $j \geq 0$, $|j_3|, |\tilde{j}_3| \leq j$. Then the Lagrangian (2.70) is given by

$$L_\phi = \sum_{j,j_3,\tilde{j}_3} \left((\partial_t - i\frac{3q-2}{2})\phi^\dagger_{j,j_3,\tilde{j}_3}(\partial_t + i\frac{3q-2}{2})\phi_{j,j_3,\tilde{j}_3} - (2j+1)^2 \phi^\dagger_{j,j_3,\tilde{j}_3}\phi_{j,j_3,\tilde{j}_3} \right) . \tag{2.75}$$

We want to compute the following index for the boson in the chiral multiplet

$$\mathrm{Tr}(-1)^F t^{R+\bar{J}_3} y^{2J_3} \ . \tag{2.76}$$

Hence, consider the following Hamiltonian

$$H_\phi = \sum_{j,j_3,\tilde{j}_3} \Big(\Pi_{j,j_3,\tilde{j}_3}\Pi^\dagger_{j,j_3,\tilde{j}_3} - i\frac{3q-2}{2}\Pi_{j,j_3,\tilde{j}_3}\phi_{j,j_3,\tilde{j}_3} + i\frac{3q-2}{2}\phi^\dagger_{j,j_3,\tilde{j}_3}\Pi^\dagger_{j,j_3,\tilde{j}_3} + (2j+1)^2 \phi^\dagger_{j,j_3,\tilde{j}_3}\phi_{j,j_3,\tilde{j}_3} \Big), \tag{2.77}$$

the R–charge

$$R_\phi = -iq \sum_{j,j_3,\tilde{j}_3} \left(\Pi_{j,j_3,\tilde{j}_3}\phi_{j,j_3,\tilde{j}_3} - \phi^\dagger_{j,j_3,\tilde{j}_3}\Pi^\dagger_{j,j_3,\tilde{j}_3} \right), \tag{2.78}$$

and the angular momentum over S^3 $(= SU(2) \times SU(2))$

$$J^3_\phi = -i \sum_{j,j_3,\tilde{j}_3} j_3 \left(\Pi_{j,j_3,\tilde{j}_3} \phi_{j,j_3,\tilde{j}_3} - \phi^\dagger_{j,j_3,\tilde{j}_3} \Pi^\dagger_{j,j_3,\tilde{j}_3} \right), \tag{2.79}$$

$$\tilde{J}^3_\phi = -i \sum_{j,j_3,\tilde{j}_3} \tilde{j}_3 \left(\Pi_{j,j_3,\tilde{j}_3} \phi_{j,j_3,\tilde{j}_3} - \phi^\dagger_{j,j_3,\tilde{j}_3} \Pi^\dagger_{j,j_3,\tilde{j}_3} \right). \tag{2.80}$$

Here we used the canonical momenta

$$\Pi_{j,j_3,\tilde{j}_3} = (\partial_t - i\frac{3q-2}{2})\phi^\dagger_{j,j_3,\tilde{j}_3} \tag{2.81}$$

and its Hermitian conjugate

$$\Pi^\dagger_{j,j_3,\tilde{j}_3} = (\partial_t + i\frac{3q-2}{2})\phi_{j,j_3,\tilde{j}_3} \ . \tag{2.82}$$

We define the ladder operators as

$$a_{1,j,j_3,\tilde{j}_3} = \frac{1}{\sqrt{4j+2}}\Big(\Pi^\dagger_{j,j_3,\tilde{j}_3} + i(2j+1)\phi\Big) \ , \tag{2.83}$$

$$a_{2,j,j_3,\tilde{j}_3} = \frac{1}{\sqrt{4j+2}}\Big(\Pi_{j,j_3,\tilde{j}_3} + i(2j+1)\phi^\dagger\Big) \ . \tag{2.84}$$

Then in terms of the ladder operators one finds that

$$R_\phi + 2J^3_\phi = -\sum_{j,j_3,\tilde{j}_3} (q+2j_3)\Big(a^\dagger_{1,j,j_3,\tilde{j}_3} a_{1,j,j_3,\tilde{j}_3} - a^\dagger_{2,j,j_3,\tilde{j}_3} a_{2,j,j_3,\tilde{j}_3}\Big). \tag{2.85}$$

$$\tilde{J}^3_\phi = -\sum_{j,j_3,\tilde{j}_3} \tilde{j}_3\Big(a^\dagger_{1,j,j_3,\tilde{j}_3} a_{1,j,j_3,\tilde{j}_3} - a^\dagger_{2,j,j_3,\tilde{j}_3} a_{2,j,j_3,\tilde{j}_3}\Big). \tag{2.86}$$

Plugging (2.85) and (2.86) into (2.76), we obtain the desired contribution of the bosonic part to the superconformal index

$$f_\phi(t,y,u) = \sum_{j_3=0}^{\infty} t^{q+j_3} \sum_{\tilde{j}_3=-j_3/2}^{j_3/2} y^{2\tilde{j}_3} \tag{2.87}$$

$$= \frac{t^q}{(1-ty)(1-\frac{t}{y})} \ . \tag{2.88}$$

Similarly, one can calculate the fermionic contribution to the index

$$f_\psi(t,y,u) = -\frac{t^{2-q}}{(1-ty)(1-\frac{t}{y})} \tag{2.89}$$

starting from the following Lagrangian

$$L_\psi = i\bar{\psi}\gamma^0\left(\partial_0 + i\frac{3q-2}{2}\right)\psi - 2i\bar{\psi}\gamma^i\left(\sigma_i^{(L)} + \frac{1}{8}\epsilon_{ijk}\gamma^{jk}\right)\psi \tag{2.90}$$

and by expanding the spinor ψ in spinor spherical harmonics. Then a free chiral multiplet contributing to the superconformal index is given by

$$f_\Phi(t,y,u) = \frac{t^q - t^{2-q}}{(1-ty)(1-\frac{t}{y})}\ . \tag{2.91}$$

Now let us consider the contribution of the gauge multiplet. First, we need to fix the gauge. In this case we choose the temporal gauge $A_0 = 0$ on $S^3 \times R$. In order to get gauge–invariant physical states we impose Gauss' law constraint for a gauge symmetry.

The supersymmetric Lagrangian describing the gauge multiplet is

$$\begin{aligned} L_g = \frac{1}{g^2}\Bigg(& 4tr\mathcal{F}_{0i}\mathcal{F}_{0i} - 8tr\mathcal{F}_{ij}\mathcal{F}_{ij} + itr\bar{\lambda}\gamma^0\mathcal{D}_0\lambda \\ & -2itr\bar{\lambda}\gamma^i\left(\mathcal{D}_i + \frac{1}{8}\epsilon_{ijk}\gamma^{jk}\right)\lambda - trD^2\Bigg), \end{aligned} \tag{2.92}$$

where λ is the chiral fermion, D is the real auxiliary field and A_μ is the gauge field

$$A = A_0\, dt + A_i\, \sigma^i_{(R)}, \tag{2.93}$$

and

$$\mathcal{F}_{0i} = \frac{1}{2}(\partial_t A_i - \sigma_i^{(L)}A_0 + A_0A_i - A_iA_0), \tag{2.94}$$

$$\mathcal{F}_{ij} = \frac{1}{2}(\sigma_i^{(L)}A_j - \sigma_j^{(L)}A_i + \epsilon_{ijk}A_k + A_iA_j - A_jA_i) \tag{2.95}$$

and the right invariant 1–forms on $SU(2)$ are given by the following form

$$\sigma^1_{(R)} = \cos\phi_2 d\phi_1 + \sin\phi_2 \sin\phi_1 d\phi_3 \ , \tag{2.96}$$
$$\sigma^2_{(R)} = \sin\phi_2 d\phi_1 + \cos\phi_2 \sin\phi_1 d\phi_3 \ , \tag{2.97}$$
$$\sigma^3_{(R)} = \cos\phi_1 d\phi_3 + d\phi_2 \ . \tag{2.98}$$

Then the contribution of a free abelian vector multiplet to the superconformal index is (for details, see [24, 93])

$$f_V(t,y) = \frac{2t^2 - t(y+\frac{1}{y})}{(1-ty)(1-\frac{t}{y})}. \tag{2.99}$$

In the case of non–abelian gauge theory with gauge group G one obtains that a single letter index for vector multiplet has the following form

$$f_V(t,y;g) = \frac{2t^2 - t(y+\frac{1}{y})}{(1-ty)(1-\frac{t}{y})}\,\chi_{adj}(g) \ . \tag{2.100}$$

Now we introduce new parameters $p = ty$ and $q = ty^{-1}$. Then the single letter particle states index gets the following form

$$\begin{aligned} \mathrm{ind}(p,q,\underline{z},\underline{y}) = &\frac{2pq - p - q}{(1-p)(1-q)}\chi_{adj}(\underline{z}) \\ &+ \sum_j \frac{(pq)^{R_j/2}\chi_{R_F,j}(\underline{y})\chi_{R_G,j}(\underline{z}) - (pq)^{1-R_j/2}\chi_{\bar{R}_F,j}(\underline{y})\chi_{\bar{R}_G,j}(\underline{z})}{(1-p)(1-q)}. \end{aligned} \tag{2.101}$$

The first term in (2.101) represents the contribution of the gauge superfields lying in the adjoint representation of the gauge group G_c. The sum over j corresponds to the contribution of chiral matter superfields φ_j transforming in the gauge group representations $R_{G,j}$ and flavor group representations $R_{F,j}$ where R_j are the field R-charges. The functions $\chi_{adj}(\underline{z})$, $\chi_{R_F,j}(\underline{y})$ and $\chi_{R_G,j}(\underline{z})$ are the characters of the corresponding representations, where $\underline{z}$ and $\underline{y}$ are the set of complex eigenvalues of matrices realizing G and F, respectively.

Finally, the full index is formed by summing over multiparticle states, i.e. by inserting the single letter index into the "plethystic" exponential PE[.] [94, 91] and integrating over the gauge group in order to get gauge–invariant quantity

$$\int_G d\mu(g)\ \mathrm{PE}[ind(\{t_i\})]\ , \tag{2.102}$$

where $\mu(g)$ is the invariant Haar measure and the plethystic exponential is defined as

$$\mathrm{PE}[f(x_i)] = \exp\left(\sum_{n=1}^{\infty} \frac{f(x_1^n, x_2^n, ...)}{n}\right)\ . \tag{2.103}$$

2.4.2 Extended supersymmetry

For the sake of completeness we write down the superconformal index for four-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories, althoug they will not be discussed in the thesis. Rather than discussing the full algebra of these theories, we will give only one particular relation in order to define the superconformal index.

$\mathcal{N} = 2$ theory

The superconformal index of a four–dimensional $\mathcal{N} = 2$ SCFT is

$$\mathrm{I}_{4d,\,\mathcal{N}=2} = \mathrm{Tr}(-1)^F \left(\frac{t}{pq}\right)^r p^{j_2+j_1}\, q^{j_2-j_1}\, t^R \prod_i a_i^{f_i}\ . \tag{2.104}$$

where $j_{1,2}$ are the Cartans of the Lorentz $SU(2)_1 \times SU(2)_2$ isometry of S^3, r is the $U(1)_r$ generator, and R the $SU(2)_R$ generator of R-symmetries. The fugacities a_i stand for flavor symmetry. Only the states with

$$\{Q^\dagger,\, Q\} = E - 2j_2 - 2R + r = 0 \tag{2.105}$$

contribute to the index. The single letter indices of the hypermultiplet and the vector multiplet have the following form

$$\begin{aligned}
\mathrm{ind}^H_{4d,\mathcal{N}=2}(p,q,t,a) &= \frac{\sqrt{t} - \frac{pq}{\sqrt{t}}}{(1-p)(1-q)}(a + a^{-1})\,, \\
\mathrm{ind}^V_{4d,\mathcal{N}=2}(p,q,t) &= -\frac{p}{1-p} - \frac{q}{1-q} + \frac{\frac{pq}{t} - t}{(1-p)(1-q)}\ .
\end{aligned} \tag{2.106}$$

$\mathcal{N} = 4$ theory

For the construction of the superconformal index for four-dimensional $\mathcal{N} = 4$ Super-Yang-Mills we need the following anti–commutation relation

$$2\{Q^{\dagger}, Q\} \; = \; E - 2j_1 - \frac{3}{2}R_1 - R_2 - \frac{1}{2}R_3, \tag{2.107}$$

where E is the energy, j_1 (and j_2) is the angular momentum corresponding to the rotation around S^3, and R_k denotes three generators of Cartan subalgebra of $SO(6)$ R–symmetry in the $SU(4)$ notation. Then one can define the superconformal index as follows

$$\mathrm{I}_{4d,\, \mathcal{N}=4}(t, y, v, w) = \mathrm{Tr}(-1)^F e^{-\beta\mathcal{H}} t^{2(E+j_1)} y^{2j_2} v^{R_2} w^{R_3}, \tag{2.108}$$

where $\mathcal{H} = 2\{Q^{\dagger}, Q\}$ and recall that only the states with $\mathcal{H} = 0$ contribute to the index. Here t, y, v and w are the additional regulators. The index (2.108) counts the number of 1/16 BPS states in the theory. An explicit computation of the superconformal index for a given gauge group gives the following result [22]

$$I_{4d,\, \mathcal{N}=4}(t, y, v, w) = \int d\mu(g) \mathrm{PE}\left[\mathrm{ind}_{4d,\mathcal{N}=4}(t, y, v, w)\chi_{adj}(G)\right], \tag{2.109}$$

with

$$\mathrm{ind}_{4d,\mathcal{N}=4}(t, y, v, w) \; = \; \frac{t^2(v + 1/w + w/v) - t^3(y + 1/y) - t^4(w + 1/v + v/w) + 2t^6}{(1 - yt^3)(1 - y^{-1}t^3)}, \tag{2.110}$$

where $d\mu(g)$ is the invariant Haar measure and $\chi_{adj}(G)$ is the character of the adjoint representation of the corresponding gauge group G. The single letter index (2.110) is the character of the $PSU(1, 2|3)$ subalgebra of the $PSU(2, 2|4)$ space–time symmetry which commutes with Q and $Q^{\dagger} = S$ [95].

3 $\mathcal{N} = 2$ SUSY in Three Dimensions

In this chapter we briefly review kinematics and dynamics of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories, including supersymmetry algebra, supersymmetric action, mirror symmetry etc. The main attention is devoted to the superconformal index and three–dimensional dualities. The subject is very broad, and we only cover the basics needed to obtain our results in the next chapters.

3.1 $3d$ $\mathcal{N} = 2$ supersymmetric theories

The three–dimensional $\mathcal{N} = 2$ gauge theory can be obtained by reducing the four–dimensional $\mathcal{N} = 1$ supersymmetry. We review aspects of $\mathcal{N} = 2$ supersymmetric gauge theories in three dimensions and introduce the notation used in the thesis. In this chapter we will closely follow the treatment in [96, 97, 36, 37, 98, 99] (see also Appendices in [100, 101]).

3.1.1 Conventions

The Clifford algebra for a $2 + 1$ - dimensional space with metric $g_{\mu\nu}$ is

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \,, \tag{3.1}$$

$$[\gamma_\mu, \gamma_\nu] = -2i\epsilon^{\mu\nu\lambda}\gamma_\lambda \,. \tag{3.2}$$

As a convenient representation we choose the matrices γ^μ as follows

$$(\gamma^1)^\alpha_\beta = i\sigma_2, \quad (\gamma^2)^\alpha_\beta = \sigma_3, \quad (\gamma^3)^\alpha_\beta = \sigma_1 \ , \tag{3.3}$$

where α, β are spinor indices in the defining representation of $SL(2,\mathbb{R})$. Spinor indices are contracted, raised and lowered with the anti–symmetric matrix $C_{\alpha\beta}$

$$C_{\alpha\beta} \ = \ -C_{\beta\alpha} \ = \ C^{\beta\alpha} \ = \ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \tag{3.4}$$

We will work in Euclidean space and consider the theories on $S^2 \times S^1$ and on a squashed S^3.

3.1.2 $\mathcal{N}=2$ SUSY Algebra

Besides the ordinary generators of the Poincare algebra the three–dimensional $\mathcal{N}=2$ SUSY algebra as for $\mathcal{N}=1$ SUSY in four dimensions has four real supercharges. These supercharges can be combined into a complex supercharge and its Hermitian conjugate

$$Q_\alpha \ \text{ and } \ \bar{Q}_\alpha \ , \tag{3.5}$$

where α is a spinor index which runs from 1 to $2(=\mathcal{N})$. The part of the $\mathcal{N}=2$ SUSY algebra involving the supercharges can be written as [36]

$$\{Q_\alpha, Q_\beta\} = \left\{\bar{Q}_\alpha, \bar{Q}_\beta\right\} = 0 \ , \tag{3.6}$$

$$\left\{Q_\alpha, \bar{Q}_\beta\right\} = 2\gamma^i_{\alpha\beta} P_i + 2i\epsilon_{\alpha\beta} Z \ , \tag{3.7}$$

where the bosonic generator P_μ is the momentum generator, Z is a central charge which can be thought of as the reduced component of four-dimensional momentum. The automorphism group of the algebra is $U(1)$ R–symmetry which rotates the supercharges as

$$[R, Q_\alpha] \ = \ -Q_\alpha \ . \tag{3.8}$$

In case of superconformal symmetry we have two additional bosonic generators: special conformal transformations K_μ and dilatations D, and two fermionic generators: S_α and $\bar{S}_\alpha$. The $\mathcal{N}=2$ superconformal algebra in three dimensions takes the form of the following supergroup[1] [14]

$$SO(3,2)_{\text{conf}} \times SO(2)_R \in OSp(2|4)\ . \tag{3.9}$$

In Euclidean signature this turns into[2]

$$SO(4,1)_{\text{conf}} \times SO(2)_R \in OSp(2|2,2)\ . \tag{3.10}$$

The first factor here is the conformal group and the second one is the R–symmetry. Note that in the superconformal case the algebra has a distinguished R–symmetry. The important relation of the superconformal algebra for our purposes is

$$\{\bar{Q}_\alpha, \bar{S}_\beta\} \;=\; M_{\mu\nu}[\gamma^\mu,\gamma^\nu]_{\alpha\beta} + 2\varepsilon_{\alpha\beta}D - 2\varepsilon_{\alpha\beta}R\ . \tag{3.11}$$

In particular we will use the following commutation relation

$$\{\bar{Q}_1, \bar{S}_1\} \;= 2D - 2R - 2j_3\ . \tag{3.12}$$

3.1.3 Multiplets

Supersymmetry representations of $3d$ $\mathcal{N}=2$ theories are closely related to representations of $\mathcal{N}=1$ theories in four-dimensions and they can be directly obtained by dimensional reduction.

A way to obtain irreducible representations is to impose constraints on superfields. In order to do so it is useful to define supercovariant derivatives:

$$D_\alpha = \frac{\partial}{\partial\theta_\alpha} - i(\gamma^\mu\bar{\theta})_\alpha\partial_\mu\ , \tag{3.13}$$

$$\bar{D}_\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu\ . \tag{3.14}$$

[1]Note that $SO(3,2)=USp(4)$ and $SO(2)=U(1)$.
[2]Note that $SO(4,1)=USp(2,2)$.

The simplest type of a constrained superfield is the chiral multiplet Φ that satisfies the following constraint

$$\bar{D}_\alpha \Phi \;=\; 0 \; . \tag{3.15}$$

As a function on the superspace it can be expanded in terms of the components: a complex scalar field ϕ, a complex Dirac fermion ψ, an auxiliary complex scalar F

$$\Phi \;=\; \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \; , \tag{3.16}$$

where θ is a Grassman coordinate and $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$.

The so-called vector multiplet consists of a real scalar field σ, a vector field A_μ, a complex Dirac fermion λ, a real auxiliary scalar field D, and its expansion in Wess-Zumino gauge is given by

$$V = -i\theta\bar{\theta}\sigma - \theta\gamma^i\bar{\theta}A_i + i\theta^2\bar{\theta}\lambda - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D; . \tag{3.17}$$

Unlike the four-dimensional $\mathcal{N} = 1$ counterpart, the three-dimensional $\mathcal{N} = 2$ vector superfield carries components which may acquire vacuum expectation values that form the Coulomb branch of the moduli space.

It is useful to define the so-called linear multiplet whose lowest component is a scalar field

$$\Sigma = \bar{D}^\alpha D_\alpha V \; . \tag{3.18}$$

It satisfies the following equations

$$D^\alpha D_\alpha \Sigma = \bar{D}^\alpha \bar{D}_\alpha \Sigma = 0 \; , \tag{3.19}$$

and

$$\Sigma = \Sigma^\dagger \; . \tag{3.20}$$

3.1.4 Supersymmetric actions

In this section we summarize actions for matter and gauge superfields.

- From a vector superfield V (3.17) one can make the gauge-invariant combination

$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V$ to construct the Yang-Mills action. Then the classical Yang-Mills kinetic terms for vector multiplets take the following form

$$\begin{aligned} S_{\mathrm{YM}} &= \frac{1}{g^2}\int d^3x\, d^2\theta\ \left(\mathrm{Tr}\, W_\alpha W^\alpha + \mathrm{c.c.}\right) \\ &= \frac{1}{g^2}\int d^3x\ \mathrm{Tr}\left(\frac{1}{4}F_{ij}F^{ij} + \mathscr{D}_i\sigma\mathscr{D}^i\sigma + D^2 + \lambda^\dagger\gamma^i\mathscr{D}_i\lambda\right). \end{aligned} \tag{3.21}$$

where the trace is performed over the fundamental representation. One may use an equivalent description in terms of the linear superfield Σ (3.18) for which the action is

$$S_{\mathrm{YM}} = \frac{1}{g^2}\int d^3x\, d^4\theta\ \mathrm{Tr}\frac{1}{4}\Sigma^2\ . \tag{3.22}$$

This is completely equivalent to (3.21) once the $d^4\theta$ integral is performed.

- In three dimensions the Yang-Mills action is not the only gauge invariant combination of the gauge fields. We also may have the Chern-Simons term which is given by

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int d^3x\ \mathrm{Tr}\left[\epsilon^{ijk}\left(A_i\partial_j A_k + i\frac{2}{3}A_iA_jA_k\right) + 2D\sigma - \lambda^\dagger\lambda\right], \tag{3.23}$$

where $k \in \mathbb{Z}$ is the Chern-Simons level. In the abelian case it can be written in the following simple form

$$S_{\mathrm{CS}} \equiv \frac{k}{4\pi}\int d^3x\, d^4\theta\ \mathrm{Tr}\Sigma V. \tag{3.24}$$

- In the case of abelian theory we can also add Fayet-Iliopoulos term to the action

$$S_{\mathrm{FI}} \equiv \int d^3x\, d^4\theta\ \xi V\ , \tag{3.25}$$

where ξ is the Fayet-Iliopoulos parameter. This term can also be written via the vector and linear multiplets

$$S_{\mathrm{FI}} \equiv \int d^3x\, d^4\theta\ \Sigma V\ , \tag{3.26}$$

where Σ has a scalar component $\sigma = \xi$ and the rest components are turned off.

- The action for chiral superfields Φ is given by

$$S_{\text{chiral}} = \int d^3x\, d^4\theta\; K(\Phi, \Phi^\dagger) + \int d^3x\, \left[d^2\theta\, W(\Phi) + \text{c.c.}\right] \tag{3.27}$$

with the Kähler potential $K(\Phi, \Phi^\dagger)$ and superpotential $W(\Phi)$. In particular, for SUSY gauge theories the Kahler potential is $\Phi e^V \Phi^\dagger$. One can expand it and obtain the kinetic term

$$\begin{aligned}\mathcal{L}_{\text{kin.}} = |\mathscr{D}_i\phi|^2 + \phi^\dagger\sigma^2\phi + i\phi^\dagger D\phi + i\psi^\dagger\gamma^i\mathscr{D}_i\psi - i\psi^\dagger\sigma\psi \\ +i\phi^\dagger\lambda^\dagger\psi - i\psi^\dagger\lambda\phi + |F|^2\,,\end{aligned} \tag{3.28}$$

where $\mathscr{D}_i$ is the Dirac operator.

- There are two different types of mass terms one may write for a chiral superfield. First we can get mass terms from non-zero vacuum expectation value of the scalar component of background vector multiplet. By modifying the Kähler potential we get

$$\int d^3x d^4\theta \Phi e^{m_{\mathbb{R}}\theta^2}\Phi^\dagger\ . \tag{3.29}$$

This mass is known as a real mass. It gives a mass to matter multiplets. We also can write down a holomorphic mass adding a quadratic term to the superpotential

$$W_{m_{\mathbb{C}}} = m_{\mathbb{C}}\Phi\bar{\Phi}\ . \tag{3.30}$$

This mass is known as a complex mass and it is the analog of the usual mass term in four dimensions. The real mass breaks parity while the complex mass does not. The physical mass of the chiral multiplet is $m = \sqrt{m_{\mathbb{R}}^2 + m_{\mathbb{C}}^2}$.

3.2 3d $\mathcal{N}=2$ mirror symmetry

Let us now turn to the so-called mirror symmetry in three-dimensional $\mathcal{N}=2$ supersymmetric gauge theories. Three dimensional mirror symmetry was first introduced for $\mathcal{N}=4$ supersymmetric gauge theories in [38] and was extended to $\mathcal{N}=2$ gauge theories resulting from supersymmetry breaking in $\mathcal{N}=4$ theories [36]. The simplest example of $\mathcal{N}=2$ mirror symmetry is the duality between supersymmetric quantum electrodynamics with one flavor and the free Wess-Zumino theory [38, 36, 102].

These two theories are defined in the UV region and flow to the same IR fixed point:

- The $\mathcal{N}=2$ supersymmetric quantum electrodynamics has one flavor consisting of two chiral fields Q, $\tilde{Q}$ and one vector multiplet V. This theory possesses extra U(1) global symmetries: one is the topological $\mathrm{U}(1)_J$, and the other is the flavor symmetry $\mathrm{U}(1)_A$.

	U(1)	$\mathrm{U}(1)_J$	$\mathrm{U}(1)_A$
Q	+1	0	+1
$\tilde{Q}$	−1	0	+1

Charges in the SQED.

- free Wess-Zumino model is the theory containing three chiral fields $q, \tilde{q}$, and S interacting through the superpotential $W = \tilde{q}Sq$. This theory has two U(1) global symmetries, named $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ [31].

	$\mathrm{U}(1)_V$	$\mathrm{U}(1)_A$
X	+1	+1
Y	−1	+1
Z	0	−2

Charges in the free Wess-Zumino theory.

In the context of mirror symmetry, we can identify $\mathrm{U}(1)_J$ and $\mathrm{U}(1)_A$ of the supersymmetric quantum electrodynamics with $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ of the Wess-Zumino model, respectively. The currents J_A associated with each $\mathrm{U}(1)_A$ are mapped with flipping the sign.

SQED		Wess-Zumino
$\mathrm{U}(1)_J$	$\leftrightarrow$	$\mathrm{U}(1)_V$
$\mathrm{U}(1)_A$	$\leftrightarrow$	$\mathrm{U}(1)_A$
J_A	$\leftrightarrow$	$-J_A$

Mirror duality.

3.3 Superconformal index

In this section, we introduce basic facts related to the three-dimensional superconformal index technique. The presentation closely follows that in [30, 29, 31].

The concept of the superconformal index was first introduced for four dimensional theories in [23, 22] and later extended to other dimensions. The superconformal index of a three–dimensional $\mathcal{N} = 2$ superconformal field theory is a twisted partition function defined on $S^2 \times S^1$ as follows [103, 79, 30]

$$I(q, \{t_i\}) = \mathrm{Tr}\left[(-1)^{\mathrm{F}} e^{-\beta\{Q,Q^\dagger\}} q^{\frac{1}{2}(\Delta+j_3)} \prod_i t_i^{F_i}\right] , \tag{3.31}$$

where

- the trace is taken over the Hilbert space of the theory on S^2.
- F plays a role of the fermion number which takes values zero on bosons and one on fermions. In presence of monopoles one needs to refine this number by shifting it by $e \times m$, where e and m are electric charge and magnetic monopole charge, respectively. See [57, 104] for a discussion of this issue in more details.
- Δ is the energy (or conformal dimension via radial quantization), j_3 is the third component of the angular momentum on S^2, R is the R-charge.
- F_i is the charge of global symmetry with fugacity t_i.
- Q is a certain supersymmetric charge in three-dimensional $\mathcal{N} = 2$ superconformal algebra with quantum numbers $\Delta = \frac{1}{2}$ and $j_3 = -\frac{1}{2}$ and $R = 1$. The supercharges $Q^\dagger = S$ and Q satisfy the following anti-commutation relation[3]

$$\{Q, S\} = \Delta - R - j_3 \; . \tag{3.32}$$

Only BPS states with $\Delta - R - j_3 = 0$ contribute to the superconformal index, therefore the index is β-independent, but becomes a non-trivial function of the fugacities t_i and q. The superconformal index counts the number of BPS states weighted by their quantum numbers.

[3]The full algebra can be found in many places, see e.g., [105].

The superconformal index can be computed exactly by the localization technique [20] and it takes the form of the following matrix integral [79, 30]

$$I(q,\{t_i\}) = \sum_{m\in\mathbb{Z}} \int \frac{1}{|W_m|} e^{-S^{(0)}_{CS}} e^{ib_0} q^{\frac{1}{2}\epsilon_0} \prod_{j}^{\text{rank}F} t_j^{q_{0j}} \\ \times \exp\left[\sum_{n=1}^{\infty} \frac{1}{n}\text{ind}(z_i^n, t^n, q^n; m)\right] \prod_{i=1}^{\text{rank}G} \frac{dz_i}{2\pi i z_i} \ . \qquad (3.33)$$

The sum in the formula is to be understood as a sum over magnetic fluxes on the two-sphere

$$m \ = \ \frac{1}{2\pi}\int_{S^2} F \ , \qquad (3.34)$$

where m parametrizes the GNO charge of the monopole configuration[4], in the examples we consider in the thesis it runs over integers.

The prefactor $|W_m| = \prod_{i=1}^{k}(\text{rank}G_i)!$ is the order of the Weyl group of G which is "broken" by the monopoles into the product $G_1 \times G_2 \times \ldots G_k$. For instance, in case of $U(N)$ gauge group $|W_m| = \prod N_k!$.

The term

$$S^{(0)}_{CS} \ = \ \frac{ik}{4\pi}\int tr_{CS}(A^{(0)}dA^{(0)} - \frac{2i}{3}A^{(0)}A^{(0)}A^{(0)}) \qquad (3.35)$$

is the contribution of the Chern–Simons term if the action contains such term and the term

$$b_0 \ = \ -\frac{1}{2}\sum_{\Phi}\sum_{\rho\in R_\Phi} |\rho(m)|\rho(g) \qquad (3.36)$$

is the 1-loop correction to the Chern–Simons term. The tr_{CS} stands for the trace containing the Chern–Simons levels, k is the Chern-Simons level and $\sum_\Phi$ and $\sum_{\rho\in R_\Phi}$ are sums over all chiral multiplets and all weights of the representation R_Φ, respectively. We give the contributions (3.35) and (3.36) for completeness, in all our examples we will consider theories without the Chern-Simons term.

The term q_{0j} in (3.33) is the zero-point contribution to the energy

$$q_{0j}(m) = -\frac{1}{2}\sum_{\Phi}\sum_{\rho\in R_\Phi} |\rho(m)| f_j(\Phi) \ . \qquad (3.37)$$

[4]The operators creating magnetic fluxes are not completely understood yet, for details, see e.g., [79].

In addition, there is a contribution from the Casimir energy of the ground state [79]

$$\epsilon_0(m) = \frac{1}{2} tr(-1)^F(\Delta + j_3) \ . \tag{3.38}$$

This quantity on a two-sphere with magnetic flux m takes the following form

$$\epsilon_0(m) = \frac{1}{2}\sum_{\Phi}(1-\Delta_\Phi)\sum_{\rho\in R_\Phi}|\rho(m)| - \frac{1}{2}\sum_{\alpha\in G}|\alpha(m)| \ , \tag{3.39}$$

where $\sum_{\alpha\in G}$ represents summation over all roots of G, and Δ_Φ is the superconformal R-charge of the chiral multiplet Φ, $\alpha(m)$ are the positive roots of the gauge group G.

One can calculate the single letter index

$$\begin{aligned} \text{ind}(z,t,q;m) &= -\sum_{\alpha\in G} e^{i\alpha(g)} q^{\frac{1}{2}|\alpha(m)|} \qquad (3.40)\\ &+ \sum_{\Phi}\sum_{\rho\in R_\Phi}\left[e^{i\rho(g)}\prod_j t_j^{f_j}\frac{q^{\frac{1}{2}|\rho(m)|+\frac{1}{2}\Delta_\Phi}}{1-q} - e^{-i\rho(g)}\prod_j t_j^{-f_j}\frac{q^{\frac{1}{2}|\rho(m)|+1-\frac{1}{2}\Delta_\Phi}}{1-q}\right] \ . \end{aligned}$$

Here the first term gives the contribution of the vector multiplets and the second line is the contribution of matter multiplets, labeled by Φ. The index j runs over the rank of the flavor symmetry group. Given the single letter index it is a combinatorical problem [94, 91] to compute the full multi-letter index. The result is given by the so-called "plethystic" exponential

$$\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\text{ind}(z^n,t^n,q^n;m)\right) \ . \tag{3.41}$$

For instance, let us consider the $\mathcal{N}=2$ theory with $U(N)$ gauge group. In this case the chiral multiplet Φ with R-charge r in the fundamental representation of the gauge group contributes to the single-letter index as

$$\sum_{i=1}^{N}\left[z t^{f(\Phi)}\frac{q^{\frac{r}{2}+\frac{|m_i|}{2}}}{1-q} - z^{-1}t^{-f(\Phi)}\frac{q^{1-\frac{r}{2}+\frac{|m_i|}{2}}}{1-q}\right] \ . \tag{3.42}$$

After the "plethystic" exponential one obtains the contribution of the chiral multiplets to the index

$$\prod_{i=1}^{N} \frac{(q^{1-\frac{r}{2}+\frac{|m_i|}{2}} t^{-f(\Phi)} z_i^{-1}; q)_\infty}{(q^{\frac{r}{2}+\frac{|m_i|}{2}} t^{f(\Phi)} z_i; q)_\infty} . \tag{3.43}$$

Similarly the contribution of the vector multiplet to the single-letter index is

$$-\sum_{i,j=1,\ldots,N,\, i\neq j} q^{\frac{1}{4}|m_i-m_j|} \frac{z_i}{z_j} , \tag{3.44}$$

and the multi-letter index gets the following form

$$q^{-\sum_{1\leq i<j\leq N} \frac{|m_i-m_j|}{2}} \prod_{i,j=1,\ldots,N,\, i\neq j} (1 - \frac{z_i}{z_j} q^{\frac{|m_i-m_j|}{2}}) . \tag{3.45}$$

Our main interest is the so-called generalized superconformal index which includes integer parameters corresponding to global symmetries. In [31] Kapustin and Willett pointed out that one can generalize the superconformal index of $3d$ $\mathcal{N}=2$ theory by considering the theory in a non–trivial background gauge field coupled to the global symmetries of the theory. As a result the superconformal index includes new discrete parameters for global symmetries and we do not sum over these parameters. In case of the generalized superconformal index the contribution (3.43) has the following form

$$\prod_{i=1}^{\text{rank}G} \frac{(q^{1-\frac{r}{2}+\frac{|m_i|+f(\Phi)n}{2}} t^{-f(\Phi)} z_i^{-1}; q)_\infty}{(q^{\frac{r}{2}+\frac{|m_i+f(\Phi)n|}{2}} t^{f(\Phi)} z_i; q)_\infty} , \tag{3.46}$$

where the parameters n_i are new discrete variables. It is convenient to express the index as a product of contributions from chiral and vector multiplets

$$I(q,\{t_a\},\{n_a\}) = \sum_{m_i} \frac{1}{|W_m|} \oint \prod_{i=1}^{\text{rank}G} \frac{dz_i}{2\pi i z_i} Z_{gauge}(z_i,m_i;q) \prod_{\Phi} Z_\Phi(z_i,m_i;t_a,n_a;q) . \tag{3.47}$$

Note that we do not write the contribution of the Chern–Simons term, since in this thesis we consider theories without this term.

It is worth to mention that the three-dimensional superconformal index can be constructed from the so-called holomorphic blocks [106] due to its factorization property [29, 107, 108]. It is possible to obtain the factorized superconformal index directly from the localization technique via the so-called Higgs branch localization [109, 110].

3.4 Supersymmetric partition function on a squashed 3-sphere

In this section we review some aspect concerning the partition function on a squashed three sphere[5] S_b^3.

Localization is the most general technique to compute supersymmetric partition functions and it was first used in [20] for the partition function on S^4 of $\mathcal{N} = 2$ four dimensional theories. The case of a three dimensional sphere was first studied in [10] for $\mathcal{N} > 2$. The extension to $\mathcal{N} = 2$ was done in [14, 12] for a round sphere and in [13] for a squashed sphere.

The general structure of the partition function on the squashed sphere has the following form

$$Z = \frac{1}{|W|} \int \prod_{i=1}^{\text{rank}G} \frac{dz_i}{i\sqrt{\omega_1\omega_2}} e^{\frac{ik\pi z_i^2}{\omega_1\omega_2} + \frac{\pi i \xi z_i}{\omega_1\omega_2}} \frac{\prod_{j=1}^{rankF} \gamma^{(2)}\left(\omega\Delta_j + \rho_j(z) + \tilde{\rho}_j(\mu); \omega_1, \omega_2\right)}{\prod_{\alpha\in R_+} \gamma^{(2)}\left(\alpha(z); \omega_1, \omega_2\right) \gamma^{(2)}\left(-\alpha(z); \omega_1, \omega_2\right)} \,. \tag{3.48}$$

The integral is performed over the Cartan subgroup of the gauge group. It is parameterized by the diagonal entries of the real scalar z in the gauge group. The exponential receives contributions from the classical action, from the Chern-Simons term at level k and from the real Fayet,ÄìIliopoulos parameter ξ. The factor of inverse $|W|$ represents the order of the Weyl group of gauge group.

The hyperbolic gamma functions $\gamma^{(2)}$ in (3.48) are obtained by computing the one loop superdeterminants of the vector and matter multiplets. The hyperbolic gamma function can be written as

$$\gamma^{(2)}(u; \omega_1, \omega_2) = e^{-\pi i B_{2,2}(u;\omega)/2} \frac{(e^{2\pi i u/\omega_1}\tilde{q}; \tilde{q})}{(e^{2\pi i u/\omega_1}; q)} \quad \text{with} \quad q = e^{2\pi i \omega_1/\omega_2}, \quad \tilde{q} = e^{-2\pi i \omega_2/\omega_1} \,, \tag{3.49}$$

[5]Preserving a $U(1)^2$ isometry of the original $SO(4)$ of the round case.

where $B_{2,2}(u;\omega)$ is the second order Bernoulli polynomial,

$$B_{2,2}(u;\omega) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}. \tag{3.50}$$

The contribution of the vector multiplet corresponds to the denominator of (3.48) and it is parameterized by the positive roots of the algebra. Actually the Vandermonde determinant in the measure exactly cancels the one loop determinant of the vector multiplet.

The contribution of the matter multiplet is the last term in the numerator of (3.48). Each term corresponds to the contribution of the j-th chiral multiplet with R charge Δ_j. Each chiral multiplet is in the corresponding representation of the gauge group G with weight $\rho_j(z)$ and in the corresponding representation of the flavor group F, with weight $\tilde{\rho}_j(\mu)$.

3.5 Compactification of $4d$ $\mathcal{N} = 1$ gauge theories on S^1

The reduction of four–dimensional supersymmetric field theories on $R^3 \times S^1$ to three–dimensional supersymmetric theories on R^3 with the same amount of supersymmetric charges was proposed in [111]. Later Dolan et al. found [112] the procedure which reduces four-dimensional $\mathcal{N} = 1$ superconformal index to three-dimensional $\mathcal{N} = 2$ partition function[6] (see also [113, 114]). A compelling physical argument for this reduction has been provided in [104] (see also [115]). The essential step in the reduction scheme is scaling of chemical potentials in the following way

$$p = e^{2\pi i v\omega_1}, \quad q = e^{2\pi i v\omega_2}, \quad z = e^{2\pi i v u}, \quad s_i = e^{2\pi i v\alpha_i} \ . \tag{3.51}$$

Then $3d$ partition function on squashed three sphere can be achieved by taking $v \to 0$ limit of $4d$ superconformal index. Geometrically, we consider $4d$ SCFT on a $S^3 \times S^1$, the limit $v \to 0$ shrinks S^1 to zero and gives rise to a three–dimensional supersymmetric theory with the same amount of supercharges on squashed S^3_b, where b is a squashing parameter. From the perspective of special functions this reduction brings elliptic gamma functions to hyperbolic gamma functions [116]

$$\Gamma(e^{2\pi \mathrm{i} v z}; e^{2\pi \mathrm{i} v\omega_1}, e^{2\pi \mathrm{i} v\omega_2}) \underset{v\to 0}{=} e^{-\pi \mathrm{i}(2z-(\omega_1+\omega_2))/24 v\omega_1\omega_2} \gamma^{(2)}(z; \omega_1, \omega_2) \ . \tag{3.52}$$

On the level of partition functions one can see that there is a duality in three dimensions coming from four-dimensional duality by this reduction procedure. However obtaining the right duality in three dimensions is more tricky. The main issue that the reduction procedure and renormalization group flow from ultraviolet to infrared does not commute with each other, because of presence of anomalous $U(1)$ symmetry in four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. One needs to break that symmetry in $3d$. The correct duality can be obtained by adding monopole operator to the $3d$ Lagrangians. To be more precise we need to add the effective superpotential $W = \eta X$ to the Lagrangian of electric theory and $W = \tilde{\eta}\tilde{X}$ to the magnetic theory, where X is a monopole operator and η is the $4d$ instanton factor.

[6]This reduction works for any number of supercharges.

3.5.1 Dualities for $SP(2N)$ gauge group

Now let us consider some examples. We start from the duality for $4d$ supersymmetric theory with the SP($2N$) group introduced by Intriligator [33]. The matter content of electric and magnetic theories are given below in tables, respectively:

	SP($2N$)	SU($2N_f$)	$U(1)_R$
Q	f	f	$2r = 1 - \frac{2(N+K)}{(K+1)N_f}$
X	T_A	1	$2s = \frac{2}{K+1}$

Matter content of the **electric** theory with the R charge assignment.

	SP($2\widetilde{N}$)	SU($2N_f$)	$U(1)_R$
q	f	$\overline{f}$	$2\widetilde{r} = 1 - \frac{2(\widetilde{N}+K)}{(K+1)N_f}$
Y	T_A	1	$2s = \frac{2}{K+1}$
M_j	1	T_A	$2r_j = 2\frac{K+j}{K+1} - 4\frac{\widetilde{N}+K}{(K+1)N_f}$

Matter content of the **magnetic** theory with the R charge assignment.

where $j = 1, \dots, K$, and $\widetilde{N} = K(N_f - 2) - N$, $K = 1, 2, \dots$

Defining $U = (pq)^s = (pq)^{\frac{1}{K+1}}$, we find the following superconformal indices for these theories [27]

$$I_E = \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!} \Gamma(U;p,q)^{N-1} \times \int_{\mathbb{T}^N} \prod_{1 \le i < j \le N} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1};p,q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1};p,q)} \prod_{j=1}^N \frac{\prod_{i=1}^{2N_f} \Gamma(s_i z_j^{\pm 1};p,q)}{\Gamma(z_j^{\pm 2};p,q)} \prod_{j=1}^N \frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{3.53}$$

$$I_M = \frac{(p;p)_\infty^{\widetilde{N}} (q;q)_\infty^{\widetilde{N}}}{2^{\widetilde{N}} \widetilde{N}!} \Gamma(U;p,q)^{\widetilde{N}-1} \prod_{l=1}^K \prod_{1 \le i < j \le 2N_f} \Gamma(U^{l-1} s_i s_j;p,q) \times \int_{\mathbb{T}^{\widetilde{N}}} \prod_{1 \le i < j \le \widetilde{N}} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1};p,q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1};p,q)} \prod_{j=1}^{\widetilde{N}} \frac{\prod_{i=1}^{2N_f} \Gamma(U s_i^{-1} z_j^{\pm 1};p,q)}{\Gamma(z_j^{\pm 2};p,q)} \prod_{j=1}^{\widetilde{N}} \frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{3.54}$$

where the balancing condition reads as follows

$$U^{2(N+K)} \prod_{i=1}^{2N_f} s_i = (pq)^{N_f} . \tag{3.55}$$

Using the asymptotic formula (3.52) for the elliptic gamma function one can proceed with the reduction of superconformal indices for a dual pair presented above. Let us reparameterize the variables in (3.53) and (3.54) in the following way $p=e^{2\pi i v\omega_1}$, $q=e^{2\pi i v\omega_2}$, $s_i=e^{2\pi i v\alpha_i}$, $z_j=e^{2\pi i v u_j}$ with $i=1,\ldots,2N_f$, $j=1,\ldots,N$. Then after taking the limit $v\to 0$, which assumes $pq\to 1$, one obtains[7]

$$I_E^{red}=\frac{1}{2^N N!}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}\big)^{N-1}\int_{-i\infty}^{i\infty}\prod_{1\le i<j\le N}\frac{\gamma\big(\frac{\omega_1+\omega_2}{K+1}\pm u_i\pm u_j\big)}{\gamma\big(\pm u_i\pm u_j\big)}\prod_{j=1}^{N}\frac{\prod_{i=1}^{2N_f}\gamma\big(\alpha_i\pm u_j\big)}{\gamma\big(\pm 2u_j\big)}\frac{du_j}{i\sqrt{\omega_1\omega_2}},\tag{3.56}$$

$$\begin{aligned}I_M^{red}=&\frac{1}{2^{\widetilde{N}}\widetilde{N}!}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}\big)^{\widetilde{N}-1}\prod_{l=1}^{K}\prod_{1\le i<j\le 2N_f}\gamma\big((l-1)\tfrac{\omega_1+\omega_2}{K+1}+\alpha_i+\alpha_j\big)\\&\times\int_{-i\infty}^{i\infty}\prod_{1\le i<j\le\widetilde{N}}\frac{\gamma\big(\frac{\omega_1+\omega_2}{K+1}\pm u_i\pm u_j\big)}{\gamma\big(\pm u_i\pm u_j\big)}\prod_{j=1}^{\widetilde{N}}\frac{\prod_{i=1}^{2N_f}\gamma\big(\frac{\omega_1+\omega_2}{K+1}-\alpha_i\pm u_j\big)}{\gamma\big(\pm 2u_j\big)}\prod_{j=1}^{\widetilde{N}}\frac{du_j}{i\sqrt{\omega_1\omega_2}},\end{aligned}\tag{3.57}$$

with the following balancing condition

$$(\omega_1+\omega_2)\frac{2(N+K)}{(K+1)}+\sum_{i=1}^{2N_f}\alpha_i=N_f(\omega_1+\omega_2)\ .\tag{3.58}$$

Here we use the following notation $\gamma(z)\equiv\gamma^{(2)}(z;\omega_1,\omega_2)$ and conventions $\gamma(a,b)\equiv\gamma(a)\gamma(b)$, $\gamma(a\pm u)\equiv\gamma(a+u)\gamma(a-u)$.

Let us consider now $\alpha_{2N_f}=\xi_1+aS$, $\alpha_{2N_f-1}=\xi_2-aS$ and take the limit $S\to\infty$, then I_E^{red} and I_M^{red} become

$$Z_E=\frac{1}{2^N N!}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}\big)^{N-1}\int_{-i\infty}^{i\infty}\prod_{1\le i<j\le N}\frac{\gamma\big(\frac{\omega_1+\omega_2}{K+1}\pm u_i\pm u_j\big)}{\gamma\big(\pm u_i\pm u_j\big)}\prod_{j=1}^{N}\frac{\prod_{i=1}^{2(N_f-1)}\gamma\big(\alpha_i\pm u_j\big)}{\gamma\big(\pm 2u_j\big)}\frac{du_j}{i\sqrt{\omega_1\omega_2}}\tag{3.59}$$

$$\begin{aligned}Z_M=&\frac{1}{2^{\widetilde{N}}\widetilde{N}!}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}\big)^{\widetilde{N}-1}\prod_{l=1}^{K}\gamma\Big((\omega_1+\omega_2)\Big(N_f-\tfrac{2N+2K-l+1}{K+1}\Big)-\textstyle\sum_{i=1}^{2(N_f-1)}\alpha_i\Big)\\&\times\prod_{l=1}^{K}\prod_{1\le i<j\le 2(N_f-1)}\gamma\Big((l-1)\tfrac{\omega_1+\omega_2}{K+1}+\alpha_i+\alpha_j\Big)\\&\times\int_{-i\infty}^{i\infty}\prod_{1\le i<j\le\widetilde{N}}\frac{\gamma\big(\frac{\omega_1+\omega_2}{K+1}\pm u_i\pm u_j\big)}{\gamma\big(\pm u_i\pm u_j\big)}\prod_{j=1}^{\widetilde{N}}\frac{\prod_{i=1}^{2(N_f-1)}\gamma\big(\frac{\omega_1+\omega_2}{K+1}-\alpha_i\pm u_j\big)}{\gamma\big(\pm 2u_j\big)}\prod_{j=1}^{\widetilde{N}}\frac{du_j}{i\sqrt{\omega_1\omega_2}}.\end{aligned}\tag{3.60}$$

[7] We omit the same divergent coefficients $\exp\left(\frac{-2\pi i(-1+K-6KN-4N^2)(\omega_1+\omega_2)}{24v\omega_1\omega_2(1+K)}\right)$.

To obtain these expressions we used the inversion relation $\gamma(z,\omega_1+\omega_2-z) = 1$ and the asymptotic formulas

$$\lim_{u\to\infty} e^{\frac{\pi i}{2}B_{2,2}(u;\omega_1,\omega_2)}\gamma(u) = 1, \quad \text{for } \arg\omega_1 < \arg u < \arg\omega_2 + \pi,$$
$$\lim_{u\to\infty} e^{-\frac{\pi i}{2}B_{2,2}(u;\omega_1,\omega_2)}\gamma(u) = 1, \quad \text{for } \arg\omega_1 - \pi < \arg u < \arg\omega_2 \,. \tag{3.61}$$

Note that the balancing condition is absent. Expressions (3.59) and (3.60) reproduces the partition functions of $3d$ $\mathcal{N}=2$ supersymemtric field theories [14, 12]. Equality of (3.59) and (3.60) gives us the duality for the $3d$ $\mathcal{N}=2$ supersymmetric gauge theories with the matter content presented in the below tables:

	SP$(2N)$	SU$(2(N_f-1))$	$U(1)_A$	$U(1)_R$
Q	f	f	1	1
X	T_A	1	0	$2/(K+1)$

Matter content of the **electric** theory with the R charge assignment.

	SP$(2(K(N_f-2)-N))$	SU$(2(N_f-1))$	$U(1)_A$	$U(1)_R$ $(j=1,\ldots,K)$
q	f	$\overline{f}$	-1	$\frac{3-K}{K+1}$
x	T_A	1	0	$\frac{2}{K+1}$
Y_j	1	1	$-2(N_f-1)$	$4N_f - \frac{4N+6K-2j+4}{K+1}$
M_j	1	T_A	2	$1+2\frac{j-1}{K+1}$

Matter content of the **magnetic** theory with the R charge assignment.

One can proceed with the reduction of flavors and take the limit $\alpha_{2N_f-2}\to\infty$ after which one gets the equality for partition functions of the Chern-Simons theories. Let us set $N_f\to N_f-2$, then the electric theory is $3d$ $\mathcal{N}=2$ Chern-Simons theory with $k=1/2$ and the magnetic theory is $3d$ $\mathcal{N}=2$ Chern-Simons theory with $k=-1/2$.

Now one can proceed further in integrating out the quarks by taking further limits $s_i\to\infty$. As the result one gets the extension for Kutasov-Schwimmer duality [117] in three dimensions: the electric theory is $3d$ $\mathcal{N}=2$ Chern-Simons theory with SP$(2N)$ gauge group and level k (such as N_f+k is even), N_f quarks (which can be also odd [118]), a chiral superfield X in adjoint representation, and the magnetic theory is $3d$

$\mathcal{N}=2$ Chern-Simons theory with $\mathrm{SP}(K(N_f+2(k-1))-2N)$ gauge group and level $-k$, N_f quarks,a chiral superfield in adjoint representation of the gauge group, mesons in T_A representation of $\mathrm{SU}(N_f)$ flavor symmetry group.

We now consider different limit for the equality between (3.56) and (3.57). Let us reparameterize the parameters in the following way $\alpha_i \to \alpha_i+\mu,\ \alpha_{i+N_f} \to \alpha_{i+N_f}-\mu,\ i=1,\dots,N_f$ and take the limit $\mu\to\infty$ after which one gets (for $K=1$ it coincides with the expression by Bult [116])

$$I_E^{red,U(N)} = \frac{1}{N!}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}\big)^{N-1}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{j=1}^{N}\frac{du_j}{\mathrm{i}\sqrt{\omega_1\omega_2}} \times \prod_{1\le i<j\le N}\frac{\gamma\big(\frac{\omega_1+\omega_2}{K+1}\pm(u_i-u_j)\big)}{\gamma\big(\pm(u_i-u_j)\big)}\prod_{j=1}^{N}\prod_{i=1}^{N_f}\gamma\big(\alpha_i+u_j,\alpha_{i+N_f}-u_j\big) \tag{3.62}$$

and

$$I_M^{red,U(N)} = \frac{1}{\widetilde{N}!}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}\big)^{\widetilde{N}-1}\prod_{l=1}^{K}\prod_{i,j=1}^{N_f}\gamma\big((l-1)\tfrac{\omega_1+\omega_2}{K+1}+\alpha_i+\alpha_{j+N_f}\big)\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{j=1}^{\widetilde{N}}\frac{du_j}{\mathrm{i}\sqrt{\omega_1\omega_2}} \times \prod_{1\le i<j\le \widetilde{N}}\frac{\gamma\big(\frac{\omega_1+\omega_2}{K+1}\pm(u_i-u_j)\big)}{\gamma\big(\pm(u_i-u_j)\big)}\prod_{j=1}^{\widetilde{N}}\prod_{i=1}^{N_f}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}-\alpha_i-u_j,\tfrac{\omega_1+\omega_2}{K+1}-\alpha_{i+N_f}+u_j\big), \tag{3.63}$$

where the balancing condition reads

$$(\omega_1+\omega_2)2\frac{N+K}{K+1}+\sum_{i=1}^{N_f}(\alpha_i+\alpha_{i+N_f}) = N_f(\omega_1+\omega_2)\ . \tag{3.64}$$

Now considering the following reparametrization

$$\alpha_{N_f-1}=\xi_1+\mu, \quad \alpha_{N_f}=\xi_3-\nu, \quad \alpha_{2N_f-1}=\xi_2-\mu, \quad \alpha_{2N_f}=\xi_4+\nu \tag{3.65}$$

with the following limit $\mu\to\infty$ and $\nu\to\infty$ one can obtain the corresponding partition functions.

4 Extended global symmetries for supersymmetric gauge theories

In this Chapter using a superconformal index technique we show evidence of a global symmetry enhancement of a supersymmetric gauge theory.

The superconformal index of a theory with a flavor group F has the Weyl group symmetry $W(F)$. The Weyl symmetry of the flavor group refers to the symmetry with respect to the exchange of the flavors defined in the suitable representation of the flavor group. In cases when the theory has a hidden symmetry, the coefficients in the decomposition of the superconformal index into characters of the flavor group give the sums of dimensions of irreducible representations of the larger symmetry group. One can use this property to study global symmetry enhancement in supersymmetric gauge theories.

In our example the superconformal index of four-dimensional $\mathcal{N} = 1$ SQCD with flavor group $SU(6)$ has the Weyl group of the exceptional root system E_6. It means that the theory with flavor group $SU(6)$ can be extended to E_6 symmetry. Indeed this is a manifestation of the four-dimensional boundary model coupled to the free five-dimensional hypermultiplet with the enhanced E_6 flavor symmetry [2].

In [26] Spiridonov and Vartanov reduced $4d\,\mathcal{N} = 1$ Super-Yang-Mills with $SU(2)$ gauge group with 8 quarks to 6 quarks and found that the index of the reduced theory has $W(E_6)$ symmetry. After this reduction in the dual theories they realized additional $SU(2)$ global symmetries, the appearance of which was unclear to the authors. In this work we give the explanation of this extended symmetry by coupling of original $N_f = 3$

theory to free $5d$ hypermultiplets[1]. This coupling bring us to E_6 global symmetry. Since we have E_6 global symmetry group, in different phases it produces us additional $SU(2)$ or $U(1)$ groups in dualities found in [26].

At the same time this E_6 symmetry can be obtained by restricting two parameters in combined $4d/5d$ index considered by Dimofte and Gaiotto [119].

4.1 Multiple duality for $SP(2)$ gauge group

In this section we consider multiple duality phenomenon for $4d$ $\mathcal{N}=1$ theory with $SP(2)$ gauge group[2] with $N_F = 4$ flavors. The duality was established in [26] and interpreted in [120]. It was shown that these dual theories are associated with the orbit of $W(E_7)$–Weyl symmetry group. The total number of dualities is $72 = 1+35+35+1$. One can classify them in four different groups in the following way [26].

The electric theory has one chiral scalar multiplet belonging to the fundamental representations (denoted as f) of $SP(2)$ and $SU(8)$, and the vector multiplet in the adjoint representation (denoted as adj) of the gauge group. The field content with global charges is given in Table 1.

	$SP(2)$	$SU(8)$	$U(1)_R$
Q	f	f	$\frac{1}{4}$
V	adj	1	$\frac{1}{2}$

Table 1. Matter content of the electric theory with the R charge assignment.

The superconformal index of the electric theory is

$$I_E = \frac{(p;p)_\infty (q;q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{i=1}^{8} \Gamma((pq)^{1/4} y_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z}. \tag{4.1}$$

[1]Note that we use the subscript F for the flavor and the subscript f for the number of quarks.
[2]This is a special case of a theory with $SP(N_c)$ gauge group and N_f flavors of matter in the fundamental representation considered in [34]. Such theory is qualitatively similar to $SU(N_c)$ gauge theories with matter in the fundamental representation considered in Chapter 2. In case of $N_c = 2$ one can consider the theory as $SU(2)$ gauge theory since $SP(2) \simeq SU(2)$.

The fugacities of $SU(8)$ flavor group y_i obey the following balancing condition

$$\prod_{i=1}^{8} y_i = 1 \ . \tag{4.2}$$

It is clear that the numerator comes from the eight chirals and eight anti–chirals, while the rest comes from the $SU(2)$ gauge multiplet and the Haar measure.

The first type of dual magnetic theory is the theory which was found by Csaki et al. in [35]. There are 35 dual theories of this type and all of them have $SU(4)_l \times SU(4)_r \times U(1)_B$ global symmetry. The field content contains two scalar chiral multiplets in the fundamental representation of the $SP(2)$ gauge group, a gauge field in the adjoint representation of the $SP(2)$ gauge group, and two singlets in the antisymmetric tensor representations of the corresponding $SU(4)$ flavor symmetry group. The field content of the theory is summarized in Table 2.

	$SP(2)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_R$
q	f	f	1	-1	$\frac{1}{4}$
$\widetilde{q}$	f	1	f	1	$\frac{1}{4}$
M	1	T_A	1	2	$\frac{1}{2}$
$\widetilde{M}$	1	1	T_A	-2	$\frac{1}{2}$
$\widetilde{V}$	adj	1	1	0	$\frac{1}{2}$

Table 2. Matter content of the first dual theory.

The superconformal index for these dual theories has the following form

$$\begin{aligned} I_M^{(1)} \;=\; & \frac{(p;p)_\infty (q;q)_\infty}{2} \prod_{1\le i<j\le 4} \Gamma((pq)^{\frac{1}{2}} y_i y_j; p, q) \prod_{5\le i<j\le 8} \Gamma((pq)^{\frac{1}{2}} y_i y_j; p, q) \\ & \times \int_{\mathbb{T}} \frac{\prod_{i=1}^{4} \Gamma((pq)^{\frac{1}{4}} v^{-2} y_i z^{\pm 1}; p, q) \prod_{i=5}^{8} \Gamma((pq)^{\frac{1}{4}} v^{2} y_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z} . \end{aligned} \tag{4.3}$$

where v is a fugacity of $U(1)_B$

$$v = \sqrt[4]{y_1 y_2 y_3 y_4}, \qquad v^{-1} = \sqrt[4]{y_5 y_6 y_7 y_8} \ . \tag{4.4}$$

The second type is the original Seiberg dual theory [32] with $SU(2)$ gauge group and $SU(4) \times SU(4) \times U(1)_B \times U(1)_R$ flavor group, one singlet in the fundamental representation of $SU(4)$ and all other matter content is the same as the theory above. The field content of the model is summarized in Table 3.

	$SP(2)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_R$
q	f	$\overline{f}$	1	1	$\frac{1}{4}$
$\widetilde{q}$	f	1	$\overline{f}$	-1	$\frac{1}{4}$
M	1	f	f	0	$\frac{1}{2}$
$\widetilde{V}$	adj	1	1	0	$\frac{1}{2}$

Table 3. Matter content of the second dual theory.

The superconformal index is

$$I_M^{(2)} = \frac{(p;p)_\infty (q;q)_\infty}{2} \prod_{i=1}^{4} \prod_{j=5}^{8} \Gamma((pq)^{\frac{1}{2}} y_i y_j; p, q) \tag{4.5}$$
$$\times \int_{\mathbb{T}} \frac{\prod_{i=1}^{4} \Gamma((pq)^{\frac{1}{4}} v^2 y_i^{-1} z^{\pm 1}; p, q) \prod_{i=5}^{8} \Gamma((pq)^{\frac{1}{4}} v^{-2} y_i^{-1} z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z}.$$

The third type of magnetic dual theory was considered by Intriligator and Pouliot in [34]. This theory has $SU(8)$ flavor symmetry group and $SU(2)$ gauge group. The field content contains a chiral scalar multiplet in the fundamental representation of the gauge group and antisymmetric representation of the flavor symmetry group, a gauge field in the adjoint representation of the gauge group and one singlet in the antisymmetric tensor representation of flavor group. The field content of the model is summarized in Table 4.

	$SP(2)$	$SU(8)$	$U(1)_R$
q	f	$\overline{f}$	$\frac{1}{4}$
M	1	T_A	$\frac{1}{2}$
$\widetilde{V}$	adj	1	$\frac{1}{2}$

Table 4. Matter content of the third dual theory.

The superconformal index of this type is

$$I_M^{(3)} = \frac{(p;p)_\infty (q;q)_\infty}{2} \prod_{1\le i<j\le 8} \Gamma((pq)^{\frac{1}{2}} y_i y_j; p,q) \int_{\mathbb{T}} \frac{\prod_{i=1}^8 \Gamma((pq)^{\frac{1}{4}} y_i^{-1} z^{\pm 1}; p,q)}{\Gamma(z^{\pm 2}; p,q)} \frac{dz}{2\pi i z}. \quad (4.6)$$

More detailed explanations about these dual theories can be found in the original paper [26] and also in [120].

The equality of all four indices follows from the following identity [92]

$$I(t_1,\ldots,t_8;p,q) = \prod_{1\le j<k\le 4} \Gamma(t_j t_k; p,q)\Gamma(t_{j+4}t_{k+4};p,q)\, I(s_1,\ldots,s_8;p,q), \quad (4.7)$$

where the complex variables s_j, $|s_j|<1$, are given in terms of t_j $(j=1,\ldots,8)$,

$$s_j = \rho^{-1} t_j,\ \ j=1,2,3,4, \quad s_j = \rho t_j,\ \ j=5,6,7,8, \quad (4.8)$$
$$\rho = \sqrt{\frac{t_1t_2t_3t_4}{pq}} = \sqrt{\frac{pq}{t_5t_6t_7t_8}}\ .$$

4.2 Enhanced flavor symmetry

All 72 dual theories are associated with the orbit of the $W(E_7)$ Weyl group. Using this fact Spiridonov and Vartanov speculated in [26], that the superconformal index may have global symmetry group E_7. In fact, Dimofte and Gaiotto explicitly showed in [119] that the theories in question, when coupled to $5d$ hypermultiplet, have an enhanced symmetry group E_7. In order to show this, they added the $5d$ hypermultiplet contributions with a specific boundary condition to the index

$$I_{4d/5d,\, N_F=4} = \prod_{1\le i<j\le 8} \frac{1}{\left(\sqrt{pq}(s_i s_j)^{-1};p,q\right)_\infty} \frac{(p;p)_\infty (q;q)_\infty}{2}$$
$$\times \oint \frac{dz}{2\pi i z} \frac{\prod_{i=1}^8 \Gamma(\sqrt[4]{pq} s_i z^{\pm}; p,q)}{\Gamma(z^{\pm 2};p,q)}. \quad (4.9)$$

where the term

$$\prod_{1\le i<j\le 8} \frac{1}{\left(\sqrt{pq}(s_i s_j)^{-1};p,q\right)_\infty} \quad (4.10)$$

corresponds to a $5d$ hypermultiplet [121]. By setting all fugacities to 1 and redefining $p = t^3 y$, $q = t^3 y^{-1}$ one can easily read off the E_7 symmetry of the index by expanding the last expression in powers of t and y

$$I_{4d/5d,\, N_F=4} = 1 + 56t^3 + 1463t^6 + 3002t^9 y + \dots\,, \tag{4.11}$$

where the coefficients 56 and 1463 are the dimensions of the irreducible representations of E_7 with highest weight $[0,0,0,0,0,0,1]$ and $[0,0,0,0,0,0,2]$, respectively and $3002 = 1539_{[0,0,0,0,0,1,0]} + 1463_{[0,0,0,0,0,0,2]}$[3].

Remarkably, the new index is invariant under the transformation of the fugacities to their duals and the expression (4.7) becomes [119]

$$I(t_1, \dots, t_8; p, q) = I(s_1, \dots, s_8; p, q). \tag{4.12}$$

If we set $s_7 s_8 = \sqrt{pq}$ in (4.9) one gets the reduction[4] of the index from $N_F = 4$ to $N_F = 3$. When we apply this reduction for the integrals $I_M^{(1)}$ and $I_M^{(2)}$, setting $s_4 s_5 = \sqrt{pq}$ and $s_7 s_8 = \sqrt{pq}$, respectively, we end up with the flavor group $SU(3)_l \times SU(3)_r \times U(1)_B \times U(1)_{add}$ for $I_M^{(1)}$ and the flavor group $SU(4) \times SU(2) \times SU(2)_{add} \times U(1)_B$ for $I_M^{(2)}$. The observation that one gets additional symmetries such as $SU(2)_{add}$ and $U(1)_{add}$ in the reduced theories, suggests that the reduced theories may also have larger symmetry than $SU(6)$, in fact E_6 flavor symmetry. Indeed it is possible to show this by adding the 5d hypermultiplet contribution to the index and apply reduction procedure. The new reduced index is

$$\begin{aligned} I_{4d/5d,\, N_F=3} = \prod_{1\le i<j\le 6} &\frac{1}{\left((pq)^{\frac{2}{3}} s_i^{-1} s_j^{-1}; p, q\right)_\infty} \prod_{i=1}^{6} \frac{1}{\left((pq)^{\frac{1}{3}} s_i^{-1} w^{\pm 1}; p, q\right)_\infty} \\ &\times \frac{(p,p)_\infty (q,q)_\infty}{2} \oint \frac{dz}{2\pi i z} \frac{\prod_{i=1}^{6} \Gamma(\sqrt[6]{pq}\, s_i z^{\pm}; p, q)}{\Gamma(z^{\pm 2}; p, q)}\,. \end{aligned} \tag{4.13}$$

Note that we have redefined the fugacities $s_i \to (pq)^{-1/12} s_i$. The balancing condition is $\prod_{i=1}^{6} s_i = 1$. Now by setting all fugacities to 1 and redefining $p = t^3 y$ and $q = t^3 y^{-1}$

[3] To find dimensions of irreducible representations of Lie algebras one can use `http://www-math.univ-poitiers.fr/~maavl/LiE/form.html`

[4] One needs to use the reflection identity for an elliptic Gamma function $\Gamma(z;p,q)\Gamma(pqz^{-1};p,q) = 1$ (see Appendix for details).

one can read off the E_6 symmetry of the index

$$I_{4d/5d,\, N_F=3} = 1 + 27t^2 + 378t^4 + 3653t^6 + 27t^5(y^{-1} + y) + \dots \tag{4.14}$$

The coefficient 27 is the dimension of the irreducible representation of E_6 with highest weight $[1,0,0,0,0,0]$ and

$$378 = 351_{[0,0,1,0,0,0]} + 27_{[1,0,0,0,0,0]}, \tag{4.15}$$

$$3653 = 3003_{[3,0,0,0,0,0]} + 650_{[1,0,0,0,0,1]}. \tag{4.16}$$

4.3 Reduction of $4d$ superconformal index to a $3d$ partition function

There is a reduction scheme [112, 113, 114] (also see [115, 104, 122, 55, 1]) of the superconformal index for a $4d$ supersymmetric theory to the partition function for a $3d$ theory considered in Chapter 3. Let us do this procedure for the index (4.13), following [112]. First we reparameterize

$$p = e^{2\pi i v\omega_1}, \quad q = e^{2\pi i v\omega_2}, \quad z = e^{2\pi i v u}, \quad s_i = e^{2\pi i v\alpha_i}, \quad w = e^{2\pi i v\alpha_7}\ , \tag{4.17}$$

and use the asymptotic formula for the elliptic gamma functions. Recall that in the limit $v \to 0$ the elliptic gamma function reduce to hyperbolic gamma function

$$\Gamma(e^{2\pi \mathrm{i} vz}; e^{2\pi \mathrm{i} v\omega_1}, e^{2\pi \mathrm{i} v\omega_2}) \underset{v\to 0}{=} e^{-\pi \mathrm{i}(2z-(\omega_1+\omega_2))/24v\omega_1\omega_2} \gamma^{(2)}(z;\omega_1,\omega_2)\ . \tag{4.18}$$

In the limit $v \to 0$ we also have

$$(z;p,q)_\infty \underset{v\to 0}{\to} \frac{1}{\Gamma_2(u;\omega_1,\omega_2)}\ , \tag{4.19}$$

where $\Gamma_2(u;\omega_1,\omega_2)$ is the Barnes double Gamma function (see Appendix for its definition and for useful properties).

To go further let us apply the limit $v \to 0$ to the index (4.13) and use the asymptotic formula above. We have also used the reflection identity and some asymptotic formulas for $\gamma^{(2)}(z)$ function (see Appendix). Here and below we will use the short-hand notations $\gamma^{(2)}(a,b;\omega_1,\omega_2) \equiv \gamma^{(2)}(a;\omega_1,\omega_2)\gamma^{(2)}(b;\omega_1,\omega_2)$, and $\gamma^{(2)}(a \pm u;\omega_1,\omega_2) \equiv \gamma^{(2)}(a+u;\omega_1,\omega_2)\gamma^{(2)}(a-u;\omega_1,\omega_2)$. Finally we arrive at

$$I_{4d/5d} \underset{v\to 0}{=} e^{\pi i(\omega_1+\omega_2)/12v\omega_1\omega_2} I^r_{4d/5d} \;, \tag{4.20}$$

where

$$\begin{aligned} I^r_{4d/5d} = &\prod_{1\le i<j\le 6} \Gamma_2\Big(\frac{\omega_1+\omega_2}{2} - (\alpha_i+\alpha_j)\Big) \prod_{i=1}^{6} \Gamma_2\Big(-\frac{\omega_1+\omega_2}{2} - (\alpha_i \pm \alpha_7)\Big) \\ &\times \frac{1}{2}\int \frac{du}{i\sqrt{\omega_1\omega_2}} \frac{\prod_{i=1}^6 \gamma^{(2)}(\alpha_i \pm u + \frac{\omega_1+\omega_2}{4};\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2u;\omega_1,\omega_2)}. \end{aligned} \tag{4.21}$$

If one considers

$$\alpha_5 = \xi_1 + aS, \qquad \alpha_6 = \xi_2 - aS, \tag{4.22}$$

and applies the additional limit $S \to \infty$, then the final answer gives an expression for the partition function of $3d$ $\mathcal{N}=2$ SYM theory [14, 12, 13]

$$Z_{3d/4d} \underset{S\to\infty}{\approx} F Z^r_{3d/4d} \;, \tag{4.23}$$

where

$$\begin{aligned} Z^r_{3d/4d} = &\Gamma_2(\frac{\omega_1+\omega_2}{2} - \xi_1 - \xi_2) \\ &\times \prod_{1\le i<j\le 4} \Gamma_2\Big(\frac{\omega_1+\omega_2}{2} - (\alpha_i+\alpha_j)\Big) \prod_{i=1}^{4} \Gamma_2\Big(-\frac{\omega_1+\omega_2}{2} - (\alpha_i \pm \alpha_7)\Big) \\ &\times \frac{1}{2}\int \frac{du}{i\sqrt{\omega_1\omega_2}} \frac{\prod_{i=1}^4 \gamma^{(2)}(\alpha_i \pm u + \frac{\omega_1+\omega_2}{4};\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2u;\omega_1,\omega_2)}. \end{aligned} \tag{4.24}$$

and for $\omega_1 = \frac{1}{\omega_2}$

$$\begin{aligned} F = &\left(-\xi_1 - \frac{5i\pi\xi_1}{6} - \xi_2 - \frac{i\pi\xi_2}{6}\right)(\omega + \frac{1}{\omega}) + (\frac{i\pi}{3} - \frac{4}{3})(\frac{1}{\omega} + \omega)^2 - i\pi \\ &- \frac{5}{2}i\pi\xi_1^2 + \frac{15\xi_2^2}{2} + \left(\frac{3}{2} - \frac{i\pi}{2}\right)\left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + 5\xi_1^2 - 2\xi_1\xi_2 + 8\alpha_7^2\right). \end{aligned} \tag{4.25}$$

From the physical point of view this reduction corresponds to adding mass terms to two quark supermultiplets and then integrating them out by sending their masses to infinity. As one can see, this theory has 4 quarks, one chiral field in the antisymmetric representation of the gauge group, and contributions from a $5d$ hypermultiplet.

4.4 Reduction to $N_f = 4$

In [119] it was shown that three-dimensional $\mathcal{N} = 2$ SQCD with $N_f = 6$ has $SO(12)$ symmetry. The authors obtained the superconformal index of the $3d$ theory by reduction from $4d$ $\mathcal{N} = 1$ theory with $N_F = 4$ inspired by [123]. We will now demonstrate that the superconformal index for the 3d $\mathcal{N} = 2$ SQCD with 4 quarks has $SO(10)$ symmetry group.

The expression for the superconformal index of the electric 3d $\mathcal{N} = 2$ supersymmetric theory with an arbitrary number of flavors N_f and fugacities s_i, t_i, $(i = 1, ..., N_f)$ is [29]

$$I_{3d,N_f} = \prod_{a,b=1}^{N_f} \frac{1}{\left(q^{\frac{1}{2}} t_a^{-1} s_b^{-1}; q\right)_\infty} \sum_{k\in\mathbb{Z}} a^{N_f|k|/2} \times \oint \frac{dz}{2\pi \mathrm{i} z} \prod_{i=1}^{N_f} \frac{(a^{1/2} q^{1/2+|k|/2} t_i^{-1} z; q)_\infty}{(a^{-1/2} q^{1/2+|k|/2} t_i z^{-1}; q)_\infty} \frac{(a^{1/2} q^{1/2+|k|/2} s_i^{-1} z^{-1}; q)_\infty}{(a^{-1/2} q^{1/2+|k|/2} s_i z; q)_\infty}, \tag{4.26}$$

with the balancing conditions $\prod_{a=1}^{N_f} t_a = 1$ and $\prod_{a=1}^{N_f} s_a = 1$. It is clear that by taking $a = q^{\frac{1}{2}}$ for $N_f = 4$ (8 quarks), we obtain the following expression

$$I_{3d} = \prod_{a,b=1}^{4} \frac{1}{\left(q^{\frac{1}{2}} t_a^{-1} s_b^{-1}; q\right)_\infty} \sum_{k\in\mathbb{Z}} q^{|k|} \times \oint \frac{dz}{2\pi \mathrm{i} z} \prod_{i=1}^{4} \frac{(q^{1/4} q^{1/2+|k|/2} t_i^{-1} z; q)_\infty}{(q^{-1/4} q^{1/2+|k|/2} t_i z^{-1}; q)_\infty} \frac{(q^{1/4} q^{1/2+|k|/2} s_i^{-1} z^{-1}; q)_\infty}{(q^{-1/4} q^{1/2+|k|/2} s_i z; q)_\infty}. \tag{4.27}$$

One can rewrite this index in the following form [119]

$$I_{3d,N_f=6} = \frac{1}{\left(q^{\frac{1}{2}} f_1 f_2 f_3 f_4 f_5 f_6; q\right)_\infty} \prod_{1\leq i<j\leq 6} \frac{1}{\left(q^{\frac{1}{2}} f_i^{-1} f_j^{-1}; q\right)_\infty}$$
$$\times \frac{1}{2} \sum_{k\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} (1-q^{|k|} z^{\pm 2}) \prod_{i=1}^{6} f_i^{-|k|} \frac{1-q^{r+\frac{1}{2}|k|+1}(q^{\frac{1}{4}} f_i z^{\pm 1})^{-1}}{1-q^{r+\frac{1}{2}|k|} q^{\frac{1}{4}} f_i z^{\pm 1}}, \tag{4.28}$$

where $f_i = t_i/\sqrt{t_1 t_2 t_3 s_4}$ and $f_{i+3} = s_i \sqrt{t_1 t_2 t_3 s_4}$ $(i = 1,2,3)$. The reduction of superconformal indices in $3d$ is similar to the $4d$ case. For the result of this section, we set $f_5 f_6 = q^{\frac{1}{2}}$ which reduces the index of the theory with 6 quarks to the index of the theory with 4 quarks

$$I_{3d,N_f=4} = \frac{(q^{1/3};q)_\infty}{\left(q f_1 f_2 f_3 f_4; q\right)_\infty} \prod_{1\leq i<j\leq 4} \frac{1}{\left(q^{\frac{1}{2}} f_i^{-1} f_j^{-1}; q\right)_\infty} \prod_{i=1}^{4} \frac{1}{\left(q^{\frac{1}{2}} f_i^{-1} q^{-\frac{1}{4}} v^{\pm 1}; q\right)_\infty}$$
$$\times \frac{1}{2} \sum_{k\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} (1-q^{|k|} z^{\pm 2}) \prod_{i=1}^{4} f_i^{-|k|} \frac{1-q^{r+\frac{1}{2}|k|+\frac{3}{4}} f_i^{-1} z^{\pm 1}}{1-q^{r+\frac{1}{2}|k|+\frac{1}{4}} f_i z^{\pm 1}}\ , \tag{4.29}$$

where the term $(q^{\frac{1}{3}};q)_\infty$ is a monopole contribution.

Note that we have chosen the representation (4.28) of the index because it is closely related to the three-dimensional $\mathcal{N}=2$ partition function (4.24). This procedure can be repeated for the initial expression of the superconformal index (4.28) in a similar way. Now one can read off the $SO(10)$–invariant operator content of the theory by expanding the last expression in powers of q and setting all fugacities to 1

$$I = 1 + 16q^{1/3} + 136q^{2/3} + 816q + 3892q^{4/3} + \dots \tag{4.30}$$

The coefficients are related to the dimensions of irreducible representations of $SO(10)$

$$16 \text{ is the dimension of the spinor representation of } SO(10) \tag{4.31}$$
$$136 = 54_{[2,0,0,0,0]} + 45_{[0,1,0,0,0]} + 16_{[0,0,0,1,0]} + 10_{[1,0,0,0,0]} + 1_{[0,0,0,0,0]}, \tag{4.32}$$
$$816 = 320_{[1,1,0,0,0]} + 210_{[0,0,0,1,1]} + 144_{[1,0,0,1,0]} + 126_{[0,0,0,2,0]} + 16_{[0,0,0,1,0]}, \tag{4.33}$$
$$3892 = 2772_{[0,0,0,4,0]} + 945_{[1,0,1,0,0]} + 120_{[0,0,1,0,0]} + 54_{[2,0,0,0,0]} + 1_{[0,0,0,0,0]}. \tag{4.34}$$

4.5 Remarks, conclusions and perspective work

- $4d$ $\mathcal{N} = 1$ Super-Yang-Mills theory with $SU(2)$ gauge group and 4 flavors has many duals whose superconformal indices are equal due to the Weyl group symmetry $W(E_7)$ [26]. This phenomenon was interpreted in [119] as a manifestation of a boundary $5d/4d$ model with the enhanced E_7 global symmetry group.
- We studied the dualities of four–dimensional $SU(2)$ supersymmetric QCD with three flavors and three–dimesional supersymmetric QCD with four quarks. Following the ideas of [26, 119] we found that a certain marginal deformation of the theory with three quark flavors can have the full E_6 flavor symmetry if coupled to a set of free 5d hypermultiplets.
- For the three–dimensional supersymmetric QCD with four quarks we provide the evidence of $SO(10)$ global symmetry.
- We also showed the connection between four-dimensional superconformal index and three-dimensional sphere partition function of the corresponding three–dimensional theory by performing dimensional reduction of the four–dimensional theory.
- It would be interesting to extend the global symmetry enhancement to full superconformal indices of $SP(2N)$ and $SU(N)$ gauge group theories with 4 flavors and some additional matter fields.
- Following these ideas one can also study the Weyl group symmetry transformations for elliptic hypergeometric integrals via global symmetry enhancement of a corresponding supersymmetric theory. In particular, it would be interesting to find a $4d$ $\mathcal{N} = 1$ theory with enhanced flavor symmetry E_8 and an elliptic hypergeometric integral with $W(E_8)$ symmetry transformation.

5 Basic hypergeometry of $3d$ dualities

In this section we study superconformal indices of three-dimensional $\mathcal{N} = 2$ supersymmetric dualities [38, 124, 125, 126]. As we mentioned before the superconformal index technique is one of the main tools for establishing and checking supersymmetric dualities.

Here we consider only confining theories [127], i.e. the theories whose infrared limit can be described in terms of gauge invariant composites (mesons and baryons) and without dual quarks. There are definitely more confining supersymmetric theories in three dimensions (for recent discussions see [101, 128]), we restrict our attention to samples of theories with $U(1)$ (supersymmetric electrodynamics) and $SU(2)$ (supersymmetric quantum chromodynamics) gauge symmetry. Note that similar works for $\mathcal{N} = 1$ supersymmetric gauge theories in four-dimensions were intensively studied in [25, 27, 28].

In our examples we give only the necessary input to compute the superconformal index and do not discuss other aspects of dual theories. As for many other dualities in physics, systematic proofs of supersymmetric dualities are absent and the superconformal index computations do not constitute a proof of the duality. There are other important arguments for three-dimensional supersymmetric dualities, i.e. study of superpotentials for interactions among chiral multiplets [104], brane construction (see e.g, [126, 129]), contact terms (see e.g., [130, 131]), etc.

The 't Hooft anomaly matching conditions [132] which played a central role in checking Seiberg dualities [32] for $\mathcal{N} = 1$ supersymmetric gauge theories become useless in three dimensions, since unlike four-dimensional gauge theories, in three dimensions there are no chiral anomalies. In three dimensions it is possible to have a classical Chern-Simons term which breaks parity and one can use the matching condition [36] for the parity anomaly [133, 134], however conditions for discrete anomalies are weaker than those for continuous anomalies.

It is worth to mention that there are other powerful methods very much in the spirit of the superconformal index such as study of partition functions on sphere [12, 135], squashed sphere [13, 15, 112, 1, 136], lens space [137, 138, 55, 139] and others.

In what follows, we omit the R–charges for chiral multiplets, since the superconformal indices of dual theories match for arbitrary assignment of the R–charge [30]. The correct R–charges for matter fields in the infrared fixed points can be obtained by the so-called Z–extremization procedure [14].

The matching of superconformal indices for dual pairs were studied mainly by expanding in terms of fugacities [30, 140, 141, 142] and only in a few works [29, 31, 3, 5] authors give rigorous proofs of the index identities.

5.1 $3d$ dualities via superconformal index technique

Example 1.

Let us consider a **theory A** and its low-energy description **theory B** which can be described purely in terms of composite gauge singlets.

- **Theory A**: Supersymmetric Quantum Chromodynamics with $SU(2)$ gauge group and $SU(6)$ flavor group, chiral multiplets in the fundamental representation of the gauge group and the flavor group, the vector multiplet in the adjoint representation of the gauge group. Note that in case of $SU(2)$ gauge theories the fundamental and antifundamental representations are equivalent, therefore we have $SU(6)$ flavor group rather than $SU(3) \times SU(3) \times U(1)$.

- **Theory B**: no gauge symmetry, fifteen chiral multiplets in the totally antisymmetric tensor representation of the flavor group.

This duality was considered in [123] where the authors presented the sphere partition functions for dual theories. It is analogues to the four-dimensional duality for similar theories [25] and can be obtained by dimensional reduction.

Using the group-theoretical data it is straightforward to compute explicitly the generalized superconformal indices, and due to the supersymmetric duality we find the following basic hypergeometric integral identity

$$\sum_{m\in\mathbb{Z}}\oint \frac{dz}{4\pi i z} q^{-|m|}(1-q^{|m|}z^2)(1-q^{|m|}z^{-2})\;(-q)^{\frac{1}{2}\sum_{i=1}^{6}(\frac{|n_i+m|}{2}+\frac{|n_i-m|}{2})} z^{-\sum_{i=1}^{6}(\frac{|n_i+m|}{2}-\frac{|n_i-m|}{2})}$$
$$\times\prod_{j=1}^{6} a_j^{-\frac{|n_j+m|}{2}-\frac{|n_j-m|}{2}} \frac{(q^{1+\frac{|n_j+m|}{2}}\frac{1}{a_j z}, q^{1+\frac{|n_j-m|}{2}}\frac{z}{a_j};q)_\infty}{(q^{\frac{|n_j+m|}{2}}a_j z, q^{\frac{|n_j-m|}{2}}\frac{a_j}{z};q)_\infty}$$
$$= q^{\frac{1}{2}\sum_{1\le j<k\le 6}\frac{|n_i+n_j|}{2}} \prod_{1\le j<k\le 6}(-a_j a_k)^{-\frac{|n_j+n_k|}{2}} \frac{(q^{1+\frac{|n_j+n_k|}{2}}a_j^{-1}a_k^{-1};q)_\infty}{(q^{\frac{|n_j+n_k|}{2}}a_j a_k;q)_\infty} \tag{5.1}$$

with the balancing conditions

$$\prod_{i=1}^{6} a_i = q \quad \text{and} \quad \sum_{i=1}^{6} n_i = 0\;. \tag{5.2}$$

This identity describes confinement without breaking of the chiral symmetry [127]. The left side of the expression (5.1) contains the contributions of twelve chirals and a vector multiplet, while the right hand side includes the contribution of fifteen chirals. From the fact that all the physical degrees of freedom of Theory B are gauge invariant there is no any integration on the right hand side.

The balancing conditions (5.2) are imposed by the effective superpotential and the theories described above are dual only in the presence of certain superpotentials. We refer the interested reader to [104] for more details related to the study of superpotentials for three-dimensional dualities.

For visual clarity, in (5.1) we used the absolute values of monopole charges as in the definition of the superconformal index. By eliminating the absolute values of the monopole charges one can rewrite the expression (5.1) as the integral identity presented in [143, 5] and formulate the following theorem:

Theorem. Let a_j be generic numbers and N_j integers satisfying $a_1 \cdots a_6 = q$ and

$N_1 + \cdots + N_6 = 0$. Then

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{6} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/a_j)_\infty}{(q^{N_j+m/2} a_j z, q^{N_j-m/2} a_j/z)_\infty} \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{6m}} \frac{dz}{2\pi i z}$$
$$= \frac{2}{\prod_{j=1}^{6} q^{\binom{N_j}{2}} a_j^{N_j}} \prod_{1\le j<k\le 6} \frac{(q/a_j a_k)_\infty}{(a_j a_k q^{N_j+N_k})_\infty}. \quad (5.3)$$

In order to get (5.1) from (5.3) one may use the following formula [57]

$$\prod_{i=0}^{\infty} \frac{1-q^{i+\frac{1}{2}|m|+1} z^{-1}}{1-q^{i+\frac{1}{2}|m|} z} = (-q^{\frac{1}{2}})^{-\frac{1}{2}(m+|m|)} z^{\frac{1}{2}(m+|m|)} \prod_{i=0}^{\infty} \frac{1-q^{i-\frac{1}{2}m+1} z^{-1}}{1-q^{i-\frac{1}{2}m} z} . \quad (5.4)$$

One also needs to use balancing conditions (5.2) when deals with such expression as

$$\prod_{1\le j<k\le 6} (a_j a_k)^{n_j+n_k} = a_1^{2n_1} (a_1 a_2 \ldots a_6)^{\frac{1}{2}n_1} \ldots a_6^{2n_6} (a_1 a_2 \ldots a_6)^{\frac{1}{2}n_6} \quad (5.5)$$
$$= \prod_{j=1} a_j^{2n_j} (a_1 a_2 \ldots a_6)^{\sum_i \frac{1}{2} n_i} \quad (5.6)$$
$$= \prod_{j=1} a_j^{2n_j} . \quad (5.7)$$

The most intriguing physical interpretation of the formula (5.1) stems from the role it plays as a star-triangle relation [54, 4] for a certain two-dimensional statistical model. We will discuss this subject in Chapter 6.

The integral identity (5.1) can be obtained by reduction [144, 55, 54] from the similar indentity for four-dimenional lens indices. In [54] such reduction was made in the context of integrable statistical models.

The issue of the $q \to 1$ limit of (5.1) was discussed in [54]. This limit also has an interpretation in terms of exactly solvable statistical models [145]. From the viewpoint of supersymmetric dualities such reduction [16] gives the equality of the sphere partition functions of dual two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories.

Example 2.

Our next example is again a supersymmetric quantum chromodynamics with a weakly coupled magnetic dual.

- **Theory A**: Supersymmetric Quantum Chromodynamics with $SU(2)$ gauge group and four flavors, chiral multiplets in the fundamental representation of the gauge group and the flavor group, the vector multiplet in the adjoint representation of the gauge group.

- **Theory B**: no gauge degrees of freedom, with six mesons and a singlet chiral field.

According to the supersymmetric duality we have the following integral identity for the generalized superconfromal indices

$$\begin{aligned} &\sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i z}(1-q^{|m|}z^2)(1-q^{|m|}z^{-2})\ (-q)^{\frac{1}{2}\sum_{i=1}^4(\frac{|n_i+m|}{2}+\frac{|n_i-m|}{2}+n_i)} \\ &\times\ z^{-\sum_{i=1}^4(\frac{|n_i+m|}{2}-\frac{|n_i-m|}{2})}\prod_{j=1}^4 a_j^{-\frac{|n_j+m|}{2}-\frac{|n_j-m|}{2}+n_j}\frac{(q^{1+\frac{|n_j+m|}{2}}\frac{1}{a_j z}, q^{1+\frac{|n_j-m|}{2}}\frac{z}{a_j};q)_\infty}{(q^{\frac{|n_j+m|}{2}}a_j z, q^{\frac{|n_j-m|}{2}}\frac{a_j}{z};q)_\infty} \\ &= (-q)^{\frac{1}{2}\sum_{1\le j<k\le 4}\frac{|n_j+n_k|}{2}-\sum_{i=1}^4 n_i} q^{-\frac{|\sum_{i=1}^4 n_i|}{4}}(a_1a_2a_3a_4)^{\frac{|\sum_{i=1}^4 n_i|-\sum_{i=1}^4 n_i}{2}} \\ &\times \frac{(q^{\frac{|\sum_{i=1}^4 n_i|}{2}}a_1a_2a_3a_4)_\infty}{(q^{1+\frac{|\sum_{i=1}^4 n_i|}{2}}/(a_1a_2a_3a_4))_\infty}\prod_{1\le j<k\le 4}(a_ja_k)^{-\frac{|n_i+n_j|+(n_i+n_j)}{2}}\frac{(q^{1+\frac{|n_j+n_k|}{2}}a_j^{-1}a_k^{-1};q)_\infty}{(q^{\frac{|n_j+n_k|}{2}}a_ja_k;q)_\infty}\ . \end{aligned} \tag{5.8}$$

The ordinary index of the theory A is considered in Chapter 4 in the context of global symmetry enhancement [2, 119].

Note that one can deform the dual theories from *Example 1* by adding mass terms for some of the quarks. After integrating out one flavor (massive modes) the theory with four flavors confines with chiral symmetry breaking [4, 146] if we keep a certain superpotential for the theory giving the balancing conditions similar to (5.2). Here the theory A has no superpotential and therefore we obtain the duality (5.8).

There is more general integral identity presented in [143]:

Theorem. For a_j and b_j generic,

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{4} \frac{(q^{1+m/2}/a_j z, q^{1-m/2}z/a_j)_\infty}{(q^{m/2}b_j z, q^{-m/2}b_j/z)_\infty} \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{4m}} \frac{dz}{2\pi i z}$$
$$= \frac{2(b_1b_2b_3b_4)_\infty}{(q/a_1a_2a_3a_4)_\infty} \prod_{1\le j<k\le 4} \frac{(q/a_ja_k)_\infty}{(b_jb_k)_\infty}. \quad (5.9)$$

One can obtain the integral identity (**??**) from (5.9) by choosing the fugacities $b_i = q^{n_i}a_i$ and using the formula (5.4).

Example 3

In contrast to four-dimensions, there exist supersymmetric dualities for abelian gauge theories in three dimensions. For details of these dualities see e.g., [147, 38]. Below we consider two examples of such dualities.

- **Theory A**: $d = 3$ $\mathcal{N} = 2$ supersymmetric electrodynamics with $U(1)$ gauge symmetry and six chiral multiplets, half of them transforming in the fundamental representation of the gauge group and another half transforming in the anti-fundamental representation.

- **Theory B**: no gauge degrees of freedom, nine gauge invariant "mesons" transforming in the fundamental representation of the flavor group.

The supersymmetric duality leads to the following identity for the generalized superconformal indices

$$\sum_{m\in Z} \oint \frac{dz}{2\pi i z} (-q)^{\frac{1}{2}\sum_{i=1}^{3}(\frac{|m_i+m|}{2}+\frac{|n_i-m|}{2})} z^{-\sum_{i=1}^{3}(\frac{|m_i+m|}{2}-\frac{|n_i-m|}{2})}$$
$$\times \prod_{i=1}^{3} a_i^{-\frac{|m_i+m|}{2}} b_i^{-\frac{|n_i-m|}{2}} \frac{(q^{1+\frac{|m_i+m|}{2}}(a_iz)^{-1};q)_\infty}{(q^{\frac{|m_i+m|}{2}}a_iz;q)_\infty} \frac{(q^{1+\frac{|n_i-m|}{2}}z/b_i;q)_\infty}{q^{\frac{|n_i-m|}{2}}(b_i/z;q)_\infty}$$
$$= (-q)^{\frac{1}{2}\sum_{i,j=1}^{3}\frac{|m_i+n_j|}{2}} \prod_{i,j=1}^{3} (a_ib_j)^{-\frac{|m_i+n_j|}{2}} \frac{(q^{1+\frac{|m_i+n_j|}{2}}(a_ib_j)^{-1};q)_\infty}{(q^{\frac{|m_i+n_j|}{2}}a_ib_j;q)_\infty}. \quad (5.10)$$

where the fugacities a_i and b_i stand for the flavor symmetry $SU(3)\times SU(3)$, z is the fugacity for the $U(1)$ gauge group and the balancing conditions are

$$\prod_{i=1}^{3} a_i = \prod_{i=1}^{3} b_i = q^{\frac{1}{2}} \text{ and } \sum_{i=1}^{3} n_i = \sum_{i=1}^{3} m_i = 0 \,. \tag{5.11}$$

In [3] we showed that the ordinary superconformal indices, obtained by setting $m_i = n_i = 0$ for $i = 1, \ldots, 3$, match and in [70] we presented the identity (5.27) without a proof.

By eliminating the absolute values of the monopole charges one can rewrite the expression (5.1) as the integral identity presented in [143] and formulate the following theorem

Theorem. Let a_j, b_j be generic numbers and M_j, N_j integers satisfying $a_1a_2a_3 = b_1b_2b_3 = q^{1/2}$ and $M_1 + M_2 + M_3 = N_1 + N_2 + N_3 = 0$. Then

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{3} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/b_j)_\infty}{(q^{M_j+m/2} a_j z, q^{N_j-m/2} b_j/z)_\infty} \frac{(-1)^m}{z^{3m}} \frac{dz}{2\pi i z}$$
$$= \frac{1}{\prod_{j=1}^{3} q^{\binom{M_j}{2}+\binom{N_j}{2}} a_j^{M_j} b_j^{N_j}} \prod_{j,k=1}^{3} \frac{(q/a_j b_k)_\infty}{(a_j b_k q^{M_j+N_k})_\infty}. \tag{5.12}$$

The identity (5.10) takes the form of (5.12) by using the formula (5.4).

Example 4.

Let us consider another example of abelian duality, namely the well-known XYZ/SQED mirror symmetry [126, 37, 36]

- **Theory A:** $\mathcal{N} = 2$ supersymmetric quantum electrodynamics, with a single $U(1)$ vector multiplet and two chiral multiplets charged oppositely under the gauge group.

- **Theory B:** free Wess–Zumino theory with three chiral multiplets. This theory often is called the XYZ model in the literature.

In this example we wish to turn on contribution to the generalized superconformal index of the topological symmetry $U(1)_J$ which is not explicit in the Lagrangian. This hidden symmetry is generated by the current

$$J^{\mu} \;=\; \varepsilon^{\mu\nu\rho}F_{\nu\rho}\ . \tag{5.13}$$

The current J^{μ} is topologically conserved due to the Bianchi identity.

In this case we have a special duality called mirror symmetry which exchanges the Coulomb branch of a theory with the Higgs branch of its mirror dual and vice versa. The duality implies the following mathematical identity [31]

$$\begin{aligned}&\sum_{s\in\mathbb{Z}}\oint\frac{dz}{2\pi iz}z^{n}w^{s}(q^{\frac14}z^{\pm1}\alpha^{-1})^{\frac{|s\mp m|}{2}}\frac{(z^{\pm1}\alpha^{-1}q^{\frac{|s\mp m|}{2}+\frac34};q)_\infty}{(z^{\pm1}\alpha q^{\frac{|s\pm m|}{2}+\frac14};q)_\infty}\\ &=(q^{\frac12}\alpha w^{\pm1})^{\frac{|m\mp n|}{2}}(\alpha)^{-2|m|}\frac{(\alpha w^{\pm1}q^{\frac{|m\mp n|}{2}+\frac34};q)_\infty}{(\alpha^{-1}w^{\pm1}q^{\frac{|m\pm n|}{2}+\frac14};q)_\infty}\frac{(\alpha^{-2}q^{|m|+\frac12};q)_\infty}{(\alpha^{2}q^{|\tilde m|+\frac12};q)_\infty}\end{aligned} \tag{5.14}$$

where the fugacity α and the monopole charge m denote the parameters for the axial $U(1)_A$ symmetry, ω and n denote the parameters for the topological $U(1)_J$ symmetry and the discrete parameter s stands for the magnetic charge corresponding to the $U(1)$ gauge group. Here we explicitly write the R-charges of chiral multiplets. Due to the permutation symmetry of the superpotential $W=\tilde{q}Sq$ for the theory B, where $q,\tilde{q}$, S are three chiral multiplets of the theory, one can fix the R-charges. The identity (5.14) was proven only for the case $m=0$ in [31].

The similar identity for ordinary superconformal indices, obtained by setting $n=m=0$ was presented in [30, 29], proven in [29] and interpreted as an integral pentagon relation in [3].

One can also consider this duality as a mirror symmetry between $\mathcal{N}=4$ supersymmetric electrodynamics with a single flavor and its dual theory with a free hypermultiplet. Then the equality (5.14) takes the following form

$$\begin{aligned}&(\alpha)^{2|m|}\frac{(\alpha^{2}q^{|m|+\frac12};q)_\infty}{(\alpha^{-2}q^{|\tilde m|+\frac12};q)_\infty}\sum_{s\in\mathbb{Z}}\oint\frac{dz}{2\pi iz}z^{n}w^{s}(q^{\frac14}z^{\pm1}\alpha^{-1})^{\frac{|s\mp m|}{2}}\frac{(z^{\pm1}\alpha^{-1}q^{\frac{|s\mp m|}{2}+\frac34};q)_\infty}{(z^{\pm1}\alpha q^{\frac{|s\pm m|}{2}+\frac14};q)_\infty}\\ &\qquad=(q^{\frac12}\alpha w^{\pm1})^{\frac{|m\mp n|}{2}}\frac{(\alpha w^{\pm1}q^{\frac{|m\mp n|}{2}+\frac34};q)_\infty}{(\alpha^{-1}w^{\pm1}q^{\frac{|m\pm n|}{2}+\frac14};q)_\infty}\ .\end{aligned} \tag{5.15}$$

5.2 Integral pentagon identities

Since a three-dimensional superconformal index can be expressed in terms of basic hypergeometric integrals [148, 65], by studying supersymmetric dualities one can get new identities for this type of special functions [5, 3, 70, 29, 31]. In this section we consider a special type of such identities, namely five term relations or the so-called pentagon identities which can be interpreted as the 2–3 Pachner move [149, 150] for triangulated 3-manifolds. The pentagon relations are interesting from different aspects, see for instance [151, 3, 152, 85, 153, 154, 155, 156, 157, 158]. Here we present some examples of integral pentagon relations relevant to the three-dimensional superconformal index.

5.2.1 Pentagon identity for hyperbolic hypergeometric functions

First we discuss some aspects of the paper [85] which are useful for the considerations in the next subsections. Let us consider the beta integral [159, 92]

$$\frac{(p;p)_\infty(q;q)_\infty}{2}\int_{\mathbb{T}}\frac{\prod_{i=1}^{6}\Gamma(t_iz;p,q)\Gamma(t_iz^{-1};p,q)}{\Gamma(z^2;p,q)\Gamma(z^{-2};p,q)}\frac{dz}{2\pi \mathrm{i} z}=\prod_{1\leq i<j\leq 6}\Gamma(t_it_j;p,q), \tag{5.16}$$

where $t_j,\ j=1,\ldots,6$ are complex parameters with the balancing condition $\prod_{j=1}^{6}t_j=pq$. This is the integral identity we discussed in Chapter 4. From the physical point of view the integral on the left hand side of the expression (6.1) is the index of the $4d$ $\mathcal{N}=1$ electric theory with $SU(2)$ gauge group and $N_F=3$ flavors, chiral scalar multiplets in the fundamental representation of the flavor group, while the expression on the right side is the index for the dual magnetic theory with chirals in the antisymmetric tensor representation of the flavor group.

Using the reduction procedure discussed in Chapter 3 (for more details, see [116]), it is straightforward to derive the integral identity for hyperbolic hypergeometric fucntions [85, 49]

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{i=1}^{3}\gamma^{(2)}(a_i-u;\omega_1,\omega_2)\gamma^{(2)}(b_i+u;\omega_1,\omega_2)\frac{du}{\mathrm{i}\sqrt{\omega_1\omega_2}}=\prod_{i,j=1}^{3}\gamma^{(2)}(a_i+b_j;\omega_1,\omega_2), \tag{5.17}$$

with the balancing condition $\sum_{i=1}^{3}(a_i+b_i)=\omega_1+\omega_2$.

Let us introduce the following function

$$\mathcal{B}(x,y)=\frac{\gamma^{(2)}(x;\omega_1,\omega_2)\gamma^{(2)}(y;\omega_1,\omega_2)}{\gamma^{(2)}(x+y;\omega_1,\omega_2)}. \tag{5.18}$$

Then from the expression (5.17) one can easily see that the function $\mathcal{B}(x,y)$ satisfies the pentagon identity [85]

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{i=1}^{3}\mathcal{B}(a_i-u,b_i+u)\frac{du}{\mathrm{i}\sqrt{\omega_1\omega_2}}=\mathcal{B}(a_2+b_1,a_3+b_2)\mathcal{B}(a_1+b_2,a_3+b_1). \tag{5.19}$$

5.2.2 Pentagon identities for basic hypergeometric functions

Our main interest is the five-term relation for the superconformal index. Such relations are interesting from the following point of view. There is a recently proposed relation called $3d-3d$ correspondence [57, 62] (see also [160, 161, 162, 163]) in similar spirit of the AGT correspondence [164]. This correspondence translates the ideal triangulation of the 3-manifold into mirror symmetry for three-dimensional supersymmetric theories. Independence of the corresponding 3-manifold invariant on the choice of triangulation corresponds to the equality of superconformal indices of mirror dual theories [57]. In this context the identity (5.29) encodes a 3–2 Pachner move for 3-manifolds.

One can express the superconformal index via the so-called tetrahedron index [57]

$$\mathcal{I}_q[m,z] \;=\; \prod_{i=0}^{\infty}\frac{1-q^{i-\frac{1}{2}m+1}z^{-1}}{1-q^{i-\frac{1}{2}m}z}, \quad \text{with } |q|<1 \text{ and } m\in Z. \tag{5.20}$$

In this subsection we will mainly express the index in terms of this function.

Example 1.

Let us consider the $d=3$ $\mathcal{N}=2$ supersymmetric quantum electrodynamics with $U(1)$ gauge group and one flavor. The superconformal index of the theory is [30, 29, 3]

$$I_e \;=\; \sum_{m\in Z}\oint\frac{dz}{2\pi iz}z^{-m}\,\mathcal{I}_q[m;q^{1/6}z^{-1}]\,\mathcal{I}_q[-m;q^{1/6}z]\;, \tag{5.21}$$

where the integration is over the unit circle with positive orientation. For simplicity we switched off[1] the topological symmetry $U(1)_J$.

The dual theory is the free Wess-Zumino theory[2] [38, 126, 36] with three chiral multiplets $q, \tilde{q}$, S interacting through the superpotential[3] $W = \tilde{q}Sq$. The index of this theory has a simpler form, since we do not need to integrate over the gauge group,

$$I_m = \Big(\mathcal{I}_q[0; q^{1/3}]\Big)^3 . \tag{5.22}$$

As we have already discussed in this Chapter, these two theories are dual under the mirror symmetry, i.e. under exchange of the Higgs and the Coulomb branches[4]. The mirror duality leads to the following integral pentagon identity [3, 70]

$$\sum_{m\in Z} \oint \frac{dz}{2\pi i z} z^{-m}\, \mathcal{I}_q[m; q^{1/6}z^{-1}]\, \mathcal{I}_q[-m; q^{1/6}z] = \Big(\mathcal{I}_q[0; q^{1/3}]\Big)^3 . \tag{5.23}$$

This is the first example of a pentagon identity for the tetrahedron index. The mathematical proof of the identity can be found in [29].

The tetrahedron index can be written in the following form:

$$\mathcal{I}_q[m, z] = \sum_{e\in Z} \mathcal{I}(m, e) z^e \tag{5.24}$$

where

$$\mathcal{I}(m, e) = \sum_{n=\frac{1}{2}(|e|-e)}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)-(n+\frac{1}{2}e)m}}{(q)_n (q)_{n+e}} . \tag{5.25}$$

This index was introduced in [57]. This function is also interesting from a mathematical point of view, see e.g. [165, 166]. The index $\mathcal{I}(m, e)$ obeys the following pentagon

[1]See, for instance, [29, 31]. We consider the influence of the topological $U(1)_J$ symmetry to the index in the next chapter, where we define the so-called generalized superconformal index.

[2]In the literature this theory sometimes is called the XYZ model.

[3]The permutation symmetry of the superpotential fixes the R-charges, but one can write the index for more general R-charge like in [30].

[4]In three-dimenisonal supersymmetric theories the Coulomb and the Higgs branch are both hyper-Kähler manifolds.

identity [57]

$$\mathcal{I}(m_1-e_2,e_1)\mathcal{I}(m_2-e_1,e_2) = \sum_{e_3} q^{e_3}\mathcal{I}(m_1,e_1+e_3)\mathcal{I}(m_2,e_2+e_3)\mathcal{I}(m_1+m_2,e_3). \quad (5.26)$$

A proof of the identity (5.26) is given in the Appendix of [165]. This pentagon relation is a counterpart of the integral pentagon identity (5.23). In order to distinguish between this type of relation and the identity of the form (5.23) we use the terminology "the *integral* pentagon identity" for the latter one.

The analogue of the pentagon identity (5.23) in terms of the generalized superconformal index is the following pentagon identity

$$\sum_{s\in Z}\int\frac{dz}{2\pi iz}(-1)^{m-\frac{|s-m|+|s+m|}{2}}z^{2n-s}\omega^m\alpha^{-m}q^{\frac{1}{4}m}\mathcal{I}_q[s+m;q^{\frac{1}{4}}\alpha z^{-1}]\mathcal{I}_q[s-m;\alpha zq^{\frac{1}{4}}]$$
$$=(-1)^{(n-\frac{|m-n|+|m+n|}{2})}\omega^{-m}\alpha^{n+2m}q^{\frac{1}{4}n}\mathcal{I}_q[m;q^{\frac{1}{4}}\alpha^{-1}\omega^{-1}]\mathcal{I}_q[-m;q^{\frac{1}{4}}\alpha^{-1}\omega]\mathcal{I}_q[2m;q^{\frac{1}{2}}\alpha^2],$$

where we switched on the background gauge field coupled to the topological $U_J(1)$ global symmetry. Here α and m denote the parameters for the axial $U(1)_A$ symmetry, ω and n denote the parameters for the topological $U_J(1)$ symmetry and the discrete parameter s stands for magnetic charge.

Example 2.

For another example, we consider the duality mentioned in Example 3 of previous section. Namely, the electric theory is the $d=3$ $\mathcal{N}=2$ superconformal field theory with $U(1)$ gauge symmetry and six chiral multiplets, half of them transforming in the fundamental representation of the gauge group and another half transforming in the anti-fundamental representation. Its mirror dual is a theory with nine chirals and without gauge degrees of freedom (the gauge symmetry is completely broken). The supersymmetric duality leads to the following identity

$$\sum_{m\in Z}\oint\frac{dz}{2\pi iz}\,(-z)^{-3m}\prod_{i=1}^{3}\mathcal{I}_q[-m,q^{\frac{1}{6}}\xi_i z]\,\mathcal{I}_q[m,q^{\frac{1}{6}}\eta_i z^{-1}]=\prod_{i,j=1}^{3}\mathcal{I}_q[0,q^{\frac{1}{3}}\xi_i\eta_j]\,, \quad (5.27)$$

where the fugacities ξ_i and η_i stand for the flavor symmetry $SU(3)\times SU(3)$ and the balancing condition is $\prod_{i=1}^3 \xi_i = \prod_{i=1}^3 \eta_i = 1$. Note that we dropped the topological symmetry $U(1)_J$. The identity (5.27) was introduced in [3], to where we refer the reader for the details and the mathematical proof of it.

Following [3] we introduce a new function

$$\mathcal{B}[m;a,b] = \frac{\mathcal{I}_q[m,a]\,\mathcal{I}_q[-m,b]}{\mathcal{I}_q[0,ab]}\ , \tag{5.28}$$

and rewrite the equality (5.27) in terms of this function. The final result is a new integral pentagon identity in terms of $\mathcal{B}[m;a,b]$ functions

$$\sum_{m\in Z}\oint \frac{dz}{2\pi i z}\,(-z)^{-3m}\prod_{i=1}^{3}\mathcal{B}[m;\xi_i z^{-1},\eta_i z] = \mathcal{B}[0;\xi_1\eta_2,\xi_3\eta_1]\ \mathcal{B}[0;\xi_2\eta_1,\xi_3\eta_2] \tag{5.29}$$

where we have redefined the flavor fugacities $\xi_i \to q^{-1/6}\xi_i$ and $\eta_i \to q^{-1/6}\eta_i$ and the new balancing condition is $\prod_{i=1}^3 \xi_i = \prod_{i=1}^3 \eta_i = q$.

We can write the analogue of the pentagon identity (5.28) in terms of the generalized superconformal index. We have already presented the integral identity for generalized superconformal indices for this duality in previous section. The result is

$$\begin{aligned}\sum_{m\in Z}\oint \frac{dz}{2\pi i z}&(-q)^{\frac{1}{2}\sum_{i=1}^3(\frac{|m_i+m|}{2}+\frac{|n_i-m|}{2})}z^{-\sum_{i=1}^3(\frac{|m_i+m|}{2}-\frac{|n_i-m|}{2})}\\ &\times\prod_{i=1}^{3} a_i^{-\frac{|m_i+m|}{2}}b_i^{-\frac{|n_i-m|}{2}}\frac{(q^{1+\frac{|m_i+m|}{2}}(a_i z)^{-1};q)_\infty}{(q^{\frac{|m_i+m|}{2}}a_i z;q)_\infty}\frac{(q^{1+\frac{|n_i-m|}{2}}z/b_i;q)_\infty}{q^{\frac{|n_i-m|}{2}}(b_i/z;q)_\infty}\\ &=(-q)^{\frac{1}{2}\sum_{i,j=1}^3\frac{|m_i+n_j|}{2}}\prod_{i,j=1}^{3}(a_i b_j)^{-\frac{|m_i+n_j|}{2}}\frac{(q^{1+\frac{|m_i+n_j|}{2}}(a_i b_j)^{-1};q)_\infty}{(q^{\frac{|m_i+n_j|}{2}}a_i b_j;q)_\infty}\ .\end{aligned} \tag{5.30}$$

Following [3] we introduce a new function

$$\begin{aligned}\mathcal{B}_m[a,n;b,m] &= (-q)^{\frac{|n|}{4}+\frac{|m|}{4}-\frac{|n+m|}{4}}a^{-\frac{|n|}{2}}b^{-\frac{|m|}{2}}(ab)^{\frac{|n+m|}{2}}\\ &\times\frac{(q^{1+\frac{|n|}{2}}a^{-1};q)_\infty}{(q^{\frac{|n|}{2}}a;q)_\infty}\frac{(q^{1+\frac{|m|}{2}}b^{-1};q)_\infty}{(q^{\frac{|m|}{2}}b;q)_\infty}\frac{(q^{\frac{|n+m|}{2}}ab;q)_\infty}{(q^{1+\frac{|n+m|}{2}}(ab)^{-1};q)_\infty}\ ,\end{aligned} \tag{5.31}$$

and rewrite the equality (5.27) in terms of this function. We obtain the following integral pentagon identity in terms of $\mathcal{B}$ functions

$$\sum_{m\in Z}\oint\frac{dz}{2\pi iz}\prod_{i=1}^{3}\mathcal{B}[a_iz,n_i+m;b_iz^{-1},m_i-m]$$
$$=\mathcal{B}[a_1b_2,n_1+m_2;a_3b_1;n_3+m_1]\,\mathcal{B}[a_2b_1,n_2+m_1;a_3b_2,n_3+m_2]\,, \tag{5.32}$$

with the balancing conditions (5.11).

5.3 Remarks, conclusions and perspective work

- Similarly to four-dimensional dualities, equality of the superconformal indices for dual theories in three dimensions leads to new non-trivial integral identities [3, 29]. Here we presented novel integral identities for basic hypergeometric integrals. More concretely, we studied the generalized superconformal index of s-confining theories in three dimensions that has a form of basic hypergeometric integral. This kind of result is crucially important for better understanding of the structure of three-dimensional supersymmetric dualities. For the most part of identities, the corresponding dualities are known in the literature but the checks of these dualities using the superconformal index technique is new. The proof of the integral identities will be presented in [5].
- We presented the so-called pentagon identities. Such identities are especially interesting from the geometrical point of view. Geometrically, the interpretation of the pentagon relation is the $3-2$ Pachner move, which relates different decompositions of a polyhedron with five ideal vertices into ideal tetrahedra.
- The results presented in this chapter rely on some physics computations. They are meant to motivate the mathematical constructions to be developed later [5, 167].

6 Integrability

In this chapter, we describe a connection between integrable statistical models and supersymmetric dualities. The investigation is restricted to two-dimensional spin models from statistical physics side and to three-dimensional supersymmetric gauge theories from other side of the correspondence. This correspondence leads to many new results.

Special functions [168] are key mathematical objects in the construction of new integrable models of lattice statistical physics and quantum field theory, see e.g. [154, 169, 170, 171, 172, 173, 174, 175, 176, 56, 58, 49, 54, 145, 177, 178, 179]. Quantum integrable systems and related Yang-Baxter equations and quantum algebras [180, 181, 182, 183] have been investigated for a long time in relation to plain hypergeometric functions, their q-analogues and elliptic functions. Fairly recently the third class of transcendental functions of hypergeometric type called elliptic hypergeometric functions has been discovered [159, 92], which strongly extended the database of classical special functions. The cornerstone of the latter functions is the following elliptic beta integral

Theorem (Spiridonov [159]). Let $t_1, \ldots, t_6, p, q \in \mathbb{C}$ with $|t_1|, \ldots, |t_6|, |p|, |q| < 1$ and $\prod_{j=1}^{6} t_j = pq$. Then

$$\frac{(p;p)_\infty (q;q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{i=1}^{6} \Gamma(t_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi \mathrm{i} z} = \prod_{1 \le i < j \le 6} \Gamma(t_i t_j; p, q), \tag{6.1}$$

where $\Gamma(z;p,q) = (pqz^{-1}; p,q)_\infty / (z;p,q)_\infty$, $(z;p,q)_\infty = \prod_{j,k=0}^{\infty} (1 - z p^j q^k)$, is the elliptic gamma function and $\mathbb{T}$ is the unit circle of positive orientation.

The first physical application of elliptic hypergeometric integrals consisted in the interpretation of some of them as wave functions or normalizations of wave functions in particular quantum mechanical problems [92]. The most important known application of identity (6.1) was found in [25] in the context of $\mathcal{N} = 1$ supersymmetric field theories within which it has the meaning of the equality of superconformal indices [22, 23, 24] in Seiberg dual theories [184, 32]. Indeed, the integral on the left-hand side of the equality (6.1) is the superconformal index of the $4d$ $\mathcal{N} = 1$ supersymmetric gauge theory with $SU(2)$ gauge group and $N_F = 6$ flavors, chiral scalar multiplets in the fundamental representation of the flavor group $SU(6)$, while the expression on the right side is the superconformal index for the dual theory without gauge degrees of freedom and the chiral fields in the 15-dimensional totally antisymmetric tensor representation of the same flavor group. In other words, the elliptic beta integral is the manifestation of the s-confinement phenomenon in gauge theories [184].

We present a new solution of the star-triangle relation and other forms of Yang-Baxter equation in terms of the basic hypergeometric identity presented in [143, 5]. We relate the Yang-Baxter equations to three-dimensional supersymmetric dualities. The new solution corresponds to the generalized superconformal index of certain $3d$ $\mathcal{N} = 2$ superconformal gauge theory having a distinguished form due to the contribution of monopoles [30, 31, 79, 29]. Detailed presentation of this correspondence is given in the last section.

6.1 Two-dimensional integrable lattice models

There have been many developments in the statistical mechanics of lattice models since Onsager‚Äôs famous solution [185] of the Ising model in 1944. Some two-dimensional examples of integrable lattice models are

- Hard-hexagon model [186]
- Fateev-Zamodchikov model [187] (the case N=2 gives the Isig model)
- Kashiwara-Miwa model [188]
- chiral Potts model [189, 190, 191]
- Faddeev-Volkov model [192, 193]

Recently Bazhanov and Sergeev [179, 176, 175] introduced an integrable spin model on a planar lattice, which generalizes all integrable lattice models mentioned above. Later

Spiridonov [49] interpreted the Bazhanov-Sergeev model in terms of four-dimensional $\mathcal{N} = 1$ quiver gauge theories. The relation to supersymmetric gauge theory was further developed by Yamazaki [50, 56, 144], who constructed the most general solution [56] containing the Bazhanov-Sergeev model as a special case.

6.2 Star-triangle relation and 3d index

6.2.1 Notation and definitions

For $q, z \in \mathbb{C}$, $|q| < 1$, we define the infinite q-product

$$(z;q)_\infty \; := \; \prod_{k=0}^{\infty}(1 - zq^k). \tag{6.2}$$

The (normalized) q-gamma function of Jackson has the form [194, 65]

$$\Gamma(z;q) \; := \frac{1}{(z;q)_\infty}. \tag{6.3}$$

Denote

$$(a,b;q)_\infty := (a;q)_\infty(b;q)_\infty, \quad (ax^{\pm 1};q)_\infty := (ax;q)_\infty(ax^{-1};q)_\infty \tag{6.4}$$

with a similar convention for other generalized gamma functions in (6.1) and other relations below.

We need the following q-hypergeometric identity.

Theorem. (Rosengren [143, 5]) Let $a_1, \ldots, a_6, q \in \mathbb{C}$ and integers $N_1, \ldots, N_6 \in \mathbb{Z}$, satisfy the constraints $|a_j|, |q| < 1$, and $\prod_{j=1}^{6} a_j = q$, $\sum_{j=1}^{6} N_j = 0$. Then

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \prod_{j=1}^{6} \frac{(q^{1+\frac{m}{2}}\frac{1}{a_j z}, q^{1-\frac{m}{2}}\frac{z}{a_j};q)_\infty}{(q^{N_j+\frac{m}{2}}a_j z, q^{N_j-\frac{m}{2}}\frac{a_j}{z};q)_\infty} \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{6m}} \frac{dz}{2\pi i z} \\ = \frac{2}{\prod_{j=1}^{6} q^{\binom{N_j}{2}} a_j^{N_j}} \prod_{1\le j<k\le 6} \frac{(q a_j^{-1} a_k^{-1};q)_\infty}{(q^{N_j+N_k} a_j a_k;q)_\infty},\end{aligned} \tag{6.5}$$

where $\mathbb{T}$ is the unit circle of positive orientation.

This is a q-beta sum-integral associated with $3d$ superconformal indices (see Chapter 5). The proof of the theorem is presented in [5].

Let us define the following generalized q-gamma function as a combination of four q-gamma functions and z^m and a^m:

$$\Gamma_q(a,n;z,m) := \frac{(q^{1+\frac{n+m}{2}}\frac{1}{az}, q^{1+\frac{n-m}{2}}\frac{z}{a}; q)_\infty}{a^n z^m (q^{\frac{n+m}{2}}az, q^{\frac{n-m}{2}}\frac{a}{z}; q)_\infty}, \tag{6.6}$$

where $a, z \in \mathbb{C}$ and $n, m \in \mathbb{Z}$.

Lemma. One has the following inversion relation:

$$\Gamma_q(a,n;z,m)\Gamma_q(b,-n;z,m) = 1, \quad ab = q. \tag{6.7}$$

Proof. Consider the explicit form of the indicated product of Γ_q-functions after the substitution $b = q/a$:

$$\begin{aligned} &\Gamma_q(a,n;z,m)\Gamma_q(\tfrac{q}{a},-n;z,m) \\ &\qquad = \frac{q^n}{z^{2m}a^{2n}} \frac{(q^{1+\frac{n+m}{2}}\frac{1}{az}, q^{1+\frac{n-m}{2}}\frac{z}{a}, q^{\frac{-n+m}{2}}\frac{a}{z}, q^{\frac{-n-m}{2}}az; q)_\infty}{(q^{\frac{n+m}{2}}az, q^{\frac{n-m}{2}}\frac{a}{z}, q^{1+\frac{-n+m}{2}}\frac{z}{a}, q^{1+\frac{-n-m}{2}}\frac{1}{az}; q)_\infty}. \end{aligned} \tag{6.8}$$

Using the relation $(a;q)_\infty = (1-a)(aq;q)_\infty$, for $n > m > 0$ we can rewrite this expression as

$$\frac{q^n}{z^{2m}a^{2n}} \prod_{i=0}^{n+m-1} \frac{1 - azq^{i-(m+n)/2}}{1 - a^{-1}z^{-1}q^{i+1-(m+n)/2}} \prod_{j=0}^{n-m-1} \frac{1 - a^{-1}zq^{i+1+(n-m)/2}}{1 - az^{-1}q^{i+(n-m)/2}} = 1. \tag{6.9}$$

For other possible values of the integers n and m one gets the same result due to the properties of q-Pochhammer symbols.

Now we can rewrite the above q-beta sum-integral in the following compact form.

$$\sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \prod_{j=1}^{6} \Gamma_q(a_j, n_j; z, m)[d_m z] = \frac{1}{\prod_{j=1}^{6} a_j^{2n_j}} \prod_{1\le j<k\le 6} \frac{(q^{1+\frac{n_j+n_k}{2}} a_j^{-1} a_k^{-1}; q)_\infty}{(q^{\frac{n_j+n_k}{2}} a_j a_k; q)_\infty}, \tag{6.10}$$

where $\prod_{j=1}^{6} a_j = q$, $\sum_{j=1}^{6} n_j = 0$, and

$$[d_m z] := \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m} \frac{dz}{4\pi i z}, \qquad [d_m z] = [d_{-m} z].$$

6.2.2 Bailey lemma and the star-triangle relation

Let us define the D-function as a product of two generalized q–gamma functions

$$D(t; a, n; z, m) := \Gamma_q(q^{\frac{1}{2}} t^{-1} a, n; z, m)\Gamma_q(q^{\frac{1}{2}} t^{-1} a^{-1}, -n; z, m). \tag{6.11}$$

It is easy to show that the function D satisfies the following properties

$$D(t^{-1}; a, n; z, m) = \frac{1}{D(t; a, n; z, m)} \tag{6.12}$$

and

$$D(1; a, n; z, m) = 1. \tag{6.13}$$

Let us introduce the integral-sum operator of the following form

$$M(t)_{x,n;z,m} f_m(z) := \frac{(t^2; q)}{(qt^{-2}; q)} \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} [d_m z]\ \Gamma_q(tx^{\pm 1}, \pm n; z, m) f_m(z), \tag{6.14}$$

where we used the following short-hands

$$\begin{aligned} \Gamma_q(tx^{\pm 1}, \pm n; z, m) &:= \Gamma_q(tx, n; z, m)\Gamma_q(tx^{-1}, -n; z, m) \\ &= D(q^{1/2} t^{-1}; x, n; z, m) \end{aligned} \tag{6.15}$$

and $f_m(z)$ is an arbitrary sequence of holomorphic functions.

We note that the following permutational symmetries hold true

$$\Gamma_q(tx^{\pm 1}, \pm n; z, m) = \Gamma_q(tz^{\pm 1}, \pm m; x, n), \tag{6.16}$$

$$D(t; a, n; z, m) = D(t; z, m; a, n). \tag{6.17}$$

Following the original integral generalization [195, 92] of the Bailey chains techniques [168], we introduce the notion of Bailey pairs in the present context.

Definition. We say that two sequences of functions $\alpha_m(z;t)$ and $\beta_m(z;t)$, of complex variables z and t and discrete variable m form a Bailey pair with respect to the parameter t if they are related by the integral-sum transform (6.14),

$$\beta_n(x;t) = M(t)_{x,n;z,m}\alpha_m(z;t). \tag{6.18}$$

Here we assume that $|tx|, |t/x| < 1$ and other regions of parameters are reached by the analytical continuation.

Bailey lemma. Suppose we have a particular Bailey pair $\alpha_k(x;t), \beta_k(x;t)$ with respect to the parameter t. Then the sequences of functions

$$\alpha'_k(x;st) = D(s;y,l;x,k)\alpha_k(x;t), \tag{6.19}$$

$$\beta'_k(x;st) = D(t^{-1};y,l;x,k)M(s)_{x,k;z,m}D(st;y,l;z,m)\beta_m(z;t), \tag{6.20}$$

where $s, y \in \mathbb{C}, l \in \mathbb{Z}$ are arbitrary new parameters, form a Bailey pair with respect to the parameter st.

Proof. Let us substitute primed sequences into the relation

$$\beta'_k(w;st) = M(st)_{w,k;x,j}\alpha'_j(x;st) \tag{6.21}$$

and use the inversion $D(t^{-1};y,l;x,k) = 1/D(t;y,l;x,k)$. This yields the operator identity

$$M(s)_{w,k;z,m}\, D(st;y,l;z,m)M(t)_{z,m;x,j} = D(t;y,l;w,k)M(st)_{w,k;x,j}D(s;y,l;x,j) \tag{6.22}$$

known as the star-triangle relation. It is a straightforward consequence of the Rosengren q-beta sum-integral. First we compute the expression on the left-hand side of

(6.22)

$$\frac{(s^2,t^2;q)}{(qs^{-2},qt^{-2};q)}\sum_{m\in\mathbb{Z}}\int_{\mathbb{T}}[d_mz]\;\Gamma_q(sw^{\pm1},\pm k;z,m)\Gamma_q(q^{\frac12}(st)^{-1}y^{\pm1},\pm l;z,m)$$
$$\times\sum_{j\in\mathbb{Z}}\int_{\mathbb{T}}[d_jx]\times\Gamma_q(tz^{\pm1},\pm m;x,j)$$
$$=\frac{(s^2,t^2;q)}{(qs^{-2},qt^{-2};q)}\sum_{j\in\mathbb{Z}}\int_{\mathbb{T}}[d_jx]\sum_{m\in\mathbb{Z}}\int_{\mathbb{T}}\prod_{j=1}^{6}\Gamma_q(a_j,n_j;z,m)[d_mz], \tag{6.23}$$

where we used the permutational symmetry of Γ_q-function and have denoted

$$a_1=sw,\quad n_1=k,\quad a_2=\frac{s}{w},\quad n_2=-k,\quad a_3=\frac{q^{1/2}y}{st},\quad n_3=l,$$
$$a_4=\frac{q^{1/2}}{sty},\quad n_4=-l,\quad a_5=tx,\quad n_5=j,\quad a_6=\frac{t}{x},\quad n_6=-j. \tag{6.24}$$

The balancing conditions hold true

$$\prod_{j=1}^{6}a_j=q\;, \tag{6.25}$$
$$\sum_{j=1}^{6}n_j=0\;, \tag{6.26}$$

and we can apply the above formula (6.10) for computing the integral over measure $[d_mz]$. This yields the expression

$$\frac{(q^{\frac{1+k+l}{2}}\frac{t}{wy},q^{\frac{1+k-l}{2}}\frac{ty}{w},q^{\frac{1-k+l}{2}}\frac{tw}{y},q^{\frac{1-k-l}{2}}twy;q)}{w^{2k}y^{2l}(q^{\frac{1+k+l}{2}}\frac{wy}{t},q^{\frac{1+k-l}{2}}\frac{w}{ty},q^{\frac{1-k+l}{2}}\frac{y}{tw},q^{\frac{1-k-l}{2}}\frac{1}{twy};q)}$$
$$\times\frac{(s^2t^2;q)}{(qs^{-2}t^{-2};q)}\sum_{j\in\mathbb{Z}}\int_{\mathbb{T}}[d_jx]\frac{(q^{1+\frac{k+j}{2}}\frac{1}{stwx},q^{1+\frac{k-j}{2}}\frac{x}{stw},q^{1+\frac{-k+j}{2}}\frac{w}{stx},q^{1-\frac{k+j}{2}}\frac{wx}{st};q)}{w^{2k}x^{2j}(q^{\frac{k+j}{2}}stwx,q^{\frac{k-j}{2}}\frac{stw}{x},q^{\frac{-k+j}{2}}\frac{stx}{w},q^{-\frac{k+j}{2}}\frac{st}{wx};q)}$$
$$\times\frac{(q^{\frac{1+l+j}{2}}\frac{s}{yx},q^{\frac{1+l-j}{2}}\frac{sx}{y},q^{\frac{1-l+j}{2}}\frac{sy}{x},q^{\frac{1-l-j}{2}}syx;q)}{y^{2l}x^{2j}(q^{\frac{1+l+j}{2}}\frac{yx}{s},q^{\frac{1+l-j}{2}}\frac{y}{sx},q^{\frac{1-l+j}{2}}\frac{x}{sy},q^{\frac{1-l-j}{2}}\frac{1}{syx};q)}$$
$$=D(t;y,l;w,k)M(st)_{w,k;x,j}D(s;y,l;x,j), \tag{6.27}$$

which proves the required identity.

We note that the derived solution of the star-triangle relation resembles structurally a different solution obtained in [54]. We stress that the parameters y and l are dummy variables in this construction, i.e. at each step of the walk along the lattice of Bailey pairs one can introduce further new parameters $y, l \to y', l' \to \ldots$.

6.2.3 Coxeter relations and the vertex type Yang-Baxter equation

Consider elementary transposition operators s_j, $j = 1, \ldots, 5$, acting on six parameters $\mathbf{t} = (t_1, \ldots, t_6)$:

$$s_j(\ldots, t_j, t_{j+1}, \ldots) = (\ldots, t_{j+1}, t_j, \ldots). \tag{6.28}$$

They generate the permutation group $\mathfrak{S}_6$ characterized by the Coxeter relations

$$s_j^2 = 1, \quad s_i s_j = s_j s_i \ \text{ for } \ |i-j| > 1, \quad s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}. \tag{6.29}$$

Define now five operators $\mathrm{S}_j(\mathbf{t})$, $j = 1, \ldots, 5$, acting on the three-index functions of three complex variables $f_{n_1,n_2,n_3}(z_1, z_2, z_3)$:

$$\begin{aligned}
[\mathrm{S}_1(\mathbf{t})f]_{n_1,n_2,n_3}(z_1, z_2, z_3) &:= M(t_1/t_2)_{z_1,n_1;z,m} f_{m,n_2,n_3}(z, z_2, z_3), \\
[\mathrm{S}_2(\mathbf{t})f]_{n_1,n_2,n_3}(z_1, z_2, z_3) &:= D(t_2/t_3; z_1, n_1; z_2, n_2) f_{n_1,n_2,n_3}(z_1, z_2, z_3). \\
[\mathrm{S}_3(\mathbf{t})f]_{n_1,n_2,n_3}(z_1, z_2, z_3) &:= M(t_3/t_4)_{z_2,n_2;z,m} f_{n_1,m,n_3}(z_1, z, z_3), \\
[\mathrm{S}_4(\mathbf{t})f]_{n_1,n_2,n_3}(z_1, z_2, z_3) &:= D(t_4/t_5; z_2, n_2; z_3, n_3) f_{n_1,n_2,n_3}(z_1, z_2, z_3), \\
[\mathrm{S}_5(\mathbf{t})f]_{n_1,n_2,n_3}(z_1, z_2, z_3) &:= M(t_5/t_6)_{z_3,n_3;z,m} f_{n_1,n_2,m}(z_1, z_2, z),
\end{aligned}$$

We stress that all these operators depend on the ratios of parameters, $\mathrm{S}_j(\mathbf{t}) = \mathrm{S}_j(t_j/t_{j+1})$. Let us prove that for an appropriate space of test functions the operators S_j generate the group $\mathfrak{S}_6$, provided their sequential action is defined via a cocycle condition $\mathrm{S}_j\mathrm{S}_k := \mathrm{S}_j(s_k(\mathbf{t}))\mathrm{S}_k(\mathbf{t})$. For this it is necessary to verify the Coxeter relations

$$\mathrm{S}_j^2 = 1, \quad \mathrm{S}_i\mathrm{S}_j = \mathrm{S}_j\mathrm{S}_i \ \text{ for } \ |i-j| > 1, \quad \mathrm{S}_j\mathrm{S}_{j+1}\mathrm{S}_j = \mathrm{S}_{j+1}\mathrm{S}_j\mathrm{S}_{j+1}. \tag{6.30}$$

Indeed, the latter relations are equivalent to algebraic properties of the Bailey lemma entries, in complete analogy with the elliptic hypergeometric case [177]. It is sufficient to establish them for S_1 and S_2, others will follow by the symmetry. So, we have

$$S_2^2 = S_2(s_2\mathbf{t})S_2(\mathbf{t}) = D(t_3/t_2; z_1, n_1; z_2, n_2)D(t_2/t_3; z_1, n_1; z_2, n_2) = 1. \tag{6.31}$$

A substantially more complicated relation is needed for S_1:

$$\begin{aligned}
[S_1^2 f]_n(x) &= [S_1(s_1\mathbf{t})S_1(\mathbf{t})f]_n(x) \\
&= M(t^{-1})_{x,n;z,m}M(t)_{z,m;y,j}f_j(y) \\
&= \sum_{j\in\mathbb{Z}} \int [d_j y]\ f_j(y)(1-t^2)(1-t^{-2}) \\
&\quad \times \sum_{m\in\mathbb{Z}} \int [d_m z]\ \Gamma_q(t^{-1}x^{\pm1}, \pm n; z, m)\Gamma_q(ty^{\pm1}, \pm j; z, m) \\
&= f_n(x), ,
\end{aligned} \tag{6.32}$$

or $S_1^2 =$, where $t = \frac{t_1}{t_2}$.

First, we claim that

$$M(1) =, \tag{6.33}$$

or

$$M(1)_{z,m;y,j}f_j(y) = f_m(z) \tag{6.34}$$

for the holomorphic test functions satisfying the reflection symmetry $f_{-m}(y^{-1}) = f_m(y)$. This fact follows from the residue calculus. For $t \to 1$ two pairs of poles approach the integration contour in $M(t)_{z,m;y,j}f_j(y)$ from two sides and pinch it. To resolve the singularity it is necessary to compute two residues which leads to the expression $(f_m(z) + f_{-m}(z^{-1}))/2$, and the reflection symmetry reduces it to one term. We now substitute in the star-triangle relation (6.22) the constraint $st = 1$. Using the inversion relation for D-function and $D(1; z_1, n_1; z_2, n_2) = 1$, the D-terms disappear on both sides and we obtain $M(t^{-1})M(t) =$.

Finally,

$$\begin{aligned}
\mathrm{S}_1\mathrm{S}_2\mathrm{S}_1 &= \mathrm{S}_1(s_2 s_1 \mathbf{t})\mathrm{S}_2(s_1\mathbf{t})\mathrm{S}_1(\mathbf{t}) \\
&= M(\tfrac{t_2}{t_3})_{z_1,n_1;z,m} D(\tfrac{t_1}{t_3}; z_2, n_2; z, m) M(\tfrac{t_1}{t_2})_{z,m;x,j} \\
&= \mathrm{S}_2\mathrm{S}_1\mathrm{S}_2 \\
&= \mathrm{S}_2(s_1 s_2 \mathbf{t})\mathrm{S}_1(s_2\mathbf{t})\mathrm{S}_2(\mathbf{t}) \\
&= D(\tfrac{t_1}{t_2}; z_1, n_1; z_2, n_2) M(\tfrac{t_1}{t_3})_{z_1,n_1;x,j} D(\tfrac{t_2}{t_3}; x, j; z_2, n_2),
\end{aligned} \tag{6.35}$$

which is precisely the star-triangle relation.

Consider the tensor product of three infinite-dimensional (equal or different) spaces $_1 \otimes_2 \otimes_3$ and associate with each space $_j$ a pair of variables: the spectral parameter u_j and the spin variable g_j, respectively. Define R-operators $\mathbb{R}_{ik}(u_i, g_i|u_k, g_k)$ acting in a non-trivial way in the subspace $_i\otimes_k$ with the unity operator action in its complement. The vertex type YBE has the form

$$\begin{aligned}
&\mathbb{R}_{12}(u_1, g_1|u_2, g_2)\,\mathbb{R}_{13}(u_1, g_1|u_3, g_3)\,\mathbb{R}_{23}(u_2, g_2|u_3, g_3) \\
&\quad = \mathbb{R}_{23}(u_2, g_2|u_3, g_3)\,\mathbb{R}_{13}(u_1, g_1|u_3, g_3)\,\mathbb{R}_{12}(u_1, g_1|u_2, g_2).
\end{aligned} \tag{6.36}$$

Actually, the R-operators depend on the difference of spectral parameters,

$$\mathbb{R}_{ik}(u_i, g_i|u_k, g_k) = \mathbb{R}_{ik}(u_i - u_j), \tag{6.37}$$

where we omitted dependence on the spin variables. Using this notation we can rewrite YBE in the more conventional form

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u-w)\,\mathbb{R}_{23}(v-w) = \mathbb{R}_{23}(v-w)\,\mathbb{R}_{13}(u-w)\,\mathbb{R}_{12}(u-v), \tag{6.38}$$

where $u = u_1, v = u_2, w = u_3$. It is convenient to single out the permutation operators from the R-operator

$$\mathbb{R}_{ik}(u) = \mathbb{P}_{ik}\,\mathrm{R}_{ik}(u), \tag{6.39}$$

where the operator $\mathbb{P}_{ik}$ interchanges the spaces, $\mathbb{P}_{ik}(\mathbb{V}_i \otimes \mathbb{V}_k) = \mathbb{V}_k \otimes \mathbb{V}_i$. Removing these permutation operators from the Yang-Baxter equation (6.36) yields the relation

$$\begin{aligned} &\mathrm{R}_{23}(u_1,g_1|u_2,g_2)\,\mathrm{R}_{12}(u_1,g_1|u_3,g_3)\,\mathrm{R}_{23}(u_2,g_2|u_3,g_3) \\ &\qquad = \mathrm{R}_{12}(u_2,g_2|u_3,g_3)\,\mathrm{R}_{23}(u_1,g_1|u_3,g_3)\,\mathrm{R}_{12}(u_1,g_1|u_2,g_2), \end{aligned} \tag{6.40}$$

where one sees only two R-operators, R_{12} and R_{23}.

Let us fix the spaces $_j$ as copies of the infinite bilateral sequences of meromorphic functions $f_j(z)$, $j \in \mathbb{Z}$. Then the triple tensor product of interest takes the form $_1 \otimes_2 \otimes_3 = f_{n_1,n_2,n_3}(z_1,z_2,z_3)$. Define now the composite operators acting in this space $\mathrm{R}_{12}(\mathbf{t})$,

$$\begin{aligned} \mathrm{R}_{12}(\mathbf{t}) = \mathrm{R}_{12}(t_1,\ldots,t_4) &= \mathrm{S}_2(s_1s_3s_2\mathbf{t})\,\mathrm{S}_1(s_3s_2\mathbf{t})\,\mathrm{S}_3(s_2\mathbf{t})\,\mathrm{S}_2(\mathbf{t}) \\ &= \mathrm{S}_2(t_1/t_4)\mathrm{S}_1(t_1/t_3)\mathrm{S}_3(t_2/t_4)\,\mathrm{S}_2(t_2/t_3), \end{aligned} \tag{6.41}$$

and $\mathrm{R}_{23}(\mathbf{t})$,

$$\begin{aligned} \mathrm{R}_{23}(\mathbf{t}) = \mathrm{R}_{23}(t_3,\ldots,t_6) &= \mathrm{S}_4(s_3s_5s_4\mathbf{t})\,\mathrm{S}_3(s_5s_4\mathbf{t})\,\mathrm{S}_5(s_4\mathbf{t})\,\mathrm{S}_4(\mathbf{t}) \\ &= \mathrm{S}_4(t_3/t_6)\,\mathrm{S}_3(t_3/t_5)\,\mathrm{S}_5(t_4/t_6)\,\mathrm{S}_4(t_4/t_5). \end{aligned} \tag{6.42}$$

Denoting

$$t_{1,2} = e^{-\pi i(u\pm g_1)}, \quad t_{3,4} = e^{-\pi i(v\pm g_2)}, \quad t_{5,6} = e^{-\pi i(w\pm g_3)}, \tag{6.43}$$

one can identify

$$\mathrm{R}_{12}(\mathbf{t}) = \mathrm{R}_{12}(u,g_1|v,g_2), \tag{6.44}$$

$$\mathrm{R}_{23}(\mathbf{t}) = \mathrm{R}_{23}(v,g_2|w,g_3) \tag{6.45}$$

and check that these operators depend only on the difference of spectral parameters $u-v$ and $v-w$, respectively.

Theorem. The R-operators (6.41) and (6.42) satisfy the vertex type Yang-Baxter relation (6.40).

Proof. Substituting the explicit forms of the R-operators into equality (6.40), we come to the relation

$$\mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4 \cdot \mathrm{S}_2\mathrm{S}_1\mathrm{S}_3\mathrm{S}_2 \cdot \mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4 = \mathrm{S}_2\mathrm{S}_1\mathrm{S}_3\mathrm{S}_2 \cdot \mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4 \cdot \mathrm{S}_2\mathrm{S}_1\mathrm{S}_3\mathrm{S}_2, \tag{6.46}$$

which is easily checked using only the cubic Coxeter relations for operators S_j in complete analogy with the cases considered in [196, 177].

6.2.4 A new two-dimensional solvable lattice model

Let us apply the operator relation (6.22) to a product of the Kronecker and Dirac delta-functions which remove integration over the x-variable and summation over the index j. This yields the functional star-triangle relation of the form

$$\begin{aligned} \sum_{m\in\mathbb{Z}} \int_0^1 \rho_m(u) \mathrm{W}_{\xi-a}(x,j;u,m)\mathrm{W}_{a+b}(y,j;u,m)\mathrm{W}_{\xi-b}(w,l;u,m)du \\ = \chi(a,b)\mathrm{W}_b(x,j;y,k)\mathrm{W}_{\xi-a-b}(x,j;w,l)\mathrm{W}_a(y,k;w,l), \end{aligned} \tag{6.47}$$

where

$$\mathrm{W}_a(x,j;u,m) = \Gamma_q(e^{2\pi i(a-\xi\pm x\pm u)}), \quad e^{-4\pi \mathrm{i}\xi} := q, \tag{6.48}$$

and

$$\rho_m(u) = \frac{(1-q^m e^{4\pi i u})(1-q^m e^{-4\pi i u})}{2q^m}, \tag{6.49}$$

$$\chi(a,b) = \frac{(qe^{4\pi i a}, qe^{4\pi i b}, e^{-4\pi i(a+b)};q)_\infty}{(e^{-4\pi i a}, e^{-4\pi i b}, qe^{4\pi i(a+b)};q)_\infty}. \tag{6.50}$$

We now define a two-dimensional lattice model associated with this relation. Consider a honeycomb lattice with the spins denoted by labels x, u, w, etc which seat in vertices. Each spin has a discrete internal degree of freedom denoted as m, j, k, l, etc (the monopole number). Neighboring spins (x,j) and (u,m) interact along the edges connecting them with the energy determined by the Boltzmann weight $\mathrm{W}_a(x,j;u,m)$. The function $\rho_m(u)$ describes the self-energy of spins, and ξ is called the crossing parameter. In this picture the “integration-plus-summation" in the star-triangle relation (6.47) means computation of the partition function for an elementary star-shaped cell with contributions coming from all possible values of the continuous spin u sitting in the central vertex and all possible values of the magnetic charge m. The honeycomb lattice can be transformed using the star-triangle relation to triangular and square lattices.

Compose now $N \times M$ sized two-dimensional square lattice of spins and associate with each horizontal edge the weight $\mathrm{W}_a(x,j;u,m)$ and with the vertical one the weight $\mathrm{W}_{\xi-a}(x,j;u,m)$. Then the partition function of such homogeneous spin system with the internal spin energy $\rho_m(u)$ has the form

$$Z = \sum_{\mathbb{Z}^{NM}} \int_{[0,1]^{NM}} \prod_{(ij)} \mathrm{W}_a(u_i,m_i;u_j,m_j) \prod_{(kl)} \mathrm{W}_{\xi-a}(u_k,m_k;u_l,m_l) \prod_s \rho_{m_s}(u_s) du_s, \quad (6.51)$$

where the first product is taken over the horizontal edges (ij), the second product goes over all vertical edges (k,l), and the third product (in s) is taken over all internal vertices of the lattice. Then one can consider the thermodynamical limit of infinite lattice, $N, M \to \infty$, and look for the free energy per spin $\kappa(a)$ found from the asymptotics

$$Z(a) \underset{N,M\to\infty}{=} e^{-NM\kappa(a)}. \quad (6.52)$$

Conjecturally, similar to the models considered in [174, 179, 49], the value of $\kappa(a)$ can be found using the reflection method [197]. Namely, one renormalizes the Bolztmann weights

$$\widetilde{\mathrm{W}}_a(x,j;u,m) = \frac{1}{m(a)} \mathrm{W}_a(x,j;u,m) \quad (6.53)$$

and chooses the multiplier $m(a)$ in such a way that the star-triangle relation takes the form

$$\sum_{m\in\mathbb{Z}} \int_0^1 \rho_m(u) \widetilde{\mathrm{W}}_{\xi-a}(x,j;u,m) \widetilde{\mathrm{W}}_{a+b}(y,j;u,m) \widetilde{\mathrm{W}}_{\xi-b}(w,l;u,m) du$$
$$= \widetilde{\mathrm{W}}_b(x,j;y,k) \widetilde{\mathrm{W}}_{\xi-a-b}(x,j;w,l) \widetilde{\mathrm{W}}_a(y,k;w,l). \quad (6.54)$$

Then,

$$Z(a) \underset{N,M\to\infty}{=} m(a)^{NM}, \quad (6.55)$$

or

$$\kappa(a) = -\log m(a). \quad (6.56)$$

Such a transformation of star-triangle relation requires

$$\frac{m(\xi-a)m(\xi-b)m(a+b)}{m(a)m(b)m(\xi-a-b)}=\chi(a,b), \tag{6.57}$$

which is possible if $m(a)$ satisfies the equation

$$\frac{m(a)}{m(\xi-a)}\frac{(e^{4\pi i(a-\xi)};q)_\infty}{(e^{-4\pi ia};q)_\infty}=1, \tag{6.58}$$

or

$$m(a+\xi)=\frac{(e^{-4\pi i(a+\xi)};q)_\infty}{(e^{4\pi ia};q)_\infty}m(-a). \tag{6.59}$$

Introduce the following infinite product

$$f(x;p,q)=(x;p,q)_\infty(pqx^{-1};p,q)_\infty, \qquad \frac{f(px;p,q)}{f(x;p,q)}=\frac{(qx^{-1};q)_\infty}{(x;q)_\infty}. \tag{6.60}$$

We note that this is the product of the numerator and denominator of the elliptic gamma function. One has the following inversion relation

$$f(x^{-1};p,q)=f(pqx;p,q). \tag{6.61}$$

Define the composite function

$$\mu(x;p,q)=\frac{f(xp\sqrt{pq};p^2,q)}{f(x\sqrt{pq};p^2,q)}. \tag{6.62}$$

It satisfies the equations

$$\mu(x;p,q)\mu(x^{-1};p,q)=1, \qquad \mu(x;p,q)\mu(p^{-1}x;p,q)=\frac{(x^{-1}p^{1/2}q^{1/2};q)_\infty}{(xp^{-1/2}q^{1/2};q)_\infty}. \tag{6.63}$$

Using these relations we can set

$$m(a)=\mu(e^{4\pi ia};q,q)=\frac{(q^2e^{4\pi ia},qe^{-4\pi ia};q,q^2)_\infty}{(qe^{4\pi ia},q^2e^{-4\pi ia};q,q^2)_\infty} \tag{6.64}$$

and see that this function satisfies the unitarity condition

$$m(-a) = \frac{1}{m(a)} \tag{6.65}$$

and the key starting equation (6.59). So, $-\log m(a)$ provides the explicit expression for the free energy per spin of the discussed two-dimensional "spin" model. For the model with the Boltzmann weights (6.53) the free energy is equal to zero.

6.2.5 Star-star relations and an IRF model Boltzmann weight

There is the ‚ÄúInteraction round a face model‚Äù (IRF) version of spin models for which four spins round a face of the lattice interact with each other. This interaction can be determined by the energy of face $\varepsilon(abcd)$ (or by the Boltzmann weigths W_{abcd}) depending on spins a, b, c, d. In the integrable case the Boltzmann weights satisfy the IRF type Yang‚ÄìBaxter equation. The hard hexagon model [186], the cyclic solid-on-solid model[198, 199, 200, 201, 202] and the restricted solid-on-solid model [194] are examples of the integrable IRF models.

Note that the IRF model considered in this subsection and vertex model from subsection 6.3.3 are equivalent to each other[1].

First we consider the simplest consequence of the Bailey chain of identities for sums of q-hypergeometric integrals described above following the elliptic hypergeometric pattern [195]. For this we use the evident explicit Bailey pair, following from the integration formula (6.10). Namely, let us choose

$$\alpha_m(z,t) = \prod_{j=1}^{4} \Gamma_q(a_j, n_j; z, m), \tag{6.66}$$

where a_j are arbitrary parameters. Substituting this expression into the integral transformation (6.18), imposing the constraint $\sum_{j=1}^{4} n_j = 0$, and choosing $t^2 = q\prod_{j=1}^{4} a_j^{-1}$,

[1] There are interesting equivalence relations between IRF and vertex models in the literature, see e.g. [203, 204, 205].

we derive from the Rosengren identity [143] that

$$\beta_n(x;t) = \frac{1}{x^{4n}\prod_{j=1}^4 a_j^{2n_j}} \prod_{1\le j<k\le 4} \frac{(q^{1+\frac{n_j+n_k}{2}} a_j^{-1}a_k^{-1};q)_\infty}{(q^{\frac{n_j+n_k}{2}} a_j a_k;q)_\infty}$$
$$\times \prod_{j=1}^4 \frac{(q^{1+\frac{n_j+n}{2}} a_j^{-1}t^{-1}x^{-1}, q^{1+\frac{n_j-n}{2}} a_j^{-1}t^{-1}x;q)_\infty}{(q^{\frac{n_j+n}{2}} a_j tx, q^{\frac{n_j-n}{2}} a_j tx^{-1};q)_\infty}. \tag{6.67}$$

We now take definitions of the Bailey lemma entries (6.19) and (6.20) and substitute them into the relation $\beta_k'(w;st) = M(st)_{w,k;x,j}\alpha_j'(x;st)$. This yields the following explicit symmetry transformation law

$$V(\underline{a},\underline{n};q) = \frac{V(\underline{\tilde{a}},\underline{n};q)}{\prod_{j=1}^8 a_j^{n_j}} \prod_{1\le j<k\le 4} \frac{(q^{1+\frac{n_j+n_k}{2}} a_j^{-1}a_k^{-1}, q^{1+\frac{n_{j+4}+n_{k+4}}{2}} a_{j+4}^{-1}a_{k+4}^{-1};q)_\infty}{(q^{\frac{n_j+n_k}{2}} a_j a_k, q^{\frac{n_{j+4}+n_{k+4}}{2}} a_{j+4}a_{k+4};q)_\infty}, \tag{6.68}$$

where

$$V(\underline{a},\underline{n};q) := \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \prod_{j=1}^8 \Gamma_q(a_j,n_j;z,m)[d_m z], \qquad \prod_{j=1}^8 a_j = q^2, \quad \sum_{j=1}^8 n_j = 0 \tag{6.69}$$

and the following notation for the parameters is used

$$a_{5,6} = stw^{\pm1}, \quad n_{5,6} = \pm k, \quad a_{7,8} = q^{1/2}s^{-1}y^{\pm1}, \quad n_{7,8} = \pm l \tag{6.70}$$

as well as

$$\tilde{a}_j = ta_j, \quad j = 1,2,3,4, \qquad \tilde{a}_j = t^{-1}a_j, \quad j = 5,6,7,8. \tag{6.71}$$

Remind also the balancing condition $t^2\prod_{j=1}^4 a_j = q$.

Conjecture. Let us take the V-function, whose parameters a_j, n_j satisfy only the balancing conditions indicated in the definition (6.69) and an additional constraint $\sum_{j=1}^4 n_j = 0$. Then we conjecture that it satisfies the symmetry transformation (6.68), where

$$\begin{cases} \tilde{a}_j = \varepsilon a_j, & j = 1,2,3,4 \\ \tilde{a}_j = \varepsilon^{-1} t_j, & j = 5,6,7,8 \end{cases} ; \quad \varepsilon = \sqrt{\frac{q}{a_1a_2a_3a_4}} = \sqrt{\frac{a_5a_6a_7a_8}{q}}. \tag{6.72}$$

Indeed, using the relation

$$\frac{(q^{1-m/2}z^{-1};q)_\infty}{(q^{-m/2}z;q)_\infty}=\frac{q^{m/2}}{(-z)^m}\frac{(q^{1+m/2}z^{-1};q)_\infty}{(q^{+m/2}z;q)_\infty},\quad m\in\mathbb{Z}, \tag{6.73}$$

one can verify that a repetition of the transformation (6.68), (6.72) returns back the original V-function, i.e. we deal with a reflection. The map $a_j \to \tilde{a}_j$ is the key reflection extending the Weyl group S_8 of the root system A_7 to the Weyl group of the exceptional root system E_7. However, because of the presence of integers n_j and the constraint $\sum_{j=1}^{4} n_j = 0$ we do not have the full $W(E_7)$ symmetry of the V-function yet. Interestingly, even in this reduced case the Bailey chains techniques yields the symmetry transformation (6.68) only when a pair of integers is forced to take particular values $n_i + n_j = n_k + n_l = 0$, $i \neq j \neq k \neq l$, which contrasts with the elliptic hypergeometric V-function case [206, 92].

Consider a $2d$ checkerboard lattice [207] where each “black" site has four “white" neighbours and, vice versa, each “white" site has four “black" neighbours. Ascribe to each edge connecting the white and black sites the Boltzmann weight W_{α_i} (6.48) with arbitrary parameters α_i subject to the constraint $\sum_{j=1}^{4} \alpha_j = 2\xi$. An IRF model is obtained when we integrate out the one-color lattice spins. The Boltzmann weight of the corresponding elementary “cell" containing four vertices determines the energy of this square face. It is given obviously by a special case of the general V-function introduced above when all integer variables n_j are paired by the relation $n_{2i-1} + n_{2i} = 0$. Then, completely similarly to [49], the symmetry transformation (6.68) has now the interpretation as a star-star relation [207]. As shown by Baxter [208] knowledge of the star-star relations automatically leads to the Yang-Baxter equation for IRF models.

6.2.6 IRF Yang-Baxter equation with spectral parameter

The Yang-Baxter equation for IRF models associated with $3d$ superconformal indices has the following form

$$\begin{aligned}&\sum_{H\in\mathbb{Z}}\int[d_Hh]\;R_{t_{41}t_{63}}\begin{pmatrix} a,A & b,B\\ h,H & c,C\end{pmatrix}\;R_{t_{63}t_{25}}\begin{pmatrix} c,C & d,D\\ h,H & e,E\end{pmatrix}\\ &\quad\times R_{t_{25}t_{41}}\begin{pmatrix} e,E & f,F\\ h,H & a,A\end{pmatrix}=\sum_{H\in\mathbb{Z}}\int[d_Hh]\;R_{t_{63}t_{25}}\begin{pmatrix} b,B & h,H\\ a,A & f,F\end{pmatrix}\\ &\quad\times R_{t_{25}t_{41}}\begin{pmatrix} d,D & h,H\\ c,C & b,B\end{pmatrix}\;R_{t_{41}t_{25}}\begin{pmatrix} f,F & h,H\\ e,E & d,D\end{pmatrix}, \qquad (6.74)\end{aligned}$$

where we introduced for convenience the shorthand notation for spectral parameters $t_{ij}=(t_i,t_j)$. The following statistical weight satisfies this equation

$$\begin{aligned}R_{(m,l)(n,r)}\begin{pmatrix} a,A & b,B\\ d,D & c,C\end{pmatrix} &= \frac{(q^{\frac{2}{3}}(n/l)^{-2},q^{\frac{2}{3}}(r/m)^{-2};q)_\infty}{(q^{\frac{1}{3}}(n/l)^{2},q^{\frac{1}{3}}(r/m)^{2};q)_\infty}\sum_{k\in\mathbb{Z}}\int[d_kz]\\ &\times\Gamma_q(q^{\frac{1}{3}}\frac{l}{n}a^{\pm1},\pm A;z,k)\Gamma_q(q^{\frac{1}{6}}\frac{r}{l}b^{\pm1},\pm B;z,k)\\ &\times\Gamma_q(q^{\frac{1}{3}}\frac{m}{r}c^{\pm1},\pm C;z,k)\Gamma_q(q^{\frac{1}{6}}\frac{n}{m}d^{\pm1},\pm D;z,k). \qquad (6.75)\end{aligned}$$

It is substantially equal to the V-function (6.69) with particular constraints on the integers $\underline{n}=(\pm A,\pm B,\pm C,\pm D)$.

For showing that function (6.75) describes a solution of equation (6.74) we use a special case of identity (6.10) associated with the star-triangle relation

$$\begin{aligned}&\sum_{m\in\mathbb{Z}}\int[d_mz]\Gamma_q(q^{\frac{1}{6}}t/sa^{\pm1},\pm A;z,m)\Gamma_q(q^{\frac{1}{6}}s/rb^{\pm1},\pm B;z,m)\Gamma_q(q^{\frac{1}{6}}r/tc^{\pm1},\pm C;z,m)\\ &=\frac{(q^{\frac{2}{3}}(t/s)^{-2},q^{\frac{2}{3}}(s/r)^{-2},q^{\frac{2}{3}}(r/t)^{-2};q)_\infty}{(q^{\frac{1}{3}}(t/s)^{2},q^{\frac{1}{3}}(s/r)^{2},q^{\frac{1}{3}}(r/t)^{2};q)_\infty}\Gamma_q(q^{\frac{1}{3}}t/ra^{\pm1},\pm A;b,B)\\ &\quad\times\Gamma_q(q^{\frac{1}{3}}r/sc^{\pm1},\pm C;a,A)\Gamma_q(q^{\frac{1}{3}}s/tb^{\pm1},\pm B;c,C). \qquad (6.76)\end{aligned}$$

We now form the following composite function defined by 6 integrations and 6 discrete summations

$$\begin{aligned}
\sum_{m_i\in\mathbb{Z}}\int\prod_{i=1}^{6}[d_{m_i}z]\ &\Gamma_q(q^{\frac{1}{6}}t_1/t_5f^{\pm1},\pm F;z_6,m_6)\Gamma_q(q^{\frac{1}{6}}t_6/t_1z_6^{\pm1},\pm m_6;z_1,m_1)\\
&\times\Gamma_q({}^{\frac{1}{6}}t_2/t_6a^{\pm1},\pm A;z_1,m_1)\ \Gamma_q(q^{\frac{1}{6}}t_1/t_2z_2^{\pm1},\pm m_2;z_1,m_1)\\
&\times\Gamma_q(q^{\frac{1}{6}}t_3/t_1b^{\pm1},\pm B;z_2,m_2)\Gamma_q(q^{\frac{1}{6}}t_2/t_3z_3^{\pm1},\pm m_3;z_2,m_2)\\
&\times\Gamma_q(q^{\frac{1}{6}}t_4/t_2c^{\pm1},\pm C;z_3,m_3)\Gamma_q(q^{\frac{1}{6}}t_3/t_4z_4^{\pm1},\pm m_4;z_3,m_3)\\
&\times\Gamma_q(q^{\frac{1}{6}}t_5/t_3d^{\pm1},\pm D;z_4,m_4)\Gamma_q(q^{\frac{1}{6}}t_4/t_5z_5^{\pm1},\pm m_5;z_4,m_4)\\
&\times\Gamma_q(q^{\frac{1}{6}}t_6/t_4e^{\pm1},\pm E;z_5,m_5)\Gamma_q(q^{\frac{1}{6}}t_5/t_6z_6^{\pm1},\pm m_6;z_5,m_5).
\end{aligned}\tag{6.77}$$

Then we integrate over z_1, z_3, and z_5 and sum over m_1, m_3, and m_5, i.e. use the star-triangle relation (6.76) for the expressions indicated in the square brackets below

$$\begin{aligned}
\sum_{m_2,m_4,m_6\in\mathbb{Z}}\int[d_{m_2}z][d_{m_4}z][d_{m_6}z]\ &\Gamma_q(q^{\frac{1}{6}}t_1/t_5f^{\pm1},\pm F;z_6,m_6)\\
&\times\Gamma_q(q^{\frac{1}{6}}t_3/t_1b^{\pm1},\pm B;z_2,m_2)\Gamma_q(q^{\frac{1}{6}}t_5/t_3d^{\pm1},\pm D;z_4,m_4)\\
&\times\Big[\sum_{m_1\in Z}\int[d_{m_1}z]\ \Gamma_q(q^{\frac{1}{6}}t_6/t_1z_6^{\pm1},\pm m_6;z_1,m_1)\\
&\times\Gamma_q(q^{\frac{1}{6}}t_2/t_6a^{\pm1},\pm A;z_1,m_1)\Gamma_q(q^{\frac{1}{6}}t_1/t_2z_2^{\pm1},\pm m_2;z_1,m_1)\Big]\\
&\times\Big[\sum_{m_3\in Z}\int[d_{m_3}z]\ \Gamma_q(q^{\frac{1}{6}}t_2/t_3z_3^{\pm1},\pm m_3;z_2,m_2)\\
&\times\Gamma_q(q^{\frac{1}{6}}t_4/t_2c^{\pm1},\pm C;z_3,m_3)\Gamma_q(q^{\frac{1}{6}}t_3/t_4z_4^{\pm1},\pm m_4;z_3,m_3)\Big]\\
&\times\Big[\sum_{m_5\in Z}\int[d_{m_5}z]\ \Gamma_q(q^{\frac{1}{6}}t_4/t_5z_5^{\pm1},\pm m_5;z_4,m_4)\\
&\times\Gamma_q(q^{\frac{1}{6}}t_6/t_4e^{\pm1},\pm E;z_5,m_5)\Gamma_q(q^{\frac{1}{6}}t_5/t_6z_6^{\pm1},\pm m_6;z_5,m_5)\Big].
\end{aligned}$$

As a result, we obtain

$$\frac{(q^{\frac{2}{3}}(t_6/t_1)^{-2}, q^{\frac{2}{3}}(t_3/t_4)^{-2}, q^{\frac{2}{3}}(t_1/t_2)^{-2}, q^{\frac{2}{3}}(t_4/t_5)^{-2}, q^{\frac{2}{3}}(t_2/t_3)^{-2}, q^{\frac{2}{3}}(t_5/t_6)^{-2}; q)_\infty}{(q^{\frac{1}{3}}(t_6/t_1)^{2}, q^{\frac{1}{3}}(t_3/t_4)^{2}, q^{\frac{1}{3}}(t_1/t_2)^{2}, q^{\frac{1}{3}}(t_4/t_5)^{2}, q^{\frac{1}{3}}(t_2/t_3)^{2}, q^{\frac{1}{3}}(t_5/t_6)^{2}; q)_\infty}$$
$$\times \frac{(q^{\frac{2}{3}}(t_6/t_4)^{-2}, q^{\frac{2}{3}}(t_4/t_2)^{-2}, q^{\frac{2}{3}}(t_2/t_6)^{-2}; q)_\infty}{(q^{\frac{1}{3}}(t_6/t_4)^{2}, q^{\frac{1}{3}}(t_4/t_2)^{2}, q^{\frac{1}{3}}(t_2/t_6)^{2}; q)_\infty} \sum_{m_2,m_4,m_6\in\mathbb{Z}} \int [d_{m_2}z][d_{m_4}z][d_{m_6}z]$$
$$\times\ \Gamma_q(q^{\frac{1}{6}}\frac{t_1}{t_5}f^{\pm 1}, \pm F; z_6, m_6)\Gamma_q(q^{\frac{1}{3}}\frac{t_6}{t_5}e^{\pm 1}, \pm E; z_4, m_4)\Gamma_q(q^{\frac{1}{3}}\frac{t_5}{t_4}e^{\pm 1}, \pm E; z_6, m_6)$$
$$\times\ \Gamma_q(q^{\frac{1}{3}}\frac{t_2}{t_1}a^{\pm 1}, \pm A; z_6, m_6)\Gamma_q(q^{\frac{1}{3}}\frac{t_1}{t_6}a^{\pm 1}, \pm A; z_2, m_2)\Gamma_q(q^{\frac{1}{6}}\frac{t_3}{t_1}b^{\pm 1}, \pm B; z_2, m_2)$$
$$\times\ \Gamma_q(q^{\frac{1}{3}}\frac{t_4}{t_3}c^{\pm 1}, \pm C; z_2, m_2)\Gamma_q(q^{\frac{1}{3}}\frac{t_3}{t_2}c^{\pm 1}, \pm C; z_4, m_4)\Gamma_q(q^{\frac{1}{6}}\frac{t_5}{t_3}d^{\pm 1}, \pm D; z_4, m_4)$$
$$\times \left[\Gamma_q(q^{\frac{1}{3}}\frac{t_6}{t_2}z_6^{\pm 1}, \pm m_6; z_2, m_2)\Gamma_q(q^{\frac{1}{3}}\frac{t_2}{t_4}z_4^{\pm 1}, \pm m_4; z_2, m_2)\Gamma_q(q^{\frac{1}{3}}\frac{t_4}{t_6}z_6^{\pm 1}, \pm m_6; z_4, m_4)\right].$$

Finally, we apply the inverse triangle-star relation to the last line product of Γ_q–functions in the square brackets and obtain the left-hand side expression in equation (6.74). The right-hand side expression of this IRF Yang-Baxter equation is obtained after performing first the integrations over z_2, z_4, z_6 and summations over m_2, m_4, m_6 and an application of a similar triangle-star transformation.

6.2.7 Star-triangle relation from supersymmetric duality

We now want to describe the two-dimensional solvable lattice models discussed above in the context of supersymmetric dualities for three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. The duality we study is very similar to the initial Seiberg duality for $\mathcal{N} = 1$ four-dimensional supersymmetric quantum chromodynamics, we have already discussed this duality in Chapter 5. The following two theories are dual to each other [5]:

- **Theory A**: $SU(2)$ gauge group with $N_f = 6$ flavors, chiral multiplets in the fundamental representation of the flavor group $SU(6)$ and in the fundamental representation of the gauge group.

- **Theory B**: without gauge degrees of freedom and the chiral fields (gauge-invariant "mesons") in the 15-dimensional totally antisymmetric tensor representation of the flavor group.

More precisely, the first interacting gauge fields theory flows in the infrared limit to the second one. This duality was considered in [123]. The authors calculated the three–dimensional ellipsoid partition functions for dual theories by applying the reduction procedure of [112, 114, 113] to the models considered in [27].

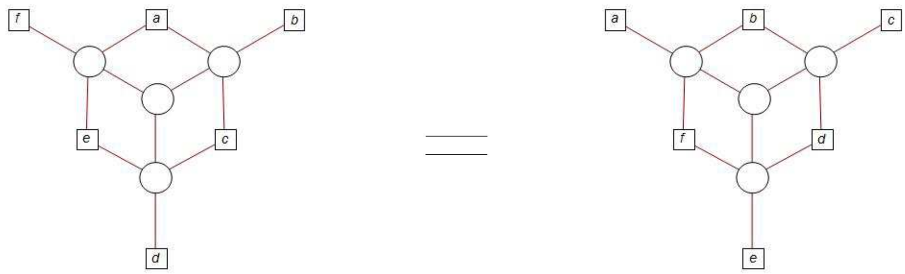

Figure 6.1: Duality of quiver diagrams.

The ordinary superconformal index of the "theory A" with enhanced symmetry was presented in [119] (see also [2] for the $N_f = 4$ case and [3, 70] for the similar theory with the broken gauge group). The duality between theories A and B leads to the equality of corresponding superconformal indices expressed by the following q-hypergeometric identity [5]

$$\sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} q^{-|m|} \prod_{j=1}^{6} \frac{(q^{1+\frac{n_j}{2}+\frac{|m|}{2}} \frac{1}{a_j z}, q^{1+\frac{n_j}{2}+\frac{|m|}{2}} \frac{z}{a_j}; q)_\infty}{(q^{\frac{n_j}{2}+\frac{|m|}{2}} a_j z, q^{\frac{n_j}{2}+\frac{|m|}{2}} \frac{a_j}{z}; q)_\infty} (1-q^{|m|}z^2)(1-q^{|m|}z^{-2}) \frac{dz}{2\pi i z}$$
$$= \frac{1}{\prod_{j=1}^{6} a_j^{n_j}} \prod_{1\le j<k\le 6} \frac{(q^{1+\frac{n_j}{2}+\frac{n_k}{2}} a_j^{-1} a_k^{-1}; q)_\infty}{(q^{\frac{n_j}{2}+\frac{n_k}{2}} a_j a_k; q)_\infty}, \tag{6.78}$$

with the balancing condition

$$\prod_{j=1}^{6} a_j = q, \quad \text{and} \quad \sum_{j=1}^{6} n_j = 0\,. \tag{6.79}$$

This condition is imposed by the effective superpotential $W = \eta X$ for the theory A, where X is a monopole operator and η is the four-dimensional instanton factor, which breaks a part of the symmetry (for details, see [104]). Using the relation [57]

$$\prod_{i=0}^{\infty} \frac{1-q^{i-\frac{1}{2}m+1}z^{-1}}{1-q^{i-\frac{1}{2}m}z} = (-q^{\frac{1}{2}})^{\frac{1}{2}(m+|m|)} z^{-\frac{1}{2}(m+|m|)} \prod_{i=0}^{\infty} \frac{1-q^{i+\frac{1}{2}|m|+1}z^{-1}}{1-q^{i+\frac{1}{2}|m|}z} \tag{6.80}$$

one can obtain the q-beta sum-integral (6.5) from (6.78).

Similarly, the full symmetry transformation (6.68) is a consequence of a duality of two $3d$ theories with $N_f = 8$. One can guess that there exist proper analogs of all elliptic hypergeometric integral identities described in [27, 28, 67] for sums of q-hypergeometric integrals associated with $3d$ dualities. Actually, the latter dualities are easily found using the reduction of $4d$ superconformal indices to $3d$ partition functions [112] which naturally leads to conjectural equalities of corresponding $3d$ superconformal indices.

By breaking the flavor symmetry to $SU(2) \times SU(2) \times SU(2)$ in (6.78) we obtain the star-triangle relation (6.76). Then the expression (6.75) corresponds to the generalized superconformal index of a $3d$ $\mathcal{N} = 2$ theory with the gauge group $G = SU(2)$ and the flavor group $F = SU(2) \times SU(2) \times SU(2) \times SU(2)$. In this picture, the IRF-type Yang-Baxter equation (6.74) is nothing else than the equality of superconformal indices of two dual $3d$ $\mathcal{N} = 2$ supersymmetric quiver gauge theories presented in Fig. 1, where the boxes correspond to $SU(2)$ flavor subgroups and the circles represent $SU(2)$ gauge subgroups.

We note that relation (6.32) describes the chiral symmetry breaking similarly to the $3d$ partition function case [146]. Indeed, it assumes the following sum-integral evaluation

$$\begin{aligned} \sum_{m\in\mathbb{Z}} \int [d_m z]\, \Gamma_q(t^{-1}x^{\pm 1}, \pm n; z, m)\Gamma_q(ty^{\pm 1}, \pm j; z, m) \\ = \frac{\delta(\phi_y+\phi_x)\delta_{n+j,0} + \delta(\phi_y-\phi_x)\delta_{n-j,0}}{q^{-j}(1-q^j y^2)(1-q^j y^{-2})(1-t^2)(1-t^{-2})}, \end{aligned} \tag{6.81}$$

where $y = e^{2\pi i \phi_y}$ and $x = e^{2\pi i \phi_x}$ and $\delta(\phi)$ is the periodic Dirac delta function with period 1, $\delta(\phi + 1) = \delta(\phi)$. On the left-hand side of equality (6.81) we have the $3d$ superconformal index of a theory with $SU(2)$ gauge group and $N_f = 4$ chiral fields with the naive flavor group $SU(2) \times SU(2)$. However, as follows from the the right-hand side expression, the true flavor group is $(SU(2) \times SU(2))_{diag}$ and the superconformal index has, actually, a non-zero support only on the corresponding subset of fugacities. This is precisely the manifestation of chiral symmetry breaking in confining theories similar to the $3d$ partition functions case [146]. A more detailed and rigorous consideration of this relation between indices and spontaneous breaking of global symmetries is needed, in particular, for the case when one has originally the full naive $SU(4)$ flavor group which is broken to $SP(4)$ group.

6.3 Remarks, conclusions and perspective work

- We presented a new solution to the star-triangle relation (Yang-Baxter equation) expressed in terms of basic hypergeometric functions. The new solution corresponds to a new solvable two-dimensional lattice model of statistical mechanics. In contrast to the Ising model, its spin variables take continuous and discrete values.
- One obtains the Kels model [145, 54] when a temperature-like parameter q tends to one in our solution.
- We describe the chiral symmetry breaking in terms of the delta-function singularities in superconformal indices for particular values of fugacities.
- It turns out that R matrix is dictated by some quantum group[2]. We wish to elucidate the origin of our solution in the framework of the representation theory of quantum group.
- There are a lot of attempts to extend the idea of integrability to three-dimensional lattice models. The Yang-Baxter equation in this case takes the form of the so-called tetrahedron equation by Zamolodchikov. It would be interesting to extend the relationship between supersymmetric dualities and integrable models and find a solution of the tetrahedron equation in this context.

[2]Roughly speaking, the quantum group is a "deformation" of a universal enveloping algebra of some Lie algebra. Almost all known solutions have been included in the quantum group scheme.

7 Appendix

7.1 Notations

For all special functions we use the notation that multiple parameters or $\pm$, $\mp$ signs in the part before the semicolon indicate a product of functions. For instance,

$$(a,b;q)_\infty = (a;q)_\infty(b;q)_\infty \tag{7.1}$$

$$\Gamma(z^\pm;p,q) = \Gamma(z;p,q)\Gamma(z^{-1};p,q) \tag{7.2}$$

The contours of all integrals appearing in the thesis are deformations of the unit circle serving to ensure that certain poles are kept inside the contour, while others are left outside.

7.2 Elliptic gamma function

In most of the formulas used in the thesis we arrived at the expressions in terms of elliptic gamma functions. For such reason we give here a definition and some properties of this function, which is appropriate generalization of Jacobi modular function.

The elliptic gamma function is a meromorphic function of three complex variables with double infinite product [209]

$$\Gamma(u;\tau,\sigma) \;=\; \prod_{i,j=0}^{\infty} \frac{1-e^{2\pi i((1+j)\tau+(1+i)\sigma-u)}}{1-e^{2\pi i(j\tau+i\sigma+u)}} \tag{7.3}$$

Here $u, \sigma, \tau \in \mathbb{C}$ and $\text{Im}\tau, \text{Im}\sigma > 0$. For our later purposes it is convenient to do the following reparametrization

$$p = e^{2\pi i\tau}, \quad q = e^{2\pi i\sigma}, \quad z = e^{2\pi i u}. \tag{7.4}$$

For generalizations of this function, see [210, 211, 212].

The elliptic gamma function satisfies many interesting properties such as symmetry under exchange of parameters p and q

$$\Gamma(z; p, q) = \Gamma(z; q, p)\,, \tag{7.5}$$

the functional relations

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \tag{7.6}$$

$$\Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q)\,, \tag{7.7}$$

and the reflection property

$$\Gamma(z; p, q)\;\Gamma(\frac{pq}{z}; p, q) = 1\,. \tag{7.8}$$

Here $\theta(z, q)$ is the theta function defined by

$$\theta(z; p) \;\; = \;\; \prod_{i=0}^{\infty}(1 - z^{-1}p^{i+1})(1 - zp^i) \tag{7.9}$$

It is related to the Jacobi theta functions. For instance, the first Jacobi theta function can be written as

$$\theta_1(\tau|z) = -iq^{1/8}y^{1/2}(q, q)_\infty\theta(y^{-1}; q), \tag{7.10}$$

$$\theta_1(\tau|z) = -iq^{1/8}y^{1/2}\prod_{k=1}^{\infty}(1 - q^k)(1 - yq^k)(1 - y^{-1}q^{k-1}), \;\; \text{with } y = e^{2\pi i z} \tag{7.11}$$

(7.12)

Th elliptic Gamma function is an automorphic form of degree 1 associated to a 2-cocycle and it has an $SL(3,Z)$ modular property [213] based on the following relations

$$\Gamma(u+\tau,\tau,\tau+\sigma)\Gamma(u,\tau+\sigma,\sigma) \; = \; \Gamma(u,\tau,\sigma) \; , \tag{7.13}$$

$$\Gamma(\frac{z}{\sigma};\frac{\tau}{\sigma},\frac{1}{\sigma}) = e^{i\pi Q(z,\tau,\sigma)}\Gamma(\frac{z-\sigma}{\tau};\frac{1}{\tau},\frac{\sigma}{\tau})\;\Gamma(z;\tau,\sigma) \tag{7.14}$$

Note that the elliptic gamma function is related to the Barnes multiple gamma function of order three [214]. Probably this relationship has connection to its modular property.

7.3 Elliptic hypergeometric functions

Elliptic hypergeometric integrals represent the top known level of special functions of hypergeometric type. They describe superconformal indices of four-dimensional supersymmetric gauge field theories and partition functions of certain two-dimensional spin systems.

A good reference for this subject is the book [215] by Gasper and Rahman and a review article [92] by Spiridonov. See also [159, 216, 206, 92].

Let c_n be complex numbers. Consider a formal power series[1]

$$\sum_{n=0}^{\infty} c_n x^n \; . \tag{7.15}$$

Depending on the following ratio

$$\frac{c_{n+1}}{c_n} \tag{7.16}$$

we define three family of hypergeometric functions.

Definition. The series (7.15) is called

- an ordinary hypergeometric series if (7.16) is a rational function of n;
- a basic hypergeometric (or simply q-hypergeometric) series if (7.16) is a trigono-

[1]We call it “formal” since we are not interested in the convergence of the series.

metric function of n;

- an elliptic hypergeometric series if (7.16) is an elliptic function of n.

The integral representations of hypergeometric functions can be defined similarly. For instance, a contour integral $\int_C \Delta(u)du$ is called elliptic hypergeometric integral[2] if the meromorphic kernel $\Delta(u)$ is the solution of the following first order finite difference equation

$$\Delta(u+a) = h(u;b,c)\Delta(u) \;, \tag{7.17}$$

where $a \in \mathbb{C}$ and $h(u;b,c)$ is an elliptic function with periods $b, c \in \mathbb{C}$ and $\mathrm{Im}(b/c) \neq 0$.

To give an example of an elliptic hypergeometric integral, let us consider the elliptic beta integral. Spiridonov [159] has evaluated the following integral as an elliptic analog of the Euler beta integral[3].

Theorem (Spiridonov). Let $t_1, \ldots, t_6, p, q \in \mathbb{C}$ with $|t_1|, \ldots, |t_6|, |p|, |q| < 1$. Then

$$\frac{(p;p)_\infty (q;q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{i=1}^{6} \Gamma(t_i z; p,q)\Gamma(t_i z^{-1}; p,q)}{\Gamma(z^2;p,q)\Gamma(z^{-2};p,q)} \frac{dz}{2\pi \mathrm{i} z} = \prod_{1\leq i<j\leq 6} \Gamma(t_i t_j; p,q), \tag{7.18}$$

where the unit circle $\mathbb{T}$ is taken in the positive orientation and we imposed the balancing condition $\prod_{i=1}^{6} t_i = pq$.

Limits of the elliptic beta integral lead to many identities for hypergeometric integrals [92, 219, 216, 220, 221, 222]. For instance, if we take the limit $p \to 0$ then (7.18) reduces to the Nassrallah–Rahman trigonometric beta integral [223][4]

$$\frac{(q,q)_\infty}{2} \int_{\mathbb{T}} \frac{(z\prod_{i=1}^{5} t_i, q)_\infty (z^{-1}\prod_{i=1}^{5} t_i, q)_\infty (z^2,q)_\infty (z^{-2},q)_\infty}{\prod_{i=1}^{5}(t_i z)_\infty (t_i z^{-1})_\infty} \frac{dz}{2\pi i z} = \frac{\prod_{j=1}^{5}(\frac{t_1t_2t_3t_4t_5}{t_j}, q)_\infty}{\prod_{1\leq i<j\leq 5}(t_it_j, q)_\infty} \tag{7.19}$$

[2]Similarly one can make a definition for multivariative case.

[3]There is a vast literature on q-beta integrals. The interested reader is referred to [217, 218].

[4]Note that the integral identity presented here was observed by Rahman in [224] as a special case of the integral found in [223]. This integral is an extension of the well-known Askey–Wilson integral [225]. If we let the q tend to 1 one obtains the corresponding ordinary hypergeometric function.

7.4 Barnes double Gamma function

The Barnes double Gamma function $\Gamma_2(u;\omega_1,\omega_2)$ is defined as

$$\log\Gamma_2(x;a,b)=\zeta_2'(0;a,b,x)+\log\rho_2(a,b), \tag{7.20}$$

where

$$\zeta_2(s;a,b,x)=\sum_{m,n=0}(am+bn+x)^{-s} \tag{7.21}$$

$$\log\rho_2(a,b)=-\lim_{x\to 0}\left(\zeta_2'(0;a,b,x)+\log x\right) \tag{7.22}$$

There is also the integral representation of this function

$$\Gamma_2(x;a,b)=\exp\left(\frac{1}{2\pi i}\int_{C_H}\frac{e^{-xt}(\log(-t)+\gamma)}{t(1-e^{-at})(1-e^{-bt})}dt\right), \tag{7.23}$$

where γ is the Euler constant and the Hankel contour C_H starts and finishes near the point $+\infty$, turning around the half–axis $[0,\infty)$ anticlockwise.

Useful reference for specific details is [226].

7.5 Hyperbolic gamma-function

The hyperbolic gamma function is defined as

$$\gamma^{(2)}(u;\omega_1,\omega_2)=e^{-\pi i B_{2,2}(u;\omega)/2}\frac{(e^{2\pi iu/\omega_1}\tilde{q};\tilde{q})}{(e^{2\pi iu/\omega_1};q)}\quad\text{with}\quad q=e^{2\pi i\omega_1/\omega_2},\quad \tilde{q}=e^{-2\pi i\omega_2/\omega_1}, \tag{7.24}$$

where $B_{2,2}(u;\omega)$ is the second order Bernoulli polynomial,

$$B_{2,2}(u;\omega)=\frac{u^2}{\omega_1\omega_2}-\frac{u}{\omega_1}-\frac{u}{\omega_2}+\frac{\omega_1}{6\omega_2}+\frac{\omega_2}{6\omega_1}+\frac{1}{2}. \tag{7.25}$$

The reflection identity for a hyperbolic gamma-function is as follows

$$\gamma^{(2)}(z, \omega_1 + \omega_2 - z; \omega_1, \omega_2) = 1, \tag{7.26}$$

and the asymptotic formulas are

$$\lim_{u\to\infty} e^{\frac{\pi i}{2} B_{2,2}(u;\omega_1,\omega_2)} \gamma^{(2)}(u; \omega_1, \omega_2) = 1, \quad \text{for } \arg \omega_1 < \arg u < \arg \omega_2 + \pi, \tag{7.27}$$

$$\lim_{u\to\infty} e^{-\frac{\pi i}{2} B_{2,2}(u;\omega_1,\omega_2)} \gamma^{(2)}(u; \omega_1, \omega_2) = 1, \quad \text{for } \arg \omega_1 - \pi < \arg u < \arg \omega_2. \tag{7.28}$$

There are different notations and modifications of hyperbolic Gamma function, relations between some of them can be found in [227, 49] (also see the appendix of [228]).

Bibliography

[1] I. B. Gahramanov and G. S. Vartanov, "Superconformal indices and partition functions for supersymmetric field theories," *XVIIth Intern. Cong. Math. Phys.* (2013) 695–703, `arXiv:1310.8507 [hep-th]`.

[2] I. Gahramanov and G. Vartanov, "Extended global symmetries for 4D $N = 1$ SQCD theories," *J.Phys.* **A46** (2013) 285403, `arXiv:1303.1443 [hep-th]`.

[3] I. Gahramanov and H. Rosengren, "A new pentagon identity for the tetrahedron index," *JHEP* **1311** (2013) 128, `arXiv:1309.2195 [hep-th]`.

[4] I. Gahramanov and V. P. Spiridonov, "The star-triangle relation and 3d superconformal indices," *JHEP* **08** (2015) 040, `arXiv:1505.00765 [hep-th]`.

[5] I. Gahramanov and H. Rosengren, "Basic hypergeometry of SUSY dualities," `arXiv:1606.08185`.

[6] M. A. Shifman, "Nonperturbative dynamics in supersymmetric gauge theories," *Prog. Part. Nucl. Phys.* **39** (1997) 1–116, `arXiv:hep-th/9704114 [hep-th]`.

[7] V. A. Novikov, "Nonperturbative QCD and supersymmetric QCD," *Phys. Usp.* **47** (2004) 109–116. [Usp. Fiz. Nauk47,113(2004)].

[8] K. Intriligator, "Aspects of supersymmetric field theories," *Int. J. Mod. Phys.* **A25** (2010) 391–400.

[9] K. Intriligator, "Lectures on 4d N=1 susy gauge theories," *Nucl. Phys. Proc. Suppl.* **216** (2011) 34–43.

[10] A. Kapustin, B. Willett, and I. Yaakov, "Exact Results for Wilson Loops in

Superconformal Chern-Simons Theories with Matter," *JHEP* **03** (2010) 089, arXiv:0909.4559 [hep-th].

[11] A. Kapustin, B. Willett, and I. Yaakov, "Nonperturbative Tests of Three-Dimensional Dualities," *JHEP* **10** (2010) 013, arXiv:1003.5694 [hep-th].

[12] N. Hama, K. Hosomichi, and S. Lee, "Notes on SUSY Gauge Theories on Three-Sphere," *JHEP* **03** (2011) 127, arXiv:1012.3512 [hep-th].

[13] N. Hama, K. Hosomichi, and S. Lee, "SUSY Gauge Theories on Squashed Three-Spheres," *JHEP* **05** (2011) 014, arXiv:1102.4716 [hep-th].

[14] D. L. Jafferis, "The Exact Superconformal R-Symmetry Extremizes Z," *JHEP* **1205** (2012) 159, arXiv:1012.3210 [hep-th].

[15] Y. Imamura and D. Yokoyama, "N=2 supersymmetric theories on squashed three-sphere," *Phys. Rev.* **D85** (2012) 025015, arXiv:1109.4734 [hep-th].

[16] F. Benini and S. Cremonesi, "Partition Functions of $\mathcal{N} = (2,2)$ Gauge Theories on S^2 and Vortices," *Commun.Math.Phys.* **334** no. 3, (2015) 1483–1527, arXiv:1206.2356 [hep-th].

[17] N. Doroud, J. Gomis, B. Le Floch, and S. Lee, "Exact Results in D=2 Supersymmetric Gauge Theories," *JHEP* **05** (2013) 093, arXiv:1206.2606 [hep-th].

[18] K. Hosomichi, R.-K. Seong, and S. Terashima, "Supersymmetric Gauge Theories on the Five-Sphere," *Nucl. Phys.* **B865** (2012) 376–396, arXiv:1203.0371 [hep-th].

[19] J. Kallen, J. Qiu, and M. Zabzine, "The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere," *JHEP* **08** (2012) 157, arXiv:1206.6008 [hep-th].

[20] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," *Commun.Math.Phys.* **313** (2012) 71–129, arXiv:0712.2824 [hep-th].

[21] E. Witten, "Constraints on Supersymmetry Breaking," *Nucl.Phys.* **B202**

(1982) 253.

[22] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, "An Index for 4 dimensional super conformal theories," *Commun.Math.Phys.* **275** (2007) 209–254, `arXiv:hep-th/0510251 [hep-th]`.

[23] C. Romelsberger, "Counting chiral primaries in N = 1, d=4 superconformal field theories," *Nucl. Phys.* **B747** (2006) 329–353, `arXiv:hep-th/0510060 [hep-th]`.

[24] C. Romelsberger, "Calculating the Superconformal Index and Seiberg Duality," `arXiv:0707.3702 [hep-th]`.

[25] F. Dolan and H. Osborn, "Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to N=1 Dual Theories," *Nucl.Phys.* **B818** (2009) 137–178, `arXiv:0801.4947 [hep-th]`.

[26] V. Spiridonov and G. Vartanov, "Superconformal indices for N = 1 theories with multiple duals," *Nucl.Phys.* **B824** (2010) 192–216, `arXiv:0811.1909 [hep-th]`.

[27] V. Spiridonov and G. Vartanov, "Elliptic Hypergeometry of Supersymmetric Dualities," *Commun.Math.Phys.* **304** (2011) 797–874, `arXiv:0910.5944 [hep-th]`.

[28] V. Spiridonov and G. Vartanov, "Elliptic hypergeometry of supersymmetric dualities II. Orthogonal groups, knots, and vortices," *Commun.Math.Phys.* **325** (2014) 421–486, `arXiv:1107.5788 [hep-th]`.

[29] C. Krattenthaler, V. Spiridonov, and G. Vartanov, "Superconformal indices of three-dimensional theories related by mirror symmetry," *JHEP* **1106** (2011) 008, `arXiv:1103.4075 [hep-th]`.

[30] Y. Imamura and S. Yokoyama, "Index for three dimensional superconformal field theories with general R-charge assignments," *JHEP* **1104** (2011) 007, `arXiv:1101.0557 [hep-th]`.

[31] A. Kapustin and B. Willett, "Generalized Superconformal Index for Three Dimensional Field Theories," `arXiv:1106.2484 [hep-th]`.

[32] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," *Nucl.Phys.* **B435** (1995) 129–146, `arXiv:hep-th/9411149 [hep-th]`.

[33] K. A. Intriligator, "New RG fixed points and duality in supersymmetric SP(N(c)) and SO(N(c)) gauge theories," *Nucl.Phys.* **B448** (1995) 187–198, `arXiv:hep-th/9505051 [hep-th]`.

[34] K. A. Intriligator and P. Pouliot, "Exact superpotentials, quantum vacua and duality in supersymmetric SP(N(c)) gauge theories," *Phys.Lett.* **B353** (1995) 471–476, `arXiv:hep-th/9505006 [hep-th]`.

[35] C. Csaki, M. Schmaltz, W. Skiba, and J. Terning, "Selfdual N=1 SUSY gauge theories," *Phys.Rev.* **D56** (1997) 1228–1238, `arXiv:hep-th/9701191 [hep-th]`.

[36] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg, and M. Strassler, "Aspects of N=2 supersymmetric gauge theories in three-dimensions," *Nucl.Phys.* **B499** (1997) 67–99, `arXiv:hep-th/9703110 [hep-th]`.

[37] J. de Boer, K. Hori, and Y. Oz, "Dynamics of N=2 supersymmetric gauge theories in three-dimensions," *Nucl.Phys.* **B500** (1997) 163–191, `arXiv:hep-th/9703100 [hep-th]`.

[38] K. A. Intriligator and N. Seiberg, "Mirror symmetry in three-dimensional gauge theories," *Phys.Lett.* **B387** (1996) 513–519, `arXiv:hep-th/9607207 [hep-th]`.

[39] G. 't Hooft, "Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking," *NATO Sci.Ser.B* **59** (1980) 135.

[40] Y. Frishman, A. Schwimmer, T. Banks, and S. Yankielowicz, "The Axial Anomaly and the Bound State Spectrum in Confining Theories," *Nucl.Phys.* **B177** (1981) 157.

[41] J. H. Brodie, P. L. Cho, and K. A. Intriligator, "Misleading anomaly matchings?," *Phys.Lett.* **B429** (1998) 319–326, `arXiv:hep-th/9802092 [hep-th]`.

[42] J. Terning, "'t Hooft anomaly matching for QCD," *Phys.Rev.Lett.* **80** (1998) 2517–2520, `arXiv:hep-th/9706074 [hep-th]`.

[43] G. Vartanov, "On the ISS model of dynamical SUSY breaking," *Phys.Lett.* **B696** (2011) 288–290, `arXiv:1009.2153 [hep-th]`.

[44] V. Spiridonov and G. Vartanov, "Elliptic hypergeometric integrals and 't Hooft anomaly matching conditions," *JHEP* **1206** (2012) 016, `arXiv:1203.5677 [hep-th]`.

[45] A. A. Ardehali, J. T. Liu, and P. Szepietowski, "Central charges from the $\mathcal{N} = 1$ superconformal index," *Phys. Rev. Lett.* **114** no. 9, (2015) 091603, `arXiv:1411.5028 [hep-th]`.

[46] L. Di Pietro and Z. Komargodski, "Cardy formulae for SUSY theories in $d = 4$ and $d = 6$," *JHEP* **12** (2014) 031, `arXiv:1407.6061 [hep-th]`.

[47] A. A. Ardehali, J. T. Liu, and P. Szepietowski, "c - a from the $\mathcal{N} = 1$ superconformal index," *JHEP* **12** (2014) 145, `arXiv:1407.6024 [hep-th]`.

[48] M. Buican, T. Nishinaka, and C. Papageorgakis, "Constraints on chiral operators in $\mathcal{N} = 2$ SCFTs," *JHEP* **12** (2014) 095, `arXiv:1407.2835 [hep-th]`.

[49] V. Spiridonov, "Elliptic beta integrals and solvable models of statistical mechanics," *Contemp.Math.* **563** (2012) 181–211, `arXiv:1011.3798 [hep-th]`.

[50] M. Yamazaki, "Quivers, YBE and 3-manifolds," *JHEP* **05** (2012) 147, `arXiv:1203.5784 [hep-th]`.

[51] J. Yagi, "Quiver gauge theories and integrable lattice models," `arXiv:1504.04055 [hep-th]`.

[52] F. Nieri, S. Pasquetti, F. Passerini, and A. Torrielli, "5D partition functions, q-Virasoro systems and integrable spin-chains," *JHEP* **12** (2014) 040, `arXiv:1312.1294 [hep-th]`.

[53] K. Costello, "Supersymmetric gauge theory and the Yangian," `arXiv:1303.2632 [hep-th]`.

[54] A. P. Kels, "New solutions of the star-triangle relation with discrete and continuous spin variables," `arXiv:1504.07074 [math-ph]`.

[55] F. Benini, T. Nishioka, and M. Yamazaki, "4d Index to 3d Index and 2d

TQFT," *Phys.Rev.* **D86** (2012) 065015, arXiv:1109.0283 [hep-th].

[56] M. Yamazaki, "New Integrable Models from the Gauge/YBE Correspondence," *J. Statist. Phys.* **154** (2014) 895, arXiv:1307.1128 [hep-th].

[57] T. Dimofte, D. Gaiotto, and S. Gukov, "3-Manifolds and 3d Indices," arXiv:1112.5179 [hep-th].

[58] M. Yamazaki and W. Yan, "Integrability from 2d $\mathcal{N} = (2, 2)$ dualities," *J. Phys.* **A48** (2015) 394001, arXiv:1504.05540 [hep-th].

[59] D. Chicherin and V. P. Spiridonov, "The hyperbolic modular double and Yang-Baxter equation," arXiv:1511.00131 [math-ph].

[60] Y. Terashima and M. Yamazaki, "3d N=2 Theories from Cluster Algebras," *PTEP* **023** (2014) B01, arXiv:1301.5902 [hep-th].

[61] F. Benini, D. S. Park, and P. Zhao, "Cluster Algebras from Dualities of 2d $\mathcal{N}$ = (2, 2) Quiver Gauge Theories," *Commun. Math. Phys.* **340** (2015) 47–104, arXiv:1406.2699 [hep-th].

[62] T. Dimofte, D. Gaiotto, and S. Gukov, "Gauge Theories Labelled by Three-Manifolds," arXiv:1108.4389 [hep-th].

[63] I. B. Frenkel and V. G. Turaev, "Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions," *The Arnold-Gelfand Mathematical Seminars* (1997) 171–204.

[64] H. Rosengren, "An elementary approach to 6j-symbols (classical, quantum, rational, trigonometric, and elliptic)," *The Ramanujan Journal* **13** no. 1, (2007) 131–166.

[65] G. Gasper and M. Rahman, "Basic Hypergeometric Series," *Cambridge University Press, 2nd ed* (2004) .

[66] V. Spiridonov and G. Vartanov, "Supersymmetric dualities beyond the conformal window," *Phys.Rev.Lett.* **105** (2010) 061603, arXiv:1003.6109 [hep-th].

[67] M. Sudano, "The Romelsberger Index, Berkooz Deconfinement, and Infinite Families of Seiberg Duals," *JHEP* **1205** (2012) 051, arXiv:1112.2996

[hep-th].

[68] V. Spiridonov and G. Vartanov, "Superconformal indices of $\mathcal{N} = 4$ SYM field theories," *Lett.Math.Phys.* **100** (2012) 97–118, arXiv:1005.4196 [hep-th].

[69] D. Kutasov and J. Lin, "N=1 Duality and the Superconformal Index," arXiv:1402.5411 [hep-th].

[70] I. Gahramanov and H. Rosengren, "Integral pentagon relations for 3d superconformal indices," arXiv:1412.2926 [hep-th].

[71] G. Festuccia and N. Seiberg, "Rigid Supersymmetric Theories in Curved Superspace," *JHEP* **06** (2011) 114, arXiv:1105.0689 [hep-th].

[72] A. Gadde, E. Pomoni, L. Rastelli, and S. S. Razamat, "S-duality and 2d Topological QFT," *JHEP* **1003** (2010) 032, arXiv:0910.2225 [hep-th].

[73] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, "Gauge Theories and Macdonald Polynomials," *Commun.Math.Phys.* **319** (2013) 147–193, arXiv:1110.3740 [hep-th].

[74] Y. Nakayama, "Index for orbifold quiver gauge theories," *Phys. Lett.* **B636** (2006) 132–136, arXiv:hep-th/0512280 [hep-th].

[75] Y. Nakayama, "Index for supergravity on AdS(5) x T**1,1 and conifold gauge theory," *Nucl. Phys.* **B755** (2006) 295–312, arXiv:hep-th/0602284 [hep-th].

[76] Y. Nakayama, "Finite N index and angular momentum bound from gravity," *Gen. Rel. Grav.* **39** (2007) 1625–1638, arXiv:hep-th/0701208 [hep-th].

[77] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and S. Raju, "Indices for Superconformal Field Theories in 3,5 and 6 Dimensions," *JHEP* **02** (2008) 064, arXiv:0801.1435 [hep-th].

[78] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, "On the Superconformal Index of N=1 IR Fixed Points: A Holographic Check," *JHEP* **1103** (2011) 041, arXiv:1011.5278 [hep-th].

[79] S. Kim, "The Complete superconformal index for N=6 Chern-Simons theory," *Nucl. Phys.* **B821** (2009) 241–284, arXiv:0903.4172 [hep-th]. [Erratum: Nucl. Phys.B864,884(2012)].

[80] Y. Nakayama, "4D and 2D superconformal index with surface operator," *JHEP* **08** (2011) 084, arXiv:1105.4883 [hep-th].

[81] D. Gaiotto, L. Rastelli, and S. S. Razamat, "Bootstrapping the superconformal index with surface defects," *JHEP* **1301** (2013) 022, arXiv:1207.3577 [hep-th].

[82] D. Gang, E. Koh, and K. Lee, "Line Operator Index on $S^1 \times S^3$," *JHEP* **05** (2012) 007, arXiv:1201.5539 [hep-th].

[83] A. Gadde and S. Gukov, "2d Index and Surface operators," *JHEP* **1403** (2014) 080, arXiv:1305.0266 [hep-th].

[84] L. F. Alday, M. Bullimore, M. Fluder, and L. Hollands, "Surface defects, the superconformal index and q-deformed Yang-Mills," *JHEP* **10** (2013) 018, arXiv:1303.4460 [hep-th].

[85] R. Kashaev, F. Luo, and G. Vartanov, "A TQFT of Turaev-Viro type on shaped triangulations," arXiv:1210.8393 [math.QA].

[86] J. Wess and J. Bagger, *Supersymmetry and supergravity*. 1992.

[87] R. Haag, J. T. Lopuszanski, and M. Sohnius, "All Possible Generators of Supersymmetries of the s Matrix," *Nucl. Phys.* **B88** (1975) 257.

[88] M. A. Shifman, "Beginning supersymmetry (supersymmetry in quantum mechanics)," *In *Shifman, M.A.: ITEP lectures on particle physics and field theory* **1** (1995) 301–344.

[89] K. Selivanov, "Counting vacua in supersymmetric Yang-Mills theory," *Phys.Usp.* **45** (2002) 675–686.

[90] M. F. Atiyah and I. M. Singer, "The index of elliptic operators on compact manifolds," *Bulletin of the American Mathematical Society* **69** no. 3, (05, 1963) 422–433.

[91] B. Feng, A. Hanany, and Y.-H. He, "Counting gauge invariants: The Plethystic program," *JHEP* **0703** (2007) 090, arXiv:hep-th/0701063 [hep-th].

[92] V. P. Spiridonov, "Essays on the theory of elliptic hypergeometric functions," *Russ. Math. Surv.* **63** no. 3, (2008) 405, arXiv:0805.3135 [math.CA].

[93] E. Gerchkovitz, "Constraints on the R-charges of free bound states from the Römelsberger index," *JHEP* **07** (2014) 071, `arXiv:1311.0487 [hep-th]`.

[94] S. Benvenuti, B. Feng, A. Hanany, and Y.-H. He, "Counting BPS Operators in Gauge Theories: Quivers, Syzygies and Plethystics," *JHEP* **0711** (2007) 050, `arXiv:hep-th/0608050 [hep-th]`.

[95] S. Nawata, "Localization of N=4 Superconformal Field Theory on S$^1 xS^3 andIndex,'' JHEP 11(2011)144, arXiv: 1104.4470[hep-th]$.

[96] I. Yaakov, "Localization of gauge theories on the three-sphere," *PhD Thesis* (2012) .

[97] B. Willett, "Localization and dualities in three-dimensional superconformal field theories," *PhD Thesis* (2012) .

[98] K. Intriligator and N. Seiberg, "Aspects of 3d N=2 Chern-Simons-Matter Theories," *JHEP* **1307** (2013) 079, `arXiv:1305.1633 [hep-th]`.

[99] K. Nii, "Three-dimensional Supersymmetric Gauge Theories and Seiberg Duality," *PhD Thesis* (2015) .

[100] I. Affleck, J. A. Harvey, and E. Witten, "Instantons and (Super)Symmetry Breaking in (2+1)-Dimensions," *Nucl.Phys.* **B206** (1982) 413.

[101] C. Csaki, M. Martone, Y. Shirman, P. Tanedo, and J. Terning, "Dynamics of 3D SUSY Gauge Theories with Antisymmetric Matter," *JHEP* **1408** (2014) 141, `arXiv:1406.6684 [hep-th]`.

[102] A. Kapustin and M. J. Strassler, "On mirror symmetry in three-dimensional Abelian gauge theories," *JHEP* **04** (1999) 021, `arXiv:hep-th/9902033 [hep-th]`.

[103] J. Bhattacharya and S. Minwalla, "Superconformal Indices for N = 6 Chern Simons Theories," *JHEP* **0901** (2009) 014, `arXiv:0806.3251 [hep-th]`.

[104] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, "3d dualities from 4d dualities," *JHEP* **1307** (2013) 149, `arXiv:1305.3924 [hep-th]`.

[105] F. Dolan, "On Superconformal Characters and Partition Functions in Three Dimensions," *J.Math.Phys.* **51** (2010) 022301, `arXiv:0811.2740 [hep-th]`.

[106] C. Beem, T. Dimofte, and S. Pasquetti, "Holomorphic Blocks in Three Dimensions," *JHEP* **1412** (2014) 177, `arXiv:1211.1986 [hep-th]`.

[107] S. Pasquetti, "Factorisation of N = 2 Theories on the Squashed 3-Sphere," *JHEP* **1204** (2012) 120, `arXiv:1111.6905 [hep-th]`.

[108] M. Taki, "Holomorphic Blocks for 3d Non-abelian Partition Functions," `arXiv:1303.5915 [hep-th]`.

[109] M. Fujitsuka, M. Honda, and Y. Yoshida, "Higgs branch localization of 3d N = 2 theories," *PTEP* **2014** no. 12, (2014) 123B02, `arXiv:1312.3627 [hep-th]`.

[110] F. Benini and W. Peelaers, "Higgs branch localization in three dimensions," *JHEP* **1405** (2014) 030, `arXiv:1312.6078 [hep-th]`.

[111] N. Seiberg and E. Witten, "Gauge dynamics and compactification to three-dimensions," *In *Saclay 1996, The mathematical beauty of physics** (1996) 333–366, `arXiv:hep-th/9607163 [hep-th]`.

[112] F. Dolan, V. Spiridonov, and G. Vartanov, "From 4d superconformal indices to 3d partition functions," *Phys.Lett.* **B704** (2011) 234–241, `arXiv:1104.1787 [hep-th]`.

[113] A. Gadde and W. Yan, "Reducing the 4d Index to the S^3 Partition Function," *JHEP* **1212** (2012) 003, `arXiv:1104.2592 [hep-th]`.

[114] Y. Imamura, "Relation between the 4d superconformal index and the S^3 partition function," *JHEP* **1109** (2011) 133, `arXiv:1104.4482 [hep-th]`.

[115] V. Niarchos, "Seiberg dualities and the 3d/4d connection," *JHEP* **1207** (2012) 075, `arXiv:1205.2086 [hep-th]`.

[116] F. J. van de Bult, "Hyperbolic hypergeometric functions," *PhD thesis* (2007) .

[117] D. Kutasov and A. Schwimmer, "On duality in supersymmetric Yang-Mills theory," *Phys. Lett.* **B354** (1995) 315–321, `arXiv:hep-th/9505004 [hep-th]`.

[118] B. Willett and I. Yaakov, "N=2 Dualities and Z Extremization in Three Dimensions," `arXiv:1104.0487 [hep-th]`.

[119] T. Dimofte and D. Gaiotto, "An E7 Surprise," *JHEP* **1210** (2012) 129,

`arXiv:1209.1404 [hep-th]`.

[120] A. Khmelnitsky, "Interpreting multiple dualities conjectured from superconformal index identities," *JHEP* **1003** (2010) 065, `arXiv:0912.4523 [hep-th]`.

[121] H.-C. Kim, S.-S. Kim, and K. Lee, "5-dim Superconformal Index with Enhanced En Global Symmetry," *JHEP* **10** (2012) 142, `arXiv:1206.6781 [hep-th]`.

[122] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, "3*d* dualities from 4*d* dualities for orthogonal groups," *JHEP* **1308** (2013) 099, `arXiv:1307.0511`.

[123] J. Teschner and G. Vartanov, "6j symbols for the modular double, quantum hyperbolic geometry, and supersymmetric gauge theories," *Lett.Math.Phys.* **104** (2014) 527–551, `arXiv:1202.4698 [hep-th]`.

[124] O. Aharony, "IR duality in d = 3 N=2 supersymmetric USp(2N(c)) and U(N(c)) gauge theories," *Phys.Lett.* **B404** (1997) 71–76, `arXiv:hep-th/9703215 [hep-th]`.

[125] A. Karch, "Seiberg duality in three-dimensions," *Phys.Lett.* **B405** (1997) 79–84, `arXiv:hep-th/9703172 [hep-th]`.

[126] J. de Boer, K. Hori, Y. Oz, and Z. Yin, "Branes and mirror symmetry in N=2 supersymmetric gauge theories in three-dimensions," *Nucl.Phys.* **B502** (1997) 107–124, `arXiv:hep-th/9702154 [hep-th]`.

[127] C. Csaki, "The Confining N=1 supersymmetric gauge theories: A Review," in *Proc. 3rd Workshop, QCD'98, 1998*, pp. 489–499. 1998. `arXiv:hep-th/9807222 [hep-th]`.

[128] A. Amariti, C. Csaki, M. Martone, and N. R.-L. Lorier, "From S-confinement to 3D Chiral Theories: Dressing the Monopoles," `arXiv:1506.01017 [hep-th]`.

[129] A. Amariti, D. Forcella, C. Klare, D. Orlando, and S. Reffert, "The braneology of 3D dualities," `arXiv:1501.06571 [hep-th]`.

[130] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, "Comments on Chern-Simons Contact Terms in Three Dimensions," *JHEP* **1209** (2012) 091, `arXiv:1206.5218 [hep-th]`.

[131] A. Amariti and C. Klare, "Chern-Simons and RG Flows: Contact with Dualities," *JHEP* **1408** (2014) 144, `arXiv:1405.2312 [hep-th]`.

[132] G. 't Hooft, "Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking," *NATO Sci. Ser. B* **59** (1980) 135.

[133] L. Alvarez-Gaume and E. Witten, "Gravitational Anomalies," *Nucl. Phys.* **B234** (1984) 269.

[134] A. N. Redlich, "Parity Violation and Gauge Noninvariance of the Effective Gauge Field Action in Three-Dimensions," *Phys. Rev.* **D29** (1984) 2366–2374.

[135] A. Dey, A. Hanany, P. Koroteev, and N. Mekareeya, "Mirror Symmetry in Three Dimensions via Gauged Linear Quivers," *JHEP* **06** (2014) 059, `arXiv:1402.0016 [hep-th]`.

[136] A. Amariti, "Integral identities for 3d dualities with SP(2N) gauge groups," `arXiv:1509.02199 [hep-th]`.

[137] Y. Imamura and D. Yokoyama, "S^3/Z_n partition function and dualities," *JHEP* **11** (2012) 122, `arXiv:1208.1404 [hep-th]`.

[138] Y. Imamura, H. Matsuno, and D. Yokoyama, "Factorization of the $S^3/\mathbb{Z}_n$ partition function," *Phys. Rev.* **D89** no. 8, (2014) 085003, `arXiv:1311.2371 [hep-th]`.

[139] F. Nieri and S. Pasquetti, "Factorisation and holomorphic blocks in 4d," `arXiv:1507.00261 [hep-th]`.

[140] A. Kapustin, H. Kim, and J. Park, "Dualities for 3d Theories with Tensor Matter," *JHEP* **1112** (2011) 087, `arXiv:1110.2547 [hep-th]`.

[141] H. Kim and J. Park, "Aharony Dualities for 3d Theories with Adjoint Matter," *JHEP* **1306** (2013) 106, `arXiv:1302.3645 [hep-th]`.

[142] J. Park and K.-J. Park, "Seiberg-like Dualities for 3d N=2 Theories with SU(N) gauge group," *JHEP* **1310** (2013) 198, `arXiv:1305.6280 [hep-th]`.

[143] H. Rosengren, "Askey-Wilson polynomials and the tetrahedron index," *talk at Conference for Dick Askey's 80th Birthday* (6-7 December, 2013) . `http://www.math.umn.edu/{~}stant001/ASKEYABS/HjalmarRosengren.pdf`.

[144] M. Yamazaki, "Four-dimensional superconformal index reloaded," *Theor.Math.Phys.* **174** (2013) 154–166.

[145] A. P. Kels, "A new solution of the star-triangle relation," *J.Phys.* **A47** no. 5, (2014) 055203, `arXiv:1302.3025 [math-ph]`.

[146] V. Spiridonov and G. Vartanov, "Vanishing superconformal indices and the chiral symmetry breaking," *JHEP* **1406** (2014) 062, `arXiv:1402.2312 [hep-th]`.

[147] M. J. Strassler, "Confining phase of three-dimensional supersymmetric quantum electrodynamics," *In *Shifman, M.A. (ed.): The many faces of the superworld** (1999) , `arXiv:hep-th/9912142 [hep-th]`.

[148] G. Gasper and M. Rahman, *Basic hypergeometric series*. Cambridge University Press, second ed., 2004.

[149] U. von Pachner, "Konstruktionsmethoden und das kombinatorische homöomorphieproblem für triangulationen kompakter semilinearer mannigfaltigkeiten," in *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 57, pp. 69–86. 1987.

[150] U. Pachner, "P.l. homeomorphic manifolds are equivalent by elementary 5hellingst," *European journal of Combinatorics* **12** no. 2, (1991) 129–145.

[151] L. Faddeev and R. Kashaev, "Quantum Dilogarithm," *Mod.Phys.Lett.* **A9** (1994) 427–434, `arXiv:hep-th/9310070 [hep-th]`.

[152] R. Kashaev, "On beta pentagon relations," `arXiv:1403.1298 [math-ph]`.

[153] J. E. Andersen and R. Kashaev, "Faddeev‚Äôs Quantum Dilogarithm and State-Integrals on Shaped Triangulations," in *Proc. Mathematical Aspects of Quantum Field Theory*, pp. 131–152. Springer, 2015.

[154] R. M. Kashaev and T. Nakanishi, "Classical and Quantum Dilogarithm Identities," *SIGMA* **7** (2011) 102, `arXiv:1104.4630 [math.QA]`.

[155] R. M. Kashaev and S. M. Sergeev, "On pentagon, ten term, and tetrahedron relations," `arXiv:q-alg/9607032 [q-alg]`.

[156] L. D. Faddeev, "Volkov's Pentagon for the Modular Quantum Dilogarithm,"

Funct. Anal. Appl. **45** (2011) 291, arXiv:1201.6464 [math.QA].

[157] A. Yu. Volkov, "Pentagon Identity Revisited I," arXiv:1104.2267 [math.QA].

[158] A. Yu. Volkov, "Beyond the 'pentagon identity'," *Lett. Math. Phys.* **39** (1997) 393–397, arXiv:q-alg/9603003 [q-alg].

[159] V. P. Spiridonov, "On the elliptic beta function," *Russ. Math. Surv.* **56** no. 1, (2001) 185.

[160] Y. Terashima and M. Yamazaki, "SL(2,R) Chern-Simons, Liouville, and Gauge Theory on Duality Walls," *JHEP* **1108** (2011) 135, arXiv:1103.5748 [hep-th].

[161] Y. Terashima and M. Yamazaki, "Semiclassical Analysis of the 3d/3d Relation," *Phys.Rev.* **D88** no. 2, (2013) 026011, arXiv:1106.3066 [hep-th].

[162] Y. Terashima and M. Yamazaki, "Emergent 3-manifolds from 4d Superconformal Indices," *Phys.Rev.Lett.* **109** (2012) 091602, arXiv:1203.5792 [hep-th].

[163] T. Dimofte, "3d Superconformal Theories from Three-Manifolds," arXiv:1412.7129 [hep-th].

[164] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," *Lett. Math. Phys.* **91** (2010) 167–197, arXiv:0906.3219 [hep-th].

[165] S. Garoufalidis, "The 3D index of an ideal triangulation and angle structures," arXiv:1208.1663 [math.GT].

[166] S. Garoufalidis, C. D. Hodgson, J. H. Rubinstein, and H. Segerman, "1-efficient triangulations and the index of a cusped hyperbolic 3-manifold," arXiv:1303.5278 [math.GT].

[167] I. Gahramanov, "Mathematical structures behind supersymmetric dualities," arXiv:1505.05656 [math-ph].

[168] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, vol. 71. Cambridge university press, 1999.

[169] V. V. Mangazeev, "On the Yang-Baxter equation for the six-vertex model," *Nucl. Phys.* **B882** (2014) 70–96, `arXiv:1401.6494 [math-ph]`.

[170] V. V. Mangazeev, "Q-operators in the six-vertex model," *Nucl. Phys.* **B886** (2014) 166–184, `arXiv:1406.0662 [math-ph]`.

[171] V. V. Bazhanov, V. V. Mangazeev, and S. M. Sergeev, "Quantum geometry of 3-dimensional lattices," *J. Stat. Mech.* **0807** (2008) P07004, `arXiv:0801.0129 [hep-th]`.

[172] V. V. Mangazeev, V. V. Bazhanov, and S. M. Sergeev, "An integrable 3D lattice model with positive Boltzmann weights," *J. Phys.* **A46** (2013) 465206, `arXiv:1308.4773 [math-ph]`.

[173] V. V. Bazhanov, V. V. Mangazeev, and S. M. Sergeev, "Exact solution of the Faddeev-Volkov model," *Phys. Lett.* **A372** (2008) 1547–1550, `arXiv:0706.3077 [cond-mat.stat-mech]`.

[174] V. V. Bazhanov, V. V. Mangazeev, and S. M. Sergeev, "Faddeev-Volkov solution of the Yang-Baxter equation and discrete conformal symmetry," *Nucl. Phys.* **B784** (2007) 234–258, `arXiv:hep-th/0703041 [hep-th]`.

[175] V. V. Bazhanov, A. P. Kels, and S. M. Sergeev, "Comment on star-star relations in statistical mechanics and elliptic gamma-function identities," *J. Phys.* **A46** (2013) 152001, `arXiv:1301.5775 [math-ph]`.

[176] V. V. Bazhanov and S. M. Sergeev, "Elliptic gamma-function and multi-spin solutions of the Yang-Baxter equation," *Nucl.Phys.* **B856** (2012) 475–496, `arXiv:1106.5874 [math-ph]`.

[177] S. E. Derkachov and V. P. Spiridonov, "Yang-Baxter equation, parameter permutations, and the elliptic beta integral," *Russ. Math. Surveys* **68** (2013) 1027–1072, `arXiv:1205.3520 [math-ph]`.

[178] D. Chicherin, S. E. Derkachov, and V. P. Spiridonov, "New elliptic solutions of the Yang-Baxter equation," `arXiv:1412.3383 [math-ph]`.

[179] V. V. Bazhanov and S. M. Sergeev, "A Master solution of the quantum Yang-Baxter equation and classical discrete integrable equations,"

arXiv:1006.0651 [math-ph].

[180] R. J. Baxter, *Exactly solved models in statistical mechanics.* Courier Corporation, 2007.

[181] L. D. Faddeev, "How algebraic Bethe ansatz works for integrable model," in *Proc. School of Physics, Les Houches, France, 1995*, pp. pp. 149–219. arXiv:hep-th/9605187 [hep-th].

[182] M. Jimbo, "Yang-Baxter equation in integrable systems," *Adv. Ser. Math. Phys.* **10** (1989) 1–715.

[183] L. A. Takhtajan and L. D. Faddeev, "The Quantum method of the inverse problem and the Heisenberg XYZ model," *Russ. Math. Surveys* **34** (1979) 11–68. [Usp. Mat. Nauk34,13(1979)].

[184] N. Seiberg, "Exact results on the space of vacua of four-dimensional SUSY gauge theories," *Phys.Rev.* **D49** (1994) 6857–6863, arXiv:hep-th/9402044 [hep-th].

[185] L. Onsager, "Crystal statistics. 1. A Two-dimensional model with an order disorder transition," *Phys. Rev.* **65** (1944) 117–149.

[186] R. J. Baxter, "Hard hexagons: exact solution," *Journal of Physics A: Mathematical and General* **13** no. 3, (1980) L61.

[187] V. A. Fateev and A. B. Zamolodchikov, "Selfdual solutions of the star triangle relations in Z(N) models," *Phys. Lett.* **A92** (1982) 37–39.

[188] M. Kashiwara and T. Miwa, "A Class of Elliptic Solutions to the Star Triangle Relation," *Nucl. Phys.* **B275** (1986) 121.

[189] G. von Gehlen and V. Rittenberg, "$Z(n)$ Symmetric Quantum Chains With an Infinite Set of Conserved Charges and $Z(n)$ Zero Modes," *Nucl. Phys.* **B257** (1985) 351.

[190] H. Au-Yang, B. M. McCoy, J. H. H. perk, S. Tang, and M.-L. Yan, "Commuting transfer matrices in the chiral Potts models: Solutions of Star triangle equations with genus > 1," *Phys. Lett.* **A123** (1987) 219–223.

[191] R. J. Baxter, J. H. H. Perk, and H. Au-Yang, "New solutions of the star

triangle relations for the chiral Potts model," *Phys. Lett.* **A128** (1988) 138–142.

[192] L. D. Faddeev, "Current - like variables in massive and massless integrable models," in *Proc. 127th Course of the International School of Physics 'Enrico Fermi', Varenna.* 1994. `arXiv:hep-th/9408041 [hep-th]`.

[193] A. Yu. Volkov, "Quantum Volterra model," *Phys. Lett.* **A167** (1992) 345–355.

[194] G. E. Andrews, R. J. Baxter, and P. J. Forrester, "Eight vertex SOS model and generalized Rogers-Ramanujan type identities," *J. Statist. Phys.* **35** (1984) 193–266.

[195] V. P. Spiridonov, "A bailey tree for integrals," *Theoretical and mathematical physics* **139** no. 1, (2004) 536–541.

[196] S. E. Derkachov and A. N. Manashov, "General solution of the Yang-Baxter equation with symmetry group SL(n,C)," *St. Petersburg Math. J.* **21** (2010) 513–577. [Alg. Anal.21N4,1(2009)].

[197] R. J. Baxter, "The inversion relation method for some two-dimensional exactly solved models in lattice statistics," *J. Statist. Phys.* **28** (1982) 1–41.

[198] R. J. Baxter, "Eight vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. 1. Some fundamental eigenvectors," *Annals Phys.* **76** (1973) 1–24.

[199] R. J. Baxter, "Eight vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. 2. Equivalence to a generalized ice-type lattice model," *Annals Phys.* **76** (1973) 25–47.

[200] R. J. Baxter, "Eight vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. 1. Eigenvectors of the transfer matrix and Hamiltonian," *Annals Phys.* **76** (1973) 48–71.

[201] P. A. Pearce and K. A. Seaton, "A solvable hierarchy of cyclic solid-on-solid lattice models," *Phys. Rev. Lett.* **60** (1988) 1347–1350.

[202] P. A. Pearce and K. A. Seaton, "Exact solution of cyclic solid-on-solid lattice models," *Annals Phys.* **193** (1989) 326–366.

[203] V. Pasquier, "Continuum Limit of Lattice Models Built on Quantum Groups,"

Nucl. Phys. **B295** (1988) 491.

[204] V. Pasquier, "Etiology of IRF Models," *Commun. Math. Phys.* **118** (1988) 355.

[205] J. Perk and F. Wu, "Nonintersecting string model and graphical approach: Equivalence with a potts model," *Journal of statistical physics* **42** no. 5-6, (1986) 727–742.

[206] V. P. Spiridonov, "Theta hypergeometric integrals," *Algebra i Analiz* **15** no. 6, (2003) 161, arXiv:math/0303205.

[207] R. J. Baxter, "Free-fermion, checkerboard and Z-invariant lattice models in statistical mechanics," *Proc. Roy. Soc. Lond.* **A404** (1986) 1–33.

[208] R. J. Baxter, "Star-triangle and star-star relations in statistical mechanics," *Int. J. Mod. Phys.* **B11** (1997) 27–37.

[209] S. N. M. Ruijsenaars, "First order analytic difference equations and integrable quantum systems," *J. Math. Phys.* **38** no. 2, (1997) 1069–1146.

[210] M. Nishizawa, "An elliptic analogue of the multiple gamma function," *J. Phys. A* **34** no. 36, (2001) 7411.

[211] A. Narukawa, "The modular properties and the integral representations of the multiple elliptic gamma functions," *Adv. Math.* **189** no. 2, (2004) 247–267, arXiv:math/0306164.

[212] L. Tizzano and J. Winding, "Multiple sine, multiple elliptic gamma functions and rational cones," arXiv:1502.05996 [math.CA].

[213] G. Felder and A. Varchenko, "The Elliptic Gamma Function and $SL(3, \neg\dagger Z) \times Z_3$," *Adv. Math.* **156** no. 1, (2000) 44 – 76, arXiv:math/9907061.

[214] E. Friedman and S. Ruijsenaars, "Shintani,Äìbarnes zeta and gamma functions," *Advances in Mathematics* **187** no. 2, (2004) 362 – 395.

[215] G. Gasper and M. Rahman, *Basic hypergeometric series.* Cambridge University Press, second ed., 2004.

[216] V. Spiridonov, "Aspects of elliptic hypergeometric functions," arXiv:1307.2876 [math.CA].

[217] W. A. Al-Salam and M. E. H. Ismail, "q-beta integrals and the q-Hermite polynomials.," *Pacific J. Math.* **135** no. 2, (1988) 209–221.

[218] R. Askey, "Ramanujan's Extensions of the Gamma and Beta Functions," *Amer. Math. Monthly* **87** no. 5, (1980) 346–359.

[219] V. P. Spiridonov, "Elliptic hypergeometric terms," in *SMF Séminaire et Congrès 23 (2011), 385-405*. 2010. `arXiv:1003.4491 [math.CA]`. `http://inspirehep.net/record/849713/files/arXiv:1003.4491.pdf`.

[220] F. J. van de Bult and E. M. Rains, "Limits of multivariate elliptic beta integrals and related bilinear forms," `arXiv:1110.1460`.

[221] F. J. van de Bult and E. M. Rains, "Limits of multivariate elliptic hypergeometric biorthogonal functions," `arXiv:1110.1458`.

[222] F. J. van de Bult and E. M. Rains, "Limits of elliptic hypergeometric biorthogonal functions," *J. Approx. Theor.* **193** no. 0, (2015) 128 – 163, `arXiv:1110.1456`.

[223] B. Nassrallah and M. Rahman, "Projection Formulas, a Reproducing Kernel and a Generating Function for q-Wilson Polynomials," *SIAM J. Math. Anal.* **16** no. 1, (1985) 186–197.

[224] M. Rahman, "An integral representation of a ${}_{10}\varphi_9$ and continuous bi-orthogonal ${}_{10}\varphi_9$ rational functions," *Canad. J. Math.* **38** no. 3, (1986) 605–618.

[225] R. Askey and J. A. Wilson, "Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials," *Mem. Amer. Math. Soc.* **319** (1985) .

[226] M. Spreafico, "On the barnes double zeta and gamma functions," *Journal of Number Theory* **129** no. 9, (2009) 2035 – 2063.

[227] D. Galakhov, A. Mironov, A. Morozov, and A. Smirnov, "Three-dimensional extensions of the Alday-Gaiotto-Tachikawa relation," *Theor.Math.Phys.* **172** (2012) 939–962, `arXiv:1104.2589 [hep-th]`.

[228] L. Hadasz, M. Pawelkiewicz, and V. Schomerus, "Self-dual Continuous Series of Representations for $U_q(sl(2))$ and $U_q(osp(1|2))$," `arXiv:1305.4596 [hep-th]`.