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A formalism to assess the accuracy of nuclear-structure weak interaction effects in precision β-decay studies

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Abstract

Multiple high precision β-decay measurements are being carried out these days on various nuclei, in search of beyond the Standard Model (SM) signatures. These measurements necessitate accurate SM theoretical predictions to be compared with. Motivated by the experimental surge, we present a formalism for such a calculation of β-decay observables, with controlled accuracy, based on a perturbative analysis of the theoretical observables related to the phenomena, including high order nuclear recoil and shape corrections. The accuracy of the corrections is analyzed by identifying a hierarchy of small parameters, related to the low-momentum transfer characterizing β-decays. Furthermore, we show that the sub-percent uncertainties, targeted by ongoing and planned experiments, entail an accuracy of the order of 10% for the solution of the nuclear many-body problem, which is well within the reach of modern nuclear theory for light to medium mass nuclei.

Keywords: beta decay, beyond the standard model signatures in nuclear physics reactions, theoretical uncertainty assessment

1. Introduction

According to the Standard Model (SM), the weak interaction of quarks and weak probes has a *V*–*A* structure [1, 2], i.e., a polar-vector minus an axial-vector current, with the same magnitude but opposite signs. This has been experimentally tested in many ways, including by studying correlations in the directions of the outgoing leptons in nuclear β-decays [3, 4].

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In the last decade, high precision angular correlation and spectrum measurements in nuclear β-decays are carefully designed to probe for signatures of deviations of the weak interaction from the V–A symmetry dictated by the SM, originating in interactions with scalar, pseudo-scalar, or tensor symmetries, which are *a priori* theoretically allowed in Lorentz invariant theories [5]. Such a signature would be an indication of beyond the Standard Model (BSM) physics. Dimensional analysis shows that the order of magnitude of such deviation is inversely proportional to the square of the new physics energy scale, e.g., a 1% deviation from the SM values of those experimental observables suggests new physics at the TeV-scale. Thus, such measurements can be viewed as low-energy counterparts for LHC measurements. New β-decay experiments are designed to have a per-mil level accuracy [6]. Such new efforts include, for example, studying the decays of ^{6}He, ^{16}N, and $^{17-23}$Ne isotopes at SARAF accelerator (Israel) [7, 8], ^{6}He decay studies at the National Superconducting Cyclotron Laboratory (USA) [9–11], Laboratoire Physique Corpusculaire de CAEN (France) [6], the University of Washington CENPA (USA) by the He6-CRES Collaboration [12], and more (for an overview of ongoing and planned experiments see [6, 13]). However, deviations from the simplistic text-book formulae are also due to finite momentum transfer, radiative corrections, and nuclear structure effects (see e.g., [14, 15] and references therein). Pin-pointing these effects demands a detailed calculation of the nuclear dynamics of the weak decay. In this paper, we present a general formalism to calculate the nuclear matrix elements of these observables, within the SM, including e.g., recoil and (spectrum-)shape corrections. The main advantage of the presented formalism is the possibility to assess a controlled theoretical accuracy, based on an identification of the parameters governing this accuracy.

This paper is structured as follows: we begin by identifying small parameters involved in the β-decay formalism (section 2). Next, we introduce the β-decay multipole expansion, analyze its properties (section 3), and employ it to formulate a general perturbative expansion of the observables, including high order corrections, for any β-decay transition (section 4). Following that, we demonstrate the corrections for allowed β-decays (section 5), and for first-forbidden decays (section 6). Finally, we discuss the expected order of magnitude of BSM signatures, then consider the accuracy required for SM calculations to detect BSM deviations in nuclear β-decay measurements (section 7), and summarize (section 8).

2. Small parameters governing nuclear structure effects of β-decays of finite nuclei

The differential distribution of a β-electron of energy E, momentum $\vec{k}$ and direction $\vec{\beta} = \frac{\vec{k}}{E}$, and a neutrino ν of momentum $\vec{\nu}$, in a $\beta^{\mp}$-decay process, can be written as [16]:

$$\frac{\mathrm{d}^5\omega}{\mathrm{d}E\frac{\mathrm{d}\hat{k}}{4\pi}\frac{\mathrm{d}\hat{\nu}}{4\pi}} = \frac{4}{\pi^2}(E_0 - E)^2 kEF^{\mp}(Z_\mathrm{f}, E)\, C_\mathrm{corr}\frac{1}{2J_\mathrm{i}+1}\Theta\left(q, \vec{\beta}\cdot\hat{\nu}\right). \tag{1}$$

Here $(E_0, \vec{q}) = (E, \vec{k}) + (\nu, \vec{\nu})$ is the momentum transfer in the process, i.e., the difference between the initial momentum and final momentum of the nuclear states. J_i is the total angular momentum of the decaying nucleus, and Z_f is the charge of the nucleus after the decay. The deformation of the β-particle wave function, due to the long-range electromagnetic interaction with the nucleus, is taken into account at the leading order by the Fermi function $F^{\mp}(Z_\mathrm{f}, E) = 4(2kR_\mathrm{f})^{2(\gamma_0-1)}\frac{|\Gamma(\gamma_0+\mathrm{i}y)|^2}{[\Gamma(2\gamma_0+1)]^2}\mathrm{e}^{\pi y}$, with $\gamma_0 = \sqrt{1-(\alpha Z_\mathrm{f})^2}$ and $y = \pm\frac{\alpha Z_\mathrm{f} E}{k}$ for a $\beta^{\mp}$-decay (in the notation of [14], that will be the traditional Fermi function F_0).

Corrections that do not originate purely in the weak matrix element, such as radiative corrections, finite-mass and electrostatic finite-size effects, and atomic and chemical effects, are represented in (1) by $C_{\rm corr}$. In the notation of, e.g., [14], these will be the finite size of the nucleus L_0, radiative corrections R, atomic exchange X, atomic mismatch (including shake-up and shake-off) r, atomic screening S and molecular screening $\Delta S_{\rm Mol}$. As these corrections are assumed known in the literature (see, e.g., [14]), and do not affect the observables we are interested in, we will not focus on them, but on effects originating from the fact that the weak interaction of a probe with a finite nucleus is affected by the complex nuclear structure of the decaying and final nuclei. These are absorbed inside $\Theta(q, \vec{\beta}\cdot\hat{\nu})$. Currently, experimental efforts focus primarily on spectrum (shape) or angular correlation measurements [6], which are affected by the nuclear structure. Our discussion will distinguish between corrections to these measurements.

In order to compare the experimental results to the theory, there is a need for a good understanding of the theoretical framework. This framework is simplified by an expansion in a set of small parameters. In the following, we review the different small parameters and their typical size in β-decays. This allows the controlled perturbative expansion of the β-decay's observables.

2.1. Small parameters related to the kinematics of the decay and the nuclear problem

The main simplification is achieved by a clear separation of scales between the typical momenta of the nucleus and the decay. I.e., the momentum transfer q is limited in β-decays, usually up to few tens MeV/c, typically up to 10 MeV/c, and it is small compared to other energy or momentum scales in the nuclear problem. In addition, the dynamics of nucleons within the nucleus is mostly non-relativistic. These allow the introduction of the following small parameters:

$$\epsilon_{\rm NR} \sim \frac{P_{\rm Fermi}}{m_{\rm N}}, \tag{2a}$$

$$\epsilon_{qr} \sim qR, \tag{2b}$$

$$\epsilon_{\rm recoil} \sim \frac{q}{m_{\rm N}}, \tag{2c}$$

$$\epsilon_{M_\pi} \sim \frac{q}{m_\pi}, \tag{2d}$$

$$\epsilon_{M} \sim \frac{\Delta M}{M_{\rm min}}, \tag{2e}$$

where $P_{\rm Fermi}$ is the Fermi momentum, $m_{\rm N}$ (m_π) is the mass of the nucleon (pion), R is the radius of the nucleus, $\Delta M \equiv M_{\rm i} - M_{\rm f}$, $M_{\rm min} \equiv \min(M_{\rm i}, M_{\rm f})$, and $M_{\rm i}$ ($M_{\rm f}$) is the mass of the initial (final) nucleus. While ϵ_{qr} comes from the expansion of the leptonic wave function, the non-relativistic expansion small parameter $\epsilon_{\rm NR}$, the recoil correction $\epsilon_{\rm recoil}$, and the pion correction ϵ_{m_π}, are all related to the low-energy assumption taken in the calculation of the hadronic currents. ϵ_M derives from the nuclear recoil of the final nucleus (as opposed to $\epsilon_{\rm recoil}$, which is related to the recoil of the nucleon and therefore larger). Inside the nucleus, the Fermi momentum is approximately 200 MeV/c, so $\epsilon_{\rm NR} \sim 0.2$, while, e.g., for an endpoint energy of $\approx$2 MeV, $\epsilon_{qr} \sim 0.02A^{\frac{1}{3}}$, $\epsilon_{\rm recoil} \sim 0.003$, $\epsilon_{m_\pi} \sim 0.02$, and $\epsilon_M \sim 0.003A^{-1}$, where A is the mass number of the nucleus. Since each small parameter has its own size, which itself varies for different momentum transfers q and different mass numbers A, one small parameter will be greater than another for one decay, and vice versa for another decay. That makes the division of the different corrections of the nuclear structure into different orders of magnitudes somewhat subject to interpretation. As will be discussed in section 7, the required accuracy of the measurements, as well as calculations, is about 10^{-3} to 10^{-4}. In addition, contemporary precision β-decay experiments are aimed at a level of 10^{-3}. In the following we tried to keep corrections that may become useful for this accuracy, leaning also on our experience with interpreting this formalism for ^{6}He [17] and ^{23}Ne [18]. Therefore, we refer to anything beyond the leading order, but still varying around 10^{-3} relative to it, as a next-to-leading order (NLO) correction include ths NLO corrections and neglect its squared value, which we label as a next-to-next-to-leading order (NNLO) correction. Assuming an endpoint of $\approx$2 MeV, those NLO corrections include the ones proportional to $\epsilon_{\text{recoil}} \sim 10^{-3}$, $\epsilon_{qr}\epsilon_{\text{NR}}$ which vary between 10^{-3} and 10^{-2} for heavy nuclei, ϵ_{qr}^2 which vary between 10^{-4} and 5×10^{-3} for heavy nuclei, and ϵ_M which can reach almost 10^{-3} for very light nuclei. Although this NLO range is quite wide, we find it useful for the purpose of interpreting different precision β-decay experiments, considering the verification of these small parameters. However, we note the order of magnitude for each correction we present, making it easy to omit any terms that are not relevant for the specific β-decay the reader is interested in.

2.2. Coulomb corrections

In the presence of the Coulomb force between the β-particle and the nucleus, the wave function of the β-particle is deformed from a plane wave [19]. This is a source of deviations from the simplistic β-decay formulas, dominated by [20]

$$\epsilon_{\text{c}} \sim \alpha Z_{\text{f}}, \tag{2f}$$

where $\alpha \approx \frac{1}{137}$ is the fine structure constant. These deviations can be divided, as described in [21], into two:

(A) corrections to the β-energy spectrum. In the leading order, they are reflected in the Fermi function (assuming an infinitely heavy point-charge model of the nucleus), whose specific form was mentioned above. Additional spectrum corrections are of the order of ϵ_{c}^2 or $\epsilon_{\text{c}}\epsilon_{qr}$ [22], and their precise form depends on the specific β-transition. Convenient forms of them for the spectrum (angular correlation) of Fermi and Gamow–Teller transitions, can be found in [14] ([23]), albeit in a homogeneous or modified Gaussian distribution nuclear charge approximations. For a general transition, their contributions at leading orders are listed in [24]. These corrections are small for very light nuclei, but as the nuclear charge grows ($Z \gtrsim 7$) they account for a substantial contribution (a per-mil of the SM value). Since current experimental efforts are focused on light nuclei, we postpone a thorough derivation of these corrections in a form that fits modern *ab initio* calculations, to future work.

(B) Corrections to the multipole operators. Two types of corrections can be distinguished here: corrections resembling the spectrum corrections, of $\epsilon_{\text{c}}\epsilon_{qr}\epsilon_{\text{recoil}}$ orders of magnitude [19], and therefore negligible at this stage, and a correction to the energy transfer E_0. In the latter, the difference between the Coulomb potentials of the decaying and final nucleus, ΔE_{c}, is added to the maximal electron energy E_0 [24] in the multipole operators (see appendix). This correction is often taken by its approximation for a homogeneously charged sphere, $\Delta E_{\text{c}} \approx \frac{6}{5}\frac{\alpha Z_{\text{f}}}{R}$ [24]. To compare with accurate experimental measurements, we recommend using the experimental value for the Coulomb displacement energy between a pair of isobaric analog levels, which is given by $\Delta E_{\text{c}} \equiv M_a(Z_>) - M_a(Z_<) + \Delta_{\text{nH}}$, where $M_a(Z_>)$ ($M_a(Z_<)$) is the atomic mass of the larger (smaller) Z member of the analog pair, and Δ_{nH} is the neutron–hydrogen mass difference [25]. These can be found in the literature with high accuracy for most nuclei and states (see, e.g., [25, 26]).

2.3. Uncertainty of the nuclear model

The nuclear model is a source of uncertainty, as it is a low-energy effective model of quantum chromodynamics (QCD). As QCD is non-perturbative, a model is written to describe the Hamiltonian and currents representing the quantum process in the nuclear regime. There are several approaches to cope with this problem. A modern one is based on the effective field theory (EFT) approach. EFT expansion relies on scale separation in the low-energy nuclear momentum–energy. As β-decays and nuclei are characterized by low energies compared to QCD scales, they can be described effectively by chiral EFT (χEFT), an EFT that includes nucleons and pions. EFT is a systematic expansion of the Hamiltonian of the fundamental theory, order-by-order, in the small parameter [6]

$$\epsilon_{\mathrm{EFT}} \sim Q/\Lambda_{\mathrm{b}}, \tag{2g}$$

where Λ_{b} is the breakdown scale of the EFT. We note that this procedure includes a systematic uncertainty, $\epsilon_{\mathrm{EFT}}^{n+1}$, where n is the order of the EFT expansion. However, χEFT is combined with a specific approach to non-relativistically expand the nuclear theory, a specific cutoff regularization, and more. This introduces an additional systematic error, $\epsilon_{\mathrm{model}}$. Finally, the resulting equations are solved numerically, introducing a third source of uncertainty, the convergence error ϵ_{conv}. The nuclear model creates significant uncertainty that arises from EFT, its many-body modeling, and the computation convergence. In conclusion, the nuclear model creates significant uncertainty that arises from EFT, its many-body modeling, and the computational convergence and approximations. The fact that the parameters in the nuclear model are calibrated using a comparison between calculation and nuclear data, creates an inherent dependence between these three sources of uncertainties. As a result, estimating the resulting total uncertainty is achieved by investigating a specific reaction. However, in many cases, there is one dominant source of uncertainty. For example, ϵ_{EFT} in light nuclei (where converged calculations with highly detailed Hamiltonians are feasible). Thus, as an order of magnitude one can use $\epsilon_{\mathrm{NM}} \equiv \max(\epsilon_{\mathrm{EFT}}, \epsilon_{\mathrm{model}}, \epsilon_{\mathrm{conv}})$ (NM is nuclear model). Estimating ϵ_{NM} should be done case by case (see ^{6}He [17] and ^{23}Ne [18] as examples). We note that a recent effort to calculate β-decays *ab initio* up to $A = 100$ has reached precision of the order of 10% of the Gamow–Teller strength [27].

In the following, we trace the scaling of different nuclear-structure-related corrections to the aforementioned dimensionless small parameters.

3. β-decay multipole expansion formalism

Assuming the SM *V–A* coupling, the function $\Theta(q, \vec{\beta} \cdot \hat{\nu})$ depends on the nuclear wave functions. Taking the nucleus to be infinitely massive by equating the lab frame to the brick wall frame, $\Theta(q, \vec{\beta} \cdot \hat{\nu})$ can be conveniently written using a multipole expansion [28],

$$\begin{aligned}
\Theta\left(q, \vec{\beta} \cdot \hat{\nu}\right) = \sum_{J=1}^{\infty} \Big[& \left(1 - (\hat{\nu} \cdot \hat{q})\left(\vec{\beta} \cdot \hat{q}\right)\right)\left(\left|\left\langle\left\|\hat{E}_J\right\|\right\rangle\right|^2 + \left|\left\langle\left\|\hat{M}_J\right\|\right\rangle\right|^2\right) \\
& \pm \hat{q} \cdot \left(\hat{\nu} - \vec{\beta}\right) 2\,\mathfrak{Re}\left(\left\langle\left\|\hat{E}_J\right\|\right\rangle\left\langle\left\|\hat{M}_J\right\|\right\rangle^*\right)\Big] \\
& + \sum_{J=0}^{\infty} \Big[\left(1 - \vec{\beta} \cdot \hat{\nu} + 2(\hat{\nu} \cdot \hat{q})\left(\vec{\beta} \cdot \hat{q}\right)\right)\left|\left\langle\left\|\hat{L}_J\right\|\right\rangle\right|^2 \\
& + \left(1 + \vec{\beta} \cdot \hat{\nu}\right)\left|\left\langle\|\hat{C}_J\|\right\rangle\right|^2 - \hat{q} \cdot \left(\hat{\nu} + \vec{\beta}\right) 2\,\mathfrak{Re}\left(\left\langle\left\|\hat{L}_J\right\|\right\rangle\left\langle\|\hat{C}_J\|\right\rangle^*\right)\Big],
\end{aligned} \tag{3}$$

where $\langle\|\hat{O}_J\|\rangle$ is the reduced matrix element of a rank J spherical tensor operator $\hat{O}_J$, between the initial- and final-nuclear state wave functions. We use a multipole operator decomposition of the nuclear current, viz the Coulomb, longitudinal, electric, and magnetic operators, defined as:

$$\hat{C}_J(q) \equiv \int \mathrm{d}^3 r\, j_J(qr) Y_J(\hat{r}) \mathcal{J}_0(\vec{r}), \tag{4a}$$

$$\hat{L}_J(q) \equiv \frac{\mathrm{i}}{q} \int \mathrm{d}^3 r \left[\vec{\nabla}(j_J(qr) Y_J(\hat{r}))\right] \cdot \vec{\mathcal{J}}(\vec{r}), \tag{4b}$$

$$\hat{E}_J(q) \equiv \frac{1}{q} \int \mathrm{d}^3 r \left[\vec{\nabla} \times (j_J(qr) \vec{Y}_{JJ1}(\hat{r}))\right] \cdot \vec{\mathcal{J}}(\vec{r}), \tag{4c}$$

$$\hat{M}_J(q) \equiv \int \mathrm{d}^3 r\, j_J(qr) \vec{Y}_{JJ1}(\hat{r}) \cdot \vec{\mathcal{J}}(\vec{r}), \tag{4d}$$

where $\vec{\mathcal{J}}(\vec{r}\,)$ ($\mathcal{J}_0(\vec{r}\,)$) is the nuclear current (charge) coupling to the probe. j_J are the spherical Bessel functions, and Y_J and $\vec{Y}_{JI1}$ are the spherical harmonics and the vector spherical harmonics, respectively. The nuclear wave functions are commonly calculated numerically using an expansion in a complete basis. In some cases, analytical expressions can be derived, for example, for an expansion in harmonic oscillator wave functions, the nuclear matrix elements of the multipole operators can be calculated using [29], limiting the interaction of the weak probes to a single nucleon, and neglecting the interaction with many-body clusters of nucleons in the nucleus.

For a vanishing parameter, the spherical Bessel functions behave as $j_J(\rho) \propto \rho^J$ (for $\rho \ll 1$). As the momentum transfer q is small for β-decays, compared to any nuclear property, $\epsilon_{qr} \sim qR$ is a small parameter. Explicit analysis shows that $\hat{C}_J, \hat{M}_J \propto \epsilon_{qr}^J$, while $\hat{L}_J, \hat{E}_J \propto \epsilon_{qr}^{J-1}$.

A further simplification is a result of the parity of the multipole operators. Since the spherical harmonics Y_J and $\vec{Y}_{JJ1}^M$ have a parity of $(-)^J$, the basic parities of the multipole operators (before taking into consideration the nuclear current involved) are $(-)^J$ for $\hat{C}_J$ and $\hat{M}_J$, and $(-)^{J+1}$ for $\hat{L}_J$ and $\hat{E}_J$. To these, we add the parity of the nuclear currents. While the weak axial charge $\mathcal{J}_0^A$, and the vector current $\vec{\mathcal{J}}^V$, have a $\frac{\vec{p}}{2m_\mathrm{N}}$ dependence at their leading order, which leads to an internal negative parity, the weak vector charge $\mathcal{J}_0^V$, and the axial current $\vec{\mathcal{J}}^A$, have no momentum dependence at their leading order, leading to an internal positive parity. In conclusion, $\hat{C}_J^V$, $\hat{L}_J^V$, $\hat{E}_J^V$ and $\hat{M}_J^A$ (the superscript V (A) denotes multipole operators calculated with the vector (axial) symmetry contribution to the weak nuclear current) have a parity of $(-)^J$, while $\hat{C}_J^A$, $\hat{L}_J^A$, $\hat{E}_J^A$ and $\hat{M}_J^V$ have a parity of $(-)^{J+1}$. A detailed non-relativistic expansion of the one-body polar-vector and axial-vector currents in powers of ϵ_NR can be found in the appendix, along with a detailed derivation of the resulting multipole operators and their accuracy estimation.

The leading dependence of the nuclear matrix element in the momentum transfer q leads also to a characterization of β-decays into allowed and forbidden transitions. The allowed transitions are, at their leading order, independent of the momentum transfer q, and characterized by no change in the orbital angular momentum L, and therefore no change in parity. A Fermi (Gamow–Teller) transition is an allowed transition in which J, the change in the total angular momentum of the nucleus, is 0 (1). Other transitions are suppressed by a q^L dependence, with $L \geqslant 1$, and therefore are much slower, and are called, for historical reasons, forbidden transitions. These are divided into different L^{th}-forbidden transitions, each with a change of orbital angular momentum L, resulting in a parity change of $(-)^L$. β-decays in which the emitted leptons pair have a total spin angular momentum of $S = 1$, and an orbital angular momentum $L = J - 1$, are called unique $L^{\rm th}$-forbidden transitions.

The multipole operators depend upon the nuclear model via the nuclear current operators, and connect between the initial (decaying) and final nuclei. This is the source of the $\epsilon_{\rm NM}$ uncertainty described in section 2. We further elaborate on this in section 7.

4. A general expression for nuclear shape and recoil corrections

Assuming the SM V–A coupling, a β-decay transition, with a $J_{\rm i}^{\pi_{\rm i}} \to J_{\rm f}^{\pi_{\rm f}}$ angular momentum and parity change, contains all the integer total angular momentum changes which uphold the selection rule $|J_{\rm i} - J_{\rm f}| \leqslant J \leqslant J_{\rm i} + J_{\rm f}$, with the exact parity $\Delta\pi \equiv \pi_{\rm i} \cdot \pi_{\rm f}$:

$$\Theta\left(q, \vec{\beta} \cdot \hat{\nu}\right) = \sum_{J=|J_{\rm i}-J_{\rm f}|}^{J_{\rm i}+J_{\rm f}} \Theta^{J^{\Delta\pi}}\left(q, \vec{\beta} \cdot \hat{\nu}\right). \tag{5}$$

In the following, we present $\Theta^{J^{\Delta\pi}}(q, \vec{\beta} \cdot \hat{\nu})$ for each $J^{\Delta\pi}$, including shape and recoil NLO corrections. BSM contributions can be found in [30].

4.1. Fermi transition

Starting from $J = 0$, Fermi transition ($J^{\Delta\pi} = 0^+$) expression, including shape and recoil NLO corrections, can be written as

$$\Theta^{0^+}\left(q, \vec{\beta} \cdot \hat{\nu}\right) = \left(1 + \delta_1^{0^+}\right)\left[1 + a^{0^+}\vec{\beta} \cdot \hat{\nu} + b^{0^+}\frac{m_{\rm e}}{E}\right]\left|\langle\|\hat{C}_0^V\|\rangle\right|^2, \tag{6}$$

with the NLO spectrum-shape correction:

$$\delta_1^{0^+} = -\frac{E_0}{q} 2\,\mathfrak{Re}\frac{\langle\|\hat{L}_0^V\|\rangle}{\langle\|\hat{C}_0^V\|\rangle} + \frac{E - \Delta M}{M_{\rm i}} + \mathcal{O}\left(\epsilon_{\rm recoil}^2, \epsilon_{qr}^2\epsilon_{\rm NR}^2, \epsilon_M^2\right). \tag{7}$$

The multipole operator $\hat{C}_0^V \propto 1$ is the Fermi leading order, and $\hat{L}_0^V \propto \epsilon_{\rm recoil}, \epsilon_{qr}\epsilon_{\rm NR}$($\epsilon_{\rm recoil}$ and $\epsilon_{qr}\epsilon_{\rm NR}$ are about the same order of magnitude for an endpoint of $\approx$2MeV, and $\hat{L}_0^V$ consists of these two forms of terms) is its NLO nuclear structure correction. $\frac{E-\Delta M}{M_i}$ is a correction of the nuclear recoil, which does not depend on the nuclear structure, and is only significant for very light nuclei. For example, for the β-decay of ^{6}He, $\epsilon_M \sim 7 \cdot 10^{-4}$ $\left(\frac{E}{M_{\rm i}} < \frac{\Delta M}{M_{\rm i}} \sim \epsilon_M\right)$. The nuclear recoil correction results from the recoiled nucleus —if it is included in the density of states, then the differential distribution at (1), and therefore $\Theta^{J^{\Delta\pi}}(q, \vec{\beta} \cdot \hat{\nu})$, should be multiplied by a nuclear recoil correction factor:

$$r = \left(1 - \frac{\Delta M}{M_{\rm i}}\right)\left[1 + \frac{E}{M_{\rm i}}\left(1 - \vec{\beta} \cdot \hat{\nu}\right) + \mathcal{O}\left(\frac{E^2}{M_{\rm i}^2}\right)\right]\left[1 + \frac{q^2}{2M_{\rm f}^2} + \mathcal{O}\left(\frac{q^4}{M_{\rm f}^4}\right)\right]. \tag{8}$$

This introduces corrections of the order of ϵ_M $\left(\frac{q^2}{M_{\rm f}^2} \leqslant \frac{(\Delta M)^2}{M_{\rm f}^2} \sim \epsilon_M^2\right)$, such as the term $\frac{E-\Delta M}{M_{\rm i}}$ in (7), and other recoiled nucleus corrections that we will present in the following. For most nuclei, these recoiled nucleus corrections are neglectable, and they are only relevant in the case of very light nuclei.

One of the parameters playing an important role in BSM experimental searches, is the Fierz interference term, which can be extracted from electron energy spectrum measurements. Fierz term vanishes in the known *V–A* differential distribution of allowed β-decay transitions. Taking into account NLO nuclear structure corrections, an m_e/E spectral behavior appears, similar to that induced by Fierz interference term if BSM effects appear, with a nuclear-structure-dependent factor,

$$b^{0^+} = \delta_b^{0^+} \equiv \frac{m_e}{q} 2\,\mathfrak{Re}\frac{\langle \| \hat{L}_0^V \| \rangle}{\langle \| \hat{C}_0^V \| \rangle} + \mathcal{O}\left(\frac{m_e}{E}\epsilon_{\text{recoil}}^2, \frac{m_e}{E}\epsilon_{qr}^2\epsilon_{\text{NR}}^2\right). \tag{9}$$

Another parameter of interest for BSM searches is the angular correlation between the emitted electron and neutrino, a, which is the coefficient of the $\vec{\beta}\cdot\hat{\nu}$ term in the differential distribution of a β-decay. The *V–A* structure of the weak interaction entails that for a Fermi transition, there are no nuclear structure corrections to this β–ν angular correlation coefficient, and it is $a^{0^+} = 1 + \tilde{\delta}_a^{0^+}$, where

$$\tilde{\delta}_a^{0^+} = -\frac{E}{M_i}\left(1 + \vec{\beta}\cdot\hat{\nu}\right) + \mathcal{O}\left(\epsilon_M^2, \epsilon_M\epsilon_{\text{recoil}}, \epsilon_M\epsilon_{qr}\epsilon_{\text{NR}}\right) \tag{10}$$

is a recoiled nucleus correction, which is only significant for very light nuclei. Therefore, in most nuclei, a^{0^+} is simply 1.

4.2. Non-unique first-forbidden transition $J^{\Delta\pi} = 0^-$

As for $J^{\Delta\pi} = 0^-$, which is a non-unique first-forbidden transition, its $\Theta^{0^-}(q, \vec{\beta}\cdot\hat{\nu})$ expression takes the form:

$$\begin{aligned}\Theta^{0^-}\left(q, \vec{\beta}\cdot\hat{\nu}\right) = \left(1 + \delta_1^{0^-}\right)\Bigg\{&\left(1 + a^{0^-}\vec{\beta}\cdot\hat{\nu} + b^{0^-}\frac{m_e}{E}\right)\Bigg[|\langle\|\hat{C}_0^A\|\rangle|^2 \\ &- \frac{E_0}{q}2\mathfrak{Re}\left(\langle\|\hat{L}_0^A\|\rangle\langle\|\hat{C}_0^A\|\rangle^*\right) + \left|\langle\|\hat{L}_0^A\|\rangle\right|^2\Bigg] \\ &+ 2\frac{E(E_0 - E)}{q^2}\left(1 + \delta_{\beta^2}^{0^-}\right)\left[\beta^2 - \left(\vec{\beta}\cdot\hat{\nu}\right)^2\left(1 + \delta_{(\beta\nu)^2}^{0^-}\right)\right]\left|\langle\|\hat{L}_0^A\|\rangle\right|^2\Bigg\},\end{aligned} \tag{11}$$

with $\hat{C}_0^A \propto \epsilon_{\text{NR}}$ and $\hat{L}_0^A \propto \epsilon_{qr}$ the leading order operators in this transition, and without any nuclear structure corrections. There are, however, recoiled nucleus corrections:

$$\delta_1^{0^-} = \frac{E - \Delta M}{M_i} + \mathcal{O}\left(\epsilon_M^2\right), \tag{12a}$$

$$\delta_{\beta^2}^{0^-} = -\frac{E}{M_i}\vec{\beta}\cdot\hat{\nu} + \mathcal{O}\left(\epsilon_M^2\right), \tag{12b}$$

$$\delta_{(\beta\nu)^2}^{0^-} = \frac{E}{M_i}\frac{|\langle\|\hat{C}_0^A\|\rangle|^2 - \frac{E_0}{q}2\,\mathfrak{Re}\left(\langle\|\hat{L}_0^A\|\rangle\langle\|\hat{C}_0^A\|\rangle^*\right) + \left|\langle\|\hat{L}_0^A\|\rangle\right|^2}{2\frac{E(E_0-E)}{q^2}\left|\langle\|\hat{L}_0^A\|\rangle\right|^2} + \mathcal{O}\left(\epsilon_M^2\right), \tag{12c}$$

which can be ignored, except for very light nuclei. Note that the term $2\frac{E(E_0-E)}{q^2}(1 + \delta_{\beta^2}^{0^-})\left[\beta^2 - (\vec{\beta}\cdot\hat{\nu})^2(1 + \delta_{(\beta\nu)^2}^{0^-})\right]\left|\langle\|\hat{L}_0^A\|\rangle\right|^2$ does not exist for $E = E_0$, except for very light nuclei, where there is, instead, a small recoiled nucleus correction term: $-\frac{E}{M_{\mathrm{i}}}(\vec{\beta}\cdot\hat{\nu})^2\left[|\langle\|\hat{C}_0^A\|\rangle|^2 - \frac{E_0}{q}2\,\mathfrak{Re}\left(\langle\|\hat{L}_0^A\|\rangle\langle\|\hat{C}_0^A\|\rangle^*\right) + \left|\langle\|\hat{L}_0^A\|\rangle\right|^2\right] + \mathcal{O}\left(\epsilon_M^2\right)$.

For this transition, the angular correlation coefficient is simply $a^{0^-} = 1$, except for very light nuclei, were $a^{0^-} = 1 + \tilde{\delta}_a^{0^-}$, with a recoiled nucleus correction:

$$\tilde{\delta}_a^{0^-} = -\frac{E}{M_{\mathrm{i}}} + \frac{m_{\mathrm{e}}}{M_{\mathrm{i}}}\frac{\frac{m_{\mathrm{e}}}{q}2\,\mathfrak{Re}\left(\langle\|\hat{L}_0^A\|\rangle\langle\|\hat{C}_0^A\|\rangle^*\right)}{|\langle\|\hat{C}_0^A\|\rangle|^2 - \frac{E_0}{q}2\,\mathfrak{Re}\left(\langle\|\hat{L}_0^A\|\rangle\langle\|\hat{C}_0^A\|\rangle^*\right) + \left|\langle\|\hat{L}_0^A\|\rangle\right|^2} + \mathcal{O}\left(\epsilon_M^2\right). \tag{12d}$$

Finally, a leading order term with similar spectral behavior as the Fierz interference term can be extracted from the spectrum,

$$b^{0^-} = \frac{\frac{m_{\mathrm{e}}}{q}2\,\mathfrak{Re}\left(\langle\|\hat{L}_0^A\|\rangle\langle\|\hat{C}_0^A\|\rangle^*\right)}{|\langle\|\hat{C}_0^A\|\rangle|^2 - \frac{E_0}{q}2\,\mathfrak{Re}\left(\langle\|\hat{L}_0^A\|\rangle\langle\|\hat{C}_0^A\|\rangle^*\right) + \left|\langle\|\hat{L}_0^A\|\rangle\right|^2}. \tag{13}$$

4.3. Gamow–Teller and unique forbidden transitions

In order to discuss the $\Theta^{J^{\Delta\pi}}(q,\vec{\beta}\cdot\hat{\nu})$ expressions for J's greater than 0, we will distinguish between two types of $0 < J^{\Delta\pi}$ transitions: $\Delta\pi = (-)^J$, and $\Delta\pi = (-)^{J-1}$. The first type, $J^{(-)^J}$, presents non-unique J^{th} forbidden transitions. The second type, $J^{(-)^{J-1}}$, presents, for $J = 1$, the allowed Gamow–Teller transition, and for $J > 1$, unique $(J-1)^{\text{th}}$ forbidden transitions (together we will refer to them as unique transitions).

Let us start with the unique transitions (e.g., Gamow–Teller transition, which is $J^{\Delta\pi} = 1^+$). A general expression, including shape and recoil NLO corrections, for any unique transition, i.e., a decay with $0 < J^{(-)^{J-1}}$, in the sum in (5), can be written as:

$$\begin{aligned}\Theta^{J^{(-)^{J-1}}}\left(q,\vec{\beta}\cdot\hat{\nu}\right) &= \frac{2J+1}{J}\left(1+\delta_1^{J^{(-)^{J-1}}}\right)\Bigg\{1 + a^{J^{(-)^{J-1}}}\vec{\beta}\cdot\hat{\nu} \\ &\quad + b^{J^{(-)^{J-1}}}\frac{m_{\mathrm{e}}}{E} + \frac{J-1}{2J+1}\frac{E(E_0-E)}{q^2}\left(1+\tilde{\delta}_{\beta^2}^{J^{(-)^{J-1}}}\right) \\ &\quad \times\left[\beta^2 - \left(\vec{\beta}\cdot\hat{\nu}\right)^2\left(1+\delta_{(\beta\nu)^2}^{J^{(-)^{J-1}}}\right)\right]\Bigg\}\left|\langle\|\hat{L}_J^A\|\rangle\right|^2,\end{aligned} \tag{14}$$

with the shape and recoil NLO corrections:

$$\begin{aligned}\delta_1^{J^{(-)^{J-1}}} &= \frac{2}{2J+1}\mathfrak{Re}\left[-J\frac{E_0}{q}\frac{\langle\|\hat{C}_J^A\|\rangle}{\langle\|\hat{L}_J^A\|\rangle} \pm \sqrt{J(J+1)}\frac{E_0-2E}{q}\frac{\langle\|\hat{M}_J^V\|\rangle}{\langle\|\hat{L}_J^A\|\rangle}\right. \\ &\quad \left. + \sqrt{J(J+1)}\frac{\langle\|\hat{E}_J^{A(\mathrm{res})}\|\rangle}{\langle\|\hat{L}_J^A\|\rangle}\right] + \frac{E-\Delta M}{M_{\mathrm{i}}} + \mathcal{O}\left(\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2, \epsilon_{\mathrm{recoil}}^2, \epsilon_M^2\right),\end{aligned} \tag{15a}$$

$$\tilde{\delta}_{\beta^2}^{J^{(-)^{J-1}}} = -\frac{2}{2J+1}\,\mathfrak{Re}\left[-J\frac{E_0}{q}\frac{\langle\|\hat{C}_J^A\|\rangle}{\langle\|\hat{L}_J^A\|\rangle} \pm \sqrt{J(J+1)}\frac{E_0-2E}{q}\frac{\langle\|\hat{M}_J^V\|\rangle}{\langle\|\hat{L}_J^A\|\rangle} + \frac{3J}{J-1}\sqrt{J(J+1)}\frac{\langle\|\hat{E}_J^{A(\mathrm{res})}\|\rangle}{\langle\|\hat{L}_J^A\|\rangle}\right] - \frac{E}{M_\mathrm{i}}\vec{\beta}\cdot\hat{\nu} + \mathcal{O}\left(\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2, \epsilon_{\mathrm{recoil}}^2, \epsilon_M^2\right), \tag{15b}$$

$$\delta_{(\beta\nu)^2}^{J^{(-)^{J-1}}} = -\frac{1}{J-1}\frac{q^2}{M_\mathrm{i}(E_0-E)} + \mathcal{O}\left(\epsilon_M^2\right), \tag{15c}$$

where $\pm$ are for $\beta^{\mp}$-decays, and $\delta_{(\beta\nu)^2}^{J^{(-)^{J-1}}}$ is neglectable except for very light nuclei. Also here there is no $\frac{J-1}{2J+1}\frac{E(E_0-E)}{q^2}\left(1+\tilde{\delta}_{\beta^2}^{J^{(-)^{J-1}}}\right)\left[\beta^2-(\vec{\beta}\cdot\hat{\nu})^2\left(1+\delta_{(\beta\nu)^2}^{J^{(-)^{J-1}}}\right)\right]$ term for $E=E_0$, except for very light nuclei, where there is, instead, a small recoiled nucleus correction term: $\frac{1}{2J+1}\frac{E}{M_\mathrm{i}}(\vec{\beta}\cdot\hat{\nu})^2+\mathcal{O}\left(\epsilon_M^2\right)$. The term $\frac{J-1}{2J+1}\frac{E(E_0-E)}{q^2}\left(1+\tilde{\delta}_{\beta^2}^{J^{(-)^{J-1}}}\right)\left[\beta^2-(\vec{\beta}\cdot\hat{\nu})^2\left(1+\delta_{(\beta\nu)^2}^{J^{(-)^{J-1}}}\right)\right]$ does not exist also for the leading order of $J=1$ (Gamow–Teller). Instead, the Gamow–Teller transition has an NLO correction term:

$$\tilde{\delta}_{\beta^2,(\beta\nu)^2}^{1^+} \equiv -\frac{2\sqrt{2}}{3}\frac{E(E_0-E)}{q^2}\,\mathfrak{Re}\frac{\langle\|\hat{E}_1^{A(\mathrm{res})}\|\rangle}{\langle\|\hat{L}_1^A\|\rangle}\left[\beta^2-\left(\vec{\beta}\cdot\hat{\nu}\right)^2\right] + \frac{1}{3}\frac{E}{M_\mathrm{i}}\left(\vec{\beta}\cdot\hat{\nu}\right)^2 + \mathcal{O}\left(\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2, \epsilon_{\mathrm{recoil}}^2, \epsilon_M^2\right). \tag{16}$$

The multipole operator $\hat{L}_J^A \propto \epsilon_{qr}^{J-1}$ is the leading order operator, while $\frac{\hat{C}_J^A}{\hat{L}_J^A}, \frac{\hat{M}_J^V}{\hat{L}_J^A} \propto \epsilon_{\mathrm{recoil}}, \epsilon_{qr}\epsilon_{\mathrm{NR}}$ (both multipoles, $\hat{C}_J^A$ and $\hat{M}_J^V$, consist of these two forms of terms. $\hat{C}_J^A$ has an additional smaller term, proportional to $\epsilon_{m_\pi}^2$) are NLO nuclear structure corrections operators. To eliminate the electric multipole operator $\hat{E}_J$, we used the relation

$$\hat{E}_J = \sqrt{\frac{J+1}{J}}\hat{L}_J - \mathrm{i}\sqrt{\frac{2J+1}{J}}\int \mathrm{d}^3 r\, j_{J+1}(qr)\vec{Y}_{J,J+1,1}(\hat{r})\cdot\vec{\mathcal{J}}(\vec{r}) \equiv \sqrt{\frac{J+1}{J}}\hat{L}_J + \hat{E}_J^{(\mathrm{res})}, \tag{17}$$

that leaves a residual correction $\hat{E}_J^{(\mathrm{res})}$ of $\hat{E}_J$ regarding $\hat{L}_J$, and for the unique transitions is reflected in another NLO correction of $\frac{\hat{E}_J^{A(\mathrm{res})}}{\hat{L}_J^A} \propto \epsilon_{qr}^2$.

The V–A structure of the weak interaction entails, for any specific unique transition, that the β–ν correlations leading order will be $a^{J^{(-)^{J-1}}} = -\frac{1}{2J+1}$. When adding the NLO corrections, the β–ν correlation becomes

$$a^{J^{(-)^{J-1}}} = -\frac{1}{2J+1}\left(1+\tilde{\delta}_a^{J^{(-)^{J-1}}}\right), \tag{18a}$$

with

$$\tilde{\delta}_a^{J^{(-)^{J-1}}} = \frac{2J}{2J+1} 2\,\mathfrak{Re}\left[(J+1)\frac{E_0}{q}\frac{\langle\|\hat{C}_J^A\|\rangle}{\left\langle\left\|\hat{L}_J^A\right\|\right\rangle} \pm \sqrt{J(J+1)}\frac{E_0-2E}{q}\frac{\left\langle\left\|\hat{M}_J^V\right\|\right\rangle}{\left\langle\left\|\hat{L}_J^A\right\|\right\rangle} + \sqrt{J(J+1)}\frac{\langle\|\hat{E}_J^{A(\mathrm{res})}\|\rangle}{\left\langle\left\|\hat{L}_J^A\right\|\right\rangle}\right] + (2J+1)\frac{E}{M_\mathrm{i}} + \mathcal{O}\left(\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2, \epsilon_{\mathrm{recoil}}^2, \epsilon_M^2\right). \tag{18b}$$

As for the Fierz term that vanishes for unique transitions in the *V–A* structure of the weak interaction, a term with similar spectral behavior can be extracted from the corrected spectrum, and our calculations indicate that it is non-zero,

$$b^{J^{(-)^{J-1}}} = \delta_b^{J^{(-)^{J-1}}} \equiv \frac{1}{2J+1}\frac{m_\mathrm{e}}{q} 2\,\mathfrak{Re}\left\{J\frac{\langle\|\hat{C}_J^A\|\rangle}{\left\langle\left\|\hat{L}_J^A\right\|\right\rangle} \pm \sqrt{J(J+1)}\frac{\left\langle\left\|\hat{M}_J^V\right\|\right\rangle}{\left\langle\left\|\hat{L}_J^A\right\|\right\rangle}\right\} + \mathcal{O}\left(\frac{m_\mathrm{e}}{E}\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2, \frac{m_e}{E}\epsilon_{\mathrm{recoil}}^2\right). \tag{19}$$

4.4. Non-unique forbidden transitions

For the case of non-unique J^{th} forbidden transitions, i.e., decays with $0 < J^{(-)^J}$, the $\Theta^{J^{(-)^J}}\left(q, \vec{\beta}\cdot\hat{\nu}\right)$ expression can be written as:

$$\Theta^{J^{(-)^J}}\left(q, \vec{\beta}\cdot\hat{\nu}\right) = \left(1+\delta_1^{J^{(-)^J}}\right)\left\{\left(1 + a^{J^{(-)^J}}\vec{\beta}\cdot\hat{\nu} + b^{J^{(-)^J}}\frac{m_\mathrm{e}}{E}\right)\xi^{J^{(-)^J}} + \frac{E(E_0-E)}{q^2}\left(1+\delta_{\beta^2}^{J^{(-)^J}}\right)\left[\beta^2 - \left(\vec{\beta}\cdot\hat{\nu}\right)^2\left(1+\delta_{(\beta\nu)^2}^{J^{(-)^J}}\right)\right] \times \left[\frac{J-1}{J}\frac{E_0^2}{q^2}\left(1+\frac{J+1}{J-1}2\,\mathfrak{Re}\,\delta^{J^{(-)^J}}\right)|\langle\|\hat{C}_J^V\|\rangle|^2 - \left|\left\langle\left\|\hat{M}_J^A\right\|\right\rangle\right|^2\right]\right\}, \tag{20}$$

where we define:

$$\xi^{J^{(-)^J}} \equiv \left[1 + \frac{1}{J}\frac{E_0^2}{q^2}\left(1-(J+1)\,2\,\mathfrak{Re}\,\delta^{J^{(-)^J}}\right)\right]|\langle\|\hat{C}_J^V\|\rangle|^2 + \left|\left\langle\left\|\hat{M}_J^A\right\|\right\rangle\right|^2 \pm \sqrt{\frac{J+1}{J}}\frac{E_0(E_0-2E)}{q^2}2\,\mathfrak{Re}\left[\left(1-\delta^{J^{(-)^J}}\right)\langle\|\hat{C}_J^V\|\rangle\left\langle\left\|\hat{M}_J^A\right\|\right\rangle^*\right], \tag{21}$$

with the NLO correction:

$$\delta^{J^{(-)^J}} = -\frac{q}{E_0}\sqrt{\frac{J}{J+1}}\frac{\langle\|\hat{E}_J^{V(\mathrm{res})}\|\rangle}{\langle\|\hat{C}_J^V\|\rangle} + \mathcal{O}\left(\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2\right), \tag{22}$$

and the recoiled nucleus corrections (relevant only for very light nuclei):

$$\delta_1^{J^{(-)^J}} = \frac{E-\Delta M}{M_\mathrm{i}} + \mathcal{O}\left(\epsilon_M^2\right), \tag{23a}$$

$$\delta_{\beta^2}^{J^{(-)^J}} = -\frac{E}{M_\mathrm{i}}\vec{\beta}\cdot\hat{\nu} + \mathcal{O}\left(\epsilon_M^2\right), \tag{23b}$$

$$\delta_{(\beta\nu)^2}^{J^{(-)^J}} = \frac{E}{M_\mathrm{i}} a^{J^{(-)^J}} + \mathcal{O}\left(\epsilon_M^2\right). \tag{23c}$$

Using the vector current conservation hypothesis (exact to relevant orders in chiral χEFT), which eliminates $\hat{L}_J^V = \frac{E_0}{q}\hat{C}_J^V$ for $J>0$ [28], the multipole operators involved in the leading order are only $\hat{C}_J^V, \hat{M}_J^A \propto \epsilon_{qr}^J$, and the NLO operator is $\frac{\hat{E}_J^{V(\mathrm{res})}}{\hat{C}_J^V} \propto \epsilon_{qr}\epsilon_\mathrm{NR}$ (the vector current is $\vec{\mathcal{J}}^V(\vec{r}) \propto \epsilon_\mathrm{NR}$).

For non-unique forbidden decays, a leading order term with similar spectral behavior as the Fierz interference term can be extracted from the spectrum:

$$b^{J^{(-)^J}} = \left[\xi^{J^{(-)^J}}\right]^{-1}\frac{m_\mathrm{e}E_0}{q^2}\left\{2|\langle\|\hat{C}_J^V\|\rangle|^2 \pm \sqrt{\frac{J+1}{J}}2\,\mathfrak{Re}\left[\left(1-\delta^{J^{(-)^J}}\right)\langle\|\hat{C}_J^V\|\rangle\left\langle\left\|\hat{M}_J^A\right\|\right\rangle^*\right]\right\}, \tag{24}$$

while the angular correlation coefficient can be recognized as:

$$\begin{aligned} a^{J^{(-)^J}} = \left[\xi^{J^{(-)^J}}\right]^{-1}\Bigg\{&\left[1-\frac{2J+1}{J}\frac{E_0^2}{q^2}\left(1-\frac{J+1}{2J+1}2\,\mathfrak{Re}\,\delta^{J^{(-)^J}}\right)\right] \\ &\times|\langle\|\hat{C}_J^V\|\rangle|^2 - \left|\left\langle\left\|\hat{M}_J^A\right\|\right\rangle\right|^2 \mp \sqrt{\frac{J+1}{J}}\frac{E_0(E_0-2E)}{q^2}2\,\mathfrak{Re} \\ &\times\left[\left(1-\delta^{J^{(-)^J}}\right)\langle\|\hat{C}_J^V\|\rangle\left\langle\left\|\hat{M}_J^A\right\|\right\rangle^*\right] + \delta_a^{J^{(-)^J}}\Bigg\}, \end{aligned} \tag{25a}$$

with an additional recoiled nucleus correction (relevant only for very light nuclei):

$$\begin{aligned} \delta_a^{J^{(-)^J}} = -\frac{E}{M_\mathrm{i}}\Bigg\{&\left(1+\frac{1}{J}\frac{E_0^2}{q^2}+2\frac{E_0}{E}\frac{m_\mathrm{e}^2}{q^2}\right)|\langle\|\hat{C}_J^V\|\rangle|^2 + \left|\left\langle\left\|\hat{M}_J^A\right\|\right\rangle\right|^2 \\ &\mp\sqrt{\frac{J+1}{J}}2\,\mathfrak{Re}\left(\langle\|\hat{C}_J^V\|\rangle\left\langle\left\|\hat{M}_J^A\right\|\right\rangle^*\right)\left[\frac{E_0(E_0-2E)}{q^2}-\frac{E_0}{E}\frac{m_\mathrm{e}^2}{q^2}\right]\Bigg\} + \mathcal{O}\left(\epsilon_M^2\right). \end{aligned} \tag{25b}$$

This concludes the review of our formalism. Aside from its suitability for modern *ab initio* calculations, its advantage over other formalisms introduced over the years, such as Holstein's form factors [31] and Behrens—Buehring's formalism [24, 32], is that we can estimate and control the uncertainty of the calculations, as well as extend them to BSM contributions, as we do in [30]. Its advantage over other formalisms which have been introduced over the years, like the form factors of Holstein [31] or the Behrens–Buehring formalism [24, 32], is the ability to estimate and control the uncertainty of the calculations, as well as the ability to expand it also for BSM contributions, as we do in [30]. As a result, it is particularly useful for current searches of BSM signatures in precision β-decay experiments. A comparison of this formalism to the others can be found in [23].

5. Nuclear shape and recoil corrections for allowed transitions

In the following, we demonstrate the use of our formalism. For example, let us consider the most general allowed transition, e.g., mixed Fermi ($J^{\Delta\pi} = 0^+$) and Gamow–Teller ($J^{\Delta\pi} = 1^+$) transitions. This requires $\Delta\pi = +$ and $J_\mathrm{i} = J_\mathrm{f} > 0$, so, following (1) and (5), the decay rate, including NLO corrections, will contain a sum of Θ^{0^+} (6) and Θ^{1^+} (14) (other Θ^{J^+}'s, with J^+s up to $J_\mathrm{i} + J_\mathrm{f}$, also participate in the sum, but they contribute only higher orders):

$$\begin{aligned}
\frac{\mathrm{d}^5\omega}{\mathrm{d}E\frac{\mathrm{d}\hat{k}}{4\pi}\frac{\mathrm{d}\hat{\nu}}{4\pi}} &= \frac{4}{\pi^2}\frac{1}{2J_\mathrm{i}+1}(E_0-E)^2 kEF^{\mp}(Z_\mathrm{f},E)C_\mathrm{corr} \\
&\quad\times\left\{\left(1+\delta_1^{0^+}\right)\left[1+\vec{\beta}\cdot\hat{\nu}+b^{0^+}\frac{m_\mathrm{e}}{E}\right]|\langle\|\hat{C}_0^V\|\rangle|^2+3\left(1+\delta_1^{1^+}\right)\right. \\
&\quad\left.\times\left[1-\frac{1}{3}\vec{\beta}\cdot\hat{\nu}\left(1+\tilde{\delta}_a^{1^+}\right)+b^{1^+}\frac{m_\mathrm{e}}{E}+\tilde{\delta}^{1^+}_{\beta^2,(\beta\nu)^2}\right]\left|\left\langle\left\|\hat{L}_1^A\right\|\right\rangle\right|^2\right\} \\
&= \frac{4}{\pi^2}\frac{1}{2J_\mathrm{i}+1}(E_0-E)^2 kEF^{\mp}(Z_\mathrm{f},E)C_\mathrm{corr}\left(|\langle\|\hat{C}_0^V\|\rangle|^2\right. \\
&\quad\left.+3\left|\left\langle\left\|\hat{L}_1^A\right\|\right\rangle\right|^2\right)\left(1+\delta_1^{\mathrm{F+GT}}\right) \\
&\quad\times\left[1+\vec{\beta}\cdot\hat{\nu}\left(1+\tilde{\delta}_a^{\mathrm{F+GT}}\right)\frac{|\langle\|\hat{C}_0^V\|\rangle|^2-\left|\left\langle\left\|\hat{L}_1^A\right\|\right\rangle\right|^2}{|\langle\|\hat{C}_0^V\|\rangle|^2+3\left|\left\langle\left\|\hat{L}_1^A\right\|\right\rangle\right|^2}\right. \\
&\quad\left.+b^{\mathrm{F+GT}}\frac{m_\mathrm{e}}{E}+\tilde{\delta}^{1^+}_{\beta^2,(\beta\nu)^2}\right],
\end{aligned} \tag{26}$$

with the NLO corrections:

$$\delta_1^{\mathrm{F+GT}} = \frac{\delta_1^{0^+}|\langle\|\hat{C}_0^V\|\rangle|^2+3\delta_1^{1^+}\left|\left\langle\left\|\hat{L}_1^A\right\|\right\rangle\right|^2}{|\langle\|\hat{C}_0^V\|\rangle|^2+3\left|\left\langle\left\|\hat{L}_1^A\right\|\right\rangle\right|^2}, \tag{27a}$$

$$\tilde{\delta}_a^{\mathrm{F+GT}} = \frac{\left(\delta_1^{0^+}+\tilde{\delta}_a^{0^+}\right)|\langle\|\hat{C}_0^V\|\rangle|^2-\left(\delta_1^{1^+}+\tilde{\delta}_a^{1^+}\right)\left|\left\langle\left\|\hat{L}_1^A\right\|\right\rangle\right|^2}{|\langle\|\hat{C}_0^V\|\rangle|^2-\left|\left\langle\left\|\hat{L}_1^A\right\|\right\rangle\right|^2}, \tag{27b}$$

where $\delta_1^{0^+}$ and $\tilde{\delta}_a^{0^+}$ are given in (7) and (10), $\tilde{\delta}^{1^+}_{\beta^2,(\beta\nu)^2}$ in (16), and $\delta_1^{1^+}$ and $\tilde{\delta}_a^{1^+}$ can be found easily from (15a) and (18b):

$$\begin{aligned}
\delta_1^{1+} &= \frac{2}{3}\mathfrak{Re}\left[-\frac{E_0}{q}\frac{\langle\|\hat{C}_1^A\|\rangle}{\left\langle\left\|\hat{L}_1^A\right\|\right\rangle}\pm\sqrt{2}\frac{E_0-2E}{q}\frac{\left\langle\left\|\hat{M}_1^V\right\|\right\rangle}{\left\langle\left\|\hat{L}_1^A\right\|\right\rangle}+\sqrt{2}\frac{\langle\|\hat{E}_1^{A(\mathrm{res})}\|\rangle}{\left\langle\left\|\hat{L}_1^A\right\|\right\rangle}\right] \\
&\quad+\frac{E-\Delta M}{M_\mathrm{i}}+\mathcal{O}\left(\epsilon_{qr}^2\epsilon_\mathrm{NR}^2,\epsilon_\mathrm{recoil}^2,\epsilon_M^2\right),
\end{aligned} \tag{28a}$$

$$\begin{aligned}
\tilde{\delta}_a^{1+} &= \frac{4}{3}\mathfrak{Re}\left[\frac{2E_0}{q}\frac{\langle\|\hat{C}_1^A\|\rangle}{\left\langle\left\|\hat{L}_1^A\right\|\right\rangle}\pm\sqrt{2}\frac{E_0-2E}{q}\frac{\left\langle\left\|\hat{M}_1^V\right\|\right\rangle}{\left\langle\left\|\hat{L}_1^A\right\|\right\rangle}+\sqrt{2}\frac{\langle\|\hat{E}_1^{A(\mathrm{res})}\|\rangle}{\left\langle\left\|\hat{L}_1^A\right\|\right\rangle}\right] \\
&\quad+3\frac{E}{M_\mathrm{i}}+\mathcal{O}\left(\epsilon_{qr}^2\epsilon_\mathrm{NR}^2,\epsilon_\mathrm{recoil}^2,\epsilon_M^2\right).
\end{aligned} \tag{28b}$$

Here, the multipole operators $\hat{C}_0^V, \hat{L}_1^A \propto 1$ are the Fermi and Gamow–Teller leading orders, while $\hat{L}_0^V, \hat{C}_1^A, \hat{M}_1^V \propto \epsilon_{\text{recoil}}, \epsilon_{qr}\epsilon_{\text{NR}}$ and $\hat{E}_1^{A(\text{res})} \propto \epsilon_{qr}^2$ are their NLO nuclear structure corrections.

As mentioned in section 4, a term with similar spectral behavior as the Fierz interference term can be extracted from the corrected spectrum, and our calculations indicate that it is non-zero. For a pure Fermi transition, its form is already described in (9), while for a pure Gamow–Teller, it is (from (19))

$$b^{1^+} = \delta_b^{1+} \equiv \frac{2m_e}{3q}\,\mathfrak{Re}\left\{\frac{\langle\|\hat{C}_1^A\|\rangle}{\langle\|\hat{L}_1^A\|\rangle} \pm \sqrt{2}\frac{\langle\|\hat{M}_1^V\|\rangle}{\langle\|\hat{L}_1^A\|\rangle}\right\} + \mathcal{O}\left(\frac{m_e}{E}\epsilon_{qr}^2\epsilon_{\text{NR}}^2, \frac{m_e}{E}\epsilon_{\text{recoil}}^2\right). \tag{29}$$

As for the angular correlation coefficient, it is, as mentioned, exactly 1 for a pure Fermi transition (with the correction (10) for very light nuclei). For a pure Gamow–Teller transition, it gets the corrected form $a^{1^+} = -\frac{1}{3}\left(1 + \tilde{\delta}_a^{1^+}\right)$ (from (18a)). Finally, for a mixed Fermi and Gamow–Teller transition one gets:

$$a^{\text{F+GT}} = \frac{|\langle\|\hat{C}_0^V\|\rangle|^2 - \left|\langle\|\hat{L}_1^A\|\rangle\right|^2}{|\langle\|\hat{C}_0^V\|\rangle|^2 + 3\left|\langle\|\hat{L}_1^A\|\rangle\right|^2}\left(1 + \tilde{\delta}_a^{\text{F+GT}}\right), \tag{30a}$$

$$b^{\text{F+GT}} = \frac{\delta_b^{0+}|\langle\|\hat{C}_0^V\|\rangle|^2 + 3\delta_b^{1+}\left|\langle\|\hat{L}_1^A\|\rangle\right|^2}{|\langle\|\hat{C}_0^V\|\rangle|^2 + 3\left|\langle\|\hat{L}_1^A\|\rangle\right|^2}, \tag{30b}$$

with the defined-above $\tilde{\delta}_a^{\text{F+GT}}$, δ_b^{0+} and δ_b^{1+}.

6. Nuclear shape and recoil corrections for first-forbidden transitions

As mentioned in section 3, first-forbidden transitions are transitions with a q^1 leading order dependence. Those will be transitions involving $J^{\Delta\pi} = 0^-, 1^-, 2^-$. Let us now assume the most general first-forbidden transition possible, including all the first-forbidden $J^{\Delta\pi}$'s. That will be any transition with $\Delta\pi = -$ and $J_\text{i} = J_\text{f} \geqslant 1$ (e.g., the transition $1^+ \to 1^-$). This way, following (1) and (5), the leading order of its decay rate, with multipoles proportional to ϵ_{NR} and ϵ_{qr}, should include a sum over $J^{\Delta\pi} = 0^-, 1^-, 2^-$ (other J^-'s can also participate in the sum, up to $J_\text{i} + J_\text{f}$, but they will contribute only higher orders):

$$\begin{aligned}
\frac{\mathrm{d}^5\omega}{\mathrm{d}E\frac{\mathrm{d}\hat{k}}{4\pi}\frac{\mathrm{d}\hat{\nu}}{4\pi}} = {} & \frac{4}{\pi^2}\frac{1}{2J_i+1}(E_0-E)^2 kEF^{\mp}\left(Z_f, E\right)C_{\text{corr}} \\
& \times\left\{\left(1+\delta_1^{0^-}\right)\left\{\left(1 + a^{0^-}\vec{\beta}\cdot\hat{\nu} + b^{0^-}\frac{m_e}{E}\right)\right.\right. \\
& \times\left[|\langle\|\hat{C}_0^A\|\rangle|^2 - \frac{E_0}{q}2\,\mathfrak{Re}\left(\langle\|\hat{L}_0^A\|\rangle\langle\|\hat{C}_0^A\|\rangle^*\right)\right. \\
& \left. + \left|\langle\|\hat{L}_0^A\|\rangle\right|^2\right] + 2\frac{E(E_0-E)}{q^2}\left(1+\delta_{\beta^2}^{0^-}\right) \\
& \times\left[\beta^2 - \left(\vec{\beta}\cdot\hat{\nu}\right)^2\left(1+\delta_{(\beta\nu)^2}^{0^-}\right)\right]\left|\langle\|\hat{L}_0^A\|\rangle\right|^2\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+\left(1+\delta_1^{1^-}\right)\left\{\left(1+a^{1^-}\vec{\beta}\cdot\hat{\nu}+b^{1^-}\frac{m_e}{E}\right)\right.\\
&\times\left\{\left[1+\frac{E_0^2}{q^2}\left(1-4\mathfrak{Re}\,\delta^{1^-}\right)\right]|\langle\|\hat{C}_1^V\|\rangle|^2\right.\\
&+\left|\left\langle\left\|\hat{M}_1^A\right\|\right\rangle\right|^2\pm\sqrt{2}\frac{E_0(E_0-2E)}{q^2}2\,\mathfrak{Re}\\
&\times\left.\left[\left(1-\delta^{1^-}\right)\langle\|\hat{C}_1^V\|\rangle\left\langle\left\|\hat{M}_1^A\right\|\right\rangle^*\right]\right\}-\frac{E(E_0-E)}{q^2}\left(1+\delta_{\beta^2}^{1^-}\right)\\
&\times\left.\left[\beta^2-\left(\vec{\beta}\cdot\hat{\nu}\right)^2\left(1+\delta_{(\beta\nu)^2}^{1^-}\right)\right]\left|\left\langle\left\|\hat{M}_1^A\right\|\right\rangle\right|^2\right\}\\
&+\frac{5}{2}\left(1+\delta_1^{2^-}\right)\left\{1+a^{2^-}\vec{\beta}\cdot\hat{\nu}+b^{2^-}\frac{m_e}{E}\right.\\
&+\frac{1}{5}\frac{E(E_0-E)}{q^2}\left(1+\tilde{\delta}_{\beta^2}^{2^-}\right)\left[\beta^2-\left(\vec{\beta}\cdot\hat{\nu}\right)^2\left(1+\delta_{(\beta\nu)^2}^{2^-}\right)\right]\Bigg\}\\
&\times\left.\left|\left\langle\left\|\hat{L}_2^A\right\|\right\rangle\right|^2\right\},
\end{aligned}
\tag{31}
$$

with the nuclear structure NLO corrections:

$$
\delta^{1^-}=-\frac{1}{\sqrt{2}}\frac{q}{E_0}\frac{\langle\|\hat{E}_1^{V(\mathrm{res})}\|\rangle}{\langle\|\hat{C}_1^V\|\rangle}+\mathcal{O}\left(\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2\right),
\tag{32a}
$$

$$
\begin{aligned}
\delta_1^{2^-}&=\frac{2}{5}\,\mathfrak{Re}\left[-\frac{2E_0}{q}\frac{\langle\|\hat{C}_2^A\|\rangle}{\left\langle\left\|\hat{L}_2^A\right\|\right\rangle}\pm\sqrt{6}\frac{E_0-2E}{q}\frac{\left\langle\left\|\hat{M}_2^V\right\|\right\rangle}{\left\langle\left\|\hat{L}_2^A\right\|\right\rangle}\right.\\
&\left.+\sqrt{6}\frac{\langle\|\hat{E}_2^{A(\mathrm{res})}\|\rangle}{\left\langle\left\|\hat{L}_2^A\right\|\right\rangle}\right]+\frac{E-\Delta M}{M_\mathrm{i}}+\mathcal{O}\left(\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2,\epsilon_{\mathrm{recoil}}^2,\epsilon_M^2\right),
\end{aligned}
\tag{32b}
$$

$$
\begin{aligned}
\tilde{\delta}_{\beta^2}^{2^-}&=-\frac{2}{5}\,\mathfrak{Re}\left[-\frac{2E_0}{q}\frac{\langle\|\hat{C}_2^A\|\rangle}{\left\langle\left\|\hat{L}_2^A\right\|\right\rangle}\pm\sqrt{6}\frac{E_0-2E}{q}\frac{\left\langle\left\|\hat{M}_2^V\right\|\right\rangle}{\left\langle\left\|\hat{L}_2^A\right\|\right\rangle}\right.\\
&\left.+6\sqrt{6}\frac{\langle\|\hat{E}_2^{A(\mathrm{res})}\|\rangle}{\left\langle\left\|\hat{L}_2^A\right\|\right\rangle}\right]-\frac{E}{M_\mathrm{i}}\vec{\beta}\cdot\hat{\nu}+\mathcal{O}\left(\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2,\epsilon_{\mathrm{recoil}}^2,\epsilon_M^2\right),
\end{aligned}
\tag{32c}
$$

$$
\begin{aligned}
\tilde{\delta}_a^{2^-}&=\frac{8}{5}\,\mathfrak{Re}\left[\frac{3E_0}{q}\frac{\langle\|\hat{C}_2^A\|\rangle}{\left\langle\left\|\hat{L}_2^A\right\|\right\rangle}\pm\sqrt{6}\frac{E_0-2E}{q}\frac{\left\langle\left\|\hat{M}_2^V\right\|\right\rangle}{\left\langle\left\|\hat{L}_2^A\right\|\right\rangle}+\sqrt{6}\frac{\langle\|\hat{E}_2^{A(\mathrm{res})}\|\rangle}{\left\langle\left\|\hat{L}_2^A\right\|\right\rangle}\right]\\
&+5\frac{E}{M_\mathrm{i}}+\mathcal{O}\left(\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2,\epsilon_{\mathrm{recoil}}^2,\epsilon_M^2\right).
\end{aligned}
\tag{32d}
$$

Here the multipole operators $\hat{C}_0^A\propto\epsilon_{\mathrm{NR}}$ and $\hat{L}_0^A,\hat{C}_1^V,\hat{M}_1^A,\hat{L}_2^A\propto\epsilon_{qr}$ are the first-forbidden leading orders, while $\frac{\hat{E}_1^{V(\mathrm{res})}}{\hat{C}_1^V}\propto\epsilon_{qr}\epsilon_{\mathrm{NR}}$, $\frac{\hat{C}_2^A}{\hat{L}_2^A},\frac{\hat{M}_2^V}{\hat{L}_2^A}\propto\epsilon_{qr}\epsilon_{\mathrm{NR}},\epsilon_{\mathrm{recoil}}$ and $\frac{\hat{E}_2^{A(\mathrm{res})}}{\hat{L}_2^A}\propto\epsilon_{qr}^2$ are their NLO corrections.

As we showed in [33], the unique first-forbidden transition, $J^{\Delta\pi} = 2^-$, is of great interest for BSM searches. Its term with similar spectral behavior as the Fierz interference term, including NLO corrections, is (from (19)):

$$b^{2^-} = \delta_b^{2^-} \equiv \frac{2m_e}{5q}\,\mathfrak{Re}\left\{2\frac{\langle\|\hat{C}_2^A\|\rangle}{\langle\|\hat{L}_2^A\|\rangle} \pm \sqrt{6}\frac{\langle\|\hat{M}_2^V\|\rangle}{\langle\|\hat{L}_2^A\|\rangle}\right\} + \mathcal{O}\left(\frac{m_e}{E}\epsilon_{qr}^2\epsilon_{\mathrm{NR}}^2, \frac{m_e}{E}\epsilon_{\mathrm{recoil}}^2\right), \tag{33}$$

and its angular correlation coefficient is (from (18a)) $a^{2^-} = -\frac{1}{5}\left(1 + \tilde{\delta}_a^{2^-}\right)$.

For completeness, we will note that for $J^{\Delta\pi} = 0^-$, a term with similar spectral behavior as the Fierz term is presented in (13), and its angular correlation is presented above it, while for $J^{\Delta\pi} = 1^-$ they are:

$$b^{1^-} = \left[\xi^{1^-}\right]^{-1}\frac{m_e E_0}{q^2}\left\{2\left|\langle\|\hat{C}_1^V\|\rangle\right|^2 \pm 2\sqrt{2}\,\mathfrak{Re}\left[\left(1-\delta^{1^-}\right)\langle\|\hat{C}_1^V\|\rangle\langle\|\hat{M}_1^A\|\rangle^*\right]\right\}, \tag{34}$$

and

$$\begin{aligned} a^{1^-} = \left[\xi^{1^-}\right]^{-1}\Bigg\{&\left[1 - 3\frac{E_0^2}{q^2}\left(1 - \frac{4}{3}\,\mathfrak{Re}\,\delta^{1^-}\right)\right]\left|\langle\|\hat{C}_1^V\|\rangle\right|^2 - \left|\langle\|\hat{M}_1^A\|\rangle\right|^2 \\ &\mp \sqrt{2}\frac{E_0(E_0-2E)}{q^2}2\,\mathfrak{Re}\left[\left(1-\delta^{1^-}\right)\langle\|\hat{C}_1^V\|\rangle\langle\|\hat{M}_1^A\|\rangle^*\right] + \delta_a^{1^-}\Bigg\}, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \xi^{1^-} \equiv &\left[1 + \frac{E_0^2}{q^2}\left(1 - 4\,\mathfrak{Re}\,\delta^{1^-}\right)\right]|\langle\|\hat{C}_1^V\|\rangle|^2 + \left|\langle\|\hat{M}_1^A\|\rangle\right|^2 \\ &\pm \sqrt{2}\frac{E_0(E_0-2E)}{q^2}2\,\mathfrak{Re}\left[\left(1-\delta^{1^-}\right)\langle\|\hat{C}_1^V\|\rangle\langle\|\hat{M}_1^A\|\rangle^*\right]. \end{aligned} \tag{36}$$

7. From nuclear structure corrections to BSM signatures

The matrix elements appearing in the formulae in the previous sections arise from the weak Hamiltonian that describes the coupling to the weak probe. Therefore, they depend on the nuclear interactions, i.e., the nuclear wave functions and the structure of the multipole operators, which are expansions of the hadronic currents and charges within the nucleus.

Nuclear interactions are the low-energy reflection of the fundamental QCD forces. Thus, one uses an effective approach to describe the nuclear Hamiltonian and the nuclear currents excited by the weak probe. A most common approach is the EFT approach, which systematically builds, order by order, the nuclear Hamiltonian and currents (the scattering operators), based on the symmetries of the fundamental theory. As EFT creates an expansion in a small parameter, Q/Λ_b, it is also a source of systematic uncertainty, which we coined ϵ_{EFT} in section 2, of the order of $\epsilon_{\mathrm{EFT}}^{n+1}$, with n the order of the expansion.

At the low-energies characterizing nuclear β-decays, this dynamics, microscopically governed by QCD, can be effectively reduced into a field theory of nucleons, pions, and short-range interactions, by the use of χEFT. χEFT results in a consistent expansion governed by the small parameter ϵ_{EFT}, which dictates the accuracy of the theory. For example, for χEFT, Q/Λ_b is determined by QCD excitations beyond the pion, while Q is determined by the nuclear dynamics, e.g., the Fermi energy. This leads to Q/Λ_b of about 0.3. In general, the weak probe interacts with ever-growing currents of clusters of nucleons. However, within χEFT, the interaction with currents of bigger clusters is weaker than smaller clusters.

Pragmatically, many calculations use the so-called 'impulse' approximation for the nuclear currents, which describes the interaction of the weak probe with the strongly interacting nucleons, and neglects the interaction of the probe with two or more nucleons simultaneously. In terms of an EFT calculation, this is a leading order approximation, which entails an $\mathcal{O}(\epsilon_{\mathrm{EFT}})$ accuracy, usually due to the leading magnetic multipole. This is due to the fact that such currents arise at NLO for the magnetic multipole of the polar-vector current, $\hat{M}_J^V$. Other significant multipoles receive two-body corrections at higher orders. The one-body currents arise consistently at leading order, and their operator structure has been developed already from a phenomenological approach, using Lorentz symmetry considerations, and thus will be the starting point for theoretical predictions. For example, for weak magnetism, $\hat{M}_1^V$, 2b-current part is suppressed by ϵ_{EFT} compared to the leading-order single-nucleon current, while the $\hat{L}_1^A$ and $\hat{C}_1^A$ 2b-current terms are associated with the next order, $\epsilon_{\mathrm{EFT}}^2$. ϵ_{EFT} is usually estimated at 10%–30%. Indeed, the magnetic moment two-body contributions are usually at this order [34], while Gamow–Teller two-body contributions are additionally suppressed [27]. When calculating the needed multipole operators within the so-called impulse approximation, i.e., single-nucleon currents weakly interacting with the β-particles, while neglecting two-body currents and bigger clusters, these ϵ_{EFT} corrections should be taken into account. A detailed derivation of the power-counting of electro-weak operators in χEFT can be found in [35] and references therein.

In this section, we outline these impulse approximation currents, and analyze their expected accuracy, from the EFT point of view. We emphasize that adding two-body currents has the potential to significantly increase the accuracy.

The one-body currents include low-energy coefficients, related to the symmetries of the probe-nucleon interactions. These are denoted by C_V and C_A for the polar-vector and axial-vector SM currents, and C_S, C_P and C_T for the scalar, pseudo-scalar and tensor BSM currents. These symmetry coefficients couple to the nuclear charges g_{sym} (sym $\in \{S, P, V, A, T\}$).

The vector and axial-vector one-body currents are (respectively) [36]:

$$\left\langle p(p_p)\right|\bar{u}\gamma_\mu d\left|n(p_n)\right\rangle = \bar{u}_p(p_p)\left[g_V(q^2)\gamma_\mu - \mathrm{i}\frac{\tilde{g}_{T(V)}(q^2)}{2m_{\mathrm{N}}}\sigma_{\mu\nu}q^\nu + \frac{\tilde{g}_S(q^2)}{2m_N}q_\mu\right]u_n(p_n), \tag{37a}$$

$$\left\langle p(p_p)\right|\bar{u}\gamma_\mu\gamma_5 d\left|n(p_n)\right\rangle = \bar{u}_p(p_p)\left[g_A(q^2)\gamma_\mu - \mathrm{i}\frac{\tilde{g}_{T(A)}(q^2)}{2m_{\mathrm{N}}}\sigma_{\mu\nu}q^\nu + \frac{\tilde{g}_P(q^2)}{2m_{\mathrm{N}}}q_\mu\right]\gamma_5 u_n(p_n). \tag{37b}$$

In the SM, $g_V = 1$ up to second order corrections in isospin breaking [37, 38], as a result of the conservation of the vector current, and $g_A \approx 1.276 g_V$ [39, 40]. The terms with $\tilde{g}_S$ and $\tilde{g}_{T(A)}$, known as second class currents, do not exist in the SM, $\tilde{g}_S$ due to current conservation, and $\tilde{g}_{T(A)}$ because of G-parity considerations. The scalar, pseudo-scalar and tensor currents are (respectively) [36]:

$$\left\langle p(p_p)\right|\bar{u}d\left|n(p_n)\right\rangle = g_S(0)\bar{u}_p(p_p)u_n(p_n) + \mathcal{O}(\epsilon_{\mathrm{recoil}}^2), \tag{37c}$$

$$\left\langle p(p_p)\right|\bar{u}\gamma_5 d\left|n(p_n)\right\rangle = g_P(0)\bar{u}_p(p_p)\gamma_5 u_n(p_n) + \mathcal{O}(\epsilon_{\mathrm{recoil}}^2), \tag{37d}$$

$$\left\langle p(p_p)\right|\bar{u}\sigma_{\mu\nu} d\left|n(p_n)\right\rangle = g_T(0)\bar{u}_p(p_p)\sigma_{\mu\nu} u_n(p_n) + \mathcal{O}(\epsilon_{\mathrm{recoil}}), \tag{37e}$$

where the scalar nuclear charge, $g_S = g_V \frac{(M_n - M_p)^{\mathrm{QCD}}}{m_d - m_u} \approx 0.8\text{–}1.2$ (M_n, M_p , m_d and m_u are the masses of the neutron, proton, down quark and up quark, respectively), the pseudo-scalar nuclear charge is $g_P = g_A \frac{M_n + M_p}{m_d + m_u} = 349(9)$ [41] (the pseudo-scalar contraction, $\bar{u}_p \gamma_5 u_n$, is $\mathcal{O}(\epsilon_{\mathrm{recoil}})$, as opposed to the others, which are $\mathcal{O}(1)$ [36]), and according to lattice QCD, the tensor nuclear charge g_T has the same order of magnitude as g_A [42]. Consequently, all these nuclear charges have the same orders of magnitude. The suppression of the symmetry coefficients results from the coefficients of the effective theory, $\epsilon_{\mathrm{sym}} \propto \left(\frac{m_W}{\Lambda}\right)^n$. These arise from the effective weak interaction Lagrangian, where Λ represents new physics scale, and $n = 0$ for sym $= V, A$, and $n \geqslant 2$ for sym $\neq V, A$. Since it is an effective theory, it will be surprising if it is accurate and contains only vector and axial currents. New experiments will have a per-mil level of precision, sensitive to new physics at the TeV scale. For the simplest BSM operator ($n = 2$), a TeV scale means $\epsilon_{\mathrm{sym}} \sim 10^{-3}$, so the needed accuracy of the calculation is about 10^{-3} to 10^{-4}.

As we have shown, the presented formalism reaches an accuracy of at least $\mathcal{O}\left(\epsilon_{qr}^2 \epsilon_{\mathrm{NR}}^2\right)$ or $\mathcal{O}\left(\epsilon_{\mathrm{recoil}}^2\right)$. For an endpoint of $\approx$2 MeV, $\epsilon_{qr} \sim 0.02 A^{\frac{1}{3}}$, $\epsilon_{\mathrm{NR}} \sim 0.2$ and $\epsilon_{\mathrm{recoil}} \sim 0.003$, this entails an order of 10% needed accuracy for the solution of the nuclear many-body problem (ϵ_{NM}). Note that although the corrections are presented by the ratio between two matrix elements, the denominator of the ratio is the decay's leading order, which can be calibrated from the time of life or other experimental data. So 10% accuracy for individual (numerator) matrix element suffices for a comparison with experimental data.

8. Summary

In this paper, we present a formalism to calculate β-decay rates and observables, including high order recoil and shape corrections, required for ongoing and planned experiments in the search for BSM physics, which aim at a per-mil level of accuracy. The formalism shows that nuclear structure corrections induce a term with spectral inverse energy dependence, imitating a finite Fierz interference term, a fact that affects the analysis of measurements of the angular correlation between the β-particle and the (anti-)neutrino. In addition, we express the nuclear-structure-related corrections to the angular correlation coefficient. Ongoing and future searches for BSM signatures are precisely characterizing the Fierz term and the angular correlation, and thus identifying the nuclear structure effects which create similar spectral behavior is essential in order to distinguish BSM signatures from effects within the SM.

We note that we do not assess here corrections to other nuclear-related observables which are used to search for BSM signatures, such as nuclear mirror transitions or isospin triplets. These are beyond the scope of the current paper.

We identify different small parameters involved in the corrections, that allow us to robustly assess the accuracy of the theoretical calculations. These are expansion parameters that originate in the low-energy character of β-decays, the non-relativistic character of the nuclear wave function, and the Coulomb interaction between the electron and the nucleus. In addition, we define a parameter that characterize the precision of the solution of the nuclear problem, ϵ_{NM}. It contains the EFT expansion order $\epsilon_{\mathrm{EFT}} \sim Q/\Lambda_{\mathrm{b}}$, the uncertainty related to the specific application, e.g., the details related to regularization of the effective theory, the non-relativistic approach to the expansion of the parameter, etc. ($\epsilon_{\mathrm{model}}$), and the error due to numerical convergence, ϵ_{conv}.

The small parameters are defined as follows:

$$\epsilon_{\rm NR} \sim \frac{P_{\rm Fermi}}{m_{\rm N}}, \tag{38a}$$

$$\epsilon_{qr} \sim qR, \tag{38b}$$

$$\epsilon_{\rm recoil} \sim \frac{q}{m_{\rm N}}, \tag{38c}$$

$$\epsilon_{m_\pi} \sim \frac{q}{m_\pi}, \tag{38d}$$

$$\epsilon_{M} \sim \frac{\Delta M}{M_{\rm min}}, \tag{38e}$$

$$\epsilon_{\rm c} \sim \alpha Z_{\rm f}, \tag{38f}$$

$$\epsilon_{\rm EFT} \sim Q/\Lambda_{\rm b}. \tag{38g}$$

These enable an analysis of the accuracy of the calculations, and demonstrate that even a solution of the nuclear many-body problem with an accuracy of about 10% for the matrix elements, can be used to significantly constrain these corrections. This demonstrates the ability of the nuclear theory to meet the needs of future and ongoing experiments looking for BSM signatures in nuclear β-decays. A complementary work, using the same formalism and matrix elements to describe BSM contributions to nuclear β-decays can be found in [30].

The current work is applicable to all nuclei, but is more advantageous for light to medium mass nuclei, where numerically controlled calculations are possible. For these nuclei, Coulomb corrections are of lesser importance, as rather good approximations for these appear in the literature (see, e.g., references [14, 23, 24], based on the Behrens–Buehring formalism, or the appendix of [43], based on the formalism of Holstein [22]). Thus, a complete study on Coulomb corrections, and their *ab initio* formulation, is out of the scope of the current work.

The application of the presented formalism to specific nuclei is already underway. The first one is ^{6}He [17], following its role in several ongoing, or soon to be initiated, precision β-decay experiments. The second is ^{23}Ne [18], in which a joint theoretical-experimental effort enabled us to reanalyze experimental data using new branching ratio measurements, and establish bounds on the presence of exotic tensor interactions.

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Data availability statement

No new data were created or analysed in this study.

Appendix. Nuclear currents and multipole operators at leading orders

To be able to discuss the nuclear dependent part, we need to examine the nuclear current. In the traditional nuclear physics picture, the electroweak current is constructed from the properties of free nucleons. Using the impulse approximation, we refer to the nuclear currents as one-body currents. Neglecting the second-class currents, we rewrite (37a) in a more detailed form:

$$\left\langle \vec{p}', \sigma', \rho' \middle| \mathcal{J}_\mu^V(0) \middle| \vec{p}, \sigma, \rho \right\rangle = \bar{u}\left(\vec{p}', \sigma'\right)\eta_{\rho'}^+\left[g_V\left(q^2\right)\gamma_\mu - \mathrm{i}\frac{\tilde{g}_{T(V)}\left(q^2\right)}{2m_\mathrm{N}}\sigma_{\mu\nu}q^\nu\right]\tau^\pm\eta_\rho u(\vec{p}, \sigma), \tag{A.1a}$$

$$\left\langle \vec{p}', \sigma', \rho' \middle| \mathcal{J}_\mu^A(0) \middle| \vec{p}, \sigma, \rho \right\rangle = \bar{u}\left(\vec{p}', \sigma'\right)\eta_{\rho'}^+\left[g_A\left(q^2\right)\gamma_\mu + \frac{\tilde{g}_P\left(q^2\right)}{2m_\mathrm{N}}q_\mu\right]\gamma_5\tau^\pm\eta_\rho u(\vec{p}, \sigma). \tag{A.1b}$$

Here $u(\vec{p}, \sigma) = \sqrt{\frac{E_p + m_\mathrm{N}}{2E_p}}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p + m_\mathrm{N}} \end{pmatrix}\chi_\sigma$ is the Dirac spinor for a free nucleon of mass m_N, $E_p = \sqrt{p^2 + m_\mathrm{N}^2}$ is the energy of the particle, χ_σ is a two-component Pauli spinor for a spin up and down along the q axis, η_ρ is a two-component Pauli isospinor, and $\tau^\pm = \mp\frac{1}{2}\left(\tau_x \pm \mathrm{i}\tau_y\right)$ are the isospin raising and lowering operators, that turn a proton into a neutron and vice versa.

After substituting the mentioned explicit form of Dirac spinors (using the convention $\bar{u} = u^+\gamma_0$, so that $u^+u = 1$), we make a non-relativistic expansion, expanding the matrix element consistently in powers of $\epsilon_\mathrm{NR} \sim \frac{P_\mathrm{fermi}}{m_\mathrm{N}}$, as momenta are assumed here up to few hundred MeV/c, and find the required matrix elements (here $P_\mu = p_\mu + p'_\mu$, $q_\mu = p_\mu - p'_\mu$ and $E_0 \equiv q_0$):

$$\left\langle \vec{p}', \sigma', \rho' \middle| \mathcal{J}_0^V(0) \middle| \vec{p}, \sigma, \rho \right\rangle = g_V\chi_{\sigma'}^+\eta_{\rho'}^+\tau^\pm\eta_\rho\chi_\sigma + \mathcal{O}\left(\epsilon_\mathrm{NR}^2\right), \tag{A.2a}$$

$$\left\langle \vec{p}', \sigma', \rho' \middle| \vec{\mathcal{J}}^V(0) \middle| \vec{p}, \sigma, \rho \right\rangle = \chi_{\sigma'}^+\eta_{\rho'}^+\frac{1}{2m_\mathrm{N}}\left[g_V\vec{P} + \left(g_V + \tilde{g}_{T(V)}\right)i\vec{q}\times\vec{\sigma}\right]\tau^\pm\eta_\rho\chi_\sigma + \mathcal{O}\left(\epsilon_\mathrm{NR}^2\right), \tag{A.2b}$$

$$\left\langle \vec{p}', \sigma', \rho' \middle| \mathcal{J}_0^A(0) \middle| \vec{p}, \sigma, \rho \right\rangle = \chi_{\sigma'}\eta_{\rho'}^+\frac{1}{2m_\mathrm{N}}\left[g_A\vec{P}\cdot\vec{\sigma} + \frac{\tilde{g}_P}{2m_N}E_0\vec{q}\cdot\vec{\sigma}\right]\tau^\pm\eta_\rho\chi_\sigma + \mathcal{O}\left(\epsilon_\mathrm{NR}^2\right), \tag{A.2c}$$

$$\left\langle \vec{p}', \sigma', \rho' \middle| \vec{\mathcal{J}}^A(0) \middle| \vec{p}, \sigma, \rho \right\rangle = \left(g_A + \frac{q^2}{\left(2m_\mathrm{N}\right)^2}\tilde{g}_P\right)\chi_{\sigma'}\eta_{\rho'}^+\vec{\sigma}\tau^\pm\eta_\rho\chi_\sigma + \mathcal{O}\left(\epsilon_\mathrm{NR}^2\right), \tag{A.2d}$$

when we kept high orders multiplied by the hadronic pseudo-scalar charge $\tilde{g}_P$, since $\tilde{g}_P \approx -\frac{\left(2m_\mathrm{N}\right)^2}{m_\pi^2 - q^2}g_A$ [44], making it two orders of magnitudes larger than the vector and axial-vector charges, g_V and g_A.

Then, we use the definition of the (second quantization) $\mathcal{J}(\vec{r})$ current matrix element as a sum over first-quantization currents $\hat{\mathcal{J}}^{(1)}$: $\langle \vec{p}', \sigma', \rho' | \mathcal{J}(\vec{r}) | \vec{p}, \sigma, \rho \rangle = \int \mathrm{d}^3x\phi_{\vec{p}'\sigma'\rho'}^+(\vec{x}) \left[\hat{\mathcal{J}}^{(1)}(\vec{x})\,\delta^{(3)}(\vec{r} - \vec{x})\right]\phi_{\vec{p}\sigma\rho}(\vec{x})$. Evaluated at $\vec{r} = 0$, we find out that $\langle \vec{p}', \sigma', \rho' | \mathcal{J}(0) | \vec{p}, \sigma, \rho \rangle = \phi_{\vec{p}'\sigma'\rho'}^+(0)\hat{\mathcal{J}}^{(1)}(0)\phi_{\vec{p}\sigma\rho}(0)$, what permits the identification of the nuclear density operators in first quantization (from (A.2a)). Finally, using the current density operator in the first quantization, $\mathcal{J}(\vec{r}) = \sum_{j=1}^A \hat{\mathcal{J}}^{(1)}(j)\delta^{(3)}(\vec{r} - \vec{r}_j)$, and based on the assumption that the first-quantization has no location dependence, so $\hat{\mathcal{J}}^{(1)}(j) = \hat{\mathcal{J}}^{(1)}(0)(j)$, one gets the following currents:

$$\mathcal{J}_0^V(\vec{r}) = g_V \sum_{j=1}^{A} \tau_j^{\pm} \delta^{(3)}(\vec{r}-\vec{r}_j) + \mathcal{O}(\epsilon_{\mathrm{NR}}^2), \tag{A.3a}$$

$$\vec{\mathcal{J}}^V(\vec{r}) = \frac{1}{2m_{\mathrm{N}}} \sum_{j=1}^{A} \left[g_V \{\vec{p}_j, \delta^{(3)}(\vec{r}-\vec{r}_j)\} + (g_V + \tilde{g}_{T(V)}) \vec{\nabla} \times \vec{\sigma}_j \delta^{(3)}(\vec{r}-\vec{r}_j) \right] \tau_j^{\pm} + \mathcal{O}(\epsilon_{\mathrm{NR}}^2), \tag{A.3b}$$

$$\mathcal{J}_0^A(\vec{r}) = \frac{1}{2m_{\mathrm{N}}} \sum_{j=1}^{A} \left[g_A \{\vec{p}_j, \delta^{(3)}(\vec{r}-\vec{r}_j)\} - \mathrm{i} \frac{\tilde{g}_P}{2m_{\mathrm{N}}} (E_0 \pm \Delta E_{\mathrm{c}}) \vec{\nabla} \delta^{(3)}(\vec{r}-\vec{r}_j) \right] \cdot \vec{\sigma}_j \tau_j^{\pm} + \mathcal{O}(\epsilon_{\mathrm{NR}}^2), \tag{A.3c}$$

$$\vec{\mathcal{J}}^A(\vec{r}) = \left[g_A + \left(\frac{q}{2m_{\mathrm{N}}} \right)^2 \tilde{g}_P \right] \sum_{j=1}^{A} \vec{\sigma}_j \tau_j^{\pm} \delta^{(3)}(\vec{r}-\vec{r}_j) + \mathcal{O}(\epsilon_{\mathrm{NR}}^2). \tag{A.3d}$$

Here, we made the operator replacements $\vec{P} \to \{\vec{p}_j, \delta^{(3)}(\vec{r}-\vec{r}_j)\}$, and $\vec{q} \to -\mathrm{i}\vec{\nabla}$, the last one based on a partial integration of Fourier transform of the transition matrix element of the current, $\int \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{r}} \langle \mathrm{f} | \mathcal{J}_\mu(\vec{r}) | \mathrm{i} \rangle$, with localized densities. We also corrected here the maximal electron energy E_0 with the Coulomb displacement energy ΔE_{c}, as mentioned in subsection 2.2.

Positioning (A.3a) into the multipole operators definition ((4) in the main text), leads to the explicit expressions for the vector and axial currents multipole operators:

$$\hat{C}_J^V(q) = g_V \sum_{j=1}^{A} M_J(q\vec{r}_j) \tau_j^{\pm} + \mathcal{O}(\epsilon_{qr}^J \epsilon_{\mathrm{NR}}^2), \tag{A.4a}$$

$$\hat{L}_J^V(q) = -\frac{q}{2m_{\mathrm{N}}} g_V \sum_{j=1}^{A} \left\{ M_J(q\vec{r}_j) - 2 \left[\frac{1}{q} \vec{\nabla} M_J(q\vec{r}_j) \right] \cdot \frac{1}{q} \vec{\nabla} \right\} \tau_j^{\pm} + \mathcal{O}(\epsilon_{qr}^{J-1} \epsilon_{\mathrm{NR}}^2), \tag{A.4b}$$

$$\hat{E}_J^V(q) = \frac{q}{m_{\mathrm{N}}} \sum_{j=1}^{A} \left\{ -\mathrm{i} g_V \left[\frac{1}{q} \vec{\nabla} \times \vec{M}_{JJ1}(q\vec{r}_j) \right] \cdot \frac{1}{q} \vec{\nabla} + \frac{g_V + \tilde{g}_{T(V)}}{2} \vec{M}_{JJ1}(q\vec{r}_j) \cdot \vec{\sigma}_j \right\} \tau_j^{\pm} + \mathcal{O}(\epsilon_{qr}^{J-1} \epsilon_{\mathrm{NR}}^2), \tag{A.4c}$$

$$\hat{M}_J^V(q) = -\frac{\mathrm{i}q}{m_{\mathrm{N}}} \sum_{j=1}^{A} \left\{ g_V \vec{M}_{JJ1}(q\vec{r}_j) \cdot \frac{1}{q} \vec{\nabla} + \mathrm{i} \frac{g_V + \tilde{g}_{T(V)}}{2} \times \left[\frac{1}{q} \vec{\nabla} \times \vec{M}_{JJ1}(q\vec{r}_j) \right] \cdot \vec{\sigma}_j \right\} \tau_j^{\pm} + \mathcal{O}(\epsilon_{qr}^J \epsilon_{\mathrm{NR}}^2), \tag{A.4d}$$

$$\hat{C}_J^A(q) = -\frac{\mathrm{i}q}{m_{\mathrm{N}}} \sum_{j=1}^{A} \left\{ g_A M_J(q\vec{r}_j) \vec{\sigma}_j \cdot \frac{1}{q} \vec{\nabla} + \frac{1}{2} \left[g_A - \frac{\tilde{g}_P}{2m_N} (E_0 \pm \Delta E_c) \right] \times \left[\frac{1}{q} \vec{\nabla} M_J(q\vec{r}_j) \right] \cdot \vec{\sigma}_j \right\} \tau_j^{\pm} + \mathcal{O}(\epsilon_{qr}^J \epsilon_{\mathrm{NR}}^2), \tag{A.4e}$$

$$\hat{L}_J^A(q) = \mathrm{i}\left(g_A + \frac{q^2}{(2m_\mathrm{N})^2}\tilde{g}_P\right)\sum_{j=1}^{A}\left[\frac{1}{q}\vec{\nabla}M_J\left(q\vec{r}_j\right)\right]\cdot\vec{\sigma}_j\tau_j^{\pm} + \mathcal{O}\left(\epsilon_{qr}^{J-1}\epsilon_\mathrm{NR}^2\right), \tag{A.4f}$$

$$\hat{E}_J^A(q) = \left(g_A + \frac{q^2}{(2m_\mathrm{N})^2}\tilde{g}_P\right)\sum_{j=1}^{A}\left[\frac{1}{q}\vec{\nabla}\times\vec{M}_{JJ1}\left(q\vec{r}_j\right)\right]\cdot\vec{\sigma}_j\tau_j^{\pm} + \mathcal{O}\left(\epsilon_{qr}^{J-1}\epsilon_\mathrm{NR}^2\right), \tag{A.4g}$$

$$\hat{M}_J^A(q) = \left(g_A + \frac{q^2}{(2m_\mathrm{N})^2}\tilde{g}_P\right)\sum_{j=1}^{A}\vec{M}_{JJ1}\left(q\vec{r}_j\right)\cdot\vec{\sigma}_j\tau_j^{\pm} + \mathcal{O}\left(\epsilon_{qr}^{J}\epsilon_\mathrm{NR}^2\right), \tag{A.4h}$$

where

$$M_J(q\vec{r}) \equiv j_J(qr)Y_J(\hat{r}), \tag{A.5a}$$

$$\vec{M}_{JL1}(q\vec{r}) \equiv j_L(qr)\vec{Y}_{JL1}(\hat{r}), \tag{A.5b}$$

which hold to the identities [,44]:

$$\frac{1}{q}\vec{\nabla}M_J(q\vec{r}) = \sqrt{\frac{J+1}{2J+1}}\vec{M}_{J,J+1,1}(q\vec{r}) + \sqrt{\frac{J}{2J+1}}\vec{M}_{J,J-1,1}(q\vec{r}), \tag{A.6a}$$

$$\frac{1}{q}\vec{\nabla}\times\vec{M}_{JJ1}(q\vec{r}) = -\mathrm{i}\sqrt{\frac{J}{2J+1}}\vec{M}_{J,J+1,1}(q\vec{r}) + \mathrm{i}\sqrt{\frac{J+1}{2J+1}}\vec{M}_{J,J-1,1}(q\vec{r})\,. \tag{A.6b}$$

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References

[1] Weinberg S 2009 V–A was the key *J. Phys.: Conf. Ser.* **196** 012002
[2] Severijns N, Beck M and Naviliat-Cuncic O 2006 Tests of the standard electroweak model in nuclear beta decay *Rev. Mod. Phys.* **78** 991–1040
[3] Wu C S, Ambler E, Hayward R W, Hoppes D D and Hudson R P 1957 Experimental test of parity conservation in beta decay *Phys. Rev.* **105** 1413–5
[4] Jackson J D, Treiman S B and Wyld H W 1957 Possible tests of time reversal invariance in beta decay *Phys. Rev.* **106** 517–21
[5] Greuling E and Meeks M L 1951 Electron-neutrino angular correlation *Phys. Rev.* **82** 531–7
[6] Cirgiliano V, Garcia A, Gazit D, Naviliat-Cuncic O, Guy S and Young A 2019 Precision beta decay as a probe of new physics (arXiv:1907.02164)
[7] Mardor I *et al* 2018 The Soreq applied research accelerator facility (SARAF): overview, research programs and future plans *Eur. Phys. J.* A **54** 91
[8] Ben O *et al* 2018 Weak interaction studies at SARAF *Hyperfine Interact.* **239** 57
[9] Huyan X *et al* 2016 Toward a measurement of weak magnetism in ^{6}He decay *Hyperfine Interact.* **237** 93

[10] Huyan X, Naviliat-Cuncic O, Voytas P, Chandavar S, Hughes M, Minamisono K and Paulauskas S V 2018 Geant4 simulations of the absorption of photons in CsI and NaI produced by electrons with energies up to 4 MeV and their application to precision measurements of the β-energy spectrum with a calorimetric technique *Nucl. Instrum. Methods Phys. Res.* A **879** 134–40
[11] Naviliat-Cuncic O 2016 Searches for exotic interactions in nuclear beta decay *AIP Conf. Proc.* **1753** 060001
[12] Asner D M *et al* 2015 Single-electron detection and spectroscopy via relativistic cyclotron radiation *Phys. Rev. Lett.* **114** 162501
[13] González-Alonso M, Naviliat-Cuncic O and Severijns N 2019 New physics searches in nuclear and neutron β decay *Prog. Part. Nucl. Phys.* **104** 165–223
[14] Hayen L, Severijns N, Bodek K, Rozpedzik D and Mougeot X 2018 High precision analytical description of the allowed β spectrum shape *Rev. Mod. Phys.* **90** 015008
[15] Hayen L and Severijns N 2019 Beta spectrum generator: high precision allowed β spectrum shapes *Comput. Phys. Commun.* **240** 152–64
[16] Lee T D and Yang C N 1956 Question of parity conservation in weak interactions *Phys. Rev.* **104** 254–8
[17] Glick-Magid A, Forssén C, Gazda D, Gazit D, Gysbers P and Navrátil P 2022 Nuclear *ab initio* calculations of ^{6}He β-decay for beyond the standard model studies *Phys. Lett.* B **832** 137259
[18] Mishnayot Y *et al* 2021 Constraining new physics with a new measurement of the ^{23}Ne branching ratio (arXiv:2107.14355)
[19] Armstrong L Jr and Kim C W 1972 Coulomb correction in nuclear β decay: elementary-particle treatment *Phys. Rev.* C **5** 672
[20] Wilkinson D H 1970 The evaluation of finite nuclear size effects in allowed β-decay *Nucl. Phys.* A **158** 476–86
[21] Holstein B R 1979 Electromagnetic effects and weak form factors *Phys. Rev.* C **19** 1467
[22] Holstein B R 1974 Electromagnetic corrections to allowed nuclear beta decay *Phys. Rev.* C **9** 1742
[23] Hayen L and Young A R 2020 Consistent description of angular correlations in β decay for beyond standard model physics searches (arXiv:2009.11364)
[24] Behrens H and Bühring W 1982 *Electron Radial Wave Functions and Nuclear Beta-decay* vol 67 (Oxford: Oxford University Press)
[25] Antony M S, Pape A and Britz J 1997 Coulomb displacement energies between analog levels for $3 \leqslant a \leqslant 239$ *At. Data Nucl. Data Tables* **66** 1–63
[26] Wang M, Huang W J, Kondev F G, Audi G and Naimi S 2021 The AME 2020 atomic mass evaluation (II). Tables, graphs and references *Chin. Phys.* C **45** 030003
[27] Gysbers P *et al* 2019 Discrepancy between experimental and theoretical β-decay rates resolved from first principles *Nat. Phys.* **15** 428–31
[28] Dirk Walecka J 2004 *Theoretical Nuclear and Subnuclear Physics* 2nd edn (Singapore: World Scientific)
[29] Haxton W and Lunardini C 2008 Sevenoperators, a mathematica script for harmonic oscillator nuclear matrix elements arising in semileptonic electroweak interactions *Comput. Phys. Commun.* **179** 345–58
[30] Glick-Magid A and Gazit D 2022 Multipole decomposition of tensor interactions of fermionic probes with composite particles and BSM signatures in nuclear reactions (arXiv:2207.01357)
[31] Barry R H 1974 Recoil effects in allowed beta decay: the elementary particle approach *Rev. Mod. Phys.* **46** 789–814
[32] Behrens H, Jänecke J and Schopper H 1969 *Numerical Tables for Beta-Decay and Electron Capture* (Berlin: Springer)
[33] Glick-Magid A, Mishnayot Y, Mukul I, Hass M, Vaintraub S, Ron G and Gazit D 2017 Beta spectrum of unique first-forbidden decays as a novel test for fundamental symmetries *Phys. Lett.* B **767** 285–8
[34] Pastore S, Pieper S C, Schiavilla R and Wiringa R B 2013 Quantum Monte Carlo calculations of electromagnetic moments and transitions in $a \leqslant 9$ nuclei with meson-exchange currents derived from chiral effective field theory *Phys. Rev.* C **87** 035503
[35] Krebs H 2020 Nuclear currents in chiral effective field theory *Eur. Phys. J.* A **56** 234
[36] Cirigliano V, Gardner S and Holstein B R 2013 Beta decays and non-standard interactions in the LHC era *Prog. Part. Nucl. Phys.* **71** 93–118
[37] Ademollo M and Gatto R 1964 Nonrenormalization theorem for the strangeness-violating vector currents *Phys. Rev. Lett.* **13** 264–6

[38] Donoghue J F and Wyler D 1990 Isospin breaking and the precise determination of V_{ud} *Phys. Lett.* B **241** 243–8
[39] Mendenhall M P *et al* 2013 Precision measurement of the neutron β-decay asymmetry *Phys. Rev.* C **87** 032501
[40] Mund D, Märkisch B, Deissenroth M, Krempel J, Schumann M, Abele H, Petoukhov A and Soldner T 2013 Determination of the weak axial vector Coupling $\lambda = g_A/g_V$ from a measurement of the β-asymmetry parameter *A* in neutron beta decay *Phys. Rev. Lett.* **110** 172502
[41] González-Alonso M and Martin Camalich J 2014 Isospin Breaking in the Nucleon Mass and the Sensitivity of β Decays to New Physics *Phys. Rev. Lett.* **112**
[42] Bhattacharya T, Cirigliano V, Cohen S D, Gupta R, Lin H-W and Yoon B 2016 Axial, scalar, and tensor charges of the nucleon from $2 + 1 + 1$-flavor lattice QCD *Phys. Rev.* D **94** 054508
[43] Calaprice F P and Holstein B R 1976 Weak magnetism and the beta spectra of ^{12}B and ^{12}N *Nucl. Phys.* A **273** 301–25
[44] Walecka J D 1975 Section 4—semileptonic weak interactions in nuclei *Muon Physics, Volume II: Weak Interactions* ed V W Hughes and C S Wu (New York: Academic) pp 113–218