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REGULAR PAPER

On the sensitivity of the purity and entropy of mixed quantum states on variations of Planck's constant

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Abstract We explore the sensitivity of quantum state characteristics, specifically purity and entropy, to variations in the chosen value of Planck's constant. This investigation begins with a novel examination of the Wigner function, framed through the displacement operator, providing fresh insights into quantum phase space analysis. Emphasizing Gaussian states, we systematically evaluate how changes in Planck's constant influence the mixedness of quantum states. By leveraging the Narcowich–Wigner spectrum, we derive key behaviors of purity and entropy under these variations. Finally, our findings are interpreted through the lens of symplectic capacity, offering a robust theoretical framework that unifies quantum state dynamics and phase-space geometry.

Keywords Mixed staes · Entropy · Gaussia · Planck's constant

1 On the variability of fundamental constants

The question of whether all constants of Nature really are constant has a long history. Paul Dirac already suggested in 1937 in his "Large Numbers Hypothesis" [5] that the gravitational constant G or the fine-structure constant $\alpha = e^2/2\hbar c$ might be subject to change over time. Since then numerous and very precise measurements seem to indicate that, indeed, α or Planck's constant h has undergone a very small and slow shifts since the Big Bang. These findings spurred further attempts to measure variations of α and whence of h, in particular using cosmological methods. In 1999, a team of astronomers headed by John Webb reported that measurements of light absorbed by quasars suggest that the value of the fine-structure constant was once slightly different from what it is today and found an upper bound for this variation at roughly 10^{-17} per year. The quest for testing this hypothesis is ongoing, and has been rekindled by recent advances (2024) by Zhang et al. [40] on nuclear clocks using laser-controllable transition in the atomic nucleus of thorium-229. Freeman Dyson had already pointed out to the author a few years ago (in a private communication) that advances in the precision of atomic clocks would be instrumental in the study of the possibly variability of constants of Nature. Perhaps the results of Zhang et al. will shed some light on these

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fascinating problems (see the review paper [33]). We mention that Duff has shown [6–8] that all the fundamental physical dimensions could be expressed using only one unity: mass.

We have collected some results on measurements of the values of h in an Appendix.

In this paper we will discuss some consequences of possible variations of Planck's constant on mixes quantum states, thus pushing the study we initiated in [17]. Our treatment is purely mathematical; we do not discuss the physical question whether Planck's constant (or other constants of Nature) are variable, or not: *Hypotheses non fingo.* As we will see the choice of the value of Planck's constant has drastic consequences for quantum states, pure or mixed. Assigning different values to h can deeply change the mixedness (and hence the purity and entropy) of a quantum state. This was already pointed out in our previous work [17] in the Gaussian case, and also discussed by Dias and Prata [4].

Notation 1 *The standard symplectic form on $\mathbb{R}^{2n}$ is $\omega(z,z') = Jz \cdot z'$ where $J = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n}\end{pmatrix}$ and $\cdot$ is the Euclidean scalar product. Let M be a symmetric matrix on $\mathbb{R}^n$; we will often use the notation $Mx\cdot x = Mx^2$. We denote by $\mathrm{Sp}(n)$ the symplectic group of $(\mathbb{R}^{2n},\omega)$ and by $\mathrm{Mp}(n)$ the corresponding metaplectic group (it is a unitary representation of the double covering of $\mathrm{Sp}(n)$). We denote by $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing test functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual (the tempered distributions). Given a tempered distribution $a \in \mathcal{S}'(\mathbb{R}^{2n})$ we denote by $\mathrm{Op_W}(a)$ the Weyl operator with symbol a.*

2 The density matrix and its Wigner distribution

2.1 From displacements to the Wigner function

The Wigner function loses much of its mystery when it is viewed as a transition amplitude between the displaced function and its reflection.

How Wigner arrived at his eponymous transform in his famous paper [38] is a mystery. Some have speculated that the idea might have come from Leo Szilard (because Wigner acknowledges his help in a footnote) but there is no firm evidence that the latter should have participated in Wigner's constructions (it is believed that Wigner wanted to boost Szilard's career by mentioning him as a collaborator). Indeed, nothing seems to motivate the introduction of the Wigner function

$$W_\hbar\psi(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}py}\psi(x+\tfrac{1}{2}y)\psi^*(x-\tfrac{1}{2}y)dy \tag{1}$$

except that it has the qualities of a quasi-probability function.

Consider now the phase space translation operator $T(z_0) : z \longmapsto z + z_0$. It acts unitarily on functions by pullback: if ψ is a function on $\mathbb{R}^n$ only then $T(z_0)\,\psi(x) = \psi(x - x_0)$ if $z_0 = (x_0, p_0)$. It turns out that $T(z_0)$ is the time-one value of the flow $t \longmapsto T(tz_0)$ determined by the Hamilton equations $\dot{x} = x_0$, $\dot{p} = p_0$ for the phase space function $\omega(z, z_0) = px_0 - p_0x$ (the "displacement Hamiltonian"). We quantize this flow to an evolution group $t \longmapsto \widehat{T}_\hbar(tz_0)$ solution of the abstract Schrödinger equation

$$i\hbar\frac{d}{dt}\widehat{T}_\hbar(tz_0) = \omega(\widehat{z}, z_0)\widehat{T}_\hbar(tz_0)$$

where $\widehat{z} = (\widehat{x}, \widehat{p})$. By definition, the time-one value $\widehat{T}_\hbar(z_0) = e^{-\frac{i}{\hbar}\omega(\widehat{z},z_0)}$ of $\widehat{T}_\hbar(tz_0)$ is the (Heisenberg–Weyl) displacement operatt0r; a simple calculation shows that its action is explicitly given by

$$\widehat{T}_\hbar(z_0)\psi(x) = e^{\frac{i}{\hbar}(p_0x - \frac{1}{2}p_0x_0)}\psi(x - x_0). \tag{2}$$

Let ψ be a square integrable function on $\mathbb{R}^n$; by definition its Wigner function (or transform) is

$$W_\hbar\psi(z) = \left(\frac{1}{\pi\hbar}\right)^n \langle \widehat{R}\widehat{T}_\hbar(z_0)\psi|\widehat{T}_\hbar(z_0)\psi\rangle \tag{3}$$

where $\widehat{R}$ is the reflection operator: $\widehat{R}\psi(x) = \psi(-x)$. This equality can be rewritten

$$W_\hbar\psi(z) = \left(\frac{1}{\pi\hbar}\right)^n \langle \widehat{R}_\hbar(z_0)\psi|\psi\rangle \tag{4}$$

where $\widehat{R}_\hbar(z_0)$ is the Grossmann–Royer reflection operator [24,34]:

$$\widehat{R}_\hbar(z_0) = \widehat{T}_\hbar(-z_0)\widehat{R}\widehat{T}_\hbar(z_0) \tag{5}$$

whose action on a function (or disruption) ψ) is given by

$$\widehat{R}_\hbar(z_0)\psi(x) = e^{\frac{2i}{\hbar}p_0(x-x_0)}\psi(2x_0 - x). \tag{6}$$

It is a simple computational exercise, using the expression using (6) to check that $W_\hbar\psi(z)$ is given by the familiar expression (1).

Definition (3)–(4) show that the dependency of the Wigner function on Planck's constant is due solely on the dimensionless phase

$$\Phi = \frac{i}{\hbar}(p_0x - \frac{1}{2}p_0x_0) \tag{7}$$

(for the signification of this phase in terms of Lagrangian submanifolds see [13]). The effect of a change of the value of Planck's constant from $\hbar$ to $\hbar'$ is straightforward:

$$W_{\hbar'}\psi(x, p) = \left(\frac{\hbar}{\hbar'}\right)^n W_\hbar\psi(x, \frac{\hbar}{\hbar'}p). \tag{8}$$

2.2 Their Wigner distribution of the density matrix

A density matrix (or operator) on a complex Hilbert space H is positive semidefinite operator $\widehat{\rho}$ on H with trace one: $\operatorname{Tr}\widehat{\rho} = 1$. he positive semi-definiteness property $\widehat{\rho} \geq 0$ means that $\langle\widehat{\rho}\psi|\psi\rangle \geq 0$ for all $\psi \in H$. Notice that since we are assuming that H is a complex Hilbert space, the positive semi-definiteness of $\widehat{\rho}$ implies that it is self-adjoint A density matrix is a compact operator, applying the spectral theorem one sees that $\widehat{\rho}$ can be written as a convex sum of orthogonal projections: $\psi_j \in H$ and corresponding positive numbers λ_j summing up to one such that

$$\widehat{\rho}_\hbar = \sum_j \lambda_j \Pi_{\psi_j} \quad , \quad \lambda_j \geq 0 \quad , \quad \sum_j \lambda_j = 1 \tag{9}$$

where $\Pi_{\psi_j} = |\psi_j\rangle\langle\psi_j$ is the orthogonal pocosin on ψ_j. Assume from now on that $H = L^2(\mathbb{R}^n)$. Then $\widehat{\rho}_\hbar$ is the Weyl operator on $\mathbb{R}^n$ and $W_{\widehat{\rho}_\hbar}$ its Wigner distribution: by definition the phase space function defined by

$$W_{\widehat{\rho}_\hbar}(z) = \sum_j \lambda_j W_\hbar \psi_j(z). \tag{10}$$

It is common in Physics to find the formula

$$W_{\widehat{\rho}_\hbar}(x, p) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}py} \langle x - \tfrac{1}{2}y|\widehat{\rho}_\hbar|x - \tfrac{1}{2}y\rangle dy. \tag{11}$$

2.3 The quantum condition

That the chosen value of Planck's constant matter is already seen on the following simple example. Suppose we have determined a state $\widehat{\rho}_\hbar$: with covariance matrix Σ (It can be determined experimentally using symplectic tomographic methods). It is well known [10,15] that if this state is quantum the covariance matrix has to satisfy the condition

$$\Sigma + \frac{i\hbar}{2}J \quad \textit{is positive semidefinite} \tag{12}$$

which implies to the Robertson–Schrödinger inequalities

$$(\Delta x_j)^2(\Delta p_j)^2 \geq (\Delta(x_j, p_j)^2 + \frac{1}{4}\hbar^2. \tag{13}$$

These conditions are sensitive to the value of $\hbar$. In fact, if (12) holds it will also hold for $\hbar' < \hbar$., and so does (13). Setting $\hbar' = r\hbar$; we have

$$\Sigma + \frac{i\hbar'}{2}J = (1 - r)\Sigma + r\left(\Sigma + \frac{i\hbar}{2}J\right);$$

since $\Sigma > 0$ and $\Sigma + (i\hbar/2)J \geq 0$ condition (12) will hold for $0 \leq r \leq 1.$, but not necessarily for $\hbar' > \hbar$. To illustrate this consider the phase space Gaussian

$$\rho_\Sigma(z) = \frac{1}{(2\pi)^n\sqrt{\det\Sigma}} e^{-\frac{1}{2}\Sigma^{-1}(z-z_0)\cdot(z-z_0)} \tag{14}$$

with covariance matrix Σ. If ρ_Σ is thew Wigner distribution of a quantum state (*i.e.* if it satisfies (12)) then the purity of this state will be will be a density operator for all $\hbar' \leq \hbar$hen its purity is ([20] or [12], p. 302)

$$\mu = \left(\frac{\hbar}{2}\right)^n \det(\Sigma^{-1/2}) \tag{15}$$

and the corresponding state will state will be pure if and only if $\det(\Sigma) = (\hbar/2)^{2n}$. It is clear that if we choose $\hbar' > \hbar$. then we will have $\mu > 1$, which is absurd.

3 Purity and Planck's constant

Since the function $(2\pi\hbar)^n W_{\widehat{\rho}_\hbar}$ is the Weyl symbol [11,15]of the operator $\widehat{\rho}_\hbar$. By definition the purity of $\widehat{\rho}_\hbar$ is

$$\mu(\widehat{\rho}_\hbar) = \mathrm{Tr}(\widehat{\rho}_\hbar^2) = \sum_j \lambda_j^2.$$

We briefly discussed in Sect. 2.3 the sensitivity of the purity of a Gaussian mixed state to variations of Planck's constant. Here is a much more general result, whose proof is based on Moyal's identity [18]: let ψ and ϕ be two square integrable functions on $\mathbb{R}^n$. We have

$$\int_{\mathbb{R}^{2n}} W_\hbar\psi(z) W_\hbar\phi(z) dz = \left(\tfrac{1}{2\pi\hbar}\right)^n |\langle\psi|\phi\rangle|^2. \tag{16}$$

Theorem 2 *(Purity)Let ρ be a phase space function. Assume that ρ is the Wigner distribution of two density matrices $\widehat{\rho}_\hbar$ and $\widehat{\rho}_{\hbar'}$corresponding to two values $\hbar$ and $\hbar'$ of Planck's constant. The purities of $\widehat{\rho}_\hbar$ and $\widehat{\rho}_{\hbar'}$ are related by the formula*

$$\mu(\widehat{\rho_{\hbar'}}) = \left(\frac{\hbar'}{\hbar}\right)^n \mu(\widehat{\rho}_\hbar). \tag{17}$$

Proof The following argument somewhat simplifies the proof in [17], &3.2. There exist (orthonormal) families (ψ_j) and (ψ_j') of square integrable functions such that

$$\rho(z) = \sum_j \lambda_j W_\hbar \psi_j(z) = \sum_j \lambda_j' W_{\hbar'}\psi_j'(z)$$

where the λ_j and λ_j' are positive numbers summing up to one. Squaring both sides of the second equality and integrating we get

$$\sum_{j,k} \lambda_j\lambda_k \int_{\mathbb{R}^{2n}} W_\hbar\psi_j(z) W_\hbar\psi_k(z) dz = \sum_{j,k} \lambda_j'\lambda_k' \int_{\mathbb{R}^{2n}} W_{\hbar'}\psi_j'(z) W_{\hbar'}\psi_k'(z) dz;$$

hence, in view of Moyal's identity and taking into account the orthonormality proprieties the ψ_j and ψ_j',

$$\left(\tfrac{1}{2\pi\hbar}\right)^n \sum_j \lambda_j^2 = \left(\tfrac{1}{2\pi\hbar'}\right)^n \sum_j \lambda_j'^2$$

which is the same thing as (17).

The equality (17) implies in particular that if $\widehat{\rho}_\hbar$ is a pure state then $\widehat{\rho_{\hbar'}}$ can be a density matrix if and only if $\hbar' < \hbar$ in which case $\widehat{\rho_{\hbar'}}$ is mixed: decreasing Planck's constant decreases purity and hence increases mixedness.

4 Entropy and Planck's constant

4.1 The von Neumann entropy: discussion

The calculations above will allow us to easily describe the behavior of a Gaussian state (36) under variations of Planck's constant. Let us briefly discuss the (von Neumann) entropy of a quantum state. It is a measure of the uncertainty or the amount of quantum information in a quantum system. By definition the (von Neumann) entropy of a density matrix $\widehat{\rho}_\hbar$ is the nonnegative number

$$\boldsymbol{S}(\widehat{\rho}_\hbar) = -\operatorname{Tr}(\widehat{\rho}_\hbar \ln \widehat{\rho}_\hbar) \tag{18}$$

where the logarithm $\ln \widehat{\rho}_\hbar$ is defined as follows: suppose that $\widehat{\rho}_\hbar$ has the spectral decomposition

$$\widehat{\rho}_\hbar = \sum_j \lambda_j \widehat{\rho}_j \quad , \quad \lambda_j > 0, \quad \sum_j \lambda_j = 1 \tag{19}$$

where the λ_j are > 0 and $\widehat{\rho}_j$ are rank-one orthogonal projections in $L^2(\mathbb{R}^n)$. Then

$$\ln \widehat{\rho}_\hbar = \sum_j (\ln \lambda_j) \widehat{\rho}_\hbar \tag{20}$$

is also a trace class operator. Note that by the change of variables $z \to S^{-1}z$ it is clear that $\mathbb{S}(\rho) = \mathbb{S}(\rho \circ S^{-1})$ for $S \in \operatorname{Sp}(n)$ (because $\det S = 1$). With the notation (23) and (24) we thus have, taking into account this symplectic invariance and the additivity of entropy, that

$$\mathbb{S}(\widehat{\rho}_\hbar) = \sum_{j=1}^n \mathbb{S}(\widehat{\rho}_{\hbar j}). \tag{21}$$

4.2 The Gaussian case

Assume that ρ is the Gaussian ρ_Σ (36); then

$$\rho_\Sigma(S^{-1}z) = \frac{1}{(2\pi)^n \sqrt{\det D}} e^{-\frac{1}{2}D^{-1}z \cdot z} \tag{22}$$

where $\Sigma = S^T DS$ is the Williamson factorization of the covariance matrix. Thus,

$$\rho_\Sigma(S^{-1}z) = \rho_1(x_1, p_1)(\otimes \rho_2(x_2, p_2) \otimes \cdot \cdot \otimes \rho_n(x_n, p_n) \tag{23}$$

with

$$\rho_j(x_j, p_j) = \frac{1}{2\pi \lambda_j^\omega} \exp\left(-\frac{1}{2\lambda_j^\omega}(x_j^2 + p_j^2)\right). \tag{24}$$

The partial entropies $\mathbb{S}\widehat{\rho}_j)$ are given by the formula

$$\mathbb{S}(\widehat{\rho}_j) = \frac{1 - \mu_j}{2\mu_j} \ln\left(\frac{1 + \mu}{1 - \mu_j}\right) - \ln\left(\frac{2\mu_j}{1 - \mu_j}\right) \tag{25}$$

(see Agarwal [1]; we have given a rigorous proof thereof in [20]. In this formula $\mu_j = \hbar'/2\lambda_j^\omega$ is the purity of $\widehat{\rho}_j$ (recall that λ_j^ω is the jth symplectic eigenvalues of the covariance matrix Σ, and that we have $\hbar' \leq 2\lambda_{\min}^\omega$. It follows that $\mathbb{S}(\widehat{\rho}_j)$ (and hence $\mathbb{S}(\widehat{\rho}_\Sigma)$) depend on the value of $\hbar$: unseeing $\mu_j = \hbar'/2\lambda_j^\omega$ in the expression (25 we get

$$\mathbb{S}(\widehat{\rho}_j) = \frac{2-\lambda_j^\omega}{2\hbar} \ln\left(\frac{\lambda_j^\omega + \hbar}{\lambda_j^\omega - \hbar}\right) - \ln\left(\frac{2\hbar}{2\lambda_j^\omega - \hbar}\right). \tag{26}$$

Using formula (21) for the sum of partial entropies we see that

$$\lim_{\hbar \to 0+} \mathbb{S}(\widehat{\rho}_\hbar) = \infty. \tag{27}$$

This can be interpreted by saying that the information about the system's precise state decreases when Planck's constant becomes smaller. Notice that the entropy is zero (pure state) if and only if all the λ_j^ω are equal to $2\hbar$ (that is if the state is a pure Gaussian).

5 The quantum Bochner theorem

Let us now address the question "*Under which conditions is a given a phase space function ρ the Wigner distribution of some density matrix*?". This question has been addressed by many authors; historically the paper [31] by Narcowich and O'Connell was the first to give a rigorous approach to the problem; their work was inspired by the so-called "KLM papers" [26–28] by Kastler, Loupias, and Miracle-Sole. The main results: can be explained as follows (we ar4e following our exposition in [18], Chapter 4; the method, different from that in KLM which relied on the theory of Banach algebras, and first appeared in our paper [3] with Cordero and Nicola. Assume that ρ is both integrable and square integrable and that

$$\int_{\mathbb{R}^{2n}} \rho(z) dz = 1. \tag{28}$$

We denote by $\rho_\Diamond$ the "diamond Fourier transform". Let N be an arbitary nonnegative integer and (z_j, z_k) an arbitrary pair of phase space points.

Theorem 3 *(Quantum Bochner)The functionρ is the Wigner distribution of a density matrix if and only if the matrices*

$$\Lambda = (\Lambda_{jk}^N)_{11 \le j,k \le N} \quad , \quad \Lambda_{jk}^N = e^{-\frac{i\hbar}{2}\omega(z_j, z_k')} \rho_\Diamond(z_j - z_k) \tag{29}$$

are all positive semidefinite.

Proof We only sketch the proof the necessity of the condition $\Lambda \ge 0$. For a complete proof and computational details see [3] or [18]. One first remarks that the conditions $\Lambda \ge 0$. are equivalent to the positivity of the polynomials in $\lambda = (\cdots, \lambda_j, \cdots) \in \mathbb{C}^N$:

$$P_N(z_j, z_k, \lambda) = \sum_{1 \le j,k \le N} \lambda_j \lambda_k^* e^{-\frac{i\hbar}{2}\omega(z_j, z_k)} \rho_\Diamond(z_j - z_k) \ge 0 \tag{30}$$

for all choices of (z_j, z_k) and N. If $\hbar \ne 0$ (which we assume from now on) these conditions are in turns equivalent to

$$P_N'(z_j, z_k, \lambda) = \sum_{1 \le j,k \le N} \lambda_j \lambda_k^* e^{-\frac{i}{2\hbar}\omega(z_j, z_k')} \rho_\omega(z_j - z_k) \ge 0. \tag{31}$$

Suppose that ρ is the Wigner distribution of a density matrix; then ρ must be of the type (10). Then, by linearity, it is sufficient to assume that $\rho = W_\hbar\psi$ for some square integrable and absolutely integrable ψ. The necessity of the conditions $\Lambda \geq 0$ and $P'(\lambda) \geq 0$ the follows from the observation that

$$P'_N(z_j, z_k, \lambda) = \left(\frac{1}{2\pi\hbar}\right)^n \left\| \sum_{1\leq j,k\leq N} \widehat{T}_\hbar(z_j - z_k)\psi \right\|^2_{L^2(\mathbb{R}^n)} \qquad \square$$

hence $P'_N(z_j, z_k, \lambda) \geq 0$ as was to be proven.

6 The Narcowich–Wigner spectrum

As discussed in Sect. 2.3 a phase space (quasi-) probability can be the Wigner distribution of a density matrix or not, depending on the value attributed to Planck's constant. The Narcowich–Wigner spectrum is a way to characterize such values. It was introduced by Narcowich in [29,31] (he called it the "Wigner spectrum"), and has been further studied by Dias and Prata [4], Bröcker and Werner [2], and the present author [17]. By definition the Narcowich–Wigner spectrum of ρ is the set $\mathrm{NW}(\rho)$ of all numbers $\hbar' \geq 0$[1] such that

$$P^{\hbar'}_N(z_j, z_k, \lambda) = \sum_{1\leq j,k\leq N} \lambda_j\lambda_k^* e^{-\frac{i\hbar'}{2}\omega(z_j,z_k)} \rho_\Diamond(z_j - z_k) \geq 0 \tag{32}$$

for all choices of N, z_j, z_k, λ. It is thus the set of all values of Planck's constant for which the phase space function ρ is the Wigner distribution of a density matrix. Put a little bit more precisely: given a phase space (quasi-)distribution ρ satisfying the normalization condition (28), for which values of $\hbar'$ is the Weyl quantization (*cf*. (11))

$$\widehat{\rho}_\hbar\psi(x) = \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar'}p(x-y)} \rho\tfrac{1}{2}(x+y), p)\psi(y)dydp \tag{33}$$

a positive semi-definite operator (and hence a density matrix)? The set of all such values of $\hbar'$ is precisely the Narcowich–Wigner spectrum

In general the Narcowich–Wigner spectrum is generally quite complicated to calculate (*cf*. the examples in [2]). We however have a partial result when the density matrix has a covariance matrix (which requires that ρ decreases sufficiently fast at infinity):

Theorem 4 *(Narcowich–Wigner, I)Let ρ be a phase space quasi-distribution centered at z_0. Assume that the covariance matrix*

$$\Sigma = \int_{\mathbb{R}^{2n}} (z - z_0)(z - z_0)^T \rho(z)dz \tag{34}$$

and is positive definite. (i) Let $\lambda^\omega_{\min}$ be the smallest symplectic eigenvalue of Σ. We have

$$\mathrm{NW}(\rho) \subseteq [0, 2\lambda^\omega_{\min}]. \tag{35}$$

(ii) The Narcowich–Wigner spectrum NW ρ) *is a compact subset of* $\mathbb{R}$.*(iii) When ρ is a Gaussian*

$$\rho_\Sigma(z) = \frac{1}{(2\pi)^n\sqrt{\det\Sigma}} e^{-\frac{1}{2}\Sigma^{-1}(z-z_0)\cdot(z-z_0)} \tag{36}$$

[1] In some texts negative values of $\hbar'$ are allowed; this corresponds to a time reversal in Schrödinger's equation.

then the equality $\mathrm{NW}(\rho) = [0, 2\lambda^{\omega}_{\min}]$ *holds.*

Proof (i) Recall [15,22,25] that the symplectic eigenvalues of a symmetric positive definite matrix are the n numbers $\lambda^{\omega}_j > 0$ such that $\pm i\lambda^{\omega}_j$ is an eigenvalue of the antisymmetric matrix $\Sigma^{1/2} J\Sigma^{1/2}$ (they are thus those of $J\Sigma$). Let D be the diagonal matrix

$$D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \quad \Lambda = \mathrm{diag}(\lambda^{\omega}_1, ..., \lambda^{\omega}_n). \tag{37}$$

In view of Williamson's diagonalization theorem,wi36 (see [15,18,25] for proofs) there exists $S \in \mathrm{Sp}(n)$ such that $\Sigma = SDS^T$. The condition $\widehat{\rho}_\hbar \geq 0$ implies that we must have $\Sigma + \frac{i\hbar'}{2}J \geq 0$ that is $S\Sigma S^T + \frac{i\hbar'}{2}SJS^T \geq 0$. Since $SJS^T = J$ we must thus have $D + \frac{i\hbar'}{2}J \geq 0$. This condition is equivalent to $(\lambda^{\omega}_j)^2 - \hbar'^2/4 \geq 0$ for all $j = 1, ..., n$, that is too $2\lambda^{\omega}_{\min} \leq \hbar'$, proving the inclusion (35). (ii) That NW ρ) is compact follows from its closedness: let (h_m) be a sequence in NW ρ) converging towards a number $\hbar'$. The polynomials (30) are all nonnegative $P^{\hbar_m}_N(z_j, z_k, \lambda) \geq 0$ hence we also have

$$P^{\hbar'}_N(z_j, z_k, \lambda) = \lim_{m\to\infty} P^{\hbar_m}_N(z_j, z_k, \lambda) \geq 0 \geq 0$$

so that $\hbar' \in$ NW ρ) . (iii) The condition $\Sigma + \frac{i\hbar'}{2}J \geq 0$ is both necessary and sufficient for a Gaussian (36) to be the Wigner distribution of a density matrix, hence every value $\hbar' \in [-2\lambda^{\omega}_{\min}, 2\lambda^{\omega}_{\min}]$ is admissible. □

7 Planck's constant and symplectic capacities

Let us briefly recall some key concepts from symplectic topology (for a detailed review see our *Phys. Reps*. paper [22] with Luef). A (normalized) symplectic capacity on the symplectic phase space $(\mathbb{R}^{2n}, \omega)$ associates to every subset $\Omega \subset \mathbb{R}^{2n}$ a number $c(\Omega) \geq 0$ or $+\infty$,satisfying the following properties:

- SC1 *Monotonicity*: If $\Omega \subset \Omega'$ then $c(\Omega) \leq c(\Omega')$;
- SC2 *Conformality*: For every $\lambda \in \mathbb{R}$ we have $c(\lambda\Omega) = \lambda^2 c(\Omega)$;
- SC3 *Symplectic invariance*: $c(f(\Omega)) = c(\Omega)$ for every symplectomorphism (= canonical transformation f of $(\mathbb{R}^{2n}, \omega)$;
- SC4 *Normalization*:Let $B^{2n}(r)$ be the phase space ball $|z| \leq r$ and $Z^{2n}_j(r)$ is the cylinder with radius r and center 0 based on the x_j, p_j plane for $1 \leq j \leq n$. We have

$$c(B^{2n}(r)) = \pi r^2 = c(Z^{2n}_j(r)). \tag{38}$$

The explicit construction of a symplectic capacity is not an easy task, especially because of the normalization axiom (SC4) which shows that a symplectic capacity is very different from volume. Actually, the conformality axiom (SC2) seems to suggest that symplectic capacities behave like areas. or, equivalently, *action*. In fact, there exits a symplectic capacity c_{HZ} (the Hofer–Zehnder capacity [25]) having the following property: if Ω is a bounded convex subset of $\mathbb{R}^{2n}$ having a smooth boundary $\partial\Omega$ then

$$c_{\mathrm{HZ}}(\Omega) = \oint_{\gamma_{\min}} pdx = \oint_{\gamma_{\min}} p_1dx_1 + \cdots + p_ndx_n \tag{CHZ}$$

where $\gamma_{\min}$ is the shortest closed Hamiltonian orbit carried by the boundary $\partial\Omega$. In fact, the existence of symplectic capacity (actually an infinity of them) follows from a celebrated theorem due to Gromov [23]. He showed that no symplectomorphism can "squeeze" a phase space ball $B^{2n}(R)$ inside a cylinder $Z^{2n}_j(r)$ unless $R \leq r$. An equivalent formulation is that if you defoam a ball $B^{2n}(R)$ using symplectomorphisms, then the orthogonal projection of that deformed ball will still have area at least πR^2. These phenomena, widely know under the nickname "principle of the symplectic camel" at first sight seems to contradict Liouville's theorem on conservation of volume; however one should not forget that conservation of volume does not imply conservation of shape! Gromov's theorem shows that the formulas

$$c_{\min}(\Omega) = \sup_{f \text{ symplecto}} \{\pi r^2 : f(B^{2n}(r)) \subset \Omega\} \tag{39}$$

$$c_{\max}(\Omega) = \inf_{f \text{ symplecto.}} \{\pi r^2 : f(\Omega) \subset Z_j^{2n}(r)\} \tag{40}$$

define symplectic capacities and we have $c_{\min}(\Omega) \leq c(\Omega) \leq c_{\max}(\Omega)$ for all symplectic capacities c. Interpolating between $c_{\min}(\Omega)$ and $c_{\max}(\Omega)$ one again obtain (infinitely many) symplectic capacities (however the Hofer–Zehnder capacity $c_{\mathrm{HZ}}(\Omega)$ is *not* obtained by such an interpolation). Now, a striking fact (and it is the only we will actually need here) is that all symplectic capacities agree on phase space ellipsoids. hey are calculated as follows: assume that Ω is the covariance ellipsoid

$$\Omega_\Sigma = \{z \in \mathbb{R}^{2n} : \tfrac{1}{2}\Sigma^{-1}z \cdot z \leq 1\}.$$

We have [14,15,22]

$$c(\Omega_\Sigma) = \sup_{S \in \mathrm{Sp}(n)} \{\pi R^2 : S(B^{2n}(R)) \subset \Omega_\Sigma\} \tag{41}$$

which implies after some straightforward calculations that

$$c(\Omega_\Sigma) = 2\pi\lambda_{\min} \tag{42}$$

where $\lambda_{\min}$ is the smallest symplectic eigenvalue of Σ. Assume that Σ satisfies the quantum condition $\hbar' \leq 2\lambda^{\omega}_{\min}$; then (42) becomes $c(\Omega_\Sigma) \geq \pi\hbar$.

It turns out that the properties of the Narcowich–Wigner spectrum can easily be derived from those of symplectic capacities:

Theorem 5 *(Narcowich–Wigner, II) Let ρ be a phase space quasi-distribution centered at* 0. *Assume that ρ has a well-defined covariance matrix Σ and let Ω_Σ be the corresponding covariance ellipsoid. The Narcowich–Wigner spectrum of ρ is*

$$\mathrm{NW}(\rho) \subset [0, c(\Omega_\Sigma)/\pi] \tag{43}$$

for every symplectic capacity c on $(\mathbb{R}^{2n}, \omega)$; equivalently

$$\mathrm{NW}(\rho) \subset [0, \hbar_{\max}] \quad , \quad \hbar_{\max} = \frac{1}{\pi}\oint_{\gamma_{\min}} p dx \tag{44}$$

where $\gamma_{\min}$ is the shortest positively oriented periodic orbit carried by the boundary $\partial\Omega_\Sigma : \frac{1}{2}\Sigma^{-1}z \cdot z = 1$ of Ω_Σ.

Proof In [14,22] we proved the following topological characterization of the uncertainty principle in terms of symplectic capacities

$$\Sigma + \frac{i\hbar'}{2}J \quad \textit{is positive semidefinite} \iff c(\Omega_\Sigma) \geq \pi\hbar'. \tag{45}$$

In [16] we introduced the notion of "quantum blob" as minimum uncertainty phase space unit.. A quantum blob is the image of a phase space ball with radius $\sqrt{\hbar'}$ by a linear symplectic transformation $S \in \mathrm{Sp}(n)$. In view of formula (41) these conditions (45) are in turn equivalent to

$$\Omega_\Sigma \quad \textit{contains a quantum blob} S(B^{2n}(\sqrt{\hbar'})). \tag{46}$$

It follows from (41) we have $c(\Omega_\Sigma) \geq \pi$ for every symplectic capacity $c\ \hbar'$

The Narcowich–Wigner spectrum can thus be redefined as consisting of all $h > 0$ such that $\oint_\gamma pdx \geq \frac{1}{2}h$ for all positively oriented periodic orbits carried by Ω_Σ.

Appendix: What is the value of Planck's constant?

On 20 May 2019 the BIPM https://www.bipm.org/en/ redefined the SI unit of mass, the kilogram, by fixing *arbitrarily* the value of Planck's constant as being

$$h = 6.626070150 \times 10^{-34}\,\mathrm{J\,s} \tag{47}$$

(see the herbage [9] of the National Institute of Standards). This *ad hoc* choice was meant to make the kilogram fit with its best known values, leading to its redefinition

$$1\ \mathrm{kg} = \frac{h}{6.626070150 \times 10^{-34}}\ \mathrm{m}^{-2}\mathrm{s} \times \frac{\nu}{c^2} \tag{48}$$

where one has used the relation $mc^2 = h\nu$ relating energy, frequency, and mass. However, previous results from the task group on fundamental constants NIST [9] in 2014 yield the interval of confidence

$$h = 6.626070150(81) \times 10^{-34}\,\mathrm{J\,s} \tag{49}$$

justifying the middle choice (47), but more recent measurements, also performed at NIST, yield the *different* result

$$h = 6.626070040(81) \times 10^{-34}\,\mathrm{J\,s} \tag{50}$$

which would logically lead to a different choice for (47).

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Declarations

Conflict of interest There have been no conflicts of interest while doing this work.

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